Characteristics of Swarms on the Edge of Fragmentation

Giuliano Punzo, Jules Simo, Derek J. Bennet and Malcolm Macdonald

giuliano.punzo@strath.ac.uk,

Advanced Space Concepts Laboratory,

Department of Mechanical and Aerospace Engineering,

University of Strathclyde, Glasgow.

(Dated: February 5, 2014)

Abstract

Fragmentation of particle swarms into isolated subgroups occurs when interaction forces are weak or restricted. In the restricted case, the swarm experiences the onset of bottlenecks in the graph of interactions that can lead to the fragmentation of the system into subgroups. This work investigates the characteristics of such bottlenecks when the number of particles in the swarm increases. It is shown, for the first time, that certain characteristics of the bottleneck can be captured by considering only the number of particles in the swarm. Considering the case of a connected communication graph constructed in the hypothesis that each particle is influenced by a fixed number of neighbouring particles, a limit case is determined for which a lower limit to the Cheeger constant can be derived analytically without the need for extensive algebraic calculations. Results show that as the number of particles increases the Cheeger constant decreases. Although ensuring a minimum number of interactions per particle is sufficient, in theory, to ensure cohesion, the swarm may face fragmentation as more particles are added to the swarm.

Extensive research into multi-particle systems and their application has been carried out, with a particular focus on providing verifiable behavioural models.[1–3] Within this context, coherent behaviour of the swarm depends upon reciprocal interactions amongst the particles. When these interactions are restricted, as they appear to be in biological systems and as must be in any practical implementation of a large scale swarm, the determinability of the behaviour of the swarm becomes highly complex. It is thus important to determine conditions in which the behaviour of the swarm becomes more or less coherent due to restricted interactions. This is particularly true for those technological applications that rely on the interactions of several independent units. In this scenario the limiting factor is constituted by the amount of information exchanged in the swarm, hence the number of active connections each unit can keep.

The effect of limited interactions has been addressed, with stability analyses that rely on the swarm staying connected and numerical simulations in support of the behaviours.[4,5] This paper considers a swarm of particles acting in a limited communication network and interacting through pairwise potentials. Associating particles to nodes of a graph, it is possible to track mutual interactions as edges of the graph and thus conclude characteristics of the system with the final aim to inform, through this analysis, the development of future multi agent systems. The particle representation, through the use of physical interactions that shape the swarm, allows an effective visualisation of the consequence of restricted interactions. A measure of how much the interaction network is bottlenecked is provided for a generic graph by the Cheeger constant, which is introduced in the following and analysed with respect to its dependence on the number of agents. Here particles are considered that interact according to a pairwise potentials, with a viscous damping that allows for relaxation to a static configuration of the system. The equations of motions of the generic agent i in the swarm are expressed as,

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i \tag{1}$$

$$m\frac{d\mathbf{v}_i}{dt} = -\nabla U_i - \sigma \mathbf{v_i} \tag{2}$$

where, $\sigma \mathbf{v_i}$ introduces the velocity dependent damping term in the dynamics, with $\sigma = 0.7$ and the mass of each particle m is taken unitary. The potential acting on the generic particle i is $U_i = \sum_j (a_{ij} U_{ij})$, with a_{ij} being the entry of the adjacency matrix which takes values "1" or "0" depending respectively on whether the particles are interacting or not. This holds for any potential considered. In particular the Morse, the quadratic and the hyperbolic potentials, which are widely spread in literature, are used here showing the wide applicability of the results regardless the kind of interaction. Relaxation to static position is achieved through the viscous dissipation term. The Morse potential is described as

$$U_{ij}^{m} = -C_a \exp\left(-\frac{|\mathbf{x}_{ij}|}{L_a}\right) + C_r \exp\left(-\frac{|\mathbf{x}_{ij}|}{L_r}\right) \tag{3}$$

where, $\mathbf{x_{ij}}$ is the relative position vector of a particle i with respect to a particle j. C_a, C_r represent the strength of the potential, while L_a and L_r govern the range of the potentials with $L_a > L_r$ to ensure stability.[2,6] As such in the proceeding analysis they will be assigned the following values: $C_a = C_r = L_a = 1$, $L_r = 0.2$. The quadratic potential, leading to accelerations proportional to the distance, is described by the expression

$$U_{ij}^{q} = (|\mathbf{x}_{ij}| - d)^2 \tag{4}$$

where, d is a reference distance between any two agents, herein set to 0.1. The hyperbolic potential[7,8] produces a distance dependent acceleration as well, which then becomes constant over large distances. This is described by the expression

$$U_{ij}^{h} = [(|\mathbf{x}_{ij}| - d)^2 + 1]^{0.5} \tag{5}$$

where, the symbols keep the same meaning and values as in Equations 3 and 4.

