Bargaining in Global Communication Networks

Abstract

We study a Rubinstein-Stahl two-player non-cooperative bargaining game played by n players connected in a communication network. We allow the players to communicate with any peer in the same component *via* the existing paths connecting the peers in a given communication network (*global* interaction). The unique stationary subgame perfect equilibrium profile characterizes the players' expected payoff as function of their *betweenness* centrality score. Secondly, we study a dynamic link-formation game which allows the players to activate new linkages or sever existing ones in order to increase their bargaining power for a given marginal cost per link. We identify the conditions under which the resultant pairwise stable network structures belong to the family of the *nested split* graphs.

JEL classification: C72; C78; D85 *Keywords:* Communication; Network; Noncooperative bargaining; Network formation

1 Introduction

In almost any socio-economic situations, agents interact and take into account the choices of other peers. However, it is also clear that agents in a community communicate only with a subset of the whole population. This fact in particular helped the development of the literature on networks where a pairwise communication possibility is represented by a connection or link between two nodes/players. This is also the starting point of Calvó-Armengol (2001). The author studies a Rubinstein-Stahl two-player non-cooperative bargaining game where n players are connected through a graph but are constrained to communicate with their direct peers; the communication constraint characterizes the bargaining power of each node composing a network and therefore their *ex-ante* expected payoff: nodes with a higher number of connections with other peers (degree) benefit from higher bargaining power than the rest of the agents in the same component.

There are cases when it seems plausible to constrain the communication possibilities exclusively to direct connections. However, in many others agents seem to benefit from exploiting their direct links to reach third peers in a chain of interactions. For example, in a business environment the value of direct connections is often not limited to the pairwise interaction within the connected peers, but also extends to the indirect connection possibilities that the same peers could reciprocally open. We first study how, in a bargaining context, the location of each agent across a communication structure could impact their relative bargaining power. In our model the agents can interact not only at a local level (with direct peers) but also with distant agents, *via* existing communication paths connecting them. However, some players could belong to relatively more paths than others and therefore, they will enjoy a higher than average chance of get interaction with other peers. We characterise the *brokerage* power of some agents observed in many social and economic environments which does not necessarily coincide with the number of their own connections (degree). The bargaining process is sequential and involves only two partners at a time. Each bargaining pair is chosen at random and the stochastic process is function of the network architecture. Ex-post payoffs are not affected by the network structure when we assume players to be homogeneous in time preferences, while ex-ante expected payoffs do.[1] Nevertheless, under our model setting we are able to derive an allocation rule which rewards agents with relatively high weighted *betweenness* centrality score and thus not necessarily those that are more connected.

Furthermore, we investigate which communication network is expected to arise as a stable structure when the players can strategically rewire their connections. In particular, when choosing to activate or sever a (costly) link, each agent will face the following trade-off: to activate a new connection is profitable if and only if the improvement of the same player's *brokerage* power is greater than its marginal cost. Interestingly, we can show that under certain conditions, the architecture of the pairwise stable network structure is *nested*: the neighbourhood of each node is contained in the neighbourhoods of nodes with higher degrees. These strongly hierarchical architectures are observed in many various real-world networks.[2]

The paper contributes to the literatures on networks and game theory in two ways. We first characterise the unique stationary equilibrium of the n-player bargaining game where players are connected in a communication network and could reach peers distant by more than one link. This result could help to explain the brokerage power of agents playing crucial intermediary roles in various existing communication networks. Finally, we enrich the analysis studying the network architectures which could endogenously arise under certain conditions as pairwise stable networks.

The paper is organised as follows. Section 2 reviews the related literature. In section 3 we introduce the model setup and characterise the subgame perfect equilibrium profile. In section 4 we study the link-formation game and define the pairwise stable networks arising for different constant marginal cost per link. Section 5 concludes.

[1] This is consistent with the results in Calvó-Armengol (2001). As pointed out by the same author, the "outside option" in the pairwise bargain is payoff equivalent to playing a Rubinstein-Stahl two-player non-cooperative bargaining game with an alternative partner, and therefore it is a non-credible threat.

[2] See for instance Uzzi (1996) who describes the network of all "better dress" firms in the New York apparel economy. Soramäki et al. (2007) finds that the Fedwire bank network structure has a particularly dissortative architecture where few hubs are largely connected while most banks have relatively few connections. Akerman and Seim (2014) study the dynamic of the global arm-trade network over the period 1950-2007 and find dissortativity over the sample indicating that few large countries with many connections trade with many small countries with few connections. De Benedictis and Tajoli (2011) analyse the world trade network over the period 1950-2000 and highlight a core-periphery architecture with strong dissortativity.

2 Related literature

Among many papers analyzing bargaining processes between agents, we mention the original contributions of Rubinstein (1982) and Morgenstern (1973). As previously pointed, our model uses a bargain protocol based on Calvó-Armengol (2001) which adapts the Rubinstein (1982) setup to a network setting. Similarly, in our model the final Nash equilibrium is defined by the players' bargaining power as function of their location in the network. However, while Calvó-Armengol (2001) constrains the communication possibilities to "local" interactions (direct neighbours), we allow for "global" interactions but constrained to the existence and length of the paths connecting the peers.[3] Calvo et al. (1999) generalize the communication between agents allowing for communication chains and thus not strictly direct connections. However, their probabilistic setting differs substantially from ours. Among many differences, our model characterizes the probability of communicating with a player not directly connected by a link as function of the network architecture, while in their setting this probability is exogenous and discretionary.

