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Value-at-Risk for fixed income portfolios: A Kalman filtering approach

We propose a way of measuring the risk of a sovereign debt portfolio by using a simple two-factor short rate model. The model is calibrated from data and then the changes in the bond prices are simulated by using a Kalman filter. The bond prices being simulated remain arbitrage-free, in contrast with principal component analysis based strategies for simulation and risk measurement of debt portfolios. In liquid sovereign debt markets, a risk measurement methodology which allows the future bond price scenarios to be arbitrage-free may be seen as a potentially more realistic way of measuring the debt portfolio risk due to interest rate fluctuations. We demonstrate the performance of this methodology with calibration and backtesting, both on simulated data as well as on a real portfolio of US government bonds.

Keywords: Keywords: Value-at-Risk, fixed income portfolios

1. Introduction

Financial institutions need to monitor and effectively manage market risk. Quantitative risk measures have become crucial management instruments for portfolio managers for this purpose. Value-at-Risk (VaR) has been chosen by the Basel Committee on Banking Supervision in Basel II as the standard risk measure for financial risk managers, see Basel Committee (2006) and Chen and Gerlach (2011) for details. VaR has received criticism by Artzner et al. (1999), Acerbi and Tasche (2002) and Szegö (2002) for not being a coherent measure of risk. A sub-additive alternative to VaR is the conditional Value-at-Risk (CVaR). Its minimization formula was first developed in Rockafellar and Uryasev (2000). VaR and CVaR at a level α are respectively given by:

$$VaR_\alpha = -q_{1-\alpha}(\mathscr{X}), \tag{1.1}$$
$$CVaR_\alpha = -\mathbb{E}\left[\mathscr{X} \,|\, \mathscr{X} \leqslant q_{1-\alpha}(\mathscr{X})\right], \tag{1.2}$$

where $q_\beta(\mathscr{X}) = \inf\{x | F_{\mathscr{X}}(x) \geqslant \beta\}$ is the β-quantile of $\mathscr{X}$.

The purpose of this paper is to measure the risk of fixed income portfolios which originates from the uncertainty in the interest rates, using VaR as a risk measure.

Several mathematical tools have been applied to model the term structure of interest rates. A first approach is using tools that smooth the yield term structure. This includes the approaches suggested in McCulloch (1975), Vasicek and Fong (1982) and Fisher et al. (1995), among others. In this article, we consider exponential affine term structure models, which is a class of models often employed to understand the dynamics of interest rates. Seminal work on this class has been provided by Vasicek (1977), Cox et al. (1985) where, respectively, Gaussian and non-Gaussian single factor models are proposed. As pointed out in Brigo and Mercurio (2006) there is evidence that the single factor models fail to explain accurately some of the features of the term structure and it is necessary to consider extensions to these models. A generalization of the single-factor models to higher dimensions have been presented e.g. in Chaplin and Sharp (1993), Chen (1995) and Duffie and Kan (1996), while Jegadeesh and Pennacchi (1996) and De Rossi (2004) focused on Gaussian multi-factor models. Babbs and Nowman (1999)

looked at one, two and three factor models on US data and found that the two-factor model performed better for the term structure of the interest rates in the US bond market. In Dai and Singleton (2000), the authors provided a classification of affine term structure models into non-nested families and explored the structural properties of each family of models. In Duffee (2002), an extension to these results was provided, by allowing the price of risk to vary independently of the interest rate volatility. Later in Ait-Sahalia and Kimmel (2010), closed-form approximations were provided for likelihood functions of one, two and three factor models under the framework developed in Dai and Singleton (2000).

The interest rate term structure can be described by single or multiple factors. A three-factor model is often preferred to describe the behaviour of economic variables. However, when it comes to out-of-sample prediction, the authors in Date and Wang (2009) state that a two-factor Gaussian model performs better than a single-factor model or a three-factor model when comparing out-of-sample one-step ahead forecasting. A two-factor model seems to offer a good compromise between the difficulty of calibrating a three factor model and the poor accuracy of a one factor model. Hence a two-factor model has been chosen for the current analysis.

The computation of Value-at-Risk for fixed income portfolios has attracted a lot of interest and a few different approaches are available in the literature. In Smith (2009), the author calculates VaR for a three zero-coupon bond portfolio by focusing on the yield volatilities and correlations. The performance of fixed-income Brazilian portfolios are assessed in Carvalhal and Daumas (2010) by computing several measures including VaR and CVaR, which are calculated assuming a normal distribution of interest rate variations. The authors conclude that the choice criteria based on minimum VaR and CVaR achieve satisfactory results. In Darbha (2001), VaR for fixed income portfolios is computed using extreme value theory. The VaR estimates so obtained are then compared to the estimates found using the variance/covariance method and the historical simulation method, concluding that the extreme value method provides the best VaR estimates. A methodology often employed to assess VaR for fixed income portfolios is scenario simulation from principal components of the yield curve. In Jamshidian and Zhu (1997) the bond prices are modeled using a small number of risk factors, the joint distribution is approximated using a multivariate discrete distribution and VaR is computed by selecting the appropriate quantile of the discrete cumulative distribution function. Gibson and Pritsker (2000) pointed out that an appropriate choice of risk factors is crucial for this methodology, and also advocated using a continuous distribution to model the extracted risk factors. In Fiori and Iannotti (2007), the authors apply principal component analysis (PCA) to Monte Carlo simulation considering the non-normality of historical observations. Another approach is suggested in Chen et al. (2007) where the authors use independent component analysis (ICA), a tool utilized in sound engineering, to calculate VaR for foreign exchange rate portfolios.

We present an alternative way to the extant works on VaR computation for fixed income portfolios. We model the instantaneous interest rate using a two-factor Vasicek model. After calibration of the model, arbitrage-free future bond prices are simulated by using a Kalman filter and are used to compute the portfolio loss quantiles of interest. We demonstrate the performance of this methodology with calibration and backtesting on simulated data as well as on a real portfolio of US government bonds.

The Kalman filter, first proposed in Kalman (1960), is a mathematical tool used to estimate the variables of interest which are not directly observable. It covers a wide range of applications in different fields including signal processing, weather sciences, econometrics and finance. In particular, in finance it has proved to be a useful instrument for a variety of purposes, including the estimation of instanta-

neous interest rate implied by the yield curve (see, e.g. Babbs and Nowman (1999) and Bolder (2001)), the estimation of the spot prices of commodities from the futures prices (see Schwartz (1997) and Manoliu and Tompaidis (2002)) and updating the uncertain drift parameters in the context of hedging in incomplete markets (Monoyios (2007)). A review of applications of filtering in financial mathematics is provided in Date and Ponomareva (2011).

The novelty of the work presented here lies in exploiting the filtering methodology for simplifying the scenario generation from the point of view of measuring the portfolio risk due to interest rate uncertainty. To authors' knowledge, this is the first time that the ability of Kalman filter to generate one step ahead conditional distribution of the short rate has been exploited for efficient simulation of arbitrage-free bond prices using a two-factor Gaussian model. The fact that only arbitrage-free scenarios are generated may be seen as an added advantage on some of the existing methods, such as those based on the principal component analysis. We present comprehensive numerical studies, including formal backtesting based on industry-standard hypothesis tests, with both simulated and real data for VaR computation on fixed income portfolios using the Kalman filter.