Consider a swarm of N particles whose number of interactions is strictly limited; in particular suppose that a particle can sense the potential of at most k other particles, the closest ones. In the following this is referred to as the k *Nearest Neighbours Rule* ($k - NNR$). When representing this into a directed sensing graph, this turns into a given node having k outgoing edges, specifically it senses the action of the closest k neighbours.

For the graph to be connected the total number of interactions, hence of connections within the swarm must be at least $N - 1$, which corresponds to a line or to a star graph. In the particular case presented here, when the connections depend upon the relative positions, and the positions of the particles depend in turn upon the forces derived through the interactions, the pairwise potential presented tends to cluster the interacting particles.[2,9,10]

This prevents the spontaneous formation of such edge-saving connected graphs.

When $N-1 \leq k$ the graph is connected, and in particular it is complete. As more particles join the system, N increases and eventually becomes greater than k. At this point the graph of the interactions is no longer complete, yet it can still be connected. It is straightforward to understand that $k = \langle N/2 \rangle$ (where $\langle \cdot \rangle$ rounds down to the nearest integer) for each particle is a sufficient, though not necessary condition for the graph to be connected. Hence, the swarm will remain cohesive. Further, in a dispersed swarm there will be at least one subgroup composed of n particles, with $n \leq \langle N/2 \rangle = k$. As the number of connections per particle is greater than the number of particles in the subgroup there must be $k-n+1$ edges for each node in this group connecting to some of the other $N-n$ nodes as Figure 1 shows. As connected particles gather together under the actions of the pairwise interactions, this will produce a cohesive group. This logic is breached when $k < \langle N/2 \rangle$. For the case $k = \langle N/2 \rangle$

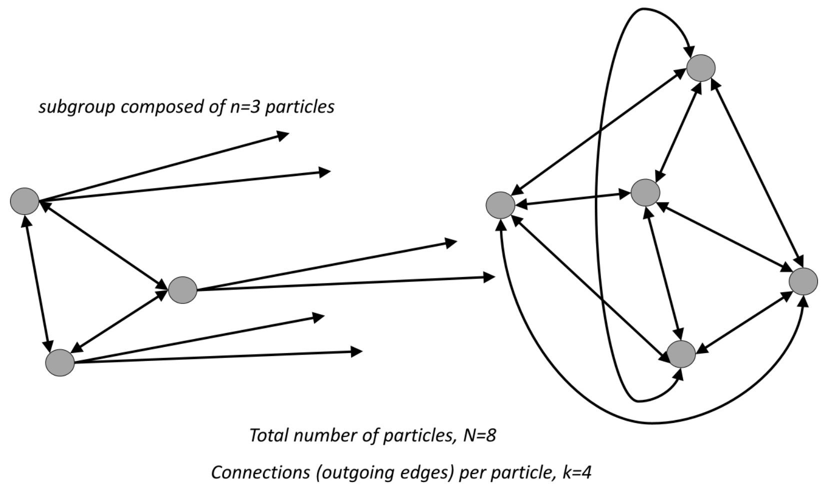


FIG. 1: The illustration shows how for $k = \langle N/2 \rangle$ the swarm must be connected. Each of the particle in the subgroup on the left must complete its 4 connections by joining the group on the right.

two clustered, but yet connected, groups arise. As the connections that a particle does not establish in its own cluster are established always on the base of closeness, these will be with some particles on the closest region of the other cluster. This gives rise to a dumbbell shape where the particles in the central, narrower part bond the two clusters together and, by symmetry, have the same number of connections to both sides of the dumbbell. Meanwhile, they are sensed by all N particles. The dumbbell shape is reflected in the adjacency matrix of the graph once each node is associated to a particle and the nodes are labeled from one-end of the dumbbell to the other. The emergence of a bottleneck when the number of particles increases, while the number of interactions per particle is held constant is shown in Figure 2 for a planar case, while in Figure 3 the final arrangement after relaxation of a 130 particle swarm with 65 connections per particle is shown together with a graphical representation of the corresponding adjacency matrix for a three-dimensional case.

