



# *On the validity of single-parcel energetics to assess the importance of internal energy and compressibility effects in stratified fluids*

Article

Accepted Version

Tailleux, R. (2015) On the validity of single-parcel energetics to assess the importance of internal energy and compressibility effects in stratified fluids. *Journal of Fluid Mechanics*, 767. R2. ISSN 0022-1120 doi: <https://doi.org/10.1017/jfm.2015.77> Available at <http://centaur.reading.ac.uk/39238/>

It is advisable to refer to the publisher's version if you intend to cite from the work.

To link to this article DOI: <http://dx.doi.org/10.1017/jfm.2015.77>

Publisher: Cambridge University Press

All outputs in CentAUR are protected by Intellectual Property Rights law, including copyright law. Copyright and IPR is retained by the creators or other copyright holders. Terms and conditions for use of this material are defined in the [End User Agreement](#).

[www.reading.ac.uk/centaur](http://www.reading.ac.uk/centaur)

## **CentAUR**

Central Archive at the University of Reading

Reading's research outputs online

# On the validity of single-parcel energetics to assess the importance of internal energy and compressibility effects in stratified fluids

By RÉMI TAILLEUX<sup>1†</sup>

<sup>1</sup>Dept. of Meteorology, University of Reading, Earley Gate, PO Box 243, Reading, RG6 6BB

(Received ?? and in revised form ??)

It is often assumed on the basis of single-parcel energetics that compressible effects and conversions with internal energy are negligible whenever typical displacements of fluid parcels are small relative to the scale height of the fluid (defined as the ratio of the squared speed of sound over gravitational acceleration). This paper shows that the above approach is flawed, however, and that a correct assessment of compressible effects and internal energy conversions requires considering the energetics of at least two parcels, or more generally, of mass conserving parcel re-arrangements. As a consequence, it is shown that it is the adiabatic lapse rate and its derivative with respect to pressure, rather than the scale height, which controls the relative importance of compressible effects and internal energy conversions when considering the global energy budget of a stratified fluid. Only when mass conservation is properly accounted for is it possible to explain why available internal energy can account for up to 40 percent of the total available potential energy in the oceans. This is considerably larger than the prediction of single-parcel energetics, according to which this number should be no more than about 2 percent.

---

## 1. Introduction

In the study of the energetics of stratified fluids, one of the criterion commonly assumed to control the importance of compressible effects and internal energy conversions relative to conversions with gravitational potential energy is the ratio  $\delta z/H_s$ , where  $\delta z$  is a typical vertical displacement and  $H_s = c_s^2/g$  is the scale height of the fluid, where  $c_s$  is the speed of sound and  $g$  the acceleration of gravity, e.g., see discussion leading to Eq. (3.6.18) of Batchelor (1967). Moreover, Batchelor (1967) suggests that the criterion  $\delta z/H_s \ll 1$  is one of out three that needs to be satisfied for a fluid to be regarded as “if it were incompressible”. Physically, this ratio can be viewed as measuring the relative change in internal energy  $\Delta IE$  over gravitational potential energy  $\Delta GPE$  experienced by a fluid parcel in a hydrostatic vertical displacement  $\delta z = z_2 - z_1$ . Indeed, if  $dm = \rho dV$  denotes the mass of the fluid parcel considered,  $\Delta GPE$  and  $\Delta IE$  are given by

$$\Delta GPE = dm g(z_2 - z_1), \quad (1.1)$$

$$\Delta IE = -dm \int_{p_1}^{p_2} p \left. \frac{\partial v}{\partial p} \right|_{adiabatic} dp = dm \frac{\bar{p}}{\rho^2 c_s^2} (p_2 - p_1) \approx dm \frac{g^2 \bar{z} (z_2 - z_1)}{c_s^2}, \quad (1.2)$$

† Present address: Dept. of Meteorology, University of Reading, Earley Gate, Reading, RG6 6BB, United Kingdom