Our network formation problem is analyzed using the definition of pair-wise stability introduced by Jackson and Wolinsky (1996). In König et al. (2009) the authors study the networks dynamically emerging when the players aim to maximize their relative centrality scores (calculated using any degree, betweenness, closeness, and Bonacich measure). As in their model, our setting implies that a potential connection with a player with a relatively high centrality score would increase the individual centrality more than a connection with a player with a lower score. However, in our case, the number of connections owned by each player is endogenous and their activation relies on a marginal benefit and marginal cost analysis.

In Sociology, communication networks and their relative implications have been widely studied (see for instance Wasserman (1994) and Skvoretz and Willer (1993) for surveys). Our model's predictions are in line with various experimental results: the most crucial players are those that

[3] We would like to mention other papers modeling the bargain in a network setting. See for instance Borm et al. (1994) for an exhaustive review.

can control the communication flow within a component.[4] As illustrated by Padgett and Ansell (1993), there are cases in which the relative *power* of an agent is not fully explained by the number of connections he owns. This could happen when there exists a player that uniquely connects different subset of agents, or in other words when the existence of paths between most nodes is conditional to the presence of one or a few critical nodes.

It is worth remarking that our interpretation of player's criticality relies on our definition of the linkage connecting a pair of agents. In particular, under our model setting, the link-structure defines the communication paths connecting a set of agents. For any agent i it is possible to reach any other agent j by passing through a chain of links which connect them. Therefore, in contrast to many papers analyzing the spread of information, disease, or any general divisible good across a network structure, the players can communicate and trade a good with any other player of the component, as if they were directly "walking" through paths.[5]

3 Setup

3.1 Communication structure

Consider a finite set of agents $N = \{1, ..., n\}$ connected in a network structure $G(N, L)$ with L the set of undirected linkages or connections between them. Each element $\{ij\} \in L$ is a binary variable; $\{ij\} = 1$ indicates a connection between the pair (i, j), while $\{ij\} = 0$ otherwise. We denote with $g \subseteq G$ a component, or a nonempty connected subnetwork $g(N', L')$ such that $\emptyset \neq N' \subseteq N$ and $L' \subseteq L$. A *path* P_{ij} between i and j belonging to g is a sequence of linkages which connect the two nodes and such that each node involved is distinct. A *geodesic* between two nodes is the shortest path between them. Let σ_{ij} be the number of geodesic paths in g which connect i and j, and $\sigma_{ij}(k)$

[4] Some of the sociological contributions define the criticality of a player as his *potential for exclusion*, or the power of some location to preclude exchanges with other agents, see for instance Pitts (1965) and Shimbel (1953).

[5] We would like to mention some important contribution from the literature on informal communication and information spread through networks. See for instance Schrader (1991), Cowan and Jonard (2004), and Allen and Cohen (1969).

be the number of such geodesics passing through node $k \neq i, j$. Let $b_{ij}(k) = \sigma_{ij}(k)/\sigma_{ij} \in [0, 1]$ be the "partial" betweenness score of the node k given the pair (i, j), or the proportion of geodesic paths connecting the pair (i, j) which pass through node k. Therefore, the general betweenness centrality score[6] of a node k is defined as $b_k = \sum_{(i,j)\in N^2} \sigma_{ij}(k)/\sigma_{ij}$. Denote $\tilde{b}_{i,k}(j) = b_{i,k}(j)/d(i, k)$ the partial "weighted" betweenness score of the node j which takes into account the distance $d(i, k)$. Thus, the general weighted betweenness centrality is measured as

$$\tilde{b}_j = \sum_{(i,k)\in N^2} \frac{b_{ik}(j)}{d(i,k)}$$

This measure shares same properties and boundaries of the general betweenness centrality but it also weights each geodesic path by its length.[7] Finally, define with $\mathbf{N}_i$ the open neighborhood of i, or the subset of nodes $j : j \in \mathbf{N}_i \to \{ij\} = 1$. The link-structure defines the communication channels between the players. In particular, we assume that an agent $i \in g$ can communicate with any other player belonging to the set $N_i \subseteq N$; N_i defines the subset of players "reachable" by i, not necessarily coinciding with N the whole set of players. Therefore, in the case of $\mathbf{N}_i = N_i \subset N$, we assume that i is able to communicate only with his direct neighbours and which are strictly less than $n-1$ players, while $N_i = N$ means that i can interact with any other player of the component g using the geodesic paths connecting them.

3.2 Payoff and bargaining protocol

Each player in N can bargain shares of a "unit pie" $\hat{x} = 1$ with another peer with whom he is directly or indirectly connected. In particular, the communication protocol follows a specific stochastic process rewarding the agents located in a higher than average number of geodesic paths connecting two players of the same component. The structural "criticality" of these players will

[6]See Freeman (1979) for details.

[7]It is clear that the measure scores its lowest value, $\tilde{b}_j = 0$, for a node in a complete network or for a peripheral node, while its highest value, $\tilde{b}_j = 1$, for the central node in a star graph.

allow them to pair-wisely bargain bigger shares of $\hat{x}$. We summarize the procedure as follows:

At each round we consider two active players in g, a proposer i and a respondent j. The player i is picked at random out of n players (with probability $1/n$). A respondent j then is picked by i with probability $\beta_{ji} \in (0,1]$ as defined below. Then i makes a proposal of share of $\hat{x}$ to player j. If j accepts the offer, the game ends with these payoffs. If j rejects the proposal, he becomes the new proposer. The game proceeds until, and if, a bilateral agreement is reached.

