Essays on risk exchanges within a collective

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You cannot win alone.

Even the strongest can be tackled and even the strongest will eventually fall if alone.

It does not matter how many times you have fallen down if you can get up fast.

Playing smart does not mean playing softly.

Thinking saves energy but it is not the only tool nor the goal.

Always keep your chin up.

Nothing is decided till the final whistle.

Give it try! Without trying, you cannot win.

Those who act like sheep will be eaten by the wolf. (Chi pecora si fa, il lupo se la mangia.)

Not all fouls are seen by the referee, therefore, be ready to go on no matter what.

Always be ready to receive the ball.

No one will pass you the ball if you do not ask for it. Speak up!

If you really want the ball, go and get it.

Although "everything said in Latin sounds profound" (Quidquid latine dictum sit, altum videtur.) and despite the fact that "nothing can be said that has not been said before" (Nil dictum quod non dictum prius.), I would like to express my gratitude to the following people who used sentences that were resonating in my head.

Jakub Pečánka: *Finished is better than perfect.*
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Mr. V. Horák: *Certainty is a machine gun. (Jistota je kulomet.)*
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Prof. J. Dupačová: *Everything is negotiable.*
Granny/Babička: *Do not be upset longer than it takes the sun to set.*
Grandpa/Děda: *Sharpen your axe before going into the forest.*
Father/Táta: *Life should be enjoyed, not spent. (Život je třeba užít, ne prožít.) — An average employee is the boss's fault.*
Mother/Máma: *Good means mediocre. (Dobře je za tři.) — The tastiest slivovica is the one you are drinking right now with your friends.*
Sara Amoroso: *The worst-case scenario: I have two hands, so I can survive.*

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To these and others, with gratitude and love,

Jaroslav Pazdera

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Chapter 1

Introduction

This thesis concerns a group of investors who have joined their endowments and formed a collective. To a certain extent, this collective is solving the same investment decisions as the individuals from which it is composed of. In addition, the collective can construct an internal market where its members exchange assets under conditions that are different to those that prevail outside of the collective. This thesis focuses on exchanges conducted on these internal markets of collectives.

Throughout this thesis, we assume that all members of the collective are rational individuals. The rationality which we have in mind is characterised by von Neumann-Morgenstern axioms and hence individuals' decision-making preferences can be described by utility functions. The individuals within a collective are simply called agents. Each agent is assumed to be endowed with initial assets, which can be exchanged or traded on the internal market. The initial endowments of each agent can also be viewed as a contribution to the collective. Furthermore, we assume that both the sizes of contributions and the preferences of the agents are common knowledge within the collective.

Examples of agents grouped in a collective with an internal market include investors participating in a closed investment fund, insurance companies participating in a reinsurance market, or a pension fund where agents are different participating generations. The agents' motivation for participating in collectives can vary greatly. Individuals participating in investment funds may be seeking investment opportunities that would be too cost prohibitive for an individual investor. Similarly, the participants in a reinsurance market could be looking to diversify their risks outside of a region. Individuals grouped into pension funds may even have been coerced into a particular pension scheme through a combination of corporate or state legislation. Irrespective of the motivation, collectives can be found in many areas of the economy. In this thesis, we focus not on the motivation behind the formation of collectives, nor the process of forming a collective itself. Rather, we consider collectives that already exist, and which contain a fixed number of agents.

In the internal market, agents are, under certain conditions, permitted to trade or conclude contracts with each other. These trades can be perceived as a redistribution of the agents' endowments; therefore the trades within the collective are often referred to as the redistribution. From agents' point of view, a natural condition on any redistribution is that the value of the agents' assets before and after any redistribution, remains unchanged. This condition is called financial fairness and it will be imposed on all internal market trades in the following two chapters. In the final chapter, this condition is not imposed, but it is rather a consequence of its setting where prices are endogenously determined.

The investment decisions of a collective differ from those of an individual not only because it includes the potential for an internal market, but also because the collective, in general, does not have its own preferences - it has to reflect those of its agents. Whereas an individual's investment decision can be viewed as a single objective optimization problem, the collective is in general solving a multi-objective optimization problem. The solutions of a multi-objective optimization are considered those allocations which are both feasible and not dominated by another. Such solutions are called Pareto efficient. Pareto efficiency condition on trades, per se, does not consider individual rationality, which is the sole determinant of the individual investment decision. Imposing Pareto efficiency on internal market trades, as applied later in this thesis, can be perceived as focusing on a collective rather than on the individual preferences of agents.

The following two chapters focus on redistributions of pooled contributions of agents that are both Pareto efficient and financially fair with respect to exogenously given pricing measure. Under mild conditions we show that such redistribution within a collective (intra-group) exists and it is unique; Chapter 2. A more general situation arises when the collective of agents is allowed to take a decision and trade some of the pooled agents' contributions on an extra-group market. Also in this situation, we can get a uniqueness of Pareto efficient and financially fair redistribution; Chapter 3. Finally, we consider a collective in a setting where there are bid-ask spreads on an extra-group market and the collective is free to use its intra-group market. In this setting we define an equilibrium similar to the competitive equilibria and we investigate under which bid-ask spreads the intra-group market is superfluous or, to the contrary, when it can accommodate all equilibrium trades; Chapter 4.

Chapter 2

The Composite Iteration Algorithm for Finding Efficient and Financially Fair Risk-Sharing Rules

Joint work with J.M. Schumacher and B.J.M. Werker

2.1 Introduction

This chapter is concerned with the design of risk sharing systems. For an example of the type of situation we have in mind, consider a collective pension fund of the type existing for instance in the Netherlands. The claim to future benefits that participants receive in return for their contributions is a contingent claim, since benefits depend on the funding status at the time of payment, and the funding status in turn depends on realized investment returns as well as on prevailing interest rates. In the design of a system of this nature, it would seem reasonable to include considerations relating to *preferences* (different degrees of risk aversion among participants) as well as considerations relating to *financial fairness* (balance between the value of agents' contributions on the one hand, and the value of the contingent claims they receive in return on the other hand). The aspect of value brings prices into play. Since the agents in the risk sharing systems we have in mind constitute only a small part of the entire economy, prices will be taken as exogenously given.

The results described in this chapter were published in Pazdera et al. (2017)

The model that we use as a basis for risk sharing design is a two-period model (time points 0 and 1) with a finite number of von Neumann-Morgenstern agents. We allow for a continuum of possible time-1 states of nature. We assume the availability of a valuation operator that is of sufficiently wide scope to determine the value of any contingent claim that might be defined as a result of risk sharing. The inputs to the design problem are (i) agents' preferences, specified by utility functions and objective probabilities, (ii) their claim values (in monetary units),[1] and (iii) the aggregate endowment (i.e., shared risk—for instance, the uncertain outcome of joint investment). The objective of the design is to find a Pareto efficient allocation of the aggregate endowment such that all agents' allotments are within their budget sets as determined by their claim values and by the given valuation operator. We refer to such allocations as being *Pareto efficient and financially fair* (PEFF).

This form of the risk sharing problem has been formulated in the literature already several decades ago (Gale, 1977; Gale and Sobel, 1979; Bühlmann and Jewell, 1978, 1979; Balasko, 1979). Results on existence and uniqueness of PEFF solutions are available in the cited papers under various assumptions. The purpose of the present chapter is to propose an effective and easily implemented computational algorithm, which may stimulate a more widespread use of the PEFF solution concept. We suggest an iterative method that is built up from simple steps. We provide a proof of convergence of the iteration, and we demonstrate that the asymptotic rate of convergence is linear. The analysis is cast in the framework of nonlinear Perron-Frobenius theory.

The model used in this chapter can be looked at from the point of view of optimal risk sharing, but it also relates to the theory of fixed-price equilibria, and to the theory of fair division. A discussion of these relationships can be given as follows.

Research on optimal risk sharing has a long history. The origins of the theory of reciprocal reinsurance treaties are traced back by Seal (1969) to de Finetti (1942). Borch (1962) obtained a parametrization of the collection of all Pareto optimal solutions to a risk sharing problem, when the preferences of agents can be described by expected utility. The value-based notion of fairness that is used in this chapter was proposed by Gale (1977) in the context of distribution of a random harvest in proportion to ownership rights. The applicability of Gale's ideas to risk sharing was noted by Bühlmann and Jewell (1978,

[1]The term "claim value" refers to the time-0 value of an agent's share. For instance, if a project with time-0 value 100 is jointly owned by two agents A and B who hold 60% and 40% of the ownership rights, respectively, then the claim value of agent A is 60 and the claim value of agent B is 40. These claim values can be achieved in many ways; for instance, agent A's claim might be 50% of the outcome of the project up to a certain threshold plus 100% of the amount by which the project outcome exceeds the threshold.

1979), who generalized the problem formulation by allowing the weights of future states that are used in the fairness condition to be different from probabilities as perceived by the agents. For the allocation problem as formulated by Gale (1977), the uniqueness of Pareto optimal and fair allocations was shown by Gale and Sobel (1979) under the assumption of a finite number of possible future states, and by Gale and Sobel (1982) in the continuous case, under somewhat restrictive conditions on utility functions. The proof of uniqueness in these papers is based on the construction of a "social welfare function", which is such that it reaches its optimum on the set of financially fair allocations at a Pareto efficient point. Bühlmann and Jewell (1979) note that essentially the same technique can be applied as well to their formulation of the problem. Sobel (1981) gives a proof of uniqueness that avoids the introduction of the social welfare function, in order to accommodate a generalization in which agents use private valuation functionals.

In recent years, formulations of the risk sharing problem in which the preferences of agents are specified by risk measures (monetary valuation functionals) have attracted considerable interest; see for instance Chateauneuf et al. (2000); Barrieu and El Karoui (2005); Acciaio (2007); Jouini et al. (2008); Filipovic and Svindland (2008); Kiesel and Rüschendorf (2008). When all agents use a translation invariant risk measure (as in Artzner et al., 1999) for evaluation, Pareto optimal solutions can only be unique up to addition of deterministic side payments which sum to zero. In such a case, the existence of Pareto optimal solutions automatically implies the existence of solutions that are both Pareto optimal and financially fair, and the question of uniqueness comes down to uniqueness of Pareto optimal solutions up to "rebalancing the cash". Uniqueness results of this type are given by Filipovic and Svindland (2008) and Kiesel and Rüschendorf (2008) under a condition of strict convexity.

A model similar to those proposed by Gale (1977) and by Bühlmann and Jewell (1979), but using more general preference specifications, was developed contemporaneously and independently by Balasko (1979). Balasko was motivated by developments in general equilibrium theory, in particular fixed-price equilibria as studied by Drèze (1975) and Benassy (1975). He used methods of differential topology to show existence of Pareto efficient and financially fair allocations. Keiding (1981) gives an existence result under a very general preference specification and mentions that, at this level of generality, uniqueness cannot be guaranteed.

In more recent work, Herings and Polemarchakis (2002, 2005) study fixed-price equilibria from the point of view of Pareto-improving interventions. The interventions discussed by these authors are based on price regulation. It may be viewed as an advantage of such interventions that they operate anonymously, by means of market variables. In this chapter, intervention takes place directly through allocation, but is subject to the constraint of respecting agents' claim values.

Fair division problems are studied extensively in social choice theory; see for instance Brams and Taylor (1996) and Brandt et al. (2016). A typical setting, as used for instance by Brams and Taylor (1996), equips agents with a linear valuation operator, which generally is different for different agents. This operator serves to define the ranking of alternatives by agents, and at the same time it also supports a notion of fairness. Fairness can be expressed as proportionality. All agents receive a share that, according to their own valuation, is at least equal to $1/n$-th of the total, where n is the number of agents. More generally, the fractions $1/n$ may be replaced by "entitlements" that are not necessarily equal to each other. These fairness constraints are expressed through inequalities, and consequently they usually do not determine a unique solution. As a stronger notion, envy-freeness is used extensively (no agent should prefer another agent's allotment to his or her own).

One way in which the setting of the present chapter is different from the framework commonly used in fair division theory is that a distinction is made between *utility* value on the one hand, and *financial* value on the other hand. Moreover, financial value is taken to be agent-independent. The notion of "claim value" used in this chapter is similar to the notion of "entitlement" (applied to financial value). However, while entitlements are used to formulate *inequality* constraints, claim values are used to specify *equality* constraints. Indeed, due to the agent-independent nature of financial value, the sum of the claim values of the agents is fixed, which makes it impossible to raise one agent's claim value without reducing the claim values of others. The separation between utility value and financial value makes it possible to define the notion of efficiency in terms of utility value, so that a distinction between efficient and inefficient solutions can still be made, even though claim values are fixed. This is analogous to the classical single-agent problem of portfolio optimization, in which the role of the claim value is played by the budget constraint, and the agent aims to maximize utility subject to the given budget.

Both in general equilibrium theory and in the literature on fair division, much attention has been paid to computational methods. In comparison, algorithms for finding PEFF solutions have not been explored as extensively. One possible approach is to employ the transformation to single-objective optimization problems from Gale (1977) that was used in subsequent papers to prove existence and uniqueness of solutions. Since the optimization problems resulting from this transformation are convex, any method for solving convex optimization problems subject to equality constraints (see for instance Boyd and Vandenberghe, 2004, Ch. 10) can qualify as a method for obtaining PEFF solutions. However, it would be of interest to make more use of the particular structure of PEFF problems.