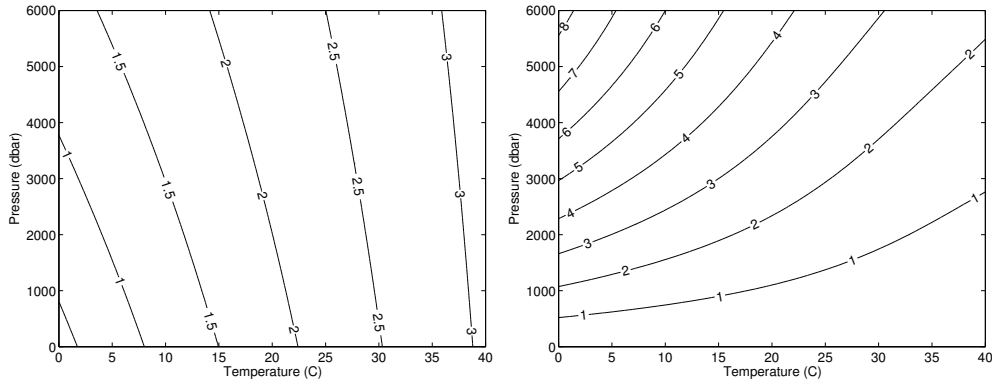


FIGURE 1. (Left panel) Adiabatic lapse rate in  $10^{-8}\text{K/Pa}$  for seawater, assuming Absolute Salinity  $S_A = 35\text{g/kg}$ . (Right panel) The second derivative  $-\partial^2 u / \partial p \partial \eta = p \partial \Gamma / \partial p$  ( $\times 10^9$ ), as a function of pressure and temperature for seawater, assuming Absolute Salinity  $S_A = 35\text{g/kg}$ , in SI units.

which implies, as stated

$$\frac{\Delta IE}{\Delta GPE} = \frac{\bar{z}}{H_s}, \quad (1.3)$$

where  $v = 1/\rho$  is the specific volume. To obtain the expression (1.2) for  $\Delta IE$ , we defined  $\bar{z}$  as the mean height/depth of the parcel considered, and used Boussinesq scaling to make the approximations  $\rho \approx \rho_0$ ,  $\bar{p} \approx -\rho_0 g \bar{z}$  and  $p_2 - p_1 \approx -\rho_0 g (z_2 - z_1)$ . In the atmosphere, the formula for the speed of sound is  $c_s = \sqrt{\gamma RT} \approx 335\text{m.s}^{-1}$ , which leads to  $H_s \approx 12\text{km}$  (based on using  $\gamma = c_p/c_v = 1006/719 = 1.4$ ,  $R = 287\text{Jkg}^{-1}\text{K}^{-1}$ ,  $T \approx 280\text{K}$ , where  $c_p$  is the specific heat capacity at constant pressure,  $c_v$  the specific capacity at constant volume,  $R$  the perfect gas constant, and  $T$  the in-situ temperature in Kelvins). As the latter value is comparable to the height of the troposphere, compressibility effects and internal energy are generally thought to be important in discussions of atmospheric energetics. In the ocean, however, the speed of sound  $c_s$  is  $O(1500\text{m.s}^{-1})$ ; the associated scale height is  $H_s = O(225\text{km})$ , which is considerably larger than even the deepest parts of the ocean. Since typical ocean depths rarely exceed 5000 m, the ratio (1.3) should rarely exceed 2%, suggesting that in contrast to the atmosphere, compressible effects and internal energy conversions should be unimportant in discussions of ocean energetics.

Although internal energy is now recognised to be essential to close the ocean energy budget, e.g., Young (2010); Tailleux (2012); Eden et al. (2014), its exact role and importance still remains unclear as is apparent in all recent reviews of ocean energetics, e.g., Wunsch & Ferrari (2004), Kuhlbrodt & al. (2007). The main difficulty appears to be the lack of rigorous approach to quantify the thermodynamic work of expansion/contraction, e.g., Tailleux (2010). The main aim of this paper is to show that single-parcel energetics is not a solution to that end, for it is inconsistent with at least three counter-arguments detailed below. Understanding the limitations of single-parcel energetics is important, for parcel-based arguments pervade the atmospheric and oceanic literature, e.g., Thorpe et al. (1989); Gade & Gustafsson (2004); McDougall (1987), while also being potentially very misleading when applied to liquids such as water or seawater.

Our first counter-argument suggests that it is the adiabatic lapse rate  $(dT/dp)_{ad} = \Gamma = \alpha T / (\rho c_p)$ , rather than the speed of sound  $c_s$  or scale height, which is the most useful fundamental quantity for correctly assessing compressibility effects. Here  $\alpha$  is the isobaric thermal expansion coefficient,  $T$  is the absolute temperature in Kelvins,  $\rho$  is the

density, and  $c_p$  is the specific heat capacity at constant pressure. Indeed, it is  $\Gamma$ , rather than  $c_s$ , which controls adiabatic temperature changes due to compression/expansion. The adiabatic lapse rate for seawater is illustrated in Fig. 1 (left panel) as a function of temperature and pressure. This figure shows that  $\Gamma$  primarily depends on temperature, and that its value typically lies between  $10^{-8}\text{K/Pa}$  and  $3.10^{-8}\text{K/Pa}$ . We can therefore estimate that a fluid parcel being displaced from the ocean surface to the ocean bottom (assumed to lie at about  $6000\text{ m} \approx 6000\text{ dbar}$ ) would experience a temperature increase due to adiabatic compression of the order  $\Delta T \approx \Gamma \Delta p = O(1^\circ\text{C})$ , based on using  $\Delta p \approx 6.10^7\text{ Pa}$  ( $= O(6000\text{ m})$ ) and  $\Gamma = 2.10^{-8}\text{K/Pa}$ . While such a temperature difference is relatively small compared to vertical temperature contrasts  $O(15 - 30^\circ)$  characteristic of equatorial and mid-latitude regions, it is comparable to the vertical temperature contrast characteristic of polar regions, so that the ratio of  $\Delta T$  over the vertical temperature contrast can occasionally be much larger than the value of 2% characterising the ratio  $\delta z/H_s$  discussed above.

