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THE THEORY OF UD DERIVATIVE AND ITS APPLICATIONS

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ABSTRACT. The sole purpose of this study is to propose a new concept to fractional derivative as Ujlayan-Dixit derivative in the classical sense using limit approach. The parameter α of the derivative is confined within a closed unit interval. It is easy to apply and generates a convex combination of function and its derivative. We observe its geometrical behavior and see, how it is different from the previous definitions of the concerned derivatives. It works as an improved form of Conformable fractional derivative and fulfill the meaningful gaps. Some basic properties of calculus like continuity, differentiability, Mean Value theorem, Comparison theorem and some numerical have discussed.

Keywords: Fractional derivative, conformable derivative, fractional differential equations, UD derivative, fractional integral.

AMS Subject Classification: 26A33, 34A08.

1. INTRODUCTION

One's nature of understanding is calculus based in which the fractional order comes from the sense of the behaviour of a function at many parameters of small scale. In modern phenomenon, research on this core concept is important to understand the nature. In the beginning it was not very popular in science and engineering being a non local property. And so this theory was considered only for non-local distributed effects. But it gives a better way to understand the natural things or phenomenon by adding some other dimension.

Actually fractional calculus is a part of real analysis that studies all the aspects taking arbitrary powers of the differential operator. In the present scenario researchers are showing more interest to work in the field of fractional calculus as the generalization of ordinary calculus, they know that it is a challenging work equipped with a number of new results. To study the history of fractional calculus one may see Podlubny [2], Miller and Ross [3], Kilbas et al. [4], Machado et al. [6].

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Fractional calculus has now been applied in areas of science and engineering, finance, bio engineering, control theory, operation research etc.. There are a number of research articles-journals (see Bagley and Torvik [1] and Baleanu et al. [5]) in which problems have been addressed involving fractional derivatives/fractional integral like fluid flow, visco elasticity, diffusive transport properties, electrical circuits etc.. As shown by Caballero et al. [8], Goodrich [9], Fec et al. [10] and Rezazadeh et al. [13], we see that a mathematical formulation based on fractional derivative has presented and analytic and experimental results are compared.

From last some decade, many explanation of fractional derivative have been investigated as a concept for modelling real world problems. No doubt a fractional derivative is difficult to compute analytically so that the researchers sometimes have to switch over to numerical methods to solve the problems like in Li et al. [7], Odibat [11] and Bolandtalat et al. [12]. The popular fractional derivatives like Riemann-Liouville, Mittag- Leffler, Caputio, Grünwald-Letnikov fractional order derivative, are not always appropriate whenever we model real world problems as physical field complications arise. Still, these definitions is used for modelling the problems. R. Khalil et al. [14] and Katugampola [15] proposed the ideas of Conformable derivative in simple form in order to get easy computation and therefore, definitions are used for fractional derivative. Atagna et al. [17], Iyiola and Nwaeze [18] and Abdeljawad [19] investigated some results and properties of Conformable derivative. Hammad and Khalil [20], Ujlayan and Dixit [21-22] and Cenesiz and Kurt [23] have shown analytical solution of some known Conformable fractional differential equations. Guebbai anh Ghait [16] has presented their ideas for different conformable fractional derivative but the computation is not easy.

The proposed derivative has been obtained from an analytic approach and not from any physical phenomenon to find a fractional derivative in the classical sense. it is simple in computation and may give better predictions with the used methodology.

This paper is organized as follows: In **section 2**, we first present relevant definitions, properties of the derivatives, propositions and that will be used to prove our main results. In **section 3** we have discussed some important theorems based on fractional calculus. In **section 4**, we establish anti-derivative corresponding to the proposed derivative. In **section 5**, discuss examples to demonstrate the results of concerned physical phenomenon. Finally, **section 6** concludes this paper.

