Using the Bootstrap to Test for Symmetry Under Unknown Dependence

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Abstract

This paper considers tests for symmetry of the one-dimensional marginal distribution of fractionally integrated processes. The tests are implemented by using an autoregressive sieve bootstrap approximation to the null sampling distribution of the relevant test statistics. The sieve bootstrap allows inference on symmetry to be carried out without knowledge of either the memory parameter of the data or of the appropriate norming factor for the test statistic and its asymptotic distribution. The small-sample properties of the proposed method are examined by means of Monte Carlo experiments, and applications to real-world data are also presented.

Key Words: Fractionally integrated process; Sieve bootstrap; Symmetry.

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1 Introduction

Testing for symmetry of a probability distribution about a specified or unspecified center is a problem that has attracted considerable attention. This is not perhaps surprising in view of the fact that many nonparametric and robust statistical procedures rely to a certain extent on the assumption of symmetry. Symmetry, or lack of it, is also important in terms of the definition and estimation of location since the center of symmetry of a distribution is its only natural location parameter. From the viewpoint of statistical model building, a check for symmetry is a useful addition to existing diagnostics since the absence of such a distributional characteristic would exclude certain families of parametric models (e.g., linear ARMA models with independent and symmetrically distributed noise) from the set of valid candidate models.

In the economics and finance literature, symmetry is an implicit or explicit assumption in some commonly used models, including, for example, many rational expectations models, the Sharpe–Lintner capital asset pricing model, and the Black–Scholes option pricing model. With many empirical studies reporting significant evidence of asymmetry in the distributions of financial and economic data, the adequacy of such models and their data coherency have become issues of concern, and extensions/modifications have been proposed to incorporate asymmetry in the models. Another prominent example from macroeconomics in which symmetry is a central issue relates to the question of whether real economic variables behave asymmetrically over the phases of the business cycle. Following the influential work of DeLong and Summers (1986), a large literature has evolved in which different types of cyclical asymmetry are identified via the distributional asymmetry of relevant economic variables. In light of favorable empirical evidence for cyclical asymmetry, economic models have been developed which are capable of generating asymmetric behavior endogenously. As Lee (2007) aptly notes, therefore, "...an appropriate test for distribution symmetry is useful not only in understanding distributional characteristics of data but also in evaluating economic hypotheses and models."

Although most of the voluminous work on the subject of testing for symmetry has focused on the case of independent, identically distributed (i.i.d.) data, a small number of studies have discussed tests which are robust to deviations from the assumption of independence; relevant references include Chen, Chou, and Kuan (2000), Psaradakis (2003, 2008), Bai and Ng (2005), Delgado and Escanciano (2007), Lee (2007), and Racine and Maasoumi (2007). These studies rely on large-sample results obtained under short-range dependence conditions, which typically imply that the autocovariances of the data decay to zero, as the lag parameter tends to infinity, sufficiently fast to be absolutely summable. It has long been recognized, however, that such dependence conditions may not accord well with the slowly decaying autocovariances that are observed in many time series.

Our aim in the present paper is to discuss tests for symmetry which are valid in the presence of not only short-range dependence but also long-range dependence and antipersistence. The defining feature of stochastic processes with such dependence structures is that their autocovariances decay to zero as a power of the lag parameter and, in the case of long-range dependence, slowly enough to be non-summable. Stochastic models exhibiting long-range dependence are not only of theoretical interest but have also been found to be useful for modelling real-world data occurring in fields as diverse as economics, geophysics, hydrology, meteorology, and telecommunications; see Doukhan, Oppenheim, and Taqqu (2003) for an extensive review.

The symmetry tests we consider exploit the fact that an odd function of centered data, which we denote by ψ, has zero expectation under distributional symmetry.[1] The tests are applied to fractionally integrated processes that may exhibit short-range dependence, long-range dependence or antipersistence, depending on the value of a memory/dependence parameter, which we denote by d. Although the test statistics are simple linear statistics, inference is complicated by the fact that their asymptotic null distributions depend on certain properties of ψ and on the (unknown) value of d, involve nuisance (and difficult to estimate) parameters, and may be non-standard. Moreover, the appropriate norming factors needed to ensure that the test statistics have a nondegenerate asymptotic distribution are also dependent upon ψ and d.

As a practical way of overcoming these obstacles, we propose to use the bootstrap to estimate the sampling distribution of the statistics of interest. Our approach

[1] A similar approach was pursued by Lee (2007) under short-range dependence conditions.

relies on the sieve bootstrap, which is based on the idea of approximating the data-generating mechanism by an autoregressive sieve, that is a sequence of autoregressive models that increase in order as the sample size diverges to infinity (Kreiss (1992); Bühlmann (1997)). The sieve bootstrap delivers tests for symmetry which are easy to implement, asymptotically valid, and require knowledge (or estimation) of neither the value of the memory parameter of the data nor of the appropriate norming factor for the test statistic. Furthermore, the resampling scheme is the same under short-range dependence, long-range dependence, and antipersistence.

The remainder of the paper is organized as follows. Section 2 formulates the problem, and introduces the test criteria and stochastic processes of interest. Section 3 describes the sieve bootstrap method for approximating the distribution of the test statistics. Section 4 examines the small-sample properties of the tests by means of simulation experiments. Section 5 illustrates the practical use of the proposed methods by presenting applications to realized stock return volatility and output growth. Section 6 summarizes and concludes.

2 Problem, Assumptions and Test Statistics

Let $\mathbf{X} := \{X_t\}_{t\in\mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables with $\mathsf{E}(X_0) = \mu$ for some $\mu \in \mathbb{R}$. The problem of interest is to test the hypothesis that the one-dimensional marginal distribution of $\mathbf{X}$ is symmetric about μ, that is,

$$\mathcal{L}(X_0 - \mu) = \mathcal{L}(\mu - X_0), \tag{1}$$

where $\mathcal{L}(V)$ denotes the distribution of a random variable V.

It is easy to see that, if (1) holds, then $\zeta_\psi := \mathsf{E}[\psi(X_0 - \mu)] = 0$ for any real, odd, Borel function ψ on $\mathbb{R}$ with $\mathsf{E}[|\psi(X_0 - \mu)|] < \infty$. Hence, ζ_ψ may be used as an index of symmetry of the distribution of X_0. Examples of functions ψ which have been used in the literature to construct tests for symmetry include $\psi(x) = x^{2b+1}$ for some $b \in \mathbb{N}$ (Gupta (1967); Bai and Ng (2005)), $\psi(x) = ax/(1 + a^2x^2)$ for some $a > 0$ (Chen, Chou, and Kuan (2000)), $\psi(x) = \arctan x$ (Premaratne and Bera (2005)), and $\psi(x) = \operatorname{sgn} x$ (Gastwirth (1971)).

A natural empirical analogue of ζ_ψ based on a sample $\mathbf{X}^n := (X_1, \ldots, X_n)$ of $n \in \mathbb{N}$ consecutive observations from $\mathbf{X}$ is

$$S_\psi := n^{-1} \sum_{t=1}^{n} \psi(X_t - \bar{X}), \tag{2}$$

where $\bar{X} := n^{-1} \sum_{t=1}^{n} X_t$. Values of S_ψ near zero would be consistent with the symmetry hypothesis (1). In the case of testing for symmetry about a specified center $\mu = \mu_0$, the summands in (2) may be replaced by $\psi(X_t - \mu_0)$.

With regard to the class of stochastic processes considered in our analysis, it will be maintained throughout that $\mathbf{X}$ is a fractionally integrated process with memory (or fractional differencing) parameter d. More precisely:

(**A.1**) $\mathbf{X}$ satisfies the equations

$$X_t - \mu = (1 - B)^{-d} Y_t, \qquad t \in \mathbb{Z}, \tag{3}$$

for some fixed $d \in (-\frac{1}{2}, \frac{1}{2})$, where $\mathbf{Y} := \{Y_t\}_{t \in \mathbb{Z}}$ is a sequence of zero-mean random variables and B denotes the backward shift operator ($BY_t := Y_{t-1}$).