Our second counter-argument relies on an important result established by Huang (2005), which suggests that the available internal energy (AIE) is negative and about 40% of the total available potential energy (APE) in the ocean. While the negative character of AIE has long been understood and rationalised, e.g., Reid et al. (1981), the result that it could represent such a large fraction of the total APE is counter-intuitive and incompatible with our current understanding of ocean energetics as discussed by Oort et al. (1989) for instance. Indeed, the APE is by definition the difference between the potential energy of the actual state and that of a reference state minimising potential energy in an adiabatic (and iso-haline) re-arrangement of the fluid parcels. It follows that the relative change in internal over gravitational potential energy of a fluid parcel moved adiabatically from its actual position to its reference position should be governed by the single-parcel energetics argument discussed above, so that it would be natural to expect the ratio  $AIE/APE$  to be  $O(2\%)$  as for the ratio  $\Delta IE/\Delta GPE$  of a single parcel.

Our last counter-argument relies on the recent results on the energetics of turbulent stratified mixing by Tailleux (2009) and Tailleux (2013) obtained from the rigorous analysis of the energetics of the fully compressible Navier-Stokes equations, which suggest that internal energy and compressibility effects in turbulent stratified fluids (especially liquids) are more important than previously assumed. Specifically, the change in background potential energy accompanying turbulent stratified mixing is found to occur at the expense of internal energy, rather than at the expense of the mechanical source of energy driving the mixing; moreover, the conversion between IE and background GPE is found to be a reversible one. As to the dissipation of APE by turbulent molecular diffusion, it is found to be fundamentally associated with the irreversible work of the divergent part of the velocity field against the pressure gradient, and to convert APE irreversibly into internal energy in the same way that viscous dissipation irreversibly converts kinetic energy into internal energy.

In view of the above evidence, it seems urgent to revisit the validity of single-parcel energetics argument for predicting the relative magnitude of compressible effects and conversions with internal energy in (turbulent) stratified fluids. In this note, we demonstrate that its key flaw is to overlook the fact that for any fluid parcel experiencing a vertical displacement, one (or more) parcel will have to move in the opposite direction to satisfy the requirement of mass conservation. [As discussed in Tailleux & Grandpeix \(2004\) in the context of moist atmospheric energetics, accounting for mass conservation can lead to a significant reduction of conditional instability estimates for an atmospheric sounding when compared to parcel-based CAPE \(Convective Available Potential Energy\) estimates.](#) Thus, for a parcel moving up whose GPE increases, there must be a parcel

moving down whose GPE decreases, so that the net change in GPE for the two-parcel system is actually much smaller than the GPE change attached to a single parcel. Obviously, the same is true for the IE changes, so that the key question is to what extent the ratio  $\Delta IE/\Delta GPE$  predicted by two-parcel energetics differ from the ratio  $\Delta IE/\Delta GPE$  predicted by single-parcel energetics? As shown in this note, the difference may occasionally turn out to be quite considerable, although it strongly depends on the equation of state of the fluid considered. Section 2 establishes that inferences about the relative importance of internal and gravitational potential energy for global energetics must involve at least two fluid parcels in order to account for mass conservation properly. Section 3 revisits the role of internal energy and compressibility effects for the oceanic and atmospheric cases in the light of the results of section 2. Section 4 summarises and discusses the results.

## 2. The importance of mass conservation for global energetics

As stated above, upward motion is in general compensated by compensating downward motion to ensure mass conservation. At least two parcels are therefore needed to investigate the effect of mass conservation on global energetics. To that end, the simplest approach is probably to consider the energetics of two parcels swapping position. Since a well known result of algebra and combinatorics is that any permutation is the product of pairwise permutations, we can expect [two-parcel energetics](#) to also shed light on the global energetics of more complex parcel rearrangements.

To compute the changes in gravitational potential and internal energy attached to each parcel requires the knowledge of height or depth for the former, and of temperature and pressure for the latter. Assuming the pressure field to be hydrostatic is therefore essential to link height or depth with pressure; as it turns out, the assumption also makes it possible to link gravitational potential energy changes to those of the thermodynamic quantity  $pv$ , where  $p$  is the hydrostatic pressure, and  $v = 1/\rho$  is the specific volume. This proves particularly useful, as  $pv = h - u$ , i.e., it is equal to the difference between the specific enthalpy  $h$  and internal energy  $u$ . This makes it possible to regard enthalpy as the effective potential energy, and  $pv = h - u$  as the effective gravitational potential energy, which considerably facilitates the discussion of the energetics of two parcels swapping position, as discussed below.

