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# Informational and Allocative Efficiency in Financial Markets with Costly Information

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**Abstract**

Costly information acquisition is introduced into a dynamic trading model of Glosten and Milgrom (1985). The market maker and some traders, called "value traders," value the asset at its fundamental value, which can be either high or low. The remaining traders, called "liquidity traders," have idiosyncratic valuations that are independent of the fundamental. At a cost, each value trader can acquire an informative, but imperfect, signal about the fundamental. In this setting, at equilibrium, each value trader acquires the signal if and only if the uncertainty about the fundamental's value conditional on publicly available information is sufficiently high. Thus, the prices quoted by the market maker are "informationally inefficient," as they do not reveal the value of the fundamental, even in the long-run. Equilibrium amount of information acquisition is either excessive or insufficient relative to the social optimum and results in an inefficient allocation of the asset among the market maker and liquidity traders.

*JEL Classification:* D80, D83, D84, G12, G14

*Keywords:* Sequential Trading, Cost of Information, Endogenous Information Acquisition.

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# 1 Introduction

This paper studies costly information acquisition in a version of sequential trading model first introduced by Glosten and Milgrom (1985). The aim of the paper is twofold: to characterize market outcomes and to undertake welfare analysis of these outcomes.

The model considers a market for a single risky asset with a binary liquidation value. It is assumed that all trades are mediated by a competitive market maker who sets the prices on the basis of the order flow. Traders arrive sequentially to the market in some pre-specified random order. There is a fixed proportion of value traders who, prior to trading, have a choice of whether or not to obtain a costly signal. The remaining traders are liquidity traders who are price sensitive and do not care about the fundamental. As the name suggests, these traders trade for liquidity reasons and thus value the asset differently from the market maker and the value traders.

It is demonstrated that, in the presence of fixed costs to information acquisition, some valuable information never reaches the market and hence prices cannot be fully informationally efficient, i.e. prices cannot converge to the liquidation value of the underlying asset. This outcome echoes the idea of Grossman and Stiglitz (1980) that there must be some equilibrium level of inefficiency in the market or otherwise equilibrium may fail to exist. In particular, in equilibrium, information must generate returns just sufficient to compensate for the costs associated with the gathering of this information. Similar informational inefficiency occurs in the sequential trading model considered here through the following mechanism. With fixed costs to information acquisition, value traders find it optimal to stop obtaining signals when the asset price is sufficiently close to one of the extremes and the benefit of a signal is outweighed by its costs. Thus there are two threshold values of the public belief about the likely asset value at which signal acquisition stops. As soon as the public belief reaches one of these thresholds, trades become uninformative and the market maker stops updating the price. As a result, the price gets trapped at a given level that may or may not be close to the liquidation value of the asset.

In the proposed model, informational inefficiency of prices has significant welfare implications. Because liquidity traders have idiosyncratic valuations, while other market participants value the asset at the fundamental, the efficient allocation of the asset depends on its liquidation value. Hence, information about the fundamental is socially valuable. However, value traders, when they decide whether to acquire informative signals, do not take into account the positive externality their decisions exert on the payoff of liquidity traders. This suggests that in the long-run, market solution may underprovide information relative to a social optimum.

For the purposes of welfare analysis, the paper introduces a utilitarian social planner who does not have access to the private information of traders but can influence their information acquisition efforts. Numerical simulations indicate that the optimal signal acquisition policy of the social planner differs qualitatively from the information acquisition strategy in the market equilibrium. The social planner either induces signal acquisition with probability 1 or prohibits it. At the market equilibrium, however, information acquisition does not stop abruptly. Instead, as the public belief regarding the asset value moves closer to the extreme values of 0 or 1, the equilibrium probability with which value traders acquire signals decreases gradually.

Welfare analysis confirms the intuition that, in the long-run, the decentralised market solution underprovides information relative to the social optimum, provided the social planner is sufficiently patient. At the same time, signal acquisition efforts induce a positive bid-ask spread, which precludes some liquidity traders from placing orders and reduces social welfare in the short-run. Consequently, if the social planner is impatient, market overprovides information relative to the social optimum.

The paper is related to rational herding applied to financial markets.[1] The literature has found that, in financial markets, accumulation of information stops in the presence of some frictions such as transaction costs as in Lee (1993) and Romano (2007), the opportunity cost of investment coupled with endogenous timing as in Chari and Kehoe (2004), or career considerations as in Dasgupta and Prat (2006, 2008). Cipriani and Guarino (2001) show that, in the long-run, trades also stop conveying private information in a sequential trading model with heterogeneous agents who differ in the utilities they derive from holding the asset.

The existing literature typically assumes that private signals are costless. The case of costly information acquisition has received surprisingly little attention. Kubler and Weizsacker (2004) present a laboratory experiment that incorporates fixed costs to signal acquisition in a canonical social learning model. In the equilibrium of this game, only the first player acquires the signal while all the other players do not buy the signal and emulate the behaviour of the first player.

Burguet and Vives (2000) consider costly information acquisition in the context of smooth and noisy versions of a canonical social learning model, where agents sequentially attempt to predict a normally distributed random variable. In their model, there is no market and, like in the early social learning models of Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992), the cost of taking an action is fixed.

This paper assigns a well-specified utility function to noise traders, which allows analysing

[1] See, for example, Avery and Zemsky (1998) or Park and Sabourian (2009).

welfare implications of market outcome. Welfare analysis is another aspect that sets this paper apart from the rest of the literature dealing with rational herding in financial markets.

The remainder of the paper is organized as follows. Section 2 outlines the model. Section 3 characterizes the market equilibrium and shows that, when information is costly, prices cannot be fully informationally efficient. Section 4 analyses welfare implications of equilibrium behaviour. Section 5 concludes.

## 2 The Setup

The model is extends a special case of the Glosten and Milgrom (1985) sequential trading model by allowing some traders to acquire information about the asset's liquidation value

Assume that there is a single risky asset with liquidation value $V \in \{0,1\}$. The prior probability of the high asset value is $p_1 \in (0,1)$ (i.e., $\mathbb{P}(V=1) = p_1$). Trading takes place in discrete time, and risk neutral traders arrive each period in a pre-specified, exogenous order. Thus, at each date $t = 1, 2, ...$ a trader is chosen from an infinite pool of risk neutral agents. Upon arrival, the trader can either remain inactive or decide to buy or sell one unit of the asset. Let $a_t \in \{\text{buy, sell, hold}\}$ be the action of the trader arriving at $t$. Assume that each agent can act only once and that he leaves the market immediately afterwards.

Ex ante, none of the traders possesses superior information. However, a proportion $\mu$, where $\mu \leq 1/2$, of traders are value traders, each of whom, before taking action $a_t$, can decide whether to acquire a symmetric binary signal $S \in \{0,1\}$ with precision $q \in \left(\frac{1}{2}, 1\right)$, meaning that $\mathbb{P}(S=1 \mid V=1) = \mathbb{P}(S=0 \mid V=0) = q$. The cost of this signal is $c$,

$0 < c \leq 1$.[2]

The remaining fraction $1 - \mu$ of traders are liquidity traders, who cannot (and do not wish to) acquire a signal. Each liquidity trader values the asset at $u$, which is privately observed and is distributed uniformly on $[0, 1]$.

Prices are set by a risk neutral market maker who, in each $t$, posts a bid price $B_t$ at which he is willing to buy the asset and an ask price $A_t$ at which he is willing to sell. As is standard in the literature (e.g., Glosten and Milgrom (1985)), it is assumed that the market maker is subject to unmodeled Bertrand competition. Consequently, in equilibrium, the market maker sets the lowest ask and the highest bid price so as to break even on every trade.

When the value of the asset is $v$, the payoff of a value trader entering the market in period $t$ is:

$$\pi^I (a_t, A_t, B_t; v) = \begin{cases} v - A_t - c\mathbb{I}\{\text{acquire}\} & \text{if } a_t = \text{buy} \\ 0 & \text{if } a_t = \text{hold} \\ B_t - v - c\mathbb{I}\{\text{acquire}\} & \text{if } a_t = \text{sell} \end{cases},$$

where $\mathbb{I}\{\text{acquire}\}$ is the indicator function that takes the value of 1 if a signal is acquired and 0 otherwise. The payoff of a liquidity trader with valuation $u$ entering the market in

[2]The parameter space of interest is $0 < c \leq 1$, as if the signal is too costly (i.e., $c > 1$), no value trader would ever find it worthwhile to acquire it. At the other extreme, if $c = 0$, the model reduces to the one considered by Avery and Zemsky (1998), Park and Sabourian (2009).

The restriction on $\mu$ ensures that if all value traders were endowed with a perfectly informative signal, the bid-ask spread would be less than one almost everywhere and the market would never break down.

period $t$ is:

$$\pi^L(a_t, A_t, B_t; v) = \begin{cases} u - A_t & \text{if } a_t = \text{buy} \\ 0 & \text{if } a_t = \text{hold} \cdot \\ B_t - u & \text{if } a_t = \text{sell} \end{cases}$$

The market maker values the asset at $V$. He has no private information and can condition prices solely on publicly available information $H_t$ and on the current period's order. The publicly available information $H_t$ contains solely the sequence of past traders' actions $a_\tau$ and the realized transaction prices $p_\tau$, i.e., $H_t = \{a_\tau, p_\tau\}_{\tau=1}^{t-1} \in \mathcal{H}_t$, where $H_1$ is a null history and $\mathcal{H}_t$ denotes the set of all $t$-period-long public histories. Whether the agent is a value trader or a liquidity trader, as well as whether he has acquired a signal, and the realization of that signal constitute his private information.

**Equilibrium Concept**

The model constitutes a sequential game of incomplete information. In this game, the relevant solution concept is the Perfect Bayesian equilibrium, which must specify a strategy for each trader and a system of beliefs such that each strategy is optimal for the trader at every history given his beliefs, and these beliefs are derived from the equilibrium strategy profile by Bayes' rule whenever possible. Due to the presence of liquidity traders, any history $H_t$ occurs with a strictly positive probability, and there are no observable deviations. Hence, in each period, each player's belief regarding the value of the underlying asset, as well as the market maker's belief regarding the type of the trader he faces are uniquely determined by Bayes' rule. In order to solve for an equilibrium, it remains to specify the market maker's belief regarding the signal acquisition strategy, equilibrium bid and ask prices after every history, the decision of each value trader whether to buy a signal, as well as the orders that information and liquidity traders place.

It is assumed that, when a trader is indifferent between placing a trading order and playing hold, he trades. This tie-breaking convention makes the notation and the analysis simpler but has no bearing on the main results. Also, the behaviour of a trader when he is indifferent between all actions in his action space $\{\text{buy, sell, hold}\}$ has no impact on the results, and thus, in this case, the trading decision can be specified in an arbitrary manner.

# 3 Equilibrium Market Outcomes

## 3.1 The Equilibrium Characterisation

Once a signal is acquired, its cost is sunk and does not affect the trading decision of a value trader. Hence, the optimal trading decision for a value trader who observes signal $S$ is to place a buy order if the expected value of the asset conditional on $S$ weakly exceeds the current ask price, i.e., $\mathbb{E}[V \mid S, H_t] \geq A_t$. The value trader prefers to sell if his conditional expectation of the asset value lies below the bid price, i.e., $\mathbb{E}[V \mid S, H_t] \leq B_t$. The trader optimally refrains from trading if $B_t < \mathbb{E}[V \mid S, H_t] < A_t$.