2. The UD derivative

Definition 2.1. *For a given function $f : [0, \infty) \to \mathbb{R}$ and $\alpha \in [0, 1]$, the UD derivative of order α is defined as*

$$D^\alpha f(x) = \lim_{\epsilon \to 0} \frac{e^{\epsilon(1-\alpha)} f(xe^{\frac{\epsilon\alpha}{x}}) - f(x)}{\epsilon}. \tag{1}$$

If this limit exists, then $D^\alpha f(x)$ is called the UD derivative of f for $\alpha \in [0, 1]$, with the understanding that $D^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha}$. Also, if f is UD differentiable in the interval $(0, x)$ for $x > 0$ and $\alpha \in [0, 1]$ such that $\lim_{x \to 0^+} f^\alpha(x)$ exist then,

$$f^\alpha(0) = \lim_{x \to 0^+} f^\alpha(x).$$

Theorem 2.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function and $\alpha \in [0, 1]$. Then, f is UD differentiable.*

Proof. By Definition 2.1, we have

$$\begin{aligned}
D^\alpha f(x) &= \lim_{\varepsilon\to 0} \frac{e^{\varepsilon(1-\alpha)} f(xe^{\frac{\varepsilon\alpha}{x}}) - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\{1+\varepsilon(1-\alpha)+o(\varepsilon^2)\}[f\{x+\varepsilon\alpha+o(\varepsilon^2)\}] - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\{1+\varepsilon(1-\alpha)\}[f(x)+f'(x)\{\varepsilon\alpha\}] - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{f(x)+\varepsilon(1-\alpha)f(x)+\varepsilon\alpha f'(x) - f(x)}{\varepsilon} \\
&= (1-\alpha)f(x) + \alpha f'(x), \qquad (2)
\end{aligned}$$

where $\alpha \in [0,1]$. □

Remark 2.1. *The UD derivatives of order α, $\alpha \in [0,1]$, of some elementary real-valued differentiable functions in $[0,\infty)$, can be given as following:*

(i) $D^\alpha(\lambda) = (1-\alpha)\lambda$ *for all constants* $\lambda \in \mathbb{R}$,
(ii) $D^\alpha((ax+b)^n) = (1-\alpha)(ax+b)^n + an\alpha(ax+b)^{n-1}$ *for all* $a,b \in \mathbb{R}$,
(iii) $D^\alpha(e^{ax+b}) = ((1-\alpha)+a\alpha)e^{ax+b}$ *for all* $a,b \in \mathbb{R}$,
(iv) $D^\alpha(\sin(ax+b)) = (1-\alpha)\sin(ax+b) + a\alpha\cos(ax+b)$ *for all* $a,b \in \mathbb{R}$,
(v) $D^\alpha(\cos(ax+b)) = (1-\alpha)\cos(ax+b) - a\alpha\sin(ax+b)$ *for all* $a,b \in \mathbb{R}$,
(vi) $D^\alpha(\log(ax+b)) = (1-\alpha)\log(ax+b) + a\alpha(ax+b)^{-1}$ *for all* $a,b \in \mathbb{R}$.

Theorem 2.2. *Let f and g be two differentiable functions in $[0,\infty)$ and $0 \le \alpha,\gamma \le 1$, then the following properties hold:*

(i) Linearity: $D^\alpha(af+bg) = aD^\alpha f + bD^\alpha g$ *for all* $a,b \in \mathbb{R}$.
(ii) Product rule: $D^\alpha(fg) = (D^\alpha f)g + \alpha(Dg)f$.
(iii) Quotient rule: $D^\alpha\left(\frac{f}{g}\right) = \frac{(D^\alpha f)g - \alpha(Dg)f}{g^2}$, *provided* $g(x) \neq 0$ *for all* $x \in [0,\infty)$.
(iv) Change of variable: Let f is a function of x, $0 \le x < \infty$, and x is function of t, $0 \le t < \infty$. Then,

$$D^\alpha f = (1-\alpha)f + \alpha\frac{df}{dx}\frac{dx}{dt}.$$

(v) Commutativity: $D^\alpha(D^\gamma)f = D^\gamma(D^\alpha)f$.

