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GENERALIZED GOLDEN RATIOS OVER INTEGER ALPHABETS

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Abstract
It is a well known result that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ and $x \in (0, \frac{1}{\beta-1})$ there exists uncountably many $(\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ such that $x = \sum_{i=1}^{\infty} \epsilon_i \beta^{-i}$. When $\beta \in (\frac{1+\sqrt{5}}{2}, 2]$ there exists $x \in (0, \frac{1}{\beta-1})$ for which there exists a unique $(\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ such that $x = \sum_{i=1}^{\infty} \epsilon_i \beta^{-i}$. In this paper we consider the more general case when our sequences are elements of $\{0, \ldots, m\}^{\mathbb{N}}$. We show that an analogue of the golden ratio exists and give an explicit formula for it.

1. Introduction

Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $I_{\beta,m} = [0, \frac{m}{\beta-1}]$. Each $x \in I_{\beta,m}$ has an expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i},$$

for some $(\epsilon_i)_{i=1}^{\infty} \in \{0, \ldots, m\}^{\mathbb{N}}$. We call such a sequence a *β-expansion* for x. For $x \in I_{\beta,m}$ we denote the set of β-expansions for x by $\Sigma_{\beta,m}(x)$, i.e.,

$$\Sigma_{\beta,m}(x) = \Big\{(\epsilon_i)_{i=1}^{\infty} \in \{0, \ldots, m\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = x\Big\}.$$

In [6] the authors consider the case when $m = 1$, they show that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ the set $\Sigma_{\beta,1}(x)$ is uncountable for every $x \in (0, \frac{1}{\beta-1})$. The endpoints of $[0, \frac{1}{\beta-1}]$ trivially have a unique β-expansion. In [5] it is shown that for $\beta \in (\frac{1+\sqrt{5}}{2}, 2]$ there exists $x \in (0, \frac{1}{\beta-1})$ with a unique β-expansion.

For $m \in \mathbb{N}$ we define $\mathcal{G}(m) \in \mathbb{R}$ to be a *generalized golden ratio for* m if for $\beta \in (1, \mathcal{G}(m))$ the set $\Sigma_{\beta,m}(x)$ is uncountable for every $x \in (0, \frac{m}{\beta-1})$, and for every $\beta \in (\mathcal{G}(m), m+1]$ there exists $x \in (0, \frac{m}{\beta-1})$ for which $|\Sigma_{\beta,m}(x)| = 1$.

In [11] the authors consider a similar setup. They consider the case where β-expansions are elements of $\{a_1, a_2, a_3\}^{\mathbb{N}}$, for some $a_1, a_2, a_3 \in \mathbb{R}$. They show that for each ternary alphabet there exists a constant $G \in \mathbb{R}$ such that, there exists nontrivial unique β-expansions if and only if $\beta > G$. Moreover they give an explicit formula for G.

Our main result is the following.

Theorem 1.1. *For each $m \in \mathbb{N}$ a generalized golden ratio exists and is equal to:*

$$\mathcal{G}(m) = \begin{cases} k+1 & \text{if } m = 2k \\ \frac{k+1+\sqrt{k^2+6k+5}}{2} & \text{if } m = 2k+1. \end{cases} \tag{1}$$

Remark 1.2. *$\mathcal{G}(m)$ is a Pisot number for all $m \in \mathbb{N}$. Recall a Pisot number is a real algebraic integer greater than 1 whose Galois conjugates are of modulus strictly less than 1.*

In section 6 we include a table of values for $\mathcal{G}(m)$. We prove Theorem 1.1 in section 3. In section 4 we consider the set of points with unique β-expansion for $\beta \in (\mathcal{G}(m), m+1]$, and in section 5 we study the growth rate and dimension theory of the set of β-expansions for $\beta \in (1, \mathcal{G}(m))$.

2. Preliminaries

Before proving Theorem 1.1 we require the following preliminary results and theory. Let $m \in \mathbb{N}$ be fixed and $\beta \in (1, m+1]$. For each $i \in \{0, \dots, m\}$ we fix $T_{\beta,i}(x) = \beta x - i$. The proof of the following lemma is trivial and therefore omitted.

Lemma 2.1. *The map $T_{\beta,i}$ satisfies the following:*

- *$T_{\beta,i}$ has a unique fixed point equal to $\frac{i}{\beta-1}$.*
- *$T_{\beta,i}(x) > x$ for all $x > \frac{i}{\beta-1}$,*
- *$T_{\beta,i}(x) < x$ for all $x < \frac{i}{\beta-1}$,*
- *$|T_{\beta,i}(x) - T_{\beta,i}(\frac{i}{\beta-1})| = \beta|x - \frac{i}{\beta-1}|$, for all $x \in \mathbb{R}$, that is $T_{\beta,i}$ scales the distance between the fixed point $\frac{i}{\beta-1}$ and an arbitrary point by a factor β.*

Understanding where in $I_{\beta,m}$ these fixed points are will be important in our later analysis.

We let

$$\Omega_{\beta,m}(x) = \Big\{(a_i)_{i=1}^{\infty} \in \{T_{\beta,0}, \dots, T_{\beta,m}\}^{\mathbb{N}} : (a_n \circ a_{n-1} \circ \dots \circ a_1)(x) \in I_{\beta,m} \text{ for all } n \in \mathbb{N}\Big\}.$$

Similarly we define

$$\Omega_{\beta,m,n}(x) = \Big\{(a_i)_{i=1}^n \in \{T_{\beta,0},\ldots,T_{\beta,m}\}^n : (a_n \circ a_{n-1} \circ \ldots \circ a_1)(x) \in I_{\beta,m}\Big\}.$$

Typically we will denote an element of $\Omega_{\beta,m,n}(x)$ or any finite sequence of maps by a. When we want to emphasise the length of a we will use the notation $a^{(n)}$. We also adopt the notation $a^{(n)}(x)$ to mean $(a_n \circ a_{n-1} \circ \ldots \circ a_1)(x)$.

Remark 2.2. *It is important to note that if for some finite sequence of maps a, $a(x) \notin I_{\beta,m}$ then we cannot concatenate a by any finite sequence of maps b, such that $b(a(x)) \in I_{\beta,m}$.*

Remark 2.3. *Let $\beta \in (1, m+1]$. For any $x \in I_{\beta,m}$ there always exists $i \in \{0,\ldots,m\}$ such that $T_{\beta,i}(x) \in I_{\beta,m}$. For $\beta > m+1$ such an i does not always exist.*

Lemma 2.4. $|\Sigma_{\beta,m}(x)| = |\Omega_{\beta,m}(x)|$.

Proof. It is a simple exercise to show that

$$\Sigma_{\beta,m}(x) = \Big\{(\epsilon_i)_{i=1}^\infty \in \{0,\ldots,m\}^{\mathbb{N}} : x - \sum_{i=1}^n \frac{\epsilon_i}{\beta^i} \in \Big[0, \frac{m}{\beta^n(\beta-1)}\Big] \text{ for all } n \in \mathbb{N}\Big\}.$$

Following [8] we observe that

$$\begin{aligned}\Sigma_{\beta,m}(x) &= \Big\{(\epsilon_i)_{i=1}^\infty \in \{0,\ldots,m\}^{\mathbb{N}} : x - \sum_{i=1}^n \frac{\epsilon_i}{\beta^i} \in \Big[0, \frac{m}{\beta^n(\beta-1)}\Big] \text{ for all } n \in \mathbb{N}\Big\} \\ &= \Big\{(\epsilon_i)_{i=1}^\infty \in \{0,\ldots,m\}^{\mathbb{N}} : \beta^n x - \sum_{i=1}^n \epsilon_i \beta^{n-i} \in I_{\beta,m} \text{ for all } n \in \mathbb{N}\Big\} \\ &= \Big\{(\epsilon_i)_{i=1}^\infty \in \{0,\ldots,m\}^{\mathbb{N}} : (T_{\beta,\epsilon_n} \circ \ldots \circ T_{\beta,\epsilon_1})(x) \in I_{\beta,m} \text{ for all } n \in \mathbb{N}\Big\}.\end{aligned}$$

Our result follows immediately. □

By Lemma 2.4 we can rephrase the definition of a generalized golden ratio in terms of the set $\Omega_{\beta,m}(x)$. This equivalent definition will be more suitable for our purposes. The set $\Omega_{\beta,m,n}(x)$ will be useful when we study the growth rate and dimension theory of the set of β-expansions.

For a point $x \in I_{\beta,m}$ we can take i to be the first digit in a β-expansion for x if and only if $\beta x - i \in I_{\beta,m}$. This is equivalent to

$$x \in \Big[\frac{i}{\beta}, \frac{i\beta + m - i}{\beta(\beta-1)}\Big],$$

as such we refer to the interval $[\frac{i}{\beta}, \frac{i\beta+m-i}{\beta(\beta-1)}]$ as the *i-th digit interval*. Generally speaking we can take i to be the j-th digit in a β-expansion for x if and only if there exists $a \in \Omega_{\beta,m,j-1}(x)$ such that, $a(x) \in [\frac{i}{\beta}, \frac{i\beta+m-i}{\beta(\beta-1)}]$. When x or an image of x is contained in the intersection of two digit intervals we have a choice of digit in our β-expansion for x. Generally speaking any two digit intervals may intersect for β sufficiently small, however for our purposes we need only consider the case when the i-th digit interval intersects the adjacent $(i-1)$-th or $(i+1)$-th digit intervals, for some $i \in \{0, \dots, m\}$. Any intersection of this type is of the form

$$\Big[\frac{i}{\beta}, \frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)}\Big],$$

for some $i \in \{1, \dots, m\}$. In what follows we refer to the interval $[\frac{i}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ as the *i-th choice interval*. Both $T_{\beta,i-1}$ and $T_{\beta,i}$ map the i-th choice interval into $I_{\beta,m}$. These intervals always exist and are nontrivial for $\beta \in (1, m+1)$.