The rest of the paper is structured as follows. Section 2 introduces the recursive equations for the Kalman filter and some of their features that will be used later. Section 3 provides the preliminary definitions for the term structure of interest rates and presents the two-factor Vasicek model, highlighting some of the key relationships. Section 4 includes details on the backtesting tools used for validation of the VaR model. Sections 5 and 6 report numerical experiments with simulated data and real data, respectively. In both these sections we compare how our method performs against the historical simulation method and the variance/covariance method for computing VaR. Section 7 concludes the article.

2. The Kalman filter

Let us consider a discrete time, linear state space system:

$$\mathbf{r}_{n+1} = A\mathbf{r}_n + \mathbf{b} + \mathbf{w}_{n+1}, \tag{2.1}$$

$$\mathbf{y}_n = C\mathbf{r}_n + \mathbf{d} + \mathbf{z}_n, \tag{2.2}$$

where $\mathbf{w}_n$ and $\mathbf{z}_n$ are zero mean, Gaussian and uncorrelated random variables at each time t_n. A, $\mathbf{b}$, C, $\mathbf{d}$, $G = \mathbb{E}(\mathbf{w}_n\mathbf{w}_n^T)$ and $H = \mathbb{E}(\mathbf{z}_n\mathbf{z}_n^T)$ are constants or are known functions of time. The variable $\mathbf{y}_n$ is the only observable variable, while $\mathbf{r}_n$ can not be directly observed and needs to be estimated. Each time-step $\Delta t = t_n - t_{n-1}$ is assumed to be constant. Equations (2.1) and (2.2) are referred to as the *transition equation* and the *measurement equation*, respectively.

There are different versions of the *Kalman filter* (KF) equations including the one reported e.g. in Date and Wang (2009). The KF consists in a set of recursive equations; the one employed in this paper involves the following:

$$\mathbf{v}_n = \mathbf{y}_n - C\hat{\mathbf{r}}_{n|n-1} - \mathbf{d}, \tag{2.3}$$

$$\Sigma_n = CV_{n|n-1}C^T + H, \tag{2.4}$$

$$K_n = V_{n|n-1}C^T\Sigma_n^{-1}, \tag{2.5}$$

$$\hat{\mathbf{r}}_{n|n} = \hat{\mathbf{r}}_{n|n-1} + K_n v_n, \tag{2.6}$$

$$\hat{\mathbf{r}}_{n+1|n} = A\hat{\mathbf{r}}_{n|n} + \mathbf{b}, \tag{2.7}$$

$$V_{n+1|n} = AV_{n|n-1}A^T + G - AV_{n|n-1}C^T\Sigma_n^{-1}CV_{n|n-1}A^T. \tag{2.8}$$

The estimation of the variable of interest $\mathbf{r}_n$ and the conditional variance of the estimate V_n based on information up to time t_{n-1} are respectively denoted by $\hat{\mathbf{r}}_{n|n-1}$ and $V_{n|n-1}$. Initial estimates $\hat{\mathbf{r}}_{0|0}$ and $V_{1|0}$ are assumed to be known or can themselves be parameterized. The innovations $\mathbf{v}_n$ and their covariance matrix Σ_n are expressed by equations (2.3) and (2.4) respectively, while K_n, often referred to as the Kalman gain, is given by the equation (2.5). Equation (2.6) represents the filtered state vector. The state vector and the covariance matrix predictions are respectively provided by equations (2.7) and (2.8). The calibration of the set of parameters which characterize the matrices A, C, G, H and the vectors $\mathbf{b}$, $\mathbf{d}$ are obtained through maximizing the likelihood of observations. Having the set of observations $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N\}$ and since $\mathbf{y}_{n+1|n} \sim \mathcal{N}(C\hat{\mathbf{r}}_{n+1|n} + \mathbf{d}, \Sigma_{n+1})$, we can express the log likelihood function, ignoring the constant terms, as:

$$\log L(Y) = -\frac{1}{2}\sum_{i=1}^{N}(\log|\Sigma_i| + \mathbf{v}_i^T \Sigma_i^{-1}\mathbf{v}_i). \tag{2.9}$$

The expression (2.9) can be maximized in MATLAB 7.9 using the inbuilt solvers such as fminsearch over the set of model parameters. Once the optimal parameters are obtained, one can forecast the successive values for the latent variable $\mathbf{r}$ as long as new observations $\mathbf{y}$ become available. This is implemented employing the recursive equations (2.3)-(2.8), with A, $\mathbf{b}$, C, $\mathbf{d}$, G and H expressed as appropriate functions of the optimized parameters.

3. Preliminary definitions for the term structure and the two-factor Vasicek model

The relationship between the interest rates and the corresponding time to maturity is called the term structure of interest rates. For clarity, some essential definitions are henceforth briefly introduced. A zero-coupon bond is a contract that pays at its maturity, T, one unit of its currency. Its value at time t is denoted by $P(t,T)$, with $t < T$. Given the price of a pure discount bond having maturity T, the bond yield (or spot rate or zero-coupon yield) associated to a particular date t is given by:

$$y(t,T) = -\frac{\ln P(t,T)}{T-t}, \tag{3.1}$$

while the instantaneous interest rate is defined by taking the limit as the time to maturity tends to zero:

$$r(t) = -\frac{\partial \ln P(t,T)}{\partial t}. \tag{3.2}$$

The price of a zero-coupon bond with maturity T at time t can be obtained by inverting this equation as:

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}\middle|\mathscr{F}_t\right],$$

where $\mathscr{F}_t$ is the natural filtration for the process and the expectation is taken under the risk neutral measure $\mathbb{Q}$. Hence modeling the variation of the instantaneous interest rate $r(t)$ over time affects the evolution of bond prices and other derivative prices, and ultimately the bond yields. The bond yields are observable quantities, while $r(t)$ is a latent variable. We need to predict $r(t)$ if we want arbitrage-free forecasts of bond yields. In exponential affine models, the yields depend affinely on the latent variable $r(t)$. In such cases, it is possible to estimate the latent variables recursively, in a computationally tractable fashion, from the observable bond yields using the Kalman filter, which was described earlier in section 2.

