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BARGAINING WITH INDEPENDENCE OF HIGHER OR IRRELEVANT CLAIMS

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<div align="center">**Abstract**</div>

This paper studies independence of higher claims and independence of irrelevant claims on the domain of bargaining problems with claims. Independence of higher claims requires that the payoff of an agent does not depend on the higher claim of another agent. Independence of irrelevant claims states that the payoffs should not change when the claims decrease but remain higher than the payoffs. Interestingly, in conjunction with standard axioms from bargaining theory, these properties characterize a new constrained Nash solution, a constrained Kalai-Smorodinsky solution, and a constrained Kalai solution.

Keywords: bargaining with claims, independence of higher claims, independence of irrelevant claims, constrained Nash solution, constrained Kalai-Smorodinsky solution, constrained Kalai solution

JEL classification: C78, D74

1   Introduction

A bargaining problem with multiple agents (cf. Nash (1950)) is described by a feasible set and a reference point inside this set. The feasible set consists of all payoff allocations in the utility space which can be jointly generated by the agents. The main question is which of these allocations will be selected by the agents or should be recommended by an arbitrator. The reference point, usually referred to as the disagreement point, serves as a lower bound, with the interpretation that it is implemented when the agents do not reach agreement.

Bargaining problems are typically studied from two perspectives. The positive or strategic approach studies solutions on the basis of their implementability, i.e. whether they result from a natural negotiation procedure. The normative or axiomatic approach studies solutions on the basis of their properties, i.e. whether they respect appealing fairness principles. Central solutions in bargaining theory are the Nash (1950) solution, which maximizes the product of the utility excesses with respect to the reference point, the Kalai and Smorodinsky (1975) solution, which maintains the ratios of the maximally possible utility excesses, and the Kalai (1977) solution, which equalizes the utility excesses.

More recently, Mariotti and Villar (2005) introduced rationing problems in which the agents share a loss instead of a surplus, described by a feasible set and a reference point outside this set. The reference point expresses rights, needs, demands, or aspirations, and serves as an upper bound for payoff allocations.

Herrero and Villar (2010) and Sudhölter and Zarzuelo (2013) merged bargaining problems and rationing problems into NTU sharing problems, where the reference point may be inside or outside the feasible set. The reference point reflects the entitlements of the agents which can be either satisfied or not.

Another line of research enriched original bargaining problems with a second exogenous reference point. Gupta and Livne (1988) analyzed problems where both reference points are inside the feasible set and serve as lower bounds. The first reference point has the classic conflict interpretation, while the second reference point emerges from pre-negotiation activities.

Chun and Thomson (1992) introduced bargaining problems with claims where one reference point is inside the feasible set and the other reference point is outside the feasible set. The inside reference point is a lower bound from which the utility excesses are measured. The outside reference point is an upper bound representing earlier commitments which cannot all be honored anymore. The main solution studied in this context is the proportional solution, which prescribes the efficient payoff allocation on the line connecting both reference points. Chun and Thomson (1992) and Lombardi and Yoshihara (2010) derived several axiomatic characterizations on domains with convex and nonconvex feasible sets, respectively. An alternative solution was studied by Bossert (1993) and Marco-Gil (1994).

This paper takes a further axiomatic approach to bargaining problems with claims. For convenience, we assume that the inside reference point equals the origin and we restrict to nonnegative feasible set allocations. In other words, we implicitly incorporate *translation invariance* and *independence of individually irrational allocations* into the definition of solutions. In this way, bargaining problems with claims can also be interpreted as bankruptcy problems with nontransferable utility (cf. Orshan, Valenciano, and Zarzuelo (2003)). This model generalizes classic bankruptcy problems as introduced by O'Neill (1982) by allowing agents to have nonlinear utility functions over their monetary payoffs. The proportional solution of Chun and Thomson (1992) and Lombardi and Yoshihara (2010) corresponds to a generalized proportional rule for bankruptcy problems. The solution studied by Bossert (1993) and Marco-Gil (1994) corresponds to a generalized constrained equal losses rule for bankruptcy problems.

On the one hand, we focus on the property *independence of higher claims*, originally appearing in the cost sharing literature (cf. Moulin and Shenker (1992)). On the domain of bargaining problems with claims, this property requires that for each pair of agents being symmetric within the feasible set, the payoff allocated to the agent with the lower claim does not depend on the higher claim of the other agent. This protects the payoffs of smaller claimants from being influenced by the big players. The proportional solution does not satisfy independence of higher claims. Interestingly, we show that, in conjunction with the standard axioms from bargaining theory, independence of higher claims characterizes a new constrained Nash solution, a constrained Kalai-Smorodinksy solution, and a constrained Kalai solution, obtained by explicitly bounding the original bargaining solutions by the claims. These three constrained bargaining solutions all correspond to a generalized constrained equal awards rule for bankruptcy problems.

On the other hand, we focus on the property *independence of irrelevant claims*, which says that the prescribed allocation should not change when the claims diminish but still dominate the allocation. For classic bankruptcy problems, this property was studied by Kibris (2012) and Stovall (2014). The proportional rule does not satisfy independence of irrelevant claims. Interestingly, we show that, in conjunction with standard axioms from bargaining theory, independence of irrelevant claims also characterizes the constrained Nash solution and the constrained Kalai solution.

This paper is organized in the following way. Section 2 provides preliminary notions for bargaining problems with claims. Section 3 formally introduces independence of higher claims and independence of irrelevant claims. In conjunction with standard axioms from bargaining theory, Section 4, Section 5, and Section 6 characterize the constrained Nash solution, the constrained Kalai-Smorodinsky solution, and the constrained Kalai solution, respectively. Section 7 formulates some concluding remarks and some suggestions for future research.