Proposition 2.5. *Suppose for any $x \in (0, \frac{m}{\beta-1})$ there always exists a finite sequence of maps that map x into the interior of a choice interval, then $\Omega_{\beta,m}(x)$ is uncountable.*

The proof of this proposition is essentially contained in the proof of Theorem 1 in [17].

Proof. Let $x \in (0, \frac{m}{\beta-1})$. Suppose there exists $n \in \mathbb{N}$ and $a \in \Omega_{\beta,m,n}(x)$ such that $a(x) \in (\frac{i}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)})$, for some $i \in \{1, \dots, m\}$. As $a(x)$ is an element of the interior of a choice interval both $T_{\beta,i-1}(a(x)) \in (0, \frac{m}{\beta-1})$ and $T_{\beta,i}(a(x)) \in (0, \frac{m}{\beta-1})$. As such our hypothesis applies to both $T_{\beta,i-1}(a(x))$ and $T_{\beta,i}(a(x))$, and we can assert that there exists a finite sequence of maps that map these two distinct images of x into the interior of another choice interval. Repeating this procedure arbitrarily many times it is clear that $\Omega_{\beta,m}(x)$ is uncountable. □

By Proposition 2.5, to prove Theorem 1.1 it suffices to show that for $\beta \in (1, \mathcal{G}(m))$ every $x \in (0, \frac{m}{\beta-1})$ can be mapped into the interior of a choice interval, and for $\beta \in (\mathcal{G}(m), m+1]$ there exists $x \in (0, \frac{m}{\beta-1})$ that never maps into a choice interval.

We define the *switch region* to be the interval

$$\Big[\frac{1}{\beta}, \frac{(m-1)\beta + 1}{\beta(\beta-1)}\Big].$$

The significance of this interval is that if a point x has a choice of digit in the j-th entry of a β-expansion, then there exists $a \in \Omega_{\beta,m,j-1}(x)$ such that $a(x) \in [\frac{1}{\beta}, \frac{(m-1)\beta+1}{\beta(\beta-1)}]$. The following lemmas are useful in understanding the dynamics of the maps $T_{\beta,i}$ around the switch region, understanding these dynamics will be important in our proof of Theorem 1.1.

Lemma 2.6. *For $\beta \in (1, \frac{m+\sqrt{m^2+4}}{2})$ and $x \in (0, \frac{m}{\beta-1})$ there exists a finite sequence of maps that map x into the interior of our switch region.*

Proof. If x is contained within the interior of the switch region we are done, let us suppose otherwise. By the monotonicity of the maps $T_{\beta,0}$ and $T_{\beta,m}$ it suffices to show that

$$T_{\beta,0}\Big(\frac{1}{\beta}\Big) < \frac{(m-1)\beta+1}{\beta(\beta-1)} \text{ and } T_{\beta,m}\Big(\frac{(m-1)\beta+1}{\beta(\beta-1)}\Big) > \frac{1}{\beta}.$$

Both of these inequalities are equivalent to $\beta^2 - m\beta - 1 < 0$, applying the quadratic formula we can conclude our result. $\square$

Remark 2.7. *When $m = 1$ the switch region is a choice interval. An application of Lemma 2.4, Proposition 2.5 and Lemma 2.6 yields the result stated in [6], i.e, for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ and $x \in (0, \frac{1}{\beta-1})$ the set $\Sigma_{\beta,1}(x)$ is uncountable.*

Lemma 2.8. *For $\beta \in (1, \frac{m+2}{2})$ every x in the interior of the switch region is contained in the interior of a choice interval.*

Proof. It suffices to show that for each $i \in \{1, 2, \ldots, m-1\}$ the $(i-1)$-th and $(i+1)$-th digit intervals intersect in a nontrivial interval. This is equivalent to

$$\frac{i+1}{\beta} < \frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)},$$

a simple manipulation yields that this is equivalent to $\beta < \frac{m+2}{2}$. $\square$

We refer the reader to Figure 1 for a diagram depicting the case where $\beta < \frac{m+2}{2}$. For $i \in \{1, 2, \ldots, m-1\}$ and $\beta \geq \frac{m+2}{2}$ the interval

$$\Big[\frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)}, \frac{i+1}{\beta}\Big]$$

is well defined. We refer to this interval as the *i-th fixed digit interval*. The significance of this interval is that if a point x is contained in the interior of the i-th fixed digit interval only $T_{\beta,i}$ maps x into $I_{\beta,m}$. Similarly we define the 0-th fixed digit interval to be $[0, \frac{1}{\beta}]$ and the m-th fixed digit interval to be $[\frac{(m-1)\beta+1}{\beta(\beta-1)}, \frac{m}{\beta-1}]$. Understanding how the different $T_{\beta,i}$'s behave on these intervals will be important when it comes to constructing generalized golden ratios in the case where m is odd.

3. Proof of Theorem 1.1

We are now in a position to prove Theorem 1.1, for ease of exposition we reduce our analysis to two cases, when m is even and when m is odd.

3.1. Case where m is even

In what follows we assume $m = 2k$ for some $k \in \mathbb{N}$.

Proposition 3.1. *For $\beta \in (1, k+1)$ every $x \in (0, \frac{m}{\beta-1})$ has uncountably many β-expansions.*

0 $\frac{1}{\beta}$ $\frac{(m-1)\beta+1}{\beta(\beta-1)}$ $\frac{m}{\beta-1}$

Figure 1: The case where $\beta \in (1, \frac{m+2}{2})$

Proof. By Lemma 2.4 and Proposition 2.5 it suffices to show that every $x \in (0, \frac{m}{\beta-1})$ can be mapped into the interior of a choice interval. It is a simple exercise to show that $\frac{m+2}{2} < \frac{m+\sqrt{m^2+4}}{2}$ for all $m \in \mathbb{N}$, therefore for $\beta \in (1, k+1)$ we can apply Lemma 2.6, therefore there exists a sequence of maps that map x into the interior of the switch region. By Lemma 2.8 every point in the interior of our switch region is contained in the interior of a choice interval. □

Proposition 3.2. *For $\beta \in (k+1, m+1]$ there exists $x \in (0, \frac{m}{\beta-1})$ with a unique β-expansion.*

Proof. It suffices to show that there exists $x \in (0, \frac{m}{\beta-1})$ that never maps into a choice interval. We consider the point $\frac{k}{\beta-1}$, we will show that this point has a unique β-expansion. This point is contained in the k-th digit interval and is the fixed point under the map $T_{\beta,k}$. To show that it has a unique β-expansion it suffices to show that it is not contained within the $(k-1)$-th or $(k+1)$-th digit intervals, this is equivalent to

$$\frac{(k-1)\beta + m - (k-1)}{\beta(\beta-1)} < \frac{k}{\beta-1} < \frac{k+1}{\beta}.$$

Both of these inequalities are equivalent to $\beta > k+1$. □

0 $\frac{k}{\beta-1}$ $\frac{m}{\beta-1}$

Figure 2: A point with unique β-expansion for $\beta \in (k+1, m+1]$.

Figure 2 describes the construction of our point with unique β-expansion for $\beta \in (k+1, m+1]$. By Proposition 3.1 and Proposition 3.2 we can conclude Theorem 1.1 in the case where m is even.

3.2. Case where m is odd

The analysis of the case where m is odd is somewhat more intricate. In what follows we assume $m = 2k+1$ for some $k \in \mathbb{N}$. Before finishing our proof of Theorem 1.1 we require the following technical results.

Lemma 3.3. *For $\beta \in (1, k+2)$ the fixed point of $T_{\beta,i}$ is contained in the interior of the choice interval $[\frac{i}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ for $i \in \{1, \ldots, k\}$, and in the interior of the choice interval $[\frac{i+1}{\beta}, \frac{i\beta+m-i}{\beta(\beta-1)}]$ for $i \in \{k+1, \ldots, m-1\}$.*

Proof. Let $i \in \{1, \ldots, k\}$. To show that the fixed point $\frac{i}{\beta-1}$ is contained in the interior of the interval $[\frac{i}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ it suffices to show that

$$\frac{i}{\beta-1} < \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}.$$

This is equivalent to $\beta < m+1-i$, which for $\beta \in (1, k+2)$ is true for all $i \in \{1, \ldots, k\}$. The case where $i \in \{k+1, \ldots, m-1\}$ is proved similarly. □

Corollary 3.4. *For $\beta \in [\frac{2k+3}{2}, k+2)$ the map $T_{\beta,i}$ satisfies $T_{\beta,i}(x) - \frac{i}{\beta-1} = \beta(x - \frac{i}{\beta-1})$ for all x contained in the i-th fixed digit interval for $i \in \{1, \ldots, k\}$, and $\frac{i}{\beta-1} - T_{\beta,i}(x) = \beta(\frac{i}{\beta-1} - x)$ for all x contained in the i-th fixed digit interval for $i \in \{k+1, \ldots, m-1\}$.*

Proof. Let $i \in \{1, \ldots, k\}$. By Lemma 3.3 the i-th fixed digit interval is to the right of the fixed point of $T_{\beta,i}$, our result follows from Lemma 2.1. The case where $i \in \{k+1, \ldots, m-1\}$ is proved similarly. □

Lemma 3.5. *Suppose $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ and x is an element of the i-th fixed digit interval for some $i \in \{1, \ldots, m-1\}$. For $i \in \{1, \ldots, k\}$*

$$T_{\beta,i}(x) < \frac{k\beta + m - k}{\beta(\beta-1)}$$

and for $i \in \{k+1, \ldots, m-1\}$

$$T_{\beta,i}(x) > \frac{k+1}{\beta}.$$

Proof. By the monotonicity of the maps $T_{\beta,i}$ it is sufficient to show that

$$T_{\beta,i}\Big(\frac{i+1}{\beta}\Big) < \frac{k\beta + m - k}{\beta(\beta-1)}$$

for $i \in \{1, \ldots, k\}$, and

$$T_{\beta,i}\Big(\frac{(i-1)\beta + m - (i-1)}{\beta(\beta-1)}\Big) > \frac{k+1}{\beta},$$

for $i \in \{k+1, \ldots, m-1\}$. Each of these inequalties are equivalent to $\beta^2 - (k+1)\beta - (k+1) < 0$. Our result follows by an application of the quadratic formula. □