An early discussion of specialized methods is given by Bühlmann and Jewell (1979). They discuss in particular the case of two agents, for which a line search suffices, and the case of exponential utility.

The iterative algorithm for finding PEFF solutions that is proposed in this chapter can be related to a matrix algorithm known as the "iterative proportional fitting procedure" (IPFP). This algorithm finds, among all matrices with given positive row and column sums, the one that is closest, in the sense of Kullback-Leibler divergence, to a given nonnegative matrix. The procedure was proposed as a matrix fitting method by Kruithof (1937) and independently by Deming and Stephan (1940), but the optimization problem solved by it was only identified decades later by Ireland and Kullback (1968). Convergence of IPFP was proved by Csiszár (1975) in the discrete case and by Rüschendorf (1995) in the continuous case.

The iterative proportional fitting procedure can be viewed as an implementation of the method of successive projections due to Bregman (1967). This method is generally applicable to convex optimization problems with equality constraints. In particular, it may be applied to the optimization problems that result from Gale's transformation of PEFF problems. Making use of the Borch parametrization (see Section 2.3.2), one then arrives at the same procedure as the one that is studied in this chapter, and that is motivated below directly from the PEFF problem.

In the framework of successive projections, one would be led to convergence analysis in the style of Csiszár (1975), based on generalized versions of the Pythagorean theorem. While convergence of Bregman projections has been discussed extensively in the literature (see for instance Censor and Lent, 1981; Bauschke and Borwein, 1996), we are unaware of a result along these lines that would apply directly to the situation considered in the present chapter. Below we use an alternative perspective, which relates PEFF solutions to positive eigenvectors of a nonlinear mapping. The analogous approach in the context of IPFP has been pioneered by Menon (1967) and Brualdi et al. (1966).

If the approximation problem solved by IPFP would be translated to the PEFF context by applying Gale's transformation backwards, it would lead to a problem in which the number of future states is finite, the probabilities of future states are agent-dependent (i.e., subjective probabilities), and agents' utility functions are logarithmic. In this chapter it is assumed that all agents assign the same probabilities to future states (since this is a standard assumption in a large part of the literature, and it simplifies notation), but extension to the case of subjective probabilities would be straightforward; cf. for instance Wilson (1968). IPFP would then become a special case of the algorithm considered here.

In the present chapter, we work within the classical framework of expected utility. We demonstrate that the problem of finding a Pareto efficient and financially fair allocation can be written as the problem of finding a positive eigenvector of a homogeneous nonlinear mapping from the nonnegative cone into itself. This leads naturally to the use of nonlinear Perron-Frobenius theory. For a general introduction to this subject, see for instance the book by Lemmens and Nussbaum (2012). We show that the homogeneous mapping associated to the fair allocation problem enjoys a number of properties that are useful within the Perron-Frobenius theory, such as continuity, monotonicity, and a more exotic property called nonsectionality.

The formulation as an eigenvector problem suggests an iterative solution method, analogous to the "power method" in the linear case (Wilkinson, 1965). We prove that convergence takes place from any given initial point within the positive cone. The iteration is built up from mappings that are easy to compute, so that it offers an attractive alternative to other methods which call for solution of large nonlinear equation systems.

This chapter takes a "social planner" point of view. We do not model a negotiation process between the agents, as for instance in Boonen (2016). State prices, which are used to determine financial fairness, are assumed to be given. The risk to be shared is taken to be given as well, as in the paper by Borch (1962). The reader may refer to Chapter 3 for the construction of a suitable homogeneous mapping in the context of risk sharing situations as in Wilson (1968), where the risk itself is subject to a decision by the collective. The uniqueness of the PEFF solution is preconditioned upon the uniqueness of state prices. In more extensive model where state prices would be obtained for instance by a bargaining process among the members of the collective, and the bargaining could have several different outcomes, the corresponding PEFF solutions would be in general different. We also do not take individual rationality of the agents into consideration. Because the choice of Pareto efficient solution is based solely upon the financial fairness constraint, there is no guarantee in general that the PEFF solution is individually rational.

This chapter is organized as follows. Notation and assumptions are covered in the section following this introduction. Section 2.3 discusses the problem formulation, and Section 2.4 presents a brief review of requisite mathematical material. The main results of this chapter are in Section 2.5. A special case is discussed in Section 2.6, followed by a discussion of the rate of convergence in Section 2.7. Finally, Section 2.8 concludes.

2.2 Notation and assumptions

The model that we use in this chapter can be understood as a two-period exchange economy under uncertainty. There is a single good and a continuum of future states of nature. State prices are supposed to be given and are described by means of a pricing measure Q. There is a finite number n of agents. The agents' preferences across distributions of future consumption are of the von Neumann-Morgenstern type: agent i ranks distributions on the basis of expected utility of future consumption, where the expectation is taken under an objective probability measure P, and utility is measured by a utility function u_i. Subject to the given pricing measure Q, the budget set of agent i is determined by a number v_i which quantifies the ownership rights of agent i, and which can be interpreted as the value (under the given pricing measure Q) of the agent's initial endowment. The aggregate endowment is denoted by X. The relationship

$$\sum_{i=1}^{n} v_i = E^Q[X] \tag{2.1}$$

holds, where the symbol E^Q denotes expectation under Q. This relation states that the sum of the claim values of the agents is equal to the time-0 value of the aggregate endowment.

The aggregate endowment X is also referred to as the total risk that is to be shared among the agents. Our sign convention is that positive values of X indicate gains and negative values indicate losses, so that the term "risk" is to be understood as "uncertain outcome" without necessarily a negative connotation.

In mathematical terms, risks are modeled as bounded random variables on a measurable space $(\Omega, \mathcal{F})$. The agents' utility functions are taken to be defined on intervals of the form (b_i, ∞) where $b_i \in [-\infty, \infty)$ and $b_i < v_i$, for $i = 1, \dots, n$, and will always be assumed to satisfy the following conditions.

Assumption 2.1 For each $i = 1, \dots, n$, the function $u_i : (b_i, \infty) \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, and strictly concave. Moreover, the following Inada conditions are satisfied:

$$\lim_{x \downarrow b_i} u_i'(x) = \infty, \qquad \lim_{x \to \infty} u_i'(x) = 0. \tag{2.2}$$

As a result of this assumption, all marginal utilities u_i' are continuous and strictly decreasing functions whose range covers the positive real axis. The inverse marginal utility of agent i will be denoted by I_i. In other words, I_i is the function from $(0,\infty)$ to (b_i,∞) that is defined implicitly by

$$u_i'(I_i(z)) = z \qquad (z > 0). \tag{2.3}$$

The inverse marginal utility is a continuous and strictly decreasing function that has the interval (b_i,∞) as its image. We write

$$b := \sum_{i=1}^{n} b_i, \qquad D := (b,\infty) \tag{2.4}$$

with the convention that $b = -\infty$ if there is an index i such that $b_i = -\infty$. The space of continuous functions from D to $\mathbb{R}$ will be denoted by $C(D,\mathbb{R})$.

If all lower bounds b_i are finite and if the total risk X is such that $P(X \le \sum_{i=1}^{n} b_i) > 0$, then it is not possible to allocate the risk in such a way that the expected utility of each agent is finite. To make the problem feasible, we need to impose that $P(X \in D) = 1$. We shall in fact work under the stronger assumption that the risk X is bounded away from the critical level.

Assumption 2.2 The total risk X takes values in a compact set $A \subset (b,\infty)$.

For vectors α, $\beta \in \mathbb{R}^n$, the notation $\alpha < \beta$ ($\alpha \le \beta$) indicates that $\alpha_i < \beta_i$ ($\alpha_i \le \beta_i$) for all i, whereas $\alpha \lneq \beta$ means $\alpha \le \beta$ and $\alpha \ne \beta$. Similar notation will be used for real-valued functions: in particular, for functions f, $g \in C(D,\mathbb{R})$ we write $f < g$ when $f(x) < g(x)$ for all $x \in D$. A mapping f from one ordered space into another will be said to be *monotone* if $x \le y$ implies $f(x) \le f(y)$, *strictly monotone* if it is monotone and $x < y$ implies $f(x) < f(y)$, and *strongly monotone* if $x \lneq y$ implies $f(x) < f(y)$.

The nonnegative cone $\{\alpha \in \mathbb{R}^n \mid \alpha \ge 0\}$ is denoted by $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$ indicates the positive cone $\{\alpha \in \mathbb{R}^n \mid \alpha > 0\}$. When α is a given vector in $\mathbb{R}^n$ and $S = \{i_1,\dots,i_k\}$ is a nonempty subset of the index set $\{1,\dots,n\}$, we write $\alpha_S := (\alpha_{i_1},\dots,\alpha_{i_k})$. If $(\alpha^k)_{k=1,2,\dots}$ is a sequence of vectors in $\mathbb{R}^n$, the notation $\alpha^k \to \infty$ means that $\alpha_i^k \to \infty$ for all $i = 1,\dots,n$.

The pricing measure that is used in the financial fairness condition is obtained from a probability measure Q defined on $(\Omega,\mathcal{F})$. In the two-period model that we consider, discounting can be dispensed with. The time-0 value of a random payoff X at time 1 will therefore simply be represented by the expectation of X with respect to the measure Q.

We do not require the valuation measure Q to be absolutely continuous with respect to the probability measure P used by agents to compute expected utility, nor do we require that the measure P should be absolutely continuous with respect to Q. The development of this chapter still applies even when the measure Q is concentrated on a single outcome of the total risk X. Such a situation may be realistic; it occurs when a group, in neglect of the stochasticity of X, has only made a decision in advance about how to divide a particular outcome. The principles of Pareto efficiency and financial fairness are then sufficient, given the agents' utility functions, to arrive at a well-defined allocation even if a different outcome is realized.

2.3 The allocation problem

2.3.1 Definitions

A *risk-sharing rule* is a vector $(y_1, \dots, y_n)$ of functions in $C(D, \mathbb{R})$ satisfying the feasibility condition

$$\sum_{i=1}^{n} y_i(x) = x \qquad (x \in D). \tag{2.5}$$

By requiring the equality to hold for all x in the domain D, which is determined by the preferences of the agents, we avoid dependence on the range of values taken by a specific risk X. It will be seen below that this extension of the problem setting affects neither existence nor uniqueness of solutions.

The risk of agent i after allocation is $Y_i := y_i(X)$, and the corresponding utility for agent i is $E^P[u_i(Y_i)]$. Risk sharing can be thought of as a particular form of allocation, so that we also sometimes use the term "allocation rule" or simply "allocation" instead of "risk-sharing rule". The functions $y_i(\,\cdot\,)$ are called *allocation functions*.

We will be looking for risk-sharing rules that are *Pareto efficient* as well as *financially fair*. The definition of Pareto efficiency is standard.

Definition 2.3 A risk-sharing rule $(y_1, \dots, y_n)$ is *Pareto efficient* (or *Pareto optimal*) if there does not exist a risk-sharing rule $(\tilde{y}_1, \dots, \tilde{y}_n)$, with associated allocated risks $(\tilde{Y}_1, \dots, \tilde{Y}_n)$, such that $(E^P[u_1(\tilde{Y}_1]), \dots, E^P[u_n(\tilde{Y}_n)]) \gneqq (E^P[u_1(Y_1)], \dots, E^P[u_n(Y_n)])$.

To state the definition of financial fairness, we use numbers v_i, for $i = 1, \dots, n$, to indicate the ownership rights of agents. These numbers may also be called claim values. The claim value specifies only the time-0 value of the allotment to be received by agent i, not the allotment itself.

Definition 2.4 A risk-sharing rule $(y_1, \dots, y_n)$ for the given total risk X is *financially fair* if, for each agent, the value of the allocated share is equal to that agent's claim value, i.e.,

$$E^Q[y_i(X)] = v_i \qquad (i = 1, \ldots, n). \tag{2.6}$$

Because the allocation functions are continuous, and because of Assumption (2.1), the random variables $Y_i = y_i(X)$ are bounded, so that their expectations under Q are indeed well defined. Feasibility of the requirement (2.6) taking into account the market clearing property (2.5) is guaranteed by the relation (2.1).

In this chapter we are interested in allocation functions that combine financial fairness with Pareto efficiency. It should be noted that the notion of Pareto efficiency that we use here is subject to feasibility (2.5), but not to financial fairness. In other words, we want to find feasible allocations that are financially fair, and that are Pareto efficient even among feasible allocations that violate financial fairness.

2.3.2 Borch's parametrization

To convert the allocation problem into a system of equations, we use the parametrization of Pareto efficient risk-sharing rules that was devised by Borch (1962).

Theorem 2.5 (Borch, 1962) *A risk-sharing rule $(y_1, \ldots, y_n)$ is Pareto efficient if and only if there exist a continuous function $J : D \to \mathbb{R}_{++}$ and positive constants $\alpha_1, \ldots, \alpha_n$ such that*

$$\alpha_i u_i'(y_i(x)) = J(x) \tag{2.7}$$

for all $x \in D$ and for all $i = 1, \ldots, n$.