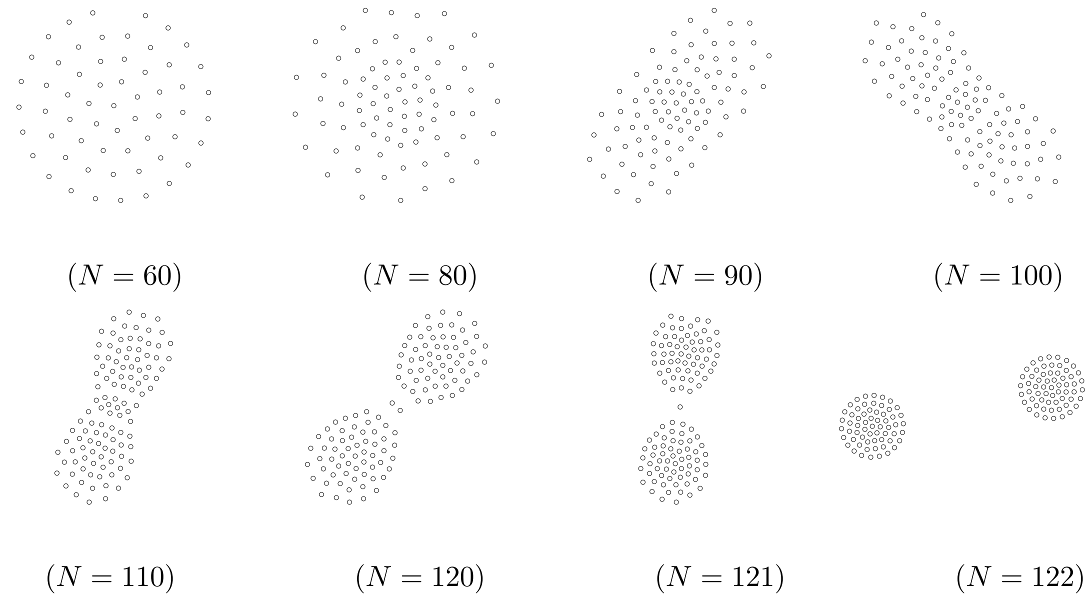
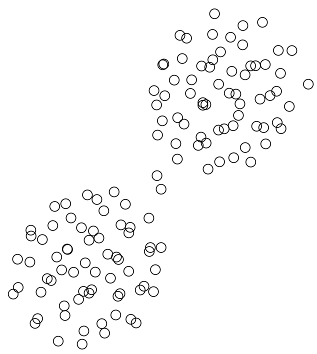


FIG. 2: Particle swarm systems in a two-dimensional space relaxing to different shapes as the number of particles N increases while the number of connections allowed per particle, k, is held constant at 60.

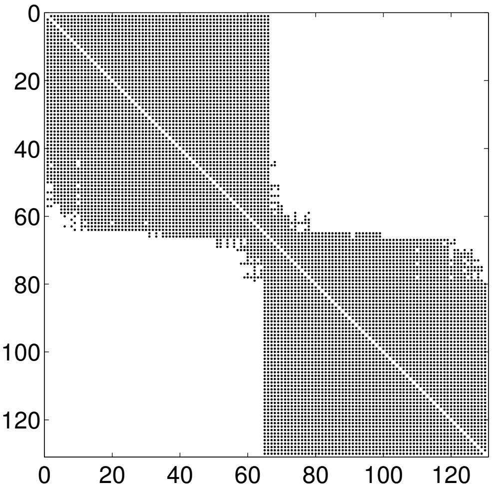
As $\langle \frac{N}{2} \rangle$ is a critical value for the number of connections per node, the bottleneck characteristics of the system for the critical case of $k = \langle N/2 \rangle$ is considered using the *Isoperimetric number*, or *Cheeger constant* of the graph. The Cheeger constant provides a measure of the flow along the edges of the graph connecting two complementary subsets of it. Null flow corresponds to the Cheeger constant being equal to zero and to the two subsets being disconnected. Meanwhile a unitary value is achieved when any two edges, belonging to the two complementary subsets, are connected, hence, the flow is the maximum possible. As such, the Cheeger constant for an oriented graph G is defined as[11],