Proposition 3.6. *For $\beta \in (1, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ every $x \in (0, \frac{m}{\beta-1})$ has uncountably many β-expansions.*

Proof. The proof where $\beta \in (1, \frac{2k+3}{2})$ is analogous to that given in the even case. As such, in what follows we assume $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. We remark that

$$\frac{k+1+\sqrt{k^2+6k+5}}{2} \leq \frac{m+\sqrt{m^2+4}}{2}$$

and

$$\frac{k+1+\sqrt{k^2+6k+5}}{2} < k+2,$$

0 $\frac{1}{\beta}$ $\frac{k+1}{\beta}$ $\frac{k\beta+m-k}{\beta(\beta-1)}$ $\frac{(m-1)\beta+1}{\beta(\beta-1)}$ $\frac{m}{\beta-1}$

Figure 3: A diagram of the case where $m = 2k+1$ and $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$

for all $k \in \mathbb{N}$. We can therefore use Lemma 2.6 and Corollary 3.4. Let $x \in (0, \frac{m}{\beta-1})$. We will show that there exists a sequence of maps that map x into the interior of a choice interval, by Lemma 2.4 and Proposition 2.5 our result follows. By Lemma 2.6 there exist a finite sequence of maps that map x into the interior of the switch region. Suppose the image of x is not contained in the interior of a choice interval, then it must be contained in the i-th fixed digit interval for some $i \in \{1, \ldots, m-1\}$. By repeatedly applying Corollary 3.4 and Lemma 3.5 the image of x must eventually be mapped into the interior of a choice interval. □

We refer the reader to Figure 3 for a diagram illustrating the case where $m = 2k+1$ and $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$.

Proposition 3.7. *For $\beta \in (\frac{k+1+\sqrt{k^2+6k+5}}{2}, m+1]$ there exists $x \in (0, \frac{m}{\beta-1})$ that has a unique β-expansion.*

Proof. We will show that the points

$$\frac{k\beta+(k+1)}{\beta^2-1} \text{ and } \frac{(k+1)\beta+k}{\beta^2-1}$$

have a unique β-expansion. The significance of these points is that

$$T_{\beta,k}\Big(\frac{k\beta+(k+1)}{\beta^2-1}\Big)=\frac{(k+1)\beta+k}{\beta^2-1}$$

and

$$T_{\beta,k+1}\Big(\frac{(k+1)\beta+k}{\beta^2-1}\Big)=\frac{k\beta+(k+1)}{\beta^2-1}.$$

To show that these points have a unique β-expansion it suffices to show that $\frac{k\beta+(k+1)}{\beta^2-1}$ and $\frac{(k+1)\beta+k}{\beta^2-1}$ belong to the k-th and $(k+1)$-th fixed digit intervals, respectively, i.e.

$$\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)}<\frac{k\beta+(k+1)}{\beta^2-1}<\frac{k+1}{\beta}, \tag{2}$$

and

$$\frac{k\beta+(m-k)}{\beta(\beta-1)}<\frac{(k+1)\beta+k}{\beta^2-1}<\frac{k+2}{\beta}. \tag{3}$$

The left hand side of (2) is equivalent to $0<\beta^2-k\beta-(k+2)$ which is equivalent to

$$\frac{k+\sqrt{k^2+4k+8}}{2}<\beta,$$

in the same time

$$\frac{k+\sqrt{k^2+4k+8}}{2}<\frac{k+1+\sqrt{k^2+6k+5}}{2}$$

for all $k\in\mathbb{N}$, therefore the left hand side of (2) holds. The right hand side of (2) is equivalent to $0<\beta^2-(k+1)\beta-(k+1)$. So (2) holds by the quadratic formula.

The right hand side of (3) is equivalent to $0<\beta^2-k\beta-(k+2)$ which we know to be true by the above. Similarly the left hand side of (3) is equivalent to $0<\beta^2-(k+1)\beta-(k+1)$, which we also know to be true. It follows that both $\frac{k\beta+(k+1)}{\beta^2-1}$ and $\frac{(k+1)\beta+k}{\beta^2-1}$ are never mapped into a choice interval and have a unique β-expansion for $\beta\in(\frac{k+1+\sqrt{k^2+6k+5}}{2},m+1]$. □

We refer the reader to Figure 4 for a diagram describing the points we constructed with unique β-expansion for $\beta\in(\frac{k+1+\sqrt{k^2+6k+5}}{2},m+1]$. By Proposition 3.6 and Proposition 3.7 we can conclude Theorem 1.1.

4. The set of points with unique β-expansion

In this section we study the set of points whose β-expansion is unique for $\beta\in(\mathcal{G}(m),m+1]$. Let

$$U_{\beta,m}=\Big\{x\in I_{\beta,m}|\ |\Sigma_{\beta,m}(x)|=1\Big\}$$

0 $\frac{k\beta+(k+1)}{\beta^2-1}$ $\frac{(k+1)\beta+k}{\beta^2-1}$ $\frac{m}{\beta-1}$

Figure 4: A point with unique β-expansion for $\beta \in (\frac{k+1+\sqrt{k^2+6k+5}}{2}, m+1]$.

and

$$W_{\beta,m} = \Big\{x \in \Big(\frac{m+1-\beta}{\beta-1}, 1\Big) \mid |\Sigma_{\beta,m}(x)| = 1\Big\}.$$

The significance of the set $W_{\beta,m}$ is that if $x \in U_{\beta,m}$, then it maps to $W_{\beta,m}$ under a finite sequence of $T_{\beta,i}$'s. In [9] the authors study the case where $m = 1$, they show that the following theorems hold.

Theorem 4.1. *The set $U_{\beta,1}$ satisfies the following:*

1. $|U_{\beta,1}| = \aleph_0$ *for* $\beta \in (\frac{1+\sqrt{5}}{2}, \beta_c)$
2. $|U_{\beta,1}| = 2^{\aleph_0}$ *for* $\beta = \beta_c$
3. $U_{\beta,1}$ *is a set of positive Hausdorff dimension for* $\beta \in (\beta_c, 2]$.

Theorem 4.2. *The set $W_{\beta,1}$ satisfies the following:*

1. $|W_{\beta,1}| = 2$ *for* $\beta \in (\frac{1+\sqrt{5}}{2}, \beta_f]$, *where* β_f *is the root of the equation*

$$x^3 - 2x^2 + x - 1 = 0, \ \beta_f = 1.75487\ldots$$

2. $|W_{\beta,1}| = \aleph_0$ *for* $\beta \in (\beta_f, \beta_c)$

3. $|W_{\beta,1}| = 2^{\aleph_0}$ *for* $\beta = \beta_c$

4. $W_{\beta,1}$ *is a set of positive Hausdorff dimension for* $\beta \in (\beta_c, 2]$.

Here $\beta_c \approx 1.78723$ is the Komornik-Loreti constant introduced in [12]. It is the smallest value of β for which $1 \in U_{\beta,1}$. Moreover β_c is the unique solution of the equation

$$\sum_{i=1}^{\infty} \frac{\lambda_i}{\beta^i} = 1,$$

where $(\lambda_i)_{i=0}^{\infty}$ is the Thue-Morse sequence (see [3]), i.e. $\lambda_0 = 0$ and if λ_i is already defined for some $i \geq 0$ then $\lambda_{2i} = \lambda_i$ and $\lambda_{2i+1} = 1 - \lambda_i$. The sequence $(\lambda_i)_{i=0}^{\infty}$ begins

$$(\lambda_i)_{i=0}^{\infty} = 0110\ 1001\ 1001\ 0110\ 1001\ \ldots.$$

In [2] it was shown that β_c is transcendental. For $m \geq 2$ we define the sequence $(\lambda_i(m))_{i=1}^{\infty} \in \{0, \ldots, m\}^{\mathbb{N}}$ as follows:

$$\lambda_i(m) = \begin{cases} k + \lambda_i - \lambda_{i-1} & \text{if } m = 2k \\ k + \lambda_i & \text{if } m = 2k+1. \end{cases}$$

We define $\beta_c(m)$ to be the unique solution of

$$\sum_{i=1}^{\infty} \frac{\lambda_i(m)}{\beta^i} = 1.$$

In [13] the authors proved that $\beta_c(m)$ is transcendental and the smallest value of β for which $1 \in U_{\beta,m}$. In section 6 we include a table of values for $\beta_c(m)$. We begin our study of the sets $U_{\beta,m}$ and $W_{\beta,m}$ by showing that the following proposition holds.

Proposition 4.3. *If* $m \geq 2$, *then* $|U_{\beta,m}| \geq \aleph_0$ *for* $\beta \in (\mathcal{G}(m), m+1]$.

In [14] the following statements were shown to hold: if $\beta \in (1, \beta_c(m))$ then $U_{\beta,m}$ is countable, $U_{\beta_c(m),m}$ has cardinality equal to that of the continuum, and for $\beta \in (\beta_c(m), m+1]$ the Hausdorff dimension of $U_{\beta,m}$ is strictly positive. Combining these results with Proposition 4.3 the following analogue of Theorem 4.1 is immediate.

Theorem 4.4. *For* $m \geq 2$ *the set* $U_{\beta,m}$ *satisfies the following:*

1. $|U_{\beta,m}| = \aleph_0$ *for* $\beta \in (\mathcal{G}(m), \beta_c(m))$

2. $|U_{\beta,m}| = 2^{\aleph_0}$ *for* $\beta = \beta_c(m)$

3. $U_{\beta,m}$ *is a set of positive Hausdorff dimension for* $\beta \in (\beta_c(m), m+1]$.