### 2.1. Quasi-thermodynamic character of gravitational potential energy

We first outline the classical proof linking gravitational potential energy to  $Pv$ . Motivated by the oceanic case, let us consider a fluid bounded above by a free surface and below by an uneven bottom respectively defined by the equations  $z = \zeta(x, y, t)$  and  $z = -H(x, y)$  respectively. Let us also assume that at the surface, the atmospheric pressure is  $p = p_a$  independent of position, and let us denote the bottom pressure by  $p = p_b(x, y, t)$ . Assuming furthermore the pressure to be hydrostatic allows one to express GPE as

$$\begin{aligned} \text{GPE} &= \int_V \rho g z \, dV = - \int_S [zp]_{-H}^{\zeta} \, dx \, dy + \int_V pv \, dm = - \int_S [\zeta p_a + H p_b] \, dx \, dy + \int_V pv \, dm \\ &= -p_a V_{total} - \int_S H(p_b - p_a) \, dx \, dy + \int_V pv \, dm = -p_a V_{tot} - M_{tot} g \bar{H} + \int_V pv \, dm \end{aligned}$$

as discussed by Oliver & Tailleux (2013) and Huang (2005), where  $V_{tot}$  and  $M_{tot}$  denote the total volume and total mass of the fluid respectively,

$$V_{tot} = \int_S (\zeta + H) dx dy, \quad M_{tot} = \frac{1}{g} \int_V (p_b - p_a) dx dy \quad (2.1)$$

It follows that

$$\Delta GPE = -p_a \Delta V_{tot} - M_{tot} g \Delta \bar{H} + \int_V \Delta(pv) dm \approx \int_V \Delta[(p - p_a)v] dm \quad (2.2)$$

Although the term  $M_{tot} g \Delta \bar{H}$  vanishes and is therefore unimportant for a flat-bottom ocean (of constant mass), it may be significant in the actual (non-flat) ocean, as discussed by Huang (2005). This effect, however, is unrelated to that discussed in this paper, and is therefore not further considered. As a result, the sum of gravitational potential energy and internal energy reads:

$$PE = \int_V h dm - p_a V_{tot} - M_{tot} g \bar{H} \quad (2.3)$$

where  $h$  is the specific enthalpy.

## 2.2. Local form of gravitational potential energy in pressure coordinates

The thermodynamic character of gravitational potential energy in the hydrostatic approximation can also be established directly by forming the evolution equation for the total energy for the hydrostatic primitive equations written in pressure coordinates, viz.,

$$\frac{D\mathbf{V}}{Dt} + \nabla_p \Phi = \mathbf{F}, \quad (2.4)$$

$$\frac{\partial \Phi}{\partial p} = -v, \quad (2.5)$$

$$\nabla_p \cdot \mathbf{V} + \frac{\partial \omega}{\partial p} = 0, \quad (2.6)$$

$$\frac{D\eta}{Dt} = \frac{\dot{q}}{T}, \quad (2.7)$$

where  $\Phi = gz$  is the geopotential,  $\eta$  is the specific entropy,  $\mathbf{V} = (U, V)$  is the horizontal velocity field,  $\omega = Dp/Dt$  is the analogue of vertical velocity, so that the substantial derivative becomes  $D/Dt = \partial/\partial t + \mathbf{V} \cdot \nabla_p + \omega \partial/\partial p$ ,  $\nabla_p$  the isobaric gradient,  $\dot{q}$  is the local rate of heating, and  $\mathbf{F}$  a representation of horizontal forcing. Thus, multiplying (2.4) by  $\mathbf{V}$  and adding it to (2.5) by  $\omega$  yields

$$\frac{D}{Dt} \frac{\mathbf{V}^2}{2} + \mathbf{V} \cdot \nabla_p \Phi + \omega \frac{\partial \Phi}{\partial p} = \mathbf{F} \cdot \mathbf{V} - \omega v. \quad (2.8)$$

Now, using the fact that the differential of the specific enthalpy is  $dh = Td\eta + vdp$  yields

$$\frac{Dh}{Dt} = T \frac{D\eta}{Dt} + v \frac{Dp}{Dt} = \dot{q} + \omega v, \quad (2.9)$$

which allows one to rewrite the local energy equation as follows

$$\frac{D}{Dt} \left( \frac{\mathbf{V}^2}{2} + h \right) + \nabla_p \cdot (\Phi \mathbf{V}) + \frac{\partial(\omega \Phi)}{\partial p} = \dot{q} + \mathbf{F} \cdot \mathbf{V}, \quad (2.10)$$

or equivalently

$$\frac{\partial}{\partial t} \left( \frac{\mathbf{V}^2}{2} + h \right) + \nabla_p \cdot (\mathcal{B}\mathbf{V}) + \frac{\partial(\mathcal{B}\omega)}{\partial p} = \dot{q} + \mathbf{F} \cdot \mathbf{V} \quad (2.11)$$

