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Polynomial Norms

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Abstract

In this paper, we study *polynomial norms*, i.e. norms that are the dth root of a degree-d homogeneous polynomial f. We first show that a necessary and sufficient condition for $f^{1/d}$ to be a norm is for f to be strictly convex, or equivalently, convex and positive definite. Though not all norms come from d^{th} roots of polynomials, we prove that any norm can be approximated arbitrarily well by a polynomial norm. We then investigate the computational problem of testing whether a form gives a polynomial norm. We show that this problem is strongly NP-hard already when the degree of the form is 4, but can always be answered by solving a hierarchy of semidefinite programs. We further study the problem of optimizing over the set of polynomial norms using semidefinite programming. To do this, we introduce the notion of *r-sos-convexity* and extend a result of Reznick on sum of squares representation of positive definite forms to positive definite biforms. We conclude with some applications of polynomial norms to statistics and dynamical systems.

Keywords: polynomial norms, sum of squares polynomials, convex polynomials, semidefinite programming
AMS classification: 90C22, 14P10, 52A27

1 Introduction

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *norm* if it satisfies the following three properties:

(i) positive definiteness: $f(x) > 0$, $\forall x \neq 0$, and $f(0) = 0$.

(ii) 1-homogeneity: $f(\lambda x) = |\lambda| f(x)$, $\forall x \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}$.

(iii) triangle inequality: $f(x + y) \leq f(x) + f(y)$, $\forall x, y \in \mathbb{R}^n$.

Some well-known examples of norms include the 1-norm, $f(x) = \sum_{i=1}^{n} |x_i|$, the 2-norm, $f(x) = \sqrt{\sum_{i=1}^{n} x_i^2}$, and the ∞-norm, $f(x) = \max_i |x_i|$. Our focus throughout this paper is on norms that can be derived from multivariate polynomials. More specifically, we are interested in establishing conditions under which the dth root of a homogeneous polynomial of degree d is a norm, where d is an even number. We refer to the norm obtained when these conditions are met as *a polynomial norm*. It is straightforward to see why we restrict ourselves to d^{th} roots of degree-d homogeneous polynomials. Indeed, nonhomogeneous polynomials cannot hope to satisfy the homogeneity condition of a norm and homogeneous polynomials of degree $d > 1$ are not 1-homogeneous unless we take their d^{th} root. The question of

when the square root of a homogeneous quadratic polynomial is a norm (i.e., when $d = 2$) has a well-known answer (see, e.g., [15, Appendix A]): a function $f(x) = \sqrt{x^T Q x}$ is a norm if and only if the symmetric $n \times n$ matrix Q is positive definite. In the particular case where Q is the identity matrix, one recovers the 2-norm. Positive definiteness of Q can be checked in polynomial time using for example Sylvester's criterion (positivity of the n leading principal minors of Q). This means that testing whether the square root of a quadratic form is a norm can be done in polynomial time. A similar characterization in terms of conditions on the coefficients are not known for polynomial norms generated by forms of degree greater than 2. In particular, it is not known whether one can efficiently test membership or optimize over the set of polynomial norms.

Outline and contributions. In this paper, we study polynomial norms from a computational perspective. In Section 2, we give two different necessary and sufficient conditions under which the dth root of a degree-d form f will be a polynomial norm: namely, that f be strictly convex (Theorem 2.2), or (equivalently) that f be convex and postive definite (Theorem 2.1). Section 3 investigates the relationship between general norms and polynomial norms: while many norms are polynomial norms (including all p-norms with p even), some norms are not (consider, e.g., the 1-norm). We show, however, that any norm can be approximated to arbitrary precision by a polynomial norm (Theorem 3.1). While it is well known that polynomials can approximate continuous functions on compact sets arbitrarily well, the approximation result here needs to preserve the convexity and homogeneity properties of the original norm, and hence does not follow, e.g., from the Stone-Weierstrass theorem. In Section 4, we move on to complexity results and show that simply testing whether the 4th root of a quartic form is a norm is strongly NP-hard (Theorem 4.1). We then provide a semidefinite programming-based hierarchy for certifying that the dth root of a degree d form is a norm (Theorem 4.4) and for optimizing over a subset of the set of polynomial norms (Theorem 4.21). The latter is done by introducing the concept of r-sum of squares-convexity (see Definition 4.7). We show that any form with a positive definite Hessian is r-sos-convex for some value of r, and present a lower bound on that value (Theorem 4.8). We also show that the level r of the semidefinite programming hierarchy cannot be bounded as a function of the number of variables and the degree only (Theorem 4.19). Finally, we cover some applications of polynomial norms in statistics and dynamical systems in Section 5. In Section 5.1, we compute approximations of two different types of norms, polytopic gauge norms and p-norms with p noneven, using polynomial norms. The techniques described in this section can be applied to norm regression. In Section 5.2, we use polynomial norms to prove stability of a switched linear system, a task which is equivalent to computing an upperbound on the joint spectral radius of a family of matrices.

2 Two equivalent characterizations of polynomial norms

We start this section with two theorems that provide conditions under which the dth root of a degree-d form is a norm. We will use these two theorems in Section 4 to establish semidefinite programming-based approximations of polynomial norms. We remark that these results are generally assumed to be known by the optimization community. Indeed, some prior work on polynomial norms has been done by Dmitriev and Reznick in [17, 18, 37, 40]. For completeness of presentation, however, and as we could not find the exact statements of these results in the form we present, we include them here with alternative proofs. Throughout this paper, we suppose that the number of variables n is larger or equal than 2 and that d is a positive even integer.

Theorem 2.1. *The dth root of a degree-d form f is a norm if and only if f is convex and positive definite.*

Proof. If $f^{1/d}$ is a norm, then $f^{1/d}$ is positive definite, and so is f. Furthermore, any norm is convex and the dth power of a nonnegative convex function remains convex.

Assume now that f is convex and positive definite. We show that $f^{1/d}$ is a norm. Positivity and homogeneity are immediate. It remains to prove the triangle inequality. Let $g := f^{1/d}$. Denote by S_f and S_g the 1-sublevel sets of f and g respectively. It is clear that

$$S_g = \{x \mid f^{1/d}(x) \leq 1\} = \{x \mid f(x) \leq 1\} = S_f,$$

and as f is convex, S_f is convex and so is S_g. Let $x, y \in \mathbb{R}^n$. We have that $\frac{x}{g(x)} \in S_g$ and $\frac{y}{g(y)} \in S_g$. From convexity of S_g,

$$g\left(\frac{g(x)}{g(x) + g(y)} \cdot \frac{x}{g(x)} + \frac{g(y)}{g(x) + g(y)} \cdot \frac{y}{g(y)}\right) \le 1.$$

Homogeneity of g then gives us

$$\frac{1}{g(x) + g(y)} g(x + y) \le 1$$

which shows that triangle inequality holds. □

Theorem 2.2. *The dth root of a degree-d form f is a norm if and only if f is strictly convex, i.e.,*

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y), \ \forall x \ne y, \ \forall \lambda \in (0, 1).$$

Proof. We will show that a degree-d form f is strictly convex if and only f is convex and positive definite. The result will then follow from Theorem 2.1.

Suppose f is strictly convex, then the first-order characterization of strict convexity gives us that

$$f(y) > f(x) + \nabla f(x)^T (y - x), \ \forall y \ne x.$$

For $x = 0$, the inequality becomes $f(y) > 0, \ \forall y \ne 0$, as $f(0) = 0$ and $\nabla f(0) = 0$. Hence, f is positive definite. Of course, a strictly convex function is also convex.