The key assumption of the two-factor Vasicek model is that the short term interest rates are given by the sum of two state variables, each of them following an Ornstein-Uhlenbeck process. Let us consider two independent state variables that follow linear, mean reverting Gaussian process under the risk neutral measure $\mathbb{Q}$:

$$r(t) = r_1(t) + r_2(t), \tag{3.3}$$

$$dr_i(t) = k_i(\theta_i - r_i(t))dt + \sigma_i dW_i(t), \qquad r_i(0) = r_{i0} \qquad i = 1,2, \tag{3.4}$$

where r_{i0}, k_i, θ_i and σ_i are positive constants, and $W_i(t)$ are uncorrelated $\mathbb{Q}$-Wiener processes. Each $r_i(t)$ conditional to $\mathscr{F}_s$ is normally distributed with mean and variance (see, e.g. Brigo and Mercurio (2006)):

$$\mathbb{E}[r_i(t)|\mathscr{F}_s] = r_i(s)e^{-k_i(t-s)} + \theta_i(1 - e^{-k_i(t-s)}), \qquad i = 1,2, \tag{3.5}$$

$$Var(r_i(t)|\mathscr{F}_s) = \frac{\sigma_i^2}{2k_i}(1 - e^{-2k_i(t-s)}), \qquad i = 1,2. \tag{3.6}$$

We discretize the two equations (3.5) and (3.6) considering evenly spaced observation times $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_N$, with $t_{n+1} - t_n = \Delta t$, obtaining (see, e.g. Bolder (2001)) the following transition equations in the same form as expressed in (2.1):

$$\begin{bmatrix} r_{1\,n+1} \\ r_{2\,n+1} \end{bmatrix} = \begin{bmatrix} e^{-k_1\Delta t} & 0 \\ 0 & e^{-k_2\Delta t} \end{bmatrix} \begin{bmatrix} r_{1\,n} \\ r_{2\,n} \end{bmatrix} + \begin{bmatrix} \theta_1(1 - e^{-k_1\Delta t}) \\ \theta_2(1 - e^{-k_2\Delta t}) \end{bmatrix} + \begin{bmatrix} w_{1\,n+1} \\ w_{2\,n+1} \end{bmatrix}, \tag{3.7}$$

where $w_{n+1|n} \sim \mathscr{N}(\mathbf{0}, G)$, with:

$$G = \begin{bmatrix} \frac{\sigma_1^2}{2k_1}(1 - e^{-2k_1\Delta t}) & 0 \\ 0 & \frac{\sigma_2^2}{2k_2}(1 - e^{-2k_2\Delta t}) \end{bmatrix}. \tag{3.8}$$

This discretisation preserves the exact conditional mean $\mathbb{E}[r_i(t+\Delta t)|\mathscr{F}_t]$ and the exact conditional variance $var[r_i(t+\Delta t)|\mathscr{F}_t]$. Let us assume that each state variable that makes up the short interest rate follows a linear, mean reverting Gaussian process with the same volatility but a different drift function under measure $\mathbb{P}$:

$$dr_i(t) = k_i(\tilde{\theta}_i - r_i(t))dt + \sigma_i d\widetilde{W}_i(t), \qquad r_i(0) = r_{i0}, \qquad i = 1,2, \tag{3.9}$$

where $\widetilde{W}_i(t)$ is a $\mathbb{P}$-Wiener process. Our assumption of an arbitrage-free market indirectly implies the existence of processes $\lambda_i(t)$ such that $\tilde{\theta}_i - \theta_i = \sigma_i\lambda_i(t)$ for $i = 1,2$ holds. It is common practice (see, e.g. De Rossi (2004) or Vasicek (1977)) to assume $\lambda_i(t) = \lambda_i$ to be constants, independent of t and $r_i(t)$.

The bond price function for the two-factor Vasicek model has the following analytical form:

$$P(t,T,r_1(t),r_2(t)) = e^{E(t,T) - F_1(t,T)r_1(t) - F_2(t,T)r_2(t)},$$

where

$$E(t,T) = \sum_{i=1}^{2} \frac{\left(k_i^2(\theta_i - \frac{\sigma_i\lambda_i}{k_i}) - \frac{\sigma_i^2}{2}\right)(F_i(t,T) - (T-t))}{k_i^2} - \frac{\sigma_i^2 F_i^2(t,T)}{4k_i}, \tag{3.10}$$

$$F_i(t,T) = \frac{1}{k_i}(1 - e^{-k_i(T-t)}), \qquad i = 1,2, \tag{3.11}$$

where λ_i is the market price of risk for the i^{th} factor.

Note that it is quite straightforward to use correlated Wiener processes in this framework, at the risk of complicating the notation as well as the calibration procedure; see, *e.g.* Date and Wang (2009) for a treatment with correlated factors. However, in authors' experience, adding correlation tends to induce numerical difficulties in likelihood maximisation, without a compensatory improvement in out-of-sample performance. Besides, even though $r_1(t)$ and $r_2(t)$ are themselves uncorrelated, the bond prices (and the yields) are (imperfectly) correlated since they are functions of $r_1(t)$ and $r_2(t)$. Hence we will continue with the use of uncorrelated random factors.

The measurement system we used involves the following relationship between zero-coupon yields and the price of zero-coupon bonds:

$$y(t,T) = -\frac{\ln P(t,T)}{T-t} = \frac{-E(t,T) + \sum_{i=1}^{2} F_i(t,T)\, r_i(t)}{T-t}. \tag{3.12}$$

Using this equation at each t_n, for a set of m bonds with maturities $T_1, \ldots, T_m$ leads to the following vector valued equation:

$$\begin{bmatrix} y(t_n,T_1) \\ y(t_n,T_2) \\ \vdots \\ y(t_n,T_m) \end{bmatrix} = \begin{bmatrix} \frac{F_1(t_n,T_1)}{T_1-t_n} & \frac{F_2(t_n,T_1)}{T_1-t_n} \\ \frac{F_1(t_n,T_2)}{T_2-t_n} & \frac{F_2(t_n,T_2)}{T_2-t_n} \\ \vdots & \vdots \\ \frac{F_1(t_n,T_m)}{T_m-t_n} & \frac{F_2(t_n,T_m)}{T_m-t_n} \end{bmatrix} \begin{bmatrix} r_1(t_n) \\ r_2(t_n) \end{bmatrix} + \begin{bmatrix} -\frac{E(t_n,T_1)}{T_1-t_n} \\ -\frac{E(t_n,T_2)}{T_2-t_n} \\ \vdots \\ -\frac{E(t_n,T_m)}{T_m-t_n} \end{bmatrix} + \begin{bmatrix} z_1(t_n) \\ z_2(t_n) \\ \vdots \\ z_m(t_n) \end{bmatrix} \tag{3.13}$$

where $z_i(t_n) \sim \mathcal{N}(0,H)$ are noise variables which reflect the deviation of bond prices from the model prices and $H = \mathrm{diag}(h_1^2, h_2^2, \ldots, h_m^2)$, where h_i are positive constants. Equation (3.7) provides the transition equation as in (2.1) and the equation (3.13) supply the measurement equation as in (2.2). Hence these equations form a linear state space system with bond yields as observable variables, so that Kalman filtering can be applied for model calibration and forecasting. These forecasts will then be used for predicting the tail losses of bond portfolios.

