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Abstract

We analyze a market populated by expected utility maximizers and smooth ambiguity-averse consumers. We study conditions under which ambiguity-averse consumers survive and affect prices in the limit. If ambiguity vanishes with time or if the economy exhibits no aggregate risk, ambiguity-averse consumers survive, but have no long-run impact on prices. In both scenarios, ambiguity-averse consumers are fully insured against ambiguity in equilibrium and, thus, behave as expected utility maximizers with correct beliefs. If ambiguity-averse consumers are not fully insured against ambiguity, they behave as expected utility maximizers with effectively wrong beliefs and an effective discount factor which might be higher or lower than their actual discount factor. Using this insight, we demonstrate that consumers with constant absolute ambiguity aversion vanish in expectations, whenever the economy faces aggregate risk. In contrast, consumers with constant relative (and thus, decreasing absolute) ambiguity aversion survive in expectation and with positive probability and have a non-trivial impact on prices in the limit.

Keywords: ambiguity, ambiguity-aversion, survival.

JEL Codes: D50, D81.

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1 Introduction

Theories of ambiguity aversion have established themselves as a viable alternative to expected utility maximization. They capture experimentally observed behavior and have been used to explain some of the empirical phenomena documented in financial markets, such as the home bias (Uppal and Wang (2003)) and the equity premium puzzle (Epstein and Scneider (2008)). Such explanations, however, are robust only if we can show that ambiguity-averse investors will exert persistent influence on market prices.

In this paper we ask about the survival of ambiguity-averse investors in a financial market. So far this question has been analyzed only for the case of max-min expected utility maximizers (as axiomatized by Gilboa and Schmeidler (1989)): Condie (2008) finds that, unless there is no aggregate risk, these investors disappear from the market. The case of no aggregate risk, however, is special in that ambiguity averse investors are fully insured and, thus, have no effect on prices. Hence even when ambiguity-averse investors survive, market outcomes look as if all investors were expected utility maximisers. Condie's approach has two main shortcomings: first, the framework of the max-min expected utility does not allow for a distinction between ambiguity and ambiguity attitude. Hence, it is not clear whether the fact that max–min investors vanish should be attributed to their ambiguity aversion, or to information asymmetries: while max-min-investors face uncertainty about the actual distribution of returns, expected utility maximizers know the correct distribution. Second, even if one were to attribute the effect to ambiguity aversion, the max-min expected utility only allows for a very extreme form of ambiguity-aversion: the decision maker always chooses the worst probability distribution to evaluate a given act. This raises the question of whether the degree of ambiguity aversion can influence survival.

Our paper addresses these issues by examining a market populated by expected utility maximizers and smooth ambiguity-averse investors, as in Klibanoff, Marinacci and Mukerji (2009), henceforth KMM (2009). We choose this model, because it allows us to separate the objective ambiguity present on the market, to which all investors are exposed, from the subjective attitude towards ambiguity. Furthermore, it also allows us to vary the degree of ambiguity aversion and relate it to the investor's chances to survive.

We assume that the market exhibits two levels of uncertainty: the first is the uncertainty about the investors' endowments, the second is the uncertainty about the probability distribution determining the evolution of endowments. We refer to the first type of uncertainty as risk and to the second type of uncertainty as ambiguity. Ambiguity is described by the set of probability distributions which can govern the endowment process. The main difference between ambiguity and risk in our model consists in the fact that the realization of the risky state (realization of endowments) is interpersonally verifiable, while the realization of the ambiguous state (the distribution of endowments) is not. Hence, asset payoffs and prices can only depend on the realization of the risky variables, but not on the realization of the ambiguous ones. Similarly, trades cannot be made contingent on the ambiguous states, i.e., on the distribution governing the endowment streams. We assume that the economy has a complete set of Arrow securities with payoffs contingent on the realization of the risky state. No assets with payoffs contingent on the realization of the ambiguous states are available.

In our model, both types of investors have the same information about the structure of uncertainty. Both ambiguity-averse investors and expected utility maximizers are averse towards risk. However, while ambiguity-averse investors prefer to reduce their exposure to ambiguity, expected utility maximizers are indifferent towards it. Hence, if both types of investors have identical discount factors and correct beliefs, then differences in their ability to survive can only be attributed to the difference in their attitude towards ambiguity.

The main finding of our paper is that if ambiguity is persistent and the economy faces aggregate risk, survival is not independent of the degree of ambiguity aversion. This is true, even though all investors in the economy are assumed to have correct beliefs and identical discount factors. The intuition behind this result is as follows: a smooth ambiguity-averse investor with correct beliefs and a constant discount factor effectively behaves as an expected utility maximizer with incorrect beliefs and a time-dependent discount factor. The factors modifying the beliefs and the discount rate depend on the decision maker's equilibrium consumption and on the function describing his attitude towards ambiguity. In particular, if the ambiguity-averse investor were completely insured against ambiguity, he would be indistinguishable from an expected utility maximizer with a constant discount factor and correct beliefs. However, we show that if the economy faces aggregate risk, the ambiguity-averse investor will not be completely insured against ambiguity and, hence, his ambiguity aversion will influence both his effective beliefs and his effective discount factor. His effectively wrong beliefs always inhibit his chances of survival compared to an expected utility maximizer with correct beliefs. However, changes in his effective discount factor can offset this effect.

We analyze three classes of functions representing the investor's attitude towards ambiguity: functions exhibiting constant absolute ambiguity aversion, $\phi(y) = -e^{-\alpha y}$; functions exhibiting constant relative ambiguity aversion and, thus, decreasing absolute ambiguity aversion, $\phi(y) = \ln y$ and $\phi(y) = y^{\gamma}$; and functions exhibiting increasing absolute ambiguity aversion of the type $\phi(y) = by - ay^{r}$. For these three classes of functions, we compute the effective discount factor of the ambiguity-averse investor. The effective discount factor is equivalent to the actual discount factor for the class of functions exhibiting constant absolute ambiguity aversion. It is larger than the actual discount factor for the class of functions with constant relative, and hence, decreasing absolute ambiguity aversion; and it is smaller for the subclass of functions with increasing absolute ambiguity aversion which have a decreasing second derivative. We then use these results to derive implications for the survival of ambiguity averse investors.

Since the effective discount factor for an ambiguity-averse investor with constant relative ambiguity aversion is larger than his actual discount factor, it forces him to save more, and thus, enhances his chances of survival. It turns out that (in expectations) this effect offsets the effect caused by wrong beliefs. Hence, investors exhibiting constant relative ambiguity aversion survive with positive probability and in expectations. Since these investors are not completely insured against risk in equilibrium, they have a non-trivial impact on prices and allocations even in the limit of those paths on which they survive. In particular, such paths will exhibit higher equity premium than predicted using standard models of expected utility maximization.

For the case of constant and increasing absolute ambiguity aversion, the effective discount factor of the ambiguity-averse investors either remains unchanged or is less than their actual discount factor, while their effective beliefs differ from the truth. Hence, unless they are fully insured against ambiguity, in expectations, such investors vanish from the market, even though their actual beliefs are correct and their discount factor is identical to the one of the expected utility maximizers.

The intuition behind these results is simple: in the smooth model of ambiguity, ambiguity aversion has an intertemporal effect, forcing the investor to save more relative to an ambiguity-neutral investor. When an investor exhibits decreasing ambiguity aversion, this effect is especially pronounced for wealth levels close to 0, thus preventing his consumption from converging to 0. In contrast, increasing absolute ambiguity aversion, leads to a reduction in savings at low levels of wealth, driving the investors out of the market.

It is important to note that the dependence of survival on ambiguity aversion arises only for cases in which: (i) the economy faces aggregate risk, and (ii) the ambiguity is persistent. To indicate the importance of aggregate risk, we analyze the case in which the total endowment of the economy is certain. We show that in this scenario, all investors will be fully insured against risk, and thus, also against ambiguity. It follows that ambiguity-averse investors with correct beliefs will survive, but their ambiguity-aversion will not matter for prices and allocations. To highlight the effect of persistent ambiguity, we study the case in which the probability distribution of asset payoffs is determined once and for all in the first period and, therefore, the investors can learn it as time evolves by observing the endowment realizations. In this case, only beliefs and discount factors determine survival. In particular, ambiguity-averse investors with correct beliefs and discount factors equal to those of the expected utility maximizers survive. However, since ambiguity vanishes in the limit, ambiguity-aversion has no long-run impact in this scenario.

The remainder of the paper is organized as follows: the next section provides a short overview of the related literature. Section 3 presents the model of a market with expected utility maximizers and smooth ambiguity-averse consumers. Section 4 defines and shows the existence of an interior equilibrium for such an economy. Section 5 analyzes the question of survival with ambiguity-aversion and states our main results. Section 6 concludes. All proofs and derivations are collected in the Appendix.

2 Related Literature

The paper which is most closely related to our work is Condie (2008), which analyzes the issue of survival of max-min expected utility maximizers. Condie shows that even when the true probability distribution is contained in the prior of a max-min consumer, this consumer vanishes, unless he is completely insured. The intuition behind this result is simple: at any period, a max-min consumer can be represented as an expected utility maximizer by choosing beliefs in such a way that they support the equilibrium consumption stream at the equilibrium prices. These effective beliefs will correspond to the truth only if the max-min consumer is completely insured, but will be wrong otherwise. Hence, max-min expected utility maximizers can survive only in economies, in which there is no persistent aggregate risk. More generally, Rigotti, Shannon and Strzalecki (2008) show how effective beliefs can be derived for all known models of ambiguity aversion. While we use a similar technique to analyze the conditions for the survival of a smooth ambiguity-averse consumer, in our infinite-horizon model there are two effects at work: ambiguity-aversion causes the consumer to behave as an expected utility maximizer with incorrect beliefs, but it may also force him to save more thereby increasing his effective discount factor. This second effect can compensate for the first one and thus result in survival. Hence, our paper extends the results by Condie (2008) by considering a more general class of ambiguity-averse consumers, clearly differentiating between objective ambiguity and subjective ambiguity attitude and highlighting the role of different degrees of ambiguity aversion for survival.

More generally, our paper contributes to the literature on survival in financial markets by reexamining the question of whether correct beliefs are the only determinant of survival. As it is well-known from the work of Sandroni (2000) and Blume and Easley (2006), in complete markets populated by expected utility maximizers, market participants with identical discount factors survive if and only if they have correct beliefs[1]. Our framework deviates from these studies in two respects: first, markets are incomplete in that they do not allow for bets on ambiguous events; second, decision makers' preferences deviate from expected utility maximization and in particular are not time-separable. The market incompleteness prevents ambiguity-averse agents from insuring completely against ambiguity. The time-inseparability of preferences leads to the difference between the actual and the effective discount factor used by ambiguity-averse agents.

We consider two special cases in which market incompleteness and time-inseparability do not matter: the case of vanishing ambiguity, in which betting on infinite endowment streams coincides with betting on the ambiguous states of the economy; and the case of no aggregate risk, in which insuring everyone

[1] The assumption of bounded endowments is also crucial for these results: Kogan, Ross, Wang and Westerfield (2008) and Yan (2008) provide models of a growing economy, in which the consumer's risk attitude influences his ability to survive, together with his discount factor and beliefs. While in our model ambiguity aversion, discount factors and beliefs jointly determine the investors' chances to survive, our results are derived in the context of bounded endowments.

against risk in equilibrium automatically guarantees that all agents are also completely insured against ambiguity. In these two cases, the only relevant characteristic for survival are the consumer's beliefs. Ambiguity-averse investors behave exactly as expected utility maximizers.

In general, however, market incompleteness and time-inseparability will have an effect on the equilibrium allocations. When the ambiguity-averse consumers are not able to fully insure themselves against ambiguity, survival is dependent on the ambiguity attitude. In this sense, our paper is related to the research on survival in incomplete financial markets. Coury and Sciubba (2012) show that it is always possible to construct an equilibrium in which an agent with incorrect beliefs survives. Beker and Chattopadhyay (2010) demonstrate that the dynamics of an economy with incomplete markets is highly non-trivial: in some cases an agent with correct beliefs can vanish, in others the economy might exhibit cycles in which the consumption of each of the agents approaches 0 infinitely often. While these papers look at rather general forms of incompleteness, in our paper the incompleteness arises from the presence of ambiguity and is, therefore, quite specific, in that it only matters for the ambiguity-averse consumers. If all consumers were expected utility maximizers, equilibrium allocations in our setting would be Pareto-optimal and the survival results for complete markets would go through.

Borovicka (2010) examines survival in the context of Epstein and Zin (1989) preferences[2]. He also shows that time-nonseparability has an effect on survival as compared to the case of time-separable preferences. Similarly to Kogan, Ross, Wang et al. (2008) and Yan (2008), his model uses a Brownian motion to model the distribution of endowments, while our results are derived for the case of bounded economy.

3 The Model

3.1 Modelling the Uncertainty

Let $\mathbb{N} = \{1, 2, ..\}$ denote the set of time periods. Uncertainty is modelled through a sequence of random variables $\{S_t\}_{t=1}^{\infty}$ which take value from a finite set $\mathcal{S}_t$. Denote by $s_t \in \mathcal{S}_t$ the realization of random variable S_t. Denote by $\Sigma = \prod_t \mathcal{S}_t$ the set of all possible observation paths, with representative element $\sigma = (s_1, s_2, ..., s_t, ...)$. Finally denote by $\Sigma_t = \prod_{\tau=1}^{t} \mathcal{S}_t$ the collection of all finite paths of length t, with representative element $\sigma_t = (s_1, s_2, ..., s_t)$. Each finite observation path σ_t identifies a decision/observation node and the set of all possible observation paths Σ can also be seen as the set of all nodes.

[2]Mathematically, Epstein-Zin preferences represent a special case of the KMM's (2009) recursive model of smooth ambiguity aversion. We thank Viktor Tsyrennikov for pointing this out.

We can represent the information revelation process in this economy through a sequence of finite partitions of the state space Σ. In particular, define the cylinder with base on $\sigma_t \in \Sigma_t$, $t \in \mathbb{N}$ as $C(\sigma_t) = \{\sigma \in \Sigma | \sigma = (\sigma_t, ..)\}$. Let $\mathbb{F}_t = \{C(\sigma_t) : \sigma_t \in \Sigma_t\}$ be a partition of the set Σ. Clearly, $\mathbb{F} = (\mathbb{F}_0, .., \mathbb{F}_t, ..)$ denotes a sequence of finite partitions of Σ such that $\mathbb{F}_0 = \Sigma$ and $\mathbb{F}_t$ is finer than $\mathbb{F}_{t-1}$. We assume that all agents have identical information and that the information revelation process is represented by the sequence $\mathbb{F}$.