2 Bargaining Problems with Claims

Let N be a nonempty and finite set of agents. For all $x, y \in \mathbb{R}^N_+$, $x < y$ denotes $x_i < y_i$ for all $i \in N$, and $x \leq y$ denotes $x_i \leq y_i$ for all $i \in N$. For all $\theta \in \mathbb{R}^N_{++}$, $x \in \mathbb{R}^N_+$, and $A \subseteq \mathbb{R}^N_+$, $\theta x \in \mathbb{R}^N_+$ denotes $\theta x = (\theta_i x_i)_{i \in N}$, and $\theta A \subseteq \mathbb{R}^N_+$ denotes $\theta A = \{\theta x \in \mathbb{R}^N_+ \mid x \in A\}$.

A *bargaining problem with claims* (cf. Chun and Thomson (1992)) is a pair (E, c), where $E \subseteq \mathbb{R}^N_+$ with $E \cap \mathbb{R}_{++} \neq \emptyset$ is a closed, bounded, convex, and strictly comprehensive[1] *feasible set* and $c \in \mathbb{R}^N_{++}$ represents the *claims* of N on E such that $c \not< x$ for all $x \in E$. Let BC^N denote the class of all bargaining problems with claims. Contrary to Chun and Thomson (1992), we restrict to *strictly* comprehensive feasible sets, but we allow for claims which exceed the maximally feasible individual payoffs.

Let $(E, c) \in \mathrm{BC}^N$. Agents $i, j \in N$ are *symmetric* in E if

$$E = \left\{ x \in \mathbb{R}^N_+ \mid \exists y \in E : y_i = x_j, y_j = x_i, y_{N \setminus \{i,j\}} = x_{N \setminus \{i,j\}} \right\}.$$

The *utopia point* $u^E \in \mathbb{R}_{++}$ is given by

$$u^E = \left(\max\{x_i \mid x \in E\} \right)_{i \in N}.$$

The *truncated feasible set* $\hat{E}^c \subseteq \mathbb{R}^N_+$ is given by

$$\hat{E}^c = \{x \in E \mid x \leq c\}.$$

A *solution* for bargaining problems with claims $\phi : \mathrm{BC}^N \to \mathbb{R}^N_+$ assigns to any bargaining problem with claims $(E, c) \in \mathrm{BC}^N$ an allocation $\phi(E, c) \in \hat{E}^c$. The *proportional solution* (cf. Chun and Thomson (1992)) $\mathbb{P} : \mathrm{BC}^N \to \mathbb{R}^N_+$ assigns to any $(E, c) \in \mathrm{BC}^N$ the allocation

$$\mathbb{P}(E, c) = \lambda^{E,c} c,$$

where $\lambda^{E,c} = \max\{\theta \in [0,1] \mid \theta c \in E\}$. In other words, the proportional solution prescribes the maximally feasible allocation on the line connecting the origin with the claims point.
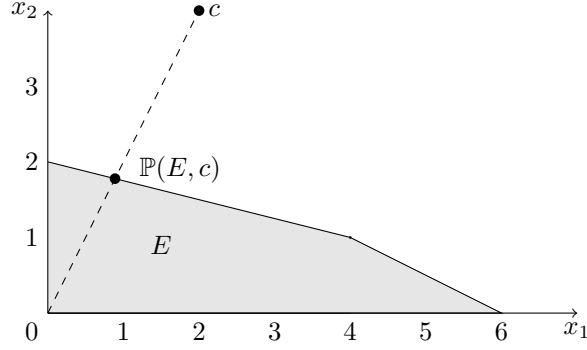
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[1]i.e. for all $x, y \in \mathbb{R}^N_+$ with $x \leq y$ and $x \neq y$, $y \in E$ implies $x \in E$ and $x < z$ for some $z \in E$.

Example 1

Let $N = \{1, 2\}$ and consider $(E, c) \in \mathrm{BC}^N$ given by

$$E = \text{c.c.h.}\, \{(6, 0), (4, 1), (0, 2)\}\,^2 \quad \text{and} \quad c = (2, 4).$$

Then $\lambda^{E,c} = \frac{4}{9}$ and $\mathbb{P}(E, c) = (\frac{8}{9}, 1\frac{7}{9})$.

$\triangle$

The proportional solution satisfies the following standard properties from bargaining theory.

Efficiency (EFF)
For all $(E, c) \in \mathrm{BC}^N$, $\phi(E, c) \not< x$ for all $x \in E$.

Efficiency states that not all agents could be made better off within the feasible set.

Symmetry (SYM)
For all $(E, c) \in \mathrm{BC}^N$ and all $i, j \in N$ symmetric in E with $c_i = c_j$,

$$\phi_i(E, c) = \phi_j(E, c).$$

Symmetry requires that symmetric agents with equal claims get equal payoffs.

Scale Covariance (SCOV)
For all $(E, c) \in \mathrm{BC}^N$ and all $\theta \in \mathbb{R}^N_{++}$,

$$\phi(\theta E, \theta c) = \theta \phi(E, c).$$

Scale covariance implies that the solution is covariant under individual rescaling of utility.

Independence of Irrelevant Alternatives (IIA)
For all $(E, c), (E', c) \in \mathrm{BC}^N$ with $E' \subseteq E$ and $\phi(E, c) \in E'$,

$$\phi(E, c) = \phi(E', c).$$

Suppose that a particular allocation is selected but the feasible set is smaller than initially thought. Fortunately, the selected allocation is still feasible. In that case, independence of irrelevant alternatives says that the selected allocation should still be implemented.

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[2]c.c.h.A denotes the smallest convex and comprehensive set containing A.

Restricted Monotonicity (RMON)
For all $(E, c), (E', c) \in \mathrm{BC}^N$ with $E \subseteq E'$ and $u^E = u^{E'}$,

$$\phi(E, c) \leq \phi(E', c).$$

Suppose that the feasible set turns out to be larger than expected but the utopia point was correctly estimated. Then restricted monotonicity ensures that all agents are allocated at least their promised payoffs.