Let $p_t$ denote the public belief which, in the case of the asset taking values in $\{0, 1\}$, is equal to the expected value of the asset given the history of trades $H_t$, $p_t := \mathbb{P}[V = 1 \mid H_t] =$

$\mathbb{E}[V \mid H_t]$ . Then, the expected value of the asset given the high signal is:

$$\begin{aligned} \mathbb{E}[V \mid S=1, H_t] &= \mathbb{P}[V=1 \mid S=1, H_t] \\ &= \frac{qp_t}{qp_t + (1-q)(1-p_t)} \geq p_t, \end{aligned} \tag{1}$$

while the expected value given the low signal is:

$$\begin{aligned} \mathbb{E}[V \mid S=0, H_t] &= \mathbb{P}[V=1 \mid S=0, H_t] \\ &= \frac{(1-q)\,p_t}{(1-q)\,p_t + q\,(1-p_t)} \leq p_t, \end{aligned} \tag{2}$$

where all conditional probabilities are obtained by Bayes' rule.

Upon acquiring a signal, value traders are always active in the market, i.e., depending on the value of the observed signal, a value trader entering the market in period $t$ places either a buy or a sell order with a strictly positive probability. This is because, if in some $t$ informed value traders were playing hold with probability 1, the market maker would break even by setting prices $A_t = B_t = p_t$. However, at such prices an informed trader with a high signal finds it optimal to buy, while an informed trader with a low signal finds it optimal to sell.

Since the market maker is assumed to break even on every trade, the ask price is the expected value of the asset conditional on history $H_t$ and a buy order, i.e., $A_t = \mathbb{E}[V \mid a_t = \text{buy at } A_t, H_t]$. Since informed value traders are always active in the market, $\mathbb{E}[V \mid a_t = \text{buy at } A_t, H_t]$ can be written as a convex combination of two terms, a liquidity trader's assessment of the asset value $p_t$ and an informed buyer's assessment. Similarly, the bid price $B_t =$

$\mathbb{E}[V \mid a_t = \text{sell at } B_t, H_t]$ is a weighted average of a liquidity trader's assessment of the asset value $p_t$ and an informed seller's assessment. It remains to identify how the value of the signal affects the direction of the trade.

By assumption, signals are informative (i.e., $q > \frac{1}{2}$). Hence, any trader who has observed a signal is better informed than the market maker, and the market maker makes an expected loss on a trade with this trader. Therefore, in equilibrium, the market maker sets bid and ask prices so that the expected profits on trades with liquidity traders are just sufficient to compensate for the expected losses on trades with informed payoff maximisers. Consequently, at each $t$, prices are such that $A_t \geq \mathbb{E}[V \mid H_t] \geq B_t$, i.e., there is a bid-and-ask spread $A_t - B_t \geq 0$. The next lemma formalises this intuition. All proofs are relegated to the appendix.

**Lemma 1.** *In equilibrium,* $A_t \geq p_t \geq B_t$.

The bid-ask spread coupled the facts that informed traders always trade and that $\mathbb{E}[V \mid S = 1, H_t] \geq p_t \geq \mathbb{E}[V \mid S = 0, H_t]$ imply that each informed trader prefers to buy after observing $S = 1$ and prefers to sell after observing $S = 0$ in every period.

As long as there is a strictly positive bid-ask spread (i.e., $A_t > B_t$), an uninformed value trader holds. This is because an uninformed trader's belief regarding the asset value coincides with the public belief $p_t$ and thus, when $A_t > p_t > B_t$, placing either a buy or a sell order involves a loss in expected terms.[3]

Given that in equilibrium, $A_t \geq B_t$, the optimal trading strategy of a liquidity trader with valuation $u$ is to buy if $u \geq A_t$, sell if $u \leq B_t$ and hold if $A_t < u < B_t$.

[3]When there is no bid and ask spread, i.e., $A_t = B_t$, trades do not have any informational content and the behaviour of uninformed traders has no influence on the market maker's belief. Hence, in this case, the behaviour of an uninformed trader can be specified in an arbitrary manner.

To solve for an equilibrium, it remains to characterise the optimal signal acquisition decision. Each value trader bases his decision of whether to acquire additional information on the expected payoff from signal acquisition, which is equal to the expected benefit of a signal minus its cost. While the cost of a signal is fixed at $c$, the expected benefit depends on the current public belief $p_t$ and the current bid and ask prices. These prices, in turn, depend of the market maker's belief regarding the information acquisition decisions of value traders. In equilibrium, the market maker must have the correct belief and value traders must behave optimally given this belief.

Suppose that the market maker believes that value traders acquire information with probability $\tilde{\sigma}$.[4] Then, the competitive market maker posts the lowest ask price and the highest bid prices satisfying the following two equations:

$$
\begin{aligned}
A_t(\tilde{\sigma}) &= \mathbb{E}\left[V \mid a_t = \text{buy at } A_t(\tilde{\sigma}), H_t\right] \\
&= \frac{\tilde{\sigma} q \mu + (1 - A_t(\tilde{\sigma}))(1-\mu)}{\tilde{\sigma}\mu(q p_t + (1-q)(1-p_t)) + (1 - A_t(\tilde{\sigma}))(1-\mu)} p_t
\end{aligned} \tag{3}
$$

and

$$
\begin{aligned}
B_t(\tilde{\sigma}) &= \mathbb{E}\left[V \mid a_t = \text{sell at } B_t(\tilde{\sigma}), H_t\right] \\
&= \frac{\tilde{\sigma}(1-q)\mu + B_t(\tilde{\sigma})(1-\mu)}{\tilde{\sigma}\mu((1-q) p_t + q(1-p_t)) + B_t(\tilde{\sigma})(1-\mu)} p_t.
\end{aligned} \tag{4}
$$

Let $\Pi(p_t, \tilde{\sigma})$ be the expected payoff from signal acquisition when the current public belief is $p_t$ and the market maker believes that value traders acquire information with probability $\tilde{\sigma}$. If a trader buys a signal and the signal turns out to be high, i.e., $S = 1$, that trader's

[4] In the notation, tilde denotes an arbitrary, not necessarily equilibrium belief of the market maker.

expected payoff is $\mathbb{E}[V \mid S=1, p_t] - A_t(\tilde{\sigma}) - c$.[5] However, if the acquired signal turns out to be low, his expected payoff is $B_t(\tilde{\sigma}) - \mathbb{E}[V \mid S=0, p_t] - c$. Thus, the overall expected payoff from signal acquisition is given by:

$$\begin{aligned} \Pi(p_t, \tilde{\sigma}) &= \mathbb{P}[S=1 \mid p_t]\left(\mathbb{E}[V \mid S=1, p_t] - A_t(\tilde{\sigma}) - c\right) \\ &\quad + \mathbb{P}[S=0 \mid p_t]\left(B_t(\tilde{\sigma}) - \mathbb{E}[V \mid S=0, p_t] - c\right). \end{aligned} \tag{5}$$

If $\Pi(p_t, \tilde{\sigma})$ is positive, a value trader entering the market in period $t$ finds it optimal to acquire the signal with probability 1. If $\Pi(p_t, \tilde{\sigma})$ is negative, the value traders finds it optimal not to acquire the signal. Finally, if $\Pi(p_t, \tilde{\sigma}) = 0$, the value trader optimally acquires the signal with probability $\sigma_t \in [0,1]$.

For any given $\tilde{\sigma}$, the expected payoff from signal acquisition varies with the current public belief regarding the value of the asset, $p_t$. When the public belief $p_t$ is either 1 or 0, $\mathbb{P}[V=1 \mid S=1, p_t] = \mathbb{P}[V=1 \mid S=0, p_t] = p_t$ and no signal can affect the belief of the informed trader. Thus, at the extreme values of $p_t$, there is no benefit to acquiring a signal. However, away from the extremes, the market maker revises his belief in response to a buy (sell) order less than the informed trader revises his belief in response to signal $S=1$ ($S=0$). Thus, a signal enables a value trader to trade with the market maker at a profit. This suggests that $\Pi(p_t, \tilde{\sigma})$ could be positive when there is sufficient uncertainty regarding the asset's value and $p_t$ is sufficiently close to $1/2$. The next lemma and its corollary confirm this intuition.

[5] Strictly speaking, expectations should be taken conditioning on the history $H_t$ and not on the public belief $p_t$. However, to interpret $H_t$, a trader needs to know the sequence of the past market maker's beliefs regarding the signal acquisition strategy of value traders, $\{\tilde{\sigma}_\tau\}_{\tau=1}^{t-1}$. In equilibrium, there is no ambiguity regarding the correct interpretation of $H_t$, as the belief of the market maker coincides with the strategy of value traders. Thus, in equilibrium, history $H_t$ can be summarised by $p_t$.

**Lemma 2.** *For any $q$ there exists $\bar{\mu}(q)$ such that for any $\mu < \bar{\mu}(q)$ and any $\tilde{\sigma}$, $\Pi(p_t, \tilde{\sigma})$ is increasing in $p_t$ for $p_t \leq 1/2$ and is decreasing in $p_t$ for $p_t \geq 1/2$.*

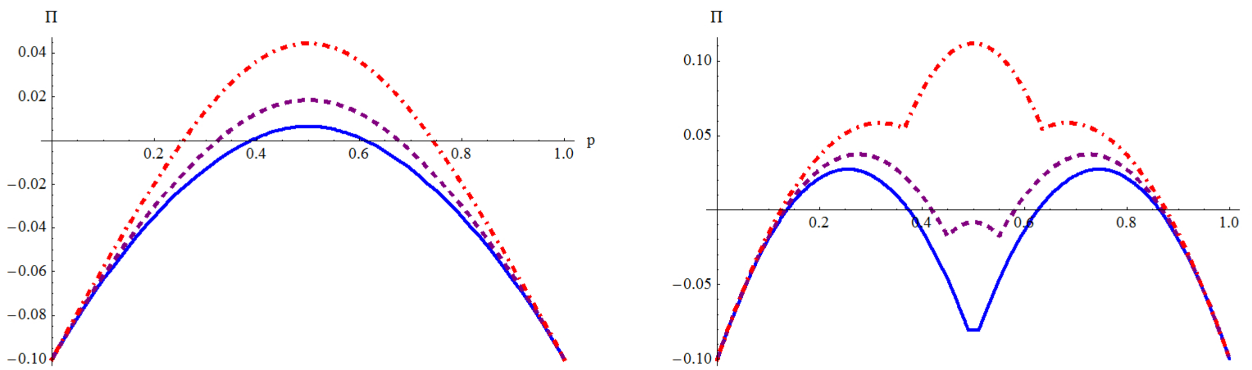
For a given belief of the market maker $\tilde{\sigma}$, the profit from signal acquisition is single peaked if there is sufficient amount of noise in the market (see Figure 1a). Thus, if each signal is highly informative (i.e., $q$ is high), the proportion ($\mu$) of value traders must be relatively low.[6]