Player j's probability of being picked as receiver by the proposer i is defined as follows,

$$\beta_{ji} = Pr[\sigma_{ij}]\frac{1}{d(i,j)} + \sum_{k \neq i \neq j} Pr[\sigma_{ik}]\frac{b_{ik}(j)}{d(i,k)} \in (0,1]$$

where $Pr[\sigma]$ is the probability of picking a specific geodesic path such as σ_{ik}, $b_{ik}(j)$ is the betweenness score of j related to σ_{ik}, and $1/d(i,k)$ is the uniform probability of being picked out of the members of the path of size $|\sigma_{ik}| = d(i,k)$. The first element on the right side describes the probability of j being picked out of the nodes composing the path P_{ij}, while the second element is the probability of being picked out of the nodes composing a path P_{ik} with $k \neq j$ and which contains j itself.[8]

To clarify the intuition we make use of an example similar to the one studied by Olken and Barron (2007). Suppose that initially each player is assigned to a distinct node which indicates a location.[9] A link between two locations indicates the presence of a road between them. At each round a randomly chosen player travels from her location to another randomly chosen destination in N. Along the path connecting the two locations, there is a chance that one of the intermediary nodes stops her and bargains a share of the unit pie. Players in locations which are relatively more "intermediary" have more opportunity to bargain than players in peripheral locations. An

[8]The β_{ji} score is strictly positive since we assume connected graphs and therefore there is always positive probability for each node j to be reached by the proposer i. The highest boundary, $\beta_{ji} = 1$ is obtained only when j is the unique node connected to the proposer i (a dyad graph), thus $Pr[\sigma_{ij}] = 1$ and $1/d(i,j) = 1$.

[9]Clearly we assume that there as many players as locations.

alternative example may interpret linkages and paths as follows. Consider two researchers of two distinct departments, say i and j, not directly connected to each other. Suppose i has an idea about a new project which he would like to start in collaboration with another scholar. The chance that the two communicate and start a new project may depend on various factors among which is the presence of third peers, say co-authors or colleagues, which introduce them. However, the longer the chain of individuals between i and j, the higher the chance that i might for example discuss the project with an intermediary scholar and not with j.[10]

To simplify the analysis without loss of generality, we assume only one geodesic path between two players and uniform probability among the possible $(n-1)$ paths from i to k, thus reducing the probability to[11]

$$\beta_{ji} = \frac{1}{n-1}\left(\frac{1}{d(i,j)} + \sum_{k\neq i\neq j} \frac{b_{ik}(j)}{d(i,k)}\right)$$

Finally, for each j we compute the measure $\beta_j = \sum\limits_{i\neq j} \beta_{ji}$, which is bounded above by $n/2$ and below by $\frac{1}{n-1}H_{n-1}$, where H_{n-1} is the $(n-1)$-th harmonic number. In particular, $n/2$ is scored by the central node of a star graph of n nodes, and $\frac{1}{n-1}H_{n-1}$ by a peripheral node of a line graph of n nodes.[12]. It is clear that

$$\beta_j = \frac{1}{n-1}\left(\sum_{i\neq j} \frac{1}{d(i,j)} + \tilde{b}_j\right)$$

or in other words, the β_j score is function of the weighted betweenness centrality score of node j.

Note that when $N_i = \mathbf{N}_i$ we replicate the setting of Calvó-Armengol (2001) since $\sum_{k\neq j} b_{ik}(j) = 0 \ \forall j \neq k \in g$, and $Pr[\sigma_{ij}|\mathbf{N_i}]\frac{1}{d(i,j)} = 1/|\mathbf{N}_i|$. On the other hand, increasing N_i such that $\mathbf{N}_i \subset N_i$, we flexibly extend communication between nodes to agents *indirectly* connected to each other.[13] Hereafter, to simplify the notation, we will assume $N_i = N$. Finally, we remark that for each

[10] Here we assume that each agent in the path is equally valuable to i as project-partner.

[11] Note that this is the same to assume that we are selecting one of the geodesic paths between two nodes (if more than one) with uniform probability. This would imply that $b_{ik}(j)$ is a binary variable, assuming value 1 when j is part of σ_{ik} or 0 otherwise.

[12] In the Appendix, we propose an illustrative example of the β_{ji} probabilities in a given connected network.

[13] We remark that also in specific case of a complete network structure, the two settings coincide.

bargaining pair (i, j), the game coincides with a Rubinstein (1982) game, with player i as the initial proposer.[14]

For simplicity and without loss of generality we assume hereafter that $\hat{x} = 1$. For any bargain-agreement and given an offerer i and a receiver j, we indicate with $x_{ij} \in [0, 1]$ the proposed share disclosed by j to i, and symmetrically $1 - x_{ij}$ the share disclosed by i to j. Consider the case $N_i = N \; \forall i \in g$. We can define the unique stationary perfect equilibrium of the bargaining game.[15]

Proposition 1. *For any g, the n-player bargaining game has a unique stationary perfect equilibrium. For all distinct players $i, j, k \in N$, the equilibrium shares are characterized by:*

$$1 - x_{ij}(g) = \delta_j \sum_{k \neq j} \beta_{kj} x_{jk}(g) \tag{1}$$

Alternatively put, the equilibrium share offered by i to a j player of the same component is equal to the expected flow obtained by j as proposer player from the rest of the receiver peers $k \in g$.