Let $\mathcal{F}_t$ be the σ-algebra generated by partition $\mathbb{F}_t$. Define $\mathcal{F}_0$ as the trivial σ-algebra. Let $\mathcal{F} = \sigma\left(\cup_{t\in\mathbb{N}} \mathcal{F}_t\right)$. It can be shown that $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ is a filtration.

We define on $(\Sigma, \mathcal{F})$ a family of probability distributions $\{\pi_n\}_{n=1}^N$ and throughout we assume $\pi_n(C(\sigma_t)) > 0, \forall \sigma_t$. In what follows, for brevity, we abuse notation slightly by denoting $\pi_n(C(\sigma_t)) = \pi_n(\sigma_t) = \pi_n(s_1, s_2, ..., s_t)$.

For any $E \in \Sigma$ define the conditional distribution of π_n given σ_t as $\pi_n(E \mid \sigma_t)$ where:

$$\pi_n(E \mid \sigma_t) = \left\{ \begin{array}{c} \frac{\pi_n(E\cap C(\sigma_t))}{\pi_n(\sigma_t)} \text{ if } t \in \mathbb{N} \\ \pi_n(E) \text{ if } t = 0 \end{array} \right. \quad \text{for any } E \in \Sigma$$

In words, $\pi_n(E \mid \sigma_t)$ is the probability under distribution π_n that the observation path will belong to E, given that we have reached node σ_t.

The one-step-ahead probability distribution $\pi_n(s_{t+1}; \sigma_t)$ at node σ_t is determined by:

$$\pi_n(s_{t+1}; \sigma_t) = \pi_n(s_1, ..s_t, s_{t+1} \mid s_1, ..s_t) = \frac{\pi_n(s_1, ..s_t, s_{t+1})}{\pi_n(s_1, ..s_t)} \text{ for any } s_{t+1} \in \mathcal{S}_{t+1}$$

In words, $\pi_n(s_{t+1}; \sigma_t)$ is the probability under distribution π_n that the next observation will be s_{t+1} given that we have reached node σ_t.

Denote by $\mu : \{\pi_n\}_{n=1}^N \to [0, 1]$ the (true) prior probability distribution over the set of probability distributions $\{\pi_n\}_{n=1}^N$, with $\mu_n = \mu(\pi_n)$ denoting the prior probability of distribution π_n. Given any $\pi_n \in \{\pi_n\}_{n=1}^N$ and any $\sigma_t \in \Sigma_t$, the posterior distribution is defined as:

$$\mu_{\sigma_t}(\pi_n) = \mu(\pi_n \mid \sigma_t) = \frac{\pi_n(\sigma_t)\mu(\pi_n)}{\sum_{j=1}^N \pi_j(\sigma_t)\mu_j}.$$

Hence there are two sources of uncertainty in the economy: uncertainty about the realization of the state of the world s_t, captured by the probability distributions π_n, and uncertainty about the actual probability measure which governs the realization of the state of the world. We will refer to the first source of uncertainty as risk, while the term ambiguity is used with regard to the second.

Two benchmark cases will be of particular interest. First, consider the situation, in which a probability distribution π_n is drawn at the beginning of period 1 according to a distribution $\mu = (\mu_1 ... \mu_n)$ and then, for each $t \in \mathbb{N}$, the variables $s_t \in S_t = S$ are distributed identically and independently according to π_n, i.e.:

$$\pi_n (s \mid \sigma_t) = \pi_n (s)$$

for all $\sigma_t \in \Sigma_t$. We will refer to this situation as the case of *vanishing ambiguity*: in this case, it is possible to learn the true probability distribution π_n by observing the state of the world s_t in each period and using Bayesian updating on the prior μ. The posterior $\mu_{\sigma_t}(\pi_n)$ converges to 1, whenever π_n is the realization of the initial draw.

Now consider a situation, in which the probability distribution $\pi_n(\sigma_t)$ determining the realization of the state of the world s_{t+1} is drawn anew at each node. Suppose that $\pi_n(\sigma_t)$ are i.i.d. according to a distribution $\mu = (\mu_1 ... \mu_n)$. We refer to this situation as the case of *persisting ambiguity*: since the distribution π which determines the state of the world changes in each period, past observations of the state of the world s_t do not provide any information about the future realizations of π. The posteriors satisfy $\mu_{\sigma_t}(\pi_n) = \mu_n$ for all nodes σ_t.

3.2 Preferences and Beliefs

There is a single good and I infinitely lived consumers, each with consumption set $\mathbb{R}_+$. A consumption plan $c : \Sigma \to \prod_{t=1}^{\infty} \mathbb{R}_+$ is a sequence of $\mathbb{R}_+$-valued functions $\{c(\sigma_t)\}_{t=1}^{\infty}$ in which each $c(\sigma_t)$ is $\mathcal{F}_t$-measurable. Each consumer is endowed with a particular consumption plan, called the endowment stream. Consumer i's endowment stream is denoted e^i.

Denote by $\mu^i : \{\pi_n\}_{n=1}^N \to [0,1]$ consumer i's prior probability distribution over the set of probability distributions $\{\pi_n\}_{n=1}^N$, with $\mu_n^i = \mu(\pi_n)$ denoting the prior probability of distribution π_n. Given any $\pi_n \in \{\pi_n\}_{n=1}^N$ and any $\sigma_t \in \Sigma_t$, agent i's posterior distribution is defined as:

$$\mu_{\sigma_t}^i(\pi_n) = \mu^i(\pi_n \mid \sigma_t) = \frac{\pi_n(\sigma_t)\mu^i(\pi_n)}{\sum_{j=1}^N \pi_j(\sigma_t)\mu_j^i}$$

Let $\succeq_i$ denote agent i's preference ordering over consumption plans. Preferences $\succeq_i$ are represented by the following recursive functional:

$$\begin{aligned} & V_{\sigma_t}^i(c^i) \\ = \quad & u_i(c^i(\sigma_t)) + \beta_i \phi_i^{-1} \left[\sum_{n=1}^N \phi_i \left(\sum_{s_{t+1} \in \mathcal{S}_{t+1}} V_{(\sigma_t, s_{t+1})}^i(c^i) \pi_n(s_{t+1}; \sigma_t) \right) \mu_{\sigma_t}^i(\pi_n) \right] \end{aligned}$$

This representation of preferences was suggested by KMM (2009). Here $\beta_i \in (0,1)$ is agent i's intertemporal discount factor; $u_i : \mathbb{R}_+ \to \mathbb{R}$ and $\phi_i : \mathbb{R} \to \mathbb{R}$ are continuous and strictly increasing functions. The interpretation of V^i is as follows: at time t, on path σ_t, consumer i receives an instantaneous utility from consumption $u^i(c^i(\sigma_t))$. From the next period on, he expects a state-contingent consumption stream which, depending on the state realization in period $t+1$, s_{t+1} will generate a discounted utility equal to $V_{\sigma_t; s_{t+1}}^i$. The consumer faces two types of uncertainty: first, he does not know which state will occur in period $t+1$, second he is uncertain on which probability distribution determines the realization of the state at $t+1$. The first type of uncertainty — risk — is captured by taking the expectation of the discounted payoffs with respect to a probability measure $\pi_n(s_{t+1};\sigma_t)$. The second type of uncertainty — ambiguity — is captured by a probability distribution over π_n, $\mu^i_{\sigma_t}(\pi_n)$ and a concave function ϕ_i. While the distribution $\mu^i_{\sigma_t}$ captures the perceived ambiguity, ϕ_i expresses consumer i's attitude towards this ambiguity. Finally, applying the inverse of ϕ_i to the expression in square brackets and multiplying by β_i corresponds to finding the certainty equivalent of the expected future consumption stream in terms of present utility. Note that when ϕ_i is a linear function (e.g., the identity), the representation above reduces to intertemporal expected utility maximization.

Our choice of the preference representation is motivated by the following considerations: first, differently from most other forms of representation of ambiguity-averse preferences, the KMM (2009) smooth model of ambiguity allows for a clear separation between ambiguity and ambiguity attitude. In particular, the function ϕ controls the degree of ambiguity aversion and allows us to compare decision makers which differ according to this characteristic. Second, the smooth model of ambiguity allows for a recursive formulation. This means that the beliefs of the decision maker are updated according to the Bayesian rule and the modelled behavior is dynamically consistent[3].

We impose the following assumptions on the primitives of the model:

Assumption 1 The functions $u_i : \mathbb{R}_+ \to \mathbb{R}$ are twice continuously differentiable, strictly concave, $u_i(0) = 0$, $\lim_{c\to 0} u_i'(c) = \infty$ and $\lim_{c\to\infty} u_i'(c) = 0$.

Assumption 2 Each of the functions $\phi_i : \mathbb{R} \to \mathbb{R}$ is either linear or strictly concave, twice continuously differentiable and $\lim_{y\to 0} \phi_i'(y) > 0$.

Assumption 3 Endowments are uniformly bounded away from zero and aggregate endowments are uniformly bounded. Formally, there is an $m > 0$ such that $e^i(\sigma_t) > m$ for all i, σ_t; moreover there is an $m' > m > 0$ such that $\sum_i e^i(\sigma_t) < m'$ for all σ_t.

Assumption 4 There is a $\delta > 0$ such that for all paths σ, dates t and states of the world $s_t \in S_t$, $\pi_n(s_t;\sigma_{t-1}) > 0$ for some $n \in \{1...N\}$ implies $\pi_n(s_t;\sigma_{t-1}) > \delta$.for all $n \in \{1...N\}$.

Assumptions 1 and 3 appear in Blume and Easley (2006). Assumption 2 is necessary, since we extend their model to the case of ambiguity aversion. Assumption 1 implies that all consumers are strictly risk-averse. Assumption 2 allows for both ambiguity-aversion and ambiguity-neutrality, hence the case of

[3]KMM (2009) provide a second formulation of the representation, which is time-separable, but violates dynamic consistency. The analysis of survival for such preferences is a question of independent interest.

expected utility maximization is covered by our model. Taken together, Assumptions 1 and 2 exclude the case in which a consumer chooses 0 consumption in an (observable) state of the world in which the consumer has a positive endowment and which has a positive probability according to this consumer's beliefs. Assumption 3 requires that each consumer's endowment in all states of the world is uniformly bounded above and uniformly bounded away from 0. Assumption 4 states that all distributions in the set $(\pi_n)_{n=1}^{N}$ are mutually absolutely continuous and that the minimal probability they can assign to a given state in the next period conditional on the history σ_{t-1} is uniformly bounded away from 0. Note that if all distributions π_n are mutually absolutely continuous, then in the two scenarios of persistent and vanishing ambiguity, the existence of a δ as specified in Assumption 4 is automatically guaranteed.

Taken together, Assumptions 1—4 guarantee that the solution to the consumer's maximization problem will be interior. Hence, they preclude the possibility that a consumer would vanish in finite time.

4 The Equilibrium of the Economy

We assume that markets are complete with respect to the observable states of the world, i.e. there is a complete system of Arrow securities contingent on the realization of σ_t for all $\sigma_t \in \Sigma_t, \forall t$. However, agents are not able to trade on the realization of the probability distribution π_n, i.e. the probability distribution over states is non-contractible. Since both endowments and consumption streams are assumed to be $\mathcal{F}_t$-measurable, at any time t, the only information available about π_n is the realization of σ_t. Hence, the restriction that trades can only be conditioned on σ_t appears fairly natural.

Definition 1 *An equilibrium of the economy is an integrable price system $(p(\sigma_t))_{\sigma_t \in \Sigma}$ and a consumption stream c^i for every consumer i such that at all nodes $\sigma_t \in \Sigma_t, \forall t$, all consumers $i \in \{1...I\}$ are maximizing their utility given the price system and markets clear:*

$$c^i = \arg\max_{c^i} V^i_{\sigma_t}(c^i) = u^i(c^i(\sigma_t)) +$$

$$+\beta_i \phi_i^{-1} \left[\sum_{n=1}^{N} \phi_i \left(\sum_{s_{t+1} \in \mathcal{S}_{t+1}} V^i_{(\sigma_t, s_{t+1})}(c^i) \pi_n(s_{t+1}; \sigma_t) \right) \mu^i_{\sigma_t}(\pi_n) \right]$$

$$s.t. \sum_{\sigma_t \in \Sigma_t} p(\sigma_t) c^i(\sigma_t) \leq \sum_{\sigma_t \in \Sigma_t} p(\sigma_t) e^i(\sigma_t), \forall t$$

$$\sum_{i=1}^{I} c^i(\sigma_t) = \sum_{i=1}^{I} e^i(\sigma_t).$$

Since markets in our economy are incomplete, we cannot directly use the Pareto-optimality conditions as in Blume and Easley (2006). Instead, we first show that an equilibrium of the economy exists and then use the properties of this equilibrium to analyze the question of survival.

Proposition 2 *An equilibrium of the economy exists.*

Our next Proposition ensures that the equilibrium can be described by a system of first-order conditions. The result of this Proposition is a direct consequence of the Inada conditions imposed on the function u, the concavity of ϕ and the mutual absolute continuity of the probability distributions $(\pi_n)_{n=1}^{N}$.

Proposition 3 *Under Assumptions 1–4, the equilibrium of the economy satisfies for all $i \in \{1...I\}$, all $t \in \mathbb{N}$ and all $\sigma_t \in \Sigma_t$ and $s_{t+1} \in S_{t+1}$ such that σ_t has a positive probability and such that $\sum_{n=1}^{N} \mu_{\sigma_t}^{i}(\pi_n)\pi_n(\sigma_t; s_{t+1}) > 0$:*

$$\frac{u_i'\left(c^i\left(\sigma_t\right)\right)}{\beta_i u_i'\left(c^i\left(\sigma_t; s_{t+1}\right)\right) \frac{\sum_{n=1}^{N} \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^{i}\left(c^i\right)\right)\right]\mu_{\sigma_t}^{i}(\pi_n)\pi_n(s_{t+1};\sigma_t)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^{i}\right)\right]\mu_{\sigma_t}^{i}(\pi_n)\right)\right)}} = \frac{p\left(\sigma_t\right)}{p\left(\sigma_t; s_{t+1}\right)} \quad (1)$$

This result allows us to use techniques similar to Blume and Easley (2006) to analyze the conditions under which ambiguity-averse consumers can survive.