where  $\mathcal{B} = \mathbf{V}^2/2 + h + \Phi$  is the Bernoulli head. Equation (2.11) states that  $\mathbf{V}^2/2 + h$  can be regarded as the total energy for the hydrostatic primitive equations, which implies that for all practical purposes,  $h$  can be regarded as the effective potential energy and hence  $pv = h - u$  as the effective gravitational potential energy, in agreement with the above integral arguments. A similar result holds for the Boussinesq seawater approximation, e.g., Young (2010); Tailleux (2012); Eden et al. (2014), and is therefore consistent with the duality between the Boussinesq and non-Boussinesq hydrostatic primitive equations demonstrated by de Szoeke & Samelson (2002).

### 2.3. Two parcels energetics

The possibility to regard the thermodynamic quantity  $pv$  as the effective quantity measuring global changes in gravitational potential energy makes it possible to assess the relative importance of internal energy and compressible effects by simply comparing any net change in internal energy  $\Delta u$  relative to any corresponding net change in enthalpy  $\Delta h$ . To simplify the analysis of the problem, we consider the following idealisation. We discretise the fluid into parcels of equal mass on a fixed grid, and assume that fluid motion can be regarded as a series of permutations of the parcels in their initial state. Since any permutation can be expressed as the product of pairwise permutations involving only two fluid parcels, it follows that the energetics of two parcels swapping positions should form the building block of our understanding of the issue.

Let us first consider the net enthalpy change associated with the swap of two fluid parcels. Each parcel  $i$ , with  $i \in \{1, 2\}$ , is initially characterised by its entropy  $\eta_i$  and pressure  $p_i$ . During the swap, each parcel retains its entropy but exchange pressure values. As a result, the net change of enthalpy resulting from the swap is

$$\begin{aligned} \Delta h_{swap} &= h(\eta_1, p_2) - h(\eta_1, p_1) + h(\eta_2, p_1) - h(\eta_2, p_2) \\ &= \int_{p_1}^{p_2} [v(\eta_1, p') - v(\eta_2, p')] dp' = \int_{p_1}^{p_2} \int_{\eta_2}^{\eta_1} \Gamma(\eta', p') d\eta' dp' \\ &= \int_{p_1}^{p_2} \int_{\eta_2}^{\eta_1} \frac{\partial^2 h}{\partial p \partial \eta}(\eta', p') d\eta' dp' = \tilde{\Gamma}(p_2 - p_1)(\eta_1 - \eta_2) \end{aligned} \quad (2.12)$$

where  $\Gamma = \partial v / \partial \eta = \alpha T / (\rho c_p)$  is the adiabatic lapse rate previously introduced, while  $\tilde{\Gamma} = \tilde{h}_{p\eta}$  is a suitably defined mean value of  $\Gamma$  that makes the above relation exact. The above result was obtained by using the fact that  $dh = v dp$  and  $dv = \Gamma d\eta$  for isentropic specific enthalpy and specific volume changes respectively, which is equivalent to say that  $\Gamma = \partial^2 h / \partial p \partial \eta$ . By the mean value theorem, we know that  $\tilde{\Gamma}$  must be bounded by the minimum and maximum values taken by  $\Gamma$  over the entropy and pressure range considered. Note that if the parcel swap is done from a fluid initially at rest and stably stratified, the pressure and entropy distribution are such that  $dP/dz = -\rho g < 0$  and  $d\eta/dz > 0$ . This implies  $(p_2 - p_1)/(\eta_2 - \eta_1) < 0$  and hence  $\Delta h_{swap} > 0$ , in which case  $\Delta h_{swap}$  represents the available potential energy acquired by the fluid as the result of the parcels swap.

The next step is to compare  $\Delta h_{swap}$  with the corresponding internal energy change  $\Delta u_{swap}$ . Using the same approach, the latter is easily shown to be given by

$$\Delta u_{swap} = u(\eta_1, p_2) - u(\eta_1, p_1) + u(\eta_2, p_1) - u(\eta_2, p_2)$$



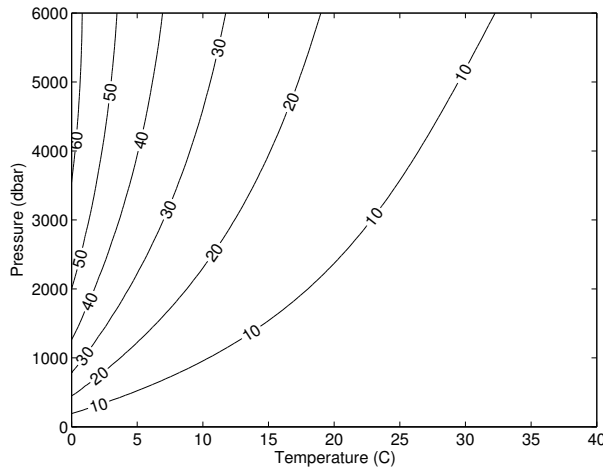


FIGURE 2. Ratio of the available internal energy over total available potential energy (with a minus sign) expressed as a percentage for seawater, assuming Absolute Salinity 35 g/kg, as defined by (3.2).