Proposition 2. *For any $g \in \mathcal{G}^*$, the allocation rule $Y : \mathcal{G}^* \to R^n_+$ corresponding to the n-player bargaining game in g is given by:*

$$Y_i(g) = \frac{1}{n} \sum_{j \in g} [\beta_{ji} x_{ij}(g) + \beta_{ij}(1 - x_{ji}(g))] \; \forall i \in N \tag{2}$$

and it is efficient.

The results above define the individual equilibrium unit shares expected by each player i in the network. It is clear from the definition of β_{ji} and β_{ij} that each equilibrium allocation $Y_i(g)$ of a player $i \in N$ is function of his relative betweenness centrality. In words, a relatively higher centrality score implies ex-ante larger shares received from the rest of the peers. Following the

[14]The bargain protocol proposed should not be confused with a n-player multilateral bargaining.
[15]All the proofs are in the Appendix section.

Corollary 1-2 in Calvó-Armengol (2001), assuming homogeneous time preferences, we can rewrite the (2) as

$$Y_i^h(g) = \frac{1}{1+\delta}\frac{1}{n}(1+\delta\beta_i)\ \forall i \in N$$

where $(1/1+\delta; \delta/1+\delta)$ is the standard agreement as shown by Rubinstein (1982) in the case of homogeneous time preferences. Finally, to highlight the impact of the structure on the final expected bargaining allocation, we take the limit of $Y_i^h(g)$ for $\delta \to 1$,

$$\lim_{\delta \to 1} Y_i^h(g) = \frac{1}{2n}(1+\beta_i), \forall i \in g \tag{3}$$

The (ex-ante) expected payoff of a player $i \in g$ is implicitly function of β_i and of the size of the component, n. The player that has the highest chance to be a receiver/proposer in the bargain process will also be the agent with highest expected payoff. These results are complementary to the ones in Calvó-Armengol (2001) since we extended the communication possibilities to any geodesic path instead of direct connections only; in other words our setting can replicate their result assuming $N_i = \mathbf{N}_i$ for all $i \in g$. A clear consequence of this is that, given a component g of order $n > 2$ and homogeneous discount factor δ,

$$Y_i^* \geq Y_j^* \iff \beta_i \geq \beta_j \qquad \forall (i,j) \in g$$

In summary, assuming a bargain protocol constrained by the communication structure interconnecting the agents, we characterize each individual optimal share as function of the player location in the network. In particular, when we allow each player in the network to reach other players distant by more than one link, agents with highest expected payoff are not necessarily the ones which are more connected (highest degree) but those with the highest weighted betweenness centrality score.[16]

[16] In the Appendix we discuss the well studied case of Florentine marriage network proposed by Padgett and Ansell (1993).

4 Stable network

Consider a third stage of the game. In particular, at this stage each player can simultaneously form new links or sever existing ones. Assume a positive marginal cost per link $c \in \mathbb{R}^{++}$, paid by the player proposing to activate a link. Any connection between two players will be activated or severed unilaterally by one of the two players involved. The intuition is that, net of activation cost, a new link is always marginally beneficial for the pair involved in terms of increasing the probability of bargaining; thus, a player would always accept a new link from another peer. Formally, each player simultaneously chooses a vector $\boldsymbol{g}_i = (g_{i1}, ..., g_{ii-1}, g_{ii+1}, ..., g_{in})$ where $g_{ij} = \{0, 1\}$. The payoff of each player i is defined by the expected utility $U_i(g) = Y_i^h(g) - c\varphi_i$, with φ_i the number of connections activated by the player i. We have $\boldsymbol{g}_i \in S_i = \{0, 1\}^{n-1}$ with S_i the set of strategies of player i and $S = \times_{i \in N} S_i$ the set of strategies of all players. Define as a *link-active* player, a node $i \in g$ such that $\exists j \neq i : \{ij\}$. We use the notion of network *stability* proposed by Jackson and Wolinsky (1996): a network is *pairwise stable* if it is such that no player has incentive to cut an existing link and no pair wants to form a new link. More formally,

Definition 1. *A network $G(N, L)$ is pairwise stable if*

1. $\forall i, j \in N \Rightarrow U_i(G \oplus \{ij\}) \geq U_i(G)$ *and* $U_j(G \oplus \{ij\}) \geq U_j(G)$

2. $\forall i, j \notin N \Rightarrow$ *if* $U_i(G) < U_i(G \oplus \{ij\}) \Rightarrow U_j(G) > U_j(G \oplus \{ij\})$

It is intuitive that players located in distinct (non-isomorphic) locations in a network G will probably face different marginal incentives to activate and/or severe linkages. The previous stages of the game characterize the equilibrium private contribution for each player as function of his criticality in G. Therefore, in the third stage, each player could modify her link structure to improve her centrality, given the existing network architecture G.

Two remarks are worth mentioning.[17] Firstly, it is crucial that the centrality of the players are only weakly monotonic with respect to their degrees; a link activation cannot decrease the betweenness centrality score of the players involved, but can potentially be neutral. Secondly, a connection with a player with relatively high betweenness centrality increases the centrality score of the node activating the link more than a connection with a player with low betweenness score; informally, a link with a relatively central player gives access to more geodesic paths than a connection with a less central agent. These two features will explain the existence and the specific architecture of some of the equilibrium network structures.