Monotonicity (MON)
For all $(E, c), (E', c) \in \mathrm{BC}^N$ with $E \subseteq E'$,

$$\phi(E, c) \leq \phi(E', c).$$

Monotonicity requires that no agent is worse off when the feasible set expands. Note that monotonicity implies restricted monotonicity. Moreover, efficiency and monotonicity together imply independence of irrelevant alternatives.

As shown by Chun and Thomson (1992), efficiency, symmetry, scale covariance, and monotonicity together characterize the proportional solution. Since their results are obtained on a slightly different domain, we provide a proof of this characterization.

Theorem (cf. Chun and Thomson (1992))
$\mathbb{P}$ *is the unique solution on* BC^N *satisfying EFF, SYM, SCOV, and MON.*

Proof. Clearly, $\mathbb{P}$ satisfies EFF, SYM, SCOV, and MON. Let $\phi : \mathrm{BC}^N \to \mathbb{R}_+^N$ be a solution satisfying EFF, SYM, SCOV, and MON. Let $(E, c) \in \mathrm{BC}^N$. By SCOV, assume without loss of generality that $\mathbb{P}_i(E, c) = 1$ for all $i \in N$. Let $\varepsilon > 0$ be small. Define $E' \subseteq \mathbb{R}_+^N$ by

$$E' = \mathrm{c.c.h.} \left\{ \left(\left(\mathbb{P}_i(E, c) + \frac{|N| - |S|}{|N| - 1} \varepsilon \right)_{i \in S}, (0)_{i \in N \setminus S} \right) \,\middle|\, S \in 2^N \setminus \{\emptyset\} \right\}.$$

Then $E' \subseteq E$. By EFF and SYM, $\phi(E', c) = \mathbb{P}(E, c)$. By MON, $\phi(E, c) \geq \phi(E', c) = \mathbb{P}(E, c)$. By EFF, $\phi(E, c) = \mathbb{P}(E, c)$. Hence, $\phi = \mathbb{P}$. $\qquad\square$

3 Independence of Higher or Irrelevant Claims

In this section, we formally introduce independence of higher claims and independence of irrelevant claims for solutions for bargaining problems with claims. Independence of higher claims plays an essential role in a characterization of the serial cost sharing mechanism (cf. Moulin and Shenker (1992)). On the domain of bargaining problems with claims, it can be defined in such a way that the payoff of an agent does not depend on the higher claim of another agent which is symmetric within the feasible set.

Independence of Higher Claims (IHC)
For all $(E, c) \in \mathrm{BC}^N$ and all $i, j \in N$, $i \neq j$, symmetric in E with $c_i \leq c_j \leq c_j'$,

$$\phi_i(E, c) = \phi_i\left(E, (c_j', c_{N \setminus \{j\}})\right).$$

Note that the proportional solution does not satisfy independence of higher claims. On the subdomain of bankruptcy problems (cf. O'Neill (1982)), independence of higher claims can be used to characterize the constrained equal awards rule.

Let $\mathrm{BANK}^N \subseteq \mathrm{BC}^N$ denote the subclass of problems $(E, c) \in \mathrm{BC}^N$ with $E = \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \leq M\}$ for some $M \in \mathbb{R}_{++}$. Such a problem arises for instance when a firm goes bankrupt and the estate should be divided among the creditors.

The *constrained equal awards rule* $\mathbb{CEA} : \mathrm{BANK}^N \to \mathbb{R}_+^N$ divides the estate as equally as possible subject to claims boundedness, so it assigns to any $(E, c) \in \mathrm{BANK}^N$ the allocation

$$\mathbb{CEA}(E, c) = \left(\min\{c_i, m^{E,c}\} \right)_{i \in N},$$

where $m^{E,c} = \sup\{\theta \in \mathbb{R}_+ \mid (\min\{c_i, \theta\})_{i \in N} \in E\}$. On the class of bankruptcy problems, efficiency, symmetry, and independence of higher claims characterize the constrained equal awards rule.

Theorem 1
$\mathbb{CEA}$ *is the unique solution on* BANK^N *satisfying EFF, SYM, and IHC.*

Proof. Clearly, $\mathbb{CEA}$ satisfies EFF, SYM, and IHC. Let $\phi : \mathrm{BANK}^N \to \mathbb{R}_+^N$ be a solution satisfying EFF, SYM, and IHC. Let $(E, c) \in \mathrm{BANK}^N$. Denote $N = \{1, \ldots, |N|\}$ such that $c_1 \leq \ldots \leq c_{|N|}$. For all $i \in N$, define $\hat{c}^i \in \mathbb{R}_{++}^N$ by

$$\hat{c}^i = (\min\{c_i, c_j\})_{j \in N}.$$

Then $\hat{c}^1 \leq \ldots \leq \hat{c}^{|N|}$. By EFF, $\phi(E, \mathbb{CEA}(E, c)) = \mathbb{CEA}(E, c)$. By sequentially applying EFF, SYM, and IHC, $\phi(E, \hat{c}^i) = \mathbb{CEA}(E, c)$ for all $i \in N$ with $\hat{c}^i \notin E$. In particular, $\phi(E, c) = \phi(E, \hat{c}^{|N|}) = \mathbb{CEA}(E, c)$. Hence, $\phi = \mathbb{CEA}$. $\qquad\square$

The constrained equal awards rule also satisfies independence of irrelevant claims. This property can be described as follows. Suppose that a particular allocation is selected but the claims are smaller than initially thought. Fortunately, all claimants were not allocated more than their real claims. In that case, independence of irrelevant claims says that the selected allocation should still be implemented.

