On the average value of the least common multiple of k positive integers

Article

Accepted Version

Hilberdink, T. and Tóth, L. (2016) On the average value of the least common multiple of k positive integers. Journal of Number Theory, 169. pp. 327-341. ISSN 0022-314X doi: https://doi.org/10.1016/j.jnt.2016.05.024 Available at http://centaur.reading.ac.uk/66057/

On the average value of the least common multiple of k positive integers

Titus Hilberdink

Department of Mathematics, University of Reading
Whiteknights, PO Box 220, Reading RG6 6AX, UK
E-mail: t.w.hilberdink@reading.ac.uk

László Tóth

Department of Mathematics, University of Pécs
Ifjúság útja 6, H-7624 Pécs, Hungary
E-mail: ltoth@gamma.ttk.pte.hu

Abstract

We deduce an asymptotic formula with error term for the sum $\sum_{n_1,\ldots,n_k\le x} f([n_1,\ldots,n_k])$, where $[n_1,\ldots,n_k]$ stands for the least common multiple of the positive integers $n_1,\ldots,n_k$ ($k\ge 2$) and f belongs to a large class of multiplicative arithmetic functions, including, among others, the functions $f(n)=n^r$, $\varphi(n)^r$, $\sigma(n)^r$ ($r>-1$ real), where φ is Euler's totient function and σ is the sum-of-divisors function. The proof is by elementary arguments, using the extension of the convolution method for arithmetic functions of several variables, starting with the observation that given a multiplicative function f, the function of k variables $f([n_1,\ldots,n_k])$ is multiplicative.

Keywords: greatest common divisor, least common multiple, arithmetic function of several variables, multiplicative function, Dirichlet series, asymptotic formula
2000 MSC: 11A05, 11A25, 11N37

1. Introduction

We use the following notation: $\mathbb{N}=\{1,2,\ldots\}$, $*$ is the Dirichlet convolution of arithmetic functions, id_r ($r\in\mathbb{R}$) is the function $\mathrm{id}_r(n)=n^r$ ($n\in\mathbb{N}$), $\mathbf{1}=\mathrm{id}_0$, $\mathrm{id}=\mathrm{id}_1$, μ denotes the Möbius function, λ is the Liouville function,

$\sigma_r = \mathbf{1} * \mathrm{id}_r$, $\sigma = \sigma_1$ is the sum-of-divisors function, $\tau = \sigma_0$ is the divisor function, $\beta_r = \lambda * \mathrm{id}_r$, $\beta = \beta_1$ is the alternating sum-of-divisors function (cf. [19]), $\varphi_r = \mu * \mathrm{id}_r$ is the generalized Euler function, $\varphi = \varphi_1$ is Euler's totient function, $\psi_r = \mu^2 * \mathrm{id}_r$ is the generalized Dedekind function, $\psi = \psi_1$ is the classical Dedekind function. If $n \in \mathbb{N}$, then $n = \prod_p p^{\nu_p(n)}$ is its prime power factorization, the product being over the primes p, where all but a finite number of the exponents $\nu_p(n)$ are zero.

Furthermore, let $(n_1, \ldots, n_k)$ and $[n_1, \ldots, n_k]$ denote the greatest common divisor (gcd) and the least common multiple (lcm) of $n_1, \ldots, n_k \in \mathbb{N}$ ($k \geq 2$), respectively.

It is easy to see that for any arithmetic function f we have the identity

$$\sum_{n_1,\ldots,n_k \leq x} f((n_1, \ldots, n_k)) = \sum_{d \leq x} (\mu * f)(d) \left\lfloor \frac{x}{d} \right\rfloor^k, \tag{1}$$

which leads to asymptotic formulas for this sum. For example, if $f = \mathrm{id}$ and $k \geq 3$, then we have

$$\sum_{n_1,\ldots,n_k \leq x} (n_1, \ldots, n_k) = \frac{\zeta(k-1)}{\zeta(k)} x^k + O(R_k(x)), \tag{2}$$

where $R_3(x) = x^2 \log x$ and $R_k(x) = x^{k-1}$ for $k \geq 4$. The case $f = \mathrm{id}$, $k = 2$ can be treated separately by writing

$$\sum_{m,n \leq x} (m,n) = 2 \sum_{m \leq n \leq x} (m,n) - \sum_{n \leq x} n$$

$$= 2 \sum_{n \leq x} (\mu * \mathrm{id}\, \tau)(n) - \frac{x^2}{2} + O(x),$$

giving, by using elementary arguments, the formula

$$\sum_{m,n \leq x} (m,n) = \frac{x^2}{\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O(x^{1+\theta+\varepsilon}), \tag{3}$$

valid for every $\varepsilon > 0$, where γ is Euler's constant and θ is the exponent appearing in Dirichlet's divisor problem.

For the lcm of k positive integers there is no formula similar to (1). However, in the case $k = 2$, the lcm of the integers $m, n \in \mathbb{N}$ can be written using their gcd as $[m,n] = mn/(m,n)$, which enables to establish the following asymptotic formula, valid for any positive real number r:

$$\sum_{m,n\le x} [m,n]^r = \frac{\zeta(r+2)}{\zeta(2)} \cdot \frac{x^{2(r+1)}}{(r+1)^2} + O(x^{2r+1}\log x). \tag{4}$$

If $r \in \mathbb{N}$, then the error term in (4) can be improved into $O(x^{2r+1}(\log x)^{2/3}(\log\log x)^{4/3})$, which is a consequence of the result of Walfisz [23, Satz 1, p. 144] for $\sum_{n\le x}\varphi(n)$.