Denote with G^s a pairwise stable network. We can present the following result,

Proposition 3. *Consider $\underline{c} > 0$ and $\bar{c} > \underline{c}$. Then,*

- *for any $c < \underline{c}$ the **complete network** is the unique pair-wise stable,*
- *for $c \geq \bar{c}$ the **empty network** is the unique pair-wise stable,*
- *for $c \in (\underline{c}, \bar{c})$ and if all the players are link-active, any pairwise stable network is a **nested split** graph.*[18]

The result points out that in a complete network, where each player has minimal betweenness score, for any two players it is beneficial to sever the link connecting them if and only if the cost to keep active the connection is higher than the particularly small gain in probability derived from it; it is always beneficial to activate a link with another player, thus a complete network is always a possible equilibrium for small enough cost per link.[19] Moreover, the marginal benefit weakly increases when the new link is activated with a node with a higher betweenness centrality score. This implies that, if a new link ij is activated by i, any link iq with $q \neq j : \tilde{b}_q > \tilde{b}_j$ must be already active. Recursively and, for intermediate cost c, this leads to a nested split structure.

[17]These observations are based on König et al. (2009). In particular, we will make use of the result in Corollary 11 and Proposition 1 of the same paper.

[18]For a formal definition of nested split graph, we refer to the definitions in the Appendix.

[19]It is particularly easy to see this result for a zero marginal cost.

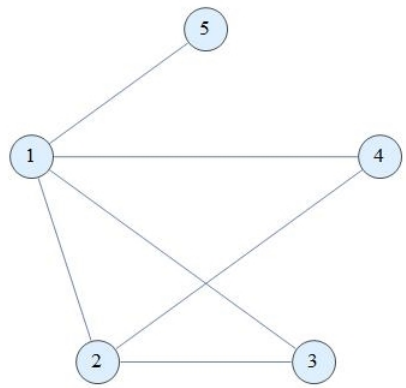

Figure 1: A nested split graph

5 Conclusions

This paper studies the impact of the structure of a given communication network on the expected payoff of players that pairwisely bargain a' la Rubinstein-Stahl. We analyse the network architecture which may arise as a pairwise stable structure when the players can strategically rewire their connections, given a fixed marginal cost per link. We build our setting over that proposed by Calvó-Armengol (2001), and in particular we extend his model in order to allow the players to communicate with peers that are distant from them by more than one connection. The (unique) subgame perfect equilibrium profile obtained for a given network structure rewards the agents with relatively higher betweenness centrality score. In words, agents which belong to more communication paths on average than other peers, expect to interact with other agents relatively more often, and therefore their ex-ante payoffs are expected to be higher. Finally, we find that for an intermediary cost per link the pairwise stable connected network belongs to the family of nested split graphs. These are network structures where the neighbourhood of each node is contained in the neighbourhoods of nodes with higher degrees.

Appendix

Proof of Proposition 1. The proof follows that of Proposition 1 in Calvó-Armengol (2001). For any player $i \in N$, define a stationary strategy σ_i consisting of a set of proposal $(x_{ij}, 1 - x_{ij})$ made by i to any other player $j \in N$, and a set of responses to the offers made to i by j such that i accepts if and only if $1 - y \geq 1 - y_{ji}$. Suppose that for some $t = 0, 1, ...$ player j rejects the player i's offer at round t. Then, at $t+1$, j offers a share $1 - y_{jk}$ to a selected player $k \neq j$ that accepts it. Thus, j gets an expected discounted payoff of $\delta_j \sum_{k \neq j} y_{jk}\beta_{kj}$. So, in order for player j's rejection at t of a share $1 - x < 1 - x_{ij}$ to be credible, we must have $1 - x_{ij} \leq \delta_j \sum_{k \neq j} y_{jk}\beta_{kj}$. At the same time, we must have $1 - x_{ij} \geq \delta_j \sum_{k \neq j} y_{jk}\beta_{kj}$. Therefore, $1 - x_{ij} = \delta_j \sum_{k \neq j} y_{jk}\beta_{kj}$ $\forall i, j, k \in N$. Player i knows that any other j player accepts a share $1 - y_{ij}$, in which case i gets y_{ij}. Then, necessarily $x_{ij} \geq y_{ij}$. Also, player j accepting a share $1 - x_{ij}$ requires that $1 - x_{ij} \geq 1 - y_{ij} \Leftrightarrow x_{ij} \leq y_{ij}$. Then $x_{ij} = y_{ij}$. Thus, $1 - x_{ij} = \delta_j \sum_{k \neq j} y_{jk}\beta_{kj}$ and $x_{ij} = y_{ij}$ $\forall i, j, k \in N$.

To prove the uniqueness, let $x_i(g) = \sum_{j \neq i} \beta_{ji} x_{ij}$ $\forall i, j \in N$. Then, (1) can be rewritten as $1 - x_{ij}(g) = \delta_j x_j(g)$. Adding up we obtain,

$$\begin{cases} x_i(g) + \sum_{j \neq i} \beta_{ji}\delta_j x_j(g) = 1 \\ x_{ij}(g) = 1 - \delta_j x_j(g) \end{cases}$$

Writing the first row of the system in matrix notation, $M \cdot A = \mathbf{1}$, with $A = [x_1(g), ..., x_n(g)]'$, $\mathbf{1} = [1, ..., 1]'$, and M is a $n \times n$ matrix given by $m_{ii} = 1$ and $m_{ij} = \beta_{ji}\delta_j$. It is easy to see that M is invertible and then the solution to the system is unique and given by $A = M^{-1}\mathbf{1}$. □

Proof of Proposition 2. A player i has the chance of bargaining a positive share of $\hat{x}$ either if picked as proposer (probability $1/n$), or as receiver or in the communication path to reach a receiver (probability β_{ij}). Thus, the payoff of each player i is the sum of the expected payoff for each type of player. An allocation rule Y is efficient if and only if $\sum_{i \in g} Y_i(g) = 1$.

