

# LOWER SEMILATTICE-ORDERED RESIDUATED SEMIGROUPS AND SUBSTRUCTURAL LOGICS

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**ABSTRACT.** We look at lower semilattice-ordered residuated semigroups and, in particular, the representable ones, i.e., those that are isomorphic to algebras of binary relations. We will evaluate expressions (terms, sequents, equations, quasi-equations) in representable algebras and give finite axiomatizations for several notions of validity. These results will be applied in the context of substructural logics.

**KEYWORDS:** finite axiomatization, relation algebras, residuation, Lambek calculus, relevance logics

## 1. INTRODUCTION

We will look at algebras of signature consisting of meet  $\cdot$ , (relation) composition  $;$ , and its right and left residuals,  $\backslash$  and  $/$ . We will provide finite axiomatizations for various classes of algebras and obtain completeness results about the corresponding notions of standard validity and semantical consequence. We will also look at alternative semantics, motivated by substructural logics, and prove finite axiomatizations in this case as well.

**1.1. The Algebras.** We start with defining abstract, axiomatically given, algebras. As usual with algebras with a (lower) semilattice reduct, we define  $x \leq y$  iff  $x \cdot y = x$ .

**Definition 1.1.** A *lower semilattice-ordered residuated semigroup* is an algebra  $\mathfrak{A} = (A, \cdot, ;, \backslash, /)$  such that  $(A, \cdot)$  is a semilattice,  $(A, ;, \backslash, /)$  is a residuated semigroup, i.e., the following equation

$$(1) \quad x ; (y ; z) = (x ; y) ; z$$

and quasi-equations

$$(2) \quad y \leq x \backslash z \text{ iff } x ; y \leq z \text{ iff } x \leq z / y$$

are satisfied, and  $;$  is monotonic, i.e., the following equation

$$(3) \quad (x \cdot x') ; (y \cdot y') \leq x ; y$$

is satisfied.

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*Date:* This is the non-refereed version of a published paper and was uploaded here for purely administrative reasons. I strongly discourage you to read it. Instead, you can find a better version at <http://www.dcs.bbk.ac.uk/~szabolcs/MeetLambek.pdf>.

We denote the class of lower semilattice-ordered residuated semigroups by LSORS.

Next we define a subclass of the abstract class by focusing on algebras of binary relations.

**Definition 1.2.** Let  $\mathfrak{A} = (A, \cdot, ;, \backslash, /) \in \text{LSORS}$ . We say that  $\mathfrak{A}$  is *representable* if  $A \subseteq \wp(U \times U)$  for some set  $U$ , the *base* of  $\mathfrak{A}$ , and the operations are interpreted as follows: meet is intersection and

$$\begin{aligned} x ; y &= \{(u, v) \in U \times U : (u, w) \in x \text{ and } (w, v) \in y \text{ for some } w\} \\ x \backslash y &= \{(u, v) \in U \times U : \text{for every } w, (w, u) \in x \text{ implies } (w, v) \in y\} \\ x / y &= \{(u, v) \in U \times U : \text{for every } w, (v, w) \in y \text{ implies } (u, w) \in x\} \end{aligned}$$

for all  $x, y \in A$ .

Usually we will not distinguish between representable LSORS and its closure under isomorphic copies.

We can extend the notion of representability to other signatures by providing interpretations for additional operations. In particular, we will look at the the *identity* constant interpreted as

$$1' = \{(u, v) \in U \times U : u = v\}$$

in an algebra with base  $U$ .

Given a signature  $\Lambda$ , we will denote the class of representable  $\Lambda$ -algebras by  $\mathbf{R}(\Lambda)$ . Thus  $\mathbf{R}(\cdot, ;, \backslash, /)$  stands for representable LSORS.

We will also consider subclasses of  $\mathbf{R}(\Lambda)$ . In particular, we define  $\mathbf{R}^c(\Lambda)$  by requiring that the *commutativity* axiom

$$(4) \quad x ; y = y ; x$$

holds in every algebra, and  $\mathbf{R}^{cd}(\Lambda)$  by the additional requirement

$$(5) \quad x \leq x ; x$$

of *density*.

We will characterize representable algebras. In particular, we prove that  $\mathbf{R}^c(\cdot, ;, \backslash, /)$  is a finitely axiomatizable variety, see Corollary 3.2. In the non-commutative case, we show in Theorem 5.2 that the variety generated by  $\mathbf{R}(\cdot, ;, \backslash, /)$  is finitely axiomatizable, while we conjecture that the representation class  $\mathbf{R}(\cdot, ;, \backslash, /)$  itself is a non-finitely axiomatizable quasi-variety.

**1.2. State Semantics.** Let  $\Lambda$  be a signature including meet, composition and its (right) residual. An important feature of the (right) residual is the following. We have  $x \leq y$  iff  $x \backslash y$  contains the identity relation:

$$(6) \quad \mathfrak{C} \models x \leq y \text{ iff } \mathfrak{C} \models 1' \leq x \backslash y$$

for every  $\mathfrak{C} \in \mathbf{R}(\Lambda)$ . Note that this makes sense even when  $1'$  is not in  $\Lambda$ , since it is meaningful whether  $\{(u, u) \in U \times U\}$  is a subset of the interpretation of a  $\Lambda$ -term (with  $U$  the base of  $\mathfrak{C}$ ).

Let  $\mathfrak{C} \in \mathbf{R}(\Lambda)$  for some signature  $\Lambda$ . We define

$$(7) \quad \mathfrak{C} \models_s t \text{ iff } \mathfrak{C} \models 1' \leq t$$

for every  $\Lambda$ -term  $t$ . We say that  $t$  is *state valid* in  $\mathbf{R}(\Lambda)$  (in symbols,  $\models_s t$ ) iff  $\mathfrak{C} \models_s t$  for every  $\mathfrak{C} \in \mathbf{R}(\Lambda)$ .<sup>1</sup>

We will prove the analogous results to the standard case. We show that there is a strongly sound and complete calculus for the state semantics over  $\mathbf{R}^c(\cdot, ;, \backslash, /)$ , see Corollary 6.3, while we have weak completeness in the non-commutative case  $\mathbf{R}(\cdot, ;, \backslash, /)$ , see Corollary 6.2.

**1.3. Connections to Substructural Logics.** The main motivation for considering state semantics originates in (substructural) logic. State semantics restricted to the commutative and dense subclass  $\mathbf{R}^{cd}(\Lambda)$  of  $\mathbf{R}(\Lambda)$  provides sound semantics for relevance logic [ABD92, RM73]. Relevance logic  $\vdash_R$  is a Hilbert-style derivation system and has the logical connectives conjunction, implication and negation. The logical connectives are interpreted in algebras of binary relations  $\mathfrak{C} \in \mathbf{R}^{cd}(\cdot, \backslash, \sim)$  as meet  $\cdot$ , (right) residual  $\backslash$  and converse negation  $\sim$  defined by

$$\sim x = \{(u, v) \in U \times U : (v, u) \notin x\}$$

respectively. State validity w.r.t. commutative and dense families of relations will be denoted by  $\models_s^{cd}$ . While this semantics is sound<sup>2</sup>

$$\vdash_R \varphi \text{ implies } \models_s^{cd} \varphi$$

completeness does not hold, since state validity for  $\mathbf{R}^{cd}(\cdot, \backslash, \sim)$  cannot be finitely axiomatized [Mik09]. See also [BDM09, Ma10] for the connection between relevance logic and state semantics.

We note that, using  $\cdot$ ,  $\backslash$  and  $\sim$ , additional connectives can be defined. For instance, we can define  $x + y$  as  $\sim(\sim x \cdot \sim y)$  and  $x ; y$  as  $\sim(y \backslash \sim x)$ .

It would be interesting to see for precisely which signatures  $\Lambda$  one can give a complete calculus capturing state validity in  $\mathbf{R}^{cd}(\Lambda)$ , since that would show (in)completeness of other fragments of relevance logic with respect to state semantics. For instance, [HM11] shows that finite axiomatization of state validities for  $\mathbf{R}^{cd}(\cdot, +, \backslash)$  is not possible, hence establishing incompleteness of the positive fragment of relevance logic.<sup>3</sup> The same non-finite axiomatizability results hold for  $\mathbf{R}^{cd}(\cdot, +, ;, \backslash)$  and  $\mathbf{R}(\cdot, +, ;, \backslash, /)$ , see [HM11] and

<sup>1</sup>The terminology ‘state semantics’ refers to the fact that truth is restricted to pairs of the form  $(u, u)$ . Note that the concept of truth uses the more general concept of interpretation, thus whether a term is true at  $(u, u)$  in general depends on whether pairs of the form  $(v, w)$  are in the interpretations of some other terms. For instance,  $x \backslash y$  is true at  $(u, u)$  iff, for every  $v$ ,  $(v, u)$  is in the interpretation of  $y$  whenever it is in the interpretation of  $x$ .