4. Backtesting for VaR models: methodology

In the case of VaR models, backtesting consists of checking whether the actual losses are in line with projected ones. Several authors recommend backtesting VaR models, including Jorion (2007), Kupiec (1995) and Christoffersen (2003). The most common method to test a VaR model has been suggested in Kupiec (1995), where the author developed a 95% confidence region for *unconditional coverage test*. The unconditional coverage test is the standard tool for backtesting models and is also recommended (see, e.g. Chen and Gerlach (2011)) by Basel II. Hence we decided to employ it throughout the numerical experiments in this paper, in addition to the conditional coverage tests which are described later in this section. According to the procedure for the unconditional coverage test, a model is correctly calibrated when the number of exceptions (i.e. the portfolio losses exceeding VaR) is in line with the confidence level. If backtesting reveals too many exceptions, then the risk is underestimated by the current model. Hence one might reserve an insufficient required capital and suffer critical losses under extreme market movements. On the other hand, too few exceptions signals an overestimated risk and that would lead to an inefficient allocation of capital, which is also not ideal for institutions that look for

maximising profits. Let us define an indicator variable $\mathscr{I}_n$ as:

$$\mathscr{I}_n = \begin{cases} 0 & \text{if } \mathscr{L}_n \leqslant VaR_{\alpha,n|n-1} \\ 1 & \text{if } \mathscr{L}_n > VaR_{\alpha,n|n-1} \end{cases}$$

where $\mathscr{L}_n = -\Delta\Pi_n$ and $VaR_{\alpha,n|n-1}$ represent respectively the loss at time t_n and the α confidence level Value-at-Risk computed at time t_n given the information at time t_{n-1}. The number of exceptions is given by $\mathscr{U} = \sum_{n=1}^{N} \mathscr{I}_n$, where N is the total number of observations. Since each weekly outcome could lead to an exception or not, the random variable $\mathscr{U}$ follows a binomial distribution:

$$f_{\mathscr{U}}(u) = \binom{N}{u} p^u (1-p)^{N-u},$$

where $p = 1-\alpha$, and α is the level for the selected VaR. Let us consider the number of exceptions in the sample, $\tilde{u}$, and define the failure rate as $\tilde{u}/N$. Null and alternative hypothesis in *Kupiec's test* (Kupiec (1995)) are as follows:

$$\begin{cases} H_0 : p = \frac{\tilde{u}}{N} \\ H_1 : p \neq \frac{\tilde{u}}{N} \end{cases}$$

We test whether the observed failure rate differs significantly from the given confidence level p. The test statistic used is:

$$LR_{uc} = -2\ln\left(\frac{(1-p)^{N-\tilde{u}} p^{\tilde{u}}}{\left[1-\frac{\tilde{u}}{N}\right]^{N-\tilde{u}} \left(\frac{\tilde{u}}{N}\right)^{\tilde{u}}}\right) \sim \chi_1^2. \tag{4.1}$$

Using a 95% confidence interval this likelihood ratio test rejects the null hypothesis if $LR_{uc} > 3.841$.

The unconditional coverage test, on its own, focusses on the number of exceptions, but it does not consider whether they are clustered. The independence test, developed in Christoffersen (2003), is capable to reject a VaR with clustered exceptions. In terms of the indicator variable $\mathscr{I}_n$, define the transition probabilities $\pi_{ij} = P(\mathscr{I}_n = i \text{ and } \mathscr{I}_{n+1} = j)$, *e.g.*, π_{01} provides the probability of having an exception tomorrow given that today there were no exception. If the exceptions sequence is independent over time, the probability of an exception tomorrow does not depend on today's outcome, i.e. $\pi_{01} = \pi_{11} = \pi$. In this case, the null and the alternative hypothesis are:

$$\begin{cases} H_0 : \pi_{01} = \pi_{11} \\ H_1 : \pi_{01} \neq \pi_{11} \end{cases}.$$

To test this hypothesis, we use the following likelihood ratio test:

$$LR_{ind} = -2\ln\left(\frac{(1-\hat{\pi})^{N_{00}+N_{10}} \hat{\pi}^{N_{01}+N_{11}}}{(1-\hat{\pi}_{01})^{N_{00}} \hat{\pi}_{01}^{N_{01}} (1-\hat{\pi}_{11})^{N_{10}} \hat{\pi}_{11}^{N_{11}}}\right) \sim \chi_1^2, \tag{4.2}$$

where $\hat{\pi} = \frac{N_{01}+N_{11}}{N_{00}+N_{01}+N_{10}+N_{11}}$, $\hat{\pi}_{01} = \frac{N_{01}}{N_{00}+N_{01}}$ and $\hat{\pi}_{11} = \frac{N_{11}}{N_{10}+N_{11}}$. N_{ij} represents the number of days when state j follows state i, and i, j can only assume values 0 and 1. Since we are interested in understanding whether simultaneously the number of exceptions is correct and VaR exceptions are independent, we can test jointly these two features using the conditional coverage test:

$$LR_{cc} = LR_{uc} + LR_{ind} \sim \chi_2^2. \tag{4.3}$$

Using a 95% confidence interval this likelihood ratio test rejects the null hypothesis if $LR_{cc} > 5.991$. Hence, the 95% level critical values for LR_{uc}, LR_{ind} and LR_{cc} are 3.841, 3.841 and 5.991 respectively.

VaR estimates are considered inadequate if at least one between LR_{uc} and LR_{ind} are above their critical value or if LR_{cc} is above its critical value.

In section 5 we simulate a weekly path for the interest rate using 'typical' values for a two-factor Vasicek model. Then we select a bond portfolio and compute weekly estimates of 95% and 99% VaR using the Monte Carlo method, historical simulation and the variance/covariance method. Next, we backtest these estimates against the series of simulated weekly losses $\mathscr{L}_n = -\Delta\Pi_n = -(\Pi_n - \Pi_{n-1})$. Section 6 will present a similar analysis employing real weekly US bond prices.

5. Numerical experiments with simulated data

In this section we assess and compare the computation of Value-at-Risk for a bond portfolio computed through Monte Carlo (MC) simulation using the Kalman filter, the historical simulation (HS) method and the variance/covariance (VC) method using simulated data. This will help to gain some insight about the performances of the three considered methods. In subsection 5.1 we present the three methods involved in our comparison and then the actual experiments are described along with their results in the next subsection.

5.1 *Monte Carlo, HS and VC*

A short term interest rate path is generated using a two risk factor model as specified by formulae (3.3) and (3.4). As mentioned earlier in sections 2 and 3, at any time t_n the simulated bond yields $\mathbf{y}_n$ are given by:

$$\mathbf{y}_n = C\mathbf{r}_n + \mathbf{d} + \mathbf{z}_n,$$

where $\mathbf{z}_n \sim \mathscr{N}(0,H)$ and $H = \mathbb{E}(\mathbf{z}_n\mathbf{z}_n^T) = \mathrm{diag}(h_1^2, h_2^2, \ldots, h_m^2)$, while C and $\mathbf{d}$ are explicitly expanded in equation (3.13). The measurement data are given by simulating a set of m bonds having different maturities whose prices are driven by the generated interest rate. The KF recursive equations (2.3) to (2.8) are used to compute the state vector and the covariance matrix predictions $\hat{\mathbf{r}}_{n+1|n}$ and $V_{n+1|n}$, according to the measurement data $\mathbf{y}_n$ provided at time t_n. Then VaR can be obtained for a specified bond portfolio by Monte Carlo simulation, using the fact that r_{n+1} is normally distributed, with its mean and covariance matrix specified by the KF predictions. The bond portfolio under study consists of J bonds having each maturity T_j and price $P_{n,j}$ at time t_n for $j = 1,2,\ldots,J$. The portfolio composition is not changing during the period considered. At time $n+1$ the value of the bond portfolio is given by:

$$\begin{aligned}\Pi_{n+1} &= \sum_{j=1}^{J} w_j P_{n+1,j}, \\ &= \sum_{j=1}^{J} w_j \exp\left(E_{n+1,j} - F_{1\,n+1,j}\, r_{1\,n+1} - F_{2\,n+1,j}\, r_{2\,n+1}\right), \end{aligned} \tag{5.1}$$