For $k = 2$ the asymptotic formulas concerning $\sum_{m,n\le x}(m,n)^r$ and $\sum_{m,n\le x}[m,n]^r$ are equivalent to those for $\sum_{n\le x} g_r(n)$ and $\sum_{n\le x}\ell_r(n)$, respectively, where $g_r(n) = \sum_{1\le j\le n}(j,n)^r$ is the gcd-sum function and $\ell_r(n) = \sum_{1\le j\le n}[j,n]^r$ is the lcm-sum function. The function $g_1(n) = \sum_{1\le j\le n}(j,n)$, investigated by S. S. Pillai [16], is also called Pillai's function in the literature.

The above and related results go back, in chronological order, to the work of E. Cesàro [6], E. Cohen [9, 10, 11], K. Alladi [1], P. Diaconis and P. Erdős [12], J. Chidambaraswamy and R. Sitaramachandrarao [7], K. A. Broughan [5], O. Bordellès [2, 3, 4], Y. Tanigawa and W. Zhai [17], S. Ikeda and K. Matsuoka [15], and others.

For example, formula (3) with the weaker error $O(x^{3/2}\log x)$ was given in [12, Th. 2, Eq. (1.4)] and was recovered in [5, Th. 4.7]. Formula (3) with the above error term was established in [7, Th. 3.1] and recovered in [2, Th. 1.1] (in both papers for Pillai's function). Formula (4) was established in [12, Th. 2, Eq. (1.6)]. The better error term for (4) in the case $r \in \mathbb{N}$ was obtained in [15, Th. 2]. Asymptotic formulas for (1) in the case $k = 2$ and for various choices of the function f, including $f = \sigma$ and $f = \varphi$ were deduced in [4, 9, 10, 11]. See also the survey paper [18].

The result

$$\sum_{m,n,q\le x} [m,n,q]^r \sim c_r \frac{x^{3(r+1)}}{(r+1)^3} \quad (x\to\infty),$$

valid for $r \in \mathbb{N}$, without any error term and with a computable constant c_r given in an implicit form, was obtained by J. L. Fernández and P. Fernández [13, Th. 3(b)]. Their proof is by an ingenious method based on the identity $[m,n,q](m,n)(m,q)(n,q) = mnq(m,n,q)$ $(m,n,q \in \mathbb{N})$ and using the dominated convergence theorem. As far as we know, there are no other asymptotic results in the literature for the sum

$$\sum_{n_1,\ldots,n_k\le x} f([n_1,\ldots,n_k]), \tag{5}$$

in the case $k \geq 3$, where f is an arithmetic function. It seems that the method of [13] can not be extended for $k \geq 3$, even in the case $f = \mathrm{id}_r$. Also, it is not possible to reduce the estimation of the sum (5) to sums of a single variable, like in (1).

In this paper we deduce an asymptotic formula with remainder term for the sum (5), where $k \geq 2$ and f belongs to a large class of multiplicative arithmetic functions, including the functions id_r with $r > -1$ real and σ_r, β_r, φ_r, ψ_r with $r \geq 1/2$ real. The proof is by elementary arguments, using the extension of the convolution method for arithmetic functions of several variables starting with the observation that given a multiplicative function f, the function of k variables $f([n_1, \ldots, n_k])$ is multiplicative and the associated multiple Dirichlet series factorizes as an Euler product. The same method was used by the second author [21] for a different problem. See the survey paper [20] of the second author for basic properties of multiplicative functions of several variables and related convolutions.

We also extend to the k dimensional case the formula

$$\sum_{m,n \leq x} \frac{[m,n]}{(m,n)} = \frac{\pi^2}{60} x^4 + O(x^3 \log x), \tag{6}$$

which can be obtained in a similar manner to the results (2) and (4). Properties of the operation $m \circ n = [m,n]/(m,n)$ were investigated by the first author [14].

Note that the following recent result of different type, concerning the lcm of several positive integers, was obtained by J. Cilleruelo, J. Rué, P. Šarka and A. Zumalacárregui [8]: $\mathrm{lcm}\{a : a \in A\} = 2^{n(1+o(1))}$ for almost all subsets $A \subset \{1, \ldots, n\}$.

2. Main results

Let $r \in \mathbb{R}$ be a fixed number. Let $\mathcal{A}_r$ denote the class of complex valued multiplicative arithmetic functions satisfying the following properties: there exist real constants C_1, C_2 such that

$$|f(p) - p^r| \leq C_1 p^{r-1/2} \quad \text{for every prime } p, \tag{i}$$

and

$$|f(p^\nu)| \leq C_2 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \geq 2. \tag{ii}$$

Note that conditions (i) and (ii) imply that

$$|f(p^\nu)| \le C_3 p^{\nu r} \quad \text{for every prime power } p^\nu \text{ with } \nu \ge 1, \tag{iii}$$

where $C_3 = \max(C_1+1, C_2)$.