Details of the proof can be found in DuMouchel (1968); Gerber and Pafumi (1998); Barrieu and Scandolo (2008). The quantity $J(x)$ can be interpreted as a Lagrange multiplier associated to the feasibility constraint (2.5). We can now state the central problem considered in this chapter as follows.

Problem 2.6 Assume given: a finite group of n agents, with utility functions u_i satisfying Assumption 2.1; a risk X satisfying Assumption 2.2; a pricing measure Q; and agents' claim values v_i satisfying (2.1). Find a vector of functions $(y_1, \ldots, y_n)$ in $C(D, \mathbb{R})$ such that the following conditions are satisfied:

- feasibility, i.e. $\sum_{i=1}^n y_i(x) = x$ for all $x \in D$;

- Pareto efficiency, i.e. there exist positive constants $\alpha_1, \dots, \alpha_n$ and a continuous function $J : D \to \mathbb{R}_{++}$ such that (2.7) holds for all $x \in D$ and for all i;
- financial fairness, i.e. $E^Q[y_i(X)] = v_i$ for all i.

The Borch condition (2.7) can be rewritten as follows in terms of the inverse marginal utilities (cf. (2.3)):

$$y_i(x) = I_i\big(J(x)/\alpha_i\big). \tag{2.8}$$

Since the functions y_i must satisfy the feasibility condition, the following condition has to be satisfied for all $x \in D$:

$$\sum_{i=1}^{n} I_i\big(J(x)/\alpha_i\big) = x. \tag{2.9}$$

For a given positive vector $(\alpha_1, \dots, \alpha_n)$ and given $x \in D$, the above equation determines $J(x)$ uniquely, since the function $z \mapsto \sum_{i=1}^n I_i(z/\alpha_i)$ is strictly decreasing. We may therefore consider the function J to be defined by the relation (2.9); to emphasize this point of view, we will sometimes write $J(x;\alpha)$ instead of $J(x)$. Conversely, if (2.9) is satisfied for a set of positive numbers $\alpha_1, \dots, \alpha_n$, then the functions $y_1, \dots, y_n$ in (2.8) determine a Pareto efficient risk-sharing rule.

In this way, Borch's theorem provides a parametrization of Pareto efficient risk-sharing rules in terms of the utility weights $\alpha_1, \dots, \alpha_n$. The effective number of parameters is in fact $n-1$ rather than n, since the allocation rule that is generated by a positive vector $(\alpha_1, \dots, \alpha_n)$ does not change if all numbers α_i are multiplied by the same positive constant. Indeed, in this case the corresponding function J is multiplied by the same constant, so that the ratios $J(x)/\alpha_i$ remain the same.

Remark 2.7 Given a vector $\alpha \in \mathbb{R}^n_{++}$, let $(y_1, \dots, y_n) = (y_1(\,\cdot\,;\alpha), \dots, y_n(\,\cdot\,;\alpha))$ denote the Pareto efficient risk-sharing rule defined through (2.8) and (2.9). The "weighted group utility" $u(\,\cdot\,;\alpha)$ corresponding to given weights $\alpha = (\alpha_1, \dots, \alpha_n)$ is defined by

$$u(x;\alpha) = \sum_{i=1}^{n} \alpha_i u_i(y_i(x)) \qquad (x \in D). \tag{2.10}$$

Under the assumption that the utility functions u_i are twice continuously differentiable and the inverse marginal utilities are continuously differentiable, it follows that the function J, being the inverse of the mapping $z \mapsto \sum_{i=1}^n I_i(z/\alpha_i)$, is differentiable as well. Consequently, the allocation functions y_i defined by (2.8) are likewise differentiable. We can then write (cf., for instance, Xia, 2004)

$$u'(x;\alpha) = \sum_{i=1}^{n} \alpha_i u_i'(y_i(x)) y_i'(x) = J(x;\alpha) \sum_{i=1}^{n} y_i'(x) = J(x;\alpha) \qquad (2.11)$$

where the second equality follows from Borch's condition (2.7) and the third uses the feasibility condition (2.5). The function $J(\,\cdot\,;\alpha)$ can thus be interpreted as the marginal group utility that corresponds to a given set of weights α.

2.3.3 Computational approaches

Consider now the problem of numerically solving the equation system consisting of the feasibility condition (2.5), the financial fairness condition (2.6), and the efficiency condition (2.7). The unknowns in these equations consist of the utility weights α_i, the marginal utility (or multipliers) $J(x)$, and the allocation functions y_i. The equation system suggests at least three broad computational approaches.

First of all, using the fact that the marginal utility J can be thought of as being defined by the utility weights, as discussed above, the system (2.5–2.6–2.7) can be rewritten as a system of n nonlinear equations in n unknowns $\alpha_1, \ldots, \alpha_n$:

$$E^Q\big[I_i(J(X;\alpha_1,\ldots,\alpha_n))/\alpha_i\big] = v_i \qquad (i = 1,\ldots,n). \qquad (2.12)$$

Subsequently, a nonlinear equation solver may be applied. This approach, which uses utility weights as the reduced set of unknowns, is analogous to Negishi's method in the theory of Arrow-Debreu equilibrium (Negishi, 1960).

Alternatively, one can express the utility weights in terms of the marginal utility by making use of the financial fairness conditions. Indeed, as will be discussed in more detail below, the equations (2.6) and (2.8) determine the weights α_i when the function J is given. Writing $\alpha = \alpha(J)$ to indicate this dependence, we can find Pareto efficient and financially fair allocations by solving the equation

$$\sum_{i=1}^{n} I_i\big(J(x)/\alpha_i(J)\big) = x \qquad (x \in D) \qquad (2.13)$$

for the unknown function J. This approach is analogous to the standard method of finding Arrow-Debreu equilibria by making use of the excess demand function; see for instance Kehoe (1991). Here, the excess demand is given by the difference of the left-hand side and the right-hand side in (2.13).

The weights that solve the equation system (2.12) can alternatively be characterized as fixed points of the composite mapping that is formed by applying the mapping $\alpha \mapsto J$ of the first method, followed by the mapping $J \mapsto \alpha$ of the second method. This leads to a third computational approach. The analogous technique in the case of Arrow-Debreu equilibrium appears in a paper by Dana (2001, p. 170) under the name “price-weight, weight-price approach”. It is natural to attempt to find fixed points by iteration of the composite mapping. While the first two methods that we discussed produce nonlinear equation systems that can be challenging to solve, the composite iteration method relies on repeated use of mappings that are easy to compute. As already mentioned in the Introduction, the iterative method can be constructed as an application of Bregman’s successive projections method via reformulation of the PEFF problem as an optimization problem; however, the motivation as given above seems more direct.

In the present chapter we focus on the “price-weight, weight-price” approach; we call it here the composite iteration algorithm. We aim to establish relevant properties of the composite iteration mapping, which allow to prove convergence of the algorithm. In particular, we prove (Theorem 2.21 below) that the composite iteration mapping can be uniquely extended to a continuous, homogeneous,[2] and monotone mapping from the nonnegative cone to itself. Moreover, it will be shown that the fixed-point problem for the composite mapping can be reformulated as the problem of finding a positive eigenvector of the mapping. These facts lead towards the use of nonlinear Perron-Frobenius theory. Application of a theorem of Oshime (1983) (see Theorem 2.9 below) allows us to conclude existence and uniqueness of solutions as well as convergence of the iterative algorithm. Before proceeding to the main results, we first review mathematical preliminaries.

2.4 Preliminaries

Order-preserving (monotone) nonlinear maps can be viewed as generalizations of positive matrices. It turns out that much of the Perron-Frobenius theory concerning eigenvalues and eigenvectors of such matrices can be extended to the nonlinear case. An extensive discussion of nonlinear Perron-Frobenius theory is provided by Lemmens and Nussbaum (2012).

A particular class of interest is the class of monotone mappings that are *homogeneous* in the sense that $\varphi(\lambda x) = \lambda\varphi(x)$ for all positive λ. For continuous homogeneous mappings from the nonnegative cone into itself, the existence of nonnegative eigenvectors follows

[2] In this chapter we always use the term “homogeneous” in the sense of “homogeneous of degree 1”.

from Brouwer's fixed-point theorem. However, for the application to Pareto efficient and financially fair allocations, we need an eigenvector with entries that are strictly positive. Conditions for existence and uniqueness of such eigenvectors form an important topic in nonlinear Perron-Frobenius theory; see, for instance, Lemmens and Nussbaum (2012, Ch. 6). Here we use a result of Oshime (1983) that guarantees the existence of a unique positive eigenvector. For ease of reference, this result is stated below. First the definition is given of a notion that can be thought of as a nonlinear variant of the irreducibility condition that is well known in linear Perron-Frobenius theory.

Definition 2.8 A mapping φ from $\mathbb{R}^n_+$ into itself is *nonsectional* if, for every decomposition of the index set $\{1, \dots, n\}$ into two complementary nonempty subsets R and S, there exists $s \in S$ satisfying

(i) $(\varphi(x))_s > (\varphi(y))_s$ for all x, $y \in \mathbb{R}^n_+$ such that $x_R > y_R$ and $x_S = y_S > 0$;

(ii) $(\varphi(x^k))_s \to \infty$ for all sequences $(x^k)_{k=1,2,\dots}$ in $\mathbb{R}^n_+$ such that $x^k_R \to \infty$ while x^k_S is fixed and positive.

Theorem 2.9 (Oshime, 1983, Thm. 8, Remark 2) *If a mapping φ from $\mathbb{R}^n_+$ into itself is continuous, monotone, homogeneous, and nonsectional, then the mapping φ has a positive eigenvector, which is unique up to scalar multiplication, with a positive associated eigenvalue. In other words, there exist a constant $\eta^* > 0$ and a vector $x^* \in \mathbb{R}^n_{++}$ such that $\varphi(x^*) = \eta^* x^*$, and if $\eta > 0$ and $x \in \mathbb{R}^n_{++}$ are such that $\varphi(x) = \eta x$, then x is a scalar multiple of x^*.*

The eigenvalue associated to the positive eigenvector in the above theorem is in fact the *maximal* eigenvalue of the mapping φ (Oshime, 1983, Thm. 3). Iteration is a standard method to find the eigenvector associated to the maximal eigenvalue. In the linear case, this technique is known as the *power method* (see for instance Wilkinson (1965, p. 570)). Due to the homogeneity of the problem, it is possible to reduce the iteration to the unit simplex. In relation to a given homogeneous mapping φ from the positive cone into itself, we can define a normalized mapping ψ from the open unit simplex $\{(x_1, \dots, x_n) \in \mathbb{R}^n_{++} \mid \sum_{i=1}^n x_i = 1\}$ into itself by

$$\psi(x) = \frac{\varphi(x)}{\|\varphi(x)\|_1} \tag{2.14}$$

where $\|v\|_1 = \sum_{i=1}^n |v_i|$ is the 1-norm of $v \in \mathbb{R}^n$. Positive eigenvectors of the mapping φ correspond to fixed points of the mapping ψ.

To prove convergence of the iterative algorithm, it is natural to use a suitable contraction mapping theorem. First of all, an appropriate metric needs to be defined. A standard metric used in nonlinear Perron-Frobenius theory is the Hilbert metric, which is defined as follows.

Definition 2.10 The *Hilbert metric* assigns to a pair (x, y) with x, $y \in \mathbb{R}^n_{++}$ the distance $d(x, y)$ given by

$$d(x, y) = \log \frac{\max_i(x_i/y_i)}{\min_i(x_i/y_i)}\,.$$

Points on the same ray are equivalent with respect to the Hilbert metric, since

$$d(ax, by) = d(x, y) \quad \text{for all } a,\, b > 0. \tag{2.15}$$

On the positive cone, the Hilbert metric is therefore only a pseudometric. Alternatively, it can be viewed as a true metric on the space of positive rays, or on the open unit simplex (cf. Lemmens and Nussbaum, 2012, Prop. 2.1.1).

The following lemma is a standard fact (see for instance Lemmens and Nussbaum (2012, Ch. 2)); for the reader's convenience, we provide a proof of the version that we need here. Recall that a mapping φ from a metric space into itself is said to be *contractive* if $d(\varphi(x), \varphi(y)) < d(x, y)$ for all x, y such that $d(x, y) > 0$.

Lemma 2.11 *If $\varphi : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is homogeneous and strongly monotone (i.e., $x \lneq y$ implies $\varphi(x) < \varphi(y)$), then φ is contractive with respect to the Hilbert metric.*

Proof. Take x, $y \in \mathbb{R}^n_{++}$ with $d(x, y) > 0$. Define $M := \max_i(x_i/y_i)$, $m := \min_i(x_i/y_i)$. We then have $my \lneq x \lneq My$, and by homogeneity and strong monotonicity of φ we obtain $m\varphi(y) < \varphi(x) < M\varphi(y)$. Therefore,

$$\min_i \frac{\varphi(x)_i}{\varphi(y)_i} > m, \qquad \max_i \frac{\varphi(x)_i}{\varphi(y)_i} < M$$

and hence $d(\varphi(x), \varphi(y)) < \log(M/m) = d(x, y)$. □

As a consequence of property (2.15), the mapping ψ that is obtained from φ by normalization to the unit simplex is contractive if φ is. Lemma 2.11 only establishes that φ is contractive, not that it is a contraction mapping; in other words, the lemma does not provide a positive number δ such that $d(\varphi(x), \varphi(y)) \leq (1 - \delta)d(x, y)$ for all x and y. Therefore, we are not in a position to apply the Banach contraction mapping theorem. Instead we use the following theorem due to Nadler, which in our application guarantees convergence as a result of the assumption that the number of agents is finite.