As usual, the fractional differencing operator $(1 - B)^{-d}$ in (3) is defined by means of a Maclaurin series expansion,

$$(1 - B)^{-d} := 1 + \sum_{j=1}^{\infty} \frac{\Gamma(j + d)}{\Gamma(d)\Gamma(j + 1)} B^j,$$

where Γ denotes the gamma function (with the convention $1/\Gamma(0) = 0$). It is further assumed that:

(**A.2**) $\mathbf{Y}$ satisfies the equations

$$Y_t = \sum_{j=0}^{\infty} \pi_j \xi_{t-j}, \qquad t \in \mathbb{Z}, \tag{4}$$

where $\{\pi_j\}_{j \in \mathbb{N}_0}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, is an absolutely summable sequence of real numbers (with $\pi_0 := 1$) such that $\pi_\infty := \sum_{j=0}^{\infty} \pi_j \neq 0$, and $\{\xi_t\}_{t \in \mathbb{Z}}$ is a strictly stationary and ergodic sequence of real-valued random variables with a finite fourth absolute moment such that $\mathsf{E}(\xi_t | \mathcal{F}_{t-1}) = 0$ and $\mathsf{E}(\xi_t^2 | \mathcal{F}_{t-1}) = \sigma^2 > 0$ for all t, $\mathcal{F}_{t-1}$ being the sigma-algebra generated by $\{\xi_{t-i}\}_{i \in \mathbb{N}}$.

Under (A.1)–(A.2), $\mathbf{X}$ is a strictly stationary and ergodic process that admits the linear representation

$$X_t - \mu = \sum_{j=0}^{\infty} \theta_j \xi_{t-j}, \qquad t \in \mathbb{Z}, \tag{5}$$

for some square-summable sequence of real numbers $\{\theta_j\}_{j\in\mathbb{N}_0}$ (with $\theta_0 := 1$) satisfying $\theta_j = \{\pi_\infty/\Gamma(d)\}j^{d-1}\{1 + o(1)\}$ ($j \to \infty$) for $d \neq 0$. Putting $\gamma_h := \mathrm{Cov}(X_0, X_h)$, $h \in \mathbb{Z}$, we have $\gamma_h = C_d\,|h|^{2d-1}\,\{1+o(1)\}$ ($|h| \to \infty$) for $d \neq 0$ and some $C_d \in \mathbb{R}\backslash\{0\}$. Hence, $\sum_{h=-\infty}^{\infty} \gamma_h = \infty$ for $d \in (0, \frac{1}{2})$ and $\mathbf{X}$ exhibits long-range dependence. When $d \in (-\frac{1}{2}, 0)$, $\mathbf{X}$ is said to be antipersistent and $\sum_{h=-\infty}^{\infty} \gamma_h = 0$. If $d = 0$, then $\mathbf{X}$ is short-range dependent with $\sum_{h=-\infty}^{\infty} \gamma_h = \sigma^2\pi_\infty^2$.

The class of stochastic processes defined by (3)–(4) is rich enough to include many processes with slowly decaying autocovariances, and is arguably the most important class of long-range dependent and antipersistent processes. A prominent example are autoregressive fractionally integrated moving average (ARFIMA) processes, obtained with $\mathbf{Y}$ in (3) being a causal ARMA process. We note, however, that rates of decay for the weighting sequence $\{\pi_j\}$ in (4) much slower than the exponential rate that is characteristic of ARMA processes are also permitted (e.g., $\pi_j = j^{-\lambda}L(j)$ for some $\lambda > 1$ and a real function L on $[1, \infty)$ that is slowly varying at infinity in Karamata's sense). It is easy to see that, if $\{\xi_t\}$ is an i.i.d. sequence, then symmetry of ξ_0 (i.e., $\mathcal{L}(\xi_0) = \mathcal{L}(-\xi_0)$) implies (1).

The asymptotic behavior of S_ψ for fractionally integrated processes depends on the value of d and on the properties of ψ and $\mathbf{Y}$, and may be determined on a case-by-case basis, under additional assumptions about ψ and $\mathbf{Y}$, by relying on the results in Ho and Hsing (1997), Koul and Surgailis (1997), and Ho (2002), inter alia. The asymptotic null distribution of S_ψ may be Gaussian or non-Gaussian, depending on the memory parameter of $\mathbf{X}$ and the properties of ψ. To appreciate why, note that, under an i.i.d. assumption about $\{\xi_t\}$, appropriate regularity conditions on the distribution of ξ_0, and smoothness conditions on ψ, S_ψ admits the representation

$$\begin{aligned} S_\psi &= n^{-1}\sum_{t=1}^{n}\psi(X_t-\mu)+\sum_{k=1}^{m}(1/k!)\Delta_\psi^{(0,k)}(0,\mu)(\bar{X}-\mu)^k \\ &\quad+\sum_{r=1}^{m-1}n^{-1}U_r\sum_{s=1}^{m-r}(1/s!)\Delta_\psi^{(r,s)}(0,\mu)(\bar{X}-\mu)^s+R_\psi, \end{aligned} \tag{6}$$

for some $m\in\mathbb{N}$, where

$$\Delta_\psi^{(r,s)}(x_0,y_0):=\left.\frac{\partial^{r+s}\mathsf{E}[\psi(X_0+x-y)]}{\partial x^r\partial y^s}\right|_{x=x_0,y=y_0},\qquad r,s\in\mathbb{N}_0,$$

$$U_r:=\sum_{t=1}^{n}\sum_{0\leqslant j_1<\cdots<j_r<\infty}\prod_{s=1}^{r}\theta_{j_s}\xi_{t-j_s},\qquad r\in\mathbb{N},$$

$U_0:=n$, and R_ψ is a remainder which converges in probability to zero, as $n\to\infty$, at a rate that depends on d and m (see Ho (2002)). In general, the first three terms in the right-hand side of (6) all contribute to the asymptotic distribution of S_ψ, and they may or may not be asymptotically normal when $\mathbf{X}$ is long-range dependent. In addition, the norming factor needed to produce a nondegenerate weak limit for the distribution of S_ψ is also dependent upon d and m. For example, under suitable moment conditions on ξ_0 and some growth conditions on ψ, the centered partial sum $T_\psi:=\sum_{t=1}^{n}\{\psi(X_t-\mu)-\zeta_\psi\}$ is asymptotically normal under $n^{-1/2}$ norming (with variance $0\leqslant\sum_{h=-\infty}^{\infty}\operatorname{Cov}(\psi(X_0-\mu),\psi(X_h-\mu))<\infty$), when either $d\in(-\frac{1}{2},0]$, or $d\in(0,\frac{1}{2})$ and $\kappa(1-2d)>1$, κ being the smallest positive integer for which $\Delta_\psi^{(\kappa,0)}(0,\mu)$ exists in $\mathbb{R}\backslash\{0\}$. If, on the other hand, $d\in(0,\frac{1}{2})$ and $\kappa(1-2d)<1$, then $n^{-1+(\kappa/2)(1-2d)}T_\psi$ converges in distribution to $(\Lambda_d^\kappa/\kappa!)\Delta_\psi^{(\kappa,0)}(0,\mu)Z_{d,\kappa}$, as $n\to\infty$, where $\Lambda_d:=\sigma\pi_\infty/\Gamma(d)$ and, for any $d\in(0,\frac{1}{2})$ and $k\in\mathbb{N}$ with $k<(1-2d)^{-1}$, the random variable $Z_{d,k}$ is defined as the k-tuple Wiener–Itô integral

$$Z_{d,k}:=\int_{\mathbb{R}^k}\left\{\int_0^1\prod_{i=1}^{k}(\max\{0,v-x_i\})^{d-1}\mathrm{d}v\right\}\mathsf{W}(\mathrm{d}x_1)\cdots\mathsf{W}(\mathrm{d}x_k),$$

W being a real-valued Gaussian random measure on $\mathbb{R}$ with Lebesgue control measure (Ho and Hsing (1997); Koul and Surgailis (1997)). Similarly, for any $r\in\mathbb{N}$ such that $|\xi_0|^{2r}$ is integrable, $n^{-1/2}U_r$ is asymptotically normal, as $n\to\infty$, when either $d\in(-\frac{1}{2},0]$, or $d\in(0,\frac{1}{2})$ and $r(1-2d)>1$; if $d\in(0,\frac{1}{2})$ and $r(1-2d)<1$, then the distribution of $n^{-1+(r/2)(1-2d)}U_r$ converges weakly to that of $(\Lambda_d^r/r!)Z_{d,r}$ as $n \to \infty$ (Surgailis (1982); Avram and Taqqu (1987)). Note that $Z_{d,k}$ is Gaussian for $k = 1$ and non-Gaussian for $k \geqslant 2$.