Independence of Irrelevant Claims (IIC)
For all $(E, c), (E, c') \in \mathrm{BC}^N$ with $c' \leq c$ and $\phi(E, c) \leq c'$,

$$\phi(E, c) = \phi(E, c').$$

Note that the proportional rule does not satisfy independence of irrelevant claims. On the class of bankruptcy problems, independence of irrelevant claims was explored by Kibris (2012) and Stovall (2014). They characterized a family of rationalizable rules and a family of monotone path rules, respectively. As the following example shows, efficiency, symmetry, and independence of irrelevant claims do not characterize the constrained equal awards rule for bankruptcy problems.

Example
Let $N = \{1, 2\}$ and let $\phi : \mathrm{BANK}^N \to \mathbb{R}_+^N$ be the solution defined by

$$\phi(E, c) = \begin{cases} (0, E) & \text{if } c_1 < \frac{1}{2}E \text{ and } c_2 \geq E; \\ \mathbb{CEA}(E, c) & \text{otherwise.} \end{cases}$$

Then ϕ satisfies efficiency, symmetry, and independence of irrelevant claims, but does not coincide with the constrained equal awards rule. $\qquad\triangle$

On the full domain of bargaining problems with claims, efficiency, symmetry, independence of higher claims, and independence of irrelevant claims are satisfied by more than one solution. However, in conjunction with other axioms from bargaining theory, these properties induce some specific solutions.

4   The Constrained Nash Solution

For classic bargaining problems, efficiency, symmetry, scale covariance, and independence of irrelevant alternatives characterize the Nash (1950) solution. This solution prescribes the feasible allocation which maximizes the product of the payoffs. For bargaining problems with claims, the proportional solution satisfies all these properties. However, if we additionally impose independence of higher claims or independence of irrelevant claims, a single alternative solution pops up, which maximizes the payoff product over the truncated feasible set.

The *constrained Nash solution* $\mathbb{CN} : \mathrm{BC}^N \to \mathbb{R}_+^N$ assigns to any $(E, c) \in \mathrm{BC}^N$ the allocation

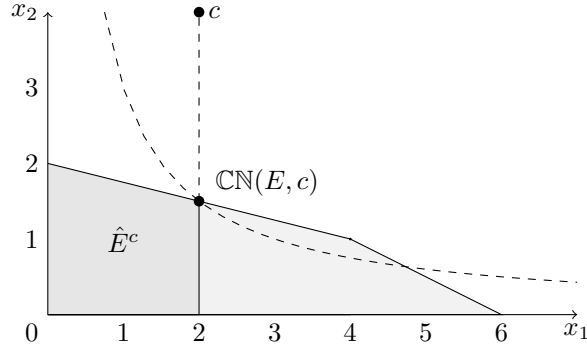$$\mathbb{CN}(E, c) = \operatorname*{argmax}_{x \in \hat{E}^c} \prod_{i \in N} x_i.$$

Note that the conditions on the feasible set imply that the constrained Nash solution is uniquely defined. Moreover, $\mathbb{CN} = \mathbb{CEA}$ on BANK^N.

Example 2
Let $N = \{1, 2\}$ and consider $(E, c) \in \mathrm{BC}^N$ given by

$$E = \text{c.c.h.} \{(6, 0), (4, 1), (0, 2)\} \quad \text{and} \quad c = (2, 4).$$

Then $\hat{E}^c = \text{c.c.h.}\{(2, 1\frac{1}{2}), (0, 2)\}$ and $\mathbb{CN}(E, c) = (2, 1\frac{1}{2})$.

$\triangle$

Theorem 2
$\mathbb{CN}$ *is the unique solution on* BC^N *satisfying EFF, SYM, SCOV, IIA, and IHC.*[3]

Proof. Clearly, $\mathbb{CN}$ satisfies EFF, SYM, SCOV, IIA, and IHC. Let $\phi : \mathrm{BC}^N \to \mathbb{R}_+^N$ be a solution satisfying EFF, SYM, SCOV, IIA, and IHC. Let $(E, c) \in \mathrm{BC}^N$. By SCOV, assume without loss of generality that $\mathbb{CN}_i(E, c) = 1$ for all $i \in N$. Define $E' \subseteq \mathbb{R}_+^N$ by

$$E' = \left\{ x \in \mathbb{R}_+^N \;\middle|\; \sum_{i \in N} x_i \leq |N| \right\}.$$

Then $\hat{E}^c \subseteq E'$. This means that $\phi(E, c) \in E'$. By IIA, $\phi(E, c) = \phi(E \cap E', c)$. Denote $N = \{1, \ldots, |N|\}$ such that $c_1 \leq \ldots \leq c_{|N|}$. For all $i \in N$, define $\hat{c}^i \in \mathbb{R}_{++}^N$ by

$$\hat{c}^i = \left(\min\{c_i, c_j\} \right)_{j \in N}.$$

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[3]This result is also valid on the domain of comprehensive feasible sets (not necessarily strictly).

Then $\hat{c}^1 \leq \ldots \leq \hat{c}^{|N|}$. By sequentially applying EFF, SYM, and IHC, $\phi(E', \hat{c}^i) = \mathbb{CN}(E, c)$ for all $i \in N$. In particular, $\phi(E', c) = \phi(E', \hat{c}^{|N|}) = \mathbb{CN}(E, c)$. Then $\phi(E', c) \in E$. By IIA, $\phi(E', c) = \phi(E \cap E', c)$. This means that

$$\phi(E, c) = \phi(E \cap E', c) = \phi(E', c) = \mathbb{CN}(E, c).$$

Hence, $\phi(E, c) = \mathbb{CN}(E, c)$. □

Theorem 3
$\mathbb{CN}$ *is the unique solution on* BC^N *satisfying EFF, SYM, SCOV, IIA, and IIC.*[3]

Proof. Clearly, $\mathbb{CN}$ satisfies EFF, SYM, SCOV, IIA, and IIC. Let $\phi : \mathrm{BC}^N \to \mathbb{R}_+^N$ be a solution satisfying EFF, SYM, SCOV, IIA, and IIC. Let $(E, c) \in \mathrm{BC}^N$. By SCOV, assume without loss of generality that $\mathbb{CN}_i(E, c) = 1$ for all $i \in N$. Define $E' \subseteq \mathbb{R}_+^N$ by

$$E' = \left\{ x \in \mathbb{R}_+^N \ \middle| \ \sum_{i \in N} x_i \leq |N| \right\}.$$