Suppose now that f is convex, positive definite, but not strictly convex, i.e., there exists $\bar{x}, \bar{y} \in \mathbb{R}^n$ with $\bar{x} \ne \bar{y}$, and $\gamma \in (0, 1)$ such that

$$f\left(\gamma \bar{x} + (1 - \gamma)\bar{y}\right) = \gamma f(\bar{x}) + (1 - \gamma)f(\bar{y}).$$

Let $g(\alpha) := f(\bar{x} + \alpha(\bar{y} - \bar{x}))$. Note that g is a restriction of f to a line and, consequently, g is a convex, positive definite, univariate polynomial in α. We now define

$$h(\alpha) := g(\alpha) - (g(1) - g(0))\alpha - g(0). \tag{1}$$

Similarly to g, h is a convex univariate polynomial as it is the sum of two convex univariate polynomials. We also know that $h(\alpha) \ge 0, \forall \alpha \in (0, 1)$. Indeed, by convexity of g, we have that $g(\alpha x + (1 - \alpha)y) \ge \alpha g(x) + (1 - \alpha)g(y), \forall x, y \in \mathbb{R}$ and $\alpha \in (0, 1)$. This inequality holds in particular for $x = 1$ and $y = 0$, which proves the claim. Observe now that $h(0) = h(1) = 0$. By convexity of h and its nonnegativity over $(0, 1)$, we have that $h(\alpha) = 0$ on $(0, 1)$ which further implies that $h = 0$. Hence, from (1), g is an affine function. As g is positive definite, it cannot be that g has a nonzero slope, so g has to be a constant. But this contradicts that $\lim_{\alpha \to \infty} g(\alpha) = \infty$. To see why this limit must be infinite, we show that $\lim_{||x|| \to \infty} f(x) = \infty$. As $\lim_{\alpha \to \infty} ||\bar{x} + \alpha(\bar{y} - \bar{x})|| = \infty$ and $g(\alpha) = f(\bar{x} + \alpha(\bar{y} - \bar{x}))$, this implies that $\lim_{\alpha \to \infty} g(\alpha) = \infty$. To show that $\lim_{||x|| \to \infty} f(x) = \infty$, let

$$x^* = \underset{||x|| = 1}{\operatorname{argmin}} f(x).$$

By positive definiteness of f, $f(x^*) > 0$. Let M be any positive scalar and define $R := (M/f(x^*))^{1/d}$. Then for any x such that $||x|| = R$, we have

$$f(x) \ge \min_{||x|| = R} f(x) \ge R^d f(x^*) = M,$$

where the second inequality holds by homogeneity of f. Thus $\lim_{||x|| \to \infty} f(x) = \infty$. □

3 Approximating norms by polynomial norms

It is easy to see that not all norms are polynomial norms. For example, the 1-norm $||x||_1 = \sum_{i=1}^{n} |x_i|$ is not a polynomial norm. Indeed, all polynomial norms are differentiable at all but one point (the origin) whereas the 1-norm is nondifferentiable whenever one of the components of x is equal to zero. In this section, we show that, though not every norm is a polynomial norm, any norm can be approximated to arbitrary precision by a polynomial norm (Theorem 3.1). A related result is given by Barvinok in [12]. In that paper, he shows that any norm can be approximated by the d-th root of a nonnegative degree-d form, and quantifies the quality of the approximation as a function of n and d. The form he obtains however is not shown to be convex. In fact, in a later work [13, Section 2.4], Barvinok points out that it would be an interesting question to know whether any norm can be approximated by the dth root of a convex form with the same quality of approximation as for d-th roots of nonnegative forms. The result below is a step in that direction, although the quality of approximation is weaker than that by Barvinok [12].

Theorem 3.1. *Let $|| \cdot ||$ be any norm on $\mathbb{R}^n$. Then,* [1] *for any even integer $d \geq 2$:*

(i) There exists an n-variate convex positive definite form f_d of degree d such that

$$\frac{d}{n+d}\left(\frac{n}{n+d}\right)^{n/d} ||x|| \leq f_d^{1/d}(x) \leq ||x||, \quad \forall x \in \mathbb{R}^n. \tag{2}$$

In particular, for any sequence $\{f_d\}$ $(d = 2, 4, 6, \ldots)$ of such polynomials one has

$$\lim_{d \to \infty} \frac{f_d^{1/d}(x)}{||x||} = 1 \quad \forall x \in \mathbb{R}^n.$$

(ii) One may assume without loss of generality that f_d in (i) is a nonnegative sum of dth powers of linear forms.

Proof of (i). Fix any norm $|| \cdot ||$ on $\mathbb{R}^n$. We denote the Euclidean inner product on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$, and the unit ball with respect to $|| \cdot ||$ by

$$B = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}.$$

We denote the polar of B with respect to $\langle \cdot, \cdot \rangle$ by

$$B^\circ = \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1 \ \forall x \in B\}.$$

Recall that B° is symmetric around the origin, because B is. One may express the given norm in terms of the polar as follows (see e.g. relation (3.1) in [12]):

$$||x|| = \max_{y \in B^\circ} \langle x, y \rangle = \max_{y \in B^\circ} |\langle x, y \rangle| \quad \forall x \in \mathbb{R}^n. \tag{3}$$

For given, even integer d, we define the polynomial

$$f_d(x) = \frac{1}{\text{vol}B^\circ} \int_{B^\circ} \langle x, y \rangle^d dy. \tag{4}$$

Note that f_d is indeed a convex form of degree d. In fact, we will show later on that f_d may in fact be written as a nonnegative sum of dth powers of linear forms.

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[1] We would like to thank an anonymous referee for suggesting the proof of part (i) of this theorem. We were previously showing that for any norm $|| \cdot ||$ on $\mathbb{R}^n$ and for any $\epsilon > 0$, there exist an even integer d and an n-variate positive definite form f_d of degree d, which is a sum of powers of linear forms, and such that

$$(1 - \epsilon)||x|| \leq f_d^{1/d}(x) \leq ||x||, \ \forall x \in \mathbb{R}^n.$$

By (3), one has
$$f_d^{1/d}(x) \le \|x\| \quad \forall x \in \mathbb{R}^n.$$

Now fix $x_0 \in \mathbb{R}^n$ such that $\|x_0\| = 1$. By (3), there exists a $y_0 \in B^\circ$ so that $\langle x_0, y_0 \rangle = 1$. Define the half-space
$$H_+ = \{y \in \mathbb{R}^n \mid \langle x_0, y \rangle \ge 0\}.$$

Then, by symmetry,
$$\mathrm{vol}\,(H_+ \cap B^\circ) = \frac{1}{2}\mathrm{vol}\,(B^\circ).$$

For any $\alpha \in (0,1)$ we now define
$$A_+(\alpha) = \{(1-\alpha)y + \alpha y_0 \mid y \in H_+ \cap B^\circ\}.$$

Then $A_+(\alpha) \subset B^\circ$, and
$$\mathrm{vol} A_+(\alpha) = \frac{1}{2}(1-\alpha)^n \mathrm{vol}\,(B^\circ).$$

Moreover
$$\langle x_0, y \rangle \ge \alpha \quad \forall y \in A_+(\alpha), \tag{5}$$
and
$$\mathrm{vol}\,(A_+(\alpha) \cap A_-(\alpha)) = 0. \tag{6}$$

Letting $A_-(\alpha) = -A_+(\alpha)$, by symmetry one has $A_-(\alpha) \subset B^\circ$, and
$$\langle x_0, y \rangle \le -\alpha \quad \forall y \in A_-(\alpha). \tag{7}$$

Thus
$$
\begin{aligned}
f^{1/d}(x_0) &= \left(\frac{1}{\mathrm{vol} B^\circ} \int_{B^\circ} \langle x_0, y \rangle^d dy \right)^{1/d} \\
&\ge \left(\frac{1}{\mathrm{vol} B^\circ} \int_{A_+(\alpha) \cup A_-(\alpha)} \langle x_0, y \rangle^d dy \right)^{1/d} \\
&\ge \left(\frac{\mathrm{vol} A_+(\alpha) + \mathrm{vol} A_-(\alpha)}{\mathrm{vol} B^\circ} \alpha^d \right)^{1/d} \quad \text{(by (5), (7), and (6))} \\
&= \alpha(1-\alpha)^{n/d}.
\end{aligned}
$$

The last expression is maximized by $\alpha = \frac{d}{n+d}$, yielding the leftmost inequality in (2) in the statement of the theorem. Finally, note that,
$$\lim_{d \to \infty} \frac{d}{n+d} \left(\frac{n}{n+d} \right)^{n/d} = \lim_{t \downarrow 0} t^t (1+t)^{-(1+t)} = 1,$$
as required. $\qquad \square$

For the proof of the second part of the theorem, we need a result concerning finite moment sequences of signed measures, given as the next lemma.

Lemma 3.2 ([42], see Lemma 3.1 in [43] for a simple proof)**.** *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue-measurable, and let μ be the normalized Lebesgue measure on Ω, i.e. $\mu(\Omega) = 1$. Denote the moments of μ by*
$$m_\mu(\alpha) = \int_\Omega x^\alpha d\mu(x) \quad \forall \alpha \in \mathbb{N}_0^n, \tag{8}$$
where $x^\alpha := \prod_i x_i^{\alpha_i}$ if $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Let $S \subset \mathbb{N}_0^n$ be finite. Then there exists an atomic probability measure, say μ', supported on at most $|S|$ points in Ω, such that
$$m_\mu(\alpha) = m_{\mu'}(\alpha) \quad \forall \alpha \in S.$$

We may now prove part 2 of Theorem 3.1.