5 Survival with Ambiguity Aversion

As it is common in the literature, we will say that a consumer i vanishes on a set of paths $\tilde{\Sigma}$ if $\lim_{t\to\infty} \inf c^i\left(\sigma_t\right) = 0$ a.s. (w.r.t. the truth) on $\tilde{\Sigma}$. Consumer i survives on $\tilde{\Sigma}$ if $\lim_{t\to\infty} \sup c^i\left(\sigma_t\right) > 0$ a.s. on $\tilde{\Sigma}$.

The survival of a consumer can in general depend on his preferences, on his discount factor and on his beliefs. In this paper, we concentrate on the impact of ambiguity aversion on survival, while keeping the discount factors and the beliefs of the decision makers identical for most of the discussion. For a given function ϕ, the coefficient of absolute ambiguity aversion is given by: $-\frac{\phi''}{\phi'}$. We distinguish between constant, decreasing and increasing absolute ambiguity aversion, depending on the monotonicity properties of $-\frac{\phi''}{\phi'}$. We will concentrate on the following classes of functions belonging to each of the three categories:

(i) $\phi\left(y\right) = -e^{-\alpha y}$ for some $\alpha > 0$: this is the class of functions ϕ which exhibit *constant absolute ambiguity aversion (CAAA),* i.e., $-\frac{\phi''(y)}{\phi'(y)}$ is constant.

(ii) $\phi\left(y\right) = \ln y$ or $\phi\left(y\right) = y^{\gamma}$ for some $\gamma \in (0;1)$: this is the class of functions ϕ which exhibit constant relative ambiguity aversion, i.e., $-y\frac{\phi''(y)}{\phi'(y)}$ is constant. All these functions also exhibit *decreasing absolute ambiguity aversion (DAAA),* i.e., $-\frac{\phi''(y)}{\phi'(y)}$ is decreasing.

(iii) $\phi\left(y\right) = by - ay^{r}$ for some a, $b > 0$ and $r \geq 2$ with$\left(\frac{b}{ra}\right)^{\frac{1}{r-1}} > \frac{m'}{(1-\beta)}$: these functions exhibit *increasing absolute ambiguity aversion (IAAA),*

i.e., $-\frac{\phi''(y)}{\phi'(y)}$ is increasing, and, in addition, have decreasing second derivative $\phi''(y)$.

We start by analyzing whether ambiguity aversion has an impact on survival for the case of vanishing ambiguity described in Section 3.

Proposition 4 *Consider an economy with vanishing ambiguity, and suppose that all consumers have identical discount factors, $\beta_i = \beta$ for all $i \in \{1...I\}$. Suppose that for a given consumer i,*

(i) i's prior μ_n^i is absolutely continuous with respect to the truth μ_n, i.e. $\mu_n > 0$ implies that $\mu_n^i > 0$;

(ii) the function defined by

$$G^i\left(\mu_{\sigma_t}(\pi_1)...\mu_{\sigma_t}(\pi_N)\right) = \frac{\sum_{n=1}^N \phi_i'(y_n)\,\mu_{\sigma_t}(\pi_n)\,\pi_n(s_{t+1})}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i(y_n)\,\mu_{\sigma_t}(\pi_n)\right)\right)},$$

where y_n are parameters bounded between $\left[0; \frac{1}{1-\beta}u(m')\right]$, is continuously differentiable and its total derivative is uniformly bounded for all values of the parameters.

Then i survives almost surely. In particular, a consumer i whose prior is absolutely continuous w.r.t. to the truth survives whenever ϕ^i is linear or belongs to any of the three categories specified above.

Proposition 4 is in line with the main result in Blume and Easley (2006). With identical discount factors only beliefs matter for survival, while preferences are immaterial. In particular, the absolute degree of ambiguity aversion plays no role in determining which of the consumers will survive, as long as the priors are absolutely continuous with respect to the truth. The additional condition (ii) we have to impose simply requires that a slight change in the posteriors, $\mu_{\sigma_t}(\pi_n)$ leads to a uniformly bounded change in the factor

$$\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_{\sigma_t}^i(\pi_n)\pi_n\left(s_{t+1};\sigma_t\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right]\mu_{\sigma_t}^i(\pi_n)\right)\right)}$$

which takes the place of beliefs in the first-order condition of the smooth ambiguity averse consumers. This implies that when μ_{σ_t} is close to the truth, this factor is close to the Dirac measure assigning a probability of 1 to the true state n. It guarantees that the factor converges to the true probability distribution at the same rate as the beliefs of an expected utility maximizer updated according to the Bayesian rule. The latter is necessary for survival, as shown in Blume and Easley (2006). As our result demonstrates, all most commonly used functional forms satisfy this condition.

Our next result establishes the survival of ambiguity averse consumers for the case of persistent ambiguity and no aggregate uncertainty. In this case, the ambiguity-averse consumer is completely insured against ambiguity and behaves as an expected utility maximizer with correct beliefs.

Proposition 5 *Suppose that all consumers have identical discount factors* $\beta_i = \beta$ *for all* $i \in \{1...I\}$ *and correct beliefs,* $\mu_n^i = \mu_n$ *for all* $n \in \{1...N\}$, $i \in \{1...I\}$. *In an economy with persistent ambiguity and no aggregate risk, i.e.* $\sum_{t=1}^{N} e^i(\sigma_{t-1}; s_t) = \sum_{t=1}^{N} e^i(\sigma_{t-1}; s'_t)$ *for all* s_t *and* $s'_t \in \mathcal{S}_t$ *and all* $t \in \mathbb{N}$, *all consumers survive.*

In the two cases discussed in Propositions (4) and (5), ambiguity-averse consumers effectively mimic expected utility maximizers, either because ambiguity vanishes with time, or because complete insurance against ambiguity coincides with complete insurance against risk, which is available to everyone in the economy.

We now turn to the case of persistent ambiguity. We simplify the notation by writing μ_n as a short-hand for $\mu_{\sigma_t}(\pi_n)$, which in this case is independent of σ_t. To understand the conditions which determine whether ambiguity-averse consumers can survive in an economy in which ambiguity matters in a non-trivial way, we start with the following Lemma:

Lemma 6 *Consider an economy with persistent ambiguity and suppose that Assumptions 1–4 are satisfied. If* i *is a smooth ambiguity-averse consumer, while* j *is an expected utility maximizer with* $\phi_j(y) = y$ *and both have correct beliefs, on any given path* $\sigma \in \Sigma$, *the equilibrium consumption streams of* i *and* j *satisfy:*

$$\lim_{T\to\infty} \frac{1}{T} \ln \frac{u'_i\left(c^i(\sigma_T; s_{T+1})\right)}{u'_j\left(c^j(\sigma_T; s_{T+1})\right)} = \left[\ln \beta_j - \ln \beta_i\right] + \tag{2}$$

$$-\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \ln \frac{\sum_{n=1}^{N} \phi'_i\left[E_{\pi_n}\left(V_i(\sigma_{t+1})\right)\right]\mu_n}{\phi'_i\left(\phi_i^{-1}\left(\sum_{n=1}^{N} \phi_i\left[E_{\pi_n}\left(V_i(\sigma_{t+1})\right)\right]\mu_n\right)\right)} +$$

$$-\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left[\ln \frac{1}{\sum_{n=1}^{N} \mu_n \pi_n(s_{t+1})} \frac{\sum_{n=1}^{N} \phi'_i\left[E_{\pi_n}\left(V_i(\sigma_{t+1})\right)\right]\mu_n \pi_n(s_{t+1})}{\sum_{n=1}^{N} \phi'_i\left[E_{\pi_n}\left(V_i(\sigma_{t+1})\right)\right]\mu_n}\right]$$

This Lemma is key to our following results. The sign of the l.h.s of (2) identifies the cases in which i survives or vanishes. Since consumption is bounded above, $u'_j\left(c^j(\sigma_T; s_{T+1})\right) \not\to 0$. It follows that $\lim_{T\to\infty} \frac{1}{T} \ln \frac{u'_i\left(c^i(\sigma_T; s_{T+1})\right)}{u'_j\left(c^j(\sigma_T; s_{T+1})\right)}$ will be positive on a given path if and only if $u'_i\left(c^i(\sigma_T; s_{T+1})\right) \to \infty$, i.e. if the consumption of i on this path converges to 0 and i disappears. If $\lim_{T\to\infty} \frac{1}{T} \ln \frac{u'_i\left(c^i(\sigma_T; s_{T+1})\right)}{u'_j\left(c^j(\sigma_T; s_{T+1})\right)}$ is negative or zero, consumer i will not disappear relative to j. The r.h.s. of (2) highlights the factors which determine whether i survives. As in Blume and Easley (2006), the first factor is the difference in the discount factors of j and i

— the higher i's discount factor β_i, the more i is going to save, hence, the more wealth he will accumulate relative to j and the higher i's chances for survival.

To understand the second and the third term on the r.h.s. of (2), it is useful to look at the MRS of an expected utility maximizer and a smooth ambiguity-averse decision maker. In an equilibrium, we have:

$$\begin{aligned} & \frac{u_i'\left(c^i\left(\sigma_t\right)\right)}{\beta_i u_i'\left(c^i\left(\sigma_t; s_{t+1}\right)\right) \frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_{\sigma_t}^i(\pi_n)\pi_n(s_{t+1};\sigma_t)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right]\mu_{\sigma_t}^i(\pi_n)\right)\right)}} \\ = & \frac{u_j'\left(c^j\left(\sigma_t\right)\right)}{\beta_j u_j'\left(c^j\left(\sigma_t; s_{t+1}\right)\right)\sum_{n=1}^N \mu_{\sigma_t}^j(\pi_n)\pi_n\left(s_{t+1};\sigma_t\right)} \end{aligned}$$

Note that the factor

$$\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_{\sigma_t}^i(\pi_n)\pi_n\left(s_{t+1};\sigma_t\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right]\mu_{\sigma_t}^i(\pi_n)\right)\right)} \tag{3}$$

in the MRS of an ambiguity-averse decision maker takes the place of the beliefs

$$\sum_{n=1}^N \mu_{\sigma_t}^j(\pi_n)\pi_n\left(s_{t+1};\sigma_t\right)$$

for an expected utility maximizer. While the expression in (3) is not necessarily a probability distribution, we can normalize it to obtain *the effective beliefs of the ambiguity-averse agent:*

$$\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_{\sigma_t}^i(\pi_n)\pi_n\left(s_{t+1};\sigma_t\right)}{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_{\sigma_t}^i(\pi_n)}. \tag{4}$$

The remaining factor is given by

$$\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_{\sigma_t}^i(\pi_n)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right]\mu_{\sigma_t}^i(\pi_n)\right)\right)} \tag{5}$$

and does not depend on the next-period-state, s_{t+1}. It can be interpreted as *an additional discount factor, which is added to the actual discount factor of i, β_i.* We will refer to the expression

$$\beta_i \frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_{\sigma_t}^i(\pi_n)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right]\mu_{\sigma_t}^i(\pi_n)\right)\right)} \tag{6}$$

as *the effective discount factor of the ambiguity-averse decision maker i.*

First consider expression (4). Note that i's effective beliefs will in general differ from i's actual beliefs. In particular, i's effective beliefs will coincide with his actual beliefs if and only if i is fully insured against ambiguity so that:

$$E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right) = E_{\pi_{n'}}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)$$

for all $n, n' \in \{1...N\}$. Hence, even if i's actual beliefs are correct, i.e. $\mu^i_{\sigma_t} = \mu_{\sigma_t}$, his effective beliefs will differ from the truth, unless he is fully insured. As in Blume and Easley (2006), i's beliefs play a crucial role for his survival relative to j. This is reflected in the last term on the r.h.s. of (2), which contains the log of the difference of the effective beliefs of i and the beliefs of j. In expectations, this term will equal the relative entropy of i and j's beliefs with respect to the true probability distribution. Note that if both i and j have correct actual beliefs, then i's effective beliefs (4) will be always wrong (unless he is insured against ambiguity). This will naturally inhibit his chances for survival relative to an expected utility maximizer with correct beliefs. In the absence of the second term on the r.h.s. of (2), we would have thus concluded that with equal discount factors $\left(\beta_i = \beta_j\right)$, ambiguity-averse decision makers would disappear from the market, unless they are fully insured against ambiguity. If they survive, the fact that they are insured against ambiguity would mean that they have no impact on prices.

Infact whenever ambiguity-averse decision makers are fully insured against ambiguity, they will not perceive the consumption stream to be ambiguous. We now define a class of economies in which agents perceive persistent ambiguity.

Definition 7 *Consumer i perceives a consumption stream $\left(e^i\left(\sigma_t\right)\right)_{\sigma_t\in\Sigma}$ to be persistently ambiguous for a given set of probability distributions $\Pi = \{\pi_1...\pi_n\}$ if there exists an $\xi > 0$ such that:*

$$\max_{n,n'\in\{1...N\}}\left|E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(e^i\right)\right) - E_{\pi_{n'}}\left(V^i_{\sigma_{t+1}}\left(e^i\right)\right)\right| \geq \xi$$

for all $t \in N$.

According to the definition, consumer i perceives a consumption stream as persistently ambiguous whenever it does not insure him against ambiguity in period t, i.e., does not provide him with the same discounted utility for all probability distributions $\pi_n \in \Pi$, even as t approaches infinity. Note that the definition incorporates both the objective characteristics of the economy such as the ambiguity characterized by the set Π, as well as subjective characteristics of the consumer, which define the function V^i.

By lemma 6, in economies where all agents have correct beliefs and perceive persistent ambiguity the effective beliefs of ambiguity-averse decision makers are distant from the truth. Hence their only chance of survival is linked to the presence of the second term in (2). Depending on whether the value of the term (5) exceeds, is equal to or is lower than 1, the effective discount factor of the ambiguity-averse decision maker will be higher, equal or lower than his actual discount factor. In particular, if $\beta_i = \beta_j$ and if (5) exceeds one, the additional discount factor will enhance the ambiguity-averse agent's ability to survive.

The decomposition, thus allows us to identify two effects which will influence the chances of survival for an ambiguity-averse agent: his effective beliefs, which in economies where agents perceive persistent ambiguity, differ from the truth and have a negative impact on survival, and, his additional discount factor, which, when larger than 1 has a positive impact on survival. The trade-off between these two effects will determine whether ambiguity-averse agents will survive and have impact on prices and allocations.