Theorem 2.12 (Nadler, 1972, Thm. 1) *When $(\mathcal{X}, d)$ is a locally compact and connected metric space, and $f : \mathcal{X} \to \mathcal{X}$ is a contractive mapping with fixed point $x^* \in \mathcal{X}$, then for every $x \in \mathcal{X}$ the sequence of iterates $(f^{(k)}(x))_{k=1,2,\ldots})$ converges to the point x^*.*

Alternatively, one might use an argument based on the fact that the open unit simplex with the Hilbert metric is a geodesic space (there is a geodesic path from any given point to any other given point); cf. Lemmens and Nussbaum (2012, Prop. 3.2.3, Thm. 6.5.1).

2.5 Main results

In this section, we prove the convergence of the composite iteration method. Our method of proof is based on application of nonlinear Perron-Frobenius theory, which calls for the verification of a number of properties of the iteration mapping. This will be done in a series of lemmas below. We also demonstrate that the approach via nonlinear Perron-Frobenius theory leads to a proof of existence and uniqueness of Pareto efficient and financially fair solutions, independent from the approach via reformulation as an optimization problem (Gale and Sobel, 1979; Bühlmann and Jewell, 1979).

First we need to introduce some notation. Recall that the domain D is defined as (b, ∞), where $b = \sum_{i=1}^{n} b_i$ and the bounds b_i are the left limits of the domains of the utility functions of the individual agents. Within the space $C(D, \mathbb{R}_+)$ of continuous functions from D to $[0, \infty)$, equipped with the topology of pointwise convergence, we define the cone of strictly decreasing functions

$$\mathcal{L} = \{f \in C(D, \mathbb{R}_+) \mid f(y) < f(x) \text{ for all } x, y \in D \text{ s.t. } y > x\} \cup \{0\}.$$

The inclusion of the zero function within this set is natural when the functions in $\mathcal{L}$ are thought of as in terms of their graphs as subsets of the region $[b, \infty] \times [0, \infty]$ in the extended two-dimensional space. The function 0 can then be viewed as a representation of the multivalued mapping whose graph is $(\{b\} \times [0, \infty]) \cup ([b, \infty] \times \{0\})$.

2.5.1 Mapping from utility weights to marginal group utility

Agents whose utility functions are defined on all of the real line will need to be distinguished from agents who can tolerate only a limited loss. We therefore introduce the index set (possibly empty)

$$U = \{i \mid b_i = -\infty\}. \tag{2.16}$$

For $\alpha \in \mathbb{R}^n_+$ such that $\alpha_U > 0$, define

$$F(z,\alpha) = \sum_{i:\alpha_i>0} I_i(z/\alpha_i) + \sum_{i:\alpha_i=0} b_i, \qquad (z>0). \tag{2.17}$$

This function is a continuous mapping from the product space $\mathbb{R}_{++} \times \{\alpha \in \mathbb{R}^n_+ \mid \alpha_U > 0\}$ to (b,∞). For fixed nonzero α, the function $F(\,\cdot\,,\alpha) : (0,\infty) \to (b,\infty)$ is continuous and strictly decreasing, and satisfies

$$\lim_{z\to\infty} F(z,\alpha) = b, \qquad \lim_{z\downarrow 0} F(z,\alpha) = \infty. \tag{2.18}$$

Therefore there is a well-defined inverse function, which is denoted by $J(\,\cdot\,,\alpha)$. Since $F(\,\cdot\,,\alpha)$ is strictly decreasing and continuous, the same properties hold for $J(\,\cdot\,,\alpha)$. For $\alpha \in \mathbb{R}^n_+$, we can therefore define the function $\varphi_1(\alpha) \in \mathcal{L}$ by

$$(\varphi_1(\alpha))(x) = \begin{cases} J(x,\alpha) & \text{if } \alpha \neq 0 \text{ and } \alpha_U > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.19}$$

for all $x \in D$. For $\alpha > 0$, the defining relationship for the mapping φ_1 may also be written in a more implicit but perhaps also more evocative form as

$$\varphi_1 : \alpha \mapsto J, \qquad \sum_{i=1}^{n} I_i\big(J(x)/\alpha_i\big) = x \qquad (x \in D). \tag{2.20}$$

We now establish various properties of this mapping such as monotonicity and continuity.

Lemma 2.13 *The mapping φ_1 is homogeneous and monotone. If $\alpha^1 \in \mathbb{R}^n_+$ and $\alpha^2 \in \mathbb{R}^n_+$ are such that $\alpha^1_U > 0$ and $\alpha^1 \gneq \alpha^2$, then we have in fact $\varphi_1(\alpha^1) > \varphi_1(\alpha^2)$.*

Proof. The homogeneity is immediate from the definitions. Concerning the monotonicity, take α^1 and α^2 in $\mathbb{R}^n_+$ such that $\alpha^1 \geq \alpha^2$. Firstly, assume that $\alpha^1_U > 0$ and $\alpha^2_U > 0$. Take $x \in D$, and let z_1 and z_2 be defined by $F(z_1,\alpha^1) = x$ and $F(z_2,\alpha^2) = x$. We then have $z_i = (\varphi_1(\alpha^i))(x)$ for $i = 1,2$. Because the function $F(\,\cdot\,,\,\cdot\,)$ is strictly increasing in each of the components of its second argument and strictly decreasing in its first argument, the vector inequality $\alpha^1 \geq \alpha^2$ and the equality $F(z_1,\alpha^1) = F(z_2,\alpha^2)$ together imply that $z_1 \geq z_2$, with strict inequality as soon as α^1 and α^2 are not equal. Secondly, assume that $\alpha^1_U > 0$ while $\alpha^2_i = 0$ for some $i \in U$, then the strict inequality $\varphi_1(\alpha^1) > \varphi_1(\alpha^2)$ trivially holds, since $\varphi_1(\alpha^1)$ takes positive values, while $\varphi_1(\alpha^2) = 0$ by definition. Finally, if there is $i \in U$ such that $\alpha^1_i = 0$, then $\varphi_1(\alpha^1) = \varphi_1(\alpha^2) = 0$.

□

To show the continuity of φ_1, we make use of the following lemma.

Lemma 2.14 *Let a topological space $\mathcal{Y}$ and a sequentially continuous mapping $f(\,\cdot\,,\,\cdot\,)$ from $\mathbb{R}_{++} \times \mathcal{Y}$ to $\mathbb{R}$ be given. Suppose that for every $y \in \mathcal{Y}$ there is exactly one $x \in \mathbb{R}_{++}$ such that $f(x, y) = 0$. Let $(y_k)_{k=1,2,\ldots}$ be a sequence in $\mathcal{Y}$ that converges to $\bar{y} \in \mathcal{Y}$. Define x_k $(k = 1, 2, \ldots)$ by the equations $f(x_k, y_k) = 0$, and let $\bar{x}$ be defined by $f(\bar{x}, \bar{y}) = 0$. If the collection $\{x_k \mid k \in \mathbb{N}\}$ is bounded, then $\lim_{k\to\infty} x_k = \bar{x}$.*

Proof. By the assumed boundedness of the collection $\{x_k \mid k \in \mathbb{N}\}$, it suffices to show that any accumulation point of this collection must coincide with $\bar{x}$. Let $\tilde{x}$ be an accumulation point, and let $(k_j)_{j=1,2,\ldots}$ satisfy $\lim_{j\to\infty} x_{k_j} = \tilde{x}$. From the continuity of the mapping f, we have $f(\tilde{x}, \bar{y}) = \lim_{j\to\infty} f(x_{k_j}, y_{k_j}) = 0$. The assumed uniqueness of the solution of the equation $f(x, y) = 0$ for given y then implies the equality $\tilde{x} = \bar{x}$. □

Lemma 2.15 *The mapping $\varphi_1 : \mathbb{R}^n_+ \to \mathcal{L}$ is continuous.*

Proof. Let $(\alpha^k)_{k=1,2,\ldots}$ be a sequence of vectors in $\mathbb{R}^n_+$ converging to a vector $\alpha \in \mathbb{R}^n_+$. Take $x \in D$; write $z_k := (\varphi_1(\alpha^k))(x)$ and $z := (\varphi_1(\alpha))(x)$. We need to show that the sequence $(z_k)_{k=1,2,\ldots}$ converges to z.

First consider the case in which $\alpha_U > 0$. In this case we also have $\alpha^k_U > 0$ for all sufficiently large k. By definition, the numbers z_k and z are positive and satisfy $F(z_k, \alpha^k) = x$ and $F(z, \alpha) = x$. Suppose there would be a subsequence $(z_{k_j})_{j=1,2,\ldots}$ that tends to infinity. For all i with $\alpha_i > 0$, the sequences $(\alpha_i^{k_j})_{j=1,2,\ldots}$ tend to finite limits, namely α_i. Consequently, the quotients $z_{k_j}/\alpha_i^{k_j}$ tend to infinity, and therefore

$$x = \lim_{j\to\infty} F(z_{k_j}, \alpha^{k_j}) = b.$$

However, we have $x \in (b, \infty)$ so that $x > b$. From this contradiction it follows that the set $\{z_k \mid k \in \mathbb{N}\}$ is bounded, and it follows from Lemma 2.14 that $\lim_{k\to\infty} z_k = z$.

Now suppose that there is an index $\ell \in U$ such that $\alpha_\ell = 0$. By definition, we then have $z = 0$. To avoid trivialities, we may assume that $\alpha^k_U > 0$ for all k. The numbers $z_k > 0$ are then given as the solutions of $F(z_k, \alpha^k) = x$. Take $\varepsilon > 0$, and suppose there would be a subsequence $(z_{k_j})_{j=1,2,\ldots}$ such that $z_{k_j} > \varepsilon$ for all j. The quotient $z_{k_j}/\alpha_\ell^{k_j}$ then tends to infinity because of the assumption that $\alpha_\ell = 0$, and the corresponding inverse marginal utility function $I_\ell(z)$ tends to $-\infty$ when its argument tends to infinity, due to the assumption that $\ell \in U$. Because $z_{k_j} > \varepsilon$ for all j and the sequences $(\alpha_i^{k_j})_{j=1,2,\ldots}$ tend to finite limits, the inverse marginal utilities $I_i(z_{k_j}/\alpha_i^{k_j})$ $(i = 1, \ldots, n)$ are bounded from above. Therefore, we obtain

$$x = \lim_{j\to\infty} F(z_{k_j}, \alpha^{k_j}) = -\infty.$$

This is a contradiction. We therefore have $\lim_{k\to\infty} z_k = 0$, as was to be proved. □

The next lemma states a property of the mapping φ_1 that relates to nonsectionality.

Lemma 2.16 *Let $(\alpha^k)_{k=1,2,\ldots}$ be a sequence in $\mathbb{R}^n_+$ that has the following property: there exist complementary nonempty index sets R and S in $\{1, \ldots, n\}$ and a vector $\alpha_S \in \mathbb{R}^{|S|}_{++}$ such that $\alpha^k_R \to \infty$ as $k \to \infty$, while $\alpha^k_S = \alpha_S$ for all k. Then $(\varphi_1(\alpha^k))(x) \to \infty$ as $k \to \infty$ for all $x \in D$.*

Proof. Take $x \in D$. Since the entries with indices in S are assumed to be positive and those with indices in R tend to infinity, we can assume that all entries of α^k are positive. Then the numbers $z_k := \varphi_1(\alpha^k)(x)$ are defined implicitly by

$$\sum_{i\in R} I_i(z_k/\alpha_i^k) + \sum_{i\in S} I_i(z_k/\alpha_i) = x. \tag{2.21}$$

Suppose that $(z_k)_{k=1,2,\ldots}$ has a bounded subsequence $(z_{k_j})_{j=1,2,\ldots}$. The quotients $z_{k_j}/\alpha_i^{k_j}$ tend to zero for $i \in R$ so that the first term on the left-hand side in (2.21) tends to infinity. The quotients z_{k_j}/α_i for $i \in S$ remain bounded, so that the second term at the left-hand side is bounded from below. Therefore the left-hand side tends to infinity as $j \to \infty$, which leads to a contradiction. The statement in the lemma follows. □

2.5.2 Mapping from marginal group utility to utility weights

We now turn to the mapping φ_2. Recall that the numbers v_i $(i = 1, \ldots, n)$ represent the claim values of the agents, and that $v_i > b_i$ for all i. We have assumed that the total risk X is bounded; consequently, for any given nonzero function $J \in \mathcal{L}$, the random variable $J(X)$ is bounded as well. For each $i = 1, \ldots, n$, the mapping $\alpha_i \mapsto E^Q I_i(J(X)/\alpha_i)$ defines a strictly increasing function with

$$\lim_{\alpha_i\to\infty} E^Q I_i(J(X)/\alpha_i) = \infty, \qquad \lim_{\alpha_i\downarrow 0} E^Q I_i(J(X)/\alpha_i) = b_i.$$

By the assumed inequality $v_i > b_i$, the equation

$$E^Q I_i(J(X)/\alpha_i) = v_i \tag{2.22}$$

therefore has a unique solution $\alpha_i > 0$. The mapping from collective marginal utility J to utility weights α can be extended to a mapping defined on $\mathcal{L}$ by

$$(\varphi_2(J))_i = \begin{cases} \alpha_i \text{ satisfying (2.22)} & \text{if } J \neq 0 \\ 0 & \text{if } J = 0 \end{cases} \tag{2.23}$$

for $i = 1, \ldots, n$.