Proof of (ii) of Theorem 3.1. Let μ be the normalized Lebesgue measure on B°, i.e.

$$d\mu(y) = \frac{1}{\mathrm{vol}B^\circ} dy.$$

We now use the multinomial theorem

$$\langle x, y \rangle^d = \sum_{|\alpha|=d} \binom{d}{\alpha} x^\alpha y^\alpha,$$

with the multinomial notation

$$|\alpha| = \sum_i \alpha_i, \quad \binom{d}{\alpha} = \frac{d!}{\alpha_1! \cdots \alpha_n!},$$

to rewrite f_d in (4) as

$$f_d(x) = \sum_{|\alpha|=d} \binom{d}{\alpha} m_\mu(\alpha) x^\alpha, \tag{9}$$

where $m_\mu(\alpha)$ is the moment of order α of μ, as defined in (8).

By Lemma 3.2, there exist $\bar{y}^{(1)}, \ldots, \bar{y}^{(p)} \in B^\circ$ with $p = \binom{d+n-1}{d}$ so that

$$m_\mu(\alpha) = \sum_{j=1}^{p} \lambda_j (\bar{y}^{(j)})^\alpha,$$

for some $\lambda_j \geq 0$ $(j = 1, \ldots, p)$ with $\sum_{j=1}^{p} \lambda_j = 1$. Substituting in (9), one has

$$
\begin{aligned}
f_d(x) &= \sum_{|\alpha|=d} \binom{d}{\alpha} \sum_{j=1}^{p} \lambda_j (\bar{y}^{(j)})^\alpha x^\alpha \\
&= \sum_{j=1}^{p} \lambda_j \left(\sum_{|\alpha|=d} \binom{d}{\alpha} (\bar{y}^{(j)})^\alpha x^\alpha \right) \\
&= \sum_{j=1}^{p} \lambda_j \langle \bar{y}^{(j)}, x \rangle^d,
\end{aligned}
$$

as required. $\qquad\square$

4 Semidefinite programming-based approximations of polynomial norms

4.1 Complexity

It is natural to ask whether testing if the dth root of a given degree-d form is a norm can be done in polynomial time. In the next theorem, we show that, unless $P = NP$, this is not the case even when $d = 4$.

Theorem 4.1. *Deciding whether the 4th root of a quartic form is a norm is strongly NP-hard.*

Proof. The proof of this result is adapted from a proof in [6]. Recall that the CLIQUE problem can be described thus: given a graph $G = (V, E)$ and a positive integer k, decide whether G contains a clique of size at least k. The CLIQUE problem is known to be NP-hard [19]. We will give a reduction from CLIQUE to the problem of testing convexity and

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positive definiteness of a quartic form. The result then follows from Theorem 2.1. Let $\omega(G)$ be the clique number of the graph at hand, i.e., the number of vertices in a maximum clique of G. Consider the following quartic form

$$b(x;y) := -2k \sum_{i,j \in E} x_i x_j y_i y_j - (1-k) \left(\sum_i x_i^2 \right) \left(\sum_i y_i^2 \right).$$

In [6], using in part a result in [30], it is shown that

$$\omega(G) \leq k \Leftrightarrow b(x;y) + \frac{n^2 \gamma}{2} \left(\sum_{i=1}^{n} x_i^4 + \sum_{i=1}^{n} y_i^4 + \sum_{1 \leq i < j \leq n} (x_i^2 x_j^2 + y_i^2 y_j^2) \right) \tag{10}$$

is convex and $b(x;y)$ is positive semidefinite. Here, γ is a positive constant defined as the largest coefficient in absolute value of any monomial present in some entry of the matrix $\left[\frac{\partial^2 b(x;y)}{\partial x_i \partial y_j} \right]_{i,j}$. As $\sum_i x_i^4 + \sum_i y_i^4$ is positive definite and as we are adding this term to a positive semidefinite expression, the resulting polynomial is positive definite. Hence, the equivalence holds if and only if the quartic on the righthandside of the equivalence in (10) is convex and positive definite. $\qquad \square$

Note that this also shows that strict convexity is hard to test for quartic forms (this is a consequence of Theorem 2.2). A related result is Proposition 3.5. in [6], which shows that testing strict convexity of a polynomial of even degree $d \geq 4$ is hard. However, this result is not shown there for *forms*, hence the relevance of the previous theorem.

Theorem 4.1 rules out the possibility of a pseudo-polynomial time characterization of polynomial norms (unless P=NP) and motivates the study of tractable sufficient conditions. The sufficient conditions we consider next are based on semidefinite programming. Semidefinite programs can be solved to arbitrary accuracy in polynomial time [44] and technology for solving this class of problems is rapidly improving [9, 33, 29, 45, 20, 41, 5].

4.2   Sum of squares polynomials and semidefinite programming review

We start this section by reviewing the notion of *sum of squares polynomials* and related concepts such as *sum of squares-convexity*. We say that a polynomial f is a *sum of squares* (sos) if $f(x) = \sum_i q_i^2(x)$, for some polynomials q_i. Being a sum of squares is a sufficient condition for being nonnegative. The converse however is not true, as is exemplified by the Motzkin polynomial

$$M(x,y) = x^4 y^2 + x^2 y^4 3 x^2 y^2 + 1 \tag{11}$$

which is nonnegative but not a sum of squares [32]. The sum of squares condition is a popular surrogate for non-negativity due to its tractability. Indeed, while testing nonnegativity of a polynomial of degree greater or equal to 4 is a hard problem, testing whether a polynomial is a sum of squares can be done using *semidefinite programming*. This comes from the fact that a polynomial p of degree d is a sum of squares if and only if there exists a positive semidefinite matrix Q such that $f(x) = z(x)^T Q z(x)$, where $z(x)$ is the standard vector of monomials of degree up to d (see, e.g., [34]). As a consequence, any optimization problem over the coefficients of a set of polynomials which includes a combination of affine constraints and sos constraints on these polynomials, together with a linear objective can be recast as a semidefinite program. These type of optimization problems are known as *sos programs* and have found widespread applications in recent years [35, 28, 24, 23].

Though not all nonnegative polynomials can be written as sums of squares, the following theorem by Artin [10] circumvents this problem using sos multipliers.

Theorem 4.2 (Artin [10])**.** *For any nonnegative polynomial f, there exists an sos polynomial q such that $q \cdot f$ is sos.*

This theorem in particular implies that if we are given a polynomial f, then we can always check its nonnegativity using an sos program that searches for q (of a fixed degree). However, this result does not allow us to optimize over

the set of nonnegative polynomials or positive semidefinite polynomial matrices using an sos program (as far as we know). This is because, in that setting, products of decision varibles arise from multiplying polynomials f and q, whose coefficients are decision variables.

By adding further assumptions on f, Reznick showed in [38] that one could further pick q to be a power of $\sum_i x_i^2$. In what follows, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.

Theorem 4.3 (Reznick [38]). *Let f be a positive definite form of degree d in n variables and define*

$$\epsilon(f) := \frac{\min\{f(u) \mid u \in S^{n-1}\}}{\max\{f(u) \mid u \in S^{n-1}\}}.$$

If $r \geq \frac{nd(d-1)}{4\log(2)\epsilon(f)} - \frac{n+d}{2}$, then $(\sum_{i=1}^n x_i^2)^r \cdot f$ is a sum of squares.

Motivated by this theorem, the notion of r-sos polynomials can be defined: a polynomial f is said to be r-sos if $(\sum_i x_i^2)^r \cdot f$ is sos. Note that it is clear that any r-sos polynomial is nonnegative and that the set of r-sos polynomials is included in the set of $(r+1)$-sos polynomials. The Motzkin polynomial in (11) for example is 1-sos although not sos.

To end our review, we briefly touch upon the concept of sum of squares-convexity (sos-convexity), which we will build upon in the rest of the section. Let H_f denote the Hessian matrix of a polynomial f. We say that f is *sos-convex* if $y^T H_f(x) y$ is a sum of squares (as a polynomial in x and y). As before, optimizing over the set of sos-convex polynomials can be cast as a semidefinite program. Sum of squares-convexity is obviously a sufficient condition for convexity via the second-order characterization of convexity. However, there are convex polynomials which are not sos-convex (see, e.g., [7]). For a more detailed overview of sos-convexity including equivalent characterizations and settings in which sos-convexity and convexity are equivalent, refer to [8].

4.2.1 Notation

Throughout, we will use the notation $H_{n,d}$ (resp. $P_{n,d}$) to denote the set of forms (resp. positive semidefinite, aka nonnegative, forms) in n variables and of degree d. We will futhermore use the falling factorial notation $(t)_0 = 1$ and $(t)_k = t(t-1)\ldots(t-(k-1))$ for a positive integer k.

4.3 Certifying validity of a polynomial norm

In this subsection, we assume that we are given a form f of degree d and we would like to prove that $f^{1/d}$ is a norm using semidefinite programming.