Then $\hat{E}^c \subseteq E'$. This means that $\phi(E, c) \in E'$. By IIA, $\phi(E, c) = \phi(E \cap E', c)$. Define $\bar{c} \in \mathbb{R}_{++}^N$ by

$$\bar{c} = \left(\max_{i \in N} \{c_i\} \right)_{j \in N}.$$

Then $c \leq \bar{c}$. By EFF and SYM, $\phi(E', \bar{c}) = \mathbb{CN}(E, c)$. By IIC, $\phi(E', c) = \phi(E', \bar{c}) = \mathbb{CN}(E, c)$. Then $\phi(E', c) \in E$. By IIA, $\phi(E', c) = \phi(E \cap E', c)$. This means that

$$\phi(E, c) = \phi(E \cap E', c) = \phi(E', c) = \mathbb{CN}(E, c).$$

Hence, $\phi(E, c) = \mathbb{CN}(E, c)$. □

5 The Constrained Kalai-Smorodinsky Solution

For classic bargaining problems, efficiency, symmetry, scale covariance, and restricted monotonicity characterize the Kalai and Smorodinsky (1975) solution. This solution prescribes the maximally feasible allocation on the line connecting the origin with the utopia point. For bargaining problems with claims, the proportional solution satisfies all these properties. However, if we additionally impose independence of higher claims, a single alternative solution pops up, which minimizes the distance to the line connecting the origin with the utopia point over the efficient allocations within the truncated feasible set.

The *constrained Kalai-Smorodinsky solution*[4] $\mathbb{CKS} : \mathrm{BC}^N \to \mathbb{R}_+^N$ assigns to any $(E, c) \in \mathrm{BC}^N$ the allocation
$$\mathbb{CKS}(E, c) = \left(\min\{c_i, \alpha^{E,c} u_i^E\} \right)_{i \in N},$$

where $\alpha^{E,c} = \max\{\theta \in [0, 1] \mid (\min\{c_i, \theta u_i^E\})_{i \in N} \in E\}$. Note that the conditions on the feasible set imply that the constrained Kalai-Smorodinsky solution is uniquely defined. Moreover, $\mathbb{CKS} = \mathbb{CEA}$ on BANK^N.
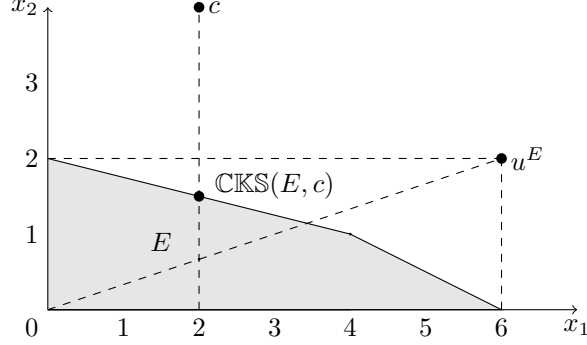
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[4]Dietzenbacher, Estévez-Fernández, Borm, and Hendrickx (2016) and Dietzenbacher and Peters (2018) studied this solution under the name *constrained relative equal awards rule*.

Example 3

Let $N = \{1, 2\}$ and consider $(E, c) \in \mathrm{BC}^N$ given by

$$E = \mathrm{c.c.h.} \{(6, 0), (4, 1), (0, 2)\} \quad \text{and} \quad c = (2, 4).$$

Then $\alpha^{E,c} = \frac{3}{4}$ and $\mathbb{CKS}(E, c) = (2, 1\frac{1}{2})$.

$\triangle$

Theorem 4

$\mathbb{CKS}$ *is the unique solution on* BC^N *satisfying EFF, SYM, SCOV, RMON, and IHC.*

Proof. Clearly, $\mathbb{CKS}$ satisfies EFF, SYM, SCOV, RMON, and IHC. Let $\phi : \mathrm{BC}^N \to \mathbb{R}_+^N$ be a solution satisfying EFF, SYM, SCOV, RMON, and IHC. Let $(E, c) \in \mathrm{BC}^N$. By SCOV, assume without loss of generality that $u_i^E = 1$ for all $i \in N$. Denote $N = \{1, \ldots, |N|\}$ such that $c_1 \leq \ldots \leq c_{|N|}$. For all $i \in N$, define $\hat{c}^i \in \mathbb{R}_{++}^N$ by

$$\hat{c}^i = (\min\{c_i, c_j\})_{j \in N}.$$

Then $\hat{c}^1 \leq \ldots \leq \hat{c}^{|N|}$. Define $E' \subseteq \mathbb{R}_+^N$ by

$$E' = \left\{ x \in \mathbb{R}_+^N \;\middle|\; \sum_{i \in N} x_i \leq 1 \right\}.$$

Then $E' \subseteq E$ and $u^{E'} = u^E$. By EFF, $\phi(E', \mathbb{CKS}(E', c)) = \mathbb{CKS}(E', c)$. By sequentially applying EFF, SYM, and IHC, $\phi(E', \hat{c}^i) = \mathbb{CKS}(E', c)$ for all $i \in N$ with $\hat{c}^i \notin E'$.