$$= \int_{p_1}^{p_2} \int_{\eta_2}^{\eta_1} \frac{\partial u}{\partial p \partial \eta}(\eta', P') d\eta' dp' \approx \tilde{u}_{p\eta}(p_2 - p_1)(\eta_1 - \eta_2)$$

where  $\tilde{u}_{p\eta}$  represents a suitably defined mean value of the following quantity (whose form is established in Appendix A)

$$\frac{\partial^2 u}{\partial p \partial \eta} = -p \frac{\partial \Gamma}{\partial p}. \quad (2.13)$$

As a result, the ratio  $\Delta u_{swap}/\Delta h_{swap}$  is given by

$$\frac{\Delta u_{swap}}{\Delta h_{swap}} = \frac{\tilde{u}_{p\eta}}{\tilde{\Gamma}} = \frac{\tilde{u}_{p\eta}}{\tilde{h}_{p\eta}}. \quad (2.14)$$

Eq. (2.14) is the main result of this paper, and shows that the ratio of internal energy to enthalpy change is controlled by the cross partial derivatives of these two thermodynamic state functions with respect to pressure and entropy. In contrast, single-parcel (sp) energetics predicts that the ratio of the change in internal energy over that enthalpy of a parcel of entropy  $\eta$  moving from pressure  $p_1$  to  $p_2$  is given by

$$\frac{\Delta u_{sp}}{\Delta h_{sp}} = \frac{\tilde{u}_p}{\tilde{h}_p} = \frac{\bar{p}}{\rho c_s^2} \approx \frac{g|z|}{c_s^2}, \quad (2.15)$$

in agreement with the result presented in the introduction. This makes it clear, therefore, the two ratios (2.14) and (2.15) obey radically different scaling, so that the key question is under what circumstances can they significantly differ from each other. As may be anticipated, the answer must depend significantly on the fluid considered and in particular on the physical properties of its equation of state, as is illustrated in more details in the next section for the particular cases of the atmosphere and ocean.

### 3. Application to the oceanic and atmospheric cases

#### 3.1. Atmospheric case

The atmosphere appears to be a special case from the viewpoint of parcel energetics, in the sense that inferences about the importance of compressibility effects and conversions

with internal energy based on single-parcel energetics are unchanged by multi-parcel energetics. This follows from that for a dry atmosphere, the internal energy and enthalpy are both linear functions of in-situ temperature  $e = c_v T$  and  $h = c_p T$  respectively, which implies that the ratio of the internal energy change over enthalpy change following a fluid parcel is equal to  $du/dh = c_v/c_p = 1/\gamma$  and therefore constant and independent of the thermodynamic process considered. Owing to the special form of internal energy and enthalpy for the perfect gas, the above constant ratio also pertains to global changes, since the total change in internal energy and enthalpy for a three-dimensional temperature change  $\delta T = \delta T(x, y, z)$  is given by

$$\Delta u = \int_V c_v \delta T dm, \quad \Delta h = \int_V c_p \delta T dm, \quad (3.1)$$

where  $\delta T = \delta T(x, y, z)$  is now a three-dimensional perturbation temperature field, and the single-parcel result  $\Delta u/\Delta h = c_v/c_p$  is recovered.

The same result can of course be recovered from the scale-height argument. To see this, first recall that for a perfect gas, the speed of sound is given by  $c_s^2 = \gamma RT$ , where  $R = c_p - c_v$  is the gas constant for dry air, yielding  $H_s = \gamma RT/g$ . Now, for a hydrostatic displacement  $\delta z = -dP/(\rho g) \approx -dh/g = -c_p dT/g$ , so that the ratio  $\delta z/H_s = -c_p dT/[\gamma RT] = c_v dT/(RT)$ . Since  $T = \theta(P/P_0)^\kappa$ , where  $\theta$  is potential temperature and  $\kappa = R/c_p = (c_p - c_v)/c_p$ , it follows that for adiabatic displacement  $\delta T/T = \kappa \delta P/P$ , as  $\theta$  is conserved, so that  $\delta z/H_s \approx (c_v/c_p) \delta P/P$ . The case of interest is  $\delta P/P = O(1)$ , in which case  $\delta z/H_s = O(c_v/c_p)$ , as expected.