Observe that $\mathrm{id}_r \in \mathcal{A}_r$ for every $r \in \mathbb{R}$, while σ_r, β_r, φ_r, $\psi_r \in \mathcal{A}_r$ for every $r \in \mathbb{R}$ with $r \ge 1/2$. The functions $f(n) = \sigma(n)^r, \beta(n)^r, \varphi(n)^r, \psi(n)^r$ also belong to the class $\mathcal{A}_r$ for every $r \in \mathbb{R}$. As other examples of functions in the class $\mathcal{A}_r$, with $r \in \mathbb{R}$, we mention $\varphi^*(n)^r$, $\sigma^*(n)^r$ and $\sigma^{(e)}(n)^r$, where $\varphi^*(n) = \prod_{p\mid n}\left(p^{\nu_p(n)}-1\right)$ is the unitary Euler totient, $\sigma^*(n) = \prod_{p\mid n}\left(p^{\nu_p(n)}+1\right)$ is the sum-of-unitary-divisors function and $\sigma^{(e)}(n) = \prod_{p\mid n}\sum_{d\mid \nu_p(n)} p^d$ denotes the sum of exponential divisors of n. Furthermore, if f is a bounded multiplicative function such that $f(p) = 1$ for every prime p, then $f \in \mathcal{A}_0$. In particular, $\mu^2 \in \mathcal{A}_0$.

We prove the following results.

Theorem 2.1. *Let $k \ge 2$ be a fixed integer and let $f \in \mathcal{A}_r$ be a function, where $r > -1$ is real. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \le x} f([n_1,\ldots,n_k]) = C_{f,k}\frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right), \tag{7}$$

and

$$\sum_{n_1,\ldots,n_k \le x} \frac{f([n_1,\ldots,n_k])}{(n_1\cdots n_k)^r} = C_{f,k}x^k + O\left(x^{k-\frac{1}{2}\min(r+1,1)+\varepsilon}\right), \tag{8}$$

where

$$C_{f,k} = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.$$

Formula (7) shows that the average order of $f([n_1,\ldots,n_k])$ is $C_{f,k}(n_1\cdots n_k)^r$, in the sense that

$$\sum_{n_1,\ldots,n_k \le x} f([n_1,\ldots,n_k]) \sim \sum_{n_1,\ldots,n_k \le x} C_{f,k}(n_1\cdots n_k)^r \quad (x \to \infty).$$

From (8) we deduce that

$$\lim_{x\to\infty}\frac{1}{x^k}\sum_{n_1,\ldots,n_k \le x} \frac{f([n_1,\ldots,n_k])}{(n_1\cdots n_k)^r} = C_{f,k},$$

representing the mean value of the function $f([n_1,\ldots,n_k])/(n_1\cdots n_k)^r$. See N. Ushiroya [22, Th. 4] and the second author [20, Prop. 19] for general results on mean values of multiplicative arithmetic functions of several variables.

Theorem 2.2. *Let $k\ge 2$ be a fixed integer and let $f\in\mathcal{A}_r$ be a function, where $r\ge 0$ is real. Then for every $\varepsilon>0$,*

$$\sum_{n_1,\ldots,n_k\le x} f\left(\frac{[n_1,\ldots,n_k]}{(n_1,\ldots,n_k)}\right) = D_{f,k}\frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}+\varepsilon}\right), \qquad (9)$$

where

$$D_{f,k} = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k)})}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.$$

In the case $f=\mathrm{id}_r$ we obtain from Theorem 2.1 the next result:

Corollary 1. *Let $k\ge 3$ and $r>-1$ be a real number. Then for every $\varepsilon>0$,*

$$\sum_{n_1,\ldots,n_k\le x} [n_1,\ldots,n_k]^r = C_{r,k}\frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right), \qquad (10)$$

and

$$\sum_{n_1,\ldots,n_k\le x} \left(\frac{[n_1,\ldots,n_k]}{n_1\cdots n_k}\right)^r = C_{r,k}x^k + O\left(x^{k-\frac{1}{2}\min(r+1,1)+\varepsilon}\right),$$

where

$$C_{r,k} = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{p^{r\max(\nu_1,\ldots,\nu_k)}}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.$$

In particular,

$$C_{r,3} = \zeta(r+2)\zeta(2r+3)\prod_p \left(1-\frac{3}{p^2}+\frac{2}{p^3}+\frac{2}{p^{r+2}}-\frac{3}{p^{r+3}}+\frac{1}{p^{r+5}}\right), \qquad (11)$$

$$C_{r,4} = \zeta(r+2)\zeta(2r+3)\zeta(3r+4)\prod_p \left(1-\frac{6}{p^2}+\frac{8}{p^3}-\frac{3}{p^4}+\frac{5}{p^{r+2}}-\frac{12}{p^{r+3}}+\frac{6}{p^{r+4}}+\frac{4}{p^{r+5}}\right.$$
$$\left.-\frac{3}{p^{r+6}}+\frac{3}{p^{2r+3}}-\frac{4}{p^{2r+4}}-\frac{6}{p^{2r+5}}+\frac{12}{p^{2r+6}}-\frac{5}{p^{2r+7}}+\frac{3}{p^{3r+5}}-\frac{8}{p^{3r+6}}+\frac{6}{p^{3r+7}}-\frac{1}{p^{3r+9}}\right). \qquad (12)$$

In the case $f = \mathrm{id}_r$ we deduce from Theorem 2.2:

Corollary 2. *Let $k \geq 3$ and $r > 0$ be a real number. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \leq x} \left(\frac{[n_1,\ldots,n_k]}{(n_1,\ldots,n_k)} \right)^r = D_{r,k} \frac{x^{k(r+1)}}{(r+1)^k} + O\left(x^{k(r+1)-\frac{1}{2}+\varepsilon} \right), \tag{13}$$

where

$$D_{r,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{p^{r(\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k))}}{p^{(r+1)(\nu_1+\cdots+\nu_k)}}.$$

In particular,

$$D_{r,3} = C_{r,3} \frac{\zeta(3r+3)}{\zeta(2r+3)}, \quad D_{r,4} = C_{r,4} \frac{\zeta(4r+4)}{\zeta(3r+4)}.$$

We remark that in the case $k = 2$ asymptotic formulas (10) and (13) reduce to (4) and (6) (case $r = 1$), respectively, but the latter ones have better error terms. Note that $D_{r,2} = \zeta(2r+2)/\zeta(2)$.