$$h(G) = \inf_{S} \frac{F(\partial S)}{\min\{F(S), F(\overline{S})\}} \tag{6}$$

where, $F = [f_{ij}]$ is a circulation, a function from the set of edges of the graph onto $\mathbb{R} - \{0\}$, S is any subset of nodes in the graph and $\overline{S}$ is its complement, while ∂S is the set of all



(a)



(b)

FIG. 3: Dumbbell emergence in a three-dimensional case. (a) A 130 particle swarm in a dumbbell shape due to the number of connections per particle being limited to 65 and (b) the corresponding adjacency matrix obtained associating each node to one particle, after having them sorted from one end to the other of the dumbbell. Dots represent non-zero entries.

edges connecting a node in S to one in $\overline{S}$. If the generic pair ij is not an edge of the graph, then $f_{ij} = 0$. A popular way to define F is on the basis of the probability distribution matrix $P = [p_{ij}]$ and its dominant left eigenvector,[11] ϕ. As such, this form of circulation is indicated with the superscript ϕ and its definition is

$$f_{ij}^{\phi} = \phi_i p_{ij} \tag{7}$$

where, i and j are indexes corresponding to generic nodes in the graph and P is a matrix whose generic entry i, j gives the probability of moving from vertex i to vertex j based on the number of links departing from i, derivable from the entries of the adjacency matrix. That is

$$p_{ij} = \frac{a_{ij}}{\sum_j a_{ij}} \quad . \tag{8}$$

Equation 7 does not directly provide information on the behaviour of the Cheeger constant as a function of the number of nodes. However, other circulations can be used to define the Isoperimetric number as long as they satisfy the condition,

$$\sum_{\substack{i \\ i \to j}} f_{ij} = \sum_{\substack{w \\ j \to w}} f_{jw} \tag{9}$$

for the generic nodes i, j and w. That is the flow in one node of the graph is null. For the matrix F this translates to having the sum over the rows equal to the sum over the columns. Therefore, an analytic expression for the Cheeger constant, which is dependent on the number of nodes in the graph, can be derived by redefining the circulation. In order to do this the connection network is imposed on the system through an adjacency matrix composed of two blocks plus two linking rows, shown in Figure 4.a. The adjacency matrix is obtained by considering that interactions between the two halves pass only through the two central nodes. This is still consistent with the $k-NNR$ for $k = \langle N/2 \rangle$ as long as each row of the idealised adjacency matrix presents $\langle N/2 \rangle$ non-zero entries, as it does. For clarity, only an even number of nodes is presented (refer to the online additional material for the case of an odd number of nodes). As the particles in the centre are sensed by both clusters, their columns in the adjacency matrix do not have any zero entries, except along the primary diagonal. This idealised approximation is shown in Figure 4.a for a graph composed of 60 nodes as opposed to one resulting from the spontaneous relaxation of 60 particles following the dynamics earlier described, with initial conditions randomly distributed in a unit sphere, shown in Figure 4.b. The idealisation of the adjacency matrix in Figure 4.a represents

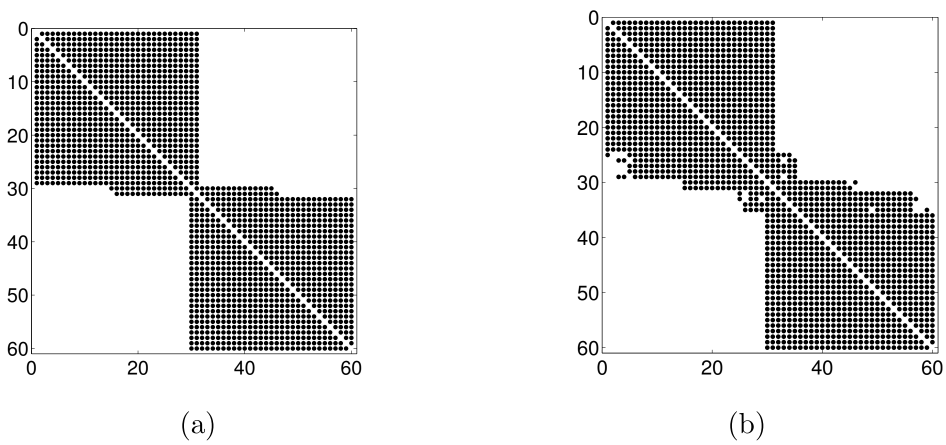


FIG. 4: Adjacency matrices for spontaneously relaxed (a) and idealised connected (b) 60 particle swarm.