<sup>2</sup>For the sake of simplicity we will not distinguish between a relevance logic formula and the corresponding algebraic term where the logical connectives are replaced by the corresponding algebraic operations.

<sup>3</sup>Analogues of commutativity and density can be defined even when composition is not in the similarity type.

[Mik11]. In contrast, we will show finite axiomatizability for state semantics over  $\mathbf{R}^{cd}(\cdot, ;, \backslash)$ , and Corollary 6.4 can be interpreted as a completeness result for relevance logic of the language consisting of conjunction, composition (or fusion) and implication.

We showed various completeness results for the (generalized) Lambek calculus,  $\mathbf{LC}$ , in [AM94]. We will expand the similarity type with meet, and consider both standard and state semantics. We provide weakly complete and sound axiomatizations using generalized sequents in both cases, see Theorem 4.3. Equivalent axiomatizations without using generalized sequents will be given in Section 5 for standard semantics and in Section 6 for state semantics.

**1.4. Organization.** The rest of the paper is organized as follows. In the next section we prove the key Lemma 2.1 that will be in the heart of the main results. In Section 3 we look at standard semantics and provide finite axiomatization in the commutative case. Next we look at the completeness of the Lambek calculus with meet (Section 4) and axiomatization for the generated variety in the non-commutative case (Section 5). In Section 6 we look at state semantics and obtain completeness results. We finish with a concluding section where we look at possible extensions of our results and state some open problems.

## 2. THE CONSTRUCTION

Let  $\mathfrak{A} = (A, \cdot, ;, \backslash, /) \in \mathbf{LSORS}$ . We describe a step-by-step construction of a chain of labelled directed graphs:  $G_0 \subseteq \dots \subseteq G_\alpha \subseteq \dots$  and let  $G_{\mathfrak{A}} = \bigcup_{\alpha} G_{\alpha}$ . Each  $G_{\alpha}$  has the form  $(U_{\alpha}, \ell_{\alpha})$  where  $U_{\alpha}$  is the set of nodes and  $\ell_{\alpha}: U_{\alpha} \times U_{\alpha} \rightarrow \wp(A)$  is the labelling function. The construction is similar to that of [AM94, Theorem 3.2].

We extend the operations of  $\mathfrak{A} \in \mathbf{LSORS}$  to subsets of elements:

$$\begin{aligned} X ; Y &= \{x ; y : x \in X, y \in Y\} \\ X \backslash Y &= \{x \backslash y : x \in X, y \in Y\} \\ X / Y &= \{x / y : x \in X, y \in Y\} \end{aligned}$$

for subsets  $X, Y$ . Given a subset  $X$ , we define the upward closure of  $X$  by

$$X^{\uparrow} = \{y : y \geq x \in X\}$$

and use the convention that if  $X = \{x\}$  is a singleton, then we write  $x^{\uparrow} = \{x\}^{\uparrow}$  and  $x ; Y = \{x\} ; Y$ , etc. By a filter of  $\mathfrak{A}$  we mean a subset  $\mathcal{F} \subseteq A$  that is closed upward (i.e.,  $\mathcal{F}^{\uparrow} = \mathcal{F}$ ) and under meet (i.e.,  $x \cdot y \in \mathcal{F}$  whenever  $x, y \in \mathcal{F}$ ). We denote the filter generated by  $X$  by  $\mathcal{F}(X)$ . Note that a singleton-generated filter  $\mathcal{F}(a) = a^{\uparrow}$  is a principal filter. We will assume that there is a distinguished filter

$$\mathcal{E} \supseteq \{a \backslash b, b / a : a, b \in A, a \leq b\}$$

of  $\mathfrak{A}$  that we will use for labelling reflexive edges.

**Initial step:** In the 0th step we construct  $G_0 = (U_0, \ell_0)$ . We define  $U_0$  by choosing distinct  $x_a, y_a$  for distinct elements  $a \in A$ , and let

$$\begin{aligned}\ell_0(x_a, x_a) &= \ell_0(y_a, y_a) = \mathcal{E} \\ \ell_0(x_a, y_a) &= \mathcal{F}(a)\end{aligned}$$

and we label all other edges by  $\emptyset$  (e.g.,  $\ell(x_a, y_b) = \emptyset$  for  $a \neq b$ ).

**Successor steps:** In the  $(\alpha + 1)$ th step we have three substeps. To deal with composition  $;$ , for every  $b; c \in \ell_\alpha(x, y)$  such that there is no  $w \in U_\alpha$  with  $b \in \ell_\alpha(x, w)$  and  $c \in \ell_\alpha(w, y)$ , we choose a fresh point  $z$ , and define

$$\begin{aligned}\ell_{\alpha+1}(z, z) &= \mathcal{E} \\ \ell_{\alpha+1}(x, z) &= \mathcal{F}(b) \\ \ell_{\alpha+1}(z, y) &= \mathcal{F}(c) \\ \ell_{\alpha+1}(r, z) &= \mathcal{F}(\ell_\alpha(r, x); b) & r \neq x, z \\ \ell_{\alpha+1}(z, s) &= \mathcal{F}(c; \ell_\alpha(y, s)) & s \neq y, z\end{aligned}$$

For all other edges  $(u, v)$ , we let  $\ell_{\alpha+1}(u, v) = \ell_\alpha(u, v)$  if  $\ell_\alpha(u, v)$  has been defined, and  $\ell_{\alpha+1}(u, v) = \emptyset$  if  $\ell_\alpha(u, v)$  is undefined. See Figure 1 for the case when  $x \neq y$  and  $\ell_\alpha(y, x) \neq \emptyset$ .

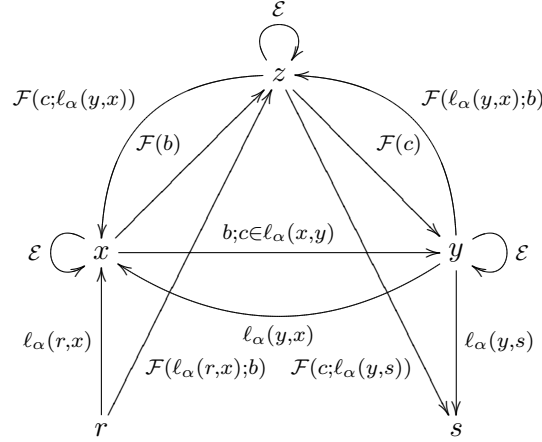


FIGURE 1. Step for composition

To deal with the residual  $\setminus$  we choose a fresh point  $z$ , for every point  $x \in U_\alpha$  and  $a \in A$ , and define

$$\begin{aligned}\ell_{\alpha+1}(z, z) &= \mathcal{E} \\ \ell_{\alpha+1}(z, x) &= \mathcal{F}(a) \\ \ell_{\alpha+1}(z, p) &= \mathcal{F}(a; \ell_\alpha(x, p)) & p \neq x, z\end{aligned}$$

For all other edges  $(u, v)$ , we let  $l_{\alpha+1}(u, v) = l_\alpha(u, v)$  if  $l_\alpha(u, v)$  has been defined, and  $l_{\alpha+1}(u, v) = \emptyset$  if  $l_\alpha(u, v)$  is undefined. See Figure 2.

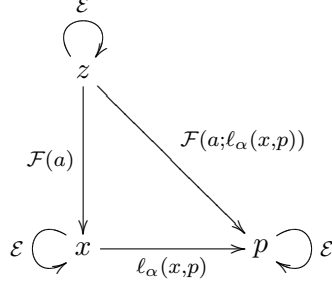


FIGURE 2. Step for the right residual

Finally we have the symmetric case for  $/$ . We choose a fresh point  $z$ , for every point  $x \in U_\alpha$  and  $a \in A$ , and define

$$\begin{aligned} l_{\alpha+1}(z, z) &= \mathcal{E} \\ l_{\alpha+1}(y, z) &= \mathcal{F}(a) \\ l_{\alpha+1}(q, z) &= \mathcal{F}(l_\alpha(q, y); a) \quad q \neq y, z \end{aligned}$$

For all other edges  $(u, v)$ , we let  $l_{\alpha+1}(u, v) = l_\alpha(u, v)$  if  $l_\alpha(u, v)$  has been defined, and  $l_{\alpha+1}(u, v) = \emptyset$  if  $l_\alpha(u, v)$  is undefined. See Figure 3.