### 3.2. Oceanic case

As mentioned in the introduction, the scale height  $H_s = c_s^2/g = O(225 \text{ km})$  in the ocean, which is much larger than the  $O(5000 \text{ m})$  mean ocean depth, so that according to single-parcel energetics arguments, internal energy changes should account for no more than 2% of total energy changes. This is to be contrasted with multi-parcel energetics considerations, which predict that  $\Delta u_{\text{swap}}/\Delta h_{\text{swap}}$  should scale as

$$-\frac{p}{\Gamma} \frac{\partial \Gamma}{\partial p} \Big|_\eta = -\frac{p}{\Gamma} \left( \Gamma \frac{\partial \Gamma}{\partial T} \Big|_p + \frac{\partial \Gamma}{\partial p} \Big|_T \right), \quad (3.2)$$

for an ocean with uniform salinity. This quantity was estimated using the Gibbs Sea Water (GSW) software library (available at <http://www.teos-10.org>, see also IOC et al. (2010); McDougall & Barker (2011)) using an Absolute Salinity value  $S_A = 35 \text{ g/kg}$ , and illustrated in Fig. 1 (right panel) as a function of temperature and pressure. This library naturally regards thermodynamic quantities as functions of (in-situ) temperature, pressure, and Absolute Salinity. Eq. (3.2) shows how to evaluate the ‘adiabatic’ derivative of  $\Gamma$  in terms of partial derivatives with respect to temperature and pressure. Fig. 2 shows that while minus the ratio (3.2) is very small for ‘shallow systems’ (associated with low pressure values), this ratio dramatically increases as the pressure increases and temperature decreases, to reach values as large as 50 or even 60 percent, which is in sharp contrast with single-parcel energetics, and which clearly establishes that compressible effects and conversions with internal energy are far from being negligible in systems as deep as the oceans, in agreement with Huang (2005).

Extending the above result to an ocean stratified both in temperature and salinity is relatively straightforward, but is not necessary for demonstrating the difficulties associated with single-parcel energetics, which is the main focus of this paper. Provisional results (Trevor McDougall, personal communication 2014) suggest that salinity gradients

re-enforce the effect established here, and will be discussed in more details in a separate oceanographic publication aimed at quantifying rigorously APE in the ocean.

#### 4. Conclusion

We have demonstrated that single-parcel energetics considerations lead to incorrect inferences about the relative importance of internal energy and compressible effects when discussing the global energetics of stratified fluids. An exception is the perfect gas, but in this case, the correct answer arguably occurs for the wrong reason. In this regard, it appears worth pointing out that McDougall & Feistel (2003) found a similar difficulty with the conventional explanation for the adiabatic lapse rate, which appears to give the 'correct' answer only for a perfect gas but not for any other general fluid. In the most general case, a correct assessment of the role of internal energy and compressible effects can only be obtained from the consideration of mass-conserving parcel re-arrangements, which need to involve two fluid parcels at the minimum. In contrast to single-parcel energetics, which assess the importance of adiabatic internal energy changes based on the smallness of the dimensionless parameter  $g\Delta z/c_s^2$ , multi-parcels energetics base such an assessment on a dimensionless parameter involving the pressure  $p$ , the adiabatic lapse rate  $\Gamma$ , and its adiabatic partial derivative with respect to pressure  $\partial\Gamma/\partial p|_\eta$ . Single- and multi-parcel energetics therefore rely on completely different physics, which seems to have been overlooked so far. This is an important result, for it provides a simple rationalisation of why the available internal energy appears to be such a large fraction of the total oceanic APE in Huang (2005) study, which had not been satisfactorily explained so far. Furthermore, two-parcel energetics confirms that  $\Gamma$ , rather than  $c_s$ , is the most fundamental quantity to assess the importance of compressibility effects, as per our first counter-argument given in the introduction.

The author gratefully acknowledges the comments of two anonymous referees, as well as detailed remarks and insights about how salinity affects the results by Trevor McDougall.

#### Appendix A. Thermodynamic relation

This appendix aims to establish the result

$$\frac{\partial^2 u}{\partial p \partial \eta} = -p \frac{\partial \Gamma}{\partial p}, \quad (\text{A } 1)$$

where  $\Gamma = \alpha T / (\rho c_p) = \partial v / \partial \eta$  is the adiabatic lapse rate. We start with the differential of internal energy  $du = T d\eta - p dv$ , and switch to  $(\eta, p)$  coordinates by regarding  $v$  as a function of specific entropy  $\eta$  and pressure  $p$ . The differential of internal energy becomes

$$du = T d\eta - p dv = \left[ T - p \frac{\partial v}{\partial \eta} \right] d\eta - p \frac{\partial v}{\partial p} dp, \quad (\text{A } 2)$$

which establishes that the first derivatives of  $e$  with respect to  $\eta$  and  $p$  are given by

$$\left. \frac{\partial u}{\partial \eta} \right|_p = T - p \left. \frac{\partial v}{\partial \eta} \right|_p = T - p\Gamma, \quad \left. \frac{\partial u}{\partial p} \right|_\eta = -p \left. \frac{\partial v}{\partial p} \right|_\eta. \quad (\text{A } 3)$$

Taking either the partial derivative of  $\partial u / \partial \eta$  with respect to  $p$  or the partial derivative of  $\partial u / \partial p$  with respect to  $\eta$  yields in both cases:

$$\frac{\partial^2 u}{\partial \eta \partial p} = \frac{\partial T}{\partial p} - p \frac{\partial \Gamma}{\partial p} - \Gamma = -p \frac{\partial \Gamma}{\partial p}, \quad (\text{A } 4)$$

as required, since by definition  $\Gamma = \partial T / \partial p$  for adiabatic transformations.