Suppose that, contrary to the lemma's hypothesis, there are too few liquidity traders in the market (i.e., $\mu \geq \bar{\mu}(q)$). In this case, when there is a lot of uncertainty regarding the asset value (i.e., $p_t$ is close to $1/2$) and the market maker believes that value traders acquire some information, the market marker responds aggressively to each trade, suspecting any trade to be an informed trade. Consequently, when $p_t$ is close to $1/2$, the bid-ask spread is high and the profit from being informed is small. However, as the public belief regarding the asset value moves closer to 0 or 1, the market maker reacts less aggressively to the information contained in each trade. Consequently, as $p_t$ moves towards the extremes, the bid-ask spread narrows and, as it narrows, more liquidity traders start placing the orders, which justifies further narrowing of the spread. As a result of this reinforced narrowing of the spread, acquiring the signal when $p_t$ is below or above $1/2$ (but not at 0 or 1) may be more profitable than acquiring the signal when $p_t = 1/2$. Figure 1 confirms that, when signals are very informative (i.e., $q$ is close to 1) and there are relatively few liquidity

[6] In a model with exogenous information and perfectly informed value traders, the bid-ask spread is less than one almost everywhere if $\mu \leq 1/2$. Lemma 2 imposes a more stringent restriction on $\mu$, as:

$$\frac{d^2\Pi}{dp_t^2}\Big|_{\substack{p=1/2\\ \mu=1/2}} = -\frac{(2q-1)\left(3\sqrt{2}-5\sqrt{2}q+8\sqrt{1-q}q)\right)}{2\sqrt{1-q}},$$

which is negative if and only if $q < 0.91$. Thus, if signals are very informative (i.e., $q$ is close to 1), restriction $\mu \leq 1/2$ may not be enough to guarantee that $\Pi(p_t, \tilde{\sigma})$ is single peaked with a global maximum at $p_t = 0.5$.




(a) When $q = 0.75$, $\Pi(p, \tilde{\sigma})$ has a unique global maximum at $p = 1/2$ for any $\tilde{\sigma}$.

(b) When $q = 1$, $\Pi(p, \tilde{\sigma})$ does not necessarily reach global maximum at $p = 1/2$.

Figure 1: $\Pi(p, \tilde{\sigma})$, a value trader's expected payoff from acquiring information when the public belief is $p_t$ and the market maker believes that value traders acquire information with probability $\tilde{\sigma}$. The solid blue line corresponds to $\tilde{\sigma} = 1$, the dashed purple line corresponds to $\tilde{\sigma} = 0.85$ and the dot-dashed red line corresponds to $\tilde{\sigma} = 0.6$. The figure assumes that $\mu = .49$ and $c = 0.1$.

traders (i.e., $\mu$ is close to $1/2$), the profit from signal acquisition may obtain maximum away from $p_t = 1/2$.

In the remainder of the paper it will be assumed that the parameters of the game satisfy $\mu < \bar{\mu}(q)$. However, Lemma 3 and Proposition 1 continue to hold even if $\mu \geq \bar{\mu}(q)$. The qualitative changes to the equilibrium characterization in the case of $\mu \geq \bar{\mu}(q)$ will be discussed briefly towards the end of the section.

The immediate consequence of Lemma 2 is that for every belief of the market maker $\tilde{\sigma}$, signal acquisition can be profitable if and only if the public belief $p_t$ is sufficiently close to 1/2.

**Corollary 1.** *Fix $q$. If $\mu < \bar{\mu}(q)$ and $0 < c < \frac{1}{2}(2q-1)$, then there exists $\bar{\sigma} \in [0,1]$ such that $\Pi(p_t, \tilde{\sigma}) \geq 0$ if and only if $\tilde{\sigma} \leq \bar{\sigma}$ and $p_t \in \left[\underline{p}(\tilde{\sigma}), \bar{p}(\tilde{\sigma})\right]$, where $\bar{p}(\tilde{\sigma}) \geq 1/2 \geq \underline{p}(\tilde{\sigma})$.*

Lemma 1 requires that the cost of signal acquisition be not too large. The restriction on the maximal size of $c$ ensures that, when $p_t = 1/2$, the expected payoff from signal acquisition is strictly positive when the market maker holds the most optimistic (from the value trader's perspective) belief regarding the signal acquisition strategy, i.e., $\tilde{\sigma} = 0$. Since at $p_t = 1/2$, profit from signal acquisition is positive when $\tilde{\sigma} = 0$, it must also be positive for other market maker's beliefs $\tilde{\sigma}$ sufficiently close to 0 as $\Pi(p_t, \tilde{\sigma})$ is continuous and decreasing in $\tilde{\sigma}$ for any $p_t$ (see Lemma 3).

From the perspective of the market maker, when value traders acquire signals with high probability, any buy or sell order has high informational content. Consequently, when $\tilde{\sigma}$ is high, the bid and ask spread is high as well. This suggests that, for any $p_t$, the expected profit from signal acquisition is decreasing in $\tilde{\sigma}$, the market maker's belief regarding the signal acquisition strategy of value traders.

**Lemma 3.** $\Pi(p_t, \tilde{\sigma})$ *is continuous in* $\tilde{\sigma}$ *and, for any* $p_t \in (0, 1)$, $\Pi(p_t, \tilde{\sigma})$ *is decreasing in* $\tilde{\sigma}$.

From Lemma 3 it follows directly that $\bar{p}(\tilde{\sigma})$ is decreasing in $\tilde{\sigma}$ and $\underline{p}(\tilde{\sigma})$ is increasing in $\tilde{\sigma}$.

**Corollary 2.** $\bar{p}(\tilde{\sigma})$ *is decreasing in* $\tilde{\sigma}$ *and* $\underline{p}(\tilde{\sigma})$ *is increasing in* $\tilde{\sigma}$.

Let $\sigma_t$ denote the signal acquisition strategy of value traders, i.e., $\sigma_t$ is the probability with which a value trader acquires information in period $t$. By definition, in equilibrium, the belief of the market maker regarding the strategy of value traders must be correct, i.e., $\tilde{\sigma} = \sigma_t$. Corollary 1 implies that, if $\Pi(1/2, 1) > 0$, when $p_t$ is sufficiently close to 1/2, information players acquire signals with probability 1 and $\tilde{\sigma} = 1$.

However, when the public belief crosses one of the thresholds $\underline{p}(1)$ or $\bar{p}(1)$, the expected (net) payoff from signal acquisition $\Pi(p_t, 1)$ becomes negative. At this point, the market maker needs to revise downwards his belief regarding the strategy of value traders.

However, according to Lemma 3, as the probability that the market maker attaches to signal acquisition goes down, the bid-ask spread decreases and, at the updated belief, the payoff from signal acquisition becomes non-negative. This suggests that, for any $p_t \in \left[\underline{p}(0), \underline{p}(1)\right] \cup [\bar{p}(1), \bar{p}(0)]$, the market maker sets prices in such a way that information players are indifferent between acquiring and not acquiring signals.

Finally, when the public belief $p_t$ either crosses $\underline{p}(0)$ from above or crosses $\bar{p}(0)$ from below, signal acquisition becomes unprofitable for any belief of the market maker $\tilde{\sigma}$. Consequently, value traders stop acquiring information and the market maker adjusts his belief regarding the signal acquisition strategy accordingly.

The discussion above characterised the bid and ask prices as well as the optimal trading and signal acquisition strategies of traders for an arbitrary belief of the market maker regarding the signal acquisition strategy of value traders. Proposition 1 combines these characterisations with the requirement that in equilibrium, the market maker's belief regarding the signal acquisition strategy should be correct to derive an equilibrium of the model. In the proposition, asterisks denote equilibrium quantities.

**Proposition 1.** *For any $0.5 < q \leq 1$ and $0 < c < (2q-1)/2$, there exists a non-empty set of sequential trading games with costly information acquisition such that the following beliefs and strategies constitute an equilibrium in those games:*

*(1) In every $t$, a value trader who has acquired a signal buys after observing $S = 1$ and sells after observing $S = 0$, and a value trader who has not acquired a signal holds if $A_t^* > B_t^*$ and behaves in an arbitrary manner otherwise.*

*(2) A liquidity trader with valuation $u$ buys if $u \geq A_t^*$, sells if $u \leq B_t^*$ and holds otherwise.*

*(3) A value trader entering the market in period $t$ acquires the signal with probability $\sigma_t^*$ such that,*

*if* $\Pi(1/2, 1) > 0$,

$$\sigma_t^* = \begin{cases} 1 & \text{if } p_t \in \left(\underline{p}(1), \bar{p}(1)\right) \\ \xi_t & \text{if } p_t \in \left[\underline{p}(0), \underline{p}(1)\right] \cup \left[\bar{p}(1), \bar{p}(0)\right], \\ 0 & \text{if } p_t \in \left[0, \underline{p}(0)\right) \cup \left(\bar{p}(0), 1\right] \end{cases} \tag{6}$$

*and, if* $\Pi(1/2, 1) \leq 0$,

$$\sigma_t^* = \begin{cases} \xi_t & \text{if } p_t \in \left[\underline{p}(0), \bar{p}(0)\right] \\ 0 & \text{if } p_t \in \left[0, \underline{p}(0)\right) \cup \left(\bar{p}(0), 1\right] \end{cases}, \tag{7}$$

*where* $\xi_t \in [0, 1]$ *solves* $\Pi(p_t, \xi_t) = 0$.

*(4) The market maker sets the lowest ask price and the highest bid prices that for* $\sigma_t^*$*, solve equations (3) and (4), respectively, i.e.:*

$$\begin{aligned} A_t^* &= \frac{1}{2(1-\mu)} \left[\sigma_t^* \mu \left(q p_t + (1-q)(1-p_t)\right) + (1-\mu)(1+p_t)\right. \\ &- \left(\left(\sigma_t^* \mu \left(q p_t + (1-q)(1-p_t)\right) + (1-\mu)(1+p_t)\right)^2\right. \\ &\quad \left.\left. -4(1-\mu)\left(\sigma_t^* \mu q p_t + (1-\mu) p_t\right)\right)^{1/2}\right] \end{aligned} \tag{8}$$

*and*

$$\begin{aligned} B_t^* &= \frac{1}{2(1-\mu)} \left[(1-\mu) p_t - \sigma_t^* \mu \left((1-q) p_t + q(1-p_t)\right)\right. \\ &+ \left(\left(\sigma_t^* \mu \left((1-q) p_t + q(1-p_t)\right) - (1-\mu) p_t\right)^2\right. \\ &\quad \left.\left. +4\sigma_t^* \mu (1-\mu)(1-q) p_t\right)^{1/2}\right]. \end{aligned} \tag{9}$$

Proposition 1 and the preceding discussion indicate that information acquisition does not stop abruptly. Instead, as the public belief regarding the asset value moves closer to the extreme values of 0 or 1, the probability with which value traders acquire signals decreases gradually and eventually falls to zero. Figure 2a illustrates the information acquisition strategy for different values of $p_t$ assuming that at $p_t = 1/2$, signals are acquired with probability 1, i.e., $\Pi(1/2, 1) > 0$. When $\Pi(1/2, 1) \leq 0$, the segment $\left[\underline{p}(1), \bar{p}(1)\right]$ does not exist, but otherwise the same figure describes signal acquisition strategy.