Our next results show that the additional discount factor (5) exactly equals 1 for consumers with correct beliefs and costant absolute ambiguity aversion; it is less than one in the case of increasing absolute ambiguity aversion; it is greater than one for the case of decreasing absolute ambiguity aversion.

Lemma 8 *For a smooth ambiguity-averse consumer with $\phi_i(y) = -e^{-\alpha y}$, the additional discount factor in (5) satisfies*

$$\frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right] \mu^i_{\sigma_t}(\pi_n)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N} \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right] \mu^i_{\sigma_t}(\pi_n)\right)\right)} = 1.$$

Hence, the effective discount factor for such a consumer is β_i.

Lemma 9 *For a smooth ambiguity-averse consumer with $\phi_i(y) = by - a(y)^r$, the additional discount factor in (5) satisfies*

$$\frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right] \mu^i_{\sigma_t}(\pi_n)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N} \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right] \mu^i_{\sigma_t}(\pi_n)\right)\right)} \leq 1. \tag{7}$$

Hence, the effective discount factor for such a consumer is smaller than β_i. More generally, condition (7) is satisfied for any function ϕ_i which exhibits IAAA and for which ϕ_i' is concave.

Lemma 10 *For a smooth ambiguity-averse consumer with $\phi_i(y) = \ln(y)$ or with $\phi_i = (y)^\gamma$ for some $\gamma \in (0;1)$, the additional discount factor in (5) satisfies*

$$\frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right] \mu^i_{\sigma_t}(\pi_n)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N} \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right] \mu^i_{\sigma_t}(\pi_n)\right)\right)} \geq 1$$

and the equality is obtained only if

$$E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right) = E_{\pi_{n'}}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)$$

for all $n \in \{1...N\}$. Hence, the effective discount factor for such a consumer is greater than β_i.

By lemmas 8 and 9 when agents are either CAAA or IAAA, the effective discount factor of the ambiguity-averse agent is either equal (in the CAAA case) or smaller (in the IAAA case) than the discount factor of the expected utility maximiser. Hence in economies where consumers perceive the consumption stream to be persistently ambiguous, a CAAA or IAAA consumer will vanish in expectations and with positive probability, even though he might be otherwise identical to the surviving expected utility maximizer. This is stated in our next proposition.

Proposition 11 *Consider an economy with persistent ambiguity and 2 consumers, an expected utility maximizer j, with $\phi_j(y) = y$ and a smooth ambiguity-averse consumer i, whose $\phi_i(y)$ is either CAAA or IAAA with a decreasing second derivative. Suppose that both consumers have identical von-Neumann-Morgenstern utility functions $u_i = u_j$, identical discount factors $\beta_i = \beta_j$, correct beliefs $\mu_n^i = \mu_n^j = \mu_n$ and identical endowments $e^i(\sigma_t) = e^j(\sigma_t)$ for all $\sigma_t \in \Sigma$. Finally, suppose that i perceives his initial endowment as persistently ambiguous. Then i will vanish with positive probability and in expectations.*

Our next result shows that the opposite result obtains for the case of DAAA (or CRAA). Lemma 10 identifies a case for which a higher effective discount factor may compensate for the wrong effective beliefs of an ambiguity averse consumer. Our next result shows that consumers with CRAA survive with positive probability and in expectations.

Proposition 12 *Consider an economy with persistent ambiguity. Let $I = \{i; j\}$. Let i be a smooth-ambiguity-averse consumer with $\phi_i(y) = \ln(y)$, or $\phi_i(y) = (y)^\gamma$ for some $\gamma \in (0; 1)$, correct beliefs and a discount factor β. Let j be an expected utility maximizer with correct beliefs and the same discount factor β. Then i survives with strictly positive probability and in expectations.*

Proposition 12 provides an instance of an ambiguity-averse consumer surviving in a market, despite having effectively wrong beliefs. The fact that his effective discount factor is higher than those of the expected utility maximizer means that the ambiguity-averse consumer saves more in equilibrium, which allows him to survive.

As Lemma 10 demonstrates, ambiguity-averse consumers will be saving more than their expected utility counterparts, even though they might have identical beliefs and identical discount factors. This implies that on those paths on which ambiguity-averse consumers survive, we will observe an excessive equity premium, which would appear to be inconsistent with the actual discount factors, but which can be attributed to the presence of ambiguity-averse investors in the economy. Hence, the equity-premium puzzle can be a persistent phenomenon if ambiguity does not vanish over time and if some of the investors exhibit constant relative ambiguity aversion.

6 Conclusion

In this paper, we analyzed the question of whether smooth ambiguity-averse consumers can survive in the presence of expected utility maximizers. We showed that the answer to this question will depend both on the nature and persistence of ambiguity and risk in the economy and on the degree of ambiguity aversion. We identified situations, in which ambiguity-averse consumers can survive by completely insuring against ambiguity and mimicking the behavior of expected utility maximizers with correct beliefs. However, in this case, ambiguity-aversion will have no impact on prices. When ambiguity in the market is persistent and ambiguity-averse consumers cannot be completely insured against it, their survival depends on the form of the function characterizing their ambiguity-aversion. In particular, consumers with constant relative ambiguity aversion will survive in expectations and with positive probability, regardless of whether they are completely insured against ambiguity. Hence, prices in a market in which ambiguity-averse investors are present can deviate from those in a market populated by expected utility maximizers with correct beliefs.

The analysis so far leaves many questions open. For most of the paper we assumed that all investors in the market have correct beliefs. It would be interesting to examine whether ambiguity-averse investors can survive when their beliefs are wrong. It is obvious that this cannot happen when they are completely insured against ambiguity or when ambiguity vanishes. However, in the case in which complete insurance against ambiguity is not available, the market incompleteness might allow them to survive, even though their predictions deviate from the truth.

Furthermore, it would be interesting to study the case in which both ambiguity-averse consumers and expected utility maximizers have wrong beliefs and examine whether the higher propensity to save of the former will ensure that they have an advantage in terms of survival.

Finally, a more explicit analysis of the price dynamics would be of interest. It would allow us to relate the empirical phenomena documented in financial markets to the results of our model and test whether the presence of ambiguity-averse consumers in the long-run can provide a better explanation to the observed patterns than the standard models based on expected utility.

Proof of Proposition 2:

An equilibrium of the economy exists under the following conditions, see Bewley (1972):

1. the consumption sets are convex, Mackey closed and contained in the set of essentially bounded measurable functions;
2. the preferences of the consumers are complete and transitive;
3. the better sets are convex and Mackey closed;
4. the worse sets are closed in the norm topology;
5. there exists a set of paths with strictly positive measure such that the preferences of all consumers satisfy strict monotonicity on this set, i.e. adding a constant to the payoff in each state and each period makes the consumer strictly better off;
6. for all consumers, the initial endowments are in the interior of the consumptions sets.

W.l.o.g., we can assume that the consumption set of a consumer $i \in \{1...I\}$ is given by the sets of all essentially bounded measurable functions and, hence, satisfies condition 1. Assumption 2 is trivially satisfied, since consumers' preferences are represented by the utility function V^i. In particular, KMM (2008) show that V^i exists and is unique for every consumption stream c^i. To prove convexity, as required by Assumption 3, first compare two streams of consumption c and c' such that $c(\sigma_t) = c'(\sigma_t)$ for all $\sigma_t \neq \sigma_1$. Consider the stream $\alpha c + (1-\alpha) c'$ for some $\alpha \in (0;1)$. Since

$$\begin{aligned}
&V^i_{\sigma_1}(\alpha c + (1-\alpha) c') = u^i(\alpha c(\sigma_1) + (1-\alpha) c'(\sigma_1)) \\
&+\beta_i \phi_i^{-1}\left[\sum_{n=1}^{N} \phi_i \left(\sum_{s_{t+1}\in\mathcal{S}_{t+1}} V^i_{(\sigma_t,s_{t+1})}(c)\pi_n(s_{t+1};\sigma_t)\right)\mu^i_{\sigma_t}(\pi_n)\right] \\
&> \alpha u^i(c(\sigma_1)) + (1-\alpha) u^i(c'(\sigma_1))) \\
&+\beta_i \phi_i^{-1}\left[\sum_{n=1}^{N} \phi_i \left(\sum_{s_{t+1}\in\mathcal{S}_{t+1}} V^i_{(\sigma_t,s_{t+1})}(c)\pi_n(s_{t+1};\sigma_t)\right)\mu^i_{\sigma_t}(\pi_n)\right] \\
&= \alpha V^i_{\sigma_1}(c) + (1-\alpha) V^i_{\sigma_1}(c')
\end{aligned}$$

it follows that the strict convexity of the better sets for such mixtures is implied by the strict concavity of $u(\cdot)$. Now let $c(\sigma_t) = c'(\sigma_t)$ for all $\sigma_t \notin \{\sigma_1\} \cup \Sigma_2$. Note that for each $\sigma_2 \in \Sigma_2$,

$$V^i_{\sigma_2}(\alpha c + (1-\alpha) c') > \alpha V^i_{\sigma_2}(c) + (1-\alpha) V^i_{\sigma_2}(c').$$

Hence, by the strict monotonicity of ϕ_i, and therefore, of ϕ_i^{-1}, we have:

$$V_{\sigma_1}^i \left(\alpha c + (1-\alpha) c'\right) > \alpha V_{\sigma_1}^i (c) + (1-\alpha) V_{\sigma_1}^i (c') .$$

Now suppose that $c(\sigma_t) = c'(\sigma_t)$ for all $\sigma_t \notin \{\sigma_1\} \cup_{t=1}^{T} \Sigma_t$. Then we know that:

$$V_{\sigma_T}^i \left(\alpha c + (1-\alpha) c'\right) > \alpha V_{\sigma_T}^i (c) + (1-\alpha) V_{\sigma_T}^i (c') ,$$

hence,

$$V_{\sigma_{T-1}}^i \left(\alpha c + (1-\alpha) c'\right) > \alpha V_{\sigma_{T-1}}^i (c) + (1-\alpha) V_{\sigma_{T-1}}^i (c') .$$

But then, since

$$u^i(\alpha c(\sigma_{T-2}) + (1-\alpha) c'(\sigma_{T-2})) > \alpha u^i \left(c(\sigma_{T-2})\right) + (1-\alpha) u^i \left(c'(\sigma_{T-2})\right))$$

it follows that

$$V_{\sigma_{T-2}}^i \left(\alpha c + (1-\alpha) c'\right) > \alpha V_{\sigma_{T-2}}^i (c) + (1-\alpha) V_{\sigma_{T-2}}^i (c') .$$

Applying the same argument by induction, we can show that convexity holds w.r.t. any two consumption streams which are constant after some time period t.

Note that each pair of consumption streams c and c' can be represented as a limit of two sequences of consumption streams $\left(c^T\right)_{T\in\mathbb{N}}$ and $\left(c'^T\right)_{T\in\mathbb{N}}$ such that for each $T \in \mathbb{N}$, c^T coincides with c on all paths of length T and is constant for all possible continuations and similarly for c'^T:

$$\begin{aligned} c^T &= \left(\left(c(\sigma_t)\right)_{t\leq T}; k...k...\right) \\ c'^T &= \left(\left(c'(\sigma_t)\right)_{t\leq T}; k...k...\right). \end{aligned}$$

We then have that the pointwise limits of the sequences satisfy:

$$\begin{aligned} \lim_{T\to\infty} c^T &= c \\ \lim_{T\to\infty} c'^T &= c' \\ \lim_{T\to\infty} \left[\alpha c^T + (1-\alpha) c'^T\right] &= \alpha c + (1-\alpha) c' \end{aligned}$$

For all $T \in \mathbb{N}$, we have:

$$V_{\sigma_1}^i \left(\alpha c^T + (1-\alpha) c'^T\right) > \alpha V_{\sigma_1}^i \alpha \left(c^T\right) + (1-\alpha) V_{\sigma_1}^i \left(c'^T\right) .$$

The function V^i is a contraction, see Marinacci and Montrucchio (2007, pp. 7-9), and hence, continuous, implying that:

$$V_{\sigma_1}^i \left(\alpha c + (1-\alpha) c'\right) > \alpha V_{\sigma_1}^i \alpha (c) + (1-\alpha) V_{\sigma_1}^i (c') .$$

We also have that V^i is uniformly continuous, hence, V^i is continuous w.r.t. the Mackey topology. This means that both the better and the worse sets are closed with respect to the Mackey topology, and, hence, also in the norm topology and assumptions 3. and 4. are satisfied.

For condition 5, take the set of paths to be Σ. Note that V^i is monotonic, see KMM (2008). Take any consumption stream c. Clearly, adding a constant $k > 0$ to $c(\sigma_1)$, strictly improves the act. But, similarly, adding a constant to each of the $c(\sigma_2)$ for $\sigma_2 \in \Sigma_2$ leads to a strict increase in $V(\sigma_2)$, and by the monotonicity of ϕ, to a strict increase in the evaluation of the act, etc. Hence, the preferences of all consumers are strictly monotonic on Σ.

Finally, Assumption 3 ensures that the endowment stream of each consumer is uniformly bounded away from 0 and from infinity, and is, therefore, in the interior of this consumer's consumption set. We conclude that an equilibrium of the economy exists. ■

Proof of Proposition 3:

If $p(\cdot)$ is an equilibrium price system, then condition (1) is the first-order condition of consumer i's maximization problem at state σ_t. Hence, it will be satisfied in any equilibrium, in which consumer i chooses an interior allocation on all finite paths with positive probabilities. We now show that Assumptions 1–4 imply that the optimal consumption streams of all consumers will be strictly positive on all finite paths which have positive probability. To show this, we demonstrate that the MRS between consumption at σ_t and at $(\sigma_t; s_{t+1})$ will always be strictly positive and finite, as long as the true probability of σ_t and the conditional probability of s_{t+1} given σ_t are both positive.

First note that since the initial endowment is uniformly bounded, then so is any of the consumption streams in equilibrium and, hence, by Assumption 1, u_i' is always strictly positive. Furthermore, setting $c(\sigma_0) = 0$ is not optimal, since endowment is uniformly bounded away from 0 and $u'(0) = \infty$.