Theorem 4.4. *Let f be a degree-d form. Then $f^{1/d}$ is a polynomial norm if and only if there exist $c > 0$, $r \in \mathbb{N}$, and an sos form $q(x)$ such that $q(x) \cdot y^T H_f(x) y$ is sos and $\left(f(x) - c(\sum_i x_i^2)^{d/2}\right)(\sum_i x_i^2)^r$ is sos. Furthermore, this condition can be checked using semidefinite programming.*

To show this result, we require a counterpart to Theorem 4.2 for matrices, which we present below.

Proposition 4.5 ([36, 21, 26]). *If $H(x)$ is a positive semidefinite polynomial matrix, then there exists a sum of squares polynomial $q(x)$ such that $q(x) \cdot y^T H(x) y$ is a sum of squares.*

Proof. This is an immediate consequence of a theorem by Procesi and Schacher [36] and independently Gondard and Ribenboim [21], reproven by Hillar and Nie [26]. This theorem states that if $H(x)$ is a symmetric polynomial matrix that is positive semidefinite for all $x \in \mathbb{R}^n$, then

$$H(x) = \sum_i A_i(x)^2,$$

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where the matrices $A_i(x)$ are symmetric and have rational functions as entries. Let $p(x)$ be the polynomial obtained by multiplying all denominators of the rational functions involved in any of the matrices A_i. Note that $p(x)^2 \cdot y^T H(x)y$ is a sum of squares as

$$p(x)^2 \cdot y^T H(x)y = \sum_i p(x)^2 y^T A_i(x)^2 y = \sum_i ||p(x)A_i(x)y||^2.$$

However, $p(x) \cdot A_i(x)$ is now a matrix with polynomial entries, which gives the result. $\qquad \square$

We remark that this result does not immediately follow from the theorem given by Artin as the multiplier q does not depend on x and y, but solely on x. We now prove Theorem 4.4.

Proof of Theorem 4.4. It is immediate to see that if there exist such a c, r, and q, then f is convex and positive definite. From Theorem 2.1, this means that $f^{1/d}$ is a polynomial norm.

Conversely, if $f^{1/d}$ is a polynomial norm, then, by Theorem 2.1, f is convex and positive definite. As f is convex, the polynomial $y^T H_f(x)y$ is nonnegative. Using Proposition 4.5, we conclude that there exists an sos polynomial $q(x)$ such that $q(x,y) \cdot y^T H_f(x)y$ is sos. We now show that, as f is positive definite, there exist $c > 0$ and $r \in \mathbb{N}$ such that $\left(f(x) - c(\sum_i x_i^2)^{d/2}\right)(\sum_i x_i^2)^r$ is sos. Let f_{min} denote the minimum of f on the sphere. As f is positive definite, $f_{min} > 0$. We take $c := \frac{f_{min}}{2}$ and consider $g(x) := f(x) - c(\sum_i x_i^2)^{d/2}$. We have that g is a positive definite form: indeed, if x is a nonzero vector in $\mathbb{R}^n$, then

$$\frac{g(x)}{||x||^d} = \frac{f(x)}{||x||^d} - c = f\left(\frac{x}{||x||}\right) - c > 0,$$

by homogeneity of f and definition of c. Using Theorem 4.3, $\exists r \in \mathbb{N}$ such that $g(x)(\sum_i x_i^2)^r$ is sos.

For fixed r, a given form f, and a fixed degree d, one can search for $c > 0$ and an sos form q of degree d such that $q(x) \cdot y^T H_f(x)y$ is sos and $\left(f(x) - c(\sum_i x_i^2)^{d/2}\right)(\sum_i x_i^2)^r$ is sos using semidefinite programming. This is done by solving the following semidefinite feasibility problem:

$$\begin{aligned}
& q(x) \text{ sos} \\
& c \geq 0 \\
& q(x) \cdot y^T H_f(x)y \text{ sos} \\
& \left(f(x) - c\left(\sum_i x_i^2\right)^{d/2}\right)\left(\sum_i x_i^2\right)^r \text{ sos},
\end{aligned} \qquad (12)$$

where the unknowns are the coefficients of q and the real number c. $\qquad \square$

Remark 4.6. *We remark that we are not imposing $c > 0$ in the semidefinite program above. This is because, in practice, especially if the semidefinite program is solved with interior point methods, the solution returned by the solver will be in the interior of the feasible set, and hence c will automatically be positive. One can slightly modify (12) however to take the constraint $c > 0$ into consideration explicitly. Indeed, consider the following semidefinite feasibility problem where both the degree of q and the integer r are fixed:*

$$\begin{aligned}
& q(x) \text{ sos} \\
& \gamma \geq 0 \\
& q(x) \cdot y^T H_f(x)y \text{ sos} \\
& \left(\gamma f(x) - \left(\sum_i x_i^2\right)^{d/2}\right)\left(\sum_i x_i^2\right)^r \text{ sos}.
\end{aligned} \qquad (13)$$

It is easy to check that (13) is feasible with $\gamma \geq 0$ if and only if the last constraint of (12) is feasible with $c > 0$. To see this, take $c = 1/\gamma$ and note that γ can never be zero.

To the best of our knowledge, we cannot use the approach described in Theorem 4.4 to optimize over the set of polynomial norms with a semidefinite program. This is because of the product of decision variables in the coefficients of f and q. The next subsection will address this issue.

4.4 Optimizing over the set of polynomial norms

In this subsection, we consider the problem of optimizing over the set of polynomial norms. To do this, we introduce the concept of r-sos-convexity. Recall that the notation H_f references the Hessian matrix of a form f.

4.4.1 Positive definite biforms and r-sos-convexity

Definition 4.7. *For an integer r, we say that a polynomial f is r-sos-convex if $y^T H_f(x)y \cdot (\sum_i x_i^2)^r$ is sos.*

Observe that, for fixed r, the property of r-sos-convexity can be checked using semidefinite programming (though the size of this SDP gets larger as r increases). Any polynomial that is r-sos-convex is convex. Note that the set of r-sos-convex polynomials is a subset of the set of $(r+1)$-sos-convex polynomials and that the case $r = 0$ corresponds to the set of sos-convex polynomials. We remark that f_d in Theorem 3.1 is in fact sos-convex, since it is a sum of squares of linear forms. Thus Theorem 3.1 implies that any norm on $\mathbb{R}^n$ may be approximated arbitrarily well by a polynomial norm that corresponds to a sos-convex form.

It is natural to ask whether any convex polynomial is r-sos-convex for some r. Our next theorem shows that this is the case under a mild assumption.

Theorem 4.8. *Let f be a form of degree d such that $y^T H_f(x)y > 0$ for $(x, y) \in S^{n-1} \times S^{n-1}$. Let*

$$\eta(f) := \frac{\min\{y^T H_f(x)y \mid (x, y) \in S^{n-1} \times S^{n-1}\}}{\max\{y^T H_f(x)y \mid (x, y) \in S^{n-1} \times S^{n-1}\}}.$$

If $r \geq \frac{n(d-2)(d-3)}{4\log(2)\eta(f)} - \frac{n+d-2}{2} - d$, then f is r-sos-convex.

Remark 4.9. *Note that $\eta(f)$ can also be interpreted as*

$$\eta(f) = \frac{\min_{x \in S^{n-1}} \lambda_{\min}(H_f(x))}{\max_{x \in S^{n-1}} \lambda_{\max}(H_f(x))} = \frac{1}{\max_{x \in S^{n-1}} \|H_f^{-1}(x)\|_2 \cdot \max_{x \in S^{n-1}} \|H_f(x)\|_2}.$$

Remark 4.10. *Theorem 4.8 is a generalization of Theorem 4.3 by Reznick. Note though that this is not an immediate generalization. First, $y^T H_f(x)y$ is not a positive definite form (consider, e.g., $y = 0$ and any nonzero x). Secondly, note that the multiplier is $(\sum_i x_i^2)^r$ and does not involve the y variables. (As we will see in the proof, this is essentially because $y^T H_f(x)y$ is quadratic in y.) It is not immediate that the multiplier should have this specific form. From Theorem 4.3, it may perhaps seem more natural that the multiplier be $(\sum_i x_i^2 + \sum_i y_i^2)^r$. It turns out in fact that such a multiplier would not give us the correct property (contrarily to $(\sum_i x_i^2)^r$) as there exist forms f whose Hessian is positive definite for all x but for which the form*

$$y^T H_f(x)y(\sum_i x_i^2 + \sum_i y_i^2)^r$$

is not sos for any r. For a specific example, consider the form f in 3 variables and of degree 8 given in [7, Theorem 3.2]. It is shown in [7] that (i) f is convex; (ii) the $(1, 1)$ entry of the Hessian of f, $H_f^{1,1}$, is not sos; and (iii) $(x_1^2 + x_2^2 + x_3^2)y^T H_f(x)y$ is sos.