Proof of Proposition 4.3. To begin with let us assume $m = 2k$ for some $k \in \mathbb{N}$, in this case $\mathcal{G}(m) = k + 1$. It is a simple exercise to show that for $\beta \in (k + 1, m + 1]$

$$T_{\beta,0}^{-n}\Big(\frac{k}{\beta - 1}\Big) = \frac{k}{\beta^n(\beta - 1)} < \frac{1}{\beta} \tag{4}$$

for all $n \in \mathbb{N}$. By the proof of Proposition 3.2 we know that $\frac{k}{\beta-1}$ has a unique β-expansion. It follows from (4) that $T_{\beta,0}^{-n}(\frac{k}{\beta-1})$ is never mapped into a choice interval and therefore has a unique β-expansion. As n was arbitrary we can conclude our result. The case where $m = 2k + 1$ is proved similarly, in this case we can consider preimages of $\frac{k\beta+(k+1)}{\beta^2-1}$. □

We also show that the following analogue of Theorem 4.2 holds.

Theorem 4.5. *If $m = 2k$ the set $W_{\beta,m}$ satisfies the following:*

1. $|W_{\beta,m}| = 1$ *for* $\beta \in (\mathcal{G}(m), \beta_f(m)]$, *where* $\beta_f(m)$ *is the root of the equation*

$$x^2 - (k + 1)x - k = 0, \ \beta_f(m) = \frac{k + 1 + \sqrt{k^2 + 6k + 1}}{2}$$

2. $|W_{\beta,m}| = \aleph_0$ *for* $\beta \in (\beta_f(m), \beta_c(m))$
3. $|W_{\beta,m}| = 2^{\aleph_0}$ *for* $\beta = \beta_c(m)$
4. $W_{\beta,m}$ *is a set of positive Hausdorff dimension for* $\beta \in (\beta_c(m), m + 1]$.

If $m = 2k + 1$ the set $W_{\beta,m}$ satisfies the following:

1. $|W_{\beta,m}| = 2$ *for* $\beta \in (\mathcal{G}(m), \beta_f(m)]$, *where* $\beta_f(m)$ *is the root of the equation*

$$x^3 - (k + 2)x^2 + x - (k + 1) = 0$$

2. $|W_{\beta,m}| = \aleph_0$ *for* $\beta \in (\beta_f(m), \beta_c(m))$
3. $|W_{\beta,m}| = 2^{\aleph_0}$ *for* $\beta = \beta_c(m)$
4. $W_{\beta,m}$ *is a set of positive Hausdorff dimension for* $\beta \in (\beta_c(m), m + 1]$.

Remark 4.6. *$\beta_f(m)$ is a Pisot number for all $m \in \mathbb{N}$.*

Using Theorem 4.4, to prove Theorem 4.5 it suffices to show that statement 1 holds in both the odd and even cases and $|W_{\beta,m}| \geq \aleph_0$ for $\beta > \beta_f(m)$ in both the odd and even cases. In section 6 we include a table of values for $\beta_f(m)$.

4.1. Proof of Theorem 4.5

The proof of Theorem 4.5 is more involved than Theorem 4.4 and as we will see requires more technical results. The following is taken from [14]. Firstly let us define the lexicographic order on $\{0,\ldots,m\}^{\mathbb{N}}$: we say that $(x_i)_{i=1}^{\infty} < (y_i)_{i=1}^{\infty}$ with respect to the lexicographic order if there exists $n \in \mathbb{N}$ such that $x_i = y_i$ for all $i < n$ and $x_n < y_n$ or if $x_1 < y_1$. For a sequence $(x_i)_{i=1}^{\infty} \in \{0,\ldots,m\}^{\mathbb{N}}$ we define $(\bar{x}_i)_{i=1}^{\infty} = (m - x_i)_{i=1}^{\infty}$. We also adopt the notation $(\epsilon_1,\ldots,\epsilon_j)^{\infty}$ to denote the element of $\{0,\ldots,m\}^{\mathbb{N}}$ obtained by the infinite concatenation of the finite sequence $(\epsilon_1,\ldots,\epsilon_j)$. Let the sequence $(d_i(m))_{i=1}^{\infty} \in \{0,\ldots,m\}^{\mathbb{N}}$ be defined as follows: let $d_1(m)$ be the largest element of $\{0,\ldots,m\}$ such that $\frac{d_1(m)}{\beta} < 1$, and if $d_i(m)$ is defined for $i < n$ then $d_n(m)$ is defined to be the largest element of $\{0,\ldots,m\}$ such that $\sum_{i=1}^{n} \frac{d_i(m)}{\beta^i} < 1$. The sequence $(d_i(m))_{i=1}^{\infty}$ is called the quasi-greedy expansion of 1 with respect to β; it is trivially a β-expansion for 1 and the largest infinite β-expansion of 1 with respect to the lexicographic order not ending with $(0)^{\infty}$. We let

$$S_{\beta,m} = \Big\{(\epsilon_i)_{i=1}^{\infty} \in \{0,\ldots,m\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} \in W_{\beta,m}\Big\},$$

it follows from the definition of $W_{\beta,m}$ that $|W_{\beta,m}| = |S_{\beta,m}|$ and to prove Theorem 4.5 it suffices to show that equivalent statements hold for $S_{\beta,m}$. The following lemma which is essentially due to Parry [15] provides a useful characterisation of $S_{\beta,m}$.

Lemma 4.7.

$$S_{\beta,m} = \Big\{(\epsilon_i)_{i=1}^{\infty} \in \{0,\ldots,m\}^{\mathbb{N}} : (\epsilon_i,\epsilon_{i+1},\ldots) < (d_{1,m},d_{2,m},\ldots) \textit{ and } (\bar{d}_{1,m},\bar{d}_{2,m},\ldots) < (\epsilon_i,\epsilon_{i+1},\ldots) \textit{ for all } i \in \mathbb{N}\Big\}$$

Remark 4.8. *If $\beta < \beta'$ then the quasi-greedy expansion of 1 with respect to β is lexicographically strictly less than the quasi-greedy expansion of 1 with respect to β'. As a corollary of this we have $S_{\beta,m} \subseteq S_{\beta',m}$ for $\beta < \beta'$.*

Proposition 4.9. *For $\beta \in (\mathcal{G}(m), \beta_f(m)]$ $|S_{\beta,m}| = 1$ when m is even, $|S_{\beta,m}| = 2$ when m is odd and $|S_{\beta,m}| \geq \aleph_0$ for $\beta \in (\beta_f(m), m+1]$.*

By the remarks following Theorem 4.5 this will allow us to conclude our result.

Proof. We begin by considering the case where $m = 2k$. When $\beta = \beta_f(m)$ we have $(d_i(m))_{i=1}^{\infty} = (k+1,k-1)^{\infty}$ and by Lemma 4.7

$$S_{\beta_f(m),m} = \Big\{(\epsilon_i)_{i=1}^{\infty} \in \{0,\ldots,m\}^{\mathbb{N}} : (\epsilon_i,\epsilon_{i+1},\ldots) < (k+1,k-1)^{\infty} \text{ and } (k-1,k+1)^{\infty} < (\epsilon_i,\epsilon_{i+1},\ldots) \text{ for all } i \in \mathbb{N}\Big\}.$$

By our previous analysis we know that for $\beta \in (\mathcal{G}(m), m+1]$ the point $\frac{k}{\beta-1}$ has a unique β-expansion, the β-expansion of this point is the sequence $(k)^{\infty}$. By Remark 4.8, to prove

$|S_{\beta,m}| = 1$ for $\beta \in (\mathcal{G}(m), \beta_f(m)]$ it suffices to show that $S_{\beta_f(m),m} = \{(k)^\infty\}$. If $(\epsilon_i)_{i=1}^\infty \in S_{\beta_f(m),m}$, then clearly ϵ_i must equal $k-1, k$ or $k+1$. If $\epsilon_i = k+1$ then by Lemma 4.7 $\epsilon_{i+1} = k-1$, similarly if $\epsilon_i = k-1$ then $\epsilon_{i+1} = k+1$. Therefore if $\epsilon_i \neq k$ for some i, then $(\epsilon_i, \epsilon_{i+1}, \ldots)$ must equal $(k-1, k+1)^\infty$ or $(k+1, k-1)^\infty$. By Lemma 4.7 this cannot happen and we can conclude that $S_{\beta_f(m),m} = \{(k)^\infty\}$. For $\beta \in (\beta_f(m), m+1]$, we can construct a countable subset of $S_{\beta,m}$; for example all sequences of the form $(k)^j(k+1, k-1)^\infty$ where $j \in \mathbb{N}$.