It is worth stressing that these difficulties also arise when testing for symmetry about a specified center $\mu = \mu_0$, even though S_ψ does not involve any estimated parameters in this case. For example, under suitable regularity conditions on ψ and $\mathbf{Y}$ (see, e.g., Ho and Hsing (1997)), $n^{\bar{\kappa}(1-2d)/2}S_\psi$ converges in distribution to $(\Lambda_d^{\bar{\kappa}}/\bar{\kappa}!)\Delta_\psi^{(\bar{\kappa},0)}(0,\mu_0)Z_{d,\bar{\kappa}}$, as $n \to \infty$ under (1), whenever $d \in (0, \frac{1}{2})$ and $\bar{\kappa}(1-2d) < 1$, $\bar{\kappa}$ being the smallest positive integer for which $\Delta_\psi^{(\bar{\kappa},0)}(0,\mu_0)$ exists in $\mathbb{R}\backslash\{0\}$.

We also note that a test of the symmetry hypothesis (1) may be based on a studentized statistic of the form $w_\psi^{-1}S_\psi$, where w_ψ^2 is a suitable estimator of the variance of S_ψ. This approach is not, however, without difficulty, even when the center of symmetry is specified, due to the fact that construction of an appropriate estimator w_ψ^2 is far from straightforward in our setting. Assuming ψ and $\mathbf{X}$ are such that the (generally nonlinear) process $\{\psi(X_t - \mu)\}$ is fourth-order stationary with autocovariances which are either absolutely summable or asymptotically proportional to $|h|^{-\beta_\psi}$, as the lag parameter $|h|$ tends to infinity, for some $\beta_\psi \in (0,2)\backslash\{1\}$, one may consider estimators w_ψ^2 of the type discussed in Berkes, Horváth, Kokoszka, and Shao (2005) and Abadir, Distaso, and Giraitis (2009). However, such estimators rely on knowledge or estimation of β_ψ (which is determined by ψ and d) and/or are sensitive to the selection of a bandwidth parameter (the optimal choice of which depends on β_ψ). For these reasons, our analysis will be based on the non-studentized statistic S_ψ.

In the next section of the paper, we discuss how the sieve bootstrap may be used as a practical way of overcoming the difficulties associated with the dependence of the behavior of S_ψ on d, m and κ. The principal advantage of the sieve bootstrap is that it can used to draw statistical inferences from S_ψ to ζ_ψ without knowledge of the memory parameter of $\mathbf{X}$ or of the properties of ψ, and is valid for all $d \in (-\frac{1}{2}, \frac{1}{2})$. Moreover, because bootstrap approximations are constructed from replicates of S_ψ, there is no need to analytically derive (nor to make assumptions about) the appropriate norming factor for S_ψ and its asymptotic null distribution.

3 Sieve Bootstrap Inference

The sieve bootstrap scheme is motivated by the observation that, if $\mathbf{Y}$ is invertible, in the sense that $\sum_{j=0}^{\infty} \pi_j z^j \neq 0$ for all complex z with $|z| \leqslant 1$, then $\mathbf{X}$ admits the autoregressive representation

$$\sum_{j=0}^{\infty} \phi_j (X_{t-j} - \mu) = \xi_t, \qquad t \in \mathbb{Z}, \tag{7}$$

for some square-summable sequence of real numbers $\{\phi_j\}_{j \in \mathbb{N}_0}$ (with $\phi_0 := 1$) satisfying $\phi_j = \{\pi_\infty \Gamma(-d)\}^{-1} j^{-d-1} \{1 + o(1)\}$ $(j \to \infty)$ for $d \neq 0$. However, as noted in Poskitt (2007), the representation (7) provides a meaningful approximation even if $\mathbf{Y}$ is not invertible. In the latter case, $\sum_{j=0}^{\infty} \phi_j z^j$ may be viewed as arising from the limit of $\sum_{j=0}^{p} \phi_{p,j} z^j$ as $p \to \infty$ $(\phi_{p,0} := 1)$, where, for a fixed $p \in \mathbb{N}$, $(-\phi_{p,1}, \ldots, -\phi_{p,p})$ are the coefficients of the best linear predictor of X_0 based on $(X_{-1}, \ldots, X_{-p})$. Under (A.1)–(A.2), the finite predictor coefficients of $\mathbf{X}$ are uniquely determined as the solution of the set of equations $\sum_{j=0}^{p} \phi_{p,j} \gamma_{k-j} = 0$ $(k = 1, \ldots, p)$ (cf. Brockwell and Davis (1991, Corollary 5.1.1)), and satisfy $\sum_{j=0}^{p} \phi_{p,j} z^j \neq 0$ for $|z| \leqslant 1$. The idea is to approximate (7) by a finite-order autoregressive model and use this as the basis of a residual-based resampling scheme. If the order of the autoregressive approximation is allowed to increase with the sample size at an appropriate rate, the distribution of the process in (7) will be matched asymptotically (cf. Kreiss (1992); Bühlmann (1997); Kapetanios and Psaradakis (2006); Poskitt (2008)).

To formalize our sieve bootstrap scheme, for some $p = p(n) \in \mathbb{N}$ with $p \ll n$, let $(\hat{\phi}_{p,1}, \ldots, \hat{\phi}_{p,p})$ and $\hat{\sigma}_p^2$ be estimators (based on $\mathbf{X}^n$) of the coefficients and noise variance, respectively, of a pth-order autoregressive model for $X_t - \bar{X}$. Further, let $\{\hat{\xi}_{p,t}\}$ be the corresponding residuals, i.e.,

$$\hat{\xi}_{p,t} := \sum_{j=0}^{p} \hat{\phi}_{p,j} (X_{t-j} - \bar{X}), \qquad t = p+1, \ldots, n,$$

with $\hat{\phi}_{p,0} := 1$. For any Borel subset A of $\mathbb{R}$, put

$$\hat{Q}_p(A) := \{2(n-p)\}^{-1} \sum_{t=p+1}^{n} \left\{ \mathbf{1}_A(\ell_p^{-1} \hat{\xi}_{p,t}) + \mathbf{1}_A(-\ell_p^{-1} \hat{\xi}_{p,t}) \right\},$$

where $\ell_p^2 := (n-p)^{-1}\sum_{t=p+1}^{n}(\hat{\xi}_{p,t}-\bar{\xi}_p)^2$, $\bar{\xi}_p := (n-p)^{-1}\sum_{t=p+1}^{n}\hat{\xi}_{p,t}$, and $\mathbf{1}_A$ denotes the indicator function of A. Bootstrap replicates $\mathbf{X}_p^* := \{X_{p,t}^*\}_{t\in\mathbb{Z}}$ of $\mathbf{X}$ are then defined via the recursion

$$\sum_{j=0}^{p}\hat{\phi}_{p,j}(X_{p,t-j}^*-\bar{X}) = \hat{\sigma}_p\xi_{p,t}^*, \qquad t\in\mathbb{Z}, \tag{8}$$

where, conditionally on $\mathbf{X}^n$, $\{\xi_{p,t}^*\}_{t\in\mathbb{Z}}$ are i.i.d. random variables with common distribution $\hat{Q}_p$. Finally, the bootstrap analogue S_ψ^* of S_ψ is obtained as

$$S_\psi^* := n^{-1}\sum_{t=1}^{n}\psi(X_{p,t}^*-\bar{X}_p^*),$$

where $\bar{X}_p^* := n^{-1}\sum_{t=1}^{n}X_{p,t}^*$.