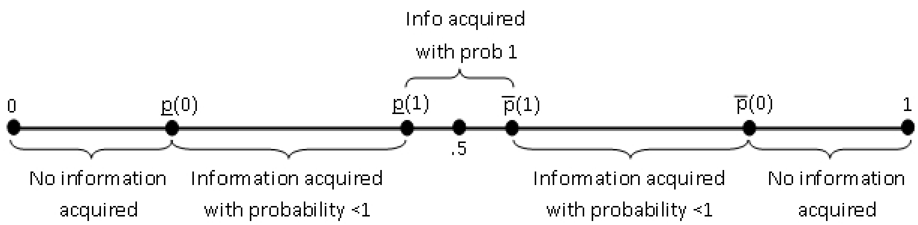
When $\mu \geq \bar{\mu}(q)$, the game continues to have an equilibrium described in Proposition 1, but in this equilibrium, the probability of acquiring the signal may not be monotonically decreasing as $p_t$ moves towards the extreme values of 0 or 1. In particular, when $\mu \geq \bar{\mu}(q)$ and $\Pi(1/2, 1) < 0$, it could be the case that signals are acquired with probability less than one for any $p_t \in \left(\underline{p}'(1), \bar{p}'(1)\right)$, where $\underline{p}'(1) > 1/2 > \bar{p}'(1)$, with probability one for any $p_t \in \left[\underline{p}(1), \underline{p}'(1)\right] \cup [\bar{p}'(1), \bar{p}(1)]$, and then again with probability less than one for any $p_t \in \left[\underline{p}(0), \underline{p}(1)\right] \cup [\bar{p}(1), \bar{p}(0)]$ (see Figure 2b).

For future reference, note that by solving $\Pi(p, 0) = 0$ for $p$, the following explicit expressions for the belief thresholds $\underline{p}(0)$ and $\bar{p}(0)$ at which traders stop acquiring information can be obtained:

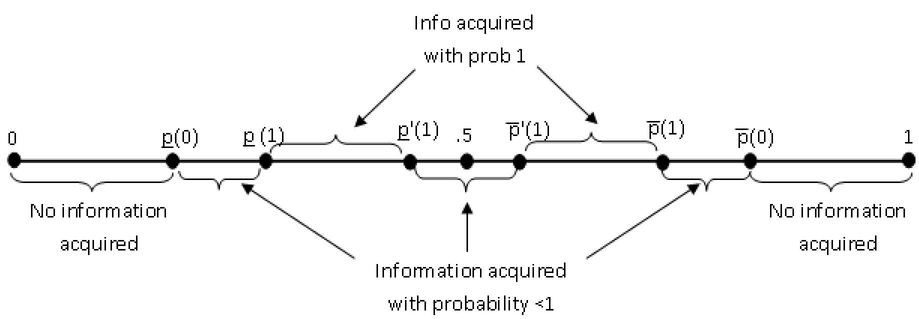
$$\underline{p}(0) = \frac{1}{2} - \sqrt{\frac{1}{4}\left(1 - \frac{2c}{2q-1}\right)}, \tag{10}$$

$$\bar{p}(0) = \frac{1}{2} + \sqrt{\frac{1}{4}\left(1 - \frac{2c}{2q-1}\right)}. \tag{11}$$

It turns out that for every $p_t$, there exists a unique mixed signal acquisition strategy of a value trader such that the market maker has the correct belief regarding this strategy and breaks even on every trade. This result can be anticipated from Lemma 3, which suggests




(a) When $\mu < \bar{\mu}(q)$, the equilibrium signal acquisition strategy $\sigma_t^*$ is weakly increasing in $p_t$ for $p_t \in [0, 1/2)$ and is weakly decreasing in $p_t$ for $p_t \in [1/2, 1]$.




(b) When $\mu \geq \bar{\mu}(q)$ and $\Pi(1/2, 1) < 0$, the equilibrium signal acquisition strategy $\sigma_t^*$ is non-monotone in $p_t$ for $p_t \in [0, 1/2)$ as well as for $p_t \in [1/2, 1]$.

Figure 2: Information acquisition strategy $\sigma_t^*$ as a function of public belief $p_t$.

that in equilibrium, traders' information acquisition decisions are strategic substitutes. According to this lemma for a given public belief $p_t$, $\Pi(p_t, \sigma)$ is a decreasing function of signal acquisition strategy $\sigma$ and thus there could be at most one solution to $\Pi(p_t, \sigma) = 0$.

The uniqueness of optimal signal acquisition strategy implies that the equilibrium in Proposition 1 is essentially unique. There could be other equilibria which differ from the one in Proposition 1 only in terms of prescribed trading behaviour for value traders in the cases of indifference, i.e., when traders are indifferent between trading and playing hold, or when they are indifferent between all actions in $\{\text{buy, sell, hold}\}$. However, these tie-breaking conventions do not affect the qualitative features of the equilibrium. Moreover, the behaviour of value traders when they are indifferent between all actions in $\{\text{buy, sell, hold}\}$ does not affect the determination of prices in any way.

**Proposition 2.** *Equilibrium prices are unique subject to the tie-breaking rule that each value trader places an order when indifferent between trading and not trading.*[7]

In the equilibrium outlined in Proposition 1, as the public belief moves closer to the extreme values of 0 or 1, the probability with which value traders acquire signals decreases. One consequence of this decline is a gradual narrowing of the bid and ask spread, $A_t^* - B_t^*$. As the public belief moves closer to the extremes, the bid and ask spread also becomes narrower in the model without costs to information acquisition. This is a basic property of Bayesian updating. When there is little uncertainty about the asset value, signals cannot have much effect on the posterior probability. However, while with costless information the bid-ask spread disappears completely only when $p_t$ is either 0 or 1, in the model with costly signals, the spread disappears away from the extremes. Thus the decrease in the

[7] The tie-breaking rule comes into play only when the bid-ask spread disappears and thus it cannot affect the equilibrium dynamics of the game with a positive probability.

probability with which signals are acquired accelerates the process of narrowing of the bid and ask spread. Intuitively, when signals are acquired less frequently, each trade has lower informational content. Hence the market maker can set bid and ask prices close to the current public belief $p_t$ and still break even. In fact, from (15) and (16) it follows that $\partial \left(A_t^* - B_t^*\right) / \partial \sigma_t^* > 0$.

In the game, the public belief $p_t$ evolves so that, given a prior $p_1$, for any $t > 1$

$$p_{t+1} = \begin{cases} A_t^* & \text{if} \quad a_t = \text{buy} \\ p_t & \text{if} \quad a_t = \text{hold} \\ B_t^* & \text{if} \quad a_t = \text{sell} \end{cases}, \tag{12}$$

where $A_t^*$ and $B_t^*$ are defined in (8) and (9). When the public belief reaches either a value (weakly) below $\underline{p}(0)$ or a value (weakly) above $\bar{p}(0)$, the expected payoff from information acquisition becomes (weakly) negative and information agents stop acquiring signals. From this point onwards, trades no longer convey any information on the asset value, and the bid-ask spread disappears. Moreover, once the belief reaches either a value weakly below $\underline{p}(0)$ or a value weakly above $\bar{p}(0)$, the market maker cannot make inferences from the actions of traders and stops updating prices. Thus prices remain fixed at the same level forever.

The result that information acquisition stops away from the extremes resembles the findings of Burguet and Vives (2000), who find that with a positive marginal cost at zero information acquisition, unbounded information accumulation is impossible in a smooth and noisy social learning model.[8] Here, a fixed cost to acquiring a signal plays the same

[8] In smooth and noisy social learning model of Burguet and Vives (2000), short-lived agents must predict a random variable. Their predictions can be based either on private information or on public information.

role as a positive cost at zero in Burguet and Vives (2000), and this cost causes information acquisition to stop. The substantial difference is that Burguet and Vives (2000) is a social learning (prediction) model, whereas the present model is an economy with prices acting as signals.

## 3.2 Comparative Statics

### The Effect of Cost to Information Acquisition on Equilibrium Behaviour

Section 3.1 has characterised the equilibrium for a fixed $c$. This section investigates how equilibrium behavior and the amount of information accumulated by the market in equilibrium change with the cost to signal acquisition.

Intuitively, a higher cost reduces incentives to acquire information for any public belief $p_t$. This intuition is formalised in the observation that the cost to signal acquisition is a shift parameter in the expected payoff function $\Pi(p,\sigma)$ : for any signal acquisition probability $\sigma$, as $c$ increases, $\Pi(p,\sigma)$ shifts down in $(p,\Pi)$ space. This implies that for any $\sigma$, $\bar{p}(\sigma)$ decreasing in $c$, while $\underline{p}(\sigma)$ is increasing in $c$.

**Proposition 3.** *For any given signal acquisition probability $\sigma$,*

$$\frac{\partial}{\partial c}\bar{p}(\sigma) < 0 \textit{ and } \frac{\partial}{\partial c}\underline{p}(\sigma) > 0.$$

*Moreover,*

$$\lim_{c\to 0}\bar{p}(\sigma) = 1 \text{ and } \lim_{c\to 0}\underline{p}(\sigma) = 0.$$

Public information is a noisy average of past predictions. The acquisition of private information is costly.

From Proposition 3, taking $\sigma = 0$, it is clear that reducing $c$ monotonically pushes out the boundaries of the public belief region where value traders acquire information. Moreover, as $c \to 0$, $\bar{p}(0) \to 1$ and $\underline{p}(0) \to 0$, and thus information acquisition never stops at the limit as costs to information acquisition disappear. Finally, since, as $c \to 0$, $\bar{p}(\sigma) \to 1$ and $\underline{p}(\sigma) \to 0$ for all $\sigma$, at the limit the region where value traders use a non-degenerate mixed signal acquisition strategy disappears. Thus, at the limit as $c \to 0$, the equilibrium behaviour of value traders in the model with costly information coincides with the behaviour of value traders in the model with no costs to information acquisition.

A higher cost of information acquisition reduces the informativeness of equilibrium in the following sense. As $c$ increases, in any given period $t$ and for any public belief $p_t$, the equilibrium probability with which value traders acquire signals (weakly) decreases.[9] This follows directly from the reduction in incentives to acquire information when it is associated with higher costs.

**Proposition 4.** *For any given public belief $p_t$, the equilibrium signal acquisition strategy of value traders, $\sigma_t^*$, is weakly decreasing in $c$, i.e., for all $p_t \in [0,1]$, if $c > c'$,$\sigma_t^*(p_t; c) \leq \sigma_t^*(p_t; c')$.*

## 3.3 Long-run Behaviour of Prices and Informational Inefficiency

As is standard, also in the current setting, the public belief is a martingale with respect to public information, as

$$\begin{aligned} \mathbb{E}(p_{t+1} \mid H_t) &= \mathbb{E}(\mathbb{E}(V \mid H_{t+1}) \mid H_t) \\ &= \mathbb{E}(V \mid H_t) = p_t, \end{aligned}$$

[9]The described reduction in information-acquisition probability affects the evolution of $p_t$ and thus has dynamic consequences. Proposition 4 does not deal with these dynamic consequences.

where the second equality follows by the Law of Iterated Expectations.[10] Since the public belief is a non-negative martingale, the conditions of the martingale convergence theorem are satisfied and the sequence of prices $\{p_t\}_{t=1}^{\infty}$ converges to a random variable $p_\infty$ almost surely.