We now consider the case where $m = 2k+1$, when $\beta = \beta_f(m)$ we have $(d_i(m))_{i=1}^\infty = (k+1, k+1, k, k)^\infty$ and

$$S_{\beta_f(m),m} = \Big\{(\epsilon_i)_{i=1}^\infty \in \{0, \ldots, m\}^{\mathbb{N}} : (\epsilon_i, \epsilon_{i+1}, \ldots) < (k+1, k+1, k, k)^\infty \text{ and } (k, k, k+1, k+1)^\infty < (\epsilon_i, \epsilon_{i+1}, \ldots) \text{ for all } i \in \mathbb{N}\Big\}.$$

By our earlier analysis we know that $\{(k, k+1)^\infty, (k+1, k)^\infty\} \subset S_{\beta,m}$ for $\beta \in (\mathcal{G}(m), m+1]$. By Remark 4.8 to prove $|S_{\beta,m}| = 2$ for $\beta \in (\mathcal{G}(m), \beta_f(m)]$ it suffices to show that $S_{\beta_f(m),m} = \{(k, k+1)^\infty, (k+1, k)^\infty\}$. By an analogous argument to that given in [9] we can show that if $(\epsilon_i)_{i=1}^\infty \in S_{\beta_f(m),m}$ then $\epsilon_i = k$ implies $\epsilon_{i+1} = k+1$, and $\epsilon_i = k+1$ implies $\epsilon_{i+1} = k$. Clearly any element of $S_{\beta_f(m),m}$ must begin with k or $k+1$ and we may therefore conclude that $S_{\beta_f(m),m} = \{(k, k+1)^\infty, (k+1, k)^\infty\}$. To see that $|W_{\beta,m}| \geq \aleph_0$ for $\beta > \beta_f(m)$ we observe that $(k+1, k)^j(k+1, k+1, k, k)^\infty \in S_{\beta,m}$ for all $j \in \mathbb{N}$, for $\beta > \beta_f(m)$. □

4.2. The growth rate of $\mathcal{G}(m)$, $\beta_f(m)$ and $\beta_c(m)$

In this section we study the growth rate of the sequences $(\mathcal{G}(m))_{m=1}^\infty$, $(\beta_f(m))_{m=1}^\infty$ and $(\beta_c(m))_{m=1}^\infty$. The following theorem summarises the growth rate of each of these sequences.

Theorem 4.10. *1.* $\mathcal{G}(2k) = k+1$ *for all* $k \in \mathbb{N}$.

2. $\beta_f(2k) - (k+2) = O(\frac{1}{k})$.

3. $\beta_c(2k) - (k+2) \to 0$ *as* $k \to \infty$.

4. $\mathcal{G}(2k+1) - (k+2) = O(\frac{1}{k})$.

5. $\beta_f(2k+1) - (k+2) \to 0$ *as* $k \to \infty$.

6. $\beta_c(2k+1) - (k+2) \to 0$ *as* $k \to \infty$.

The proof of this theorem is somewhat trivial but we include it for completion. To prove this result we firstly require the following lemma.

Lemma 4.11. *The sequence* $\beta_c(m)$ *is asymptotic to* $\frac{m}{2}$*, i.e.,* $\lim_{m\to\infty} \frac{\beta_c(m)}{m/2} = 1$.

Proof. Suppose $m = 2k$. It is a direct consequence of the definition of $\lambda_i(m)$ and $\beta_c(m)$ that the following inequalties hold

$$\sum_{i=0}^{\infty} \frac{k-1}{\beta_c(m)^i} \leq \beta_c(m) \leq \sum_{i=0}^{\infty} \frac{k+1}{\beta_c(m)^i},$$

which is equivalent to

$$\frac{k-1}{1-\frac{1}{\beta_c(m)}} \leq \beta_c(m) \leq \frac{k+1}{1-\frac{1}{\beta_c(m)}}.$$

Dividing through by $m/2$ and using the fact that $\beta_c(m) \to \infty$ we can conclude our result. The case where $m = 2k+1$ is proved similarly. □

We are now in a position to prove Theorem 4.10.

Proof of Theorem 4.10. Statements $1, 2$ and 4 are an immediate consequence of Theorem 1.1 and Theorem 4.5. It remains to show statements 3 and 6 hold; statement 5 will follow from the fact that $\mathcal{G}(2k+1) < \beta_f(2k+1) < \beta_c(2k+1)$. It is immediate from the definition of $\lambda_i(m)$ that if $m = 2k$ then

$$\beta_c(m) = k + 1 + \frac{k}{\beta_c(m)} + \sum_{i=2}^{\infty} \frac{\lambda_{i+1}(m)}{\beta_c(m)^i}.$$

It is a straighforward consequence of $1 \in U_{\beta_c(m),m}$ that $|\sum_{i=1}^{\infty} \frac{\lambda_{i+j}(m)}{\beta_c(m)^i}| \leq 1$, for all $j, m \in \mathbb{N}$. Therefore $\sum_{i=2}^{\infty} \frac{\lambda_{i+1}(m)}{\beta_c(m)^i} \to 0$ as $m \to \infty$, combining this statement with Lemma 4.11 we may conclude our result when $m = 2k$. The case where $m = 2k+1$ is proved similarly.

□

5. The growth rate and dimension theory of $\Sigma_{\beta,m}(x)$

To describe the growth rate of β-expansions we consider the following. Let

$$\mathcal{E}_{\beta,m,n}(x) = \Big\{(\epsilon_1, \ldots, \epsilon_n) \in \{0, \ldots, m\}^n | \exists (\epsilon_{n+1}, \epsilon_{n+2}, \ldots) \in \{0, \ldots, m\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} = x\Big\}.$$

We define an element of $\mathcal{E}_{\beta,m,n}(x)$ to be a *n-prefix* for x. Moreover, we let

$$\mathcal{N}_{\beta,m,n}(x) = |\mathcal{E}_{\beta,m,n}(x)|$$

and define the *growth rate of* $\mathcal{N}_{\beta,m,n}(x)$ to be

$$\lim_{n\to\infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n},$$

when this limit exists. When this limit does not exist we can consider the *lower and upper growth rates of* $\mathcal{N}_{\beta,m,n}(x)$, these are defined to be

$$\liminf_{n\to\infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n} \text{ and } \limsup_{n\to\infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n}$$

respectively.

In this paper we also consider $\Sigma_{\beta,m}(x)$ from a dimension theory perspective. We endow $\{0,\ldots,m\}^{\mathbb{N}}$ with the metric $d(\cdot,\cdot)$ defined as follows:

$$d(x,y) = \begin{cases} (m+1)^{-n(x,y)} & \text{if } x \neq y, \text{ where } n(x,y) = \inf\{i : x_i \neq y_i\} \\ 0 & \text{if } x = y. \end{cases}$$

We will consider the Hausdorff dimension of $\Sigma_{\beta,m}(x)$ with respect to this metric. It is a simple exercise to show that the following inequalities hold:

$$\dim_H(\Sigma_{\beta,m}(x)) \leq \liminf_{n\to\infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n} \leq \limsup_{n\to\infty} \frac{\log_{m+1} \mathcal{N}_{\beta,m,n}(x)}{n}. \tag{5}$$

The case where $m = 1$ is studied in [4], [8] and [10]. In [4] and [8] the authors show that for $\beta \in (1, \frac{1+\sqrt{5}}{2})$ and $x \in (0, \frac{1}{\beta-1})$ we can bound the lower growth rate and Hausdorff dimension of $\Sigma_{\beta,1}(x)$ below by some strictly positive function depending only on β, in [10] the growth rate is studied from a measure theoretic perspective. Our main result is the following.

Theorem 5.1. *For $\beta \in (1, \mathcal{G}(m))$ and $x \in (0, \frac{m}{\beta-1})$ the Hausdorff dimension of $\Sigma_{\beta,m}(x)$ can be bounded below by some strictly positive constant depending only on β.*

By (5) a similar statement holds for both the lower and upper growth rates of $\mathcal{N}_{\beta,m,n}(x)$. Replicating the proof of Lemma 2.4 it can be shown that the following result holds.

Proposition 5.2. $\mathcal{N}_{\beta,m,n}(x) = |\Omega_{\beta,m,n}(x)|$

By Proposition 5.2 we can identify elements of $\Omega_{\beta,m,n}(x)$ with elements of $\mathcal{E}_{\beta,m,n}(x)$, as such we also define an element of $\Omega_{\beta,m,n}(x)$ to be a *n-prefix* for x. To prove Theorem 5.1 we will use a method analogous to that given if [4]. We construct an interval $\mathcal{I}_\beta \subset I_{\beta,m}$ such that, for each $x \in \mathcal{I}_\beta$ we can generate multiple prefixes for x of a fixed length depending on β that map x back into $\mathcal{I}_\beta$. As we will see Theorem 5.1 will then follow by a counting argument. As was the case in our previous analysis we reduce the proof of Theorem 5.1 to two cases.

5.1. Case where m is even

In what follows we assume $m = 2k$ for some $k \in \mathbb{N}$. To prove Theorem 5.1 we require the following technical lemma.

Lemma 5.3. *For each $\beta \in (1, k+1)$ there exists $\epsilon_0(\beta) > 0$ such that, if $x \in [\frac{1}{\beta}, \frac{1}{\beta} + \epsilon_0(\beta))$ then $T_{\beta,0}(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$, and similarly if $x \in (\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)}]$ then $T_{\beta,m}(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$.*

Proof. This follows from Lemma 2.6 and a continuity argument. □

For each $i \in \{1, \ldots, m-1\}$ we let $\epsilon_i(\beta) = \frac{1}{2}(\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \frac{i+1}{\beta})$. If $\beta \in (1, k+1)$ then $\epsilon_i(\beta) > 0$ for each $i \in \{1, \ldots, m-1\}$. We define the interval $\mathcal{I}_\beta = [L(\beta), R(\beta)]$ where $L(\beta)$ and $R(\beta)$ are defined as follows:

$$L(\beta) = \min\left\{T_{\beta,1}\Big(\frac{1}{\beta} + \epsilon_0(\beta)\Big), \min_{i\in\{1,\ldots,m-1\}} T_{\beta,i+1}\Big(\frac{i+1}{\beta} + \epsilon_i(\beta)\Big)\right\}$$

and

$$R(\beta) = \max\left\{T_{\beta,m-1}\Big(\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)\Big), \max_{i\in\{1,\ldots,m-1\}} T_{\beta,i-1}\Big(\frac{i+1}{\beta} + \epsilon_i(\beta)\Big)\right\}.$$

We refer to Figure 5 for a diagram illustrating the interval $\mathcal{I}_\beta$ in the case where $m = 2$ and $\beta \in (1, 2)$.