Let $k \in N$ be such that $\hat{c}^i \in E'$ for all $i \in N$ with $i < k$ and $\hat{c}^i \notin E'$ for all $i \in N$ with $i \geq k$. Let $\ell \in N$ be such that $\hat{c}^i \in E$ for all $i \in N$ with $i \leq \ell$ and $\hat{c}^i \notin E$ for all $i \in N$ with $i > \ell$. For all $i \in N$ with $k \leq i \leq \ell$, define $E^i \subseteq \mathbb{R}_+^N$ by

$$E^i = \mathrm{c.c.h.} \left(E' \cup \left\{ \left(\left(\hat{c}_j^i + \frac{|N| - |S|}{|N| - 1} \varepsilon \right)_{j \in S}, (0)_{j \in N \setminus S} \right) \;\middle|\; S \in 2^N \setminus \{\emptyset\} \right\} \right).$$

Then $E' \subseteq E^k \subseteq \ldots \subseteq E^\ell$ and $u^{E'} = u^{E^k} = \ldots = u^{E^\ell}$. By RMON, $\phi_i(E^k, \hat{c}^j) = c_i = \hat{c}_i^k$ for all $i, j \in N$ with $i < k \leq j$. By sequentially applying EFF, SYM, and IHC, $\phi(E^k, \hat{c}^j) = \hat{c}^k$ for all $j \in N$ with $j \geq k$. By RMON, $\phi_k(E^i, \hat{c}^j) = c_k$ for all $i, j \in N$ with $k \leq i \leq j$ and $k \leq i \leq \ell$. By sequentially applying this whole argument, $\phi(E^i, \hat{c}^j) = \hat{c}^i$ for all $i, j \in N$ with $k \leq i \leq j$ and $k \leq i \leq \ell$.

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Define $E'' \subseteq \mathbb{R}_+^N$ by

$$E'' = \text{c.c.h.} \left(E' \cup \left\{ \left(\left(\mathbb{CKS}_i(E,c) + \frac{|N| - |S|}{|N| - 1} \varepsilon \right)_{i \in S}, (0)_{i \in N \setminus S} \right) \ \middle| \ S \in 2^N \setminus \{\emptyset\} \right\} \right).$$

Then $E^\ell \subseteq E'' \subseteq E$ and $u^{E^\ell} = u^{E''} = u^E$. By RMON, $\phi_i(E'', \hat{c}^j) = c_i = \mathbb{CKS}_i(E,c)$ for all $i, j \in N$ with $i \le \ell < j$. By sequentially applying EFF, SYM, and IHC, $\phi(E'', \hat{c}^j) = \mathbb{CKS}(E,c)$ for all $j \in N$ with $j > \ell$. In particular, $\phi(E'', c) = \phi(E'', \hat{c}^{|N|}) = \mathbb{CKS}(E,c)$. By RMON, $\phi(E,c) \ge \phi(E'', c) = \mathbb{CKS}(E,c)$. By EFF, $\phi(E,c) = \mathbb{CKS}(E,c)$. Hence, $\phi = \mathbb{CKS}$. $\qquad\square$

The constrained Kalai-Smorodinsky solution also satisfies independence of irrelevant claims. However, as the following example shows, efficiency, symmetry, scale covariance, restricted monotonicity, and independence of irrelevant claims do not characterize the constrained Kalai-Smorodinsky solution.

Example
Let $N = \{1, 2\}$ and let $\phi : \text{BC}^N \to \mathbb{R}_+^N$ be the solution defined by

$$\phi(E,c) = \begin{cases} (0, u_2^E) & \text{if } c_1 < \frac{1}{2} u_1^E \text{ and } c_2 \ge u_2^E; \\ \mathbb{CKS}(E,c) & \text{otherwise.} \end{cases}$$

Then ϕ satisfies efficiency, symmetry, scale covariance, restricted monotonicity, and independence of irrelevant claims, but does not coincide with the constrained Kalai-Smorodinsky solution. $\qquad\triangle$

6 The Constrained Kalai Solution

For classic bargaining problems, efficiency, symmetry, and monotonicity characterize the Kalai (1977) solution. This solution prescribes the maximally feasible allocation with equal payoffs. For bargaining problems with claims, the proportional solution satisfies all these properties. However, if we additionally impose independence of higher claims or independence of irrelevant claims, a single alternative solution pops up, which minimizes the distance to equal payoff allocations over the efficient allocations within the truncated feasible set.

The *constrained Kalai solution* $\mathbb{CK} : \text{BC}^N \to \mathbb{R}_+^N$ assigns to any $(E,c) \in \text{BC}^N$ the allocation

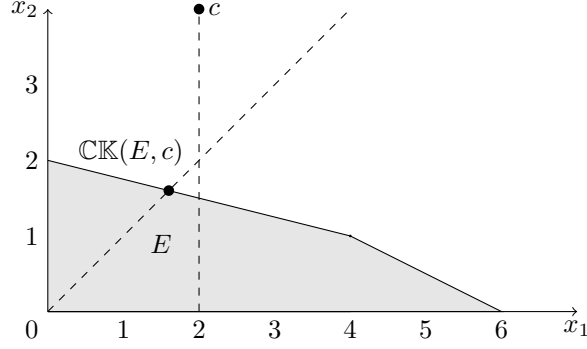$$\mathbb{CK}(E,c) = \left(\min\{c_i, \mu^{E,c}\} \right)_{i \in N},$$

where $\mu^{E,c} = \sup\{\theta \in \mathbb{R}_+ \mid (\min\{c_i, \theta\})_{i \in N} \in E\}$. Note that the conditions on the feasible set imply that the constrained Kalai solution is uniquely defined. Moreover, $\mathbb{CK} = \mathbb{CEA}$ on BANK^N.

Example 4
Let $N = \{1, 2\}$ and consider $(E,c) \in \text{BC}^N$ given by

$$E = \text{c.c.h.} \{(6,0), (4,1), (0,2)\} \quad \text{and} \quad c = (2,4).$$

Then $\mu^{E,c} = 1\frac{3}{5}$ and $\mathbb{CK}(E,c) = (1\frac{3}{5}, 1\frac{3}{5})$.