The conditional distribution of S_ψ^*, given $\mathbf{X}^n$, constitutes the sieve bootstrap approximation to the null sampling distribution of S_ψ. Note that the empirical distribution $\hat{Q}_p$ is symmetric about zero and, in consequence, the conditional distribution of $X_{p,t}^*$, given $\mathbf{X}^n$, is symmetric with $\bar{X}$ as the center of symmetry. This means that $\mathbf{X}_p^*$ is constructed in a way which reflects the symmetry hypothesis under test even when $\mathbf{X}$ does not satisfy (1). This is important for ensuring that the bootstrap test of the hypothesis (1) has reasonable power against departures from symmetry (cf. Hall and Wilson (1991); Lehmann and Romano (2005, Section 15.6)).

To examine the asymptotic properties of the sieve bootstrap for S_ψ, the following additional assumptions are made:

(**A**.**3**) $(\hat{\phi}_{p,1},\ldots,\hat{\phi}_{p,p})$ and $\hat{\sigma}_p^2$ satisfy the empirical pth-order Yule–Walker equations $\sum_{j=0}^{p}\hat{\phi}_{p,j}\hat{\gamma}_{k-j} = \delta_{0,k}\hat{\sigma}_p^2$ $(k=0,1,\ldots,p)$, where $\delta_{0,k}$ is Kronecker's delta and $\hat{\gamma}_h := n^{-1}\sum_{t=1}^{n-|h|}(X_{t+|h|}-\bar{X})(X_t-\bar{X})$ for $|h|<n$.

(**A**.**4**) $p = p(n)\to\infty$ as $n\to\infty$ so that $p(n) = O\left(\{\log n\}^\tau\right)$ for some $\tau\geqslant 1$.

(**A**.**5**) ψ is continuously differentiable almost everywhere on $\mathbb{R}$.

The proposition that follows (the proof of which appears in the Appendix) establishes strong consistency of the sieve bootstrap approximation to the null distribution of S_ψ under the resampling scheme described earlier, thereby justifying the use of our bootstrap-based inferential procedures. Closeness of two distributions Q_1 and Q_2 on $\mathbb{R}$ having finite second moments is described in terms of their Mallows–Wasserstein distance, which is defined as $\rho(Q_1, Q_2) := \inf\{\mathsf{E}(|V_1 - V_2|^2)\}^{\frac{1}{2}}$, the infimum being taken over all pairs of random variables (V_1, V_2) with $\mathcal{L}(V_1) = Q_1$ and $\mathcal{L}(V_2) = Q_2$. It is well known (see, e.g., Bickel and Freedman (1981, Lemma 8.3)) that ρ metrizes weak convergence together with convergence of second moments. (In what follows, $\mathcal{L}(V|\mathbf{X}^n)$ denotes the conditional distribution of V given $\mathbf{X}^n$).

Proposition 1 *Suppose* (A.1)–(A.5) *and* (1) *hold. Then,* $\rho\left(\mathcal{L}(S_\psi^*|\mathbf{X}^n), \mathcal{L}(S_\psi)\right) \to 0$ *almost surely as* $n \to \infty$*.*

It is worth noting that assumption (A.3) is made for the sake of technical convenience because it ensures that the polynomial $\sum_{j=0}^{p} \hat{\phi}_{p,j} z^j$ has no zeros in the disk $\{z : |z| \leqslant 1\}$. However, the Yule–Walker estimator in (A.3) may be replaced by the least-squares estimator without changing the conclusion of Proposition 1, for the two estimators are asymptotically equivalent under our regularity conditions (cf. Poskitt (2007, Corollary 1)). Similarly, (A.4) is used because of its appealing feature that the requirement on the relative asymptotic rates of p and n does not depend on the (unknown) memory parameter d, but the assertion of Proposition 1 also holds for any choice of p that diverges to infinity with n at the rate $o(\{n/\log n\}^{\frac{1}{2}-\max\{0,d\}})$.

In practice, analytical computation of the bootstrap distribution of S_ψ^* is typically intractable, but an approximation (of any desired accuracy) can be obtained by Monte Carlo simulation. Specifically, if $S_{\psi,1}^*, \ldots, S_{\psi,N}^*$ are N conditionally independent copies of S_ψ^*, obtained by repeating N times the resampling procedure described earlier, then the empirical distribution of $(S_{\psi,1}^*, \ldots, S_{\psi,N}^*)$ serves as an approximation to the bootstrap distribution of S_ψ^*. Hence, the bootstrap P-value for a test which rejects for large values of $|S_\psi|$ is obtained as $P_\psi^* := N^{-1} \sum_{i=1}^{N} \mathbf{1}_{(-\infty,0]}(|S_\psi| - |S_{\psi,i}^*|)$, and the null hypothesis of symmetry is rejected at a given level of significance $\alpha \in (0,1)$ if $P_\psi^* \leqslant \alpha$. Another possibility is to reject symmetry whenever $\bar{P}_\psi^* \leqslant \alpha$, where $\bar{P}_\psi^* := \min\{2N^{-1} \sum_{i=1}^{N} \mathbf{1}_{(-\infty,0]}(S_\psi - S_{\psi,i}^*), 2N^{-1} \sum_{i=1}^{N} \mathbf{1}_{(0,\infty)}(S_\psi - S_{\psi,i}^*)\}$; this corresponds to an equal-tailed (rather than symmetrical) test of nominal level α. As we did not find any significant differences between the properties of tests based on $\bar{P}_\psi^*$ and P_ψ^*, we shall hereafter focus on the latter.

The sieve bootstrap may also be used to construct confidence intervals for ζ_ψ based on S_ψ. For example, for a fixed $\alpha \in (0,1)$, an (approximate) $100(1-\alpha)\%$ two-sided confidence interval for ζ_ψ is obtained as

$$\left[2S_\psi - K^*_\psi(1-\tfrac{\alpha}{2}),\ 2S_\psi - K^*_\psi(\tfrac{\alpha}{2})\right], \tag{9}$$

where K^*_ψ is the quantile function associated with $\mathcal{L}(S^*_\psi|\mathbf{X}^n)$. An approximation to K^*_ψ can be obtained by modifying the bootstrap scheme described earlier so that, conditionally on $\mathbf{X}^n$, each $\xi^*_{p,t}$ in (8) is distributed according to the empirical distribution $\hat{Q}^\dagger_p(A) := (n-p)^{-1}\sum_{t=p+1}^n \mathbf{1}_A(\ell_p^{-1}\hat{\xi}_{p,t} - \ell_p^{-1}\bar{\xi}_p)$ instead of its symmetrized counterpart $\hat{Q}_p$ (this implies that the bootstrap replicates $\mathbf{X}^*_p$ are not constrained to be symmetrically distributed). When (A.1)–(A.5) hold, asymptotic validity of the sieve bootstrap for S_ψ under the resampling scheme based on $\hat{Q}^\dagger_p$ follows from Lemma 1, Theorem 2 and Remark 2 of Poskitt (2008).

In the implementation of the sieve bootstrap in practice, bootstrap replicates may be obtained according to (8) by setting $(X^*_{p,-p+1},\ldots,X^*_{p,0}) = (X_{q-p+1},\ldots,X_q)$, where q is an integer chosen randomly from the set $\{p, p+1, \ldots, n\}$. Another possibility is to set $X^*_{p,-p+1} = \cdots = X^*_{p,0} = \bar{X}$, generate $n + n_0$ replicates for some large positive integer n_0, and then discard the initial n_0 replicates to eliminate start-up effects (this procedure, with $n_0 = 100$, is used in the remainder of the paper).

Another important practical consideration is the choice of the order p of the autoregressive sieve. A data-driven selection procedure may be based on minimization (over a suitable range of values of p) of an objective function of the form

$$\mathcal{C}(p) := \log \hat{\sigma}_p^2 + n^{-1} p f(n), \tag{10}$$

where $f(n)$ is a nondecreasing function of n that determines the strength of the penalty term associated with any given order p. The well-known Akaike information criterion (AIC), Schwarz Bayesian criterion, and Hannan–Quinn criterion are obtained from (10) with $f(n) = 2$, $f(n) = \log n$, and $f(n) = c \log\log n$ $(c > 2)$, respectively. The following proposition (proved in the Appendix) provides the theoretical justification for the use of order selection criteria such as $\mathcal{C}(p)$ in our setting by giving conditions under which a data-dependent choice of p based on (10) meets, with probability 1, the requirements of (A.4).