where $P_{n+1,j}$ and Π_{n+1} are, respectively, the price of the j^{th} bond and the portfolio net worth at time t_{n+1} and w_j is the quantity of the j^{th} bond held. r_1 and r_2 are the factors and E, F_1 and F_2 are the known functions depending on $T_j - t_{n+1}$ as specified in (3.10) and (3.11). Since $\mathbf{r}_{n+1|n} \sim \mathscr{N}(\hat{\mathbf{r}}_{n+1|n}, V_{n+1|n})$, one can perform a full Monte Carlo simulation to obtain an approximate distribution of the predicted loss $\Pi_{n+1} - \Pi_n$ and obtain the Value-at-Risk at a specific confidence level. Once 95% and 99% VaR are computed, it is possible to obtain their nonparametric confidence intervals based on finite sample

theory[1]. Table 1 displays the index of ordered statistics to obtain 95% confidence intervals for 95% and 99% VaR using 10,000 draws, which is the number of draws used in all individual Monte Carlo simulation experiments in this paper.

Table 1. Nonparametric 95% confidence intervals for 95% and 99% VaR using 10,000 Monte Carlo simulations.

Number of Draws	95% VaR		99% VaR	
	Lower Bound	Upper Bound	Lower Bound	Upper Bound
10,000	457	544	81	120

The historical simulation (HS) is a nonparametric procedure for computing the Value-at-Risk where no specific assumptions about the distribution of risk factors are made. It considers the lower quantile of the distribution of the actual historical returns and assumes that history will repeat itself. Let us consider the time series of rates of return $R_n = \frac{\Pi_n - \Pi_{n-1}}{\Pi_{n-1}}$ at any time t_n, for $n = 1, 2, \ldots, N_1$. Let now be $\mathscr{R}$ the random variable gathering all the computed rates of returns. We assume that the lower quantiles of $\mathscr{R}$ will remain constant in future samples of R_n for $n = N_1 + 1, N_1 + 2, \ldots, N_2$. Hence we can write:

$$\Pr\left[\Pi_n - \Pi_{n-1} > q_{1-\alpha}\Pi_{n-1}\right] = \alpha.$$

$q_{1-\alpha}$ is thus a number such that the historic portfolio return exceeds that number with probability α.

Similar to the HS method, the variance/covariance (VC) method considers the lower quantile of the distribution of the actual historical returns and assumes that history will repeat itself. However, VC method assumes that rates of return are normally distributed with its mean and variance being the historical sample mean and sample variance: $\mathscr{R} \sim \mathscr{N}(\mu_{\mathscr{R}}, \sigma^2_{\mathscr{R}})$. Hence we have

$$\Pr\left[\mathscr{R} > \mu_{\mathscr{R}} - q^{\mathscr{N}}_{1-\alpha}\sigma_{\mathscr{R}}\right] = \alpha, \tag{5.2}$$

where $q^{\mathscr{N}}_{1-\alpha}$ represents the $(1-\alpha)$ quantile of the normal random variable $\mathscr{R}$ (i.e. 1.645 and 2.326 for the 95% and 99% VaR, respectively), and α represents the level of confidence of VaR. Again, we assume that the lower quantiles of $\mathscr{R}$ will remain constant in future samples of R_n for $n = N_1 + 1, N_1 + 2, \ldots, N_2$. Hence we can write:

$$\Pr\left[\Pi_n - \Pi_{n-1} > (\mu_{\mathscr{R}} - q^{\mathscr{N}}_{1-\alpha}\sigma_{\mathscr{R}})\Pi_{n-1}\right] = \alpha.$$

Thus the mean and the variance of the portfolio return completely specify the VaR at all confidence levels for the VC method.

5.2 *Simulation experiments*

In this subsection, we consider an interest rate path, compute bond portfolio values and assess the reliability of 95% and 99% VaR estimates for the MC method, the HS method and the VC method using the backtesting procedure reported in section 4. The simulation has been run choosing 'typical' values for the parameters (see Castellanos Pinzon (2008)) and it involves computing a weekly interest rate using the Euler discretization of stochastic differential equations (3.3) and (3.4). The procedure for computing the VaR using Monte Carlo simulation consists of the following four steps:

[1]See Pritsker (1997) for more details on finite sample theory.

- Considering five yields obtained by simulating five batches of bonds, each having a different maturity;
- Estimating the KF parameters using an in-sample subset, by implementing the equations (2.3)-(2.8) and maximising the likelihood function as described earlier;
- Computing 95% and 99% VaR using Monte Carlo simulation as described in subsection 5.1.

The choice of the dataset is related to the standard of its desired backtesting. To achieve an adequate level of reliability, one requires to consider a sufficiently large number of values. We opted for 250 values. The procedure adopted considers an in-sample subset consisting of 200 yields for each of the five batches of simulated bonds to estimate the vector of parameters and then uses the estimated values to calculate 50 one-step ahead yield predictions, that will be compared with the corresponding out-of-sample actual values. The choice of employing 50 one-step ahead yield predictions comes from empirical evidence suggesting that out-of-sample fitting using real data is rather good for a number of time steps in a range of 50–75 data, while outside this range results of fitting are poorer. Repeating this procedure five times, shifting both the in-sample and the out-of-sample by 50 values as explained next, allow us to compute the required 250 non-overlapping values:

In-sample	Out-of-sample
$1-200$	$201-250$
$51-250$	$251-300$
$101-300$	$301-350$
$151-350$	$351-400$
$201-400$	$401-450$

The time-step $\Delta t = \frac{1}{52}$ (i.e. weekly data is used), while the whole interest rate path generated consists of 450 simulations.

The procedure for the HS method and the VC method involves using the in-sample subsets to obtain the quantiles $q_{1-\alpha}$ and $q^{\mathcal{N}}_{1-\alpha}$, respectively, that will be employed to compute 95% and 99% VaR in the out-of-sample subsets. Bond prices included in the considered portfolio to calculate the relevant statistics $q_{1-\alpha}$ and $q^{\mathcal{N}}_{1-\alpha}$ are computed using the formula:

$$P_{n,j} = e^{-y_{n,j}(T_j - t_n)}, \qquad j = 1,2,\ldots,J,$$

where $y_{n,j}$ represents the realization of the j^{th} bond yield simulated at time t_n. The values to simulate the two-factor Vasicek model described by equations (3.3) and (3.4) are reported in table 2:

Table 2. Coefficients used for the simulation of the two-factor Vasicek model.

i	x_{i0}	k_i	θ_i	σ_i	λ_i
1	0.015	0.375	0.044	0.015	-0.18
2	0.025	0.02	0.014	0.01	-0.0001

The starting values set for the initialization of the KF algorithm are:

$$\hat{\mathbf{r}}_{0|0} = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix} \qquad \text{and} \qquad V_{1|0} = 5 \times 10^{-3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that changes in the initial values do not change the qualitative aspects of results. The observable measurements are supplied by five bonds whose features are reported in table 3:

Table 3. Features of the bonds providing the measurement values.

j	$T_j(years)$	h_j^2
1	0.5	0.0009^2
2	1	0.0011^2
3	1.5	0.0010^2
4	2	0.0012^2
5	5	0.0006^2