Suppose for the sake of contradiction that

$$q_r(x, y) := y^T H_f(x) y (\sum_i x_i^2 + \sum_i y_i^2)^r$$

is sos for some r. This implies that the polynomial

$$q_r(x, \alpha, 0, 0) = \alpha^2 H_p^{1,1}(x) \left(\sum_{i=1}^{3} x_i^2 + \alpha^2 \right)^r$$

should be sos for any α as it is obtained from q_r by setting $y = (\alpha, 0, 0)^T$. Expanding this out, we get

$$q_r(x, \alpha, 0, 0) = \alpha^2 H_f^{1,1}(x) \sum_{k=0}^{r} \binom{r}{k} \left(\sum_{i=1}^{3} x_i^2 \right)^k \alpha^{2r-2k}$$

$$= \alpha^{2r+2} H_f^{1,1}(x) + \sum_{k=1}^{r} \binom{r}{k} \left(\sum_{i=1}^{3} x_i^2 \right)^k \alpha^{2r-2k+2} H_f^{1,1}(x)$$

$$= \alpha^{2r+2} \left(H_f^{1,1}(x) + \sum_{k=1}^{r} \binom{r}{k} \left(\sum_{i=1}^{3} x_i^2 \right)^k \frac{1}{\alpha^{2k}} H_f^{1,1}(x) \right).$$

From arguments (ii) and (iii) above, we know that $H_f^{1,1}(x)$ is not sos but

$$\sum_{k=1}^{r} \binom{r}{k} \left(\sum_{i=1}^{3} x_i^2 \right)^k \frac{1}{\alpha^{2k}} H_f^{1,1}(x)$$

is sos. Fixing α large enough, we can ensure that $q_r(x, \alpha, 0, 0)$ is not sos. This contradicts our previous assumption.

Remark 4.11. *Theorem 4.8 can easily be adapted to biforms of the type $\sum_j f_j(x) g_j(y)$ where f_j's are forms of degree d in x and g_j's are forms of degree $\tilde{d}$ in y. In this case, there exist integers s, r such that*

$$\sum_j f_j(x) g_j(y) \cdot (\sum_i x_i^2)^r \cdot (\sum_i y_i^2)^s$$

is sos. For the purposes of this paper however and the connection to polynomial norms, we will show the result in the particular case where the biform of interest is $y^T H_f(x) y$.

We associate to any form $f \in H_{n,d}$, the d-th order differential operator $f(D)$, defined by replacing each occurence of x_j with $\frac{\partial}{\partial x_j}$. For example, if $f(x_1, \ldots, x_n) := \sum_i c_i x_1^{a_1^i} \ldots x_n^{a_n^i}$ where $c_i \in \mathbb{R}$ and $a_j^i \in \mathbb{N}$, then its differential operator will be

$$f(D) = \sum_i c_i \frac{\partial^{a_1^i}}{\partial x_1^{a_1^i}} \ldots \frac{\partial^{a_n^i}}{\partial x_n^{a_n^i}}.$$

Our proof will follow the structure of the proof of Theorem 4.3 given in [38] and reutilize some of the results given in the paper which we quote here for clarity of exposition.

Proposition 4.12 ([38], see Proposition 2.6). *For any nonnegative integer r, there exist nonnegative rationals λ_k and integers α_{kl} such that*

$$(x_1^2 + \ldots + x_n^2)^r = \sum_k \lambda_k (\alpha_{k1} x_1 + \ldots + \alpha_{kn} x_n)^{2r}.$$

For simplicity of notation, we will let $\alpha_k := (\alpha_{k1}, \ldots, \alpha_{kl})^T$ and $x := (x_1, \ldots, x_n)^T$. Hence, we will write $\sum_k \lambda_k (\alpha_k^T x)^{2r}$ to mean $\sum_k \lambda_k (a_{k1} x_1 + \ldots + a_{kn} x_n)^{2r}$.

Proposition 4.13 ([38], see Proposition 2.8). *If $g \in H_{n,e}$ and $h = \sum_k \lambda_k (\alpha_k^T x)^{d+e} \in H_{n,d+e}$, then*

$$g(D)h = (d+e)_e \sum_k \lambda_k g(\alpha_k)(\alpha_k^T x)^d.$$

Proposition 4.14 ([38], see Theorem 3.7 and 3.9). *For $f \in H_{n,d}$ and $s \geq d$, we define $\Phi_s(f) \in H_{n,d}$ by*

$$f(D)(x_1^2 + \ldots + x_n^2)^s =: \Phi_s(f)(x_1^2 + \ldots + x_n^2)^{s-d}. \tag{14}$$

The inverse $\Phi_s^{-1}(f)$ of $\Phi_s(f)$ exists and this is a map verifying $\Phi_s(\Phi_s^{-1}(f)) = f$.

Proposition 4.15 ([38], see Theorem 3.12). *Suppose f is a positive definite form in n variables and of degree d and let*

$$\epsilon(f) = \frac{\min\{f(u) \mid u \in S^{n-1}\}}{\max\{f(u) \mid u \in S^{n-1}\}}.$$

If $s \geq \frac{nd(d-1)}{4\log(2)\epsilon(f)} - \frac{n-d}{2}$, then $\Phi_s^{-1}(f) \in P_{n,d}$.

We will focus throughout the proof on biforms of the following structure

$$F(x; y) := \sum_{1 \leq i,j \leq n} y_i y_j p_{ij}(x), \tag{15}$$

where $p_{ij}(x) \in H_{n,d}$, for all i, j, and some even integer d. Note that the polynomial $y^T H_f(x) y$ (where f is some form) has this structure. We next present three lemmas which we will then build on to give the proof of Theorem 4.8.

Lemma 4.16. *For a biform $F(x; y)$ of the structure in (15), define the operator $F(D; y)$ as*

$$F(D; y) = \sum_{ij} y_i y_j p_{ij}(D).$$

If $F(x; y)$ is positive semidefinite (i.e., $F(x; y) \geq 0$, $\forall x, y$), then, for any $s \geq 0$, the biform

$$F(D; y)(x_1^2 + \ldots + x_n^2)^s$$

is a sum of squares.

Proof. Using Proposition 4.12, we have

$$(x_1^2 + \ldots + x_n^2)^s = \sum_l \lambda_l (\alpha_{l1} x_1 + \ldots \alpha_{ln} x_n)^{2s},$$

where $\lambda_l \geq 0$ and $\alpha_l \in \mathbb{Z}^n$. Hence, applying Proposition 4.13, we get

$$F(D; y)(x_1^2 + \ldots + x_n^2)^s = \sum_{i,j} y_i y_j (p_{ij}(D)(x_1^2 + \ldots + x_n^2)^s)$$

$$= \sum_{i,j} y_i y_j \left((2s)_d \sum_l \lambda_l p_{ij}(\alpha_l)(\alpha_l^T x)^{2s-d} \right)$$

$$= (2s)_d \sum_l \lambda_l (\alpha_l^T x)^{2s-d} \sum_{i,j} y_i y_j p_{ij}(\alpha_l). \tag{16}$$

Notice that $\sum_{i,j} y_i y_j p_{ij}(\alpha_l)$ is a quadratic form in y which is positive semidefinite by assumption, which implies that it is a sum of squares (as a polynomial in y). Furthermore, as $\lambda_l \geq 0$ $\forall l$ and $(\alpha_l^T x)^{2s-d}$ is an even power of a linear form, we have that $\lambda_l (\alpha_l^T x)^{2s-d}$ is a sum of squares (as a polynomial in x). Combining both results, we get that (16) is a sum of squares. $\square$

We now extend the concept introduced by Reznick in Proposition 4.14 to biforms.