Let σ_t have a positive probability and be such that $u(c(\sigma_t)) > 0$. By the argument above, at least one such σ_t exists. KMM (2008) demonstrate that if the consumption stream is bounded, so is $V^i(c)$, hence, $E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)$ are bounded as well. It follows, by Assumption 2, that $\phi_i'\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right]$ and $\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu^i_{\sigma_t}(\pi_n)\right)\right)$ are also strictly positive. We first show that it is not optimal to choose a consumption path on which $E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right) = 0$ for some, and, hence, by the mutual absolute continuity of the distributions π_n postulated in Assumption 4, for all $n \in \{1...N\}$.

Indeed, assume that in the optimum, $E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right) = 0$ for all $n \in \{1...N\}$. It follows that the continuation of the consumption stream c entails $c(\sigma_t; s) = 0$ for all $s \in S_{t+1}$ and $c(\sigma_t; s; s_{t+2}...s_{t+k}) = 0$ for any continuation of the path $(\sigma_t; s)$. Hence, at node σ_t, consumer i envisions a constant consumption of 0 at all following nodes. Consider a deviation at node σ_t and at all nodes $(\sigma_t; s)$ wit $s \in \mathcal{S}_t$ such that consumption at σ_t is given by $c^i(\sigma_t) - \bar{\epsilon}$ and instead, $c^i(\sigma_t; s) = \epsilon > 0$, with $\bar{\epsilon} p(\sigma_t) = \epsilon \sum_{s \in \mathcal{S}_{t+1}} p(\sigma_t; s)$. Assume that consumption from s_{t+2} on remains at 0 for all continuation paths. Hence, consumer i trades some of his (positive) consumption is σ_t for some strictly positive consumption in all one-step-ahead states of the world. The utility of such a consumption stream at node σ_t is given by:

$$\begin{aligned}
& u_i\left(c^i\left(\sigma_t\right)\right)+\beta_i \phi_i^{-1}\left[\sum_{n=1}^N \phi_i\left(\sum_{s \in \mathcal{S}_{t+1}} V_{\left(\sigma_t, s\right)}^i(c) \pi_n\left(s ; \sigma_t\right)\right) \mu_{\sigma_t}^i\left(\pi_n\right)\right] \\
= & \; u_i\left(c^i\left(\sigma_t\right)\right)+\beta_i \phi_i^{-1}\left[\sum_{n=1}^N \phi_i\left(\sum_{s \in \mathcal{S}_{t+1}} u_i(\epsilon) \pi_n\left(s ; \sigma_t\right)\right) \mu_{\sigma_t}^i\left(\pi_n\right)\right] \\
= & \; u_i\left(c^i\left(\sigma_t\right)\right)+\beta_i \phi_i^{-1}\left[\sum_{n=1}^N \phi_i\left(u_i(\epsilon)\right) \mu_{\sigma_t}^i\left(\pi_n\right)\right] \\
= & \; u_i\left(c^i\left(\sigma_t\right)\right)+\beta_i \phi_i^{-1}\left[\phi_i\left(u_i(\epsilon)\right)\right] \\
= & \; u_i\left(c^i\left(\sigma_t\right)-\bar{\epsilon}\right)+\beta_i u_i(\epsilon) .
\end{aligned}$$

It is obvious that the derivative w.r.t. ϵ at $\epsilon=0$ is ∞, hence any small ϵ represents an improvement over the original plan, in contradiction to the assumption made above.

It follows that in the optimum, $E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)>0$ for all $n \in\{1 \ldots N\}$, and, hence, we can exclude the case in which

$$\sum_{n=1}^N \phi_i^{\prime}\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right] \mu_{\sigma_t}^i\left(\pi_n\right) \pi_n\left(s_{t+1} ; \sigma_t\right) \tag{8}$$

or

$$\phi_i^{\prime}\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right] \mu_{\sigma_t}^i\left(\pi_n\right)\right)\right) \tag{9}$$

equals ∞. Indeed, by Assumption 2, (8) can be ∞ only if $E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)=0$ for some (and, thus all) $n \in\{1 \ldots N\}$ and, similarly, (9) can be ∞ only if

$$\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right] \mu_{\sigma_t}^i\left(\pi_n\right)\right)=0,$$

or $E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)=0$ for all $n \in\{1 \ldots N\}$. It follows that both (8) and (9) are less than ∞. Furthermore, both expressions are strictly positive for all s_{t+1} such that $\pi_n\left(s_{t+1}\right)>0$, which, by Assumption 4 is true, whenever $\sum_{n=1}^N \mu_{\sigma_t}^i\left(\pi_n\right) \pi_n\left(s_{t+1} ; \sigma_t\right)>0$. Since $u^{\prime}(0)=\infty$, this implies that $c^i\left(\sigma_t ; s_{t+1}\right) \neq 0$, whenever $\sum_{n=1}^N \mu_{\sigma_t}^i\left(\pi_n\right) \pi_n\left(s_{t+1} ; \sigma_t\right)>0$. According to assumption 4, however, this is only true if $\sum_{n=1}^N \mu_{\sigma_t}\left(\pi_n\right) \pi_n\left(s_{t+1} ; \sigma_t\right)=0$, or state s_{t+1} indeed has a probability of 0 conditional on σ_t. It follows that, conditional on being in a node σ_t to which i assigns positive consumption, consumer i assigns positive consumption to all nodes $(\sigma_t; s_{t+1})$ which have positive one-step-ahead conditional probabilities given σ_t. Since i will enjoy positive consumption in period 0, forwards induction implies that i will have strictly positive consumption on all finite paths which have positive probability with respect to the truth. This, in turn implies that the first order condition will hold on all such paths. ■

Proof of Proposition 4:

Let $\pi^* \in \{\pi_1 ... \pi_n\}$ be the true distribution of returns. Note that for a constant consumption stream c^i,

$$\frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \left(c^i \right) \right) \right] \mu_{\sigma_t} (\pi_n) \pi_n (s_{t+1})}{\phi_i' \left(\phi_i^{-1} \left(\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \right) \right] \mu_{\sigma_t} (\pi_n) \right) \right)} = \sum_{n=1}^{N} \mu_{\sigma_t} (\pi_n) \pi_n (s_{t+1}).$$

We denote by ϵ_t the sequence describing the rate of convergence of Bayesian updating on μ:

$$\epsilon_t (\sigma) = \left| \mu_{\sigma_t} - \delta_{\pi^*} \right|.$$

Hence, if we can show that

$$\frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \left(c^i \right) \right) \right] \mu_{\sigma_t} (\pi_n) \pi_n (s_{t+1})}{\phi_i' \left(\phi_i^{-1} \left(\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \right) \right] \mu_{\sigma_t} (\pi_n) \right) \right)} \tag{10}$$

converges uniformly to π^* at a rate of at most $\epsilon_t(\sigma)$ on the set of all consumption streams, we would have shown that an ambiguity averse investor learns the truth at least as fast as a Bayesian and, hence, according to Theorem 4 in Blume and Easley (2006) survives almost surely.

Suppose first that for every $n \in \{1...N\}$, the total derivative of the function (10) with respect to $\mu_{\sigma_t}(\pi_n)$ is continuous and uniformly bounded on the set of all possible values of $\phi_i' \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \left(c^i \right) \right) \right]$. Then, G^i is Lipschitz, see Lee (2003, p. 595), and there exists a constant K such that

$$\begin{aligned} & \left| \begin{array}{c} \frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \left(c^i \right) \right) \right] \mu_{\sigma_t} (\pi_n) \pi_n (s_{t+1})}{\phi_i' \left(\phi_i^{-1} \left(\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \right) \right] \mu_{\sigma_t} (\pi_n) \right) \right)} \\ - \frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \left(c^i \right) \right) \right] \tilde{\mu}_{\sigma_t} (\pi_n) \pi_n (s_{t+1})}{\phi_i' \left(\phi_i^{-1} \left(\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \right) \right] \tilde{\mu}_{\sigma_t} (\pi_n) \right) \right)} \end{array} \right| \\ \leq \quad & K \left| \mu_{\sigma_t} - \tilde{\mu}_{\sigma_t} \right| \end{aligned}$$

It follows that for each path σ and for all t,

$$\begin{aligned} & \left| \frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \left(c^i \right) \right) \right] \mu_{\sigma_t} (\pi_n) \pi_n (s_{t+1})}{\phi_i' \left(\phi_i^{-1} \left(\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \right) \right] \mu_{\sigma_t} (\pi_n) \right) \right)} - \frac{\phi_i' \left[E_{\pi_{n^*}} \left(V^i_{\sigma_{t+1}} \left(c^i \right) \right) \right] \pi_{n^*} (s_{t+1})}{\phi_i' \left[E_{\pi_{n^*}} \left(V^i_{\sigma_{t+1}} \right) \right]} \right| \\ = \quad & \left| \frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \left(c^i \right) \right) \right] \mu_{\sigma_t} (\pi_n) \pi_n (s_{t+1})}{\phi_i' \left(\phi_i^{-1} \left(\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V^i_{\sigma_{t+1}} \right) \right] \mu_{\sigma_t} (\pi_n) \right) \right)} - \pi_{n^*} (s_{t+1}) \right| \\ \leq \quad & K \epsilon_t (\sigma) \end{aligned}$$

Since $K\epsilon_t(\sigma)$ has a rate of convergence of $\epsilon_t(\sigma)$, it follows that the beliefs of the smooth ambiguity-averse agents converge to the truth at the same rate as those of the Bayesian expected utility maximizers.

It remains to show that for the special cases considered in this paper, $\phi(y) = -e^{-\alpha y}$, $\phi(y) = \ln y$, $\phi(y) = y^{\gamma}$ and $\phi(y) = \frac{2\alpha}{1-2\alpha y}$, the condition on (10) is indeed satisfied. We start with $\phi(y) = -e^{-\alpha y}$: in this case,

$$\begin{aligned} & \frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n} \left(V_{\sigma_{t+1}}^i \left(c^i \right) \right) \right] \mu_{\sigma_t}(\pi_n) \pi_n(s_{t+1})}{\phi_i' \left(\phi_i^{-1} \left(\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma_{t+1}}^i \right) \right] \mu_{\sigma_t}(\pi_n) \right) \right)} \\ = & \frac{\sum_{n=1}^{N} \alpha e^{-\alpha[E_{\pi_n}(V_i)]} \mu_{\sigma_t}(\pi_n) \pi_n(s_{t+1})}{\sum_{n=1}^{N} \alpha e^{-\alpha[E_{\pi_n}(V_i)]} \mu_{\sigma_t}(\pi_n)}. \end{aligned}$$

The derivative with respect to $\mu_{\sigma_t}(\pi_{\bar{n}})$ is:

$$\begin{aligned} & \frac{\alpha e^{-\alpha\left[E_{\pi_{\bar{n}}}(V_i)\right]} \pi_{\bar{n}}(s_{t+1}) \sum_{n=1}^{N} \alpha e^{-\alpha[E_{\pi_n}(V_i)]} \mu_{\sigma_t}(\pi_n)}{\left(\sum_{n=1}^{N} \alpha e^{-\alpha[E_{\pi_n}(V_i)]} \mu_{\sigma_t}(\pi_n) \right)^2} + \\ & - \frac{\alpha e^{-\alpha\left[E_{\pi_{\bar{n}}}(V_i)\right]} \sum_{n=1}^{N} \alpha e^{-\alpha[E_{\pi_n}(V_i)]} \mu_{\sigma_t}(\pi_n) \pi_n(s_{t+1})}{\left(\sum_{n=1}^{N} \alpha e^{-\alpha[E_{\pi_n}(V_i)]} \mu_{\sigma_t}(\pi_n) \right)^2} \\ = & \frac{\alpha e^{-\alpha\left[E_{\pi_{\bar{n}}}(V_i)\right]} \sum_{n \neq \bar{n}} \alpha e^{-\alpha[E_{\pi_n}(V_i)]} \mu_{\sigma_t}(\pi_n) \left[\pi_{\bar{n}}(s_{t+1}) - \pi_n(s_{t+1}) \right]}{\left(\sum_{n=1}^{N} \alpha e^{-\alpha[E_{\pi_n}(V_i)]} \mu_{\sigma_t}(\pi_n) \right)^2} \end{aligned}$$

Note that $E_{\pi_n}(V)$ is bounded between 0 and an upper bound, M, given by the discounted value of the consumption stream assigning the maximal total endowment of the economy, m' to consumer i. Hence, $\alpha e^{-\alpha\left[E_{\pi_{\bar{n}}}(V_i)\right]}$ is bounded between $\alpha e^{-\alpha M}$ and α, and so the expression in the denominator is bounded between $\left[\left(N e^{-\alpha M} \right)^2 ; (N\alpha)^2 \right]$. Since the numerator is a finite sum of uniformly bounded terms, it is also uniformly bounded. Hence, the derivative with respect to $\mu_{\sigma_t}(\pi_{\bar{n}})$ is indeed finite and uniformly bounded on the set of all possible values of $\phi_i' \left[E_{\pi_n} \left(V_{\sigma_{t+1}}^i \left(c^i \right) \right) \right]$, and, hence, on every path σ. It follows that the total derivative of (10) is also uniformly bounded.