T_j and h_j^2 represent the maturity and the variance of the zero mean noise term that perturbs the measured j^{th} bond yield, respectively. The goodness of fit reached for both the in-sample subset and the out-of-sample subset was assessed considering the relative absolute error (RAE), defined as:

$$\text{RAE} = \frac{|\text{simulated rate - predicted rate}|}{\text{simulated rate}}.$$

The features of the bonds included in the portfolio under study are reported in table 4:

Table 4. Features of the bonds included in the simulated portfolio.

j	$T_j(years)$	w_j
1	1	15,000
2	2	35,000
3	5	30,000

where T_j are the maturities and w_j is the initial number of units of the j^{th} bond held. Table 5 reports the parameter estimates, along with the standard errors for the estimates given in brackets, for the five in-sample subset considered, and their corresponding mean of the RAE for both the in-sample (indicated as $MRAE$) and the out-of-sample (indicated as $MRAE^*$). The standard errors are computed using the Hessian matrix of the log likelihood function; see Bolder (2001), for example. The true values of the parameters used in simulation were listed earlier in this section in table 2.

Table 5. Estimated parameters for the subset considered, and their corresponding *MRAE* and $MRAE^*$.

Subset	1	2	3	4	5
k_1	0.3127	0.2987	0.2942	0.3025	0.3211
	(0.0219)	(0.0212)	(0.0204)	(0.0264)	(0.0178)
θ_1	0.0551	0.0335	0.0005	0.0229	0.0536
	(0.0016)	(0.0016)	(0.0024)	(0.0037)	(0.0012)
σ_1	0.0161	0.0176	0.0177	0.0166	0.0145
	(0.0014)	(0.0013)	(0.0019)	(0.0022)	(0.0016)
λ_1	-0.2821	-0.5838	-0.9375	-0.6707	-0.0944
	(0.0351)	(0.0298)	(0.0305)	(0.0319)	(0.0323)
k_2	0.0586	0.0452	0.0222	0.0105	0.0733
	(0.0325)	(0.0297)	(0.0295)	(0.0329)	(0.0201)
θ_2	0.0131	0.0005	0.0083	0.0062	0.0076
	(0.0043)	(0.0045)	(0.0057)	(0.0039)	(0.0021)
σ_2	0.0109	0.0100	0.0093	0.0082	0.0110
	(0.0019)	(0.0022)	(0.0029)	(0.0026)	(0.0015)
λ_2	-0.0002	-0.0014	-0.0014	-0.0001	-0.0014
	(3.11e-04)	(5.60e-04)	(2.99e-04)	(3.10e-04)	(1.92e-04)
h_1	0.0006	0.0006	0.0007	0.0007	0.0007
	(6.93e-05)	(5.78e-05)	(5.91e-05)	(3.61e-05)	(5.20e-05)
h_2	0.0011	0.0011	0.0010	0.0010	0.0011
	(6.05e-05)	(2.99e-05)	(6.67e-05)	(5.56e-05)	(5.56e-05)
h_3	0.0009	0.0009	0.0009	0.0008	0.0008
	(6.30e-05)	(5.88e-05)	(6.01e-05)	(5.51e-05)	(6.89e-05)
h_4	0.0011	0.0011	0.0011	0.0011	0.0011
	(6.46e-05)	(4.92e-05)	(3.96e-05)	(7.23e-05)	(5.47e-05)
h_5	0.0009	0.0009	0.0010	0.0010	0.0010
	(6.13e-05)	(4.80e-05)	(5.33e-05)	(4.98e-05)	(7.22e-05)
$MRAE(\%)$	0.89	0.91	1.13	1.36	1.40
$MRAE^*(\%)$	1.28	1.46	1.15	1.09	0.83

Local modeling of the interest rate allows to obtain an overall good fit, as highlighted by low *MRAE* and $MRAE^*$ values reported in table 5. We used the estimated parameters to perform a Monte Carlo simulation as described in subsection 5.1 to obtain the estimates of one-step ahead 95% and 99% VaR. We also computed the 95% and 99% VaR using the HS method and the VC method which were previously

described.

Table 6. Summary of test results for the considered simulation.

Method	α	$\tilde{x}$	N_{00}	N_{01}	N_{10}	N_{11}	LR_{uc}	LR_{ind}	LR_{cc}	A/R
MC	95%	12	228	10	10	2	0.0213	2.5109	2.5322	A
	99%	4	242	4	4	0	0.7691	0.1301	0.8992	A
HS	95%	13	225	12	12	1	0.0208	0.1528	0.1736	A
	99%	**7**	236	7	7	0	**5.4970**	0.4033	**5.9003**	R
VC	95%	15	221	14	14	1	0.4961	0.0122	0.5082	A
	99%	**9**	232	9	9	0	**10.2290**	0.6724	**10.9014**	R

Table 6 reports the 95% and 99% VaR backtesting outcomes for an instance of the process described by equations (3.3) and (3.4), having values reported in table 2. Here, $\tilde{x}$ is the number of exceptions over out-of-sample data (i.e. the total number of exceptions over five non-overlapping out-of-sample data subsets) and the rest of the notation is the same as in section 4. The values in bold represents either the values outside the non-rejection confidence intervals or the values above the corresponding critical value. In particular, the last column of the table indicates whether the VaR, at the specified level of confidence, estimated using either the MC simulation, the HS method or the VC method is accepted (A) or rejected (R). While 95% and 99% VaR estimates obtained using the MC simulation are both accepted, 95% VaR estimates using the HS method and the VC method are accepted but the 99% VaR estimates using the HS and the VC estimates are both rejected.

Other instances of parameter values (not reported here) of the considered interest rate simulation lead to the conclusion that HS and VC methods, at least in this context, are not reliable for the estimation of 95% and especially of 99% VaR since, as reported here, their backtesting occasionally fails to be accepted. The assumption that the past provides a fair representation of the immediate future seems to be not always true. The ability to estimate VaR using HS and VC methods seems to depend on the specific path simulated. This might also depend on the (implicit) assumption of portfolio returns being independent and identically distributed, which appears to be unrealistic. Furthermore all the rates of returns are given equal weights, where it might be more appropriate to assign different weights according to the fact that data further away from the present have a lower predicting influence compared to the closer ones. Monte Carlo simulation, provided that the calibration is sufficiently accurate, seems reliable independently from the single path realization and leads to VaR estimates that are not rejected by the unconditional and conditional tests. Varying the coefficients for the simulation of the interest rate and/or varying the features of the bonds included in the considered portfolio leads to similar conclusions and are not reported here for brevity.

One can also carry out CVaR tests on the same data using equation (1.2). The expectation in the CVaR definition can be approximately evaluated as a tail probability weighted summation of VaRs for the MC and the HS methods and is given in closed-form for the VC method. CVaR, being coherent, is mathematically a far better measure of loss. However, there are no rigorous hypothesis tests available to validate a CVaR model, nor are there any measures, such as the number of exceptions in the case of VaR, with which to compare different models. Finding VaR at multiple confidence levels can give a qualitative idea of whether the model is suitable to evaluate CVaR as well; *e.g.* a model which is accepted for 95%, 97.5% and 99% VaR through backtesting is likely to be acceptable for 95% CVaR as well. Since reporting the CVaR will not lead to any additional information about the quality of the

models, we have restricted the comparison to finding the VaR and testing it via tests based on the number and the clustering of exceptions.