Lemma 4.17. *For a biform $F(x; y)$ of the structure as in (15), we define the biform $\Psi_{s,x}(F(x; y))$ as*

$$\Psi_{s,x}(F(x; y)) := \sum_{i,j} y_i y_j \Phi_s(p_{ij}(x)),$$

where Φ_s is as in (14). Define

$$\Psi_{s,x}^{-1}(F(x; y)) := \sum_{i,j} y_i y_j \Phi_s^{-1}(p_{ij}(x)),$$

where Φ_s^{-1} is the inverse of Φ_s. Then, we have

$$F(D; y)(x_1^2 + \ldots + x_n^2)^s = \Psi_{s,x}(F)(x_1^2 + \ldots + x_n^2)^{s-d} \tag{17}$$

and

$$\Psi_{s,x}(\Psi_{s,x}^{-1}(F)) = F. \tag{18}$$

Proof. We start by showing that (17) holds:

$$F(D; y)(x_1^2 + \ldots + x_n^2)^s = \sum_{i,j} y_i y_j p_{ij}(D)(x_1^2 + \ldots + x_n^2)^s$$

$$\underset{\text{using (14)}}{=} \sum_{i,j} y_i y_j \Phi_s(p_{ij}(x))(x_1^2 + \ldots x_n^2)^{s-d}$$

$$= \Psi_{s,x}(F)(x_1^2 + \ldots + x_n^2)^{s-d}.$$

We now show that (18) holds:

$$\Psi_{s,x}(\Psi_{s,x}^{-1}(F)) = \Psi_{s,x}\left(\sum_{i,j} y_i y_j \Phi_s^{-1}(p_{ij}(x))\right) = \sum_{i,j} y_i y_j \Phi_s \Phi_s^{-1}(p_{ij}) = \sum_{i,j} y_i y_j p_{ij} = F.$$

$\square$

Lemma 4.18. *For a biform $F(x; y)$ of the structure in (15), which is positive on the bisphere, let*

$$\eta(F) := \frac{\min\{F(x; y) \mid (x, y) \in S^{n-1} \times S^{n-1}\}}{\max\{F(x; y) \mid (x, y) \in S^{n-1} \times S^{n-1}\}}.$$

If $s \geq \frac{nd(d-1)}{4\log(2)\eta(F)} - \frac{n-d}{2}$, then $\Psi_{s,x}^{-1}(F)$ is positive semidefinite.

Proof. Fix $y \in S^{n-1}$ and consider $F_y(x) = F(x; y)$, which is a positive definite form in x of degree d. From Proposition 4.15, if

$$s \geq \frac{nd(d-1)}{4\log(2)\epsilon(F_y)} - \frac{n-d}{2},$$

then $\Phi_s^{-1}(F_y)$ is positive semidefinite. As $\eta(F) \leq \epsilon(F_y)$ for any $y \in S^{n-1}$, we have that if

$$s \geq \frac{nd(d-1)}{4\log(2)\eta(F)} - \frac{n-d}{2},$$

then $\Phi_s^{-1}(F_y)$ is positive semidefinite, regardless of the choice of y. Hence, $\Psi_{s,x}^{-1}(F)$ is positive semidefinite (as a function of x and y).

$\square$

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Proof of Theorem 4.8. Let $F(x; y) = y^T H_f(x)y$, let $r \geq \frac{n(d-2)(d-3)}{4\log(2)\eta(f)} - \frac{n+d-2}{2} - d$, and let

$$G(x; y) = \Psi_{r+d,x}^{-1}(F).$$

We know by Lemma 4.18 that $G(x; y)$ is positive semidefinite. Hence, using Lemma 4.16, we get that

$$G(D, y)(x_1^2 + \ldots + x_n^2)^{r+d}$$

is sos. Lemma 4.17 then gives us:

$$G(D; y)(x_1^2 + \ldots + x_n^2)^{r+d} \underset{\text{using (17)}}{=} \Psi_{r+d,x}(G)(x_1^2 + \ldots + x_n^2)^r$$

$$\underset{\text{using (18)}}{=} F(x; y)(x_1^2 + \ldots + x_n^2)^r.$$

As a consequence, $F(x; y)(x_1^2 + \ldots + x_n^2)^r$ is sos.

$\square$

The last theorem of this section shows that one cannot bound the integer r in Theorem 4.8 as a function of n and d only.

Theorem 4.19. *For any integer $r \geq 0$, there exists a form f in 3 variables and of degree 8 such that $H_f(x) \succ 0, \forall x \neq 0$, but f is not r-sos-convex.*

Proof. Consider the trivariate octic:

$$f(x_1, x_2, x_3) = 32x_1^8 + 118x_1^6x_2^2 + 40x_1^6x_3^2 + 25x_1^4x_2^2x_3^2 - 35x_1^4x_3^4 + 3x_1^2x_2^4x_3^2 - 16x_1^2x_2^2x_3^4 + 24x_1^2x_3^6$$
$$+ 16x_2^8 + 44x_2^6x_3^2 + 70x_2^4x_3^4 + 60x_2^2x_3^6 + 30x_3^8.$$

It is shown in [7] that f has positive definite Hessian, and that the $(1,1)$ entry of $H_f(x)$, which we will denote by $H_f^{(1,1)}(x)$, is 1-sos but not sos. We will show that for any $r \in \mathbb{N}$, one can find $s \in \mathbb{N}\backslash\{0\}$ such that

$$g_s(x_1, x_2, x_3) = f(x_1, sx_2, sx_3)$$

satisfies the conditions of the theorem.

We start by showing that for any s, g_s has positive definite Hessian. To see this, note that for any $(x_1, x_2, x_3) \neq 0, (y_1, y_2, y_3) \neq 0$, we have:

$$(y_1, y_2, y_3)H_{g_s}(x_1, x_2, x_3)(y_1, y_2, y_3)^T = (y_1, sy_2, sy_3)H_f(x_1, sx_2, sx_3)(y_1, sy_2, sy_3)^T.$$

As $y^T H_f(x)y > 0$ for any $x \neq 0, y \neq 0$, this is in particular true when $x = (x_1, sx_2, sx_3)$ and when $y = (y_1, sy_2, sy_3)$, which gives us that the Hessian of g_s is positive definite for any $s \in \mathbb{N}\backslash\{0\}$.

We now show that for a given $r \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that $(x_1^2 + x_2^2 + x_3^2)^r y^T H_{g_s}(x)y$ is not sos. We use the following result from [39, Theorem 1]: for any positive semidefinite form p which is not sos, and any $r \in \mathbb{N}$, there exists $s \in \mathbb{N}\backslash\{0\}$ such that $(\sum_{i=1}^n x_i^2)^r \cdot p(x_1, sx_2, \ldots, sx_n)$ is not sos. As $H_f^{(1,1)}(x)$ is 1-sos but not sos, we can apply the previous result. Hence, there exists a positive integer s such that

$$(x_1^2 + x_2^2 + x_3^2)^r \cdot H_f^{(1,1)}(x_1, sx_2, sx_3) = (x_1^2 + x_2^2 + x_3^2)^r \cdot H_{g_s}^{(1,1)}(x_1, x_2, x_3)$$

is not sos. This implies that $(x_1^2 + x_2^2 + x_3^2)^r \cdot y^T H_{g_s}(x)y$ is not sos. Indeed, if $(x_1^2 + x_2^2 + x_3^2)^r \cdot y^T H_{g_s}(x)y$ was sos, then $(x_1^2 + x_2^2 + x_3^2)^r \cdot y^T H_{g_s}(x)y$ would be sos with $y = (1, 0, 0)^T$. But, we have

$$(x_1^2 + x_2^2 + x_3^2)^r \cdot (1, 0, 0)H_{g_s}(x)(1, 0, 0)^T = (x_1^2 + x_2^2 + x_3^2)^r \cdot H_{g_s}^{(1,1)}(x),$$

which is not sos. Hence, $(x_1^2 + x_2^2 + x_3^2)^r \cdot y^T H_{g_s}(x)y$ is not sos, and g is not r-sos-convex.

$\square$

Remark 4.20. *Any form f with $H_f(x) \succ 0, \forall x \neq 0$ is strictly convex but the converse is not true.*

To see this, note that any form f of degree d with a positive definite Hessian is convex (as $H_f(x) \succeq 0, \forall x$) and positive definite (as, from a recursive application of Euler's theorem on homogeneous functions, $f(x) = \frac{1}{d(d-1)} x^T H_f(x) x$). From the proof of Theorem 2.2, this implies that f is strictly convex.

To see that the converse statement is not true, consider the strictly convex form $f(x_1, x_2) := x_1^4 + x_2^4$. We have

$$H_f(x) = 12 \cdot \begin{bmatrix} x_1^2 & 0 \\ 0 & x_2^2 \end{bmatrix}$$

which is not positive definite e.g., when $x = (1,0)^T$.

4.4.2 Optimizing over a subset of polynomial norms with r-sos-convexity

In the following theorem, we give a semidefinite programming-based hierarchy for optimizing over the set of forms f with $H_f(x) \succ 0, \forall x \neq 0$. Comparatively to Theorem 4.4, this theorem allows us to impose as a constraint that the dth root of a form be a norm, rather than simply testing whether it is. This comes at a cost however: in view of Remark 4.20 and Theorem 2.2, we are no longer considering all polynomial norms, but a subset of them whose d^{th} power has a positive definite Hessian.