Now consider the case of $\phi(y) = \ln y$. For this case,

$$\begin{aligned} & \frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n} \left(V_{\sigma_{t+1}}^i \left(c^i \right) \right) \right] \mu_{\sigma_t}(\pi_n) \pi_n(s_{t+1})}{\phi_i' \left(\phi_i^{-1} \left(\sum_{n=1}^{N} \phi_i \left[E_{\pi_n} \left(V_{\sigma_{t+1}}^i \right) \right] \mu_{\sigma_t}(\pi_n) \right) \right)} \\ = & \frac{\sum_{n=1}^{N} \frac{\mu_{\sigma_t}(\pi_n) \pi_n(s_{t+1})}{E_{\pi_n}\left(V_{\sigma_{t+1}}^i (c^i) \right)}}{\prod_{n=1}^{N} \left(\frac{1}{E_{\pi_n}\left(V_{\sigma_{t+1}}^i (c^i) \right)} \right)^{\mu_{\sigma_t}(\pi_n)}}. \end{aligned}$$

The first derivative w.r.t. $\mu_{\sigma_t}(\pi_{\bar{n}})$ is:

$$\left[\begin{array}{c} \left(\frac{\pi_{\bar{n}}(s_{t+1})}{E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}(c^i)\right)} - \frac{\pi_N(s_{t+1})}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)\cdot \\ \cdot \prod_{n=1}^{N-1}\left(\frac{1}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)^{\mu_{\sigma_t}(\pi_n)}\left(\frac{1}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)^{1-\sum_{n=1}^{N-1}\mu_{\sigma_t}(\pi_n)} + \\ -\prod_{n=1}^{N-1}\left(\frac{1}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)^{\mu_{\sigma_t}(\pi_n)}\left(\frac{1}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)^{1-\sum_{n=1}^{N-1}\mu_{\sigma_t}(\pi_n)}\cdot \\ \cdot\left(\ln\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}{E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)\sum_{n=1}^{N}\frac{\mu_{\sigma_t}(\pi_n)\pi_n(s_{t+1})}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)} \end{array}\right]\cdot$$

$$\cdot\left[\prod_{n=1}^{N-1}\left(\frac{1}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}\right)^{\mu_{\sigma_t}(\pi_n)}\left(\frac{1}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}\right)^{1-\sum_{n=1}^{N-1}\mu_{\sigma_t}(\pi_n)}\right]^{-2}$$

$$= \frac{\left(\frac{\pi_{\bar{n}}(s_{t+1})}{E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}(c^i)\right)} - \frac{\pi_N(s_{t+1})}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right) - \left(\ln\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}{E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)\sum_{n=1}^{N}\frac{\mu_{\sigma_t}(\pi_n)\pi_n(s_{t+1})}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)}}{\prod_{n=1}^{N-1}\left(\frac{1}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)^{\mu_{\sigma_t}(\pi_n)}\left(\frac{1}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)^{1-\sum_{n=1}^{N-1}\mu_{\sigma_t}(\pi_n)}}$$

$$= \frac{\left(\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}{E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\pi_{\bar{n}}\left(s_{t+1}\right) - \pi_N\left(s_{t+1}\right)\right)}{\prod_{n=1}^{N-1}\left(\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)^{\mu_{\sigma_t}(\pi_n)}} + \tag{11}$$

$$-\frac{\left(\ln\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}{E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)\sum_{n=1}^{N}\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\mu_{\sigma_t}\left(\pi_n\right)\pi_n\left(s_{t+1}\right)}{\prod_{n=1}^{N-1}\left(\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}(c^i)\right)}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)}\right)^{\mu_{\sigma_t}(\pi_n)}}. \tag{12}$$

Now note that for any $n \in \{1...N-1\}$,

$$\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}{E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)} = \frac{\sum_{s\in S}\pi_N\left(s\right)V^i_{\sigma_t;s}\left(c^i\right)}{\sum_{s\in S}\pi_n\left(s\right)V^i_{\sigma_t;s}\left(c^i\right)}.$$

Observe that by Assumption 4,

$$\frac{\delta}{1-\delta} \leq \frac{\sum_{s\in S}\pi_N\left(s\right)V^i_{\sigma_t;s}\left(c^i\right)}{\sum_{s\in S}\pi_n\left(s\right)V^i_{\sigma_t;s}\left(c^i\right)} \leq \frac{1-\delta}{\delta}$$

holds for all possible values of $\left(V^i_{\sigma_t;s}\left(c^i\right)\right)_{s\in S}$. Thus, examining (11), we find that it is uniformly bounded on all paths σ. It follows that the total derivative of (10) with respect to $\left(\mu_{\sigma_t}\left(\pi_n\right)\right)_{n=1}^{N}$ is uniformly bounded, as asserted.

For the case of $\phi(y) = y^{\gamma}$,

$$\begin{aligned}
&\frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right] \mu_{\sigma_t}(\pi_n)\, \pi_n(s_{t+1})}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N} \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right] \mu_{\sigma_t}(\pi_n)\right)\right)} \\
= \quad & \frac{\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma-1} \mu_{\sigma_t}(\pi_n)\, \pi_n(s_{t+1})}{\left(\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma} \mu_{\sigma_t}(\pi_n)\right)^{\frac{\gamma-1}{\gamma}}}.
\end{aligned}$$

The first derivative w.r.t. $\mu_{\sigma_t}(\pi_{\bar{n}})$ is:

$$\begin{bmatrix}
\left(\left[E_{\pi_{\bar{n}}}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma-1} \pi_{\bar{n}}(s_{t+1}) - \left[E_{\pi_N}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma-1} \pi_N(s_{t+1})\right) \cdot \\
\cdot \left(\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma} \mu_{\sigma_t}(\pi_n)\right)^{\frac{\gamma-1}{\gamma}} + \\
-\left(\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma-1} \mu_{\sigma_t}(\pi_n)\, \pi_n(s_{t+1})\right) \cdot \\
\cdot \frac{\gamma-1}{\gamma}\left(\left[E_{\pi_{\bar{n}}}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma} - \left[E_{\pi_N}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma}\right)^{\frac{\gamma-1}{\gamma}} \left(\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma} \mu_{\sigma_t}(\pi_n)\right)^{-\frac{1}{\gamma}}
\end{bmatrix}.$$

$$\cdot \left(\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma} \mu_{\sigma_t}(\pi_n)\right)^{-2\frac{\gamma-1}{\gamma}}$$

$$\begin{bmatrix}
\left(\left[E_{\pi_{\bar{n}}}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma-1} \pi_{\bar{n}}(s_{t+1}) - \left[E_{\pi_N}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma-1} \pi_N(s_{t+1})\right) \cdot \\
\cdot \left(\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma} \mu_{\sigma_t}(\pi_n)\right) + \\
-\frac{\gamma-1}{\gamma}\left(\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma-1} \mu_{\sigma_t}(\pi_n)\, \pi_n(s_{t+1})\right) \cdot \\
\cdot \left(\left[E_{\pi_{\bar{n}}}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma} - \left[E_{\pi_N}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma}\right)^{\frac{\gamma-1}{\gamma}}
\end{bmatrix}.$$

$$\cdot \left(\sum_{n=1}^{N} \left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]^{\gamma} \mu_{\sigma_t}(\pi_n)\right)^{-\left(2-\frac{1}{\gamma}\right)}$$

$$= \left[\begin{array}{c} \left(\left[\frac{E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)} \right]^{\gamma-1} \pi_{\bar{n}}\left(s_{t+1}\right) - \left[\frac{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)} \right]^{\gamma-1} \pi_N\left(s_{t+1}\right) \right) \cdot \\ \cdot \left(\sum_{n=1}^N \left[\frac{E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)} \right]^{\gamma} \mu_{\sigma_t}\left(\pi_n\right) \right) + \\ -\frac{\gamma-1}{\gamma} \left(\sum_{n=1}^N \left[\frac{E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)} \right]^{\gamma-1} \mu_{\sigma_t}\left(\pi_n\right) \pi_n\left(s_{t+1}\right) \right) \left(\left[\frac{E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)} \right]^{\gamma} - 1 \right)^{\frac{\gamma-1}{\gamma}} \end{array} \right] \cdot$$

$$\cdot \left(\sum_{n=1}^N \left[\frac{E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)} \right]^{\gamma} \mu_{\sigma_t}\left(\pi_n\right) \right)^{-\left(2-\frac{1}{\gamma}\right)}$$

By the same argument as above, on any given path σ, all the terms $\frac{E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}{E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)}$ are uniformly bounded above and below. Hence, the total derivative w.r.t. $\left(\mu_{\sigma_t}\left(\pi_n\right)\right)_{n=1}^N$ is also uniformly bounded.

Finally, let $\phi\left(y\right) = by - ay^r$ (where $\left(\frac{b}{ra}\right)^{\frac{1}{r-1}} > \frac{1}{1-\beta} u\left(m'\right)$). Then, $\phi'\left(y\right) = b - ary^{r-1}$ is bounded below by $b - ar\left(\frac{1}{1-\beta}u\left(m'\right)\right)^{r-1}$ and bounded above by b. $\phi''\left(y\right) = -\left(r-1\right) ray^{r-2}$ and is also bounded above and below. The derivative of (10) with respect to $\mu_{\sigma_t}\left(\pi_{\bar{n}}\right)$ has the form:

$$\left(\frac{\sum_{n=1}^N \phi_i' \left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right] \mu_{\sigma_t}\left(\pi_n\right) \pi_n\left(s_{t+1}\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_{\sigma_t}\left(\pi_n\right)\right)\right)} \right)'_{\mu_{\sigma_t}\left(\pi_{\bar{n}}\right)}$$

$$\begin{aligned} = \; & \phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_{\sigma_t}\left(\pi_n\right)\right)\right) \cdot \\ & \cdot \frac{\left[\phi_i'\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right]\pi_{\bar{n}}\left(s_{t+1}\right) - \phi_i'\left[E_{\pi_N}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right]\pi_N\left(s_{t+1}\right)\right]}{\left[\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_{\sigma_t}\left(\pi_n\right)\right)\right)\right]^2} \\ & - \left[\phi_i\left[E_{\pi_{\bar{n}}}\left(V^i_{\sigma_{t+1}}\right)\right] - \phi_i\left[E_{\pi_N}\left(V^i_{\sigma_{t+1}}\right)\right]\right]\phi_i''\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_{\sigma_t}\left(\pi_n\right)\right)\right) \cdot \\ & \cdot \frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right]\mu_{\sigma_t}\left(\pi_n\right)\pi_n\left(s_{t+1}\right)}{\left[\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_{\sigma_t}\left(\pi_n\right)\right)\right)\right]^3} \end{aligned}$$

It is easy to see that since the denominators are uniformly bounded away from 0, and since all other terms are uniformly bounded, the derivative itself is uniformly bounded for all values of $\phi_i'\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\left(c^i\right)\right)\right]$, thus giving the desired result. ■

Proof of Proposition 5:

We showed in the proof of Proposition 2 that the preferences of all consumers are strictly convex. Hence, in the absence of aggregate risk, all consumers will be completely insured and the price ratios will satisfy:

$$\frac{p\left(\sigma_t; s_{t+1}\right)}{p\left(\sigma_t; s'_{t+1}\right)} = \frac{\sum_{n=1}^{N} \mu_{\sigma_t}(\pi_n)\pi_n\left(s_{t+1}; \sigma_t\right)}{\sum_{n=1}^{N} \mu_{\sigma_t}(\pi_n)\pi_n\left(s'_{t+1}; \sigma_t\right)}$$

for all $t \in \mathbb{N}$, all $\sigma_t \in \Sigma$ and all $s_{t+1},\ s'_{t+1} \in \mathcal{S}_{t+1}$. This implies that for all $t \in \mathbb{N}$,

$$V^i_{(\sigma_t, s_{t+1})}(c^i) = V^i_{(\sigma_t, s'_{t+1})}(c^i) = V^i_t\left(c^i\right)$$

for all $s_{t+1},\ s'_{t+1} \in \mathcal{S}_{t+1}$. Hence,

$$\phi_i^{-1}\left[\sum_{n=1}^{N} \phi_i\left(\sum_{s_{t+1}\in\mathcal{S}_{t+1}} V^i_{(\sigma_t, s_{t+1})}(c^i)\pi_n(s_{t+1}; \sigma_t)\right)\mu^i_{\sigma_t}(\pi_n)\right] = V^i_t(c^i)$$

It follows that all consumers in the economy effectively behave as expected utility maximizers with correct beliefs and, therefore, survive almost surely with respect to the truth. ■

Proof of Lemma 6:

By Proposition 3, we have

$$\frac{u'_i\left(c^i\left(\sigma_t\right)\right)}{\beta_i u'_i\left(c^i\left(\sigma_t; s_{t+1}\right)\right)\frac{\sum_{n=1}^{N}\phi'_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)\right]\mu_n\pi_n(s_{t+1})}{\phi'_i\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_n\right)\right)}} = \frac{u'_j\left(c^j\left(\sigma_t\right)\right)}{\beta_j u'_j\left(c^j\left(\sigma_t; s_{t+1}\right)\right)\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}.$$

Hence,

$$\begin{aligned}
\prod_{t=1}^{T}\frac{u'_i\left(c^i\left(\sigma_t\right)\right)}{u'_i\left(c^i\left(\sigma_t; s_{t+1}\right)\right)}\frac{u'_j\left(c^j\left(\sigma_t; s_{t+1}\right)\right)}{u'_j\left(c^j\left(\sigma_t\right)\right)} &= \prod_{t=1}^{T}\frac{\beta_i\frac{\sum_{n=1}^{N}\phi'_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)\right]\mu_n\pi_n(s_{t+1})}{\phi'_i\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_n\right)\right)}}{\beta_j\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)} \\
\frac{u'_i\left(c^i\left(\sigma_1\right)\right)}{u'_i\left(c^i\left(\sigma_T; s_{T+1}\right)\right)}\frac{u'_j\left(c^j\left(\sigma_T; s_{T+1}\right)\right)}{u'_j\left(c^j\left(\sigma_1\right)\right)} &= \prod_{t=1}^{T}\frac{\beta_i\frac{\sum_{n=1}^{N}\phi'_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)\right]\mu_n\pi_n(s_{t+1})}{\phi'_i\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_n\right)\right)}}{\beta_j\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}
\end{aligned}$$

$$\frac{u'_j\left(c^j\left(\sigma_T; s_{T+1}\right)\right)}{u'_i\left(c^i\left(\sigma_T; s_{T+1}\right)\right)} = \frac{u'_j\left(c^j\left(\sigma_1\right)\right)}{u'_i\left(c^i\left(\sigma_1\right)\right)}\prod_{t=1}^{T}\frac{\beta_i\frac{\sum_{n=1}^{N}\phi'_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}(c^i)\right)\right]\mu_n\pi_n(s_{t+1})}{\phi'_i\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V^i_{\sigma_{t+1}}\right)\right]\mu_n\right)\right)}}{\beta_j\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}$$

We are interested in the term $\lim_{T\to\infty}\frac{1}{T}\ln\frac{u'_i\left(c^i\left(\sigma_T; s_{T+1}\right)\right)}{u'_j\left(c^j\left(\sigma_T; s_{T+1}\right)\right)}$. Since by Proposition 3 both $u'_j\left(c^j\left(\sigma_1\right)\right)$ and $u'_i\left(c^i\left(\sigma_1\right)\right)$ are strictly positive and finite, it follows that