Proposition 2 *Suppose* (A.1)–(A.3) *hold and let* $\hat{p} := \arg\min_{1 \leqslant p \leqslant M} \mathcal{C}(p)$*, with* $M = M(n) \to \infty$ *and* $M(n) = O(\{\log n\}^{\tau})$*, as* $n \to \infty$*, for some* $\tau \geqslant 1$*,* $f(n) > 0$ *for all* $n \in \mathbb{N}$*, and* $f(n) = o(n\{\log n\}^{-\tau-\varepsilon})$*, as* $n \to \infty$*, for some* $\varepsilon > 0$*. Then,* $\hat{p}$ *satisfies* (A.4) *almost surely.*

We note that the result stated in Proposition 2 remains true if the Yule–Walker estimator $\hat{\sigma}_p^2$ in (10) is replaced by the corresponding least-squares estimator. We also note that, under mild regularity conditions (cf. Poskitt (2007, Theorem 9)), the autoregressive order selected by $\mathcal{C}(p)$ with $f(n) = 2$ is asymptotically efficient, in the sense defined by Shibata (1980), for all $d \in (-\frac{1}{2}, \frac{1}{2})$.

We conclude this section by remarking that the linear structure imposed on $\mathbf{Y}$ by (A.2) is admittedly somewhat restrictive. However, the results of Bickel and Bühlmann (1996, 1997) indicate that linearity may not be too onerous a requirement since the closure (with respect to the total variation metric) of the class of linear processes is quite large; roughly speaking, for any stationary nonlinear process, there exists another process in the closure of linear processes having identical sample paths with probability exceeding $e^{-1} \simeq 0.37$. This suggests that the sieve bootstrap is likely to yield reasonably good approximations within a class of processes larger than that associated with (5).

4 Monte Carlo Experiments

In this section, we examine the small-sample properties of the proposed symmetry tests by means of Monte Carlo experiments. The data-generating mechanism is an ARFIMA process satisfying the equations

$$X_t - \mu = (1 + \varphi B)^{-1}(1 + \vartheta B)(1 - B)^{-d}\xi_t, \qquad t \in \mathbb{Z}, \tag{11}$$

with $\mu = 0$, $\varphi = -0.7$, $\vartheta = -0.3$, and $d \in \{-0.1, 0, 0.25, 0.4\}$.[2] The i.i.d. noise $\{\xi_t\}$ in (11) is drawn from the following distributions, standardized to have zero mean and unit variance (β_1 and β_2 are the classical measures of skewness and kurtosis based on the standardized third and fourth central moments, respectively):

[2] The values of the ARFIMA parameters are taken from Nordman, Sibbertsen, and Lahiri (2007).

(S1) Normal ($\beta_1 = 0, \beta_2 = 3$).

(S2) Double exponential ($\beta_1 = 0, \beta_2 = 6$).

(S3) Student's t with 5 degrees of freedom ($\beta_1 = 0, \beta_2 = 9$).

(S4) Generalized lambda with parameters $\lambda_1 = 0$, $\lambda_2 = -0.397912$, $\lambda_3 = \lambda_4 = -0.16$ ($\beta_1 = 0, \beta_2 = 11.6$).

(S5) Generalized lambda with parameters $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = \lambda_4 = -0.24$ ($\beta_1 = 0, \beta_2 = 126$).

(A1) Chi-square with 4 degrees of freedom ($\beta_1 = 1.4142, \beta_2 = 6$).

(A2) Exponential ($\beta_1 = 2, \beta_2 = 9$).

(A3) Lognormal with median 1 and shape parameter 0.7 ($\beta_1 = 2.888, \beta_2 = 20.79$).

(A4) Generalized lambda with parameters $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -0.001$, $\lambda_4 = -0.13$ ($\beta_1 = 3.16, \beta_2 = 23.8$).

(A5) Generalized lambda with parameters $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -0.0001$, $\lambda_4 = -0.17$ ($\beta_1 = 3.88, \beta_2 = 40.7$).[3]

For $d \neq 0$, artificial data are generated via the infinite-order moving-average representation of the ARFIMA process (11) truncated after the first 1,000 terms.[4] We consider two sample sizes $n \in \{100, 300\}$, and take the order of the sieve approximation to be the minimizer of the AIC over the range $1 \leqslant p \leqslant \lfloor 2(\log n)^2 \rfloor$, $\lfloor x \rfloor$ denoting the integral part of x. The approximating autoregression is fitted by the method of least squares (which is preferred here over the Yule–Walker method because it is known to produce estimates that exhibit smaller finite-sample bias).

[3] The generalized lambda distribution is most easily specified via its quantile function, which is $K(u) = \lambda_1 + \lambda_2^{-1}\{u^{\lambda_3} - (1-u)^{\lambda_4}\}$, $u \in (0,1)$ (Ramberg and Schmeiser (1974)). The parameter values for (S4), (S5), (A4) and (A5) are taken from Randles, Fligner, Policello, and Wolfe (1980).

[4] The stationary solution of (11) satisfies (5) with $\theta_1 = -\varphi + \vartheta + \eta_1$ and $\theta_j = -\theta_{j-1}\varphi + \eta_{j-1}\vartheta + \eta_j$ for $j \geqslant 2$, where $\eta_i := \{\Gamma(1+i)\Gamma(d)\}^{-1}\Gamma(d+i)$ for $i \in \mathbb{N}$.

Bootstrap approximations are constructed from $N = 500$ bootstrap replicates, while the number of Monte Carlo replications per experiment is 1,000.

The tests we consider are based on the statistic S_ψ defined in (2) with the following four functions ψ: $\psi_1(x) := x^3$, $\psi_2(x) := x/(1+x^2)$, $\psi_3(x) := \arctan x$, and $\psi_4(x) := \operatorname{sgn} x$. We note that ψ_1 is related to a skewness-type test, ψ_2 is associated with the test proposed by Chen, Chou, and Kuan (2000), ψ_3 is associated with the test of Premaratne and Bera (2005), and ψ_4 is related to the sign test of Gastwirth (1971).

The Monte Carlo Type-I error probabilities of tests of nominal level $\alpha = 0.05$ are shown in Table 1. The bootstrap tests perform reasonably well across the values of d and the different noise distributions considered. The error in the rejection probability of tests based on ψ_1, ψ_2 and ψ_3 is more pronounced for highly leptokurtic distributions such as (S4) and (S5) when $n = 100$. Even in such cases, however, deviations of the empirical rejection probabilities from the nominal level are not so large as to render the tests unattractive for applications.

It may be useful to note that one possible way of improving the reliability of the tests in terms of small-sample Type-I error probability may be by calibrating their level, an idea that dates back to Loh (1987) and Beran (1988). Choi and Hall (2000) demonstrated that, in the presence of short-range dependence, such calibration can deliver sieve bootstrap confidence intervals that are second-order accurate. To my knowledge, analogous results are currently unavailable for long-range dependent or antipersistent processes.

Table 2 contains the Monte Carlo rejection rates of the tests when the null hypothesis of symmetry is false. It is evident that rejection rates improve with increasing sample size and smaller values of the memory parameter. Asymmetry of the marginal distribution of $\{X_t\}$ is detected with high probability when $d = -0.1$ or $d = 0$. In the presence of long-range dependence, however, all the tests considered generally suffer a loss in power, the results being quite sensitive to the strength of dependence. Tests based on ψ_2 and ψ_3 still have respectable rejection rates for $d = 0.25$ and $n = 300$ (and noise distributions other than (A1)), but no test is particularly successful at detecting deviations from symmetry when $d = 0.4$.

5 Empirical Illustrations

In this section, we illustrate the practical use of the proposed methods by analyzing two real data sets.

5.1 Realized Volatility

In our first application, we examine the symmetry properties of empirical measures of stock return variability. The use of model-free volatility measures constructed from high-frequency intra-day returns on financial assets has become very popular in recent years. Such measures are typically reported to exhibit long-range dependence and to have an asymmetric marginal distribution which becomes approximately symmetric (or even Gaussian) after a logarithmic transformation (see, e.g., Andersen, Bollerslev, Diebold, and Ebens (2001); Andersen, Bollerslev, Diebold, and Labys (2001)).