Theorem 4.21. *Let f be a degree-d form. Then $H_f(x) \succ 0, \forall x \neq 0$ if and only if $\exists c > 0, r \in \mathbb{N}$ such that $f(x) - c(\sum_i x_i^2)^{d/2}$ is r-sos-convex. Furthermore, this condition can be imposed using semidefinite programming.*

Proof. If there exist $c > 0, r \in \mathbb{N}$ such that $g(x) = f(x) - c(\sum_i x_i^2)^{d/2}$ is r-sos-convex, then $y^T H_g(x) y \geq 0, \forall x, y$. As the Hessian of $(\sum_i x_i^2)^{d/2}$ is positive definite for any nonzero x and as $c > 0$, we get $H_f(x) \succ 0, \forall x \neq 0$.

Conversely, if $H_f(x) \succ 0, \forall x \neq 0$, then $y^T H_f(x) y > 0$ on the bisphere (and conversely). Let

$$f_{\min} := \min_{||x|| = ||y|| = 1} y^T H_f(x) y.$$

We know that $f_{\min}$ is attained and is positive. Take $c := \frac{f_{\min}}{2d(d-1)}$ and consider

$$g(x) := f(x) - c\left(\sum_i x_i^2\right)^{d/2}.$$

Then

$$y^T H_g(x) y = y^T H_f(x) y - c \cdot \left(d(d-2)\left(\sum_i x_i^2\right)^{d/2-2}\left(\sum_i x_i y_i\right)^2 + d \sum_i (x_i^2)^{d/2-1}\left(\sum_i y_i^2\right) \right).$$

Note that, by Cauchy-Schwarz, we have $(\sum_i x_i y_i)^2 \leq ||x||^2 ||y||^2$. If $||x|| = ||y|| = 1$, we get

$$y^T H_g(x) y \geq y^T H_f(x) y - c(d(d-1)) > 0.$$

Hence, $H_g(x) \succ 0, \forall x \neq 0$ and there exists r such that g is r-sos-convex from Theorem 4.8.

For fixed r, the condition that there be $c > 0$ such that $f(x) - c(\sum_i x_i^2)^{d/2}$ is r-sos-convex can be imposed using semidefinite programming. This is done by searching for coefficients of a polynomial f and a real number c such that

$$y^T H_{f-c(\sum_i x_i^2)^{d/2}} y \cdot \left(\sum_i x_i^2\right)^r \text{ sos}$$

$$c \geq 0.$$

(19)

Note that both of these conditions can be imposed using semidefinite programming. $\square$

Remark 4.22. *Note that we are not imposing $c > 0$ in the above semidefinite program. As mentioned in Section 4.3, this is because in practice the solution returned by interior point solvers will be in the interior of the feasible set.*

In the special case where f is completely free[2] (i.e., when there are no additional affine conditions on the coefficients of f), one can take $c \geq 1$ in (19) instead of $c \geq 0$. Indeed, if there exists $c > 0$, an integer r, and a polynomial f such that $f - c(\sum_i x_i^2)^{d/2}$ is r-sos-convex, then $\frac{1}{c}f$ will be a solution to (19) with $c \geq 1$ replacing $c \geq 0$.

5 Applications

5.1 Norm approximation and regression

In this section, we study the problem of approximating a (non-polynomial) norm by a polynomial norm. We consider two different types of norms: p-norms with p noneven (and greater than 1) and gauge norms with a polytopic unit ball. For p-norms, we use as an example $||(x_1, x_2)^T|| = (|x_1|^{7.5} + |x_2|^{7.5})^{1/7.5}$. For our polytopic gauge norm, we randomly generate an origin-symmetric polytope and produce a norm whose 1-sublevel corresponds to that polytope. This allows us to determine the value of the norm at any other point by homogeneity (see [15, Exercise 3.34] for more information on gauge norms, i.e., norms defined by convex, full-dimensional, origin-symmetric sets). To obtain our approximations, we proceed in the same way in both cases. We first sample $N = 200$ points $x_1, \ldots, x_N$ uniformly at random on the sphere S^{n-1}. We then solve the following optimization problem with d fixed:

$$\min_{f \in H_{2,d}} \quad \sum_{i=1}^{N}(||x_i||^d - f(x_i))^2 \tag{20}$$

$$\text{s.t.} \quad f \text{ sos-convex.}$$

Problem (20) can be written as a semidefinite program as the objective is a convex quadratic in the coefficients of f and the constraint has a semidefinite representation as discussed in Section 4.2. The solution f returned is guaranteed to be convex. Moreover, any sos-convex form is sos (see [22, Lemma 8]), which implies that f is nonnegative. One can numerically check to see if the optimal polynomial is in fact positive definite (for example, by checking the eigenvalues of the Gram matrix of a sum of squares decomposition of f). If that is the case, then, by Theorem 2.1, $f^{1/d}$ is a norm. Futhermore, note that we have

$$\left(\sum_{i=1}^{N}(||x_i||^d - f(x_i))^2\right)^{1/d} \geq \frac{N^{1/d}}{N}\sum_{i=1}^{N}(||x_i||^d - f(x_i))^{2/d}$$

$$\geq \frac{N^{1/d}}{N}\sum_{i=1}^{N}(||x_i|| - f^{1/d}(x_i))^2,$$

where the first inequality is a consequence of concavity of $z \mapsto z^{1/d}$ and the second is a consequence of the inequality $|x - y|^{1/d} \geq |\,||x|^{1/d} - |y|^{1/d}|$. This implies that if the optimal value of (20) is equal to ϵ, then the sum of the squared differences between $||x_i||$ and $f^{1/d}(x_i)$ over the sample is less than or equal to $N \cdot (\frac{\epsilon}{N})^{1/d}$.

It is worth noting that in our example, we are actually searching over the entire space of polynomial norms of a given degree. Indeed, as f is bivariate, it is convex if and only if it is sos-convex [8]. In Figure 1, we have drawn the 1-level sets of the initial norm (either the p-norm or the polytopic gauge norm) and the optimal polynomial norm obtained via (20) with varying degrees d. Note that when d increases, the approximation improves.

A similar method could be used for *norm regression*. In this case, we would have access to data points $x_1, \ldots, x_N$ corresponding to noisy measurements of an underlying unknown norm function. We would then solve the same optimization problem as the one given in (20) to obtain a polynomial norm that most closely approximates the noisy data.

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[2]This is the case of our two applications in Section 5.

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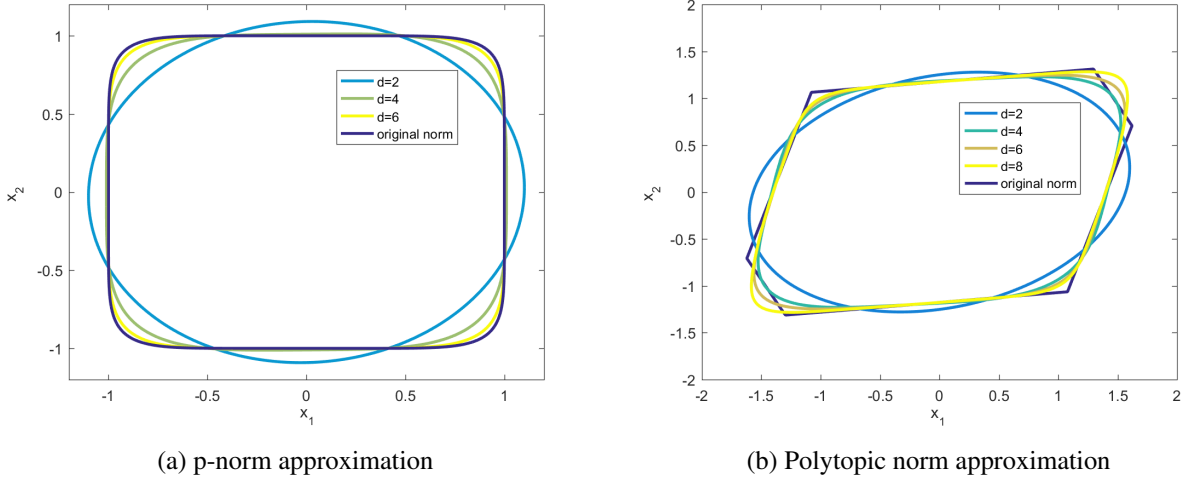(a) p-norm approximation   (b) Polytopic norm approximation

Figure 1: Approximation of non-polynomial norms by polynomial norms

5.2 Joint spectral radius and stability of linear switched systems

As a second application, we revisit a result from one of the authors and Jungers from [1, 3] on finding upperbounds on the joint spectral radius of a finite set of matrices. We first review a few notions relating to dynamical systems and linear algebra. The spectral radius ρ of a matrix A is defined as

$$\rho(A) = \lim_{k \to \infty} ||A^k||^{1/k}.$$

The spectral radius happens to coincide with the eigenvalue of A of largest magnitude. Consider now the discrete-time linear system $x_{k+1} = Ax_k$, where x_k is the $n \times 1$ state vector of the system at time k. This system is said to be *asymptotically stable* if for any initial starting state $x_0 \in \mathbb{R}^n$, $x_k \to 0$, when $k \to \infty$. A well-known result connecting the spectral radius of a matrix to the stability of a linear system states that the system $x_{k+1} = Ax_k$ is asymptotically stable if and only if $\rho(A) < 1$.