Proposition 5.4. *Let $\beta \in (1, k+1)$. There exists $n(\beta) \in \mathbb{N}$ such that, for each $x \in \mathcal{I}_\beta$ there exists two elements $a, b \in \Omega_{\beta,m,n(\beta)}(x)$ such that $a(x) \in \mathcal{I}_\beta$ and $b(x) \in \mathcal{I}_\beta$.*

Proof. Let $x \in \mathcal{I}_\beta$. Without loss of generality we may assume that $\epsilon_0(\beta)$ is sufficiently small such that $\mathcal{I}_\beta$ contains the switch region. By Lemma 2.6 there exists a sequence of maps a that map x into the interior of our switch region. By Lemma 5.3 we may assume that $a(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$.

The distance between the endpoints of $\mathcal{I}_\beta$ and the endpoints of $I_{\beta,m}$ (the fixed points of the maps $T_{\beta,0}$ and $T_{\beta,m}$,) can be bounded below by some positive constant, by Lemma 2.1 $T_{\beta,0}$ and $T_{\beta,m}$ both scale the distance between their fixed points and a general point by a factor β, therefore we can bound the length of our sequence a above by some constant $n_s(\beta) \in \mathbb{N}$ that does not depend on x. We will show that we can take $n(\beta) = n_s(\beta) + 1$.

We remark that:

$L(\beta)$ $\frac{1}{\beta}+\epsilon_0(\beta)$ $\frac{2}{\beta}+\epsilon_1(\beta)$ $\frac{\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)$ $R(\beta)$

Figure 5: The interval $\mathcal{I}_\beta$ in the case where $m=2$ and $\beta\in(1,2)$.

$$\begin{aligned}\Big[\frac{1}{\beta}+\epsilon_0(\beta),\frac{(m-1)\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)\Big] =&\Big[\frac{1}{\beta}+\epsilon_0(\beta),\frac{2}{\beta}\Big]\\ &\bigcup\Big[\frac{(m-2)\beta+2}{\beta(\beta-1)},\frac{(m-1)\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)\Big]\\ &\bigcup_{i=1}^{m-2}\Big[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)},\frac{i+2}{\beta}\Big]\\ &\bigcup_{i=1}^{m-1}\Big[\frac{i+1}{\beta},\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}\Big].\end{aligned}$$

We now proceed via a case analysis.

- If $a(x)\in[\frac{1}{\beta}+\epsilon_0(\beta),\frac{2}{\beta}]$ then $T_{\beta,0}(a(x))\in\mathcal{I}_\beta$ and $T_{\beta,1}(a(x))\in\mathcal{I}_\beta$.
- If $a(x)\in[\frac{(m-2)\beta+2}{\beta(\beta-1)},\frac{(m-1)\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)]$ then $T_{\beta,m-1}(a(x))\in\mathcal{I}_\beta$ and $T_{\beta,m}(a(x))\in$

$\mathcal{I}_\beta$.

- If $a(x) \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}, \frac{i+2}{\beta}]$ for some $i \in \{1, \ldots, m-2\}$ then $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i+1}(a(x)) \in \mathcal{I}_\beta$.
- We reduce the case where $a(x) \in [\frac{i+1}{\beta}, \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ for some $i \in \{1, \ldots, m-1\}$ to two subcases. If $a(x) \in [\frac{i+1}{\beta}, \frac{i+1}{\beta} + \epsilon_i(\beta)]$ then by the monotonicity of our maps, both $T_{\beta,i-1}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$. Similarly, in the case where $a(x) \in [\frac{i+1}{\beta}+\epsilon_i(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}]$ both $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i+1}(a(x)) \in \mathcal{I}_\beta$.

We have shown that for any $x \in \mathcal{I}_\beta$ there exists $n(x) \leq n_s(\beta) + 1$ such that two distinct elements of $\Omega_{\beta,m,n(x)}(x)$ map x into $\mathcal{I}_\beta$. If $n(x) < n_s(\beta)+1$ then we can concatenate our two elements of $\Omega_{\beta,m,n(x)}(x)$ by an arbitrary choice of maps of length $n_s(\beta) + 1 - n(x)$ that map the image of x into $\mathcal{I}_\beta$. This ensures that we can take our sequences of maps to be of length $n_s(\beta) + 1$. □

For $\beta \in (1, k+1)$ and $x \in (0, \frac{m}{\beta-1})$ we may assume that there exists a sequence of maps a that maps x into $\mathcal{I}_\beta$. We denote the minimum number of maps required to do this by $j(x)$. Replicating arguments given in [4] we can use Proposition 5.4 to construct an algorithm by which we can generate two prefixes of length $n(\beta)$ for $a^{(j(x))}$. Repeatedly applying this algorithm to succesive images of $a^{(j(x))}$ we can generate a closed subset of $\Sigma_{\beta,m}(x)$. We denote this set by $\sigma_{\beta,m}(x)$ and the set of n-prefixes for x generated by this algorithm by $\omega_{\beta,m,n}(x)$. Replicating the proofs given in [4] we can show that the following lemmas hold.

Lemma 5.5. *Let $x \in (0, \frac{m}{\beta-1})$. Assume $n \geq j(x)$ then*

$$|\omega_{\beta,m,n}(x)| \geq 2^{\frac{n-j(x)}{n(\beta)}-1}.$$

Lemma 5.6. *Let $x \in (0, \frac{m}{\beta-1})$. Assume $l \geq j(x)$ and $b \in \omega_{\beta,m,l}(x)$, then for $n \geq l$*

$$|\{a = (a_i)_{i=1}^n \in \omega_{\beta,m,n}(x) : a_i = b_i \text{ for } 1 \leq i \leq l\}| \leq 2^{\frac{n-l}{n(\beta)}+2}.$$

With these lemmas we are now in a position to prove Theorem 5.1 in the case where m is even. The argument used is analogous to the one given in [4], which is based upon Example 2.7 of [7].

Proof of Theorem 5.1 when $m = 2k$. By the monotonicity of Hausdorff dimension with respect to inclusion it suffices to show that $\dim_H(\sigma_{\beta,m}(x))$ can be bounded below by a strictly positive constant depending only on β. It is a simple exercise to show that $\sigma_{\beta,m}(x)$ is a compact set; by this result we may restrict to finite covers of $\sigma_{\beta,m}(x)$. Let $\{U_n\}_{n=1}^N$ be a finite cover of $\sigma_{\beta,m}(x)$. Without loss of generality we may assume that all elements of our cover satisfy $\mathrm{Diam}(U_n) < (m+1)^{-j(x)}$. For each U_n there exists $l(n) \in \mathbb{N}$ such that

$$(m+1)^{-(l(n)+1)} \leq \mathrm{Diam}(U_n) < (m+1)^{-l(n)}.$$

It follows that there exists $z^{(n)} \in \{0, \ldots, m\}^{l(n)}$ such that, $y_i = z_i^{(n)}$ for $1 \leq i \leq l(n)$, for all $y \in U_n$. We may assume that $z^{(n)} \in \omega_{\beta,m,l(n)}(x)$, if we supposed otherwise then $\sigma_{\beta,m}(x) \cap U_n = \emptyset$ and we can remove U_n from our cover. We denote by C_n the set of sequences in $\{0, \ldots, m\}^{\mathbb{N}}$ whose first $l(n)$ entries agree with $z^{(n)}$, i.e.

$$C_n = \Big\{(\epsilon_i)_{i=1}^{\infty} \in \{0, \ldots, m\}^{\mathbb{N}} : \epsilon_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n)\Big\}.$$

Clearly $U_n \subset C_n$ and therefore the set $\{C_n\}_{n=1}^N$ is a cover of $\sigma_{\beta,m}(x)$.

Since there are only finitely many elements in our cover there exists J such that $(m+1)^{-J} \leq \text{Diam}(U_n)$ for all n. We consider the set $\omega_{\beta,m,J}(x)$. Since $\{C_n\}_{n=1}^N$ is a cover of $\sigma_{\beta,m}(x)$ each $a \in \omega_{\beta,m,J}(x)$ satisfies $a_i = z_i^{(n)}$ for $1 \leq i \leq l(n)$, for some n. Therefore

$$|\omega_{\beta,m,J}(x)| \leq \sum_{n=1}^{N} \left|\{a \in \omega_{\beta,m,J}(x) : a_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n)\}\right|.$$

By counting elements of $\omega_{\beta,m,J}(x)$ and Lemmas 5.5 and 5.6 we observe the following;

$$\begin{aligned}
2^{\frac{J-j(x)}{n(\beta)}-1} &\leq |\omega_{\beta,m,J}(x)| \\
&\leq \sum_{n=1}^{N} \left|\{a \in \omega_{\beta,m,J}(x) : a_i = z_i^{(n)} \text{ for } 1 \leq i \leq l(n)\}\right| \\
&\leq \sum_{n=1}^{N} 2^{\frac{J-l(n)}{n(\beta)}+2} \\
&= 2^{\frac{J+1}{n(\beta)}+2} \sum_{n=1}^{N} 2^{\frac{-(l(n)+1)}{n(\beta)}} \\
&\leq 2^{\frac{J+1}{n(\beta)}+2} \sum_{n=1}^{N} \text{Diam}(U_n)^{\frac{\log_{m+1} 2}{n(\beta)}}.
\end{aligned}$$

Dividing through by $2^{\frac{J+1}{n(\beta)}+2}$ yields

$$\sum_{n=1}^{N} \text{Diam}(U_n)^{\frac{\log_{m+1} 2}{n(\beta)}} \geq 2^{\frac{-j(x)-3n(\beta)-1}{n(\beta)}},$$

the right hand side is a positive constant greater than zero that does not depend on our choice of cover. It follows that $\dim_H(\sigma_{\beta,m}(x)) \geq \frac{\log_{m+1} 2}{n(\beta)}$, our result follows. □

5.2. Case where m is odd

In what follows we assume $m = 2k+1$ for some $k \in \mathbb{N}$. For $\beta \in (1, \frac{2k+3}{2})$ the proof of Theorem 5.1 is analogous to the even case for $\beta \in (1, k+1)$. As such, in what follows we assume $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. The significance of $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ is that for $i \in \{1, \ldots, m-1\}$ the i-th fixed digit interval is well defined.