$\triangle$

Theorem 5

$\mathbb{CK}$ *is the unique solution on* BC^N *satisfying EFF, SYM, MON, and IHC.*

Proof. Clearly, $\mathbb{CK}$ satisfies EFF, SYM, MON, and IHC. Let $\phi : \mathrm{BC}^N \to \mathbb{R}_+^N$ be a solution satisfying EFF, SYM, MON, and IHC. Let $(E, c) \in \mathrm{BC}^N$. Denote $N = \{1, \ldots, |N|\}$ such that $c_1 \leq \ldots \leq c_{|N|}$. For all $i \in N$, define $\hat{c}^i \in \mathbb{R}_{++}^N$ by

$$\hat{c}^i = (\min\{c_i, c_j\})_{j \in N}.$$

Then $\hat{c}^1 \leq \ldots \leq \hat{c}^{|N|}$. Let $k \in N$ be such that $\hat{c}^i \in E$ for all $i \in N$ with $i \leq k$ and $\hat{c}^i \notin E$ for all $i \in N$ with $i > k$. For all $i \in N$ with $i \leq k$, define $E^i \subseteq \mathbb{R}_+^N$ by

$$E^i = \text{c.c.h.} \left\{ \left(\left(\hat{c}_j^i + \frac{|N| - |S|}{|N| - 1} \varepsilon \right)_{j \in S}, (0)_{j \in N \setminus S} \right) \,\middle|\, S \in 2^N \setminus \{\emptyset\} \right\}.$$

Then $E^1 \subseteq \ldots \subseteq E^k$. By EFF, $\phi(E^1, \hat{c}^1) = \hat{c}^1$. By sequentially applying EFF, SYM, and IHC, $\phi(E^1, \hat{c}^j) = \hat{c}^1$ for all $j \in N$. By MON, $\phi_1(E^i, \hat{c}^j) = c_1$ for all $i, j \in N$ with $i \leq j$ and $i \leq k$. By sequentially applying this whole argument, $\phi(E^i, \hat{c}^j) = \hat{c}^i$ for all $i, j \in N$ with $i \leq j$ and $i \leq k$.

Define $E' \subseteq \mathbb{R}_+^N$ by

$$E' = \text{c.c.h.} \left\{ \left(\left(\mathbb{CK}_i(E, c) + \frac{|N| - |S|}{|N| - 1} \varepsilon \right)_{i \in S}, (0)_{i \in N \setminus S} \right) \,\middle|\, S \in 2^N \setminus \{\emptyset\} \right\}.$$

Then $E^k \subseteq E' \subseteq E$. By MON, $\phi_i(E', \hat{c}^j) = c_i = \mathbb{CK}_i(E, c)$ for all $i, j \in N$ with $i \leq k < j$. By sequentially applying EFF, SYM, and IHC, $\phi(E', \hat{c}^j) = \mathbb{CK}(E, c)$ for all $j \in N$ with $j > k$. In particular, $\phi(E', c) = \phi(E', \hat{c}^{|N|}) = \mathbb{CK}(E, c)$. By MON, $\phi(E, c) \geq \phi(E', c) = \mathbb{CK}(E, c)$. By EFF, $\phi(E, c) = \mathbb{CK}(E, c)$. Hence, $\phi = \mathbb{CK}$. $\square$

Theorem 6

$\mathbb{CK}$ *is the unique solution on* BC^N *satisfying EFF, SYM, MON, and IIC.*

Proof. Clearly, $\mathbb{CK}$ satisfies EFF, SYM, MON, and IIC. Let $\phi : \mathrm{BC}^N \to \mathbb{R}_+^N$ be a solution satisfying EFF, SYM, MON, and IIC. Let $(E, c) \in \mathrm{BC}^N$. Denote $N = \{1, \ldots, |N|\}$ such that $c_1 \leq \ldots \leq c_{|N|}$. For all $i \in N$, define $\bar{c}^i \in \mathbb{R}_{++}^N$ by

$$\bar{c}_j^i = \begin{cases} c_j & \text{for all } j \in N \text{ with } j < i. \\ c_{|N|} & \text{for all } j \in N \text{ with } j \geq i; \end{cases}$$

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Then $\bar{c}^1 \geq \ldots \geq \bar{c}^{|N|}$. For all $i \in N$, define $\hat{c}^i \in \mathbb{R}^N_{++}$ by

$$\hat{c}^i = (\min\{c_i, c_j\})_{j \in N}.$$

Then $\hat{c}^1 \leq \ldots \leq \hat{c}^{|N|} = c = \bar{c}^{|N|} \leq \ldots \leq \bar{c}^1$. Let $k \in N$ be such that $\hat{c}^i \in E$ for all $i \in N$ with $i \leq k$ and $\hat{c}^i \notin E$ for all $i \in N$ with $i > k$. For all $i \in N$ with $i \leq k$, define $E^i \subseteq \mathbb{R}^N_+$ by

$$E^i = \text{c.c.h.} \left\{ \left(\left(\hat{c}^i_j + \frac{|N| - |S|}{|N| - 1} \varepsilon \right)_{j \in S}, (0)_{j \in N \setminus S} \right) \,\middle|\, S \in 2^N \setminus \{\emptyset\} \right\}.$$

Then $E^1 \subseteq \ldots \subseteq E^k$. By EFF and SYM, $\phi(E^1, \bar{c}^1) = \hat{c}^1$. By IIC, $\phi(E^1, \bar{c}^j) = \phi(E^1, \bar{c}^1) = \hat{c}^1$ for all $j \in N$. By MON, $\phi_1(E^i, \bar{c}^j) = c_1$ for all $i, j \in N$ with $i \leq j$ and $i \leq k$. By sequentially applying this whole argument, $\phi(E^i, \bar{c}^j) = \hat{c}^i$ for all $i, j \in N$ with $i \leq j$ and $i \leq k$.