Lemma 2.17 *The mapping φ_2 is homogeneous and strictly monotone, i.e., $\varphi_2(J_1) \geq \varphi_2(J_2)$ when $J_1 \geq J_2$ and $\varphi_2(J_1) > \varphi_2(J_2)$ when $J_1 > J_2$.*

Proof. The homogeneity is immediate from the definition. The strict monotonicity follows from the fact that all inverse marginal utilities I_i are strictly decreasing; in case $J_2 = 0$, the strict monotonicity is immediate from the definition. □

Lemma 2.18 *The mapping φ_2 is sequentially continuous.*

Proof. Let $(J_k)_{k=1,2,\ldots}$ be a sequence in $\mathcal{L}$, converging pointwise to $J \in \mathcal{L}$, and fix $i \in \{1, \ldots, n\}$. Write $\alpha_i^k := (\varphi_2(J_k))_i$ and $\alpha_i := (\varphi_2(J))_i$. We want to show that $(\alpha_i^k)_{k=1,2,\ldots}$ converges to α_i.

First assume that the limit function J is nonzero; we can then assume that all elements of the sequence J_k are nonzero as well. Note that the collection of random variables $J_k(X)$ is bounded above by $\sup_k J_k(\inf X)$ and below by $\inf_k J_k(\sup X)$. Therefore, if there would exist a subsequence $(\alpha_i^{k_j})_{j=1,2,\ldots}$ converging to infinity, we would have

$$v_i = \lim_{j\to\infty} E^Q I_i(J_{k_j}(X)/\alpha_i^{k_j}) = \infty. \tag{2.24}$$

This would contradict the assumptions. Consequently, the collection $\{\alpha_i^k \mid k \in \mathbb{N}\}$ is bounded. Consider the function $G : \mathbb{R}_{++} \times \mathcal{L} \to \mathbb{R}$ defined by

$$G(\alpha_i, J) = E^Q I_i(J(X)/\alpha_i).$$

It follows from the bounded convergence theorem that this function is sequentially continuous. The relation $\lim \alpha_i^k = \alpha_i$ follows from Lemma 2.14.

Consider now the case in which $J = 0$. In this case, we have by definition $\alpha_i = 0$. Take $\varepsilon > 0$ and suppose that there would exist a subsequence $(\alpha_i^{k_j})_{j=1,2,\ldots}$ such that $\alpha_i^{k_j} > \varepsilon$ for all $j = 1, 2, \ldots$. The convergence of $(J_k)_{k=1,2,\ldots}$ to $J = 0$ would then imply the same conclusion as in (2.24). Consequently, we have $\lim_{k\to\infty} \alpha_i^k = 0$, as was to be shown. $\square$

The final lemma establishes a property that will be used in a nonsectionality argument.

Lemma 2.19 *Let $(J_k)_{k=1,2,\ldots}$ be a sequence in $\mathcal{L}$ such that $J_k(x) \to \infty$ for all $x \in D$ as $k \to \infty$. Then $(\varphi_2(J_k))_i \to \infty$ for all $i = 1, \ldots, n$.*

Proof. Choose $i \in \{1, \ldots, n\}$. Suppose that the i-th entry of $\alpha^k := \varphi_2(J_k)$ does not tend to infinity. Then there exist a finite number M and a subsequence $(\alpha_i^{k_j})_{j=1,2,\ldots}$ such that $\alpha_i^{k_j} < M$ for all j. We would then have

$$v_i = \lim_{j\to\infty} E^Q I_i(J^{k_j}(X)/\alpha_i^{k_j}) = b_i. \tag{2.25}$$

This is a contradiction, since it is assumed that $v_i > b_i$. Therefore the statement of the lemma follows. $\square$

2.5.3 The complete iteration mapping

With the mappings $\varphi_1 : \mathbb{R}^n_+ \to \mathcal{L}$ and $\varphi_2 : \mathcal{L} \to \mathbb{R}^n_+$ in hand, one can define a mapping φ from $\mathbb{R}^n_+$ into itself in the obvious way by

$$\varphi(\alpha) = \varphi_2(\varphi_1(\alpha)). \tag{2.26}$$

It follows from the development above and the Borch parametrization (2.8) that Pareto efficient and financially fair solutions of the risk sharing problem are in one-to-one correspondence with vectors $\alpha \in \mathbb{R}^n_{++}$ such that $\varphi(\alpha) = \alpha$. The proposition below implies that it is in fact sufficient to look for positive eigenvectors of the mapping φ. A similar argument was used by Menon (1967) in an analysis of the IPFP.

Proposition 2.20 *The mapping φ can only have 1 as an eigenvalue corresponding to a positive eigenvector. In other words, if $\alpha \in \mathbb{R}^n_{++}$ is such that $\varphi(\alpha) = \lambda\alpha$, then $\lambda = 1$.*

Proof. Let $\alpha > 0$ be such that $\varphi(\alpha) = \lambda\alpha$. Since φ maps the positive cone into itself, the eigenvalue λ must be positive. Define $J = \varphi_1(\alpha)$; then $\varphi_2(J) = \lambda\alpha$. Note that $J(x) > 0$

for all $x \in D$. By definition, we have

$$\sum_{i=1}^{n} I_i(J(x)/\alpha_i) = x \qquad (x \in D)$$
$$E^Q I_i(J(X)/(\lambda\alpha_i)) = v_i \qquad (i = 1, \dots, n).$$

Therefore,

$$\sum_{i=1}^{n} E^Q I_i(J(X)/(\lambda\alpha_i)) = \sum_{i=1}^{n} v_i = E^Q X = \sum_{i=1}^{n} E^Q I_i(J(X)/\alpha_i).$$

The claim follows by noting that the function $\lambda \mapsto I_i(J(x)/(\lambda\alpha_i))$, for fixed x and fixed i, is strictly increasing in λ. □

Theorem 2.21 *The mapping φ defined by (2.19, 2.23, 2.26) has a unique continuous extension to a mapping from the nonnegative cone to itself. This extension is homogeneous, monotone, and nonsectional. On the positive cone, the mapping φ is strongly monotone.*

Proof. The continuity follows from Lemmas 2.15 and 2.18. Monotonicity and homogeneity follow from Lemmas 2.13 and 2.17. These lemmas also imply strong monotonicity of φ on the positive cone. Consider now two nonempty complementary subsets R and S of the index set $\{1, \dots, n\}$ as in Definition 2.8. If α^1 and α^2 are such that $\alpha_S^2 > 0$, $\alpha_S^1 = \alpha_S^2$, and $\alpha_R^1 > \alpha_R^2$, then it follows from Lemma 2.13 that $\varphi_1(\alpha^1) > \varphi_1(\alpha^2)$. The strict inequality is preserved by the mapping φ_2 according to Lemma 2.17, so that item (i) in Definition 2.8 is satisfied. The condition in item (ii) in Definition 2.8 is fulfilled due to Lemma 2.16 and Lemma 2.19. □

By Oshime's theorem (Theorem 2.9), we can now conclude the following.

Corollary 2.22 *Problem 2.6 has a unique solution. The unique allocation rule that is Pareto efficient and financially fair is given by*

$$y_i(x) = I_i(J(x)/\alpha_i) \tag{2.27}$$

for $i = 1, \dots, n$ and $x \in D$, where I_i is the inverse marginal utility function of agent i, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a positive eigenvector (unique up to multiplication by a positive scalar) of the mapping φ defined in (2.26), and J is given by $J = \varphi_1(\alpha)$ through the mapping φ_1 defined in (2.19).

The mapping φ induces a normalized mapping ψ from the open unit simplex into itself via (2.14). Under the assumptions of the above theorem, this mapping is contractive and has a fixed point. Using the fact that the open simplex in finite dimensions is a locally compact and connected metric space, we can therefore apply Theorem 2.12 to conclude the global convergence of the composite iteration algorithm.

Corollary 2.23 *Under Assumptions 2.1 and 2.2, the mapping ψ defined by (2.19, 2.23, 2.26) and (2.14) has the following property: for every α^0 in the open unit simplex $\{\alpha \in \mathbb{R}^n_{++} \mid \sum_{i=1}^n \alpha_i = 1\}$, the sequence of vectors $(\alpha^0, \alpha^1, \dots)$ defined iteratively by*

$$\alpha^{i+1} = \psi(\alpha^i) \tag{2.28}$$

converges to the unique eigenvector in the open unit simplex of the mapping φ defined by (2.19, 2.23, 2.26).

Concerning implementation, it can be noted that the equations in (2.19) and (2.23) can be solved in parallel for different $x \in D$ and different $i \in \{1, \dots, n\}$, respectively, and that in each case the problem comes down to determining the root of a strictly monotone scalar function. The normalization is used above to simplify the proof of convergence. The fact that the eigenvalue associated to the positive eigenvector is equal to 1 suggests that normalization is not really needed. Computational experience indeed indicates that the composite iteration algorithm performs just as well, or perhaps even better, when normalization is not applied.

2.6 Equicautious HARA collectives

The class of utility functions with hyperbolic absolute risk aversion (HARA class) consists of the functions $u(\,\cdot\,)$ for which the corresponding coefficient of risk aversion $-u''(x)/u'(x)$ is of the form $1/(\sigma x + \tau)$, where σ and τ are constants. Special cases are the exponential utility functions ($\sigma = 0$) and the power utilities ($\tau = 0$). The coefficient σ is called *cautiousness* by Wilson (1968); it measures how quickly the coefficient of risk aversion increases as wealth goes down. As noted by Wilson, collectives of agents who have identical cautiousness enjoy special properties. Such collectives are called *equicautious* by Amershi and Stoeckenius (1983). Examples are collectives of power utility agents who all have the same degree of relative risk aversion, and collectives consisting of exponential utility agents. The proposition below shows that, in the equicautious HARA case, the composite iteration mapping φ enjoys special properties too; the normalized iteration based on this mapping converges in one step.

Proposition 2.24 *In the case of an equicautious HARA collective, the mapping ψ defined by (2.19, 2.23, 2.26) and (2.14) satisfies $\psi(\psi(\alpha)) = \psi(\alpha)$ for all $\alpha \in \mathbb{R}^n_{++}$.*

Proof. Suppose that $u_1, \ldots, u_n$ are utility functions of an equicautious HARA collective, and let $-u_i'(x)/u_i''(x) = \sigma x + \tau_i$. By definition of the functions I_i we have, for all $z > 0$, $u_i'(I_i(z)) = z$ and hence $u_i''(I_i(z))I_i'(z) = 1$, so that

$$-zI_i'(z) = -\frac{u_i'(I_i(z))}{u_i''(I_i(z))} = \sigma I_i(z) + \tau_i.$$

For any given weight vector $\alpha \in \mathbb{R}^n_{++}$, the function I defined by $I(z) = \sum_{i=1}^n I_i(z/\alpha_i)$ satisfies

$$-zI'(z) = -\sum_{i=1}^{n}(z/\alpha_i)I_i'(z/\alpha_i) = \sum_{i=1}^{n}\big(\sigma I_i(z/\alpha_i) + \tau_i\big) = \sigma I(z) + \sum_{i=1}^{n}\tau_i. \tag{2.29}$$

Write $\tau := \sum_{i=1}^n \tau_i$. Since J as defined in (2.19) is the inverse function of I, we have

$$-J(x)I'(J(x)) = \sigma x + \tau. \tag{2.30}$$

From the relation $I(J(x)) = x$ it follows that $I'(J(x))J'(x) = 1$; therefore (2.30) implies that $-J(x)/J'(x) = \sigma x + \tau$. This shows that the function J defined by (2.19) depends on the coefficients $\alpha_1, \ldots, \alpha_n$ only through a multiplicative factor. Consequently, the coefficients $\alpha_1, \ldots, \alpha_n$ that are determined from the function J via (2.22) represent a positive eigenvector of φ, so that convergence of the iteration (2.28) is achieved in one step. □

2.7 Rate of convergence

The asymptotic rate of convergence of the composite iteration algorithm is governed by the linearization of the iteration map around the fixed point of the iteration. As is well known (Wilkinson, 1965, Ch. 9), the power method for finding the eigenvector corresponding to the dominant eigenvalue of a matrix has a linear rate of convergence, and the speed of convergence is determined by the ratio of the absolute value of the second largest eigenvalue with respect to the absolute value of the largest eigenvalue. In our case the largest eigenvalue is equal to 1, and therefore the asymptotic convergence speed of the composite iteration algorithm is simply given by the size of the second largest eigenvalue of the Jacobian of this matrix at the fixed point of the iteration. The Jacobian matrix can be computed on the basis of the following proposition. First we introduce some notation. Given a utility function u that is twice differentiable, strictly increasing, and strictly concave, the corresponding *coefficient of absolute risk tolerance* is the function T defined by

$$T(x) = -\frac{u'(x)}{u''(x)}\,. \tag{2.31}$$

This is the inverse of the usual Arrow-Pratt coefficient of absolute risk aversion.