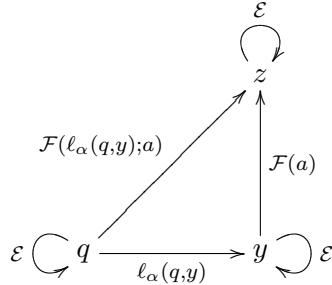


FIGURE 3. Step for the left residual

**Limit step:** We take the union  $G_\beta$  of the constructed graphs  $G_\alpha$  ( $\alpha < \beta$ ).

We define  $G_{\mathfrak{A}} = (U_{\mathfrak{A}}, l_{\mathfrak{A}}) = \bigcup_\alpha G_\alpha$ . Observe that every label  $l_{\mathfrak{A}}(x, y)$  with  $x \neq y$  is in fact a principal filter. Furthermore, it is not difficult to see that these steps can be scheduled so that  $G_{\mathfrak{A}}$  satisfies the following *saturation conditions*.

**Sat(·):** For every  $x$  and  $y$ , if  $b ; c \in \ell_{\mathfrak{A}}(x, y)$ , then there is  $z$  such that  $b \in \ell_{\mathfrak{A}}(x, z)$  and  $c \in \ell_{\mathfrak{A}}(z, y)$ .

**Sat(\):** For every  $x$ , there is  $z$  such that  $\ell_{\mathfrak{A}}(z, x) = \mathcal{F}(a)$ , and for every  $p$ , we have  $\ell_{\mathfrak{A}}(z, p) = \mathcal{F}(a ; \ell_{\mathfrak{A}}(z, p))$ .

**Sat(/):** For every  $y$ , there is  $z$  such that  $\ell_{\mathfrak{A}}(y, z) = \mathcal{F}(a)$  and for every  $q$ , we have  $\ell_{\mathfrak{A}}(q, z) = \mathcal{F}(\ell_{\mathfrak{A}}(q, y) ; a)$ .

We define the *coherence condition* as follows.

**Coh(·):** If  $a \in \ell_{\mathfrak{A}}(x, z)$  and  $b \in \ell_{\mathfrak{A}}(z, y)$ , then  $a ; b \in \ell_{\mathfrak{A}}(x, y)$ .

Note that coherence may not hold for an arbitrary  $\mathfrak{A} \in \text{LSORS}$ .

We are ready to formulate our key lemma.

**Lemma 2.1.** *Let  $\mathfrak{A} \in \text{LSORS}$  and assume that  $G_{\mathfrak{A}}$  is coherent. Then  $\mathfrak{A}$  is representable:  $\mathfrak{A} \in \mathbf{R}(\cdot, ;, \setminus, /)$ .*

*Proof.* We define

$$(8) \quad \text{rep}(a) = \{(u, v) \in U_{\mathfrak{A}} \times U_{\mathfrak{A}} : a \in \ell_{\mathfrak{A}}(u, v)\}$$

for every  $a \in A$ . We claim that  $\text{rep}$  is an isomorphism from  $\mathfrak{A}$  into the full algebra  $(\wp(U_{\mathfrak{A}} \times U_{\mathfrak{A}}), \cdot, ;, \setminus, /)$ . Injectivity of  $\text{rep}$  is guaranteed by the initial step (and the fact that we do not alter labels in later steps). Since we used filters as labels,  $\text{rep}$  respects meet. Checking that  $\text{rep}$  preserves composition is easy by using the conditions  $\text{Coh}(\cdot)$  and  $\text{Sat}(\cdot)$ .

Finally we check the residuals. First assume that  $(u, v) \in \text{rep}(a) \setminus \text{rep}(b)$ , i.e., for all  $w$ , if  $(w, u) \in \text{rep}(a)$ , then  $(w, v) \in \text{rep}(b)$ . By  $\text{Sat}(\setminus)$  there is  $z \in U_{\mathfrak{A}}$  such that  $\ell_{\mathfrak{A}}(z, u) = \mathcal{F}(a) = a^{\uparrow}$  and  $\ell_{\mathfrak{A}}(z, v) = \mathcal{F}(a ; \ell_{\mathfrak{A}}(z, v))$ . First consider the case  $u = v$ . Then  $\ell_{\mathfrak{A}}(z, v) = \ell_{\mathfrak{A}}(z, u) = \mathcal{F}(a)$ . Since  $b \in \ell_{\mathfrak{A}}(z, v)$ , we have  $a \leq b$ , whence  $a \setminus b \in \ell_{\mathfrak{A}}(u, u) = \mathcal{E}$ . When  $u \neq v$  we argue as follows. Since  $b \in \ell_{\mathfrak{A}}(z, v)$ , we have  $a ; x \leq b$  for some  $x \in \ell_{\mathfrak{A}}(u, v)$  (recall that  $\mathcal{F}(a ; \ell_{\mathfrak{A}}(u, v))$  is a principal filter), whence  $x \leq a \setminus b$  by (2). Thus  $a \setminus b \in \ell_{\mathfrak{A}}(u, v)$ . On the other hand, if  $a \setminus b \in \ell_{\mathfrak{A}}(u, v)$ , then  $\ell_{\mathfrak{A}}(w, u) ; (a \setminus b) \subseteq \ell_{\mathfrak{A}}(w, v)$  for all  $w \in U_{\mathfrak{A}}$  by  $\text{Coh}(\cdot)$ . Hence  $(w, u) \in \text{rep}(a)$  implies  $(w, v) \in \text{rep}(a ; (a \setminus b)) \subseteq \text{rep}(b)$ , by (2) and that  $\text{rep}$  respects  $\leq$ . Checking the other residual is completely analogous.  $\square$

### 3. COMMUTATIVE REPRESENTABLE ALGEBRAS

We will apply the above construction to achieve finite axiomatizations for various theories of  $\mathbf{R}(\cdot, ;, \setminus, /)$ .

First we note that, in the presence of meet, the quasi-equations (2) can be formulated as equations. In fact, V. Pratt showed that the equations

$$(9) \quad x \setminus (y \cdot y') \leq x \setminus y \qquad (x \cdot x') / y \leq x / y$$

$$(10) \quad x ; (x \setminus y) \leq y \qquad (x / y) ; y \leq x$$

$$(11) \quad y \leq x \setminus (x ; y) \qquad x \leq (x ; y) / y$$

imply (2), see [Pr90]. Thus LSORS is in fact a variety.

But there are valid equations in  $R(\cdot, ;, \backslash, /)$  that are not derivable from the axioms for LSORS. We will say that  $x$  is a *residuated term* if it has the form  $y \backslash z$  or  $y / z$ , and a residuated term is *reflexive* if  $y = z$  (since terms of the form  $y \backslash y$  and  $y / y$  include the identity relation in representable algebras, hence their interpretations are reflexive). Then the following are valid equations in  $R(\cdot, ;, \backslash, /)$ .

“Reflexivity”:

$$(12) \quad x ; y \geq y \leq y ; x$$

if  $x$  is a reflexive residuated term.

“Idempotency”:

$$(13) \quad (x \cdot y) \backslash (x \cdot y) \leq x \cdot y \leq (x \cdot y) / (x \cdot y)$$

if  $x, y$  are reflexive residuated terms.

It is easily checked that these axioms are indeed valid. We just note that the interpretations of reflexive residuated elements  $x$  must include the identity (they are reflexive) and they are transitive ( $x ; x \leq x$ ).

Let  $Ax(\cdot, ;, \backslash, /)$  be the collection of the following axioms: semilattice axioms for meet, semigroup axiom for composition (1), monotonicity (3), the axioms for the residuals (9), (10) and (11), “reflexivity” (12) and “idempotency” (13).

**Lemma 3.1.** *Let  $\mathfrak{A} \models Ax(\cdot, ;, \backslash, /)$ . Assume that  $\mathfrak{A}$  additionally satisfies commutativity (4). Then  $\mathfrak{A} \in R(\cdot, ;, \backslash, /)$ .*

*Proof.* Let  $\mathfrak{A} = (A, \cdot, ;, \backslash, /)$  satisfy the conditions of the lemma. We show that  $\mathfrak{A} \in R(\cdot, ;, \backslash, /)$  by applying Lemma 2.1.

First note that the collection  $\{x \backslash x, x / x : x \in A\}$  of reflexive residuated elements of  $A$  is closed under meet by axiom (13). Thus the set  $\mathcal{E} = \{x \backslash x, x / x : x \in A\}^\uparrow$  is a filter. Also note that  $\mathcal{E}$  is closed under composition by axioms (3) and (12).