When there are no costs associated with information acquisition, the public belief settles almost surely at the liquidation value of the asset (see, for example, Avery and Zemsky (1998) and Glosten and Milgrom (1985)). In a model with costly signals, however, convergence to the true value of the asset is impossible and the public belief might even settle "far away" from the true value. Proposition 5 demonstrates that in the case of costly signals, in the limit as time tends to infinity, the bid-ask spread disappears as in the case where $c = 0$. This is just a consequence of the fact that the public belief is a uniformly integrable martingale and thus, in addition to converging almost surely, it also converges in $L^1$.

Equilibrium characterisation in Section 3.1 indicates that, with costly signals, the bid-ask spread disappears when the public belief crosses one of the thresholds $\underline{p}(0)$ or $\bar{p}(0)$ (and value traders no longer find it optimal to acquire additional information).[11]Consequently, the fact that as $t \to \infty$, the bid-ask spread disappears implies that in the long-run information acquisition information acquisition stops, i.e., the public belief converges to a random variable $p_\infty$ taking values either "close to," but weakly below $\underline{p}(0)$, or "close to," but weakly above $\bar{p}(0)$.[12] Since $\underline{p}(0) > 0$ and $\bar{p}(0) < 1$, in equilibrium, the market price

[10]More formally, expectations should be conditioned on the $\sigma$-algebra generated by $H_t$.

[11]Due to Bayesian updating, the price moves in discrete steps. Thus in the long-run, the posterior belief can overshoot either $\underline{p}(0)$ or $\bar{p}(0)$, instead of landing at the threshold precisely. Consequently, the support of random variable $p_\infty$ could be comprised of more than just two points.

[12]If the bid-ask spread did not disappear, the support of random variable $p_\infty$ would have been contained in interval $\left(\underline{p}(0), \bar{p}(0)\right)$.

(with with probability 1) gets stuck away from both 0 and 1, never converging to the true value of the underlying asset even at the limit. Moreover, it is possible for transaction prices to get trapped close to $\underline{p}(0)$ even when $V = 1$, or close to $\bar{p}(0)$ even when $V = 0$.

**Proposition 5.** *When $c > 0$, the sequence of public beliefs converges to a random variable $p_\infty$ that takes values either close to, but weakly below $\underline{p}(0)$ or close to, but weakly above $\bar{p}(0)$.*

Proposition 5 highlights the difference between the case of no costs to information acquisition and the case of positive signal cost. When $c = 0$, the public belief converges to the true value of the asset, while with arbitrarily small but positive $c$, public belief could converge to a value far away from the liquidation value of the asset. Nevertheless, there is no discontinuity in equilibrium at the limit as $c \to 0$. Proposition 3 establishes that as $c$ tends to zero, thresholds $\underline{p}(0)$ and $\bar{p}(0)$ at which information acquisition stops tend to 0 and 1 respectively and thus, at the limit as $c \to 0$, market must learn the true value of the asset. Proposition 6 and its Corollary 3 confirm that the approximate probability of price getting trapped far away from the true value of the asset decreases as $c$ decreases and, at the limit as $c \to 0$, it is equal to zero.

For concreteness, Proposition 6 assumes that $V = 0$ and computes the probability

$$\mathbb{P}\left(p_\infty \geq \bar{p}(0) \mid V = 0\right)$$

of the public belief converging to a value (weakly) above $\bar{p}(0)$.[13] This probability is com-

[13] This is without loss of generality as, given the symmetry of the environment, $\mathbb{P}\left(p_\infty \leq \underline{p}(0) \mid V = 1\right) = \mathbb{P}\left(p_\infty \geq \bar{p}(0) \mid V = 0\right)$.

puted using the fact that conditional on $V = 0$, the odds ratio

$$r_t \;:=\; \frac{p_t}{1 - p_t}$$

is a bounded martingale with respect to public history $H_t$ (see Lemma 4 in Appendix A) and thus its unconditional expected value at $t = 0$ is equal to $p_1/(1 - p_1)$. Moreover, according to Proposition 5, as $t \to \infty$, the public belief converges either to a value close to, but weakly below, $\underline{p}(0)$ or to a value close to, but weakly above, $\bar{p}(0)$. This implies that asymptotically the odds ratio is approximately equal either to $\underline{p}(0) / \left(1 - \underline{p}(0)\right)$ or to $\bar{p}(0) / (1 - \bar{p}(0))$.[14] Equating the asymptotic expected value of the odds ratio to its unconditional expected value allows deriving the approximate probability of the public belief getting stuck far away from the underlying value of the asset as a function of $p_1$, $\underline{p}(0)$, and $\bar{p}(0)$. Since $\bar{p}(0)$ is decreasing, while $\underline{p}(0)$ is increasing in $c$ (see Proposition 3), its functional form implies that the approximate probability of public belief being trapped far away from the true value is increasing in $c$.

**Proposition 6.** *Suppose that* $0 < c < (2q - 1)/2$, $V = 0$ *and* $p_1 \in \left(\underline{p}(0), \bar{p}(0)\right)$. *Then:*

$$\mathbb{P}\left(p_\infty \geq \bar{p}(0) \mid V = 0\right) \;=\; \frac{\left(p_1 - \underline{p}(0)\right)(1 - \bar{p}(0))}{\left(\bar{p}(0) - \underline{p}(0)\right)(1 - p_1)}$$

*and is increasing in* $c$.

Since as $c$ tends to 0, $\underline{p}(0) \to 0$ and $\bar{p}(0) \to 1$, it immediately follows that the approximate probability of price getting trapped far away from the true value of the asset tends to 0 at

[14] The public belief converges to values close to, but not exactly equal to the thresholds $\underline{p}(0)$ and $\bar{p}(0)$ because, due to Bayesian updating, the price moves in discrete steps.

the limit.

**Corollary 3.**

$$\lim_{c\to 0} \mathbb{P}\left(p_\infty \geq \bar{p}\left(0\right) \mid V=0\right)=0$$

# 4 Welfare

The previous section characterised equilibrium in a decentralised market. This section compares information acquisition efforts of decentralised-market participants to information acquisition policy that a utilitarian social planner would wish to implement. The interest of this question lies in the fact that, while there are no gains from trade between the market maker and value traders, there are allocative gains to trade between the market maker and liquidity traders. Hence, the informational inefficiency of prices identified in Section 3.3 leads to allocative inefficiency in the long-run. However, because of the costs of information acquisition, long-run allocative inefficiency does not immediately imply that the total surplus is not maximized at the maket solution.

Consider a utilitarian social planner who does not have access to the private information of traders, but can control the decisions to acquire this information through, e.g., the use of taxes or subsidies. In period $t$ when public belief regarding the asset value is $p_t$ and value traders acquire signals with probability $\sigma$, the instantaneous social surplus is defined to be:

$$\begin{aligned} S\left(p_t,\sigma\right) &= (1-\mu)\left(\int_{A_t^*}^{1}(u-p)\,du+\int_0^{B_t^*}(p-u)\,du\right)-\mu\sigma c \\ &= (1-\mu)\left(\left(A_t^*+B_t^*-1\right)p_t-\frac{1}{2}\left(A_t^{*2}+B_t^{*2}-1\right)\right)-\mu\sigma c, \end{aligned} \tag{13}$$

where $B_t^*$ and $A_t^*$ the bid and ask prices, respectively, defined in (9) and (8).

When information acquisition ceases and bid ask spread disappears, (13) reduces to:

$$S(p,0) = (1-\mu)\left(\frac{1}{2} - p(1-p)\right), \tag{14}$$

which is maximised at $p \in \{0,1\}$. This implies that in the long-run, allocative gains from trade are maximised if market learns the true value of the asset. However, information acquisition is costly and it also increases the bid-ask spread in the current period, which, in turn, reduces the current-period allocative efficiency as liquidity traders with valuations in $[B_t^*, A_t^*]$ do not trade. Thus, the social planner faces a trade-off between long-term gains and short-term costs.

Suppose the planner discounts future at rate $\delta \in (0,1)$. The planner's objective is to maximise the expected total surplus over the infinite horizon:

$$\max_{\{\sigma_0,\sigma_1,\ldots\}} (1-\delta)\,\mathbb{E}\sum_{t=0}^{\infty} \delta^t S(p_t,\sigma_t).$$

Discounting arises naturally in the model if in every trading period, a public announcement that resolves uncertainty regarding the asset value (e.g., announcement of a merger or earnings of a company) is expected to come with probability $(1-\delta)$. Following such release of public information, information-based trading ceases. Thus in the environment with frequent news, the social planner is likely to display impatience, while in the environment where news is scarce, the social planner is likely to attach higher weight to payoffs of future traders.

**Impatient social planner**

Because the bid-ask spread precludes some liquidity traders from trading, an impatient social planner with discount factor $\delta = 0$ would prefer no information acquisition in every period and for any $p_t$. Thus, when there is significant uncertainty regarding the asset value, at the decentralised market solution, there is overproduction of information relative to the social optimum.

**Proposition 7.** *Suppose equilibrium described in Proposition 1 exists and in each $t$, social planner solves:*

$$\max_{\sigma_t} S\left(p_t, \sigma_t\right).$$

*Then, market participants acquire too much information for any $p_t \in \left[\underline{p}(0), \bar{p}(0)\right]$ relative the social optimum.*

**Patient social planner**

A very patient social planner cares mostly about the expected future payoffs of traders. From (14), it follows that expected future welfare is maximized when the social planner learns the true value of the asset. Consequently, regardless of the (fixed) belief $p \in (0,1)$, terminating information acquisition is suboptimal for an infinitely patient social planner (i.e., when $\delta \to 1$).[15]

**Proposition 8.** *For every period $t$ and public belief $p_t \in (0,1)$, there exists a $\bar{\delta}(p_t) \in (0,1)$ such that if the discount factor of the social planner exceeds $\bar{\delta}(p_t)$, $\sigma_t = 0$ is not socially optimal.*

[15] This result is similar to the result in Bose et al. (2006), who study a dynamic pricing by a monopolist selling to buyers who learn from each other's purchases. The price posted in each period serves to extract rent from the current buyer, as well as to control the amount of information transmitted to future buyers. Information increases future rent extraction, and thus an infinitely patient monopolist would never find it optimal to stop providing buyers with some information and charge a pooling price.

It immediately follows from Proposition 8 that a sufficiently patient social planner would continue acquiring information beyond the signal-acquisition thresholds of the decentralised market. This is because while deciding whether to acquire a signal, value traders do not take into account the externality their decision exerts on the payoffs of future liquidity traders.