Proof. Using the equation (2), we get

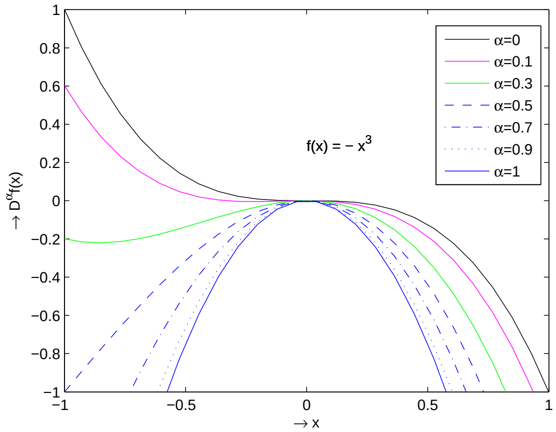
$$\begin{aligned}
D^\alpha(D^\gamma)f &= (1-\alpha)(1-\gamma)f + \alpha(1-\gamma)f' + \gamma(1-\alpha)f' + \alpha\gamma f'' \\
&= D^\gamma(D^\alpha)f.
\end{aligned}$$

This completes the proof of part (v) and the proof of the rest of the parts are obvious. □

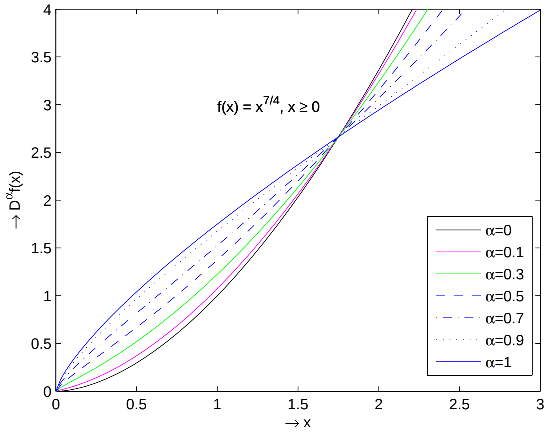
Remark 2.2. *The UD derivative of order α, $\alpha \in [0,1]$, as given in Definition 2.1, violets the Leibnitz's rule for fractional derivatives, $D^\alpha(fg) \neq gD^\alpha f + fD^\alpha g$. It also violets the law of indices, $D^\alpha(D^\gamma)f \neq D^{\alpha+\gamma}f$.*

Remark 2.3. *The equation* (2) *asserts that the UD derivative of order α, $\alpha \in [0,1]$, of a differentiable function $f : [0,\infty) \to \mathbb{R}$, is a convex combination of the function and the first derivative itself. Also, $D^\alpha f(x) = f(x)$, for $\alpha = 0$ and $D^\alpha f(x) = f'(x)$, for $\alpha = 1$, i.e., the UD derivative posses conformable property of conformable fractional derivatives.*

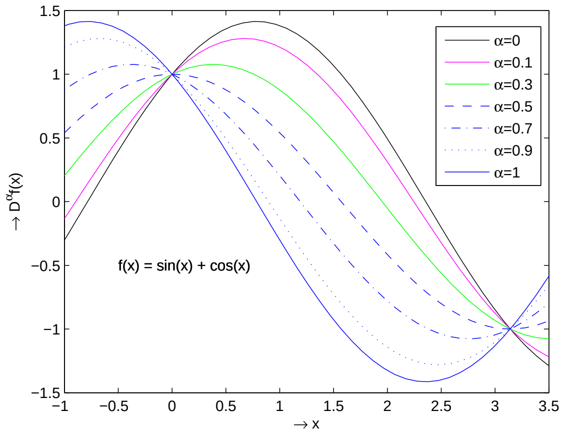
The geometrical interpretation of the UD derivative of order α, $\alpha \in [0,1]$, of some real-valued differentiable functions can be visualized as follows:



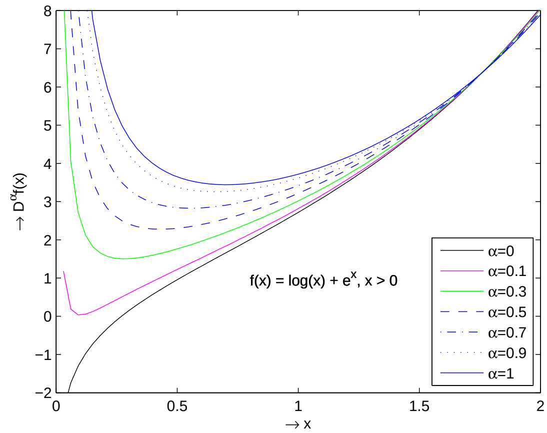
(A) D^α operating on $f(x) = -x^3$



(B) D^α operating on $f(x) = x^{7/4}, x \geq 0$



(C) D^α operating on $f(x) = \sin x + \cos x$



(D) D^α operating on $f(x) = \log x + e^x, x > 0$

FIGURE 1. For $\alpha \in [0, 1]$, the figures in this graph show that $D^\alpha f(x) = f(x)$, for $\alpha = 0$ and $D^\alpha f(x) = f'(x)$, for $\alpha = 1$. Also, $D^\alpha f(x)$ tends to $f'(x)$ uniformly as α tends from 0 to 1.

Definition 2.2. *Let $f : [0, \infty) \to \mathbb{R}$ is a n times differentiable function. Then, the UD derivative of of order α, $\alpha \in (n, n+1]$, is defined as*

$$D^\alpha f(x) = \lim_{\varepsilon \mapsto 0} \frac{e^{\varepsilon(1-\alpha)} f^{\lceil\alpha\rceil - 1}(x e^{\frac{\varepsilon\alpha}{x}}) - f^{\lceil\alpha\rceil-1}(x)}{\varepsilon},$$

where $\lceil\alpha\rceil$ represents the smallest integer greater than or equal to α.

3. Some additional properties of the UD derivative

Theorem 3.1. *Let the function f is not unbounded in $[0, \infty)$. If f is UD differentiable for some $\alpha \in [0, 1]$ at $x = a$, then f continuous at $x = a$.*

Proof. We show that $\lim\limits_{\epsilon\mapsto 0} f(x+\epsilon\alpha) = f(x)$.

$$\begin{aligned}
\lim_{\epsilon\mapsto 0} f(x+\epsilon\alpha) - f(x) &= \lim_{\epsilon\mapsto 0}\left(\frac{(1+\epsilon(1-\alpha))f(x+\epsilon\alpha) - \epsilon(1-\alpha)f(x+\epsilon\alpha) - f(x)}{\epsilon}\right)\epsilon \\
&= \lim_{\epsilon\mapsto 0}\left(\frac{(1+\epsilon(1-\alpha))f(x+\epsilon\alpha) - f(x)}{\epsilon}\right)\epsilon - \lim_{\epsilon\mapsto 0}\epsilon(1-\alpha)f(x+\epsilon\alpha) \\
&= \lim_{\epsilon\mapsto 0}\left(D^\alpha f\right)\epsilon - \lim_{\epsilon\mapsto 0}\epsilon(1-\alpha)f(x+\epsilon\alpha) \\
&= 0 \text{ (as } f \text{ is not unbounded for all } 0 \le x < \infty).
\end{aligned}$$

□

Theorem 3.2 (Rolle's theorem for the UD derivative)**.** *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ is a given function such that*

(a) f is continuous in $[a, b]$,
(b) f is differentiable in $]a, b[$,
(c) $f(a) = f(b)$.

Then, there exists a point $c \in\,]a, b[$, such that

$$D^\alpha f(c) = (1-\alpha)f(c),$$

where $\alpha \in [0, 1]$.

Proof. As we know that $D^\alpha(c) = (1-\alpha)f(c) + \alpha f'(c)$ and from classical Rolle's theorem of f, $f'(c) = 0$ implies $D^\alpha f(c) = (1-\alpha)f(c)$. □

Theorem 3.3 (Mean Value theorem for the UD derivative)**.** *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ is a given function such that*

(a) f is continuous in $[a, b]$,
(b) f is differentiable in $]a, b[$.

Then, there exists a point $c \in\,]a, b[$, such that

$$D^\alpha f(c) = (1-\alpha)f(c) + \alpha\frac{f(b)-f(a)}{b-a},$$

where $\alpha \in [0, 1]$.

Proof. Result follows from classical Lagrange's mean value theorem of f. □

Theorem 3.4. *Let f,g are two functions such that*

(a) f, g are continuous in $[a, b]$,
(b) f,g are UD differentiable in $]a, b[$ for $0 \le \alpha \le 1$,
(c) $D^\alpha f(x) = D^\alpha g(x)$ for all x in (a, b) and $0 < \alpha \le 1$.