The analysis is based on individual stocks of five companies in the Dow Jones Industrial Average index, namely Boeing Co. (BA), Caterpillar Inc. (CAT), Coca-Cola Co. (KO), Merck & Co. Inc. (MRK), and Pfizer Inc. (PFE). The raw data consist of tick-by-tick quotes, extracted from the NYSE Trade and Quote database, for the period from January 2, 2003 to December 31, 2007, a total of 1,258 trading days. The data are used to construct two nonparametric measures of daily volatility: the standard realized volatility (RV) measure (sum of squared intra-day returns) at a five-minute sampling frequency and the two-scales realized volatility (TSRV) measure of Zhang, Mykland, and Aït-Sahalia (2005) with five-minute grids. RV is extensively used in the literature and generally performs well in data-based comparisons of alternative volatility estimators (see Liu, Patton, and Sheppard (2012)). TSRV is known to have good statistical properties (as an estimator of the latent quadratic variation of the efficient logarithmic price process) in the presence of market microstructure noise.

In Tables 3 and 4, we show summary statistics for each of the two raw volatility measures, its natural logarithm and its multiplicative inverse. The inverse transformation was recommended by Gonçalves and Meddahi (2011) as a means of improving the accuracy of confidence intervals for quadratic variation based on RV. A semiparametric estimate ($\hat{d}$) of the memory parameter of each volatility series is also reported. This is obtained using the local Whittle estimator (Künsch (1987); Robinson (1995)) with bandwidth set equal to $\lfloor\{16(-2.19\hat{c})^2\}^{-1/5}n^{4/5}\rfloor$, where $\hat{c}$ is the least-squares estimate of the third coefficient in the pseudo-regression of $\log I(\omega_i)$ on $(1, -2\log\omega_i, \frac{1}{2}\omega_i^2)$, for $i = 1, 2, \ldots, \lfloor 0.3n^{8/9}\rfloor$, $I(\omega_i)$ being the periodogram ordinate of the observations $\mathbf{X}^n$ at the Fourier frequency $\omega_i := 2\pi i/n$ (cf. Henry and Robinson (1996); Andrews and Sun (2004)).

The marginal distributions of RV and TSRV are leptokurtic and skewed for all stocks. The logarithmic transformation reduces skewness and kurtosis substantially, a finding which is in line with empirical observations made in other studies; the same is also true for the inverse transformation in the majority of cases. The estimated value of the memory parameter is significantly different from zero for most of the series and, in the case of logarithmic and inverse volatility measures, fairly close to the "typical value" of 0.4 that is frequently reported in the literature (see, e.g., Andersen, Bollerslev, Diebold, and Ebens (2001); Andersen, Bollerslev, Diebold, and Labys (2001)).

Bootstrap P-values for tests of symmetry based on the four functions ψ used previously in Section 4 are shown in Tables 5 and 6; the cases in which a 90% bootstrap confidence interval for ζ_ψ, constructed as in (9), contains the value zero are also indicated. Bootstrap approximations are computed from $N = 2,000$ bootstrap replicates, with the sieve order estimated as the minimizer of the AIC over the range $1 \leqslant p \leqslant \lfloor 2(\log n)^2\rfloor$. Symmetry is rejected by tests based on S_{ψ_2}, S_{ψ_3} and S_{ψ_4} for all RV and TSRV series. The skewness-type statistic S_{ψ_1} and the confidence interval for ζ_{ψ_1} do not provide strong evidence against symmetry for two RV series (KO and PFE) and two TSRV series (BA and KO). In the case of logarithmic TSRV, symmetry is rejected for all series but KO (and also BA when using ψ_1, which is not perhaps surprising since skewness is relatively small for BA). The evidence against symmetry is even stronger for logarithmic RV, the only marginal non-rejection being obtained for KO when using the ψ_1 function. Findings are similar for the inverse volatility measures, with evidence in favor of symmetry being provided by the test based on S_{ψ_1} and the confidence interval for ζ_{ψ_1} only in the case of the inverse TSRV measure for BA. These results are especially convincing in light of the low power of tests when the memory parameter is large and positive, and indicate that distributional symmetry may not be as typical a characteristic of logarithmic and inverse volatility measures as is often suggested in the literature.

We end by noting that, although realized measures of volatility such as RV and TSRV are treated as essentially observable, they do contain a sampling error and should not therefore be thought of as being the same as the true (latent) return variability. This must be borne in mind when interpreting empirical evidence from any inferential procedure that makes use of realized volatility measures, not least because sampling error (and other measurement errors) can potentially conceal the true features of the latent integrated volatility (cf. Hansen and Lunde (2014); Rossi and de Magistris (2014)).

5.2 Output Growth

As a second illustration, we examine the symmetry properties of real output growth rates of G7 economies. DeLong and Summers (1986) characterized business cycle asymmetry by the asymmetry of the marginal distribution of the growth rate of a measure of output. This type of asymmetry is referred to by Sichel (1993) as 'steepness' (contractions are steeper than expansions, or vice versa), and is an example of what Ramsey and Rothman (1996) classified as 'longitudinal' asymmetry (that is, asymmetry in the direction of the movement of the business cycle). Nieuwerburgha and Veldkamp (2006) developed a dynamic stochastic general equilibrium model which can explain steepness of the business cycle. Although there is a considerable body of literature on cyclical asymmetry, the empirical findings are generally mixed, depending on the particular measure of output used and the statistical testing procedure employed.

The data used in our analysis consist of 139 annual observations, from 1870 through 2008, on real GDP per capita, and are taken from Angus Maddison's well-known database (available at `www.ggdc.net/maddison/oriindex.htm`). Growth rates are computed by first-differencing the logarithmically transformed raw data.[5]

In Table 7, we report the sample skewness coefficient and estimated memory parameter of the output growth series, as well as bootstrap P-values for tests of symmetry. All quantities appearing in the table are computed in the same way as in the application discussed in Section 5.1. The estimated values of the memory parameter indicate that GDP growth rates exhibit short-range dependence in the case of France, Italy, Japan and the U.K., but are antipersistent in the case of Canada, U.S. and (possibly) Germany. This means that the estimated memory parameters for the (logarithmically transformed) GDP series for the latter three economies are in the region $(\frac{1}{2}, 1)$, suggesting that the underlying stochastic processes are nonstationary but with impulse responses that decay towards zero, albeit at a polynomial rate. The possibility of antipersistent behavior resulting from 'overdifferencing' nonstationary time series whose order of integration is fractional and less than unity is an issue that has received virtually no attention in the literature on growth-rate asymmetry.

Looking at the P-values for the tests based on the four ψ functions, symmetry is rejected for Canada by all four tests and for the U.K. by tests based on statistics other than S_{ψ_1}. Confidence intervals for ζ_ψ suggest the presence of asymmetry in the Canadian, French and U.K. GDP growth rates.

6 Summary

In this paper, we have discussed tests for symmetry (around a specified or unspecified center) of the marginal distribution of a fractionally integrated process. As a practical means of implementing the tests, we have proposed using a symmetric sieve bootstrap procedure to estimate finite-sample P-values and/or critical values. The sieve bootstrap delivers tests which are asymptotically valid for short-range dependent, long-range dependent and antipersistent time series, and does not require

[5]Needless to say, with such a long span of data issues related to structural stability and regime change deserve serious consideration; however, for the purposes of this illustration, we abstract from such issues.

knowledge (or estimation) of the memory parameter of the data or of the appropriate norming factor for the test statistic. Simulation results have shown that the bootstrap tests performs satisfactorily, although they tend to lack power when the memory parameter is large and positive. Applications to realized measures of stock return volatility and real output growth illustrated the practical use of the proposed procedures.