**Corollary 4.** *There exists a $\bar{\delta} \in (0,1)$ such that if the discount factor of the social planner exceeds $\bar{\delta}$, he would find it optimal to continue acquiring information at public belief thresholds $\underline{p}(0)$ and $\bar{p}(0)$, defined in (10) and (11), respectively.*

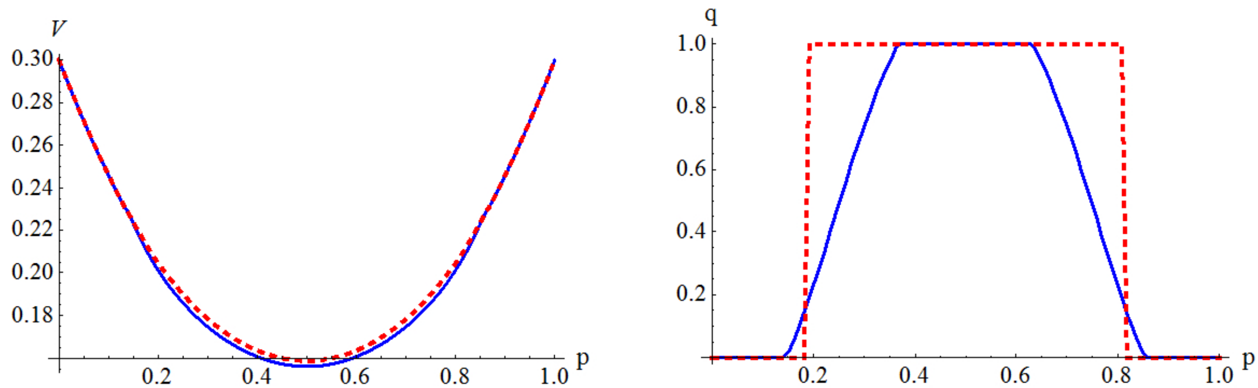
**Long-run information acquisition of the social planner**

Since information acquisition is costly, when there is little uncertainty regarding the asset's value, the social planner finds it optimal to stop acquiring additional signals.

**Proposition 9.** *For every discount factor $\delta \in (0,1)$, there exist $\underline{p}^S$ and $\bar{p}^S$, $\bar{p}^S > 1/2 > \underline{p}^S$, such that it is socially optimal to acquire some information if and only if $p \in \left[\underline{p}^S, \bar{p}^S\right]$.*

**Short-term behaviour**

At one extreme, a very patient social planner is ready to endure the immediate cost to information acquisition in order to improve future welfare. At the other extreme, a very impatient social planner would rather prohibit information acquisition so as not to exclude liquidity traders in the short-run. For a fixed discount factor $\delta$, it is possible to construct examples in which the social planner's information acquisition interval $\left[\underline{p}^S, \bar{p}^S\right]$ is not wider than the market's information acquisition interval $\left[\underline{p}(0), \bar{p}(0)\right]$. At the same




(a) Value Functions

(b) Policy Functions

Figure 3: Planner's solution v. Market Equilibrium

time, it could be the case that for some part of the market's information acquisition interval $\left[\underline{p}(0), \bar{p}(0)\right]$, the social planner induces higher information acquisition effort than that in the market equilibrium. This is because in the market equilibrium, the probability with which value traders acquire signals declines gradually, while the social planner either induces signal acquisition with probability 1 or prohibits it.

Figure 3 depicts an example where $\left[\underline{p}(1), \bar{p}(1)\right] \subset \left[\underline{p}^S, \bar{p}^S\right] \subset \left[\underline{p}(0), \bar{p}(0)\right]$ and in $\left[\underline{p}^S, \underline{p}(1)\right] \cup \left[\bar{p}(1), \bar{p}^S\right]$ the social planner induces higher information acquisition effort than the signal acquisition probability in the unsubsidised market equilibrium. Panel (a) depicts value functions associated with the market solution (blue dots) and the planner's problem (red dots); panel (b) depicts equilibrium signal acquisition probabilities (blue line) and planner's optimal policy function (red dashed line). The value functions are generated through value function iteration (on 100 point grid) with the following parameters:[16] $\delta = 0.94$, $q = .7$, $\mu = 0.4$ and $c = 0.1$.

[16]I thank Romans Pancs for providing the Mathematica algorithm for generating these graphs.

# 5 Concluding Remarks

It has been demonstrated that, at the unique equilibrium of a sequential trade model with costly information acquisition, value traders stop obtaining signals when the asset price is sufficiently close to one of the extremes and the benefit of a signal is outweighed by its cost. Thus there are two threshold values of the public belief about the likely asset value at which signal acquisition stops. As a result, the price does not converge to the liquidation value of the asset. Moreover, conditional on the true asset value, the probability of price getting stuck far away from the liquidation value of the asset is strictly positive, indicating that costs to information acquisition may bring about informational inefficiency thereby prices do not reveal the value of the fundamental, even in the long-run.[17]

The paper also studies welfare implications of the termination of information acquisition. In the proposed sequential trading model, liquidity traders value the asset differently from the market maker and informed traders. Because of this difference in valuations, there are allocative gains from trade and it matters whether the price approaches the liquidation value of the asset in the long-run. However, value traders, when they decide whether to acquire informative signals, do not take into account the positive externality their decisions exert on the welfare of liquidity traders. This implies that, when the social discount factor is sufficiently high, the decentralised market solution underprovides information relative to a social optimum. At the same time, if the social discount factor is low, market overprovides information as signal acquisition efforts induce a positive bid-ask spread, which in turn, prices some of the liquidity traders out of the market in the short-run.

[17] It can be demonstrated that qualitatively similar results prevail in a more general setting where value traders can choose the precision of the signal to acquire.

# A Proofs

**Proof of Lemma 1:** Follows by Proposition 1 in Glosten and Milgrom (1985).

**Proof of Lemma 2:** Fix $1 > q > 1/2$. After some algebra $\Pi(p_t, \tilde{\sigma})$ can be written as follows:

$$\Pi(p_t, \tilde{\sigma}) = (2q-1)p_t - (qp_t + (1-q)(1-p_t))A_t(\tilde{\sigma}) + ((1-q)p_t + q(1-p_t))B_t(\tilde{\sigma}) - c.$$

Taking the derivative with respect to $p_t$ and evaluating the limit as $\mu \to 0$ :

$$\lim_{\mu \to 0} \frac{d\Pi}{dp_t} = 2(2q-1)(1-2p) \begin{cases} > 0 & \text{if } p < 1/2 \\ = 0 & \text{if } p = 1/2 \\ < 0 & \text{if } p > 1/2 \end{cases}.$$

Note that $A_t(\tilde{\sigma})$ and $B_t(\tilde{\sigma})$ are continuous functions of $\mu$and thus $\Pi(p_t, \tilde{\sigma})$ is continuous in $\mu$. Consequently, for sufficiently small $\mu$, $\Pi(p_t, \tilde{\sigma})$ is increasing in $p_t$ for $p_t \leq 1/2$ and is decreasing in $p_t$ for $p_t \geq 1/2$, reaching maximum at $p_t = 1/2$.

**Proof of Corollary 1:** By Lemma 2 any $\tilde{\sigma}$, $p_t = 1/2$ is a unique point at which $\Pi(p_t, \tilde{\sigma})$ reaches the maximum. Note that

$$\Pi(1/2, 0) = q - \frac{1}{2} - c > 0,$$

by hypothesis of the lemma. Since by Lemma 3 $\Pi(1/2, \tilde{\sigma})$ is continuous and decreasing in $\tilde{\sigma}$, there must exist some $\bar{\sigma} \in (0, 1]$ such that $\Pi(1/2, \tilde{\sigma}) \geq 0$ if and only if $\tilde{\sigma} \leq \bar{\sigma}$. If $\Pi(1/2, 1) > 0$, let $\bar{\sigma} = 1$; otherwise define $\bar{\sigma}$ as a solution to $\Pi(1/2, \bar{\sigma}) = 0$.

Consider $\tilde{\sigma} \leq \bar{\sigma}$. Note that for any such $\tilde{\sigma}$, $\Pi(1, \tilde{\sigma}) = \Pi(0, \tilde{\sigma}) = -c$ and $\Pi(1/2, \tilde{\sigma}) \geq 0$. Since for $p_t \in [0, 1/2]$, $\Pi(p_t, \tilde{\sigma})$ is continuous and strictly increasing function of $p_t$, by intermediate-value theorem, there exists $\underline{p}(\tilde{\sigma}) \in [0, 1/2]$ such that $\Pi(p_t, \tilde{\sigma}) < 0$ for all $p_t \in \left[0, \underline{p}(\tilde{\sigma})\right)$ and $\Pi(p_t, \tilde{\sigma}) \geq 0$ for all $p_t \in \left[\underline{p}(\tilde{\sigma}), 1/2\right]$. Similarly, since for $p_t \in [1/2, 1]$, $\Pi(p_t, \tilde{\sigma})$ is continuous and strictly decreasing function of $p_t$, by intermediate-value theorem, there exists $\bar{p}(\tilde{\sigma}) \in [1/2, 1]$ such that $\Pi(p_t, \tilde{\sigma}) < 0$ for all $p_t \in (\bar{p}(\tilde{\sigma}), 1]$ and $\Pi(p_t, \tilde{\sigma}) \geq 0$ for all $p_t \in [1/2, \bar{p}(\tilde{\sigma})]$.

**Proof of Lemma 3:** The fact that $\Pi(p_t, \tilde{\sigma})$ is continuous in $\tilde{\sigma}$ follows directly from the expression (5).

$$\frac{\partial}{\partial\tilde{\sigma}}\Pi\left(p_t,\tilde{\sigma}\right) = -\left(qp_t+(1-q)(1-p_t)\right)\frac{\partial A_t}{\partial\tilde{\sigma}}+\left((1-q)p_t+q(1-p_t)\right)\frac{\partial B_t}{\partial\tilde{\sigma}},$$

where, for any $p_t \in (0,1)$,

$$\begin{aligned}\frac{\partial A_t}{\partial\tilde{\sigma}} &= \frac{1}{2(1-\mu)} \\ &\times\left[\frac{2(1-\mu)\mu q p_t}{\sqrt{\left(\tilde{\sigma}\mu\left(qp_t+(1-q)(1-p_t)\right)+(1-\mu)(1+p_t)\right)^2-4(1-\mu)\left(\tilde{\sigma}\mu q p_t+(1-\mu)p_t\right)}}\right. \\ &+\mu\left(qp_t+(1-q)(1-p_t)\right)\times\Bigg(1- \\ &\left.\left.\frac{\tilde{\sigma}\mu\left(qp_t+(1-q)(1-p_t)\right)+(1-\mu)(1+p_t)}{\sqrt{\left(\tilde{\sigma}\mu\left(qp_t+(1-q)(1-p_t)\right)+(1-\mu)(1+p_t)\right)^2-4(1-\mu)\left(\tilde{\sigma}\mu q p_t+(1-\mu)p_t\right)}}\right)\right] \\ &> 0 \end{aligned} \tag{15}$$

and

$$\begin{aligned}\frac{\partial B_t}{\partial\tilde{\sigma}} &= \frac{1}{2(1-\mu)} \\ &\times\left[\frac{2\mu(1-\mu)(1-q)p_t}{\sqrt{\left(\tilde{\sigma}\mu\left((1-q)p_t+q(1-p_t)\right)-(1-\mu)p_t\right)^2+4\tilde{\sigma}\mu(1-\mu)(1-q)p_t}}\right. \\ &+\mu\left((1-q)p_t+q(1-p_t)\right) \\ &\left.\times\left(\frac{\tilde{\sigma}\mu\left((1-q)p_t+q(1-p_t)\right)-(1-\mu)p_t}{\sqrt{\left(\tilde{\sigma}\mu\left((1-q)p_t+q(1-p_t)\right)-(1-\mu)p_t\right)^2+4\tilde{\sigma}\mu(1-\mu)(1-q)p_t}}-1\right)\right] \\ &< 0 \end{aligned} \tag{16}$$

Since $\partial A_t/\partial\tilde{\sigma} > 0$ and $\partial B_t/\partial\tilde{\sigma} < 0$ for every $\tilde{\sigma}$ and every $p_t \in (0,1)$, $\partial\Pi\left(p_t,\tilde{\sigma}\right)/\partial\tilde{\sigma} < 0$ for every $p_t \in (0,1)$ as required.