Proposition 2.25 *Consider the mappings φ_1, φ_2, and the composite mapping φ as defined in (2.26) on the basis of a given group of agents with utility functions $u_i(\,\cdot\,)$ and claim values v_i. The agents' coefficients of absolute risk tolerance are denoted by $T_i(\,\cdot\,)$. Let $\alpha \in \mathbb{R}^n_{++}$ be such that $\varphi(\alpha) = \alpha$, and let $y_i(\,\cdot\,)$ denote the corresponding allocation functions as determined by (2.8). The linearization of the mapping φ_1 at the point α is given by*

$$D\varphi_1(\alpha) : \Delta\alpha \mapsto \Delta J, \quad \Delta J(x) = \frac{J(x)}{\sum_{i=1}^n T_i(y_i(x))} \sum_{i=1}^n T_i(y_i(x))\,\frac{\Delta\alpha_i}{\alpha_i} \qquad (x \in D). \tag{2.32}$$

The linearization of the mapping φ_2 at $J = \varphi_1(\alpha)$ is given by

$$D\varphi_2(J) : \Delta J \mapsto \Delta\alpha, \quad \Delta\alpha_i = \frac{\alpha_i}{E^Q[T_i(y_i(X))]}\, E^Q\Big[T_i(y_i(X))\frac{\Delta J(X)}{J(X)}\Big] \qquad (i = 1, \dots, n). \tag{2.33}$$

The Jacobian at α of the composite mapping φ is given by

$$\big(D\varphi(\alpha)\big)_{ik} = \frac{\alpha_i/\alpha_k}{E^Q[T_i(y_i(X))]}\, E^Q\left[\frac{T_i(y_i(X))T_k(y_k(X))}{\sum_{i=1}^n T_i(y_i(X))}\right] \quad (i = 1, \dots, n;\ k = 1, \dots, n). \tag{2.34}$$

Proof. The linearization of the defining relationship (2.20) of the mapping φ_1 around a given point $\alpha \in \mathbb{R}^n_{++}$ is given by

$$\sum_{i=1}^n \frac{1}{\alpha_i}\, I_i'(J(x)/\alpha_i)\Delta J(x) - \sum_{i=1}^n \frac{J(x)}{\alpha_i^2}\, I_i'(J(x)/\alpha_i)\Delta\alpha_i = 0. \tag{2.35}$$

In terms of the allocation functions that are associated to the point α by means of Borch's condition for Pareto efficiency (2.7), we can write

$$I_i'(J(x)/\alpha_i) = I_i'(u_i'(y_i(x))) = \frac{1}{u_i''(y_i(x))} = \frac{\alpha_i}{J(x)}\,\frac{u_i'(y_i(x))}{u_i''(y_i(x))} = -\frac{\alpha_i}{J(x)}\,T_i(y_i(x)). \tag{2.36}$$

Together with (2.35), this leads to (2.32). The defining relationship (2.22) of the mapping φ_2 is linearized as follows:

$$E^Q\Big[\frac{1}{\alpha_i}\, I_i'(J(X)/\alpha_i)\Delta J(X)\Big] - E^Q\Big[\frac{J(X)}{\alpha_i^2}\, I_i'(J(X)/\alpha_i)\Delta\alpha_i\Big] = 0. \tag{2.37}$$

Together with (2.36), this leads to (2.33). Finally, the expression (2.34) is obtained by combining (2.32) and (2.33). □

The quantity $T_i(y_i(x))/\sum_{i=1}^n T_i(y_i(x))$ might be called the *tolerance share* of agent i at outcome x, within the allocation scheme defined by the functions y_i. Under the efficiency condition (2.7), this function can be given an alternative interpretation as follows. Since (2.7) implies the equality $J'(x) = \alpha_i u_i''(y_i(x))y_i'(x)$, we can write

$$T_i(y_i(x)) = -\frac{u_i'(y_i(x))}{u_i''(y_i(x))} = -\frac{J(x)}{J'(x)}\, y_i'(x).$$

Therefore we have

$$\sum_{i=1}^n T_i(y_i(x)) = -\frac{J(x)}{J'(x)}, \tag{2.38}$$

and hence

$$\frac{T_i(y_i(x))}{\sum_{i=1}^n T_i(y_i(x))} = y_i'(x). \tag{2.39}$$

In other words, in Pareto efficient allocations the tolerance share of each agent is equal, at every outcome x, to the derivative of that agent's allocation function at the point x. Given the interpretation of $J(x)$ as a group marginal utility, the right hand side of (2.38) can be viewed as a group risk tolerance, which agrees with the natural interpretation of the left hand side.

The expression for the Jacobian can be simplified further by introducing a probability measure Q_i, which is associated to agent i under a given allocation scheme, as follows:

$$E^{Q_i}[Z] = \frac{E^Q[T_i(y_i(X))Z]}{E^Q[T_i(y_i(X))]} \qquad (Z \in L^\infty(\Omega, \mathcal{F}, Q)). \tag{2.40}$$

Using also (2.39), we can then write the elements of the Jacobian at the fixed point of the iteration mapping as

$$\big(D\varphi(\alpha)\big)_{ik} = \frac{\alpha_i}{\alpha_k}\, E^{Q_i}[y_k'(X)]. \tag{2.41}$$

While this is a short formula, an advantage of the expression (2.34) as given in the proposition above is that it does not require computing derivatives of the allocation functions.

Remark 2.26 In the case of equicautious HARA utilities, it is well known that efficient allocation rules must be linear (Amershi and Stoeckenius, 1983, Thm. 5). Suppose the allocation functions are given by $y_i(x) = a_i x + b_i$ where a_i and b_i $(i = 1, \ldots, n)$ are constants. In this case, it follows from the expression (2.33) that the Jacobian of the iteration mapping at the fixed point is given by

$$\big(D\varphi(\alpha)\big)_{ik} = \frac{\alpha_i}{\alpha_k}\, a_k.$$

This implies that the Jacobian has rank 1, as expected from Section 2.6.

We conclude this section with a small numerical example in which we illustrate the convergence behavior of the composite iteration algorithm.

Example 2.27 Suppose a risk is to be divided between three agents who are referred to as "senior" (S), "mezzanine" (M), and "equity" (E). The agents use power utility $u_i(x) = x^{1-\gamma_i}/(1-\gamma_i)$, with different coefficients of relative risk aversion γ_i (10, 5, and 2). The agents have agreed on a pricing functional that gives positive weights to nine possible outcomes of the risk X. These outcomes are of the form $\exp z$, with $z = -2, -1.5, \ldots, 2$, and the corresponding weights (state prices) are proportional to $\exp(-\frac{1}{2}z^2)$. In other words, under the pricing measure, the risk X follows a discrete approximation to a lognormal distribution; numerical values are given in the second row of Table 2.1. There is no need to specify the probability measure P since the PEFF solution does not depend on it, due to the assumption that agents all use the same probabilities to compute expected utility. The three agents have equal ownership rights.

The composite iteration algorithm, initialized at the point $\alpha^0 = \big(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\big)$, without renormalization of iterates α^i, produces after four iterations a solution that satisfies the feasibility constraint (2.5) up to an error that is less than 0.5% (i.e. $\max \big|\big(\sum_{i=1}^{3} y_i(X) - X\big)\big| / E^Q[X] < 0.005$). The fairness constraints (2.6) are satisfied up to machine precision by the design of the algorithm. After three more iterations, the error in the feasibility constraint is less than 0.01%. The fixed point (corresponding to the scaling as given in the definition of the utility functions) is $\alpha = (0.03, 0.41, 0.56)$. The resulting allocation rule is shown in Table 2.1 for the outcomes of X that receive positive weights under Q. It can be verified that the claims held by the agents all have equal value when the value is computed by taking the weighted sum of the payoffs, with weights given in the row labeled q.

X	0.1353	0.2231	0.3679	0.6065	1.0000	1.6487	2.7183	4.4817	7.3891
q	0.0276	0.0663	0.1238	0.1802	0.2042	0.1802	0.1238	0.0663	0.0276
S	0.1138	0.1730	0.2554	0.3627	0.4888	0.6221	0.7554	0.8881	1.0228
M	0.0214	0.0495	0.1080	0.2178	0.3956	0.6408	0.9447	1.3060	1.7321
E	0.0001	0.0006	0.0045	0.0260	0.1155	0.3858	1.0182	2.2876	4.6342

Table 2.1: Pareto efficient and financially fair allocation of an approximately lognormal risk among three power utility agents labeled S, M, and E, with different coefficients of risk aversion, and with equal ownership rights. The row labeled X shows possible payoffs that are to be divided among the agents; the row labeled q shows valuation weights (measure Q).

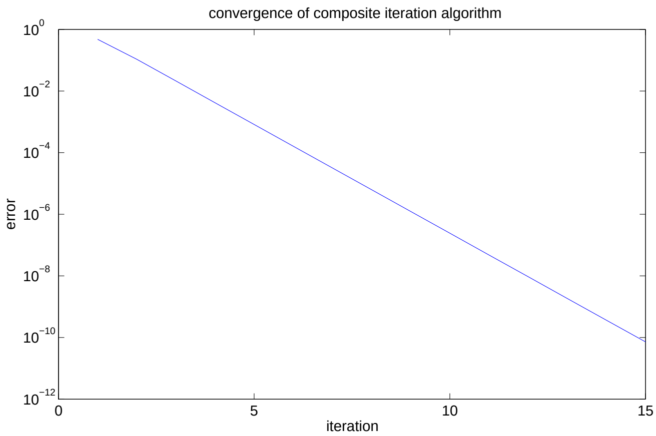

Figure 2.1: Error in the feasibility constraint as a function of the number of iterations

The size of the error as a function of the number of iterations is shown in Figure 2.1. It is seen that the asymptotic regime sets in almost immediately. The slope of the line in the figure closely matches the magnitude of the second largest eigenvalue of the Jacobian, which can be found from (2.34) and which for the data as given equals 0.197. The convergence becomes only marginally slower when the grid size is increased to get a closer approximation to a lognormal variable under Q. Reducing the differences between the risk aversion coefficients of the agents tends to make the convergence faster, and the same holds when the differences between the ownership rights of the agents are made larger.

2.8 Conclusions and further research

In this chapter we have studied the application of the composite iteration method to a fair division problem under a linear notion of fairness. The application features agents with concave and additively separable preferences. In this setting, the composite iteration map can be easily computed. We have established a number of relevant properties of the map, which allow to prove existence and uniqueness of solutions and global convergence of the corresponding iteration map.

We have assumed that the total risk which is to be allocated among the agents is given. However, in many situations, the collective can decide to a certain extent how much risk it wants to take. Problems of collective investment decisions have been considered for instance by Wilson (1968) and Xia (2004). The notion of financial fairness would seem to be relevant in this context, but has not received much attention in the literature so far. A treatment of collective investment along the lines of the present chapter is given in the following chapter.

Multiperiod allocation problems have been considered for instance by Gale and Machado (1982); Barrieu and Scandolo (2008); Gollier (2008); Cui et al. (2011). Existence and uniqueness of Pareto efficient and financially fair allocation rules in the multiperiod context has been shown by Bao et al. (2017) using methods analogous to the ones in the present chapter.

The composite iteration algorithm can be applied analogously (cf. Dana, 2001) in the case of Arrow-Debreu equilibrium. Among the known sufficient conditions for the composite iteration map to be strongly monotone in this case, the most important one is additive separability with low risk aversion; see Dana (2001) for details. The algorithm can be formulated for general preferences under suitable concavity assumptions, but simplifies notably in the case of additively separable preferences as considered in this chapter.

It would be of interest to give an economic meaning to the magnitude of the second largest eigenvalue of the Jacobian of the composite iteration mapping at the fixed point. As noted in Remark 2.26, this number equals zero in the equicautious case. The question therefore arises to what extent the magnitude of the second largest eigenvalue could be viewed as a quantitative measure of nonequicautiousness, or (since equicautiousness makes it possible to aggregate preferences) as a quantitative measure of inaggregability of preferences.

Chapter 3

Cooperative Investment in Incomplete Markets Under Financial Fairness

Joint work with J.M. Schumacher and B.J.M. Werker

3.1 Introduction

A large literature has emanated from the seminal paper by Borch (1962), which characterized Pareto optimality for risk sharing among a group of agents. In the model of Borch's theorem, the only essential attributes of agents are their preferences across risky prospects. In particular, the framework does not include a concept of ownership rights, which potentially could be used to provide additional guidelines for risk sharing agreements. An extension of Borch's framework in this direction was carried out by Bühlmann and Jewell (1978, 1979). These authors define the notion of a "fair Pareto optimal risk exchange" (FAIRPOREX). Fairness is understood here in the sense of a financial valuation operator (expectation under a risk-neutral measure). The basic presumption is that it is possible to assign a financial value both to what each agent contributes to the collective and to the stochastic payoff that the agent receives from the collective. Fairness then means that, for each agent separately, equality should hold between the two values.