7 Appendix

Proof of Proposition 1. Noting that, under (A.5), S_ψ satisfies the conditions of Lemma 1 of Poskitt (2008), the required result follows by arguments almost identical to those used by Poskitt in the proof of his Theorem 2. Specifically, in view of Poskitt's Remark 2 regarding mean correction, the only modification to his proof that needs to be made relates to $\{\xi^*_{p,t}\}$ in (8) obeying $\mathcal{L}(\xi^*_{p,t}|\mathbf{X}^n) = \hat{Q}_p$ instead of $\mathcal{L}(\xi^*_{p,t}|\mathbf{X}^n) = \hat{Q}^\dagger_p$, where $\hat{Q}^\dagger_p(A) := (n-p)^{-1}\sum_{t=p+1}^n \mathbf{1}_A(\ell_p^{-1}\hat{\xi}_{p,t} - \ell_p^{-1}\bar{\xi}_p)$. To accommodate this modification, let $(\phi_{p,1},\ldots,\phi_{p,p},\sigma_p^2)$ be the solution of the pth-order Yule–Walker equations $\sum_{j=0}^p \phi_{p,j}\gamma_{k-j} = \delta_{0,k}\sigma_p^2$ $(k=0,1,\ldots,p)$, with $\phi_{p,0} := 1$, and put $\xi_{p,t} := \sum_{j=0}^p \phi_{p,j}(X_{t-j}-\mu)$, $t \in \mathbb{Z}$, so that $\mathsf{E}(\xi_{p,t}^2) = \sigma_p^2$. Upon observing that $\{\xi_{p,t} - \hat{\sigma}_p\xi^*_{p,t}\}$ are independent under the conditional probability measure carrying $\mathbf{X}^*_p$, a straightforward calculation yields

$$\mathsf{E}^*\left(\left\{\sum_{j=0}^{\infty}\hat{\phi}_{p,j}\left(\xi_{p,t-j} - \hat{\sigma}_p\xi^*_{p,t-j}\right)\right\}^2\right) = (n-p)^{-1}\sum_{s=p+1}^{n}(\xi_{p,s} - \hat{\sigma}_p\ell_p^{-1}\hat{\xi}_{p,s})^2 \cdot \sum_{j=0}^{\infty}\hat{\phi}_{p,j}^2,$$

where $\hat{\phi}_{p,j} := 0$ for $j > p$ and E^* denotes expectation with respect to $\mathbf{X}^*_p$, conditional on $\mathbf{X}^n$. Furthermore, since $\ell_p^{-1}\hat{\sigma}_p = 1 + o(1)$ $(n \to \infty)$ almost surely (Poskitt (2008, p. 247)), we have

$$(n-p)^{-1}\sum_{s=p+1}^{n}(\xi_{p,s} - \hat{\sigma}_p\ell_p^{-1}\hat{\xi}_{p,s})^2 = (n-p)^{-1}\sum_{s=p+1}^{n}(\xi_{p,s} - \hat{\xi}_{p,s})^2 + o(1), \quad \text{as } n \to \infty,$$

with the $o(1)$ term being of that order almost surely. Then, it is easy to see that the arguments in the proof of Poskitt's Theorem 2 go through unchanged under the resampling scheme based on $\hat{Q}_p$, and the assertion of the proposition follows. ■

Proof of Proposition 2. The second condition of (A.4) holds for $\hat{p}$ on account of $M(n) = O(\{\log n\}^\tau)$ for some $\tau \geqslant 1$. Hence, it remains to show that $\hat{p}$ diverges to infinity with n almost surely. To this end, for each $p \in \mathbb{N}$, let $\xi_{p,t} := \sum_{j=0}^{p} \phi_{p,j}(X_{t-j} - \mu)$, $t \in \mathbb{Z}$, $\phi_{p,0} := 1$, be the error associated with the best linear predictor of X_t based on $(X_{t-1}, \ldots, X_{t-p})$. By Theorem 36.4 in Billingsley (1995, p. 495) and the pointwise ergodic theorem, as $n \to \infty$, $n^{-1} \sum_{t=1}^{n} \xi_{p,t}^2$ and $n^{-1} \sum_{t=1}^{n} \xi_t^2$ converge almost surely to $\sigma_p^2 = \sum_{j=0}^{p} \phi_{p,j}\gamma_j > 0$ and σ^2, respectively, with $\sigma_p^2 - \sigma^2 \to 0$ as $p \to \infty$. Therefore, in view of Corollary 1 and Theorem 8 of Poskitt (2007),

$$\mathcal{C}(p) = \log \sigma^2 + \log\left(1 + \frac{\sigma_p^2 - \sigma^2}{\sigma^2}\right) + \frac{pf(n)}{n} + o(1), \quad \text{as } n \to \infty,$$

uniformly in p, with the $o(1)$ term being of that order almost surely. It can now be seen that the asserted property of $\hat{p}$ is true, for, if $\hat{p}$ does not diverge as $n \to \infty$, then $(\sigma_p^2 - \sigma^2)/\sigma^2$ remains bounded away from zero for values of p along the $\hat{p}$ sequence and, since $n^{-1}pf(n) \to 0$ as $n \to \infty$, the minimum of $\mathcal{C}(p)$ with respect to p is not attained. ■

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Table 1. Percentage Rejection Rates at the 5% Nominal Level

		$n = 100$				$n = 300$			
	$\psi \backslash d$	-0.1	0.0	0.25	0.4	-0.1	0.0	0.25	0.4
(S1)	ψ_1	3.1	5.2	4.2	6.4	4.8	4.8	4.6	4.6
	ψ_2	5.3	5.0	6.2	8.7	4.9	5.4	5.6	6.3
	ψ_3	4.8	4.7	6.2	8.6	4.8	5.2	5.5	6.4
	ψ_4	2.3	4.0	3.8	5.5	3.4	3.5	4.6	5.1
(S2)	ψ_1	7.3	6.2	4.8	5.6	5.5	5.4	4.3	3.9
	ψ_2	6.6	6.9	6.9	9.2	5.2	5.2	6.3	6.2
	ψ_3	6.9	6.1	6.6	8.7	5.3	5.5	5.7	6.0
	ψ_4	5.4	4.3	4.9	7.1	4.9	5.5	4.6	4.6
(S3)	ψ_1	8.4	5.8	5.5	6.7	5.3	6.2	4.4	6.3
	ψ_2	9.1	5.4	8.0	8.5	7.6	6.2	6.4	8.7
	ψ_3	8.2	5.0	7.7	8.2	7.9	6.2	6.1	8.6
	ψ_4	4.4	3.7	5.8	6.0	4.6	4.5	5.4	7.3
(S4)	ψ_1	10.7	12.0	8.0	5.3	7.7	8.4	5.6	4.5
	ψ_2	8.1	9.4	9.1	8.0	6.6	7.0	6.2	8.1
	ψ_3	8.8	10.1	9.0	8.1	6.1	7.2	6.1	7.2
	ψ_4	6.4	7.1	6.2	6.1	4.6	5.6	5.5	7.4
(S5)	ψ_1	11.0	9.0	7.3	5.6	7.2	7.2	5.0	4.0
	ψ_2	8.8	7.2	9.4	6.7	6.0	6.7	6.5	8.6
	ψ_3	9.3	8.0	8.3	7.4	5.9	7.1	6.2	7.2
	ψ_4	5.0	5.5	6.0	5.3	4.5	4.7	4.8	7.5

Note: The entries have approximate standard error 0.7.