Before defining the interval $\mathcal{I}_\beta$ we require the following. We let

$$\epsilon_i(\beta) = \begin{cases} \frac{1}{2}\Big(\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \frac{i}{\beta-1}\Big) & \text{if } i \in \{1, \ldots, k\} \\ \frac{1}{2}\Big(\frac{i}{\beta-1} - \frac{i+1}{\beta}\Big) & \text{if } i \in \{k+1, \ldots, m-1\} \end{cases}$$

By Lemma 3.3, $\epsilon_i(\beta) > 0$ for all $i \in \{1, \ldots, m-1\}$ for $\beta \in (1, k+2)$. Before proving an analogue of Proposition 5.4 we require the following technical lemmas. It is a simple exercise to show that the following analogue of Lemma 5.3 holds.

Lemma 5.7. *For each $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ there exists $\epsilon_0(\beta) > 0$ such that, if $x \in [\frac{1}{\beta}, \frac{1}{\beta} + \epsilon_0(\beta))$ then $T_{\beta,0}(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$, and similarly if $x \in (\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)}]$ then $T_{\beta,m}(x) \in [\frac{1}{\beta} + \epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)]$.*

Lemma 5.8. *Let $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. For each $i \in \{1, \ldots, k-1\}$ there exists $\epsilon_i^*(\beta) > 0$ such that, if $x \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)]$ then $T_{\beta,i}(x) < \frac{k+2}{\beta} + \epsilon_{k+1}$. Similarly for $i \in \{k+2, \ldots, m-1\}$ there exists $\epsilon_i^*(\beta) > 0$ such that, if $x \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta), \frac{i+1}{\beta} + \epsilon_i(\beta)]$ then $T_{\beta,i}(x) > \frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)} - \epsilon_k$.*

Proof. By the analysis given in the proof of Lemma 3.5 for $i \in \{1, \ldots, k-1\}$, we have $T_{\beta,i}(\frac{i+1}{\beta}) < \frac{k\beta+m-k}{\beta(\beta-1)}$ for $\beta \in (1, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. However, for $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$ $\frac{k\beta+m-k}{\beta(\beta-1)} \leq \frac{k+2}{\beta}$. The existence of $\epsilon_i^*(\beta)$ then follows by a continuity argument and the monotonicity of the maps $T_{\beta,i}$. The case where $i \in \{k+2, \ldots, m-1\}$ is proved similarly. □

We are now in a position to define the interval $\mathcal{I}_\beta$. Let $\mathcal{I}_\beta = [L(\beta), R(\beta)]$ where

$$L(\beta) = \min\Bigg\{T_{\beta,1}\Big(\frac{1}{\beta} + \epsilon_0(\beta)\Big), T_{\beta,k+1}\Big(\frac{k\beta+k+1}{\beta^2-1}\Big), \min_{i\in\{2,\ldots,k\}}\Big\{T_{\beta,i}\Big(\frac{i}{\beta} + \epsilon_{i-1}^*(\beta)\Big)\Big\}, \min_{i\in\{k+2,\ldots,m\}}\Big\{T_{\beta,i}\Big(\frac{i}{\beta} + \epsilon_{i-1}(\beta)\Big)\Big\}\Bigg\}$$

and

$$R(\beta) = \max\Bigg\{T_{\beta,k}\Big(\frac{(k+1)\beta+k}{\beta^2-1}\Big), T_{\beta,m-1}\Big(\frac{(m-1)\beta+1}{\beta(\beta-1)} - \epsilon_0(\beta)\Big), \max_{i\in\{1,\ldots,k\}}\Big\{T_{\beta,i-1}\Big(\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta)\Big) \max_{i\in\{k+2,\ldots,m-1\}}\Big\{T_{\beta,i-1}\Big(\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta)\Big)\Big\}\Bigg\}.$$

$L(\beta)$ $R(\beta)$

$\frac{1}{\beta}+\epsilon_0(\beta)$ $\frac{3}{\beta(\beta-1)}-\epsilon_1(\beta)$ $\frac{\beta+2}{\beta^2-1}$ $\frac{2\beta+1}{\beta^2-1}$ $\frac{3}{\beta}+\epsilon_2(\beta)$ $\frac{2\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)$

Figure 6: The interval $\mathcal{I}_\beta$ in the case where $m=3$ and $\beta \in [\frac{5}{2}, 1+\sqrt{3})$.

For ease of exposition in Figure 6 we give a diagram illustrating the interval $\mathcal{I}_\beta$, in the case where $m=3$ and $\beta \in [\frac{5}{2}, 1+\sqrt{3})$.

Proposition 5.9. *Let* $\beta \in [\frac{2k+3}{2}, \frac{k+1+\sqrt{k^2+6k+5}}{2})$. *There exists* $n(\beta) \in \mathbb{N}$ *such that, for each* $x \in \mathcal{I}_\beta$ *there exists two elements* $a, b \in \Omega_{\beta,m,n(\beta)}(x)$ *such that* $a(x) \in \mathcal{I}_\beta$ *and* $b(x) \in \mathcal{I}_\beta$.

Proof. Without loss of generality we may assume that $\epsilon_0(\beta)$ is sufficiently small such that $\mathcal{I}_\beta$ contains the switch region. By Lemma 2.6 there exists a sequence of maps a that map x into the switch region. As the endpoints of $\mathcal{I}_\beta$ are bounded away from the endpoints of $I_{\beta,m}$ we can bound the length of a above by some $n_s(\beta) \in \mathbb{N}$. Moreover, by Lemma 5.7 we may assume that $a(x) \in [\frac{1}{\beta}+\epsilon_0(\beta), \frac{(m-1)\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)]$. As in the even case it is useful to treat $[\frac{1}{\beta}+\epsilon_0(\beta),\frac{(m-1)\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)]$ as the union of subintervals. We observe that

$$\begin{aligned}\Big[\frac{1}{\beta}+\epsilon_0(\beta),\frac{(m-1)\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)\Big] =&\Big[\frac{1}{\beta}+\epsilon_0(\beta),\frac{m}{\beta(\beta-1)}-\epsilon_1(\beta)\Big]\\ &\bigcup\Big[\frac{m}{\beta}+\epsilon_{m-1}(\beta),\frac{(m-1)\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)\Big]\\ &\bigcup\Big[\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)}-\epsilon_k(\beta),\frac{k+2}{\beta}+\epsilon_{k+1}(\beta)\Big]\\ &\bigcup_{i=2}^{k}\Big[\frac{i}{\beta}+\epsilon^*_{i-1}(\beta),\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}-\epsilon_i(\beta)\Big]\\ &\bigcup_{i=k+2}^{m-1}\Big[\frac{i}{\beta}+\epsilon_{i-1}(\beta),\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}-\epsilon^*_i(\beta)\Big]\\ &\bigcup_{i=1}^{k-1}\Big[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}-\epsilon_i(\beta),\frac{i+1}{\beta}+\epsilon^*_i(\beta)\Big]\\ &\bigcup_{i=k+2}^{m-1}\Big[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}-\epsilon^*_i(\beta),\frac{i+1}{\beta}+\epsilon_i(\beta)\Big].\end{aligned}$$

Without loss of generality we may assume that $\epsilon_0(\beta),\epsilon_i(\beta),\epsilon^*_i(\beta)$ are all sufficiently small such that each of the above intervals in our union are well defined and nontrivial. We now proceed via a case analysis.

- If $a(x)\in[\frac{1}{\beta}+\epsilon_0(\beta),\frac{m}{\beta(\beta-1)}-\epsilon_1(\beta)]$ then $T_{\beta,0}(a(x))\in\mathcal{I}_\beta$ and $T_{\beta,1}(a(x))\in\mathcal{I}_\beta$.
- If $a(x)\in[\frac{m}{\beta}+\epsilon_{m-1}(\beta),\frac{(m-1)\beta+1}{\beta(\beta-1)}-\epsilon_0(\beta)]$ then $T_{\beta,m-1}(a(x))\in\mathcal{I}_\beta$ and $T_{\beta,m}(a(x))\in\mathcal{I}_\beta$.
- Suppose $a(x)\in[\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)}-\epsilon_k(\beta),\frac{k+2}{\beta}+\epsilon_{k+1}(\beta)]$. If $a(x)\in[\frac{k\beta+k+1}{\beta^2-1},\frac{(k+1)\beta+k}{\beta^2-1}]$ then $T_{\beta,k}(a(x))\in\mathcal{I}_\beta$ and $T_{\beta,k+1}(a(x))\in\mathcal{I}_\beta$. If $a(x)\in[\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)}-\epsilon_k(\beta),\frac{k\beta+k+1}{\beta^2-1}]$ then we are a bounded distance away from the fixed point of the map $T_{\beta,k}$, by Lemma 2.1 we know that $T_{\beta,k}$ scales the distance between $a(x)$ and the fixed point of $T_{\beta,k}$ by a factor β, therefore we can bound the number of maps required to map $a(x)$ into $[\frac{k\beta+k+1}{\beta^2-1},\frac{(k+1)\beta+k}{\beta^2-1}]$. By a similar argument, if $a(x)\in[\frac{(k+1)\beta+k}{\beta^2-1},\frac{k+2}{\beta}+\epsilon_{k+1}(\beta)]$ we can bound the number of maps required to map $a(x)$ into $[\frac{k\beta+k+1}{\beta^2-1},\frac{(k+1)\beta+k}{\beta^2-1}]$. By the above we can assert that when $a(x)\in[\frac{(k-1)\beta+m-(k-1)}{\beta(\beta-1)}-\epsilon_k(\beta),\frac{k+2}{\beta}+\epsilon_{k+1}(\beta)]$ there exists two distinct sequences of maps whose length we can bound above by some $n_c(\beta)\in\mathbb{N}$ that map $a(x)$ into $\mathcal{I}_\beta$.
- If $a(x)\in[\frac{i}{\beta}+\epsilon^*_{i-1}(\beta),\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)}-\epsilon_i(\beta)]$ for some $i\in\{2,\ldots,k-1\}$ then $T_{\beta,i-1}(a(x))\in\mathcal{I}_\beta$ and $T_{\beta,i}(a(x))\in\mathcal{I}_\beta$.