Among other special cases we consider the functions $\sigma, \varphi \in \mathcal{A}_1$ and $\mu^2 \in \mathcal{A}_0$.

Corollary 3. *Let $k \geq 2$. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \leq x} \sigma([n_1,\ldots,n_k]) = C_{\sigma,k} \frac{x^{2k}}{2^k} + O\left(x^{2k-1/2+\varepsilon} \right),$$

and

$$\sum_{n_1,\ldots,n_k \leq x} \frac{\sigma([n_1,\ldots,n_k])}{n_1 \cdots n_k} = C_{\sigma,k} x^k + O\left(x^{k-1/2+\varepsilon} \right),$$

where

$$C_{\sigma,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{\sigma(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{2(\nu_1+\cdots+\nu_k)}}.$$

In particular,

$$C_{\sigma,2} = \zeta(3)\zeta(4) \prod_p \left(1 + \frac{1}{p^2} - \frac{2}{p^3} - \frac{2}{p^5} + \frac{2}{p^6}\right).$$

Corollary 4. *Let $k \geq 2$. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \leq x} \varphi([n_1,\ldots,n_k]) = C_{\varphi,k} \frac{x^{2k}}{2^k} + O\left(x^{2k-1/2+\varepsilon}\right),$$

and

$$\sum_{n_1,\ldots,n_k \leq x} \frac{\varphi([n_1,\ldots,n_k])}{n_1 \cdots n_k} = C_{\varphi,k} x^k + O\left(x^{k-1/2+\varepsilon}\right),$$

where

$$C_{\varphi,k} = \prod_p \left(1 - \frac{1}{p}\right)^k \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{\varphi(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{2(\nu_1+\cdots+\nu_k)}}.$$

In particular,

$$C_{\varphi,2} = \zeta(3) \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3} - \frac{1}{p^4} + \frac{2}{p^5} - \frac{1}{p^6}\right).$$

Corollary 5. *Let $k \geq 2$. Then for every $\varepsilon > 0$,*

$$\sum_{n_1,\ldots,n_k \leq x} \mu^2([n_1,\ldots,n_k]) = \frac{x^k}{\zeta(2)^k} + O\left(x^{k-1/2+\varepsilon}\right).$$

Remark 1. It would be interesting to find the best possible error, especially in particular cases. For example, for $r = 1$ in Corollary 1, the relative error is $O(x^{-1/2+\epsilon})$. Can we improve the exponent further and if so, by how much?

3. Proofs

An arithmetic function g of k variables is called multiplicative if

$$g(m_1 n_1, \ldots, m_k n_k) = g(m_1, \ldots, m_k) g(n_1, \ldots, n_k),$$

provided that $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. Hence

$$g(n_1, \ldots, n_k) = \prod_p g\left(p^{\nu_p(n_1)}, \ldots, p^{\nu_p(n_k)}\right)$$

for every $n_1, \ldots, n_k \in \mathbb{N}$. In this case the multiple Dirichlet series of the function g can be expanded into an Euler product:

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{g(n_1,\ldots,n_k)}{n_1^{z_1} \cdots n_k^{z_k}} = \prod_p \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{g(p^{\nu_1},\ldots,p^{\nu_k})}{p^{\nu_1 z_1 + \cdots + \nu_k z_k}}.$$

We need the following lemmas.

Lemma 3.1. *If $k \ge 2$ and $f \in \mathcal{A}_r$ with $r > -1$ real, then*

$$L_{f,k}(z_1,\ldots,z_k) := \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f([n_1,\ldots,n_k])}{n_1^{z_1}\cdots n_k^{z_k}} = \zeta(z_1-r)\cdots\zeta(z_k-r)H_{f,k}(z_1,\ldots,z_k),$$

where the multiple Dirichlet series $H_{f,k}(z_1,\ldots,z_k)$ is absolutely convergent for

$$\Re z_1,\ldots,\Re z_k > A := \begin{cases} r+\frac{1}{2}, & \text{if } r \ge 0, \\ \frac{r+1}{2}, & \text{if } -1 < r < 0. \end{cases} \tag{14}$$

Proof. If f is a multiplicative function of a single variable, then the arithmetic function of k variables $f([n_1,\ldots,n_k])$ is multiplicative. It follows that

$$L_{f,k}(z_1,\ldots,z_k) = \prod_p \sum_{\nu_1,\ldots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}} \tag{15}$$

Case I. Assume that $r \ge 0$. Grouping the terms of the sum in (15) according to the values $\nu_1+\cdots+\nu_k$ we have

$$L_{f,k}(z_1,\ldots,z_k) = \prod_p \left(1 + \frac{f(p)}{p^{z_1}} + \cdots + \frac{f(p)}{p^{z_k}} + \sum_{\nu_1+\cdots+\nu_k\ge 2} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}\right). \tag{16}$$