the connection structure, where N is even, that produces the smallest possible bottleneck compliant with the physical restriction of having at least 2 central communicating particles for N even. Consequently, the associated Cheeger constant is the minimum achievable, as a smaller value would imply a smaller number of particles bridging the two halves, leading to fragmentation. This is the reason for which the configuration in Figure 4.a is considered here as limit case before the swarm fragmentation. Conversely, Figure 4.b shows a typical adjacency matrix resulting from spontaneous relaxation using Morse potential, however, as in this case the adjacency matrix is not imposed on the swarm, spontaneous relaxation can reduce to anything from this to the limiting case described.

Defined only for existing edges, the circulation used has a matrix representation with the same non-zero entries as the adjacency matrix. The actual value for each entry is found by requiring that entries belonging to columns with the same number of non-zero elements take the same value. This, together with the condition expressed by Equation 9, provides a number of relations that are sufficient to fully define the circulation for the matrix in Figure 4.(a). This circulation takes four possible values, as there are four different column lengths in the matrix; call these a, b, c and d. Four equations are hence derived by requiring the sums along the rows and the columns of the circulation matrix to be equal, as per Equation 9, one for each column length as

$$\begin{cases} \left(\frac{N}{2}-2\right)a &= \left(\frac{N}{4}-2\right)a+b+\left(\frac{N}{4}-1\right)c+2d \\ \left(\frac{N}{2}-1\right)b &= \left(\frac{N}{4}-1\right)a+\left(\frac{N}{4}-1\right)c+2d \\ \frac{N}{2}c &= \left(\frac{N}{4}-1\right)a+b+\left(\frac{N}{4}-2\right)c+2d \\ (N-1)\,d &= b+2\left(\frac{N}{4}-1\right)c+d \end{cases} \tag{10}$$

where, again N is the number of particles/nodes in the graph. Each of the equations in the linear system expresses the equality between row and column sum. For instance in the first equation the sum along any of the first or last $(\frac{N}{4}-1)$ columns is $(\frac{N}{2}-2)a$ as all these columns have $(\frac{N}{2}-2)$ nonzero entries for which the value a is imposed. This is compared to the sum along any of the first or last $(\frac{N}{4}-1)$ rows featuring the first $(\frac{N}{4}-2)$ entries equal to a, one entry equal to b, $(\frac{N}{2}-1)$ entries equal to c and 2 equal to d. To allow a solution other than the zero solution, a is considered known and $a \in \Re^{+}$. The solution of the system in Equation 10 is then,

$$\begin{cases} b &= \frac{N-2}{N}a \\ c &= \frac{\frac{N}{4}a-b-2d}{\frac{N}{4}-1} \\ d &= \frac{-\frac{N-2}{N}+\frac{N}{2}}{N+2}a \end{cases} . \tag{11}$$

The Cheeger constant can thus be derived from its definition in Equation 6 as,

$$h(G)=h(N)=\frac{\frac{N}{2}d+\left(\frac{N}{4}-1\right)c}{\left(\frac{N}{4}-1\right)\left(\frac{N}{2}-2\right)a+\left(\frac{N}{2}-1\right)b+\left(\frac{N}{4}-1\right)\frac{N}{2}c+(N-1)\,d} , \tag{12}$$

that, after some algebraic manipulation can be reduced to

$$h(G) = h(N) = \frac{2N\left(N^2 - 4N + 6\right)}{N^4 - 2N^3 - 6N^2 + 20N + 8} \quad . \tag{13}$$

Equation 13 is obtained in the hypothesis of N being a multiple of 4, otherwise more complicated expressions are to be defined that take into account the shifting of the rows of one position depending on the value of N; for clarity this is not done here. Equation 13 can be proved to be a decreasing function of N, in particular $h(N)$ tends to zero as N approaches infinity, that is,

$$\lim_{N \to \infty} h(N) = 0 \tag{14}$$

The Cheeger constant so defined does not depend on any of the coefficients as they are all proportional to the variable a, which eventually cancels out through simple algebraic manipulations.