We claim that the graph  $G_{\mathfrak{A}}$  yielded by the step-by-step construction of Section 2 is coherent. It is easy to check that the initial graph  $G_0$  is coherent. For instance, if  $x, y \in \ell_{\mathfrak{A}}(u_a, u_a) = \mathcal{E}$ , then  $x ; y \in \mathcal{E}$  and also  $a \leq x ; a ; y$ , whence  $\ell_0(u_a, u_a) \supseteq \ell_0(u_a, u_a) ; \ell_0(u_a, u_a)$  and  $\ell_0(u_a, v_a) \supseteq \ell_0(u_a, u_a) ; \ell_0(u_a, v_a) ; \ell_0(v_a, v_a)$ .

Now assume, inductively, that  $G_\alpha$  is coherent. We have to show that the graph after the successor steps is coherent. Most instances of  $\text{Coh}(\cdot)$  are easy to check, we just work out the most complicated case. Consider the case of composition when  $b ; c \in \ell_\alpha(x, y)$  and  $\ell_\alpha(y, x) \neq \emptyset$ , see Figure 4. We need  $c ; d ; b \in \ell_{\alpha+1}(z, z) = \mathcal{E}$  for every  $d \in \ell_\alpha(y, x)$ . By induction, we have that  $b ; c ; d \in \ell_\alpha(x, x) = \mathcal{E}$ , i.e.,  $e \leq b ; c ; d$  for some  $e \in \mathcal{E}$ . By commutativity (4), we get  $e \leq c ; d ; b$ , whence  $c ; d ; b \in \mathcal{E} = \ell_{\alpha+1}(z, z)$ , as desired.

Thus we can apply Lemma 2.1, whence  $\mathfrak{A}$  is representable.  $\square$

We are ready to formulate the finite axiomatizability of the commutative subclass of representable algebras. Recall that  $R^c(\cdot, ;, \backslash, /)$  denotes that



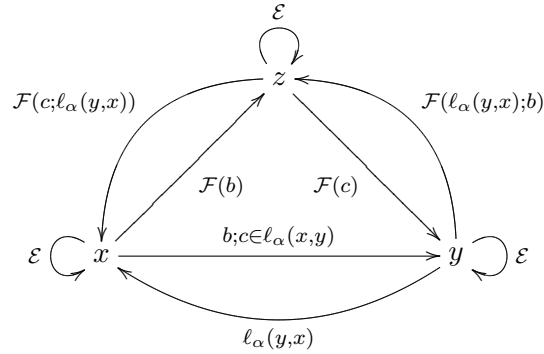


FIGURE 4. Checking coherence

subclass of  $\mathbf{R}(\cdot, ;, \backslash, /)$  where every algebra is commutative. Analogously let  $\mathbf{Ax}^c(\cdot, ;, \backslash, /)$  denote  $\mathbf{Ax}(\cdot, ;, \backslash, /)$  plus commutativity (4). By a straightforward application of Lemma 3.1 we get the following.

**Corollary 3.2.** *The class  $\mathbf{R}^c(\cdot, ;, \backslash, /)$  is a finitely axiomatizable variety. In fact,*

$$\mathfrak{A} \models \mathbf{Ax}^c(\cdot, ;, \backslash, /) \text{ iff } \mathfrak{A} \in \mathbf{R}^c(\cdot, ;, \backslash, /)$$

for every algebra  $\mathfrak{A} = (A, \cdot, ;, \backslash, /)$ .

**Remark 3.3.** In commutative algebras the interpretations of the two residuals coincide:  $x \backslash y = y / x$ . Thus we could drop one of the residuals, say,  $/$ , and state Corollary 3.2 for  $\mathbf{R}^c(\cdot, ;, \backslash)$ . Then we would have the obvious simplification in the axioms (drop those mentioning  $/$ ) and in the construction (we do not need the step dealing with  $/$ ).

We will see that in the non-commutative case,  $\mathbf{Ax}(\cdot, ;, \backslash, /)$  provides finite axiomatization for the variety generated by the representable algebras. We will establish this by showing that the free algebra of the variety defined by  $\mathbf{Ax}(\cdot, ;, \backslash, /)$  is representable. To this end first we give an axiomatization using sequents.

#### 4. LAMBEK CALCULUS WITH MEET

We will now define a sequent calculus that will be an extension of the (generalized) Lambek calculus [La58]. Let us recall that a *sequent* is  $x_1, \dots, x_n \Rightarrow x_0$  where every  $x_i$  is a term. Sequences of terms will be denoted by capitals  $T, U, V, \dots$ , and  $U, V$  denotes the juxtaposition of  $U$  and  $V$  with the convention that it means  $U$  when  $V$  is empty, and it means  $V$  when  $U$  is empty.

We will need the axioms and derivation rules of Figure 5, where lower case letters denote non-empty terms and capital letters denote (possibly empty) sequences of terms. The Lambek calculus LC has the single axiom (I) and derivation rules ( $;L$ ), ( $;R$ ), ( $\backslash L$ ), ( $\backslash R$ ), ( $/L$ ), ( $/R$ ) and (*Cut*) with

<b>Axioms:</b>	
(I)	$x \Rightarrow x$
(Com)	$x ; y \Rightarrow y ; x$
(Den)	$x \Rightarrow x ; x$
<b>Derivation rules:</b>	
(;L)	$\frac{U, x, y, V \Rightarrow z}{U, x ; y, V \Rightarrow z}$
(\L)	$\frac{T \Rightarrow x \quad U, y, V \Rightarrow z}{U, T, x \setminus y, V \Rightarrow z}$
(/L)	$\frac{T \Rightarrow x \quad U, y, V \Rightarrow z}{U, y / x, T, V \Rightarrow z}$
(\cdot L1)	$\frac{U, x, V \Rightarrow y}{U, x \cdot z, V \Rightarrow y}$
(\cdot R)	$\frac{T \Rightarrow x \quad T \Rightarrow y}{T \Rightarrow x \cdot y}$
(Cut)	$\frac{T \Rightarrow x \quad U, x, V \Rightarrow y}{U, T, V \Rightarrow y}$
(;R)	$\frac{P \Rightarrow x \quad Q \Rightarrow y}{P, Q \Rightarrow x ; y}$
(\R)	$\frac{x, T \Rightarrow y}{T \Rightarrow x \setminus y}$
(/R)	$\frac{T, x \Rightarrow y}{T \Rightarrow y / x}$
(\cdot L2)	$\frac{U, x, V \Rightarrow y}{U, z \cdot x, V \Rightarrow y}$

FIGURE 5. Sequent formalism

the restriction that the sequences  $T$ ,  $P$  and  $Q$  are not empty. We get a generalization  $\text{LC}_0$  of the Lambek calculus by allowing sequents with the empty term on the left of  $\Rightarrow$ , i.e., the sequences  $T$ ,  $P$  and  $Q$  are allowed to be empty in this case — we will call these ‘empty-headed’ sequents.

The empty-headed version is more general in the sense that there are more derivable sequents, an example is  $x \Rightarrow x ; (y \setminus y)$ , see [AM94]. On the other hand, restricting derivability to empty-headed sequents does not lead to the loss of expressive power. Indeed, we have

$$\vdash x_1, \dots, x_n \Rightarrow x_0 \text{ implies } \vdash \Rightarrow (x_1 ; \dots ; x_n) \setminus x_0$$

by applying (;L)  $n - 1$  times and then applying (\R) (with empty  $T$ ).

We will extend the Lambek calculus with meet. Terms and sequents are defined analogously to the meet-free case. We will need the derivation rules ( $\cdot L1$ ), ( $\cdot L2$ ) and ( $\cdot R$ ), see Figure 5.

The semantics for sequents is defined as follows. Let  $\mathfrak{C} \in \mathcal{R}(\cdot, ;, \setminus, /)$ . Then

$$(14) \quad \mathfrak{C} \models x_1, \dots, x_n \Rightarrow x_0 \text{ iff } \mathfrak{C} \models x_1 ; \dots ; x_n \leq x_0$$

for every sequent  $x_1, \dots, x_n \Rightarrow x_0$  including the empty-headed case  $n = 0$  when

$$(15) \quad \mathfrak{C} \models \Rightarrow x_0 \text{ iff } \mathfrak{C} \models 1' \leq x_0$$

by convention.

**4.1. The Non-commutative Case.** The sequent calculus  $\text{SC}_0$  has the single axiom  $(I)$  and derivation rules  $(;L)$ ,  $(;R)$ ,  $(\backslash L)$ ,  $(\backslash R)$ ,  $(/L)$ ,  $(/R)$ ,  $(Cut)$ ,  $(\cdot L1)$ ,  $(\cdot L2)$  and  $(\cdot R)$ , see Figure 5. The subscript 0 in  $\text{SC}_0$  indicates that sequents are allowed to be empty headed.