Let $\Re z_1,\ldots,\Re z_k \ge \delta > r$. By using condition (i) from the definition of the class $\mathcal{A}_r$,

$$\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) \quad (1 \le j \le k).$$

Also, by condition (iii) following the definition of the class $\mathcal{A}_r$ and by using that $r \ge 0$ we deduce that

$$\left|\frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}\right| \le C_3 \frac{p^{r\max(\nu_1,\ldots,\nu_k)}}{p^{\delta(\nu_1+\cdots+\nu_k)}} \le C_3 \frac{1}{p^{(\delta-r)(\nu_1+\cdots+\nu_k)}}.$$

Thus the sum in (16) over $\nu_1+\cdots+\nu_k \ge 2$ is $O\left(p^{-2(\delta-r)}\right)$. We obtain

$$L_{f,k}(z_1,\ldots,z_k)\zeta^{-1}(z_1-r)\cdots\zeta^{-1}(z_k-r)$$

$$= \prod_p \left(1-\frac{1}{p^{z_1-r}}\right)\cdots\left(1-\frac{1}{p^{z_k-r}}\right)\left(1+\frac{1}{p^{z_1-r}}+\cdots+\frac{1}{p^{z_k-r}}+O\left(\frac{1}{p^{\delta-r+1/2}}\right)\right)$$

$$+O\left(\frac{1}{p^{2(\delta-r)}}\right)\Bigg) = \prod_p \left(1+O\left(\frac{1}{p^{\delta-r+1/2}}\right)+O\left(\frac{1}{p^{2(\delta-r)}}\right)\right),$$

since $\Re z_j \ge \delta$ $(1 \le j \le k)$, where the terms $\pm\frac{1}{p^{z_j-r}}$ $(1 \le j \le k)$ cancel out. Here the latter product converges absolutely when $\delta - r + 1/2 > 1$ and $2(\delta - r) > 1$, that is, for $\delta > r + 1/2$.

Case II. Assume that $-1 < r < 0$. Now we group the terms of the sum in (15) according to the values $\max(\nu_1, \ldots, \nu_k)$:

$$L_{f,k}(z_1,\ldots,z_k) = \prod_p \left(1 + \sum_{\max(\nu_1,\ldots,\nu_k)=1} \frac{f(p)}{p^{\nu_1 z_1+\cdots+\nu_k z_k}} + \sum_{\max(\nu_1,\ldots,\nu_k)\ge 2} \frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}\right). \tag{17}$$

Let $\Re z_1, \ldots, \Re z_k \ge \delta \ge 0$. Consider the sum in (17) over $\max(\nu_1, \ldots, \nu_k) = 1$ and suppose that $\nu_i = 1$ for m $(1 \le m \le k)$ distinct values of i. If $m = 1$, then by condition (i) from the definition of the class $\mathcal{A}_r$ we have

$$\frac{f(p)}{p^{z_j}} = \frac{1}{p^{z_j-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) \quad (1 \le j \le k).$$

If $m \ge 2$, then

$$\left|\frac{f(p)}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}\right| \le \frac{(C_1+1)p^r}{p^{m\delta}} = O\left(\frac{1}{p^{2\delta-r}}\right).$$

This shows that the sum in (17) over $\max(\nu_1, \ldots, \nu_k) = 1$ is

$$\frac{1}{p^{z_1-r}} + \cdots + \frac{1}{p^{z_k-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right).$$

Furthermore, by condition (ii) we deduce that for $\max(\nu_1, \ldots, \nu_k) \ge 2$,

$$\left|\frac{f(p^{\max(\nu_1,\ldots,\nu_k)})}{p^{\nu_1 z_1+\cdots+\nu_k z_k}}\right| \le C_2 \frac{p^{r\max(\nu_1,\ldots,\nu_k)}}{p^{\delta(\nu_1+\cdots+\nu_k)}} \le C_2 \frac{1}{p^{(\delta-r)\max(\nu_1,\ldots,\nu_k)}}$$

$(\delta \ge 0)$ and it follows that the sum in (17) over $\max(\nu_1, \ldots, \nu_k) \ge 2$ is $O\left(p^{-2(\delta-r)}\right) = O\left(p^{-(2\delta-r)}\right)$, since $r < 0$.

We obtain that

$$L_{f,k}(z_1,\ldots,z_k) = \prod_p \left(1 + \frac{1}{p^{z_1-r}} + \cdots + \frac{1}{p^{z_k-r}} + O\left(\frac{1}{p^{\delta-r+1/2}}\right) + O\left(\frac{1}{p^{2\delta-r}}\right)\right)$$

and

$$L_{f,k}(z_1,\ldots,z_k)\zeta^{-1}(z_1-r)\cdots\zeta^{-1}(z_k-r)$$

$$=\prod_p\left(1-\frac{1}{p^{z_1-r}}\right)\cdots\left(1-\frac{1}{p^{z_k-r}}\right)\prod_p\left(1+\frac{1}{p^{z_1-r}}+\cdots+\frac{1}{p^{z_k-r}}\right.$$

$$\left.+O\left(\frac{1}{p^{\delta-r+1/2}}\right)+O\left(\frac{1}{p^{2\delta-r}}\right)\right)$$

$$=\prod_p\left(1+O\left(\frac{1}{p^{\delta-r+1/2}}\right)+O\left(\frac{1}{p^{2\delta-r}}\right)\right),$$

since $\Re z_j\ge\delta$ $(1\le j\le k)$, where the terms $\pm\frac{1}{p^{z_j-r}}$ $(1\le j\le k)$ cancel out, similar to Case I. Here the latter product converges absolutely when $\delta-r+1/2>1$ and $2\delta-r>1$, that is, for $\delta>(r+1)/2>0$. □