- If $a(x) \in [\frac{i}{\beta} + \epsilon_i(\beta), \frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta)]$ for some $i \in \{k+2, \ldots, m-1\}$ then $T_{\beta,i-1}(a(x)) \in \mathcal{I}_\beta$ and $T_{\beta,i}(a(x)) \in \mathcal{I}_\beta$.

- If $a(x) \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)]$ for some $i \in \{1, \ldots, k-1\}$ then $a(x)$ is a bounded distance away from the fixed point of the map $T_{\beta,i}$, by Lemma 2.1 we know that $T_{\beta,i}$ scales the distance between $a(x)$ and its fixed point by a factor β, therefore we can bound the number of maps required to map $a(x)$ outside of the interval $[\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)]$ by some $n_i(\beta) \in \mathbb{N}$. If $a(x)$ has been mapped into an interval covered by one of the above cases we are done, if not it has to be mapped into another interval of the form $[\frac{(j-1)\beta+m-(j-1)}{\beta(\beta-1)} - \epsilon_j(\beta), \frac{j+1}{\beta} + \epsilon_j^*(\beta)]$. By Corollary 3.4 and Lemma 5.8 we know that $i < j \leq k+1$. Repeating the previous step as many times as is necessary we can ensure that within $\sum_{i=1}^{k-1} n_i(\beta)$ maps, $a(x)$ has to be mapped into an interval that was addressed in one of our previous cases.

- The case where $a(x) \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i^*(\beta), \frac{i+1}{\beta} + \epsilon_i(\beta)]$ for some $i \in \{k+2 \ldots, m-1\}$ is analogous to the case where $a(x) \in [\frac{(i-1)\beta+m-(i-1)}{\beta(\beta-1)} - \epsilon_i(\beta), \frac{i+1}{\beta} + \epsilon_i^*(\beta)]$ for some $i \in \{1, \ldots, k-1\}$.

We have shown that for any $x \in \mathcal{I}_\beta$ there exists $n(x) \in \mathbb{N}$ such that, two distinct elements of $\Omega_{\beta,m,n(x)}(x)$ map x into $\mathcal{I}_\beta$, moreover $n(x) \leq n_s(\beta) + n_c(\beta) + \sum_{i=1}^{k-1} n_i(\beta)$. We take $n(\beta)$ to equal $n_s(\beta) + n_c(\beta) + \sum_{i=1}^{k-1} n_i(\beta)$. If $n(x) < n(\beta)$ then as in the even case we concatenate our image of x by an arbitrary sequence of maps of length $n(\beta) - n(x)$ that map x into $\mathcal{I}_\beta$, this ensures our sequences of maps are of length $n(\beta)$.

□

Repeating the analysis given in the case where m is even we can conclude Theorem 5.1 in the case where m is odd.

6. Open questions and a table of values for $\mathcal{G}(m)$, $\beta_f(m)$ and $\beta_c(m)$

We conclude with a few open questions and a table of values for $\mathcal{G}(m)$, $\beta_f(m)$ and $\beta_c(m)$.

- In [1] the authors study the order in which periodic orbits appear in the set of uniqueness. When $m = 1$ they show that as $\beta \nearrow 2$ the order in which periodic orbits appear in the set of uniqueness is intimately related to the classical Sharkovskii ordering. It is natural to ask whether a similar result holds in our general case.

- In [18] it is shown that when $m = 1$ and $\beta = \frac{1+\sqrt{5}}{2}$ the set of numbers: $x = \frac{(1+\sqrt{5})n}{2} (\mathrm{mod}\ 1)$ for some $n \in \mathbb{N}$ have countably many β-expansions, while the other elements of $(0, \frac{1}{\beta-1})$ have uncountably many β-expansions. Does an analogue of this statement hold in the case of general m?

- Let $p_1,\ldots,p_k$ be points in $\mathbb{R}^d$ such that the polyhedra Π with these vertices is convex. Let $\{f_i\}_{i=1}^k$ be the one parameter family of maps given by

$$f_i(x)=\lambda x+(1-\lambda)p_i,$$

where $\lambda\in(0,1)$ is our parameter. As is well know there exists a unique compact non-empty S_λ such that $S_\lambda=\cup_{i=1}^k f_i(S_\lambda)$. We say that $(\epsilon_i)_{i=1}^\infty\in\{1,\ldots,k\}^{\mathbb{N}}$ is an address for $x\in S_\lambda$ if $\lim_{n\to\infty}(f_{\epsilon_n}\circ\ldots\circ f_{\epsilon_1})(\mathbf{0})=x$. We ask whether an analogue of the golden ratio exists in this case, i.e, does there exists λ^* such that for $\lambda\in(\lambda^*,1)$ every $x\in S_\lambda\setminus\{p_1,\ldots,p_k\}$ has uncountably many addresses, but for $\lambda\in(0,\lambda^*)$ there exists $x\in S_\lambda\setminus\{p_1,\ldots,p_k\}$ with a unique address. In [16] the author shows that an analogue of the golden ratio exists in the case when $d=2$ and $k=3$.

Table 1: Table of values for $\mathcal{G}(m)$, $\beta_f(m)$ and $\beta_c(m)$

m	$\mathcal{G}(m)$	$\beta_f(m)$	$\beta_c(m)$
1	$\frac{1+\sqrt{5}}{2}\approx 1.61803\ldots$	$1.75488\ldots$	$1.78723\ldots$
2	2	$1+\sqrt{2}=2.41421\ldots$	$2.47098\ldots$
3	$1+\sqrt{3}\approx 2.73205\ldots$	2.89329...	2.90330...
4	3	$\frac{3+\sqrt{17}}{2}=3.56155\ldots$	$3.66607\ldots$
5	$\frac{3+\sqrt{21}}{2}\approx 3.79129\ldots$	3.93947	$3.94583\ldots$
6	4	$2+\frac{\sqrt{28}}{2}=4.64575\ldots$	$4.75180\ldots$
7	$2+2\sqrt{2}\approx 4.82843\ldots$	4.96095...	$4.96496\ldots$
8	5	$\frac{5+\sqrt{41}}{2}=5.70156\ldots$	$5.80171\ldots$
9	$\frac{5+\sqrt{45}}{2}\approx 5.85410\ldots$	$5.97273\ldots$	$5.97537\ldots$
10	6	$3+\sqrt{14}=6.74166\ldots$	$6.83469\ldots$

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References

[1] J,-P. Allouche, M. Clarke and N. Sidorov, Periodic unique beta-expansions: the Sharkovskii ordering, Ergodic Theory Dynam. Systems 29 (2009), no. 4, 1055–1074.

[2] J,-P. Allouche and M. Cosnard, The Komornik-Loreti constant is transcendental, Amer. Math. Monthly 107 (2000), no. 5, 448–449.

[3] J,-P. Allouche and J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in C. Ding, T. Helleseth, and H. Niederreiter, eds., Sequences and their applications: Proceedings of SETA '98, Springer-Verlag, 1999, pp. 1–16.

[4] S. Baker, The growth rate and dimension theory of beta-expansions, Fund. Math. 219 (2012), 271–285

[5] Z. Daróczy and I. Katai, Univoque sequences, Publ. Math. Debrecen 42 (1993), no. 3–4, 397–407.

[6] P. Erdős, I. Joó and V. Komornik, Characterization of the unique expansions $1 = \sum_{i=1}^{\infty} q^{-n_i}$ and related problems, Bull. Soc. Math. Fr. 118 (1990), 377–390.

[7] K. Falconer. Fractal Geometry: Mathematical Foundation and Applications. John Wiley, Chichester, 1990.

[8] D. J. Feng, N. Sidorov, Growth rate for beta-expansions, Monatsh. Math. 162 (2011), no. 1, 41–60.

[9] P. Glendinning and N. Sidorov, Unique representations of real numbers in non-integer bases, Math. Res. Letters 8 (2001), 535–543.

[10] T. Kempton, Counting β-expansions and the absolute continuity of Bernoulli convolutions, Preprint.

[11] V. Komornik, A. Lai, M. Pedicini, Generalized golden ratios of ternary alphabets, J. Eur. Math. Soc. (JEMS) 13 (2011), no. 4, 1113–1146.

[12] V. Komornik and P. Loreti, Unique developments in non-integer bases, Amer. Math. Monthly 105 (1998), no. 7, 636–639.

[13] V. Komornik and P. Loreti, Subexpansions, superexpansions and uniqueness properties in non-integer bases, Period. Math. Hungar. 44 (2002), no. 2, 197–218.

[14] D. Kong, W. Li, F. Dekking, Intersections of homogeneous Cantor sets and beta-expansions, Nonlinearity 23 (2010), no. 11, 2815–2834.

[15] W. Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hungar. 11 (1960) 401–416.

[16] N. Sidorov, Combinatorics of linear iterated function systems with overlaps. Nonlinearity 20 (2007), no. 5, 1299–1312.

[17] N. Sidorov, Expansions in noninteger bases, Lecture notes of a graduate course at the summer school at Queen Mary, University of London, July, 2010.

[18] N. Sidorov and A. Vershik, Ergodic properties of the Erdos measure, the entropy of the golden shift, and related problems. Monatsh. Math. 126 (1998), no. 3, 215–261.