$$\begin{aligned}\textstyle\sum_{i\in g} Y_i(g) &= \frac{1}{n}\left[\sum_{i\in g}\sum_{j\in g}\beta_{ji}x_{ij}(g)+\sum_{i\in g}\sum_{j\in g}\beta_{ij}(1-x_{ji})\right]\\ &\Leftrightarrow \textstyle\sum_{i\in g} Y_i(g) = \frac{1}{n}\left[\sum_{\{i:j\}\in g}\beta_{ji}x_{ij}(g)+\sum_{\{j:i\}\in g}\beta_{ij}(1-x_{ji})\right]\\ &\Leftrightarrow \textstyle\sum_{i\in g} Y_i(g) = \frac{1}{n}\sum_{\{i:j\}\in g}\beta_{ji} = \frac{1}{n}\sum_{i\in g}\sum_{j\in g}\beta_{ji} = 1\end{aligned}$$

since $\sum_{j\in g}\beta_{ji} = 1$. Therefore the allocation is efficient. □

Proof of Proposition 3. Consider a complete graph K_n of $n > 2$. We know that by construction $b_i = 0\ \forall i \in K_n$. Therefore, $\beta_i = 1$ and $Y_i(K_n)^h = \frac{1}{1+\delta}\frac{1}{n}(1+\delta) = \frac{1}{n}\ \forall i \in K_n$. This also implies that each player gets a payoff equal to $U_i(K_n) = \frac{1}{n} - c\varphi_i$. If a player severs an active link obtains a gain of c but a loss in terms of probability equal to $(1-\frac{5}{2}\frac{1}{n-1})$: the net impact in terms of utility is $\frac{1}{n} - \frac{1}{1+\delta}\frac{1}{n}(1+\delta\frac{5}{2}\frac{1}{n-1})$. Define $\underline{c} = \frac{1}{n} - \frac{1}{1+\delta}\frac{1}{n}(1+\delta\frac{5}{2}\frac{1}{n-1})$. It follows that for any $c \le \underline{c}$, it is not beneficial to any player to cut a link. Moreover, since the net impact $\frac{1}{n} - \frac{1}{1+\delta}\frac{1}{n}(1+\delta\frac{5}{2}\frac{1}{n-1})$ is the smallest possible, if $c \le \underline{c}$ and $G \ne K_n$, each player will have the incentive to activate a new link (if it is possible). Thus, for $c \le \underline{c}$, the complete network is the unique pairwise stable structure.

Consider the empty network G^e. By construction, $\tilde{b}_i = 0\ \forall i \in G^e$, and $U_i(G^e) = 0\ \ \forall i \in G^e$. Define $G' = G^e + \{ij\}$. Then, still $\tilde{b}_i = \tilde{b}_j = 0$, but $Y_i(G')^h = Y_j(G')^h = \frac{1}{2}$. Therefore, $U_i(G') = \frac{1}{2} - c$ for player i who activates the link. Define $\bar{c} = 1/2$. It follows that for $c > \bar{c}$ no player in G^e benefits from activating a link, i.e. G^e is pairwise stable. Moreover, since $1/2 - c$ is the highest marginal benefit which a player could obtain by activating a new link, in any graph different than G^e, the players with active link would benefit from severing their connections. Thus, for $c \ge \bar{c}$, the empty network is the unique pairwise stable structure.

Consider a general pairwise stable graph G^s of $n \ge 3$ players and intermediate cost. Rank the players according to their degree and label them such that $\{1, 2, ..., n\}$ is the ordered set of n players with $\varphi_1 \ge \varphi_2 \ge ... \ge \varphi_n$. We start showing that for any player i the following logic statement is true:

$$\exists\{ij\} \Longrightarrow \exists\{\bar{ij'}\}\ \ \forall j' < j \qquad (4)$$

Suppose it is not the case and $\exists\{ij\}$ but $\nexists\{i\bar{j}'\}$. Then, by definition $\nexists\{i\bar{j}'\} \Rightarrow \nexists\{ij'\} \wedge \nexists\{j'i\}$. However, this is a contradiction to G^s since for $j' < j$, i can at least optimally cut the link $\{ij\}$ and activate $\{ij'\}$ weakly improving his utility. Thus in a stable network G^s, condition (4) must be true. Recall that if all the players are link-active, it means that $\forall i \in G^s \; \exists j \neq i : \{ij\} \in L$. We show recursively that the structure of G^s is nested. Start from player 1. Then, by (4) any player $j \neq 1$ will be connected to 1. Take player 2. Recall that the marginal benefit of a linkage weakly increases with the centrality score of the target player. Therefore, player 2 can only have at most the same neighbors of 1; any link received by 2 is also active with 1 by (4). In other words, $N_2 \subseteq N_1$. Continue until player n and we obtain $N_n \subseteq N_{n-1} \subseteq ... \subseteq N_1$, or a nested split graph.