## REFERENCES

- BATCHELOR, G.K. 1967 An introduction to Fluid Dynamics. Cambridge University Press. 615 pp.
- DE SZOEKE, R. A. & SAMELSON, R. M. 2002 The duality between Boussinesq and non-Boussinesq hydrostatic equations of motion. *J. Phys. Oceanogr.* **30**, 2194–2203.
- EDEN, C., L. CZESCHEL & D. OLBERS 2014 Toward energetically consistent ocean models. *J. Phys. Oceanogr.* **44**, 3160–3184.
- GADE, H. G. & GUSTAFSSON, K. E. 2004 Application of classical thermodynamic principles to the study of oceanic overturning circulation. *Tellus* **56A**, 371–386.
- HUANG, R. X. 2005 Available potential energy in the world’s oceans. *J. Mar. Res.* **63**, 141–158.
- IOC, SCOR & IAPSO 2010 The international thermodynamic equation of seawater — 2010: Calculations and use of thermodynamic properties. *Intergovernmental Oceanographic Commission, Manuals and Guides No 56, UNESCO (English)*, 196 pp.
- KUHLBRODT, T., A. GRIESEL, M. MONTOYA, A. LEVERMANN, M. HOFMANN, & S. RAHMSTORF 2007 On the driving processes of the Atlantic meridional overturning circulation. *Rev. Geophys.*, **45**, RG2001 doi:10.1029/2004RG000166.
- MCDUGALL, T. J. 1987 Neutral surfaces. *J. Phys. Oceanogr.* **17**, 1950–1964.
- MCDUGALL, T. J. & P. M. BARKER 2011 Getting started with TEOS-10 and the Gibbs Seawater (GSW) Oceanographic Toolbox. 28 pp. SCOR/IAPSO WG127, ISBN 978-0-646-55621-5.
- MCDUGALL, T. J. & R. FEISTEL 2003 What causes the adiabatic lapse rate? *Deep-Sea Res. I* **50**, 1523–1535.
- OLIVER, K.I. & R. TAILLEUX 2013 Thermobaric control of gravitational potential energy generation by diapycnal mixing in the deep ocean. *Geophys. Res. Lett.*, **40**, 327–331.
- OORT, A.H., S.C. ASCHER, S. LEVITUS & J. P. PEIXOTO 1989 New estimates of the available potential energy in the ocean. *J. Geophys. Res.*, **94**, 3187–3200.
- REID, R.O., B.A. ELLIOTT & D.B. OLSON 1981 Available potential energy: a clarification. *J. Phys. Oceanogr.* **11**, 15–29.
- TAILLEUX, R. & GRANDPEIX, J. Y. 2004 On the seemingly incompatible parcel and globally integrated views of the energetics of triggered atmospheric deep convection over land. *Q. J. Roy. Met. Soc.* **130**, 3223–3243.
- TAILLEUX, R. 2009 On the energetics of stratified turbulent mixing, irreversible thermodynamics, Boussinesq models, and the ocean heat engine controversy. *J. Fluid Mech.* **638**, 339–382.
- TAILLEUX, R. 2010 On the buoyancy power input in the oceans energy cycle. *Geophys. Res. Lett.*, **37**, L22603, doi:10.1029/2010GL044962.
- TAILLEUX, R. 2012 Thermodynamics/dynamics coupling in weakly compressible turbulent stratified fluids. *IRSRN Thermodyn.* 2012:609701.
- TAILLEUX, R. 2013 Irreversible compressible work and available potential energy dissipation in turbulent stratified fluids. *Physica Scripta*, T155, 014033 doi:10.1088/0031-8949/2013/T155/014033
- THORPE, A.,J., B.J. HOSKINS & V. INNOCENTINI 1989 The parcel method in baroclinic atmosphere. *J. Atm. Sci.*, **46**, 1274–1284.
- WUNSCH, C. & FERRARI, R. 2004 Vertical mixing, energy, and the general circulation of the oceans. *Ann. Rev. of Fluid Mech.*, **36**, 281–314.
- YOUNG, W. R. 2010 Dynamic enthalpy, conservative temperature, and the seawater Boussinesq approximation. *J. Phys. Oceanogr.*, **40**, 394–400.