We will show that  $\text{SC}_0$  is weakly complete and sound w.r.t.  $\mathbf{R}(\cdot, ;, \backslash, /)$ . We need the following technical lemma.

**Lemma 4.1.** *Let  $\vdash_0$  denote derivability in  $\text{SC}_0$ . Assume  $\vdash_0 \Rightarrow x; y$ . Then  $\vdash_0 \Rightarrow x$  and  $\vdash_0 \Rightarrow y$  as well.*

*Proof.* We will show that the cut rule can be eliminated from  $\text{SC}_0$ . In the proof of the cut elimination for  $\text{SC}_0$  we closely follow the ‘‘Proof of Gentzen’s theorem’’ in [La58, Section 9], the cut elimination for the original Lambek calculus. The proof is almost literally the same, we just have to consider the case of terms with main connective  $\cdot$  as well.

Let  $d(T)$  be the number of separate occurrences of the operations in the sequence of terms  $T$ . The *degree* of the cut rule

$$\frac{T \Rightarrow x \quad U, x, V \Rightarrow y}{U, T, V \Rightarrow y}$$

is defined to be  $d(T) + d(x) + d(U) + d(V) + d(y)$ . We will show that in any application of cut, whose premises have been proved without using cut, either the conclusion can be proved without cut, or else the cut can be replaced by one or two cuts of smaller degree. There are the following (not necessarily mutually exclusive) cases.

*Case 1:*  $T \Rightarrow x$  is an instance of the axiom  $(I)$ . Then  $x = T$ , and the other premise coincide with the conclusion.

*Case 2:*  $U, x, V \Rightarrow y$  is an instance of axiom  $(I)$ . Then  $U$  and  $V$  are empty, and the conclusion coincides with the other premise.

*Case 3:* The last step in the proof of  $T \Rightarrow x$  is a rule that does not introduce the main operation of  $x$ . The proof is essentially the same as that of Case 3 in [La58, pp.167–168].

*Case 4:* The last step in the proof of  $U, x, V \Rightarrow y$  is a rule that does not introduce the main operation of  $x$ . The proof is essentially the same as that of Case 4 in [La58, p.168].

*Cases 5, 6, 7:* The last steps in the proof of both premises introduce the main operation of  $x = x_1; x_2$  or  $x = x_1 \backslash x_2$  or  $x = x_1 / x_2$ . The proof is the same as that of Case 5, Case 6 or Case 7 in [La58, p.168–169], respectively.

*Case 8:* The last step in the proofs of both premises is a rule that introduces the main operation of  $x = x_1 \cdot x_2$ . That is, we have

$$\frac{T \Rightarrow x_1 \quad T \Rightarrow x_2}{T \Rightarrow x_1 \cdot x_2}$$

and either

$$\frac{U, x_1, V \Rightarrow y}{U, x_1 \cdot x_2, V \Rightarrow x} \quad \text{or} \quad \frac{U, x_2, V \Rightarrow y}{U, x_1 \cdot x_2, V \Rightarrow y}$$

and then the application of the cut

$$\frac{T \Rightarrow x_1 \cdot x_2 \quad U, x_1 \cdot x_2, V \Rightarrow y}{U, T, V \Rightarrow y}$$

Then we can apply one of the cut rules

$$\frac{T \Rightarrow x_1 \quad U, x_1, V \Rightarrow y}{U, T, V \Rightarrow y} \quad \text{or} \quad \frac{T \Rightarrow x_2 \quad U, x_2, V \Rightarrow y}{U, T, V \Rightarrow y}$$

both of which has a smaller degree.

Thus we have established the following.

**Claim 4.2.** *The cut rule can be eliminated from  $\text{SC}_0$ .*

Now assume that  $\vdash \Rightarrow x; y$  in  $\text{SC}_0$ . By Claim 4.2,  $\Rightarrow x; y$  has a cut-free proof in  $\text{SC}_0$ . Then the last step in the proof was an application of rule  $(;R)$  with empty  $P$  and  $Q$ , since the empty-headed conclusion of all other rules have a main operation different from  $;$ . That is,  $\vdash \Rightarrow x$  and  $\vdash \Rightarrow y$  in  $\text{SC}_0$ , and we are done.  $\square$

**Theorem 4.3.** *The derivation system  $\text{SC}_0$  is weakly complete and sound w.r.t. both the standard and the state semantics for  $\mathbf{R}(\cdot, ;, \backslash, /)$ . That is, for every non-empty term  $a$  and sequence of terms  $T$ ,*

$$\vdash_0 T \Rightarrow a \text{ iff } \models T \Rightarrow a$$

where  $\vdash_0$  denotes derivability in  $\text{SC}_0$ , and  $\models$  denotes standard validity in  $\mathbf{R}(\cdot, ;, \backslash, /)$ . In particular, when  $T$  is the empty sequence, we get

$$\vdash_0 \Rightarrow a \text{ iff } \models_s a$$

where  $\models_s$  denotes state validity in  $\mathbf{R}(\cdot, ;, \backslash, /)$ .

*Proof.* We take the Lindenbaum–Tarski algebra  $\mathfrak{A}$  of  $\text{SC}_0$ , and show that  $\mathfrak{A} \in \mathbf{R}(\cdot, ;, \backslash, /)$  by applying Lemma 2.1.

We define the ordering  $\leq$  of  $(\cdot, ;, \backslash, /)$ -terms by

$$a \leq b \text{ iff } \vdash_0 a \Rightarrow b$$

and the equivalence  $\equiv$  by

$$a \equiv b \text{ iff } a \leq b \text{ and } b \leq a$$

for terms  $a, b$ . The elements of  $\mathfrak{A}$  are the  $\equiv$ -equivalence classes of terms. It is easy to show that  $\equiv$  is in fact a congruence relation, see [AM94, Lemma 2.1]

for a similar proof. Then the operations on  $\mathfrak{A}$  can be defined in the usual way. It is routine to check that  $\mathfrak{A}$  satisfies the LSORS-axioms.

We will need

$$\mathcal{E} = \{a : \vdash_0 \Rightarrow a\}$$

Note that  $\mathcal{E}$  is a filter, since it is closed upward by (*Cut*), and it is closed under meet by ( $\cdot R$ ) (applied to empty  $T$ ). Furthermore,  $\mathcal{E}$  is closed under  $;$  because of the following. Assume that  $a, b \in \mathcal{E}$ , i.e.,  $\Rightarrow a$  and  $\Rightarrow b$  are derivable. Then applying ( $;$ *R*) with empty  $P$  and  $Q$  gives us

$$\frac{\Rightarrow a \quad \Rightarrow b}{\Rightarrow a ; b}$$

as desired. Also

$$\mathcal{E} \supseteq \{a \setminus b, b / a : a \leq b\}$$

since  $\vdash_0 \Rightarrow a \setminus b$  and  $\vdash_0 \Rightarrow b / a$  for  $a \leq b$ , by (*R*\) and (*R*/).

In addition, for every  $a \in A$  and  $e \in \mathcal{E}$ ,

$$(16) \quad a ; e \geq a \leq e ; a$$

holds in  $\mathfrak{A}$  because of the following. We can apply ( $;$ *R*) with empty  $Q$ :

$$\frac{a \Rightarrow a \quad \Rightarrow e}{a \Rightarrow a ; e}$$

whence  $a \leq a ; e$  by the definition of  $\leq$ . The derivation of  $a \Rightarrow e ; a$  is similar.

We claim that the constructed graph  $G_{\mathfrak{A}}$  of Section 2 is coherent. The initial graph  $G_0$  is easily seen to be coherent, since  $e, e' \in \mathcal{E}$  implies  $e ; e' \in \mathcal{E}$  and  $e ; a \geq a \leq a ; e'$  for every  $a \in A$ . For the successor steps, observe that in the composition case of the construction, we can assume that  $x \neq y$ : for if  $b ; c \in \ell_{\alpha}(x, x) = \mathcal{E}$ , then we have  $b \in \ell_{\alpha}(x, x) = \mathcal{E}$  and  $c \in \ell_{\alpha}(x, x) = \mathcal{E}$ , by Lemma 4.1. Hence we do not have to find witnesses for labels on reflexive edges, and thus we can assume that  $\ell_{\alpha}(y, x) = \emptyset$  when  $b ; c \in \ell_{\alpha}(x, y)$ , see Figure 1. Thus the set of edges of  $G_{\mathfrak{A}}$  with non-empty labels is antisymmetric:

$$\ell_{\mathfrak{A}}(u, v) \neq \emptyset \neq \ell_{\mathfrak{A}}(v, u) \text{ implies } u = v$$

for every  $u$  and  $v$ . Because of the simple structure of the labelled graph, it is easy to check that coherence is preserved during the successor steps.