□

Definition 2. *(from Mahadev and Peled (1995)) Let G = (N,L) be a graph whose distinct positive degrees are* $\upsilon_1 < \upsilon_2 < ... < \upsilon_k$*, and let* $\upsilon_0 = 0$ *(even if no agent with degree 0 exists in G). Further, define* $\Upsilon_i = \{v \in N : \upsilon_v = \upsilon_i\}$ *for* $i = 0, ..., k$*. Then the vector* $\Upsilon = (\Upsilon_0, \Upsilon_1, ..., \Upsilon_k)$ *is called the degree partition of G.*

Definition 3. *(from Mahadev and Peled (1995)) Consider a nested split graph* $G = (N, L)$ *and let* $\Upsilon=(\Upsilon_0, \Upsilon_1, ..., \Upsilon_k)$ *be its degree partition. Then the nodes N can be partitioned in independent sets* Υ_i*,* $i, ..., [\frac{f}{2}]$ *and dominating sets* Υ_i*,* $[\frac{k}{2}] + 1, ..., k$*. Moreover, the neighborhoods of the nodes are nested. In particular, for each node* $v \in \Upsilon_i$*,* $i, ..., k$*,*

$$N_v = \begin{cases} \bigcup_{j=1}^{i} \Upsilon_{k+1-j}, & \text{if } i, ..., [\frac{f}{2}], \\ \bigcup_{j=1}^{i} \Upsilon_{k+1-j} \setminus v & \text{if } [\frac{k}{2}] + 1, ..., k \end{cases}$$

Impact of a new linkage

We study the impact of a new link on the equilibrium configuration of a given network structure. Intuitively, a new connection potentially modifies the bargaining power of the new paired players as much as of third players belonging to the same component. We know that, given a connected component g, activating a linkage between two nodes $i, j \in g$ creating the component $g \oplus ij$ implies that $(b'_i, b'_j) \geq (b_i, b_j)$ with $b'_{i,j}$ the centrality scores of $i, j \in g \oplus ij$. In words, a new linkage between two nodes i, j of the same component g weakly increases the centrality scores of the same i and j. Moreover, for any new linkage connecting (i, j), $b'_{j'} \leq b_{j'}$ with $j' \neq i, j$, i.e. the impact of a new connection on the centrality scores of the rest of the nodes in the component is weakly negative. Denote with $\bar{\tilde{B}}(g)$, we indicate the average weighted betweenness of g. We can present the following theorem.[20]

Theorem 1. *The average weighted betweenness score of g is function of n and the distance between the pairs of nodes d. In particular, $\bar{\tilde{B}}(g) = (n-1)(\bar{H}_d - 1)$ where H_d is the dth harmonic number.*

Proof. Consider two nodes $u, v \in N$ distant $d(u, v) = d$. Define the set $P^h_{u,v}(w) = \{w|\ d(u, w) = h\ \wedge\ d(w, v) = d - h\}$ with $0 \leq h \leq d$, and their union $\cup^d_{h=0} P^h_{u,v}$. Then, for $h \in [2, d]$ we have $\sum_{w \in P^h_{u,v}} \tilde{b}_{u,v}(w) = \sum_{w \in P^h_{u,v}} \frac{1}{h}$. Therefore, we derive that

$$\sum_{w \in N} \tilde{b}_{u,v}(w) = \sum_{w \in P_{u,v}} \tilde{b}_{u,v}(w) = \sum_{h=0}^{d} \sum_{w \in P_{u,v}} \tilde{b}_{u,v}(w) = \sum_{h=2}^{d} \sum_{w \in P_{u,v}} \frac{1}{h} = \sum_{w \in P_{u,v}} (H_d - 1)$$

with H_h the hth Harmonic number, with h from 2 to d. Then, we can rewrite the graph average

[20] We remark that Comellas and Gago (2007) shows that, given a connected graph g of order n, $\bar{B}(g) = (n-1)(\bar{L}(g) - 1)$, where $\bar{B}(g) = \frac{1}{n} \sum_{i \in g} b_i$ is the average betweenness score in g, and $\bar{L}(g) = \frac{1}{n(n-1)} \sum_{(i,j) \in N^2} d(i, j)$ is the average distance between two nodes in g. The main difference with our result is due to the fact that the modified betweenness score discounts the length of a path between two nodes, or alternatively said, partial centrality scores related to relatively long paths have small impact on the final probability measure.

modified betweenness as

$$\tilde{\bar{B}}(g) = \frac{1}{n}\sum_{w\in N}\tilde{B}_w = \frac{1}{n}\sum_{w\in N}\sum_{(u,v)\in N^2}\tilde{b}_{u,v}(w) = \frac{1}{n}\sum_{(u,v)\in N^2}\sum_{w\in N}\tilde{b}_{u,v}(w) = \frac{1}{n}\sum_{(u,v)\in N^2}(H_d - 1) = (n-1)\bar{H}_d$$

with $\bar{H}_d$ is the average of $H_d - 1$ values for each d. Note that H_d is monotonic and concave with respect to d, and therefore, $\tilde{\bar{B}}(g)$ increases at a decreasing rate with $\bar{L}$. However, we remark that for relatively small order n, the impact of small distances d could be relatively high on the final probability. This is due to the fact that the Harmonic number is more affected by smaller distances d. □

In words, the average weighted betweenness score in a component g is function of the number of nodes and the average length of the geodesic paths between a pair of nodes. Consequently, we can also state that given a non-regular component g of order $n > 2$, increasing the density of the graph weakly decreases the average bargaining power. This leads to lower average variance across the bargained shares.