$\lim_{T\to\infty}\frac{1}{T}\ln\frac{u_j'\left(c^j(\sigma_1)\right)}{u_i'(c^i(\sigma_1))}=0$. Note that

$$\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right]\mu_n\right)\right)} = \frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\right)\right]\mu_n\right)\right)}\cdot$$
$$\cdot\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_{\sigma_{t+1}}^i\left(c^i\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n},$$

and hence,

$$\lim_{T\to\infty}\frac{1}{T}\ln\frac{u_i'\left(c^i\left(\sigma_T;s_{T+1}\right)\right)}{u_j'\left(c^j\left(\sigma_T;s_{T+1}\right)\right)} = \left[\ln\beta_j-\ln\beta_i\right]-$$
$$-\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T\ln\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\right)\right)}$$
$$+\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T\left[\begin{array}{c}\ln\left[\sum_{n=1}^N\mu_n\pi_n\left(s_{t+1}\right)\right]\\ -\ln\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n}\end{array}\right].$$

■

Proof of Lemma 8:

For $\phi(y)=-e^{-\alpha y}$, we have $\phi'(y)=\alpha e^{-\alpha y}$, $\phi^{-1}(y)=\ln\left(-\frac{1}{y}\right)^{\frac{1}{\alpha}}$, $\phi'\left(\phi^{-1}(y)\right)=\alpha e^{\ln(-y)}=-\alpha y$. Then,

$$\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\right)\right]\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_i\right)\right]\mu_n\right)\right)}$$
$$=\frac{\sum_{n=1}^N \alpha e^{-\alpha\left[E_{\pi_n}\left(V_i\right)\right]}\mu_n}{\sum_{n=1}^N \alpha e^{-\alpha\left[E_{\pi_n}\left(V_i\right)\right]}\mu_n}=1.$$

■

Proof of Lemma 9:

$\phi(y)=by-ay^r$ (where $\left(\frac{b}{ra}\right)^{\frac{1}{r-1}}>\frac{1}{1-\beta}u(m')$). Then, $\phi'(y)=b-ary^{r-1}$ and $\phi''(y)=-(r-1)ray^{r-2}$ is decreasing in y, implying that ϕ' is concave. Then, since ϕ is concave,

$$\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\right)\right]\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^N \phi_i\left[E_{\pi_n}\left(V_i\right)\right]\mu_n\right)\right)}$$
$$\le \frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\right)\right]\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\phi_i\left(\sum_{n=1}^N E_{\pi_n}\left(V_i\right)\mu_n\right)\right)\right)}=\frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\right)\right]\mu_n}{\phi_i'\left(\sum_{n=1}^N E_{\pi_n}\left(V_i\right)\mu_n\right)}$$
$$\le \frac{\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\right)\right]\mu_n}{\left(\sum_{n=1}^N \phi_i'\left[E_{\pi_n}\left(V_i\right)\right]\mu_n\right)}=1.$$

$\blacksquare$

Proof of Lemma 10:

For the case of $\phi(y) = \ln y$, we have $\phi'(y) = \frac{1}{y}$, $\phi^{-1}(y) = e^y$, $\phi'\left(\phi^{-1}(y)\right) = \frac{1}{e^y} = e^{-y}$. Hence, the additional discount factor (5) can be written as

$$\frac{\sum_{n=1}^{N} \phi_i'(y_n)\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i(y_n)\mu_n\right)\right)} = \frac{\sum_{n=1}^{N}\frac{1}{y_n}\mu_n}{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n}} \geq 1,$$

where the inequality immediately follows from the relation between the arithmetic and the geometric mean.

For the case of $\phi(y) = y^\gamma$, we have

$$\frac{\sum_{n=1}^{N} \phi_i'(y_n)\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i(y_n)\mu_n\right)\right)} = \frac{\sum_{n=1}^{N} y_n^{\gamma-1}\mu_n}{\left(\sum_{n=1}^{N} y_n^{\gamma}\mu_n\right)^{\frac{\gamma-1}{\gamma}}}$$

Since for $\gamma \in (0;1)$

$$\left(\sum_{n=1}^{N} y_n^{\gamma-1}\mu_n\right)^{\frac{1}{\gamma-1}} \geq \left(\sum_{n=1}^{N} y_n^{\gamma}\mu_n\right)^{\frac{1}{\gamma}},$$

$$\frac{\sum_{n=1}^{N} y_n^{\gamma-1}\mu_n}{\left(\sum_{n=1}^{N} y_n^{\gamma}\mu_n\right)^{\frac{\gamma-1}{\gamma}}} \geq 1$$

obtains. In both cases, the equality holds if and only if $y_n = y_{n'}$ for all n, $n' \in \{1...N\}$. $\blacksquare$

Proof for Proposition 11:

We will show that i vanishes relative to j, i.e., the ratio of i's consumption relative to j's converges to 0, which would imply that i vanishes. Since $\beta_i = \beta_j$, expression (2) reduces to

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\ln\frac{u_i'(\sigma_T; s_{T+1})}{u_j'(\sigma_T; s_{T+1})} &= -\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\ln\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i(\sigma_{t+1})\right)\right]\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V_i(\sigma_{t+1})\right)\right]\mu_n\right)\right)} \\
&\quad + \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left[\begin{array}{c}\ln\left[\sum_{n=1}^{N}\mu_n\pi_n(s_{t+1})\right] \\ -\ln\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}(V_i(\sigma_{t+1}))\right]\mu_n\pi_n(s_{t+1})}{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}(V_i(\sigma_{t+1}))\right]\mu_n}\end{array}\right]
\end{aligned}$$

By lemma 8, for a CAAA consumer:

$$\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i(\sigma_{t+1})\right)\right]\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V_i(\sigma_{t+1})\right)\right]\mu_n\right)\right)} = 1$$

By lemma 9, for a IAAA consumer:

$$\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\right)\right)}\leq 1$$

Hence we conclude that, for a CAAA consumer:

$$\lim_{T\to\infty}\frac{1}{T}\ln\frac{u_i'\left(\sigma_T;s_{T+1}\right)}{u_j'\left(\sigma_T;s_{T+1}\right)}=\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left[\ln\frac{\left[\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)\right]\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n}{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}\right]. \tag{13}$$

while for a IAAA consumer:

$$\lim_{T\to\infty}\frac{1}{T}\ln\frac{u_i'\left(\sigma_T;s_{T+1}\right)}{u_j'\left(\sigma_T;s_{T+1}\right)}\leq\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left[\ln\frac{\left[\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)\right]\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n}{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}\right]$$

Note that since

$$\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n}$$

is a probability distribution, we have that its relative entropy with respect to the true probability distribution, $\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)$ is positive:

$$\sum_{s_{t+1}\in\mathcal{S}_{t+1}}\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)\left[\ln\frac{\left[\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)\right]\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n}{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}\right]\geq 0.$$

This value will be strictly positive on all paths, on which consumer i is not fully insured. So suppose that i is not fully insured. Then, since each $\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]$ is bounded between 0 and a maximal value, obtained when consumer i receives the maximal value of the entire endowment of the economy m' in each state and in each period, it follows that each of the terms $\phi_i'\left(\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\right)$ is bounded. It follows that all summation terms on the r.h.s. of (13) are uniformly bounded, and so is their average. We conclude that $\frac{1}{T}\ln\frac{u_i'\left(\sigma_T;s_{T+1}\right)}{u_j'\left(\sigma_T;s_{T+1}\right)}$ behaves as a submartingale and, hence, a.s. converges to a random variable with a positive expected value. Hence, i disappears in expectations and with positive probability. ■

Proof of Proposition 12:
First, we rewrite condition (2) as:

$$\lim_{T\to\infty}\frac{1}{T}\ln\frac{u_i'\left(c^i\left(\sigma_T;s_{T+1}\right)\right)}{u_j'\left(c^j\left(\sigma_T;s_{T+1}\right)\right)}=\left[\ln\beta-\ln\beta\right]$$

$$-\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\ln\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\right)\right)}\frac{1}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}$$

$$+\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left[\ln\left[\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)\right]+\ln\left[\frac{1}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}\right]\right]$$

$$=-\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\ln\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\right)\right)}\frac{1}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)} \tag{14}$$

Suppose that i disappears on σ. Since endowment is uniformly bounded below, it follows that the consumption of j is uniformly bounded below. Furthermore, endowment is also bounded above, and hence, $u_j'\left(c^j\left(\sigma_T;s_{T+1}\right)\right)\in\left(u'\left(m'\right);u'\left(m\right)\right)$. Since $\lim_{T\to\infty}c^i\left(\sigma_T\right)=0$,

$$\lim_{T\to\infty}u_i'\left(c^i\left(\sigma_T;s_{T+1}\right)\right)=\infty.$$

It follows that if i disappears, the r.h.s. of the equation must be positive in the limit. Conversely, if the r.h.s. of the equation is positive, $\lim_{T\to\infty}u_i'\left(c^i\left(\sigma_T;s_{T+1}\right)\right)=\infty$.

We proceed in two steps. First, we show that all of the terms on the r.h.s. are non-positive in expectations. Second, we show that the term on the r.h.s. is bounded. This implies that we can apply the Martingale convergence theorem: the r.h.s. converges pointwise to a random variable which is negative in expectation. Since i survives on all paths, on which the r.h.s. of (14) is non-positive, this implies that i survives with positive probability and in expectations.

Step 1: We want to show that:

$$E\left[\ln\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\right)\right)}\frac{1}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}\right]\geq 0. \tag{15}$$

Denote by $y_n=:E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)$.

Case 1: For $\phi\left(y\right)=\ln\left(y\right)$, we have $\phi'\left(y\right)=\frac{1}{y}$, $\phi^{-1}\left(y\right)=e^y$, $\phi'\left(\phi^{-1}\left(y\right)\right)=$

$\frac{1}{e^y} = e^{-y}$. The expression above then reduces to:

$$\begin{aligned}
& E \ln \left[\frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n \left(s_{t+1}\right)}{\exp\left(-\left(\sum_{n=1}^{N} \mu_n \ln y_n\right)\right)} \frac{1}{\sum_{n=1}^{N} \mu_n \pi_n \left(s_{t+1}\right)} \right] \\
= \quad & E \ln \left[\frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n \left(s_{t+1}\right)}{\exp\left(\sum_{n=1}^{N} \mu_n \ln \frac{1}{y_n}\right)} \frac{1}{\sum_{n=1}^{N} \mu_n \pi_n \left(s_{t+1}\right)} \right] \\
= \quad & E \ln \left[\frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n \left(s_{t+1}\right)}{\prod_{n=1}^{N} \left(\frac{1}{y_n}\right)^{\mu_n}} \frac{1}{\sum_{n=1}^{N} \mu_n \pi_n \left(s_{t+1}\right)} \right] \\
= \quad & \sum_{k=1}^{K} \sum_{n=1}^{N} \mu_n \pi_n \left(s_k\right) \ln \left[\frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n \left(s_k\right)}{\prod_{n=1}^{N} \left(\frac{1}{y_n}\right)^{\mu_n}} \frac{1}{\sum_{n=1}^{N} \mu_n \pi_n \left(s_k\right)} \right]
\end{aligned}$$

Note that this term exceeds 0 if and only if

$$\ln \prod_{k=1}^{K} \left[\frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n \left(s_k\right)}{\prod_{n=1}^{N} \left(\frac{1}{y_n}\right)^{\mu_n}} \frac{1}{\sum_{n=1}^{N} \mu_n \pi_n \left(s_k\right)} \right]^{\sum_{n=1}^{N} \mu_n \pi_n (s_k)} \geq 0,$$

or

$$\prod_{k=1}^{K} \left[\frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n \left(s_k\right)}{\prod_{n=1}^{N} \left(\frac{1}{y_n}\right)^{\mu_n}} \frac{1}{\sum_{n=1}^{N} \mu_n \pi_n \left(s_k\right)} \right]^{\sum_{n=1}^{N} \mu_n \pi_n (s_k)} \geq 1$$

We know that the arithmetic mean is larger than the geometric mean:

$$\sum_{n=1}^{N} \frac{1}{y_n} \frac{\mu_n \pi_n \left(s_k\right)}{\sum_{n=1}^{N} \mu_n \pi_n \left(s_k\right)} \geq \prod_{n=1}^{N} \left(\frac{1}{y_n}\right)^{\frac{\mu_n \pi_n (s_k)}{\sum_{n=1}^{N} \mu_n \pi_n (s_k)}}$$

Hence:

$$\prod_{k=1}^{K}\left[\frac{\sum_{n=1}^{N}\frac{1}{y_n}\mu_n\pi_n\left(s_k\right)}{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n}}\frac{1}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_k\right)}\right]^{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}$$

$$\geq \prod_{k=1}^{K}\left[\frac{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\frac{\mu_n\pi_n(s_k)}{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}}}{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n}}\right]^{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}$$

$$= \prod_{k=1}^{K}\frac{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n\pi_n(s_k)}}{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n\sum_{n=1}^{N}\mu_n\pi_n(s_k)}} =$$

$$= \frac{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\sum_{k=1}^{K}\mu_n\pi_n(s_k)}}{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n\sum_{k=1}^{K}\sum_{n=1}^{N}\mu_n\pi_n(s_k)}} = \frac{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n\sum_{k=1}^{K}\pi_n(s_k)}}{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n\sum_{k=1}^{K}\sum_{n=1}^{N}\mu_n\pi_n(s_k)}} =$$

$$= \frac{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n}}{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n}} = 1$$

Case 2: For $\phi\left(y\right) = y^{\gamma}$, (15) reduces to:

$$\sum_{n=1}^{N}\sum_{s_{t+1}\in S}\mu_n\pi_n\left(s_{t+1}\right)\ln\left[\frac{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_{t+1}\right)}\left(\sum_{n=1}^{N}y_n^{\gamma}\mu_n\right)^{\frac{\gamma-1}{\gamma}}\right] \leq 0$$

$$\sum_{n=1}^{N}\sum_{s_{t+1}\in S}\mu_n\pi_n\left(s_{t+1}\right)\ln\left[\frac{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_{t+1}\right)}\right] \leq \ln\left(\sum_{n=1}^{N}y_n^{\gamma}\mu_n\right)^{\frac{1-\gamma}{\gamma}} \quad (16)$$