Table 2. Percentage Rejection Rates at the 5% Nominal Level

		$n = 100$				$n = 300$			
	$\psi \backslash d$	-0.1	0.0	0.25	0.4	-0.1	0.0	0.25	0.4
(A1)	ψ_1	67.9	51.9	12.1	8.20	96.2	93.4	29.9	9.40
	ψ_2	92.4	77.5	20.3	9.90	100	100	44.3	14.4
	ψ_3	93.0	80.3	22.0	10.3	100	100	47.8	16.2
	ψ_4	58.5	44.0	12.9	7.00	98.9	93.3	30.0	11.3
(A2)	ψ_1	71.4	60.8	16.5	8.20	94.3	92.9	39.5	12.0
	ψ_2	99.6	95.7	30.3	11.8	100	100	64.0	16.7
	ψ_3	98.4	94.1	32.2	12.5	100	100	70.4	20.2
	ψ_4	87.8	72.6	17.7	7.60	100	99.9	46.6	13.5
(A3)	ψ_1	58.7	48.5	17.3	12.1	80.3	78.7	33.4	10.6
	ψ_2	98.6	93.0	29.2	14.5	100	99.9	64.4	18.6
	ψ_3	94.9	88.4	28.9	15.6	99.7	99.8	68.8	22.0
	ψ_4	85.5	71.6	19.8	8.40	100	99.6	51.9	14.9
(A4)	ψ_1	62.4	53.3	19.4	11.6	79.7	79.0	33.8	11.8
	ψ_2	99.4	95.8	34.9	12.5	100	100	73.0	20.5
	ψ_3	95.4	91.1	34.0	14.7	100	99.8	75.1	24.2
	ψ_4	91.6	79.9	22.2	8.10	100	100	56.6	16.9
(A5)	ψ_1	59.0	50.5	20.5	11.2	74.8	71.2	35.5	11.4
	ψ_2	99.8	97.4	34.4	13.2	100	100	76.9	22.2
	ψ_3	95.1	91.4	33.7	14.4	99.9	99.8	77.6	25.1
	ψ_4	95.2	83.2	22.2	8.00	100	100	63.2	18.1

Note: The entries have approximate standard error at most 1.6.

Table 3. Summary Statistics for RV

	Mean	Median	Std. Dev.	Skewness	Kurtosis	$\hat{d}$
				RV		
BA	1.8507	1.3335	2.0493	9.2524	148.80	0.4341 (0.0687)
CAT	1.9228	1.5100	1.6789	5.9644	68.157	0.3486 (0.0542)
KO	0.9866	0.7157	2.2414	27.900	896.74	0.1977 (0.0495)
PFE	1.5312	1.1079	2.4442	13.119	222.50	0.1373 (0.0508)
MRK	1.7754	1.2095	2.5515	8.5515	101.57	0.1419 (0.0527)
				Logarithmic RV		
BA	0.3501	0.2878	0.6789	0.5099	3.6885	0.4606 (0.0530)
CAT	0.4508	0.4121	0.5991	0.4921	3.9016	0.4224 (0.0556)
KO	-0.2982	-0.3345	0.6521	0.6028	5.5246	0.4416 (0.0442)
PFE	0.1586	0.1025	0.6253	0.9071	6.1126	0.3609 (0.0488)
MRK	0.2679	0.1902	0.6783	0.9119	5.4261	0.3353 (0.0359)
				Inverse RV		
BA	0.8680	0.7499	0.5698	1.7310	8.2696	0.4171 (0.0495)
CAT	0.7518	0.6622	0.4400	1.4846	6.2517	0.4118 (0.0516)
KO	1.6400	1.3972	1.1266	3.4940	35.807	0.3938 (0.0521)
PFE	1.0124	0.9026	0.5844	1.3099	5.7063	0.3499 (0.0387)
MRK	0.9328	0.82680	0.6005	2.3568	15.895	0.3487 (0.0493)

Note: Figures in parentheses are asymptotic standard errors.

Table 4. Summary Statistics for TSRV

	Mean	Median	Std. Dev.	Skewness	Kurtosis	$\hat{d}$
				TSRV		
BA	1.6155	1.2234	1.3427	4.0552	41.069	0.4252 (0.0539)
CAT	1.7537	1.4371	1.2460	2.9464	16.602	0.3794 (0.0546)
KO	0.8555	0.6491	1.2217	20.882	589.15	0.3191 (0.0552)
PFE	1.3466	1.0082	2.0408	14.861	288.11	0.1605 (0.0483)
MRK	1.5921	1.1442	2.2274	10.227	141.32	0.1396 (0.0516)
				Logarithmic TSRV		
BA	0.2449	0.2017	0.6682	0.1944	3.4242	0.4598 (0.0536)
CAT	0.3846	0.3626	0.5749	0.3319	3.3536	0.4358 (0.0556)
KO	-0.4041	-0.4322	0.6550	0.2630	4.5477	0.4613 (0.0453)
PFE	0.0423	0.0082	0.6320	0.6167	5.4661	0.3594 (0.0454)
MRK	0.1842	0.1347	0.6636	0.7049	5.1932	0.3630 (0.0447)
				Inverse TSRV		
BA	0.9754	0.8174	0.8586	11.092	245.57	0.3535 (0.0556)
CAT	0.7957	0.6959	0.4565	1.4819	6.3034	0.4202 (0.0522)
KO	1.8505	1.5407	1.4533	4.8089	48.669	0.3607 (0.0449)
PFE	1.1524	0.9918	0.7432	2.2671	13.343	0.3200 (0.0374)
MRK	1.0146	0.8740	0.6886	3.2067	29.004	0.3478 (0.0427)

Note: Figures in parentheses are asymptotic standard errors.

Table 5. P-Values for Symmetry Tests on RV

	ψ_1	ψ_2	ψ_3	ψ_4
		RV		
BA	0.0255	0.0000	0.0000	0.0000
CAT	0.0635	0.0000	0.0000	0.0000
KO	0.1365*	0.0030	0.0270	0.0000
PFE	0.1015*	0.0000	0.0000	0.0000
MRK	0.0325	0.0000	0.0000	0.0000
		Logarithmic RV		
BA	0.0130	0.0005	0.0005	0.0005
CAT	0.0110	0.0005	0.0005	0.0110
KO	0.1040*	0.0400	0.0410	0.0595
PFE	0.0005	0.0005	0.0005	0.0005
MRK	0.0015	0.0000	0.0000	0.0005
		Inverse RV		
BA	0.0000	0.0000	0.0000	0.0000
CAT	0.0000	0.0000	0.0000	0.0000
KO	0.1060	0.0000	0.0000	0.0000
PFE	0.0000	0.0000	0.0000	0.0000
MRK	0.0030	0.0000	0.0000	0.0000

Note: An asterisk indicates that a 90% confidence interval for ζ_ψ contains the value 0.

Table 6. P-Values for Symmetry Tests on TSRV

	ψ_1	ψ_2	ψ_3	ψ_4
		TSRV		
BA	0.1200*	0.0000	0.0000	0.0000
CAT	0.0000	0.0000	0.0000	0.0000
KO	0.1655*	0.0150	0.0425	0.0090
PFE	0.0215	0.0000	0.0030	0.0000
MRK	0.0360	0.0000	0.0000	0.0000
		Logarithmic TSRV		
BA	0.2075*	0.0160	0.0345	0.0115
CAT	0.0135	0.0085	0.0070	0.0085
KO	0.2680*	0.2425*	0.2520*	0.2820*
PFE	0.0070	0.0165	0.0115	0.0205
MRK	0.0055	0.0000	0.0005	0.0005
		Inverse TSRV		
BA	0.1270*	0.0045	0.0195	0.0010
CAT	0.0000	0.0000	0.0000	0.0000
KO	0.0130	0.0000	0.0000	0.0000
PFE	0.0000	0.0000	0.0000	0.0000
MRK	0.0380	0.0000	0.0000	0.0000

Note: An asterisk indicates that a 90% confidence interval for ζ_ψ contains the value 0.

Table 7. GDP Growth Rates

	Skewness	$\hat{d}$	P-Value			
			ψ_1	ψ_2	ψ_3	ψ_4
Canada	-0.8116	-0.3690 (0.1213)	0.0045	0.0235	0.0080	0.0355
France	0.5279	-0.1072 (0.1508)	0.3980*	0.1665	0.1295	0.1930
Germany	-5.3317	-0.3176 (0.1291)	0.1025	0.3160*	0.2060*	0.2595*
Italy	-0.9993	0.0449 (0.1213)	0.4075*	0.5275*	0.2800*	0.4475*
Japan	-5.7143	0.0659 (0.1250)	0.1330*	0.6855*	0.6430*	0.9510*
U.K.	-0.8865	-0.1681 (0.1213)	0.1425	0.0065	0.0240	0.0410
U.S.	-0.7533	-0.4725 (0.1043)	0.2180*	0.5460*	0.5945*	0.4880*

Note: An asterisk indicates that a 90% confidence interval for ζ_ψ contains the value 0. Figures in parentheses are asymptotic standard errors.