Thus we can apply Lemma 2.1, whence

$$\text{rep}(a) = \{(u, v) \in U_{\mathfrak{A}} \times U_{\mathfrak{A}} : a \in \ell_{\mathfrak{A}}(u, v)\}$$

is an isomorphism. Thus the *rep*-image  $\mathfrak{B}$  of  $\mathfrak{A}$  is in  $\mathbf{R}(\cdot, ;, \setminus, /)$ .

Now assume  $T = a_1, \dots, a_n$  and  $\vdash_0 T \Rightarrow a$ . Then  $\mathfrak{A} \models a_1 ; \dots ; a_n \leq a$ , whence  $\mathfrak{B} \models a_1 ; \dots ; a_n \leq a$  as desired.

For state semantics we argue as follows. Assume  $\vdash_0 \Rightarrow a$ . Since  $\text{rep}(a) \supseteq \{(u, u) : u \in U_{\mathfrak{A}}\}$  whenever  $a \in \mathcal{E}$ , we have that  $\mathfrak{B} \models_s a$ . This finishes the proof of Theorem 4.3.  $\square$

The cut elimination also shows that the equational theory is decidable. Since every other rule introduces a connective, it is enough to check a bounded number of proof attempts whether at least one of them is successful for a given sequent/equation.

**Corollary 4.4.** *The equational theory of  $R(\cdot, ;, \backslash, /)$  is decidable.*

**4.2. With Commutativity.** Let  $SC_0^c$  be  $SC_0$  augmented with the commutativity axiom (*Com*) of Figure 5.

**Theorem 4.5.** *The derivation system  $SC_0^c$  is strongly complete and sound w.r.t. both the standard and the state semantics for  $R^c(\cdot, ;, \backslash, /)$ . That is, for any set of sequents  $\Gamma$ , non-empty term  $a$  and sequence of terms  $T$ ,*

$$\Gamma \vdash_0^c T \Rightarrow a \text{ iff } \Gamma \models^c T \Rightarrow a$$

where  $\vdash_0^c$  denotes derivability in  $SC_0^c$ , and  $\models^c$  denotes standard semantical consequence in  $R^c(\cdot, ;, \backslash, /)$ . In particular, when  $T$  is the empty sequence, we get

$$\Gamma \vdash_0^c \Rightarrow a \text{ iff } \Gamma \models_s^c a$$

where  $\models_s^c$  denotes semantical consequence in state semantics for  $R^c(\cdot, ;, \backslash, /)$ .

*Proof.* The proof is similar to that of Theorem 4.3. We take the Lindenbaum–Tarski algebra  $\mathfrak{A} = (A, \cdot, ;, \backslash, /)$  of  $SC_0^c$  and show that  $\mathfrak{A} \in R^c(\cdot, ;, \backslash, /)$ . Note that  $\mathfrak{A}$  is commutative:  $x;y = y;x$  holds in  $\mathfrak{A}$ . Thus we can apply Lemma 3.1, whence  $\mathfrak{A}$  is representable.  $\square$

## 5. NON-COMMUTATIVE REPRESENTABLE ALGEBRAS

Next we define a sequent calculus  $SC$  that is equivalent to  $SC_0$  and does not use empty-headed sequents.

The sequent calculus  $SC$  has the derivation rules  $(;L)$ ,  $(;R)$ ,  $(\backslash L)$ ,  $(\backslash R)$ ,  $(/L)$ ,  $(/R)$ , (*Cut*),  $(\cdot L1)$ ,  $(\cdot L2)$ ,  $(\cdot R)$  (see Figure 5) and the following axioms in addition to axiom (*I*).

“Reflexivity”:

$$(Ref1) \quad y \Rightarrow x ; y \quad (Ref2) \quad y \Rightarrow y ; x$$

if  $x$  is a reflexive residuated term.

“Idempotency”:

$$(Ide1) \quad (x \cdot y) \backslash (x \cdot y) \Rightarrow x \cdot y \quad (Ide2) \quad (x \cdot y) / (x \cdot y) \Rightarrow x \cdot y$$

if  $x$  and  $y$  are reflexive residuated terms.

The same restriction applies as in the case of  $LC$ : the sequences  $T$ ,  $U$  and  $V$  are non-empty.

**Lemma 5.1.** *The derivation systems  $SC$  and  $SC_0$  are equivalent: for any term  $a$  and non-empty sequence of terms  $T$ ,*

$$\vdash T \Rightarrow a \text{ iff } \vdash_0 T \Rightarrow a$$

where  $\vdash$  denotes derivability in  $\mathbf{SC}$  and  $\vdash_0$  denotes derivability in  $\mathbf{SC}_0$ .

*Proof.* We showed in the proof of Theorem 4.3 that  $(Ref1)$  and  $(Ref2)$  are derivable in  $\mathbf{SC}_0$ , see (16), since reflexive residuated terms are in  $\mathcal{E}$ . Indeed,  $\Rightarrow x \setminus x$  and  $\Rightarrow x / x$  are derivable in  $\mathbf{SC}_0$  from the axiom  $x \Rightarrow x$  by the application of  $(\setminus R)$  and  $(/R)$ , respectively. Then so are  $\Rightarrow (x \setminus x) \cdot (y / y)$  and other meets of reflexive residuated terms, by  $(\cdot R)$ . Then we have

$$\frac{\Rightarrow (x \setminus x) \cdot (y / y) \quad (x \setminus x) \cdot (y / y) \Rightarrow (x \setminus x) \cdot (y / y)}{[(x \setminus x) \cdot (y / y)] \setminus [(x \setminus x) \cdot (y / y)] \Rightarrow (x \setminus x) \cdot (y / y)}$$

by  $(\setminus L)$ . Other instances of  $(Ide1)$  and  $(Ide2)$  are derived similarly. Thus all the axioms of  $\mathbf{SC}$  are derivable in  $\mathbf{SC}_0$ .

Now assume that  $\vdash_0 T \Rightarrow a$ . We need  $\vdash T \Rightarrow a$ . We will use induction on the number of rules of the form

$$\frac{\dots \Rightarrow x \quad \dots}{U \Rightarrow y}$$

(i.e., where one of the premises is an empty-headed sequent  $\Rightarrow x$ ) applied in the derivation of  $T \Rightarrow a$  in  $\mathbf{SC}_0$ .

If the derivation of  $T \Rightarrow a$  in  $\mathbf{SC}_0$  does not use empty-headed sequents, then the same derivation works in  $\mathbf{SC}$ . Otherwise the derivation uses empty-headed sequents  $\Rightarrow x$ . Such a sequent can be derived in four different ways: from other empty-headed sequents  $\Rightarrow y$  and  $\Rightarrow z$

- by  $(;R)$  in which case  $x = y ; z$  or
- by  $(\cdot R)$  in which case  $x = y \cdot z$ ,

or from a sequent  $y \Rightarrow z$

- by  $(\setminus R)$  in which case  $x = y \setminus z$  or
- by  $(/R)$  in which case  $x = z / y$ ,

for some terms  $y$  and  $z$ . Thus  $x$  is built up from terms  $x_0, \dots, x_n$  of the form  $y \setminus z$  or  $z / y$  with  $y \Rightarrow z$  by using  $;$  and  $\cdot$ . Let  $x'_i$  be  $y \setminus y$  if  $x_i$  is  $y \setminus z$  and  $z / z$  if  $x_i$  is  $z / y$ . Note that  $x'_i \Rightarrow x_i$ , since  $\setminus$  is antimonotone in the second and  $/$  is monotone in the first argument (derivable already in  $\mathbf{LC}$ ). Define  $x' = x'_0 \cdot \dots \cdot x'_n$ . Thus  $x'$  is a meet of reflexive residuated elements. Observe that  $x' \Rightarrow x$  is derivable in  $\mathbf{SC}$ . Indeed, this easily follows from  $x'_i \cdot x'_j \Rightarrow x'_i ; x'_j$  (by “reflexivity”) and  $x'_i \Rightarrow x_i$ . Also  $\Rightarrow x'$  is derivable in  $\mathbf{SC}_0$  by simply exchanging every occurrence of  $(;R)$  by  $(\cdot R)$  in the derivation of  $\Rightarrow x$  from  $\Rightarrow x_i$  (for  $0 \leq i \leq n$ ).