In 1960, Rota and Strang introduced a generalization of the spectral radius to a *set* of matrices. The *joint spectral radius (JSR)* of a set of matrices $\mathcal{A} := \{A_1, \ldots, A_m\}$ is defined as

$$\rho(\mathcal{A}) := \lim_{k \to \infty} \max_{\sigma \in \{1, \ldots, m\}^k} ||A_{\sigma_k} \ldots A_{\sigma_1}||^{1/k}. \tag{21}$$

Analogously to the case where we have just one matrix, the value of the joint spectral radius can be used to determine stability of a certain type of system, called *a switched linear system.* A switched linear system models an uncertain and time-varying linear system, i.e., a system described by the dynamics

$$x_{k+1} = A_k x_k,$$

where the matrix A_k varies at each iteration within the set $\mathcal{A}$. As done previously, we say that a switched linear system is asymptotically stable if $x_k \to \infty$ when $k \to \infty$, for any starting state $x_0 \in \mathbb{R}^n$ and any sequence of products of matrices in $\mathcal{A}$. One can establish that the switched linear system $x_{k+1} = A_k x_k$ is asymtotically stable if and only if $\rho(\mathcal{A}) < 1$ [27].

Though they may seem similar on many points, a key difference between the spectral radius and the joint spectral radius lies in difficulty of computation: testing whether the spectral radius of a matrix A is less than equal (or strictly less) than 1 can be done in polynomial time. However, already when $m = 2$, the problem of testing whether $\rho(A_1, A_2) \leq 1$ is undecidable [14]. An active area of research has consequently been to obtain sufficient conditions

for the JSR to be strictly less than one, which, for example, can be checked using semidefinite programming. The theorem that we revisit below is a result of this type. We start first by recalling a theorem linked to stability of a linear system.

Theorem 5.1 (see, e.g., Theorem 8.4 in [25]). *Let $A \in \mathbb{R}^{n \times n}$. Then, $\rho(A) < 1$ if and only if there exists a contracting quadratic norm; i.e., a function $V : \mathbb{R}^n \to \mathbb{R}$ of the form $V(x) = \sqrt{x^T Q x}$ with $Q \succ 0$, such that $V(Ax) < V(x), \forall x \neq 0$.*

The next theorem (from [1, 3]) can be viewed as an extension of Theorem 5.1 to the joint spectral radius of a finite set of matrices. It is known that the existence of a contracting quadratic norm is no longer necessary for stability in this case. This theorem show however that the existence of a contracting polynomial norm is.

Theorem 5.2 (adapted from [1, 3], Theorem 3.2). *Let $\mathcal{A} := \{A_1, \ldots, A_m\}$ be a family of $n \times n$ matrices. Then, $\rho(A_1, \ldots, A_m) < 1$ if and only if there exists a contracting polynomial norm; i.e., a function $V(x) = f^{1/d}(x)$, where f is an n-variate sos-convex and positive definite form of degree d, such that $V(A_i x) < V(x), \forall x \neq 0$ and $\forall i = 1, \ldots, m$.*

We remark that in [2], the authors show that the degree of f cannot be bounded as a function of m and n. This is expected from the undecidability result mentioned before.

Example 5.3. *We consider a modification of Example 5.4. in [4] as an illustration of the previous theorem. We would like to show that the joint spectral radius of the two matrices*

$$A_1 = \frac{1}{3.924} \begin{bmatrix} -1 & -1 \\ 4 & 0 \end{bmatrix}, \quad A_2 = \frac{1}{3.924} \begin{bmatrix} 3 & 3 \\ -2 & 1 \end{bmatrix}$$

is strictly less that one.

To do this, we search for a nonzero form f of degree d such that

$$f - (\sum_{i=1}^{n} x_i^2)^{d/2} \text{ sos-convex}$$

$$f(x) - f(A_i x) - (\sum_{i=1}^{n} x_i^2)^{d/2} \text{ sos, for } i = 1, 2. \tag{22}$$

If problem (22) is feasible for some d, then $\rho(A_1, A_2) < 1$. A quick computation using the software package YALMIP [31] and the SDP solver MOSEK [9] reveals that, when $d = 2$ or $d = 4$, problem (22) is infeasible. When $d = 6$ however, the problem is feasible and we obtain a polynomial norm $V = f^{1/d}$ whose 1-sublevel set is the outer set plotted in Figure 2. We also plot on Figure 2 the images of this 1-sublevel set under A_1 and A_2. Note that both sets are included in the 1-sublevel set of V as expected. From Theorem 5.2, the existence of a polynomial norm implies that $\rho(A_1, A_2) < 1$ and hence, the pair $\{A_1, A_2\}$ is asymptotically stable.

Remark 5.4. *As mentioned previously, problem (22) is infeasible for $d = 4$. Instead of pushing the degree of f up to 6, one could wonder whether the problem would have been feasible if we had asked that f of degree $d = 4$ be r-sos-convex for some fixed $r \geq 1$. As mentioned before, in the particular case where $n = 2$ (which is the case at hand here), the notions of convexity and sos-convexity coincide; see [8]. As a consequence, one can only hope to make problem (22) feasible by increasing the degree of f.*

6 Future directions

In this paper, we provided semidefinite programming-based hierarchies for certifying that the dth root of a given degree-d form is a polynomial norm (Section 4.3), and for optimizing over the set of forms with positive definite Hessians (Section 4.4). A clear gap emerged between forms which are strictly convex and those which have a positive definite Hessian, the latter being a sufficient (but not necessary) condition for the former. This leads us to consider the following two open problems.
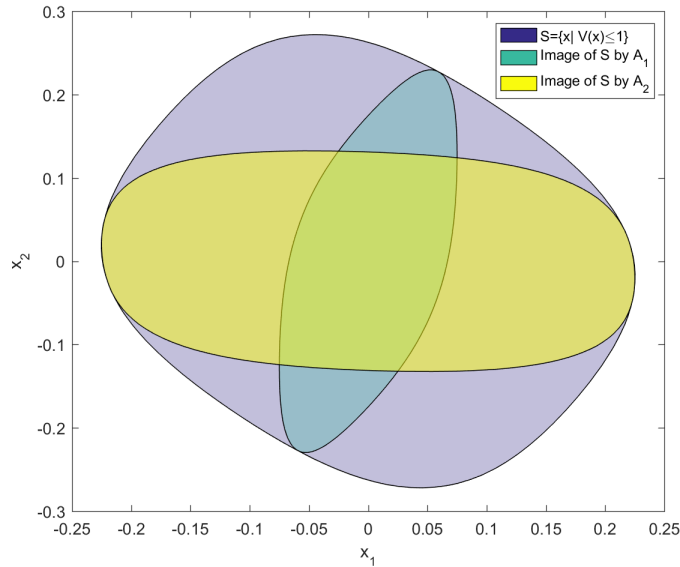
Figure 2: Image of the sublevel set of V under A_1 and A_2

Open Problem 1. *Does there exist a family of cones $K_{n,2d}^r$ that have the following two properties: (i) for each r, optimization of a linear function over $K_{n,2d}^r$ can be carried out with semidefinite programming, and (ii) every strictly convex form f in n variables and degree $2d$ belongs to $K_{n,2d}^r$ for some r? We have shown a weaker result, namely the existence of a family of cones that verify (i) and a modified version of (ii), where strictly convex forms are replaced by forms with a positive definite Hessian.*

Open Problem 2. *Helton and Nie have shown in [22] that one can optimize a linear function over sublevel sets of forms that have positive definite Hessians with semidefinite programming. Is the same statement true for sublevel sets of all polynomial norms?*

On the application side, it might be interesting to investigate how one can use polynomial norms to design *regularizers* in machine learning applications. Indeed, a very popular use of norms in optimization is as regularizers, with the goal of imposing additional structure (e.g., sparsity or low-rankness) on optimal solutions. One could imagine using polynomial norms to design regularizers that are based on the data at hand in place of more generic regularizers such as the 1-norm. Regularizer design is a problem that has already been considered (see, e.g., [11, 16]) but not using polynomial norms. This can be worth exploring as we have shown that polynomial norms can approximate any norm with arbitrary accuracy, while remaining differentiable everywhere (except at the origin), which can be beneficial for optimization purposes.

Acknowledgement

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