Then,

$$(f-g)(x) = \eta e^{\frac{(\alpha-1)x}{\alpha}},$$

where η is a constant.

Proof. Let $y(x) = f(x) - g(x)$ for all $x \in (a, b)$, then

$$\begin{aligned}
& D^\alpha(f-g)(x) = 0 \\
\Rightarrow\ & D^\alpha y = 0 \\
\Rightarrow\ & y(x) + \alpha D y(x) = 0 \\
\Rightarrow\ & y(x) = \eta e^{\frac{(\alpha-1)x}{\alpha}}.
\end{aligned}$$

□

Corollary 3.1. *If $D^\alpha f(x) = 0$, then f is not constant, infact $f(x) = \eta e^{\frac{(\alpha-1)}{\alpha}x}$.*

Theorem 3.5. *Let y is UD differentiable for $\alpha \in (0, 1]$ and $p(x), q(x)$ are continuous functions in $x \in (a, b)$. Then, the initial value problem*

$$D^\alpha y + p(x)y = q(x); \quad y(x_0) = y_0 \tag{3}$$

has unique solution in the interval (a, b) for $x_0 \in (a, b)$.

Proof. Using the equation (2), the equation (3) can be written as:

$$\alpha y' + (p(x) + (1 - \alpha))y = q(x)$$

and result follows from the Uniqueness theorem of ordinary differential equations. □

Theorem 3.6. *Let y_1, y_2 be two linearly independent solutions of*

$$D^\alpha D^\alpha y(x) + p(x)D^\alpha y(x) + q(x)y(x) = 0, \tag{4}$$

where $p(x), q(x)$ are continuous functions on (a, b) and $\alpha \in (0, 1]$. Then y_1 has a zero between any two consecutive zeroes of y_2. that is, zeros of y_1, y_2 occur alternately.

Proof. Using the equation (2), the equation (4) can be written as

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0,$$

where $P(x) = \frac{1}{\alpha}(2(1-\alpha) + p(x))$, $Q(x) = \frac{1}{\alpha^2}((1-\alpha)^2 + (1-\alpha)p(x) + q(x))$ and result follows from Sturm separation theorem of ordinary differential equations. □

4. The UD integral or anti-UD derivative of order α

In this section, we define a UD integral of order α, $\alpha \in (0, 1]$, which is an inverse operator of the proposed UD derivative as defined in Definition 2.1.

Let g be a UD differentiable function as needed and $0 < \alpha \le 1$. Then,

$$\begin{aligned}
&D^\alpha g(x) = f(x), \text{ where } D^\alpha \equiv \frac{d^\alpha}{dx^\alpha} \\
\Rightarrow\ &(1-\alpha)g(x) + \alpha D(g(x)) = f(x); \text{ where } D \equiv \frac{d}{dx} \\
\Rightarrow\ &\frac{dg(x)}{dx} + \frac{(1-\alpha)}{\alpha}g(x) = \frac{1}{\alpha}f(x) \\
\Rightarrow\ &g(x) = \frac{1}{\alpha}e^{\frac{(\alpha-1)}{\alpha}x}\int f(x).e^{\frac{(1-\alpha)}{\alpha}x}dx + Ce^{\frac{(\alpha-1)}{\alpha}x},
\end{aligned}$$

where C is constant.
Also $g(x) = I^\alpha(f(x))$, is called anti-UD derivative of $f(x)$ for $\alpha \in (0, 1]$. One may verify that, at $\alpha = 1$, this integral coincides with the classical integral.

Definition 4.1. *Let f be a continuous function in $[a, b]$. The UD integral, $I_a^\alpha f$, is defined as follows:*

$$I_a^\alpha f(x) = \frac{1}{\alpha}\int_a^x e^{\frac{(1-\alpha)}{\alpha}(t-x)}f(t)dt, \quad \textit{where } \alpha \in (0, 1].$$

5. Numerical Problems

In this section, it is assumed that $D^\alpha \equiv \frac{d^\alpha}{dx^\alpha}$, the functions involved in the considered fractional differential equations are all differentiable in $[0,\infty)$, range of the fractional order lies in $(0,1]$, and the UD derivative is used as given by the equation (2).