Consider a step in the derivation of  $T \Rightarrow a$  in  $\mathbf{SC}_0$  that uses a premise  $\Rightarrow x$ . One option is that we apply  $(;R)$ , say,

$$\frac{\Rightarrow x \quad Q \Rightarrow y}{Q \Rightarrow x ; y}$$

whence we have the derivation

$$\frac{\Rightarrow x' \quad Q \Rightarrow y}{Q \Rightarrow x' ; y}$$

as well. By the induction hypothesis (IH),  $Q \Rightarrow y$  is derivable in SC. Then so is  $Q \Rightarrow (x' \setminus x') ; y$  by (Ref1). Since  $x'$  is a meet of reflexive residuated terms, we can apply (Ide1), whence we get that  $Q \Rightarrow x' ; y$  is derivable. We noted above that  $x' \Rightarrow x$  is derivable in SC. Hence so is  $Q \Rightarrow x ; y$  as desired.

Another option is that we apply ( $\setminus L$ ), say,

$$\frac{\Rightarrow x \quad U, y, V \Rightarrow z}{U, x \setminus y, V \Rightarrow z}$$

Then we have the derivation

$$\frac{\Rightarrow x' \quad U, y, V \Rightarrow z}{U, x' \setminus y, V \Rightarrow z}$$

as well. By IH,  $U, y, V \Rightarrow z$  is derivable in SC. Then so is  $U, (x' \setminus x') \setminus y, V \Rightarrow z$ , since  $(x' \setminus x') \setminus y \Rightarrow (x' \setminus x') ; [(x' \setminus x') \setminus y] \Rightarrow y$  is derivable by (Ref1) and ( $\setminus L$ ). Since  $x' \setminus x' \Rightarrow x'$  is derivable by (Ide1) and  $\setminus$  is antimonotone in the first argument, we get that  $U, x' \setminus y, V \Rightarrow z$  is derivable. We noted above that  $x' \Rightarrow x$  is derivable in SC. Hence so is  $U, x \setminus y, V \Rightarrow z$ , by applying the antimonotonicity of  $\setminus$  in its first argument again. The case when we apply ( $/L$ ) is completely symmetric.

Thus we managed to “replace” the rule used in the derivation of  $T \Rightarrow a$  in  $\text{SC}_0$  by a derivation in SC. It follows that the whole derivation of  $T \Rightarrow a$  in  $\text{SC}_0$  can be “translated” to a derivation on SC.  $\square$

**Theorem 5.2.** *The equational theory of  $\text{R}(\cdot, ;, \setminus, /)$  is finitely axiomatized by  $\text{Ax}(\cdot, ;, \setminus, /)$ .*

*Proof.* By Theorem 4.3 and Lemma 5.1, SC is a weakly complete and sound derivation system for  $\text{R}(\cdot, ;, \setminus, /)$ -validity. Similarly to the equivalence of the algebraic and sequent formalizations of the Lambek calculus, see [La58, Section 8], one can show that SC is equivalent to the LSORS-axioms augmented with axioms (12) and (13), which is in turn equivalent to  $\text{Ax}(\cdot, ;, \setminus, /)$ .  $\square$

**Remark 5.3.** The reader may wonder whether there is a finite axiomatization for the quasi-variety of representable algebras  $\text{R}(\cdot, ;, \setminus, /)$ . The problem with representing an arbitrary algebra  $\mathfrak{C}$  satisfying the axioms is as follows. Assume that  $a \setminus a \leq b ; c$  in  $\mathfrak{C}$  and we are in a step-by-step construction dealing with composition for  $a \setminus a \in \ell_\alpha(u, u)$ . Then we need  $v$  such that  $b \in \ell_{\alpha+1}(u, v)$  and  $c \in \ell_{\alpha+1}(v, u)$ . These labels are not difficult to find, but we need an appropriate label for  $(v, v)$  as well. The label  $\ell_{\alpha+1}(v, v)$  should include  $c ; b$  and all reflexive residuated terms, and hence their meets as well. There are valid quasi-equations that guarantee the existence of suitable labels, see below, but we conjecture that there is no finite equational base for all these quasi-equations.

Consider the following quasi-equations  $q_n$  for  $n \in \omega$ :

$$(17) \quad a \setminus a \leq b ; c \text{ implies } d \leq d ; (b ; [(c ; b) \cdot (a \setminus a)]^n ; c)$$



We claim that for every  $n \geq 1$ ,  $R(\cdot, ;, \setminus) \models q_n$ . Let  $\mathfrak{C} \in R(\cdot, ;, \setminus)$  be an algebra represented on a base set  $U$ . Assume that  $(u, v) \in d$ . Since  $a \setminus a$  contains the identity on  $U$ , we have  $(v, v) \in a \setminus a$ . By  $a \setminus a \leq b ; c$ , we get  $(v, w) \in b$  and  $(w, v) \in c$  for some  $w \in U$ . Also,  $(w, w) \in a \setminus a$ . Then  $(w, w) \in [(c ; b) \cdot (a \setminus a)]^n$  for every  $n \geq 1$ . Hence  $(v, v) \in b ; [(c ; b) \cdot (a \setminus a)]^n ; c$ , whence  $(u, v) \in d ; (b ; [(c ; b) \cdot (a \setminus a)]^n ; c)$  as desired.

**Conjecture 5.4.**

- (1) *The set  $\{q_n : 1 \leq n \in \omega\}$  is “independent”.*
- (2) *The representation classes  $R(\cdot, ;, \setminus)$  and  $R(\cdot, ;, \setminus, /)$  are not finitely axiomatizable.*

## 6. STATE SEMANTICS

Next we consider the state semantics for  $(\cdot, ;, \setminus, /)$ -expressions. Recall that we defined this in (7) as

$$\mathfrak{C} \models_s t \text{ iff } \mathfrak{C} \models 1' \leq t$$

for every term  $t$  and algebra  $\mathfrak{C}$ .

Next we define term formalisms, see Figure 6. We use the convention that  $x ; \epsilon = \epsilon ; x = x \setminus \epsilon = \epsilon \setminus x = x / \epsilon = \epsilon / x = x$  if  $\epsilon$  is the empty term. The idea, roughly, is that we replace every  $\Rightarrow$  in the sequent axioms and rules of  $SC_0$  by  $\setminus$  in the term formalism. We do not need  $(;L)$  and  $(;R)$  but we have to state associativity of  $;$  explicitly as axioms.

**6.1. Without Commutativity.** The term derivation system  $TC$  is defined by the axioms  $(I')$ ,  $(A1')$  and  $(A2')$  and derivation rules  $(\setminus 1')$ ,  $(\setminus 2')$ ,  $(/1')$ ,  $(/2')$ ,  $(\cdot L1')$ ,  $(\cdot L2')$ ,  $(\cdot R')$  and  $(Cut')$  with the restriction that the terms on the right of the main  $\setminus$  and  $x$  in  $(\cdot L1')$  and  $(\cdot L2')$  are not empty.

Similarly to the equivalence of the algebraic and sequent formalizations of the Lambek calculus, see [La58, Section 8], one can establish the following.

**Claim 6.1.** *The derivation systems  $TC$  and  $SC_0$  are equivalent:*

$$\vdash \Rightarrow x \text{ in } SC_0 \text{ iff } \vdash x \text{ in } TC$$

for every term  $x$ .

Then the following is a straightforward consequence of Theorem 4.3.

**Corollary 6.2.** *The term calculus  $TC$  is weakly complete and sound w.r.t. state semantics for  $R(\cdot, ;, \setminus, /)$ : for every term  $t$ ,*

$$\vdash t \text{ iff } \models_s t$$

where  $\vdash$  denotes derivability in  $TC$  and  $\models_s$  denotes state validity in  $R(\cdot, ;, \setminus, /)$ .

<b>Axioms:</b>			
$(I')$	$x \setminus x$		
$(Com')$	$(x; y) \setminus (y; x)$		
$(Den')$	$x \setminus (x; x)$		
$(A1')$	$((x; y); z) \setminus (x; (y; z))$		
$(A2')$	$(x; (y; z)) \setminus ((x; y); z)$		
 <b>Derivation rules:</b>			
$(\setminus L')$	$\frac{y \setminus (x \setminus z)}{(x; y) \setminus z}$	$(\setminus R')$	$\frac{(x; y) \setminus z}{y \setminus (x \setminus z)}$
$(/L')$	$\frac{x \setminus (z / y)}{(x; y) \setminus z}$	$(/R')$	$\frac{(x; y) \setminus z}{x \setminus (z / y)}$
$(\cdot L1')$	$\frac{x \setminus z}{(x \cdot y) \setminus z}$	$(\cdot L2')$	$\frac{x \setminus z}{(y \cdot x) \setminus z}$
$(\cdot R')$	$\frac{x \setminus y \quad x \setminus z}{x \setminus (y \cdot z)}$		
$(Cut')$	$\frac{x \setminus y \quad y \setminus z}{x \setminus z}$		

FIGURE 6. Term formalism

**6.2. The Commutative Case.** We already noted that the interpretations of the residuals coincide in commutative algebras. That is why we restrict the language to  $(\cdot, ;, \setminus)$ . The derivation system  $\text{TC}^c$  is given by the axioms  $(I')$ ,  $(A1')$ ,  $(A2')$  and  $(Com')$ , and derivation rules  $(\setminus L')$ ,  $(\setminus R')$ ,  $(\cdot L1')$ ,  $(\cdot L2')$ ,  $(\cdot R')$  and  $(Cut')$  with the restriction that  $x$  in  $(\cdot L1')$ ,  $(\cdot L2')$  and the terms on the right of the main  $\setminus$  cannot be empty.