Lemma 3.2. *If $k\ge 2$ and $f\in\mathcal{A}_r$ with $r\ge 0$, then*

$$\overline{L}_{f,k}(z_1,\ldots,z_k):=\sum_{n_1,\ldots,n_k=1}^{\infty}\frac{f\left(\frac{[n_1,\ldots,n_k]}{(n_1,\ldots,n_k)}\right)}{n_1^{z_1}\cdots n_k^{z_k}}=\zeta(z_1-r)\cdots\zeta(z_k-r)\overline{H}_{f,k}(z_1,\ldots,z_k),$$

where the multiple Dirichlet series $\overline{H}_{f,k}(z_1,\ldots,z_k)$ is absolutely convergent for $\Re z_1,\ldots,\Re z_k>r+1/2$.

Proof. Similar to the proof of Lemma 3.1, Case I. If f is multiplicative, then the function $f([n_1,\ldots,n_k]/(n_1,\ldots,n_k))$ is also multiplicative and we have

$$\overline{L}_{f,k}(z_1,\ldots,z_k)=\prod_p\sum_{\nu_1,\ldots,\nu_k=0}^{\infty}\frac{f(p^{\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k)})}{p^{\nu_1z_1+\cdots+\nu_kz_k}}$$

$$=\prod_p\left(1+\frac{f(p)}{p^{z_1}}+\cdots+\frac{f(p)}{p^{z_k}}+\sum_{\nu_1+\cdots+\nu_k\ge 2}\frac{f(p^{\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k)})}{p^{\nu_1z_1+\cdots+\nu_kz_k}}\right).\tag{18}$$

If $\Re z_1,\ldots,\Re z_k\ge\delta>r$, then it follows that

$$\left|\frac{f(p^{\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k)})}{p^{\nu_1z_1+\cdots+\nu_kz_k}}\right|\le C\frac{p^{r(\max(\nu_1,\ldots,\nu_k)-\min(\nu_1,\ldots,\nu_k))}}{p^{\delta(\nu_1+\cdots+\nu_k)}}\le C\frac{1}{p^{(\delta-r)(\nu_1+\cdots+\nu_k)}},$$

thus the sum in (18) over $\nu_1+\cdots+\nu_k\ge 2$ is $O\left(p^{-2(\delta-r)}\right)$. Furthermore, we use the same arguments as in the previous proof. □

Proof of Theorem 2.1. From Lemma 3.1 we deduce the convolutional identity

$$f([n_1,\dots,n_k]) = \sum_{j_1d_1=n_1,\dots,j_kd_k=n_k} j_1^r\cdots j_k^r h_{f,k}(d_1,\dots,d_k),$$

where

$$\sum_{n_1,\dots,n_k=1}^{\infty} \frac{h_{f,k}(n_1,\dots,n_k)}{n_1^{z_1}\cdots n_k^{z_k}} = H_{f,k}(z_1,\dots,z_k).$$

Therefore

$$\sum_{n_1,\dots,n_k\le x} f([n_1,\dots,n_k]) = \sum_{j_1d_1\le x,\dots,j_kd_k\le x} j_1^r\cdots j_k^r h_{f,k}(d_1,\dots,d_k)$$

$$= \sum_{d_1,\dots,d_k\le x} h_{f,k}(d_1,\dots,d_k) \sum_{j_1\le x/d_1} j_1^r \cdots \sum_{j_k\le x/d_k} j_k^r$$

$$= \sum_{d_1,\dots,d_k\le x} h_{f,k}(d_1,\dots,d_k) \left(\frac{x^{r+1}}{(r+1)d_1^{r+1}} + O(\frac{x^R}{d_1^R})\right)\cdots\left(\frac{x^{r+1}}{(r+1)d_k^{r+1}} + O(\frac{x^R}{d_k^R})\right),$$

where $R := \max(r,0)$. We deduce that

$$\sum_{n_1,\dots,n_k\le x} f([n_1,\dots,n_k]) = \frac{x^{k(r+1)}}{(r+1)^k} \sum_{d_1,\dots,d_k\le x} \frac{h_{f,k}(d_1,\dots,d_k)}{d_1^{r+1}\cdots d_k^{r+1}} + S_{k,r}(x), \quad (19)$$

with

$$S_{k,r}(x) \ll \sum_{u_1,\dots,u_k} x^{u_1+\cdots+u_k} \sum_{d_1,\dots,d_k\le x} \frac{|h_{f,k}(d_1,\dots,d_k)|}{d_1^{u_1}\cdots d_k^{u_k}}, \quad (20)$$

where the first sum is over $u_1,\dots,u_k \in \{r+1,R\}$ such that at least one u_i is R. Let $u_1,\dots,u_k$ be fixed and assume that $u_i = R$ for t ($1\le t\le k$) values of i, we take the first t values of i. Then $x^{u_1+\cdots+u_k}$ times the inner sum of (20) is, using the notation A given by (14),