While the purely preference-based setting of Borch typically leads to an infinite num-

The results described in this chapter were published in Pazdera et al. (2016)

ber of solutions (the "contract curve"), Bühlmann and Jewell (1978, 1979) show that their framework, which employs preferences as well as financial values, produces a unique solution under quite general conditions. Their proof makes use of an ingenious argument due to Gale (1977) (cf. Gale and Sobel, 1979). The problem considered by Gale (1977) is to find a rule for the division of the uncertain proceeds of a jointly owned enterprise, in such a way that Pareto efficiency holds and the expected sizes (under the real-world measure) of the portions received by the shareholders are proportional to their ownership rights.

Analytical solutions of the problem of finding the Pareto efficient and financially fair risk-sharing rule are only available in special cases. These cases include in particular the situation of an equicautious HARA collective (Amershi and Stoeckenius, 1983), in which it is known (Thm. 5 in the cited paper) that Pareto efficient risk sharing rules consist of affine functions of the realized outcome. In other cases, a numerical solution procedure must be followed. Bühlmann and Jewell (1978, 1979) have hinted at an iterative method to compute the fair Pareto optimal risk exchange, but they did not work out the details. A study of an iterative algorithm to compute the Pareto efficient and financially fair solution was undertaken in Chapter 2. The algorithm reduces the problem to a series of steps, each of which requires only to determine the root of a continuous and monotonic scalar function. The approach is based on the characterization of the Pareto efficient and financially fair allocation in terms of the positive eigenvector of an associated homogeneous nonlinear mapping. In this way, it also leads to an existence and uniqueness proof via nonlinear Perron-Frobenius theory, different from the original argument of Gale (1977).

In the models studied by Borch (1962), Gale (1977), and Bühlmann and Jewell (1978, 1979), the risk that agents are confronted with is given exogenously. Such models are applicable for instance to the pooling of underwriting risks of insurance companies. In other situations, agents may have a certain amount of freedom in selecting their exposure to risk. Such situations arise for instance in collective pension funds, where trustees must decide on the investment policy of the collective as well as on benefit policy (Gollier, 2008; Cui et al., 2011). An extension of Borch's framework covering endogenous risk was undertaken by Wilson (1968). For a discussion of financial fairness within this context, we need to distinguish between complete and incomplete markets.

When the market is complete, agents can achieve individual optimality subject to their budget constraints by trading just for themselves. This solution is automatically financially fair, and one can verify that it is also Pareto efficient under only the global budget constraint (Arrow, 1951; Xia, 2004). Consequently, when the market is complete there is no reason for agents to form a collective (other than reasons of convenience which we do not discuss here). Therefore, the complete markets are not our prime interest. However, when markets are incomplete, it may be beneficial for agents to trade among each other, in addition to trading at the market. On the other hand, when markets are incomplete, it is more difficult to address financial fairness, because the market does not fully specify a valuation operator.

A simple example may be helpful here. Suppose two neighbors both use a particular brand of chinaware. Suppose that one neighbor needs to replace a cup, while the other has suffered the loss of a saucer. At the shop, they find out that cups and saucers are only sold together, not separately. Knowing of each other's predicament, they decide to form a collective. Together, they purchase a set consisting of a cup and saucer; subsequently, the cup goes to one neighbor, and the saucer to the other. To make the agreement financially fair, the contribution of each participant to the collective should be equal in value to what that participant receives from the collective. This boils down to the question which part of the total cost of a cup and saucer should be borne by each of the two neighbors. Due to incompleteness, market prices by themselves do not provide the values of cups and saucers separately. In spite of this lack of information, the two neighbors may still find ways to arrive at the required valuation. For instance, they might negotiate between themselves in order to arrive at a common pricing rule, or they might agree to lay the matter in the hands of an expert who can make a statement about what market prices for cups and saucers would be if they would be traded on their own.

In this chapter, we consider a collective in which participants have agreed on a common pricing rule. We do not model the process by which they arrive at such a common rule; it may be based on negotiation, or on the advice of an authority. Instead, we simply take the common pricing rule as a primitive of the model. One of the main questions we are interested in is whether, once the pricing rule has been fixed, the requirements of Pareto efficiency and financial fairness (in the sense of the chosen rule) are sufficient to determine both the joint investment decision and the allocation rule uniquely. It is understood here that uniqueness is conditional on the chosen pricing rule, which, as noted before, is not fully objectively determined in an incomplete market.

We show that the approach from Chapter 2 can be used in this case as well. The iterative algorithm that we present here is quite similar to the one in Chapter 2, but it includes an additional optimization step in each cycle of the iteration. The optimization problem to be solved is of the convex type, and can be reformulated in such a way to take advantage of either the presence of few constraints (close-to-complete markets) or of many constraints (few degrees of freedom in the investment decision). The situation with respect to uniqueness is different, and we are not able to prove a uniqueness theorem at a level of generality similar to Corollary 2.22. We discuss the source of the difficulties, and we give several sufficient conditions under which uniqueness does hold.

The model of this chapter can be reinterpreted in terms of production economies. Instead of different future states, we then have different commodities that can be produced on the basis of inputs (such as labor) provided by a collective. The collective needs to take a decision both on the amounts of goods produced and on the allocation of these goods to participants. While in the financial interpretation of the model it is natural to work with linear budget constraints, these may be replaced by nonlinear constraints in the context of production. A model of this type has been considered by Gale and Machado (1982). These authors state that the solution is not unique in general, and they provide a proof of uniqueness under a condition which they call “separability of production sets”. The condition implies that production sets can be characterized by a single constraint. In the financial interpretation, this comes down to market completeness, the single constraint being the budget constraint. Below we provide (Proposition 3.10) a short proof of uniqueness in the complete market case, which differs from the one due to Gale and Machado (1982).

The approach of this chapter is different from the traditional approach in game theory where typically the collection of Pareto optimal solutions is restricted by coalitional stability, rather than by a notion of financial fairness. In fact, we do not take individual rationality into consideration. There is no guarantee that a Pareto efficient and financially fair solution is utility-improving for every agent in a given collective. Differently from traditional game theoretical approach we do not consider that agents can choose to participate or not in the collective. We consider that agents in the collective are fixed.

We work in the simple setting of a one-period model and a finite probability space. We consider collectives consisting of agents whose preferences can be described by expected utility, and we assume that the agents have common beliefs in the sense that they all assign the same probabilities to events. The utility functions, which may be different for different agents, will be assumed to satisfy a number of standard conditions. We assume that investments take place in a liquid financial market so that the possible investment decisions are parametrized by a subspace of the payoff space.

This chapter is organized as follows. Assumptions and notations are specified in Section 3.2. Section 3.3 presents the main results of this chapter. Special cases which can be solved analytically are mentioned in Section 3.4, and a numerical example is presented in Section 3.5. Conclusions follow in Section 3.6. Most proofs are relegated to the appendix.

3.2 Notation, assumptions, and definitions

3.2.1 Mathematical conventions

Notation and terminology relating to vector inequalities will be as follows. Inequalities are interpreted componentwise; in other words, the expression $a \geq b$ for $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ means that $a_i \geq b_i$ for all $i = 1, \ldots, n$, and $a > b$ means $a_i > b_i$ for all i. The expression $a \gneq b$ indicates that $a \geq b$ and $a \neq b$. Analogous notation will be used in the case of function spaces; when f and g are functions from a set D to $\mathbb{R}$, we write $f \geq g$ when $f(x) \geq g(x)$ for all $x \in D$, $f > g$ when $f(x) > g(x)$ for all $x \in D$, and $f \gneq g$ when $f \geq g$ while $f \neq g$.

A function f from (a subset of) $\mathbb{R}^n$ into $\mathbb{R}^m$ will be said to be *strictly increasing* if, for x and y in the domain of f, the inequality $x \gneq y$ implies that $f(x) \gneq f(y)$, and *strongly increasing* when $x \gneq y$ implies that $f(x) > f(y)$. A sequence of vectors $(a^k)_{k=1,2,\ldots}$ in $\mathbb{R}^n$ is said to *converge to infinity* if all scalar sequences $(a_i^k)_{k=1,2,\ldots}$ $(i = 1, \ldots, n)$ tend to infinity.

The *nonnegative orthant* and the *positive orthant* in $\mathbb{R}^n$ are defined by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n \mid x > 0\}$, respectively. A *cone* in a real vector space V is a subset of V that is closed under multiplication by nonnegative scalars. The term *positive cone* refers to the set $\mathbb{R}^n_{++} \cup \{0\}$. The *closed unit simplex* (or simply *unit simplex*) in $\mathbb{R}^n$ is the set $\{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$. The restriction of this set to the positive orthant is called the *open unit simplex*. We use the term "homogeneous" as shorthand for "positively homogeneous of degree 1"; that is, a function f defined on a cone in $\mathbb{R}^n$ is *homogeneous* if the equality $f(\lambda x) = \lambda f(x)$ holds for all $\lambda > 0$ and all x in the domain of f.

3.2.2 Economic assumptions

The risks in the economy that we consider are modeled as random variables on a finite outcome space $\Omega = \{\omega_1, \ldots, \omega_m\}$. The space $\mathbb{R}^\Omega$ of random variables defined on Ω (i.e. functions from Ω to $\mathbb{R}$) will correspondingly be written as $\mathbb{R}^m$. The probability of outcome ω_j, $j = 1, \ldots, m$, is denoted by $p_j > 0$, and the measure that is defined in this way is written as P. The expectation operator under this measure is denoted by E, as opposed to E^Q which is the expectation under the pricing measure to be introduced below. Elements of $\mathbb{R}^m$ will also be referred to as *assets* or *contingent claims*.

We consider a group of a finite number of agents, n, in the economy. Each agent is endowed with a utility function $u_i : (0, \infty) \to \mathbb{R}$ being strictly increasing, strictly concave, and twice continuously differentiable, and which moreover satisfies the following Inada conditions:

$$\lim_{x \downarrow 0} u_i'(x) = \infty, \qquad \lim_{x \to \infty} u_i'(x) = 0. \tag{3.1}$$

A common class of utility functions fulfilling these conditions is the class of power utility functions. Objective functions described by the utility functions are defined by

$$U_i(X) = \sum_{j=1}^{m} p_j u_i(X_j) = E[u_i(X)], \qquad X \in \mathbb{R}^m_{++}. \tag{3.2}$$

The inverse of the marginal utility function u_i' is denoted by I_i; this is a strictly decreasing and continuous function defined on $(0, \infty)$ with

$$\lim_{z \downarrow 0} I_i(z) = \infty, \qquad \lim_{z \to \infty} I_i(z) = 0.$$

A pricing functional is defined on the asset space $\mathbb{R}^m$ by expectation under a measure Q that is equivalent to measure P; we set the discount factor equal to unity without loss of generality. The Q-probability of an outcome $\omega_j \in \Omega$ is denoted by q_j, so that the expression

$$E^Q X = \sum_{j=1}^{m} q_j X_j$$

gives the price of an asset $X \in \mathbb{R}^m$.

We assume furthermore that a vector $X^{\text{in}} \in \mathbb{R}^m$ and a subspace $\mathcal{X}_0 \subset \{X \in \mathbb{R}^m \mid E^Q X = 0\}$ are given, which are interpreted as the *initial endowment of the group of agents* (collective risk) and the *space of zero-cost trades that are available to the group*, respectively. Under the assumption that the collective as a whole has agreed on prices of all assets in X, the space $\mathcal{X}_0$ is well-defined even in an incomplete market setting. By its investment decision, the collective modifies its position from X^{in} to $X^{\text{in}} + X$, with $X \in \mathcal{X}_0$. We write

$$\mathcal{X} := \{X^{\text{in}} + X \mid X \in \mathcal{X}_0\}. \tag{3.3}$$

The space of accessible trades $\mathcal{X}_0$ is a subspace of the space of zero-price assets $\{X \mid E^Q X = 0\}$; it is allowed, and actually it is the main situation of interest below, that the inclusion is strict, because the strict inclusion corresponds to an incomplete market setting. A standing assumption is that the initial endowment of the collective X^{in} and the space of accessible trades $\mathcal{X}_0$ are such that the shifted space $\mathcal{X} = \{X^{\text{in}}\} + \mathcal{X}_0$ has nonempty intersection with the positive orthant. In other words, on the basis of its initial endowment it is possible for the collective to take positions that produce positive payoffs in every state of nature. This condition is required because the utility functions that we use are defined only on the positive half line.

Finally we assume that for each agent i a number v_i is given, which is referred to as the *claim value* of the agent or the *value of the contribution* of that agent to the group. These numbers will be used to define the notion of financial fairness for risk sharing rules. The numbers v_i must satisfy the consistency requirement

$$\sum_{i=1}^{n} v_i = E^Q X^{\text{in}}. \tag{3.4}$$

Moreover, the numbers v_i should be positive for the allocation problem to be meaningful.

For the purpose of the design of risk sharing systems along the lines of this chapter, no further information concerning the agents is required. However, to be able to compare the outcomes of collective schemes to individual schemes, one needs to define initial endowments as well as accessible trading spaces for the individual agents. If $X_i^{\text{in}} \in \mathbb{R}^m$ denotes the initial endowment of agent i ($i = 1, \ldots, n$), then one may define

$$X^{\text{in}} = \sum_{i=1}^{n} X_i^{\text{in}} \quad \text{and} \quad v_i = E^Q X_i^{\text{in}}$$

so that the consistency requirement (3.4) is satisfied automatically.