Define $E' \subseteq \mathbb{R}^N_+$ by

$$E' = \text{c.c.h.} \left\{ \left(\left(\mathbb{CK}_i(E, c) + \frac{|N| - |S|}{|N| - 1} \varepsilon \right)_{i \in S}, (0)_{i \in N \setminus S} \right) \,\middle|\, S \in 2^N \setminus \{\emptyset\} \right\}.$$

Then $E^k \subseteq E' \subseteq E$. By MON, $\phi_i(E', \bar{c}^{k+1}) = c_i = \mathbb{CK}_i(E, c)$ for all $i \in N$ with $i \leq k$. By EFF and SYM, $\phi(E', \bar{c}^{k+1}) = \mathbb{CK}(E, c)$. By IIC, $\phi(E', \bar{c}^j) = \phi(E', \bar{c}^{k+1}) = \mathbb{CK}(E, c)$ for all $j \in N$ with $j > k$. In particular, $\phi(E', c) = \phi(E', \bar{c}^{|N|}) = \mathbb{CK}(E, c)$. By MON, $\phi(E, c) \geq \phi(E', c) = \mathbb{CK}(E, c)$. By EFF, $\phi(E, c) = \mathbb{CK}(E, c)$. Hence, $\phi = \mathbb{CK}$. $\qquad\square$

7 Concluding Remarks

In this paper, we axiomatically studied bargaining problems with claims and focused on the properties independence of higher claims and independence of irrelevant claims. In particular, we compared the proportional solution, the constrained Nash solution, the constrained Kalai-Smorodinsky solution, and the constrained Kalai solution and illustrated them with the following example.

Example 5
Let $N = \{1, 2\}$ and consider $(E, c) \in \text{BC}^N$ given by

$$E = \text{c.c.h.} \{(6, 0), (4, 1), (0, 2)\} \quad \text{and} \quad c = (2, 4).$$

Then $\mathbb{P}(E, c) = (\frac{8}{9}, 1\frac{7}{9})$, $\mathbb{CN}(E, c) = (2, 1\frac{1}{2})$, $\mathbb{CKS}(E, c) = (2, 1\frac{1}{2})$, and $\mathbb{CK}(E, c) = (1\frac{3}{5}, 1\frac{3}{5})$.

$\triangle$

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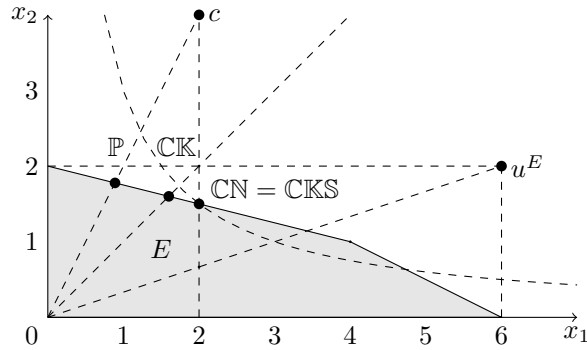
We studied the implications of independence of higher claims and independence of irrelevant claims in conjunction with the standard axioms efficiency, symmetry, scale covariance, independence of irrelevant alternatives, restricted monotonicity, and monotonicity from bargaining theory. The following table presents an overview of the properties and indicates the axiomatic characterizations.

	$\mathbb{P}$	$\mathbb{CN}$	$\mathbb{CN}$	$\mathbb{CKS}$	$\mathbb{CK}$	$\mathbb{CK}$
EFF	$+^*$	$+^*$	$+^*$	$+^*$	$+^*$	$+^*$
SYM	$+^*$	$+^*$	$+^*$	$+^*$	$+^*$	$+^*$
SCOV	$+^*$	$+^*$	$+^*$	$+^*$	$-$	$-$
IIA	$+$	$+^*$	$+^*$	$-$	$+$	$+$
RMON	$+$	$-$	$-$	$+^*$	$+$	$+$
MON	$+^*$	$-$	$-$	$-$	$+^*$	$+^*$
IHC	$-$	$+^*$	$+$	$+^*$	$+^*$	$+$
IIC	$-$	$+$	$+^*$	$+$	$+$	$+^*$

Alternatively, in line with the work of Dagan and Volij (1993), bargaining problems with claims could be solved by converting them into regular bargaining problems with the truncated feasible sets. The Nash solution for these converted bargaining problems would coincide with the constrained Nash solution for bargaining problems with claims. The Kalai-Smorodinsky solution for these bargaining problems would coincide with the truncated proportional solution for bargaining problems with claims, which prescribes efficient allocations proportional to the utopia point of the truncated feasible set. The truncated proportional solution also satisfies efficiency, symmetry, scale covariance, and restricted monotonicity, but does not satisfy independence or irrelevant alternatives, independence of higher claims, and independence of irrelevant claims. The Kalai solution for converted bargaining problems corresponds to a solution for bargaining problems with claims which satisfies symmetry, monotonicity, independence of higher claims, independence or irrelevant claims, but does not satisfy efficiency and scale covariance.

This paper only considers fixed population axioms, i.e. none of the axioms are based on changes in the population. Future research could study independence of higher claims and independence of irrelevant claims in conjunction with variable population axioms from bargaining theory, e.g. *population monotonicity* (cf. Thomson (1983a), Thomson (1983b)), *consistency* (cf. Lensberg (1988)), and *converse consistency* (cf. Chun (2002)). Population monotonicity, which requires that no remaining agent is worse off when some agents leave with zero payoff, is satisfied by the proportional solution, the constrained Kalai-Smorodinsky solution, and the constrained Kalai solution, but is not satisfied by the constrained Nash solution. Consistency, which requires invariance when some agents leave with their allocated payoffs, is satisfied by the proportional solution, the constrained Nash solution, and the constrained Kalai solution, but is not satisfied by the constrained Kalai-Smorodinsky solution. Converse consistency, which requires selection of a certain allocation when it is selected for all reduced problems, is satisfied by the proportional solution and the constrained Kalai solution, but not satisfied by the constrained Nash solution and the constrained Kalai-Smorodinsky solution.

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