Similarly to the non-commutative case, we can apply the corresponding completeness result for the sequent formalism, Theorem 4.5.

**Corollary 6.3.** *The derivation system  $\text{TC}^c$  is strongly sound and complete w.r.t. state semantics for  $\text{R}^c(\cdot, ;, \setminus)$ :*

$$\Gamma \vdash^c t \text{ iff } \Gamma \models_s^c t$$

for any set  $\Gamma$  of terms and term  $t$ , where  $\vdash^c$  denotes derivability in  $\text{TC}^c$  and  $\models_s^c$  denotes consequence in the state semantics for  $\text{R}^c(\cdot, ;, \setminus)$ .

Let  $\text{TC}^{cd}$  be the derivation system given by augmenting  $\text{TC}^c$  with the density axiom  $(Den')$ , and  $\text{R}^{cd}(\cdot, ;, \setminus)$  be the dense subclass of  $\text{R}^c(\cdot, ;, \setminus)$ . The reader should not have any problem in establishing the following.

**Corollary 6.4.** *The derivation system  $\text{TC}^{cd}$  is strongly sound and complete w.r.t. state semantics for  $\mathbf{R}^{cd}(\cdot, ;, \backslash)$ .*

## 7. CONCLUSIONS

We have seen that (state) validities are finitely axiomatizable for  $\mathbf{R}^{cd}(\cdot, ;, \backslash)$  and  $\mathbf{R}(\cdot, ;, \backslash, /)$ . The reader may wonder whether similar results could be achieved with join  $+$  (interpreted as union) instead of meet. In this case no finite axiomatization is possible. (Below  $0$ ,  $1$  and  $\smile$  denote the bottom and top elements and relation converse, respectively.)

**Theorem 7.1.** *Let  $\{+, ;, \backslash, /\} \subseteq \Lambda \subseteq \{+, ;, \backslash, /, \smile, 0, 1', 1\}$ . Then state validities for  $\mathbf{R}^{cd}(\Lambda)$  and  $\mathbf{R}(\Lambda)$  are not finitely axiomatizable (by axioms and rules that are state valid).*

*Proof.* The heart of the proof is the following [AMN12, Theorem 3.2].

**Theorem 7.2.** *Let  $\{+, ;, \backslash, /\} \subseteq \Lambda \subseteq \{+, ;, \backslash, /, \smile, 0, 1', 1\}$ . The equational theory of  $\mathbf{R}(\Lambda)$  is not finitely axiomatizable.*

*Moreover, there is no first-order logic formula valid in  $\mathbf{R}(+, ;, \backslash, /, \smile, 0, 1', 1)$  which implies all the equations valid in  $\mathbf{R}(\Lambda)$ .*

In the proof of Theorem 7.2, for every  $n \in \omega$ , we had a  $\{+, ;, \backslash, /, \smile, 0, 1', 1\}$ -algebra  $\mathfrak{A}_n$  and  $\{+, ;, \backslash, /\}$ -terms  $\tau_n$  and  $\sigma_n$  such that

- (1)  $\mathfrak{A}_n$  is not representable, i.e., it is not isomorphic to a family of relations,
- (2) any non-trivial ultraproduct  $\mathfrak{A}$  of  $(\mathfrak{A}_n : n \in \omega)$  is representable,
- (3)  $\tau_n \leq \sigma_n$  fails in  $\mathfrak{A}_n$ ,
- (4)  $\tau_n \leq \sigma_n$  is valid in representable algebras.

From these facts Theorem 7.2 easily follows.<sup>4</sup>

Relation composition is defined so that commutativity and density hold in  $\mathfrak{A}_n$  (hence the two residuals coincide), and  $\mathfrak{A} \in \mathbf{R}^{cd}(+, ;, \backslash, /, \smile, 0, 1', 1)$ . Thus the equational theory of  $\mathbf{R}^{cd}(\Lambda)$  is not finitely axiomatizable.

Finally, using the displayed formulas (6) and (7), for every  $\mathfrak{C} \in \mathbf{R}^{cd}(\Lambda)$ , we have  $\mathfrak{C} \models \tau_n \leq \sigma_n$  iff  $\mathfrak{C} \models_s \tau_n \backslash \sigma_n$ . Thus the theory

$$\{\rho : \models_s^{(cd)} \rho\}$$

is not finitely axiomatizable, finishing the proof of Theorem 7.1.  $\square$

We conclude with the following problem.

**Problem 7.3.** *Are the validities for  $\mathbf{R}^{cd}(\cdot, ;, \backslash, 1')$  and  $\mathbf{R}(\cdot, ;, \backslash, /, 1')$  finitely axiomatizable?*

<sup>4</sup>In passing we note that the sequence of algebras  $(\mathfrak{A}_n : n \in \omega)$  were used in [AM11]. We proved that the quasi-variety  $\mathbf{R}(+, ;)$  is not finitely axiomatizable by establishing items (1)–(2) above and showing items (3)–(4) for quasi-equations instead of equations  $\tau_n \leq \sigma_n$ . The novelty in [AMN12] is that the quasi-equations can be replaced by equations provided that we include the residuals into the signature.

Note that in this case we can have an explicit use of the identity constant  $1'$ , while we only had an implicit use of  $1'$  via state semantics. But if we have ordered monoids instead of semigroups, then additional problems arise in the quest for axiomatization, see [HM07]. In the light of Conjecture 5.4, the answer is probably negative.

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#### REFERENCES

- [ABD92] A.R. Anderson, N.D. Belnap and J.M. Dunn, *Entailment. The Logic of Relevance and Necessity. Vol. II.*, Princeton University Press. 1992.
- [AM94] H. Andr eka and Sz. Mikul as, “Lambek calculus and its relational semantics: completeness and incompleteness”, *Journal of Logic, Language and Information*, 3:1–37, 1994.
- [AM11] H. Andr eka and Sz. Mikul as, “Axiomatizability of positive algebras of binary relations”, *Algebra Universalis*, 66:7–34, 2011.
- [AMN12] H. Andr eka, Sz. Mikul as and I. N emeti, “Residuated Kleene algebras”, in R.L. Constable and A. Silva (eds.), *Kozen Festschrift*, pages 1–11, Springer-Verlag, 2012.
- [BDM09] K. Bimb o, J.M. Dunn and R.D. Maddux, “Relevance logic and relation algebras”, *Review of Symbolic Logic*, 2(1):102–131, 2009.
- [HM07] R. Hirsch and Sz. Mikul as, “Representable semilattice-ordered monoids”, *Algebra Universalis*, 57:333–370, 2007.
- [HM11] R. Hirsch and Sz. Mikul as, “Positive fragments of relevance logic and algebras of binary relations”, *Review of Symbolic Logic*, 4(1):81–105, 2011.
- [La58] J. Lambek, “The mathematics of sentence structure” *American Mathematical Monthly*, 65:154–170, 1958.
- [Ma10] R.D. Maddux, “Relevance logic and the calculus of relations”, *Review of Symbolic Logic*, 3(01):41–70, 2010.
- [Mik09] Sz. Mikul as, “Algebras of relations and relevance logic”, *Journal of Logic and Computation*, 19:305–321, 2009.
- [Mik11] Sz. Mikul as, “On representable ordered residuated semigroups”, *Logic Journal of the IGPL*, 19(1):233–240, 2011.
- [Pr90] V. Pratt, “Action logic and pure induction”, in J. van Eijck (editor), *Logics in AI: European Workshop JELIA '90*, pages 97–120. Springer-Verlag, 1990.
- [RM73] R. Routley and R.K. Meyer, “The semantics of entailment (I)”, in H. Leblanc (ed.), *Truth, Syntax and Modality*, pages 199–243, North-Holland, 1973.

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