$$\ll x^{(k-t)(r+1)+tR} \sum_{d_1,\dots,d_k\le x} \frac{|h_{f,k}(d_1,\dots,d_k)|}{d_1^R\cdots d_t^R d_{t+1}^{r+1}\cdots d_k^{r+1}}$$

$$= x^{(k-t)(r+1)+tR} \sum_{d_1,\dots,d_k\le x} \frac{|h_{f,k}(d_1,\dots,d_k)| d_1^{A-R+\varepsilon}\cdots d_t^{A-R+\varepsilon}}{d_1^{A+\varepsilon}\cdots d_t^{A+\varepsilon} d_{t+1}^{r+1}\cdots d_k^{r+1}}$$

$$\le x^{(k-t)(r+1)+tR} x^{t(A-R+\varepsilon)} \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{A+\varepsilon}\cdots d_t^{A+\varepsilon} d_{t+1}^{r+1}\cdots d_k^{r+1}}$$
$$= x^{k(r+1)-t(r+1-A)+t\varepsilon} H_{f,k}(A+\varepsilon,\ldots,A+\varepsilon,r+1,\ldots,r+1)$$
$$\ll x^{k(r+1)-t(r+1-A)+t\varepsilon},$$

since the latter series is convergent by Lemma 3.1. Using that $r+1-A = \frac{1}{2}\min(r+1,1) > 0$, the obtained error is maximal for $t=1$ giving

$$O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right).$$

Furthermore, for the sum in the main term of (19) we have

$$\sum_{d_1,\ldots,d_k\le x} \frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{r+1}\cdots d_k^{r+1}}$$
$$= \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{r+1}\cdots d_k^{r+1}} - \sum_{\emptyset\ne I\subseteq\{1,\ldots,k\}} \sum_{\substack{d_i>x,\, i\in I\\ d_j\le x,\, j\notin I}} \frac{h_{f,k}(d_1,\ldots,d_k)}{d_1^{r+1}\cdots d_k^{r+1}}, \tag{21}$$

where the series is convergent by Lemma 3.1, and its sum is $H_{f,k}(r+1,\ldots,r+1)$.

Let I be fixed and assume that $I=\{1,2,\ldots,s\}$, that is $d_1,\ldots,d_s>x$ and $d_{t+1},\ldots,d_k\le x$, where $s\ge 1$. We deduce, by noting that $A-(r+1) = -\frac{1}{2}\min(r+1,1) < 0$,

$$\sum_{\substack{d_1,\ldots,d_s>x\\ d_{s+1},\ldots,d_k\le x}} \frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{r+1}\cdots d_k^{r+1}}$$
$$= \sum_{\substack{d_1,\ldots,d_s>x\\ d_{s+1},\ldots,d_k\le x}} \frac{|h_{f,k}(d_1,\ldots,d_k)| d_1^{A-(r+1)+\varepsilon}\cdots d_s^{A-(r+1)+\varepsilon}}{d_1^{A+\varepsilon}\cdots d_s^{A+\varepsilon} d_{s+1}^{r+1}\cdots d_k^{r+1}}$$
$$\le x^{s(A-(r+1)+\varepsilon)} \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{|h_{f,k}(d_1,\ldots,d_k)|}{d_1^{A+\varepsilon}\cdots d_s^{A+\varepsilon} d_{s+1}^{r+1}\cdots d_k^{r+1}}$$
$$= x^{s(A-(r+1)+\varepsilon)} H_{f,k}(A+\varepsilon,\ldots,A+\varepsilon,r+1,\ldots,r+1)$$
$$\ll x^{-\frac{s}{2}\min(r+1,1)+s\varepsilon},$$

the latter series (the same as before) being convergent, and the obtained error is maximal for $s = 1$ giving, according to (19) and (21), the same error

$$O\left(x^{k(r+1)-\frac{1}{2}\min(r+1,1)+\varepsilon}\right).$$

This proves asymptotic formula (7) with the constant $C_{f,k} = H_{f,k}(r+1,\dots,r+1)$. Here, according to Lemma 3.1,

$$C_{f,k} = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{\nu_1,\dots,\nu_k=0}^{\infty} \frac{f(p^{\max(\nu_1,\dots,\nu_k)})}{p^{(r+1)(\nu_1+\dots+\nu_k)}}.$$

The proof of (8) is similar, based on Lemma 3.1 and the convolutional identity

$$\frac{f([n_1,\dots,n_k])}{(n_1\cdots n_k)^r} = \sum_{j_1d_1=n_1,\dots,j_kd_k=n_k} \frac{h_{f,k}(d_1,\dots,d_k)}{d_1^r\cdots d_k^r},$$

which implies that

$$\sum_{n_1,\dots,n_k\le x} \frac{f([n_1,\dots,n_k])}{(n_1\cdots n_r)^r} = \sum_{d_1,\dots,d_k\le x} \frac{h_{f,k}(d_1,\dots,d_k)}{d_1^r\cdots d_k^r} \sum_{j_1\le x/d_1} 1 \cdots \sum_{j_k\le x/d_k} 1.$$

□

Proof of Theorem 2.2. Formula (9) is obtained by using Lemma 3.2, in exactly the same way as (7) (here $r \ge 0$ and $R = \max(r,0) = r$), with the constant $D_{f,k} = \overline{H}_{f,k}(r+1,\dots,r+1)$. □