Comparison of the above analytically determined Cheeger constant as a function of only the number of nodes, and the Cheeger constant obtained through numerical integration of the spontaneous relaxation of particles driven by pairwise potential within a network of links based on the $k - NNR$, is presented in Figure 5.

Numerical integration is conducted in a three-dimensional space with initial positions and velocities chosen randomly within a unit sphere and for sufficient time to have the swarm relaxed into a static pattern. Numerical integration is by an explicit Euler scheme, with an integration step of 10^{-3} seconds. The Cheeger constant is then calculated using Equation 6 by inserting the values of circulation obtained through Equation 7. The two partitions S and $\overline{S}$ are identified on the base of the spatial arrangement the formation attains after relaxation, i.e. the dumbbell. It is found that the Cheeger constant, obtained from the adjacency matrix with two linking rows, closely tracks that obtained from numerical simulations averaged over 100 runs for each data point. As expected, the prediction is found to be always below, or at most equal to, the lowest value found within the numerical simulation data set confirming that the analytic curve provides a lower bound for the prediction of the Cheeger constant. This can be easily determined for very large swarms where calculation of Cheeger constant using eigenvalues becomes problematic.

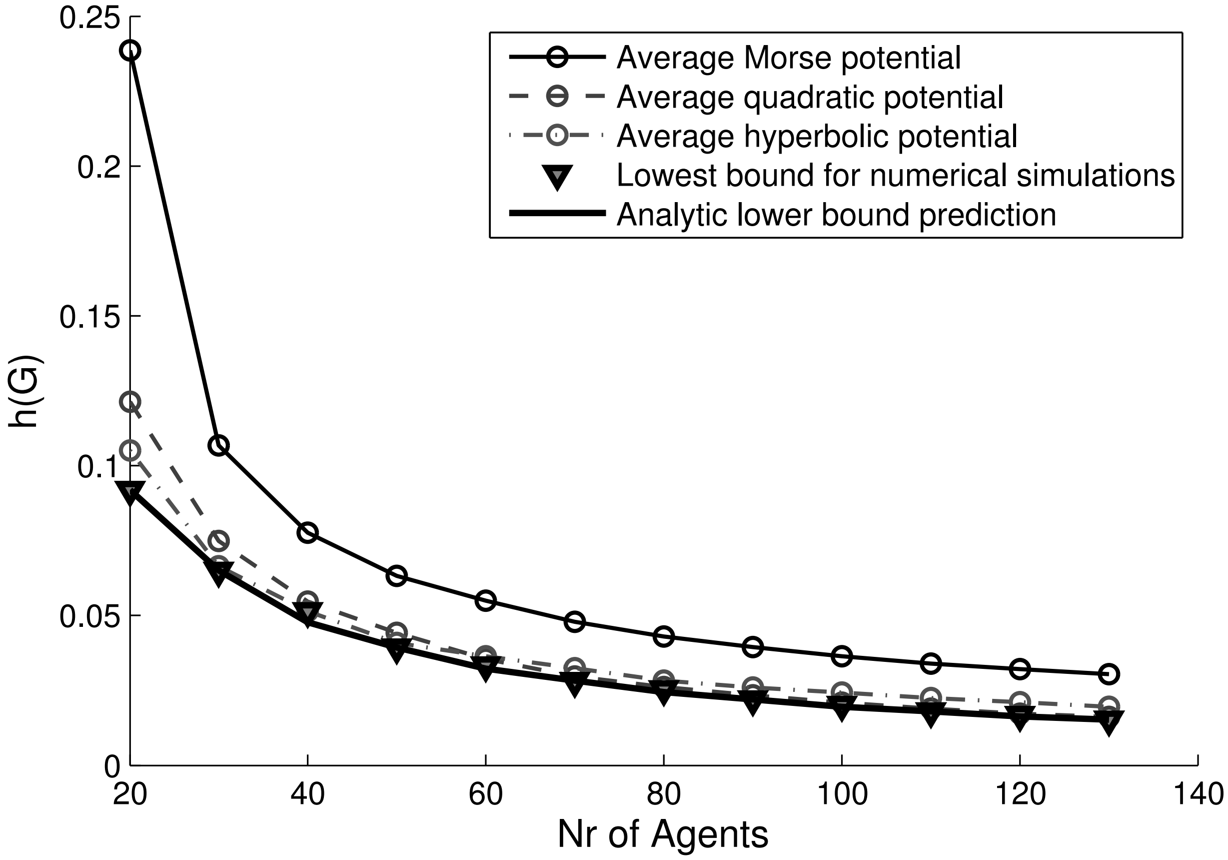


FIG. 5: Comparison of Cheeger constant obtained by the analytic expression simply based on the number of particles and that obtained through numerical simulations averaged over 100 runs with random initial conditions.

Results show that, in the limit case of $k = \langle N/2 \rangle$, the spontaneous relaxation produces a graph of interactions with a number of crossing links between the two clusters, in excess of those strictly needed to ensure cohesion as the numerical simulations on average returns values above the minimum attainable. Notwithstanding, when swarms grow larger the number of links across the two clusters reduces compared to the total number of interactions within the swarm. This is confirmed by the fact the Cheeger constant is a monotonically decreasing function. Thus, as the Cheeger constant is a measure of the bottleneck characteristics of a graph, the results show how a swarm of particles that interact on the base of the $k-NNR$, with $k = \langle N/2 \rangle$, tends to become more and more bottlenecked as the number of particles increases justifying the assertion that an increase in the number