Suppose a component g of $n > 2$ players and the activation of a new link. We can define two "types" of players: the ones activating the new link, say $(i, j') \in g : \{ij'\} \notin g$, and the rest of the peers, say $j \in g : j \neq i, j'$. Define $g' = g \oplus \{ij'\}$. Then, it is easy to see that

$$\Delta\beta_{ij}(g') = \frac{1}{n-1}\sum_{(j,k)\in N^2}\Delta\tilde{b}_{jk}(i) \geq 0$$

and symmetrically for j'. Moreover, the activation of the link $\{ij'\}$ can only weakly increase the centrality scores of i and j', or $\Delta b_i(g') \geq 0$, but always increases the final probability of i and j' of $1/n-1$ due to the new direct connection between them; the final change on the probabilities β_i and $\beta_{j'}$ is strictly bigger than zero. Symmetric results could be obtained for the case of link-severing. On the other hand, the impact of $\{ij'\}$ on j players depends on each individual j's location in the component. In particular, given cardinality $n > 2$, any activation $\{ij'\}$ implies a reduction of centrality for at least one player j and at least by $1/(n-1)$. However, the new linkage $\{ij'\}$ could also positively affect the centrality of a player $j \neq i, j'$. We expect that the final total average impact will be negative, but the local impact on each j will depend on their respective location in g.

Example: Computation of β scores

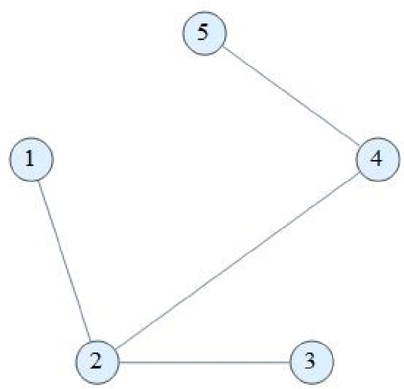

Figure 2

Consider the network of $n = 5$ players in Figure 2 and assume $N_i = N$. Each geodesic path σ_{ik} is picked with uniform probability 1/4. We report in Table 1 the relative probabilities (each element defines the probability β_{ji} with j the row-player and i the column-player). Player 2 expects to be a respondent with higher probability compared to the other nodes, given almost any receiver player, while player 5 with the smallest chance. In terms of centrality score, the vector of betweenness scores associated to the graph is $\mathbf{b} = (0, 5, 0, 3, 0)$.

Table 1

j/i	1	2	3	4	5	β_j
1	0	0.25	0.125	0.125	0.083	0.583
2	0.583	0	0.583	0.5	0.292	1.958
3	0.125	0.25	0	0.125	0.083	0.583
4	0.208	0.375	0.417	0	0.542	1.542
5	0.083	0.125	0.083	0.25	0	0.542
$\sum_j \beta_{ji}$	1	1	1	1	1	

5.1 Example: the Florentine marriage network in 1430's

Padgett and Ansell (1993) analysed in detail for the first time the social network which connected the most powerful families in Florence in the 1430. In particular, the authors proposed an alternative explanation of the rise of the political influence of the Medici family which took into account their relative "position" in the network.

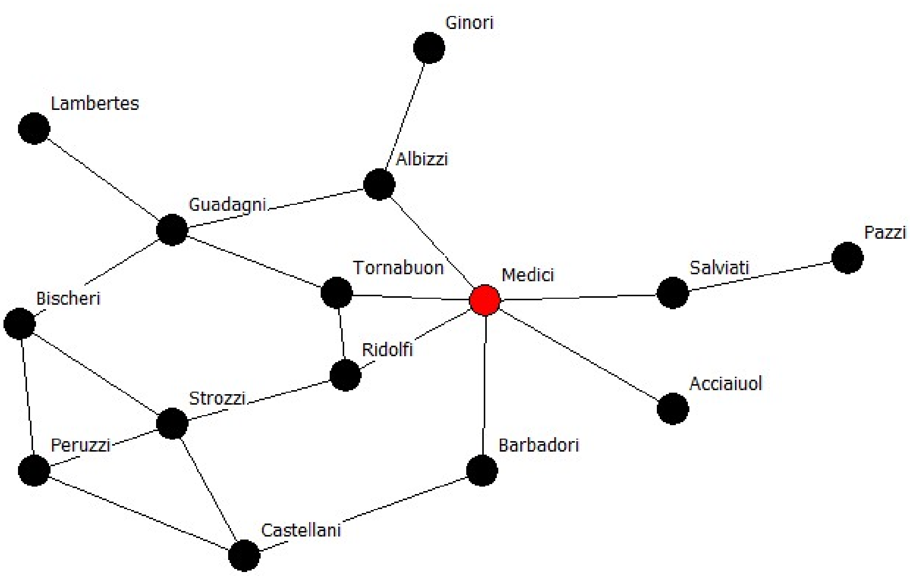

Figure 3: Florentine marriage network in 1430's. The Medici family (red) scores the highest betweenness centrality.

Analysing only the number of connections owned by each of these families is not particularly insightful. The Medici family scores the highest degree but the difference with families like the Guadagni is not striking. Looking at the betweenness centrality score instead, the Medici family scores the highest value, 0.522, much higher than the Guadagni's score 0.255. As pointed out by the same authors, the Medici were in a high brokerage position between different factions, and this could have consolidated their political power over time. Interpreting this in our context, the Medici family was often an essential step for families from different factions wishing to communicate and exchange information. In our setting, a high betweenness score implies high β scores, or high ex-ante expect payoff from the bargain process. The β scores for the Medici and Guadagni families are respectively $\beta_M = 2.97$ and $\beta_G = 1.35$. To illustrate the asymmetry in the partial β measures, we observe for example that $\beta_{MG} = 0.125$ and $\beta_{GM} = 0.06$; Members of the Guadagni family would cross 12.5% of the times members of the Medici family in order to communicate with other families of the network, while members of the Medici family meet the Guadagni family less than half of this time in order to reach other families of the community.

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