Proof of Corollary 1. Apply Theorem 2.1 for $f = \mathrm{id}_r$. Here

$$C_{r,3} = \prod_p \left(1-\frac{1}{p}\right)^3 \sum_{a,b,c=0}^{\infty} \frac{p^{r\max(a,b,c)}}{p^{(r+1)(a+b+c)}}$$

$$= \prod_p \left(1-\frac{1}{p}\right)^3 (6S_1+3S_2+3S_3+S_4),$$

with

$$S_1 = \sum_{0\le a<b<c} \frac{p^{rc}}{p^{(r+1)(a+b+c)}}, \quad S_2 = \sum_{0\le a=b<c} \frac{p^{rc}}{p^{(r+1)(2a+c)}},$$

$$S_3 = \sum_{0\le a<b=c} \frac{p^{rc}}{p^{(r+1)(a+2c)}}, \quad S_4 = \sum_{0\le a=b=c} \frac{p^{rc}}{p^{(r+1)3c}},$$

which gives (11). Formula (12) for the constant $C_{r,4}$ can be computed in a similar manner. □

Proof of Corollary 2. Apply Theorem 2.2 for $f = \mathrm{id}_r$. The constants $D_{r,3}$ and $D_{r,4}$ can be evaluated like above. □

Proof of Corollaries 3, 4, 5. Apply Theorem 2.1 for $f = \sigma$, $f = \varphi$ with $r = 1$, resp. $f = \mu^2$ with $r = 0$. □

References

[1] K. Alladi, On generalized Euler functions and related totients, in vol. *New Concepts in Arithmetic Functions*, Matscience Report 83, The Institute of Mathematical Sciences, Madras, 1975.

[2] O. Bordellès, A note on the average order of the gcd-sum function, *J. Integer Seq.* **10** (2007), Article 07.3.3, 4 pp.

[3] O. Bordellès, Mean values of generalized gcd-sum and lcm-sum functions, *J. Integer Seq.* **10** (2007), Article 07.9.2, 13 pp.

[4] O. Bordellès, The composition of the gcd and certain arithmetic functions, *J. Integer Seq.* **13** (2010), Article 10.7.1, 22 pp.

[5] K. A. Broughan, The gcd-sum function, *J. Integer Seq.* **4** (2001), Article 01.2.2, 16 pp, errata added in 2007.

[6] E. Cesàro, Ètude moyenne du plus grand commun diviseur de deux nombres, *Annali di Matematica Pura ed Applicata* **13** (1885), 235–250.

[7] J. Chidambaraswamy and R. Sitaramachandrarao, Asymptotic results for a class of arithmetical functions, *Monatsh. Math.* **99** (1985), 19–27.

[8] J. Cilleruelo, J. Rué, P. Šarka, and A. Zumalacárregui, The least common multiple of random sets of positive integers, *J. Number Theory* **144** (2014), 92–104.

[9] E. Cohen, Arithmetical functions of a greatest common divisor. I, *Proc. Amer. Math. Soc.* **11** (1960), 164–171.

[10] E. Cohen, Arithmetical functions of a greatest common divisor. II. An alternative approach, *Boll. Un. Mat. Ital.* (3) **17** (1962), 349–356.

[11] E. Cohen, Arithmetical functions of a greatest common divisor. III. Cesàro's divisor problem, *Proc. Glasgow Math. Assoc.* **5** (1961), 67–75 .

[12] P. Diaconis and P. Erdős, On the distribution of the greatest common divisor, Technical Report No. 12, Department of Statistics, Stanford University, Stanford, 1977; Reprinted in *A festschrift for Herman Rubin, IMS Lecture Notes Monogr. Ser., Inst. Math. Statist.*, **45**, (2004), 56–61.

[13] J. L. Fernández and P. Fernández, On the probability distribution of the gcd and lcm of r-tuples of integers, Preprint, 2013, 24 pp, arXiv:1305.0536 [math.NT].

[14] T. Hilberdink, The group of squarefree integers, *Linear Algebra Appl.* **457** (2014), 383–399.

[15] S. Ikeda and K. Matsuoka, On the lcm-sum function, *J. Integer Seq.* **17** (2014), Article 14.1.7, 11 pp.

[16] S. S. Pillai, On an arithmetic function, *J. Annamalai Univ.* **2** (1933), 243–248.

[17] Y. Tanigawa and W. Zhai, On the gcd-sum function, *J. Integer Seq.* **11** (2008), Article 08.2.3, 11 pp.

[18] L. Tóth, A survey of gcd-sum functions, *J. Integer Seq.* **13** (2010), Article 10.8.1, 23 pp.

[19] L. Tóth, A survey of the alternating sum-of-divisors function, *Acta Univ. Sapientiae, Math.* **5** (2013), 93–107.

[20] L. Tóth, Multiplicative Arithmetic Functions of Several Variables: A Survey, in vol. *Mathematics Without Boundaries, Surveys in Pure Mathematics*, T. M. Rassias, P. M. Pardalos (eds.), Springer, New York, 2014, pp. 483–514, arXiv:1310.7053 [math.NT].

[21] L. Tóth, Counting r-tuples of positive integers with k-wise relatively prime components, *J. Number Theory* **166** (2016), 105–116.

[22] N. Ushiroya, Mean-value theorems for multiplicative arithmetic functions of several variables, *Integers* **12** (2012), 989–1002.

[23] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte, XV. VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.