**Proof of Proposition 1:** The specified trading strategy maximises the payoff of liquidity traders. Consider the specified trading strategy of value traders. For any $p_t \in [0,1]$ and any $\sigma_t^* \in [0,1]$,

$$\mathbb{E}[V \mid S=1, H_t] \geq \quad A_t^* \geq p_t \geq B_t^* \quad \geq \mathbb{E}[V \mid S=0, H_t],$$

where $A_t^*$ and $B_t^*$ are defined in (8) and (9) respectively and $\mathbb{E}[V \mid S=1, H_t]$ and $\mathbb{E}[V \mid S=0, H_t]$ are given in (1) and (2) respectively. Thus the expected payoff of an informed trader who has observed $S=1$ is weakly greater when he buys than when he sells:

$$\begin{aligned}
\mathbb{E}\pi\left(\text{buy}, A_t^*, B_t^*\right) &= \mathbb{E}[V \mid S=1, H_t] - A_t^* - c \geq -c \\
&\geq B_t^* - \mathbb{E}[V \mid S=1, H_t] - c = \mathbb{E}\pi\left(\text{sell}, A_t^*, B_t^*\right).
\end{aligned}$$

Hence it is optimal to buy after observing $S=1$. Similarly, the expected payoff of an informed trader who has observed $S=0$ is weakly greater when he sells than when he buys:

$$\begin{aligned}
\mathbb{E}\pi\left(\text{buy}, A_t^*, B_t^*\right) &= \mathbb{E}[V \mid S=0, H_t] - A_t^* - c \leq -c \\
&\leq B_t^* - \mathbb{E}[V \mid S=1, H_t] - c = \mathbb{E}\pi\left(\text{sell}, A_t^*, B_t^*\right).
\end{aligned}$$

Hence it is optimal to sell after observing $S=0$.

Given that a value trader entering the market in period $t$ acquires a signal with probability $\sigma_t^*$, $A_t^*$, which is defined in (8), is equal to $\mathbb{E}[V \mid a_t = \text{buy at } A_t^*, H_t]$ and $B_t^*$, which is defined in (9), is equal to $\mathbb{E}[V \mid a_t = \text{sell at } B_t^*, H_t]$. Hence the market maker breaks even on every trade.

The expected payoff from abstaining from signal acquisition is 0 for any $p_t \in [0,1]$. This is because $A_t^* \geq p_t \geq B_t^*$ and thus an uniformed value trader either holds, when $A_t^* > p_t > B_t^*$, or trades at a price equal to his assessment of the asset value, when $A_t^* = p_t = B_t^*$. In either case, an uniformed value trader earns an expected payoff of 0. Given $A_t^*$ and $B_t^*$ defined in (8) and (9) respectively, for any $p_t \in \left(\underline{p}(1), \bar{p}(1)\right)$, the expected payoff from signal acquisition is strictly positive by Corollary 1. Hence, in this range of $p_t$, it is optimal to acquire a signal with probability 1. For any $p_t \in \left[\underline{p}(0), \underline{p}(1)\right] \cup [\bar{p}(1), \bar{p}(0)]$, the expected payoff from signal acquisition is 0 by construction.[18] Hence a value trader is indifferent between acquiring and not acquiring the signal and acquiring a signal with probability $\xi_t$ constitutes an optimal behaviour. Finally, for any $p_t \in \left[0, \underline{p}(0)\right) \cup (\bar{p}(0), 1]$, the expected payoff from signal acquisition is negative by Lemma 1. Hence, in this range of $p_t$, it is optimal not to acquire a signal.

It follows that, after every history, liquidity and value traders do not have profitable deviations and the market maker breaks even and has the correct belief regarding the signal acquisition strategy of value traders. Hence the proposed prices and strategy for information and liquidity traders constitute an equilibrium.

**Proof of Proposition 2:** Suppose in period $t$ there exist two equilibrium ask prices: $A_t^*$ and $\hat{A}_t^*$ such that $A_t^* < \hat{A}_t^*$. Given the market maker's belief regarding the signal acquisition strategy of value traders, the ask price is uniquely determined by the lowest solution to $A = \mathbb{E}[V \mid a_t = \text{buy at } A, H_t]$. In equilibrium, the belief of the market maker is correct and equal to the probability with which a value trader acquires a signal in period $t$. Hence the difference between $A_t^*$ and $\hat{A}_t^*$ could stem only from non-uniqueness of the optimal signal acquisition strategy of a value trader entering the market in period $t$.

Let $\hat{\sigma}_t$ and $\sigma_t^*$ be the probability with which a value trader entering the market in period $t$ acquires a signal in an equilibrium with an ask price of $\hat{A}_t^*$ and in an equilibrium with an ask price of $A_t^*$ respectively. $A_t^* < \hat{A}_t^*$ implies that $\hat{\sigma}_t > \sigma_t^* \geq 0$ by 15. Suppose that $\Pi(1/2, 1) \geq 1$. When $p_t \in \left(\underline{p}(1), \bar{p}(1)\right)$, by Lemma 1, the expected payoff from signal acquisition is strictly positive irrespective of the belief of the market maker. Hence in this range of $p_t$, it is optimal for a value trader to acquire a signal with probability 1 and it is impossible to have $\hat{\sigma}_t$ and $\sigma_t^*$ such that $\hat{\sigma}_t \neq \sigma_t^*$ and both of which are consistent with equilibrium behaviour. Similarly, when $p_t \in \left[0, \underline{p}(0)\right) \cup (\bar{p}(0), 1]$, by Corollary 1 the expected payoff from signal acquisition is strictly negative irrespective of the belief of the market maker. Hence, in this range of $p_t$, it is optimal for a value trader not to acquire a signal and both $\hat{\sigma}_t$ and $\sigma_t^*$, where $\hat{\sigma}_t \neq \sigma_t^*$, cannot be consistent with equilibrium behaviour. Finally, suppose that $p_t \in \left[\underline{p}(0), \underline{p}(1)\right] \cup [\bar{p}(1), \bar{p}(0)]$. If $\Pi(p_t, \sigma_t^*) = 0$ then, by

[18]If at $p_t = 1/2$, it is profitable to acquire signal and $\Pi(p_t, 1) \geq 0$, by Corollary 2 $\underline{p}(0) < \underline{p}(1)$, $\bar{p}(1) < \bar{p}(0)$ and intervals $\left[\underline{p}(0), \underline{p}(1)\right]$ and $[\bar{p}(1), \bar{p}(0)]$ are non-empty. If $\Pi(p_t, 1) < 0$, interval $\left[\underline{p}(0), \bar{p}(0)\right]$ is non-empty by Corollary 1.

Lemma 3, $\Pi(p_t, \hat{\sigma}_t) < 0$. Thus at ask price $\hat{A}_t^*$, value traders find it optimal not to acquire a signal and $\hat{\sigma}_t = 0$, which contradicts the assumption that $\hat{\sigma}_t > \sigma_t^* \geq 0$. If $\Pi(p_t, \hat{\sigma}_t) = 0$ then, by Lemma 3, $\Pi(p_t, \sigma_t^*) > 0$. Thus at ask price $\hat{A}_t^*$, value traders acquire a signal with probability 1 and $\sigma_t^* = 1$, which contradicts the assumption that $\hat{\sigma}_t > \sigma_t^*$. Similar arguments show that ask price is unique also if $\Pi(1/2, 1) \leq 0 < \Pi(1/2, 0)$.

The uniqueness of the bid price can be proved in a similar manner.

**Proof of Proposition 3:** Note that (1) $c$ shifts $\Pi(p, \sigma)$ up and down in $(p, \Pi)$ space; (2) for any $\sigma$, viewed as a function of $p$, $\Pi(p, \sigma)$ has finite slope for any $p$, and (3) $\Pi(p, \sigma)$ is strictly increasing for $p < 1/2$ and is strictly decreasing for $p > 1/2$. Hence, it follows that

$$\frac{\partial}{\partial c}\bar{p}(\sigma) < 0 \text{ and } \frac{\partial}{\partial c}\underline{p}(\sigma) > 0.$$

Moreover,

$$\lim_{c \to 0} \Pi(0, \sigma) = \lim_{c \to 0} \Pi(1, \sigma) = 0.$$

Since for any given $\sigma$, $\Pi(0, \sigma) = \Pi(1, \sigma)$ constitutes the minimum of $\Pi(p, \sigma)$ as a function of $p$, it follows that for any $p$ and any $\sigma$, $\lim_{c \to 0} \Pi(p, \sigma) \geq 0$. Consequently,

$$\lim_{c \to 0} \bar{p}(\sigma) = 1 \text{ and } \lim_{c \to 0} \underline{p}(\sigma) = 0.$$

**Proof of Proposition 4:** Fix $p_t$ and consider $c > c'$, so that any $\sigma$, $\Pi(p_t, \sigma; c)$ lies below $\Pi(p_t, \sigma; c')$ in $(p, \Pi)$ space. Let $\sigma = \sigma_t^*(p_t; c')$. There are three possibilities to consider:

1. $\Pi(p_t, \sigma; c') > 0$ and $\Pi(p_t, \sigma; c) > 0$. In this case, $\sigma_t^*(p_t; c') = \sigma_t^*(p_t; c) = 1$.

2. $\Pi(p_t, \sigma; c') < 0$ and $\Pi(p_t, \sigma; c) < 0$. In this case, $\sigma_t^*(p_t; c') = \sigma_t^*(p_t; c) = 0$.

3. $\Pi(p_t, \sigma; c') \geq 0$ and $\Pi(p_t, \sigma; c) \leq 0$. In this case, $\sigma_t^*(p_t; c) \leq \sigma_t^*(p_t; c') \leq 1$.