Since

$$\ln\left(\sum_{n=1}^{N}y_n^{\gamma}\mu_n\right)^{\frac{1-\gamma}{\gamma}} \geq \ln\left(\prod_{n=1}^{N}\left(y_n^{\gamma}\right)^{\mu_n}\right)^{\frac{1-\gamma}{\gamma}},$$

we will show that:

$$\sum_{n=1}^{N}\sum_{s_{t+1}\in S}\mu_n\pi_n\left(s_{t+1}\right)\ln\left[\frac{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_{t+1}\right)}\right]\leq\ln\left(\prod_{n=1}^{N}\left(y_n^{\gamma}\right)^{\mu_n}\right)^{\frac{1-\gamma}{\gamma}},$$

which implies the inequality in (16).

$$\begin{aligned}
\sum_{n=1}^{N}\sum_{s_{t+1}\in S}\mu_n\pi_n\left(s_{t+1}\right)\ln\left[\frac{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_{t+1}\right)}\right] &\leq \ln\prod_{n=1}^{N}y_n^{(1-\gamma)\mu_n}\\
\sum_{n=1}^{N}\sum_{s_{t+1}\in S}\mu_n\pi_n\left(s_{t+1}\right)\ln\left[\frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}\right] &\geq \ln\prod_{n=1}^{N}y_n^{(\gamma-1)\mu_n}
\end{aligned}$$

$$\prod_{k=1}^{K}\left[\frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_k\right)}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_k\right)}\right]^{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}\geq\prod_{n=1}^{N}y_n^{(\gamma-1)\mu_n}$$

$$\frac{\left(\prod_{k=1}^{K}\frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n(s_k)}{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}\right)^{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}}{\prod_{n=1}^{N}y_n^{(\gamma-1)\mu_n}}\geq 1$$

Since the arithmetic mean is larger than the geometric mean, we have:

$$\left[\frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_k\right)}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_k\right)}\right]\geq\left[\prod_{n=1}^{N}\left(y_n^{\gamma-1}\right)^{\frac{\mu_n\pi_n(s_k)}{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}}\right]$$

and we obtain:

$$\frac{\left(\prod_{k=1}^{K}\frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n(s_k)}{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}\right)^{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}}{\prod_{n=1}^{N}y_n^{(\gamma-1)\mu_n}}\geq\frac{\left[\prod_{k=1}^{K}\prod_{n=1}^{N}\left(y_n^{\gamma-1}\right)^{\frac{\mu_n\pi_n(s_k)}{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}}\right]^{\sum_{n=1}^{N}\mu_n\pi_n(s_k)}}{\prod_{n=1}^{N}y_n^{(\gamma-1)\mu_n}}=$$

$$\frac{\prod_{k=1}^{K}\prod_{n=1}^{N}\left(y_n^{\gamma-1}\right)^{\mu_n\pi_n(s_k)}}{\prod_{n=1}^{N}y_n^{(\gamma-1)\mu_n}}=\frac{\prod_{n=1}^{N}\left(y_n^{\gamma-1}\right)^{\sum_{k=1}^{K}\mu_n\pi_n(s_k)}}{\prod_{n=1}^{N}y_n^{(\gamma-1)\mu_n}}=\frac{\left[\prod_{n=1}^{N}\left(y_n^{\gamma-1}\right)^{\mu_n}\right]}{\prod_{n=1}^{N}y_n^{(\gamma-1)\mu_n}}=1.$$

Hence, the desired inequality obtains.

Step 2: We want to show that the term

$$-\ln\frac{\sum_{n=1}^{N}\phi_i'\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\pi_n\left(s_{t+1}\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N}\phi_i\left[E_{\pi_n}\left(V_i\left(\sigma_{t+1}\right)\right)\right]\mu_n\right)\right)}\frac{1}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)}$$

is uniformly bounded.

We begin by showing that each of the terms

$$-\ln \frac{\sum_{n=1}^{N} \phi_i' \left[E_{\pi_n}\left(V_i\left(s_k;\sigma_t\right)\right)\right] \mu_n \pi_n\left(s_k\right)}{\phi_i'\left(\phi_i^{-1}\left(\sum_{n=1}^{N} \phi_i\left[E_{\pi_n}\left(V_i\left(s_k;\sigma_t\right)\right)\right] \mu_n\right)\right)}$$

is bounded from above.

Case 1: For $\phi(y) = \ln y$, rewrite

$$\ln \frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n\left(s_k\right)}{\exp\left(-\left(\sum_{n=1}^{N} \mu_n \ln y_n\right)\right)} \tag{17}$$

as

$$\ln \frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n\left(s_k\right)}{\exp\left(-\left(\sum_{n=1}^{N} \mu_n \ln y_n\right)\right)} = \ln \frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n\left(s_k\right)}{\prod\limits_{n=1}^{N} \left(\frac{1}{y_n}\right)^{\mu_n}} \geq$$

$$\geq \quad \ln \left[\min_{\substack{n\in\{1\ldots N\}\\ k\in\{1\ldots K\}}} \pi_n\left(s_k\right) \frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n}{\prod\limits_{n=1}^{N} \left(\frac{1}{y_n}\right)^{\mu_n}}\right] \geq \ln \left[\min_{\substack{n\in\{1\ldots N\}\\ k\in\{1\ldots K\}}} \pi_n\left(s_k\right)\right] = \ln \delta,$$

where the second weak inequality follows from the relation between the arithmetic and the geometric mean and the last equality results from Assumption 4 combined with the fact that condition (1) holds on those nodes σ_t which are reached with positive probability. We conclude that

$$-\ln \frac{\sum_{n=1}^{N} \frac{1}{y_n} \mu_n \pi_n\left(s_k\right)}{\exp\left(-\left(\sum_{n=1}^{N} \mu_n \ln y_n\right)\right)} \leq -\ln \delta$$

Case 2: For $\phi(y) = y^{\gamma}$, note as above that

$$\ln \frac{\sum_{n=1}^{N} \mu_n \pi_n\left(s_k\right) \left(\sum_{n=1}^{N} y_n^{\gamma} \mu_n\right)^{\frac{\gamma-1}{\gamma}}}{\sum_{n=1}^{N} y_n^{\gamma-1} \mu_n \pi_n\left(s_k\right)} \tag{18}$$

$$\leq \frac{\sum_{n=1}^{N} \mu_n \pi_n\left(s_k\right) \left(\prod_{n=1}^{N} \left(y_n^{\gamma}\right)^{\mu_n}\right)^{\frac{\gamma-1}{\gamma}}}{\sum_{n=1}^{N} y_n^{\gamma-1} \mu_n \pi_n\left(s_k\right)}.$$

Hence,

$$\begin{aligned}\ln \frac{\sum_{n=1}^{N}\mu_n\pi_n\left(s_k\right)\left(\sum_{n=1}^{N}y_n^{\gamma}\mu_n\right)^{\frac{\gamma-1}{\gamma}}}{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_k\right)} &\leq \ln \frac{\sum_{n=1}^{N}\mu_n\pi_n\left(s_k\right)\left(\prod\left(y_n^{\gamma}\right)^{\mu_n}\right)^{\frac{\gamma-1}{\gamma}}}{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_k\right)} = \\ -\ln \frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_k\right)}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_k\right)\prod y_n^{(\gamma-1)\mu_n}} &\leq -\ln\left[\min_{\substack{n\in\{1\ldots n\}\\ s_{t+1}\in S_{t+1}}}\pi_n\left(s_k\right)\frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_k\right)\prod y_n^{(\gamma-1)\mu_n}}\right] \leq -\ln\delta.\end{aligned}$$

We thus conclude that

$$-\ln \frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_k\right)}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_k\right)\left(\sum_{n=1}^{N}y_n^{\gamma}\mu_n\right)^{\frac{\gamma-1}{\gamma}}} \leq -\ln\delta.$$

It follows that for all T,

$$-\frac{1}{T}\sum_{t=1}^{T}\ln\frac{\sum_{n=1}^{N}\frac{1}{y_n}\mu_n\pi_n\left(s_{t+1}\right)}{\exp\left(-\left(\sum_{n=1}^{N}\mu_n\ln y_n\right)\right)} \leq -\ln\delta$$

and

$$-\frac{1}{T}\sum_{t=1}^{T}\ln\frac{\sum_{n=1}^{N}y_n^{\gamma-1}\mu_n\pi_n\left(s_{t+1}\right)}{\sum_{n=1}^{N}\mu_n\pi_n\left(s_{t+1}\right)\left(\sum_{n=1}^{N}y_n^{\gamma}\mu_n\right)^{\frac{\gamma-1}{\gamma}}} \leq -\ln\delta,$$

thus establishing a uniform upper bound on the r.h.s. of (14).

Next, we show that the r.h.s. of (14) is also uniformly bounded below. Note that the l.h.s. of (14) will be negative for a given T only if $\frac{u_i'\left(c^i(\sigma_T;s_{T+1})\right)}{u_j'\left(c^j(\sigma_T;s_{T+1})\right)} < 1$, hence only if $\frac{u_i'\left(c^i(\sigma_T;s_{T+1})\right)}{u_j'\left(2m-c^i(\sigma_T;s_{T+1})\right)} < 1$, where m is the lower bound on the initial endowment of each of the consumers. Since u_i' and u_j' are decreasing, and by Assumption 1, $u_i'(0) = u_j'(0) = \infty$, the function $\frac{u_i'\left(c^i\right)}{u_j'\left(2m-c^i\right)}$ is strictly decreasing in c^i, converges towards ∞ for $c^i = 0$ and towards 0 for $c^i = 2m$. It follows that there exists a unique $\bar{c}^i \in (0;2m)$ which satisfies $\frac{u_i'\left(\bar{c}^i\right)}{u_j'\left(2m-\bar{c}^i\right)} = 1$ such that $\frac{u_i'\left(c^i(\sigma_T;s_{T+1})\right)}{u_j'\left(c^j(\sigma_T;s_{T+1})\right)} < 1$ if and only if $c^i\left(\sigma_T;s_{T+1}\right) \geq \bar{c}^i > 0$. This, in turn, implies a strictly positive lower bound on every $y_n = E_{\pi_n}\left[V^i\left(\sigma_{T+1}\right)\right]$, which is obtained when i's consumption stream assigns $\bar{c}_i$ to the node $\left(\sigma_T;s_{T+1}\right)$ and 0-consumption on all consecutive nodes. Simultaneously, there is also an upper bound on y_n defined by the maximal consumption i can enjoy in a given period, m'. Denote by $y_{\min}$ and $y_{\max}$ the minimal and maximal values of y_n. Then, we obtain:

$$-\ln\frac{\sum_{n=1}^{N}\frac{1}{y_n}\mu_n\pi_n\left(s_{T+1}\right)}{\prod_{n=1}^{N}\left(\frac{1}{y_n}\right)^{\mu_n}\sum_{n=1}^{N}\mu_n\pi_n\left(s_{T+1}\right)} \geq -\ln\frac{y_{\max}}{y_{\min}}$$

for every T for which $\frac{u_i'\left(c^i(\sigma_T;s_{T+1})\right)}{u_j'(c^j(\sigma_T;s_{T+1}))} < 1$.

Similarly, for the case $\phi(y) = y^\gamma$, we obtain:

$$-\ln \frac{\sum_{n=1}^N \mu_n \pi_n (s_{t+1}) \left(\sum_{n=1}^N y_n^\gamma \mu_n\right)^{\frac{\gamma-1}{\gamma}}}{\sum_{n=1}^N y_n^{\gamma-1} \mu_n \pi_n (s_{t+1})} \geq -\ln \left(\frac{y_{\max}}{y_{\min}}\right)^{1-\gamma}$$

for every T for which $\frac{u_i'\left(c^i(\sigma_T;s_{T+1})\right)}{u_j'(c^j(\sigma_T;s_{T+1}))} < 1$. Choose an arbitrary T: we have that either

$$-\frac{1}{T}\sum_{t=1}^{T} \ln \frac{\sum_{n=1}^N \frac{1}{y_n}\mu_n \pi_n (s_{t+1})}{\exp\left(-\left(\sum_{n=1}^N \mu_n \ln y_n\right)\right)} \geq 0$$

or

$$-\frac{1}{T}\sum_{t=1}^{T} \ln \frac{\sum_{n=1}^N \frac{1}{y_n}\mu_n \pi_n (s_{t+1})}{\exp\left(-\left(\sum_{n=1}^N \mu_n \ln y_n\right)\right)} < 0.$$

If the latter is true, choose the largest T' smaller than T such that:

$$-\frac{1}{T'}\sum_{t=1}^{T'} \ln \frac{\sum_{n=1}^N \frac{1}{y_n}\mu_n \pi_n (s_{t+1})}{\exp\left(-\left(\sum_{n=1}^N \mu_n \ln y_n\right)\right)} \geq 0$$

and note that $\ln \frac{\sum_{n=1}^N \frac{1}{y_n}\mu_n \pi_n (s_{T+1})}{\exp\left(-\left(\sum_{n=1}^N \mu_n \ln y_n\right)\right)} \geq -\ln \frac{y_{\max}}{y_{\min}}$ for all $\tilde{T} \in \{T'+1...T\}$. It follows that for each T,

$$-\frac{1}{T}\sum_{t=1}^{T} \ln \frac{\sum_{n=1}^N \frac{1}{y_n}\mu_n \pi_n (s_{t+1})}{\exp\left(-\left(\sum_{n=1}^N \mu_n \ln y_n\right)\right)} \geq \min\left\{-\ln \frac{y_{\max}}{y_{\min}}; 0\right\}.$$

By analogy, we also obtain:

$$-\frac{1}{T}\sum_{t=1}^{T} \ln \frac{\sum_{n=1}^N y_n^{\gamma-1} \mu_n \pi_n (s_{t+1})}{\sum_{n=1}^N \mu_n \pi_n (s_{t+1}) \left(\sum_{n=1}^N y_n^\gamma \mu_n\right)^{\frac{\gamma-1}{\gamma}}} \geq \min\left\{-\ln \left(\frac{y_{\max}}{y_{\min}}\right)^{1-\gamma}; 0\right\}.$$

Combining Steps 1 and 2, we conclude that $-\frac{1}{T}\ln \frac{u_i'\left(c^i(\sigma_T;s_{T+1})\right)}{u_j'(c^j(\sigma_T;s_{T+1}))}$ is a bounded supermartingale. Hence, it converges almost surely to a random variable, the expectation of which is negative. Therefore, $\lim_{T\to\infty} u_i'\left(c^i(\sigma_T; s_{T+1})\right) \neq \infty$ holds with strictly positive probability and in expectations. ■

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