of particles is not entirely compensated by an increase in the number of cross-links between the two clusters of the dumbbell that the system eventually relaxes into. In this frame the minimum value achievable by the Cheeger constant, related to the narrowest bottleneck, can always be bound from below knowing only the number of particles. Thinking towards the engineering of multi-agent systems, in the case of a very large number of agents, possible consequences arising from the behaviour described are even more incisive. Emergence of a bottleneck restricts sensing and information flow, hence updates of the system's state, which agents need for cohesion, is delayed. In-turn this can directly results in fragmentation into sub-groups even in the case of k close to, but still greater than, $\langle N/2 \rangle$.

---

[1] M. H. Mabrouk and C. R. McInnes. Nonlinear stability of vortex formation in swarms of interacting particles. *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, 78(1):012903, 2008.

[2] M. R. D'Orsogna, Y. L. Chuang, A. L. Bertozzi, and L. S. Chayes. Self-propelled particles with soft-core interactions: Patterns, stability, and collapse. *Physical Review Letters*, 96(10):104302, 2006.

[3] V. Gazi and K. M. Passino. Stability analysis of swarms. *IEEE Transactions on Automatic Control*, 48(Compendex):692–697, 2003.

[4] H. G. Tanner, A. Jadbabaie, and G. J. Pappas. Flocking in fixed and switching networks. *Automatic Control, IEEE Transactions on*, 52(5):863–868, 2007.

[5] R. Sepulchre, D. A. Paley, and N. E. Leonard. Stabilization of planar collective motion with limited communication. *IEEE Transactions on Automatic Control*, 53(Compendex):706–719, 2008.

[6] C. R. McInnes. Vortex formation in swarms of interacting particles. *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, 75(3):032904, 2007.

[7] A. Badawy and C.R. McInnes. Small spacecraft formation-flying using potential functions. *Acta Astronautica*, 65(11-12):1783–1788, 2009.

[8] D. J. Bennet and C. R. McInnes. Verifiable control of a swarm of unmanned aerial vehicles. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, 223(7):939–953, 2009.

[9] D. J. Bennet and C. R. McInnes. Verifiable control of a swarm of unmanned aerial vehicles. *Pro-*

ceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering, 223(7):939–953, 2009.

10 W. M. Spears and D. F. Gordon. Using artificial physics to control agents. In *in IEEE International Conference on Information, Intelligence, and Systems*, pages 281–288, 1999.

11 F. Chung. Laplacians and the cheeger inequality for directed graphs. *Annals of Combinatorics*, 9(1):1–19, 2005.