**Proof of Proposition 5:** Let $c > 0$. Since

$$\begin{aligned} \mathbb{E}(p_{t+1} \mid H_t) &= \mathbb{E}(\mathbb{E}(V \mid H_{t+1}) \mid H_t) \\ &= \mathbb{E}(V \mid H_t) = p_t, \end{aligned}$$

the belief process $\{p_t\}_{t=1}^{\infty}$ is a non-negative martingale with respect to public history $H_t$ and, by the Martingale Converge Theorem, as $t \to \infty$, it converges almost surely to a finite limit $p_\infty$. Moreover, since for every $t$, $p_t \leq 1$, $\{p_t\}_{t=1}^{\infty}$ is uniformly integrable and thus it also converges in $L^1$, i.e.,

$$\mathbb{E}\left|p_t - p_\infty\right| \to 0.$$

Note that by the Minkowski inequality,

$$\begin{aligned} \mathbb{E}\left|p_{t+1} - p_t\right| &= \mathbb{E}\left|p_{t+1} - p_\infty + p_\infty - p_t\right| \\ &\leq \mathbb{E}\left|p_{t+1} - p_\infty\right| + \mathbb{E}\left|p_\infty - p_t\right|. \end{aligned}$$

This combined with $L^1$ convergence implies that, as $t \to \infty$,

$$\mathbb{E}\left|p_{t+1} - p_t\right| \to 0,$$

where

$$\begin{aligned} \mathbb{E}\left|p_{t+1} - p_t\right| &= \mathbb{E}\left(\mathbb{E}\left(\left|p_{t+1} - p_t\right| \mid H_t\right)\right) \\ &= \mathbb{E}\left(\mathbb{P}\left(a_t = \text{buy} \mid H_t\right)\left|A_t^* - p_t\right| + \mathbb{P}\left(a_t = \text{sell} \mid H_t\right)\left|B_t^* - p_t\right|\right) \end{aligned}$$

It follows that, as $t \to \infty$, both $\left|A_t^* - p_t\right| \to 0$ and $\left|B_t^* - p_t\right| \to 0$. Hence, asymptotically the bid-and-ask spread disappears and $\{p_t\}_{t=1}^{\infty}$ converges almost surely to $p_\infty$ that takes values either close to, but weakly below $\underline{p}(0)$ or close to, but weakly above $\bar{p}(0)$.

The proof of Proposition 6 makes use of the following lemma.

**Lemma 4.** *Conditional on $V = 0$, the odds ratio $r_t$, defined as*

$$r_t := \frac{p_t}{1 - p_t},$$

*is a martingale.*

*Proof.* If $p_1 \in \left[0, \underline{p}(0)\right] \cup \left[\bar{p}(0), 1\right]$, $p_t = p_1$ in every period $t$, the proposition trivially follows. Hence assume that $p_1 \in \left(\underline{p}(0), \bar{p}(0)\right)$.

Let $\hat{\mathbb{P}}$ be the probability measure on the space of outcomes $\{0,1\} \times (\{\text{buy, sell, hold}\} \times [0,1])^{\infty}$ induced by the prior belief $p_1$, random order of trader arrival and the equilibrium strategy profile, conditional on $V = 0$.

Under the probability measure $\hat{\mathbb{P}}$, $\{r_t\}_{t=1}^{\infty}$ is a martingale because

$$
\begin{aligned}
\hat{\mathbb{E}}\left(\frac{p_{t+1}}{1-p_{t+1}} \mid H_t\right) &= \hat{\mathbb{P}}\left(a_t = \text{buy} \mid H_t\right)\frac{A_t^*}{1-A_t^*} \\
&\quad +\hat{\mathbb{P}}\left(a_t = \text{hold} \mid H_t\right)\frac{p_t}{1-p_t} \\
&\quad +\hat{\mathbb{P}}\left(a_t = \text{sell} \mid H_t\right)\frac{B_t^*}{1-B_t^*} \\
&= \left(\sigma_t^*\mu\left(1-q\right)+\left(1-A_t^*\right)\left(1-\mu\right)\right)\frac{\sigma_t^* q\mu+\left(1-A_t^*\right)\left(1-\mu\right)}{\sigma_t^*\mu\left(1-q\right)+\left(1-A_t^*\right)\left(1-\mu\right)}\frac{p_t}{1-p_t} \\
&\quad +\left(\left(1-\sigma_t^*\right)\mu+\left(A_t^*-B_t^*\right)\left(1-\mu\right)\right)\frac{p_t}{1-p_t} \\
&\quad +\left(\sigma_t^*\mu q+B_t^*\left(1-\mu\right)\right)\frac{\sigma_t^*\left(1-q\right)\mu+B_t^*\left(1-\mu\right)}{\sigma_t^*\mu q+B_t^*\left(1-\mu\right)}\frac{p_t}{1-p_t} \\
&= \frac{p_t}{1-p_t}.
\end{aligned}
$$

□

**Proof of Proposition 6:** Let $T$ be the time when the belief process reaches either $\bar{p}\left(0\right)$ or $\underline{p}\left(0\right)$, i.e.,

$$T = \min\left\{T_{\bar{p}}, T_{\underline{p}}\right\},$$

where

$$T_{\bar{p}} = \inf\left\{t \geq 1:\ p_t \geq \bar{p}\left(0\right)\right\}$$

and

$$T_{\underline{p}} = \inf\left\{t \geq 1:\ p_t \leq \underline{p}\left(0\right)\right\}.$$

Let $\hat{\mathbb{P}}$ be the probability measure on the space of outcomes $\{0,1\} \times (\{\text{buy, sell, hold}\} \times [0,1])^{\infty}$ induced by the prior belief $p_1$, random order of trader arrival and the equilibrium strategy profile, conditional on $V = 0$. Denote by $\hat{\mathbb{E}}(\cdot)$ expectations taken with respect to this measure. With this notation,

$$\mathbb{P}\left(p_{\infty} \geq \bar{p}(0) \mid V = 0\right) = \hat{\mathbb{P}}\left(T = T_{\bar{p}}\right).$$

Since on the equilibrium path, for every $t$, $r_t$ is bounded above by $\bar{p}(0) / (1 - \bar{p}(0)) < \infty$, by Lemma 4 $\{r_t\}_{t=1}^{\infty}$ is a uniformly integrable martingale. It follows that the Optional Sampling Theorem applies and the odds ratio process stopped at $T$ is also a martingale and $\hat{\mathbb{E}}(r_T \mid H_t) = r_t$ for every $t \leq T$. Thus $\hat{\mathbb{E}}(r_T) = r_1$. Since at $T$, $r_T \simeq \bar{p}(0) / (1 - \bar{p}(0))$ or $r_T \simeq \underline{p}(0) / \left(1 - \underline{p}(0)\right)$,

$$\begin{aligned}
\hat{\mathbb{E}}(r_T) &\simeq \left(1 - \hat{\mathbb{P}}\left(T = T_{\bar{p}}\right)\right) \frac{\underline{p}(0)}{1 - \underline{p}(0)} + \hat{\mathbb{P}}\left(T = T_{\bar{p}}\right) \frac{\bar{p}(0)}{1 - \bar{p}(0)} \\
&= \frac{p_1}{1 - p_1}.
\end{aligned}$$

Rearranging the expression above, the desired expression is obtained:

$$\hat{\mathbb{P}}\left(T = T_{\bar{p}}\right) \simeq \frac{\left(p_1 - \underline{p}(0)\right)(1 - \bar{p}(0))}{\left(\bar{p}(0) - \underline{p}(0)\right)(1 - p_1)}.$$

Taking the derivative:

$$\frac{\partial}{\partial c}\hat{\mathbb{P}}\left(T = T_{\bar{p}}\right) = -\frac{(\bar{p}(0) - p_1)(1 - \bar{p}(0)) \frac{\partial \underline{p}(0)}{\partial c} + \left(p_1 - \underline{p}(0)\right)\left(1 - \underline{p}(0)\right) \frac{\partial \bar{p}(0)}{\partial c}}{(1 - p_1)\left(\bar{p}(0) - \underline{p}(0)\right)^2}.$$

Note that the expressions for $\underline{p}(0)$ and $\bar{p}(0)$ are obtained in (10) and (11), respectively, by solving $\Pi(p, 0) = 0$ for $p$. Thus:

$$\bar{p}(0) = 1 - \underline{p}(0)$$

and

$$\frac{\partial \underline{p}(0)}{\partial c} = -\frac{\partial \bar{p}(0)}{\partial c} = \frac{2c}{2q-1}\left(1-\frac{2c}{2q-1}\right)^{-1/2} > 0.$$

The expression of the derivative of $\hat{\mathbb{P}}\left(T = T_{\bar{p}}\right)$ with respect to $c$ simplifies to:

$$\frac{\partial}{\partial c}\hat{\mathbb{P}}\left(T = T_{\bar{p}}\right) = \frac{\left(p_1 - \underline{p}(0)\right)^2}{(1-p_1)\left(1-2\underline{p}(0)\right)^2}\frac{\partial \underline{p}(0)}{\partial c} > 0.$$

**Proof of Proposition 7:** Taking the derivative w.r.t. $\sigma$ of the social surplus defined in (13):

$$\frac{\partial S}{\partial \sigma} = (1-\mu)\left((p_t - B_t^*)\frac{\partial B_t^*}{\partial \sigma} - (A_t^* - p_t)\frac{\partial A_t^*}{\partial \sigma}\right) - \mu c < 0,$$

where the inequality follows as $p_t - B_t^* \geq 0$, $\partial B_t^*/\partial\sigma < 0$ from (16), $A_t^* - p_t \geq 0$ and $\partial A_t^*/\partial\sigma > 0$ from (15). Thus, the per-period social surplus is decreasing in information acquisition efforts and it is optimal to set $\sigma = 0$ in every $t$.

**Proof of Proposition 8:** Consider two alternative strategies a social planner could pursue:

1. The social planner sets $\sigma_t = \sigma > 0$ and $\sigma_\tau = 0$ in every period $\tau = t+1, t+2, \ldots$ .
2. The social planner sets $\sigma_\tau = 0$ in the current period $t$ and in every period thereafter.

The difference in payoffs from the two strategies is:

$$(1-\delta)\left[S(p_t,\sigma) - S(p_t,0)\right] + \delta\left[\mathbb{E}S(p_{t+1},0) - S(p_t,0)\right].$$

The first term in square brackets is finite and independent of $\delta$. The term in the second square brackets is positive:

$$\mathbb{E}S(p_{t+1},0) \geq S(\mathbb{E}p_{t+1},0) = S(p_t,0),$$

where the inequality follows by Jensen's inequality as $S(p,0)$ is convex in $p$ and the equality follows as the belief process $\{p_\tau\}_{\tau=t}^{\infty}$ is a martingale. Consequently for large enough $\delta$, the first strategy yields higher payoff than the second strategy.

**Proof of Proposition 9:** Suppose the current public belief regarding the asset value is $p$ and consider two alternative strategies for the social planner:

1. The social planner sets $\sigma > 0$ in the current period and $\sigma = 0$ in every subsequent period.

2. The social planner ceases to acquire information immediately.

Since for any $p$, $S(p,0) \leq (1-\mu)/2$, the continuation payoff from the first policy is bounded above by:

$$\delta(1-\mu)\frac{1}{2}.$$

The payoff from the second strategy is

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t S(p,0) = (1-\delta)S(p,0) + \delta(1-\mu)\left(\frac{1}{2} - p(1-p)\right)$$

by (14). It follows that the difference in payoffs between the two policies is bounded above by:

$$(1-\delta)\left[S(p,\sigma) - S(p,0)\right] + \delta(1-\mu)p(1-p).$$

Since as $p \to 0$ or 1, the bid and the ask prices tend to $p$, it follows that:

$$\lim_{p\to\{0,1\}}(1-\delta)\left[S(p,\sigma) - S(p,0)\right] + \delta(1-\mu)p(1-p) = -(1-\delta)c\sigma\mu < 0.$$

Consequently only $\sigma = 0$ maximises social welfare when $p$ is sufficiently close to 0 or 1.

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