Example 5.1. *Consider the fractional differential equation,*

$$D^\gamma(D^\alpha)y(x) = 0. \tag{5}$$

Using UD derivative, the equation (5) *can be written as,*

$$\alpha\gamma y''(x) + (\alpha+\gamma-2\alpha\gamma)y'(x) + (1-\alpha)(1-\gamma)y(x) = 0.$$

This yields the solution,

$$y(x) = Ae^{\left(\frac{\alpha-1}{\alpha}\right)x} + Be^{\left(\frac{\gamma-1}{\gamma}\right)x}, \tag{6}$$

where A, B are arbitrary constants.
It should be noted that solution (6) *depends on parameter α only (when $x \neq 0$) if one use Conformable derivative (see [15],[16]).*

Example 5.2. *Consider the fractional differential equation,*

$$D^{1/2}y(x) + y(x) = xe^{-3x}. \tag{7}$$

Using Conformable derivative, $D^{1/2}y(x) = x^{1/2}\frac{dy(x)}{dx}$, the equation (7) *can be written as,*

$$\frac{dy(x)}{dx} + \frac{1}{\sqrt{x}}y(x) = \sqrt{x}e^{-3x}$$

having the solution as an integral problem

$$y(x) = e^{-2\sqrt{x}}\left(\int \sqrt{x}e^{2\sqrt{x}}e^{-3x}dx + k_1\right), \tag{8}$$

where k_1 is an arbitrary constant.
But with UD derivative, the equation (7) *can be written as,*

$$\frac{dy(x)}{dx} + 3y(x) = 2xe^{-3x}$$

and solution in closed form is obtained as

$$y(x) = k_2e^{-3x} + x^2e^{-3x}, \tag{9}$$

where k_2 is an arbitrary constant.

Example 5.3. *The relaxed equation in fractional space is described by the equation*

$$\frac{d^\alpha y(t)}{dt^\alpha} + c^\alpha y(t) = 0, \tag{10}$$

where $c > 0, t > 0$ and $0 < \alpha < 1$ has the solution,

$$y(t) = E_\alpha(-c^\alpha t^\alpha) \tag{11}$$

in terms of Mittag-Leffler function.
And, using Conformable derivative, the solution of the fractional differential equation (10) *is given as*

$$y(t) = Aexp\left(-\frac{(ct)^\alpha}{\alpha}\right) \tag{12}$$

where A is an arbitrary constant.
Again, using the proposed UD derivative, the solution of (10) *is given as*

$$y(t) = Bexp\left(\frac{(\alpha - 1) - c^{\alpha}}{\alpha}\right)t, \tag{13}$$

where B is an arbitrary constant.
Above all the obtained solutions coincide at $\alpha = 1$.

Example 5.4. *Using UD derivative, the fractional initial value problem for damped simple harmonic oscillator*

$$\frac{d^2y(t)}{dt^2} + b\frac{d^{\alpha}y(t)}{dt^{\alpha}} + \omega_0^2 y(t) = f(t);\ 0 < \alpha < 1, t > 0, with$$

$$y(0) = c_0,\ \frac{d^{\alpha}y(0)}{dt^{\alpha}} = c_1$$

may be wtitten as,

$$\frac{d^2y(t)}{dt^2} + \lambda\frac{dy(t)}{dt} + \mu y(t) = f(t).$$

In which complementary function of the solution will be

$$e^{-\lambda t/2}\left(A\sin\sqrt{\mu - \frac{\lambda^2}{4}}\ t + B\cos\sqrt{\mu - \frac{\lambda^2}{4}}\ t\right),$$

where $\lambda = \alpha b$ *and* $\mu = b(1-\alpha) + \omega_0^2$. *As well, the particular integral and the values of arbitrary constants* A, B *can be find using initial conditions when* $f(t)$ *is known.*

6. Conclusion

We have a different idea to deal with fractional derivative as UD derivative to solve the related problems. A numerical method or series solution is used in the absence of an appropriate analytic method, which raises an error in the result. Present work is analytic thoroughly and generalizes the ordinary results. The novelty of the work reflects from the methodology and examples. One may obtain UD derivative of a function defined on interval $[a, b]; a, b \in \mathbb{R}$.

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