Having seen the performance of our method with a simulated portfolio, we carry out the same tests using real bond prices.

6. Numerical experiments with real data

The aim of this section is to compute 95% and 99% VaR using the Monte Carlo simulation for an actual portfolio of bonds and to compare its performance against the HS method and the VC method, as described in the section 5. This is achieved in three steps:

1. Calibrating a two-factor Vasicek term structure model using the Kalman filter for the chosen dataset;
2. Computing the VaR at the required confidence level, as described in section 5;
3. Backtesting the one-step ahead forecasting, as described in subsection 4.

Unlike in the simulation experiments, the interest rate values are unknown. Hence we need to compare the bond yields to judge the accuracy of calibration. The goodness of fit can be assessed considering the RAE, which is defined for the experiments using real data as:

$$\text{RAE} = \frac{|\text{observed yield - predicted yield}|}{\text{observed yield}}.$$

Subsection 6.1 introduces the data employed and explains how it is used to calibrate the parameters, while subsection 6.2 reports the results of backtesting the estimates obtained with the proposed method.

6.1 *Data*

The dataset employed for this experiment consists of 450 weekly yields computed on three groups of US government bonds from $28/06/2001$ to $11/10/2012$: three batches of very short term bonds (5 to 7 months maturity), two batches of short term bonds (4.5 to 5.5 years maturity) and one batch of medium term bonds (10 years maturity). All the data was retrieved from Datastream. This data was split into five in-sample and five out-of-sample data sets using exactly the same procedure as used in section 5.2 for the simulated data set of the same size. We calibrated five two-factor Vasicek models on the five different in-sample yield data-sets mentioned above, with each model having 14 parameters. Each in-sample estimation is used to produce one-step ahead forecasting of yields. The procedure adopted for the experiment using real data is similar to the one chosen for the experiment using simulated data. We used five in-sample subsets and follow the four steps described in subsection 5.2. Computation of statistics $q_{1-\alpha}$ and $\mu_{\mathscr{R}} - q^{\mathscr{N}}_{1-\alpha}\sigma_{\mathscr{R}}$ to obtain VaR through the HS and the VC methods are calculated using the actual bond values.

6.2 *Results*

Table 7 displays the estimated parameters along with their standard errors, using the five in-sample data subsets mentioned.

Table 7. Estimated parameters for the considered financial data.

Subset	1	2	3	4	5
k_1	0.7030	0.7118	0.7095	0.7023	0.6891
	(0.0341)	(0.0351)	(0.0277)	(0.0186)	(0.0213)
θ_1	0.0056	0.0047	0.0049	0.0048	0.0045
	(9.37e-04)	(0.0013)	(0.0019)	(0.0015)	(0.0012)
σ_1	0.0321	0.0314	0.0332	0.0327	0.0288
	(0.0076)	(0.0084)	(0.0058)	(0.0079)	(0.0052)
λ_1	-0.4591	-0.4606	-0.4842	-0.4751	-0.4553
	(0.0924)	(0.0790)	(0.1315)	(0.0819)	(0.0518)
k_2	0.0255	0.0275	0.0231	0.0241	0.0219
	(0.0116)	(0.0091)	(0.0137)	(0.0104)	(0.0085)
θ_2	0.0035	0.0035	0.0028	0.0030	0.0029
	(0.0015)	(5.88e-04)	(0.0012)	(0.0018)	(0.0011)
σ_2	0.0142	0.0154	0.0138	0.0144	0.0127
	(5.98e-04)	(1.07e-04)	(3.05e-04)	(5.24e-04)	(5.72e-04)
λ_2	-0.2652	-0.2639	-0.2528	-0.2509	-0.2629
	(0.0510)	(0.0225)	(0.0131)	(0.0337)	(0.0794)
h_1	0.0009	0.0011	0.0010	0.0009	0.0013
	(8.84e-05)	(7.02e-05)	(6.46e-05)	(9.03e-05)	(8.30e-05)
h_2	0.0012	0.0006	0.0011	0.0011	0.0011
	(1.03e-04)	(9.38e-05)	(6.33e-05)	(9.98e-05)	(7.13e-05)
h_3	0.0013	0.0010	0.0006	0.0009	0.0009
	(1.22e-04)	(1.27e-04)	(1.11e-04)	(1.19e-04)	(7.38e-05)
h_4	0.0007	0.0010	0.0007	0.0011	0.0007
	(9.13e-05)	(9.08e-05)	(5.33e-05)	(1.08e-04)	(4.92e-05)
h_5	0.0009	0.0004	0.0012	0.0011	0.0009
	(1.21e-04)	(9.77e-05)	(1.06e-04)	(9.23e-05)	(9.04e-05)
h_6	0.0010	0.0008	0.0009	0.0008	0.0009
	(6.95e-05)	(5.54e-05)	(7.44e-05)	(1.64e-04)	(8.06e-05)
$MRAE(\%)$	1.22	1.34	1.18	1.36	1.15
$MRAE^*(\%)$	1.16	1.09	1.29	1.12	1.20

The estimated values were used to calculate 50 one-step ahead interest rate predictions, and used to carry out the Monte Carlo simulation as described in subsection 5.1. The portfolio considered includes an initial number of $50,000$ of each of the six bonds. Table 8 reports the conditional and the unconditional tests for the Monte Carlo simulation, the HS method and the VC method.

Table 8. Summary of test results for the experiment with real data.

Method	α	$\tilde{x}$	N_{00}	N_{01}	N_{10}	N_{11}	LR_{uc}	LR_{ind}	LR_{cc}	A/R
MC	95%	16	219	15	15	1	0.9514	0.0006	0.9520	A
	99%	5	240	5	5	0	1.9568	0.2041	2.1609	A
HS	95%	17	218	15	15	2	1.5403	0.5996	2.1399	A
	99%	7	237	6	6	1	**5.4970**	1.8520	**7.3490**	R
VC	95%	16	220	14	14	2	0.9514	1.6762	2.6276	A
	99%	6	238	6	6	0	3.5554	0.2952	3.8506	A

The Monte Carlo simulation and the VC method provide acceptable estimations of both 95% and 99% VaR, while the HS method just provides an acceptable 95% VaR but fails to provide a valid 99% VaR. As highlighted in the previous section, the assumption that the past provides a fair representation of the immediate future might be not realistic. In this instance the HS method produces too many exceptions in estimates of 99% VaR, therefore the evaluation of 99% VaR is rejected. The calibrated model seems accurate enough for the purpose of estimating the quantiles, since the conditional and unconditional tests are non-rejected for both 95% and 99% VaR estimates. The distribution of the portfolio returns seems to be close enough to normality in this particular case for the VC method to be deemed acceptable, although this is not always the case (as seen in the simulation experiments in the previous section). Further, note that VC method does not provide any intuition about the riskiness of the individual components of the portfolio. It is easy in our KF based methodology to carry out such exploratory analysis, *e.g.* it is quite straightforward to find the VaR estimates for hypothetical short-dated and long dated portfolios, from the results of the same Monte Carlo experiment as above, and assess whether the short end or the long end of the yield curve currently poses more risk.

7. Conclusions

In this article, we propose a way of measuring the Value-at-Risk of fixed income portfolios and we backtest it for both simulated and real data. Our method provides scenarios which are arbitrage-free and which (arguably) better reflect the market conditions for highly liquid government securities. The KF-based method requires simulating only a vector of two random variables for one-step ahead forecasts and is hence computationally cheaper as compared to principal components analysis using more than two principal components. Numerical experiments with simulated data as well as real treasuries data confirm the utility of our method in measuring the tail risk.

The focus of this paper is limited to compute the Value-at-Risk for a bond portfolio. A possible future research direction could be to extend the portfolio composition to include other nonlinear interest instruments, such as interest rate caps and floors.

References

Acerbi, C. and Tasche, D. (2002), 'Expected shortfall: a natural coherent alternative to Value at Risk', *Econ. Notes* **31**, 379–388.

Ait-Sahalia, Y. and Kimmel, R. (2010), 'Estimating affine multifactor term structure models using closed-form likelihood expansions', *J. Financ.Econ.* **98**, 113–144.

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999), 'Coherent measures of risk', *Math. Financ.* **9**(3), 203–228.

Babbs, S. H. and Nowman, K. B. (1999), 'Kalman filtering of generalized Vasicek term structure models', *J. Financ. Quant. Anal.* **34**, 115–130.

Basel Committee (2006), 'Basel Committee on banking supervision international convergence of capital measurement and capital standards'.

Bolder, D. J. (2001), 'Affine term-structure models: theory and implementation', *Bank of Canada, Working Paper No. 2001-15* .

Brigo, D. and Mercurio, F. (2006), *Interest Rate Models – Theory and Practice*, Springer Finance, Springer, Berlin.

Carvalhal, A. and Daumas, S. P. (2010), 'A minimizing risk strategy for Brazilian fixed-income portfolios', *Lat. Am. Bus. Rev.* **11**(1), 1–18.

Castellanos Pinzon, J. B. (2008), 'Estimating multifactor uncorrelated Vasicek and CIR model for the UK term structure', *Centre for Computational Finance and Economic Agents, University of Essex. Working Paper Series, WP024-08* .

Chaplin, G. B. and Sharp, K. P. (1993), 'Analytic solutions for bonds and bond options under *n* correlated stochastic processes', *Institute of Insurance and Pension Research Working Paper, University of Waterloo* .

Chen, Q. and Gerlach, R. H. (2011), 'The two-sided Weibull distribution and forecasting of financial risk', *OME Working Paper* pp. 1–36.

Chen, R.-R. (1995), 'A two-factor preference-free model for interest rate sensitive claims equilibrium model of the term structure of interest rates', *J. Futures Mark.* **15**(3), 345–372.

Chen, Y., Härdle, W. and Spokoiny, V. (2007), 'Portfolio value at risk based on independent component analysis', *J. Comput. Appl. Math.* **205**(1), 594–607.

Christoffersen, P. F. (2003), *Elements of Financial Risk Management*, Academic Press.

Cox, J. C., Ingersoll, J. E. and Ross, S. A. (1985), 'A theory of the term structure of interest rates', *Econom.* **53**(2), 385–407.

Dai, Q. and Singleton, K. (2000), 'Specification analysis of affine term structure models', *J. Financ.* **55**, 1943–1978.

Darbha, G. (2001), 'Value-at-Risk for fixed income portfolios. A compari-son of alternative models', *National Stock Exchange, Mumbai, India* .

Date, P. and Ponomareva, K. (2011), 'Linear and non-linear filtering in mathematical finance: a review.', *IMA J. Manag. Math.* **22**(3), 195–211.

Date, P. and Wang, C. (2009), 'Linear Gaussian affine term structure models with unobservable factors: Calibration and yield forecasting', *Eur. J. Oper. Res.* **195**(1), 156–166.

De Rossi, G. (2004), 'Kalman filtering of consistent forward rate curves: a tool to estimate and model dynamically the term structure', *J. Empir. Financ.* **11**(2), 277–308.

Duffee, G. (2002), 'Term premia and interest rate forecasts in affine models', *J. Financ.* **57**, 405–443.

Duffie, D. and Kan, R. (1996), 'A yield-factor model of interest rates', *Math. Financ.* **6**(4), 379–406.

Fiori, R. and Iannotti, S. (2007), 'Scenario based principal component Value-at-Risk: an application to Italian banks' interest rate risk exposure', *J. Risk* **9**(3), 63–99.

Fisher, M., Nychka, D. and Zervos, D. (1995), 'Fitting the term structure of interest rates with smoothing splines', *Board of Governors of the Federal Reserve System (U.S.), Finance and Economics Discussion Series* **95**(1), 1–30.

Gibson, M. S. and Pritsker, M. (2000), 'Improving grid-based methods for estimating Value at Risk of fixed-income portfolios', *Board of Governors of the Federal Reserve System (U.S.). Finance and Economics Discussion Series* **25**, 1–29.

Jamshidian, F. and Zhu, Y. (1997), 'Scenario simulation: theory and methodology', *Financ. Stoch.* **1**(1), 43–67.

Jegadeesh, N. and Pennacchi, G. G. (1996), 'The behavior of interest rates implied by the term structure of Eurodollar futures', *J. Money Credit Bank.* **28**(3), 426–446.

Jorion, P. (2007), *Value at Risk: the new benchmark for managing financial risk*, McGraw-Hill.

Kalman, R. E. (1960), 'A new approach to linear filtering and prediction problems', *J. Basic Eng.* **82**, 35–45.

Kupiec, P. H. (1995), 'Techniques for verifying the accuracy of risk measurement models', *J. Deriv.* **3**(2), 73–84.

Manoliu, M. and Tompaidis, S. (2002), 'Energy futures prices: term structure models with Kalman filter estimation', *Appl. Math. Financ.* **9**(1), 21–43.

McCulloch, J. H. (1975), 'The tax-adjusted yield curve', *J. Financ.* **30**(3), 811–830.

Monoyios, M. (2007), 'Optimal hedging and parameter uncertainty', *IMA J. Manag. Math.* **18**(4), 331–351.

Pritsker, M. (1997), 'Evaluating Value-at-Risk methodologies: accuracy versus computational time', *J. Financ. Serv. Res.* **12**(2-3), 201–242.

Rockafellar, R. T. and Uryasev, S. (2000), 'Optimization of conditional Value-at-Risk', *J. Risk* **2**, 21–41.

Schwartz, E. S. (1997), 'The stochastic behavior of commodity prices: implications for valuation and hedging', *J. Financ.* **52**(3), 923–973.

Smith, D. J. (2009), 'A primer on bond portfolio value at risk', *Adv. Financ. Educ.: J. Financ. Educ. Assoc.* **7**(1-2), 1–14.

Szegö, G. (2002), 'Measures of risk', *J. Bank. Financ.* **26**(7), 1253–1272.

Vasicek, O. (1977), 'An equilibrium characterisation of the term structure', *J. Financ. Econ.* **5**, 177–188.

Vasicek, O. and Fong, G. (1982), 'Term structure modeling using exponential splines', *J. Financ.* **37**(2), 339–348.