3.2.3 Risk sharing schemes

A risk sharing scheme is a combination of a collective investment decision and a rule for the allocation of investment returns to the participants in the collective. In more formal terms, a risk sharing scheme is a tuple $(X; y)$ where $X \in \mathbb{R}^m$ and $y = (y_1, \ldots, y_n)$ is a vector of functions from $(0, \infty)$ to $(0, \infty)$. The corresponding *agent shares* are defined by $Y_i := y_i(X)$. The risk sharing scheme is said to be *feasible* if the vector X belongs to the constraint set $\mathcal{X}$ and is moreover positive, and the allocation functions $y_1, \ldots, y_n$ satisfy the redistribution property

$$\sum_{i=1}^{n} y_i(x) = x \quad \text{for all } x > 0. \tag{3.5}$$

Positivity of the position X taken by the collective is needed, because the agent's utilities are defined only on the positive halfline. If there would be a state of nature in which the payoff of the collective would be nonpositive, then no feasible allocation assigns positive payoffs to all agents.

Above, we have introduced the values v_i of agent's contributions. These are used below to define the notion of financial fairness. The definition below follows Bühlmann and Jewell (1979).

Definition 3.1 Let a pricing measure Q, a collective risk X, and values of agent's contributions v_i be given, with $E^Q X = \sum_{i=1}^n v_i$. An allocation rule $(y_1, \ldots, y_n)$ is *financially fair* if $E^Q[y_i(X)] = v_i$ for all $i = 1, \ldots, n$.

We work with a notion of Pareto efficiency that is constrained only by feasibility, not by financial fairness.

Definition 3.2 A feasible risk sharing scheme $(X; y)$ for n agents with objective functions U_i is *Pareto efficient* if there does not exist another feasible scheme $(\tilde{X}; \tilde{y})$ such that

$$\big[U_1\big(\tilde{y}_1(\tilde{X})\big), \ldots, U_n\big(\tilde{y}_n(\tilde{X})\big)\big] \gneq \big[U_1\big(y_1(X)\big), \ldots, U_n\big(y_n(X)\big)\big].$$

In this chapter we consider preferences to be given by expected utility as in (3.2). In (3.2), expectations are taken with respect to the objective probability measure P, whereas the financial fairness constraint involves the pricing measure Q.

Remark 3.3 Uniqueness of risk sharing schemes will always be understood in the sense of uniqueness with probability 1. In other words, two risk sharing schemes (X, y) and (X', y') (both for n agents) are considered to be the same if the random variables $Y_i := y_i(X)$ and $Y_i' := y_i'(X')$ are equal with probability 1 for all $i = 1, \ldots, n$.

3.3 Main result

In this section we establish, under some conditions, the existence of a unique Pareto efficient and financially fair (PEFF) risk sharing scheme. The result applies to situations in which the collective is to a certain extent free in choosing its risk exposure. The classical work of Borch (1962) is concerned with the situation in which a collective faces a *given* risk, such as when a number of insurance companies form a pool with respect to the claims that may be received in a given year. It was shown by Borch that, in this case, the collection of all Pareto optimal solutions for risk sharing between the agents can be parametrized in terms of positive parameters $\alpha_1, \ldots, \alpha_n$ by the prescription

$$y_i(x) = I_i\big(J(x; \alpha)/\alpha_i\big), \tag{3.6}$$

where the notation α is used to refer to the vector of parameters $(\alpha_1, \ldots, \alpha_n)$, and where, for $\alpha > 0$, the function $J(x; \alpha)$ is defined implicitly by

$$J(\,\cdot\,; \alpha) : x \mapsto z \quad \text{s.t.} \quad \sum_{i=1}^n I_i\big(z/\alpha_i\big) = x \qquad (x > 0). \tag{3.7}$$

As is easily verified, the equation (3.7) indeed uniquely defines $J(x;\alpha)$ for each given $x > 0$ and given $\alpha > 0$. The proof of Borch's result (see for instance DuMouchel (1968), Barrieu and Scandolo (2008) for details) is based on the fact that, under convexity assumptions, all points in the Pareto efficient set can be found by solving weighted-sum optimization problems of the form

$$\sum_{i=1}^{n} \alpha_i E\big[u_i\big(y_i(X)\big)\big] \to \max \quad \text{subject to } \sum_{i=1}^{n} y_i(X) = X, \tag{3.8}$$

where X is the given total risk. The restriction to positive (rather than nonnegative) coefficients α_i can be motivated by noting that the weighted-sum optimization problem that results from setting one or more (but not all) of the α_i's equal to zero does not give rise to boundary points of the set of achievable joint preference levels of the agents, since the levels of agents who are taken into account in the weighted sum can always be improved when the allocations to the agents with zero weight are made smaller.

When the total risk X is not fixed but can be chosen to a given extent by the collective, the idea of weighted-sum optimization still applies, but this time the optimization is carried out not only with respect to the allocation functions but also with respect to X. The optimal allocation functions for a given risk are provided by Borch's result; in fact, as noted by Borch, these functions do not depend on the distribution of the risk X because the problem (3.8) can be solved for each realization of X separately. As a consequence, the weighted-sum optimization problem can be formulated directly in terms of the risk X that is to be chosen by the collective:

$$\sum_{i=1}^{n} \alpha_i E\big[u_i\big(I_i(J(X)/\alpha_i)\big)\big] \to \max \quad \text{subject to } X \in \mathcal{X}, \tag{3.9}$$

where the set $\mathcal{X} \subset \mathbb{R}^m$ represents the positions that can be taken. Given the above, it is convenient to introduce the function

$$u(x;\alpha) = \sum_{i=1}^{n} \alpha_i u_i\big(I_i(J(x)/\alpha_i)\big). \tag{3.10}$$

Calculation shows (Xia, 2004) that in fact

$$u'(x;\alpha) = J(x;\alpha) \tag{3.11}$$

so that, if $u(x;\alpha)$ is interpreted as a weighted group utility, then $J(x;\alpha)$ is the corresponding weighted marginal group utility. It can be verified that the function $u(\,\cdot\,;\alpha)$ inherits all properties that were assumed for the utility functions u_i in Section 3.2.2. It then follows that the optimization problem in (3.9) has a unique solution. Consequently (Xia, 2004, Thm. 3.1), a risk sharing scheme $(\hat{X}; y)$ is Pareto efficient if and only if there exists a set of positive parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that

$$y_i(x) = I_i\big(J(x;\alpha)/\alpha_i\big) \quad (i = 1, \ldots, n) \tag{3.12a}$$

$$\hat{X} = \arg\max_{X \in \mathcal{X}} E[u(X;\alpha)], \tag{3.12b}$$

where the weighted group utility $u(x;\alpha)$ is defined in (3.10) and the constraint set $\mathcal{X}$ is given by (3.3). Financial fairness for the scheme $(\hat{X}; y)$ holds if

$$E^Q[y_i(\hat{X})] = v_i. \tag{3.12c}$$

It can be verified immediately that the solutions of (3.12a–3.12b) that correspond to $(\alpha_1, \ldots, \alpha_n)$ and $(\lambda\alpha_1, \ldots \lambda\alpha_n)$ are identical for any positive λ, so that we are free to impose the constraint

$$\sum_{i=1}^{n} \alpha_i = 1 \tag{3.13}$$

on parameter vectors. Our parameter set then becomes the open unit simplex.

In the analysis below, an important role is played by the mapping from the open unit simplex to $\mathbb{R}^m$ that assigns to a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ the random variable $J(\hat{X})$, where $\hat{X}$ is defined by the optimization problem (3.12b) and J is defined in (3.7). For a fixed random variable X, the mapping $\alpha \mapsto J(X;\alpha)$ is strictly (even strongly) increasing. This follows from the fact that the marginal utilities u_i', and hence also their inverses I_i, are strictly decreasing functions of their arguments. However, the same may not hold for the mapping $\alpha \mapsto J(\hat{X})$, since the location of the optimum $\hat{X}$ depends on the parameter vector α. An example of nonmonotonicity is shown in the appendix. Whether or not monotonicity holds depends on the specifications of the preferences of the agents, as well as on the restrictions that are imposed on the positions that may be taken by the collective. We state the monotonicity below as an assumption.

Assumption 3.4 The mapping $\alpha \mapsto J(\hat{X};\alpha)$, where J is defined by (3.7) and $\hat{X}$ by (3.12b), is strictly increasing.

One sufficient condition for this assumption to be satisfied is that the agents form a so called equicautious HARA collective; see the discussion in Section 3.4.3 below. While that is a condition in terms of the agents' utilities, the proposition below (which is proved in the appendix) gives a sufficient condition that is independent of the utilities and instead is stated in terms of the constraint set.

Proposition 3.5 *Assumption 3.4 is satisfied if there exists a random variable G defined on Ω such that the space of accessible zero-cost trades $\mathcal{X}_0$ is given by*

$$\mathcal{X}_0 = \{X \in \mathbb{R}^m \mid E^Q[X \mid G] = 0\}. \tag{3.14}$$

The condition appearing in (3.14) may be rewritten in a more explicit form as

$$\sum_{j \in S(g)} q_j X_j = 0 \quad \text{for all } g \text{ such that } P(G = g) > 0,$$

where

$$S(g) = \{j \mid G(\omega_j) = g\}.$$

The random variable G can be thought of as a device by which the outcomes ω_j are grouped into non-overlapping categories. Situations in which there indeed exists such a categorizer include the case of no choice ($\mathcal{X}_0 = \{0\}$; take $G = X$) and the complete-market case (take $G =$ constant). An example of a situation that does not correspond to one of these extremes is the following.

Example 3.6 Let Ω consist of four points, and assume that the space $\mathcal{X}_0$ is spanned by the vectors $[1, 1, -1, -1]$ and $[1, -1, 1, -1]$. This situation is obtained when the accessible assets are bets on the results of two Bernoulli trials, both with equal Q-probabilities of the two outcomes "heads" and "tails". The space Ω may then be described as $\{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$. Introduce a new random variable G by $G(\mathtt{HH}) = G(\mathtt{TT}) = 1$ and $G(\mathtt{HT}) = G(\mathtt{TH}) = 2$. The condition (3.14) is equivalent to the two conditions $X(\mathtt{HH}) + X(\mathtt{TT}) = 0$ and $X(\mathtt{HT}) + X(\mathtt{TH}) = 0$, which together characterize the space $\mathcal{X}_0$.

At this point we can state the main result of this chapter and give an outline of its proof. Details of the proof are provided in the appendix. For the reader's convenience, we recapitulate the setting. Our context is a single-period financial market, given by a finite probability space with objective probability measure P and pricing measure Q. A collective is formed by n agents whose characteristics are given by their utility functions u_i (of the type described in Section 3.2.2) and by the market values $v_i > 0$ of their contributions. A risk sharing scheme consists of a combination of a collective investment decision and a rule for allocation of the investment returns to the participants. The position that can be taken by the collective is constrained to lie in a given affine subset $\mathcal{X}$ of the asset space $\mathbb{R}^m$ such that $E^Q X = v := \sum_{i=1}^n v_i$ for all $X \in \mathcal{X}$. We are looking for risk sharing schemes that are both financially fair in the sense of Definition 3.1 and unconstrained Pareto efficient in the sense of Definition 3.2 (i.e. Pareto efficiency should hold even among schemes that are not financially fair). Our main result is as follows.

Theorem 3.7 *In the setting as described above, if Assumption 3.4 holds, then there exists a unique risk sharing scheme that is both Pareto efficient and financially fair.*

The proof is based on the parametrization of Pareto efficient solutions in terms of the unit simplex in $\mathbb{R}^n$. Each parameter vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{++}$ with $\sum_{i=1}^n \alpha_i = 1$ gives rise to a Pareto efficient risk sharing scheme through the specification (3.12). The resulting individual risks of the agents are given by $Y_i = y_i(\hat{X})$, where the function y_i and the random variable $\hat{X}$ are defined in (3.12a) and (3.12b), respectively. The corresponding weighted group marginal utility is $\hat{Z}(\alpha) := J(\hat{X}; \alpha)$, and we have $Y_i = I_i(\hat{Z}(\alpha)/\alpha_i)$. On the other hand, by the fact that the individual inverse marginal utilities I_i are strictly decreasing functions traversing all positive values, any given positive random variable Z defined on Ω gives rise to a uniquely determined vector $\hat{\alpha} \in \mathbb{R}^n_{++}$ through the equations

$$E^Q[I_i(Z/\hat{\alpha}_i)] = v_i \qquad (i = 1, \ldots, n). \tag{3.15}$$

The Pareto efficient risk sharing scheme associated to a given parameter vector α is financially fair if and only if the mapping $Z \mapsto \hat{\alpha}(Z)$ defined by (3.15) takes $\hat{Z}(\alpha)$ to α. The problem of finding a Pareto efficient and financially fair risk sharing scheme can therefore be phrased as a fixed-point problem for the composite mapping

$$\varphi : \alpha \mapsto \hat{Z} \mapsto \hat{\alpha}. \tag{3.16}$$

This fixed-point problem is addressed in the appendix. For convenience, the mapping $\alpha \mapsto \hat{Z}(\alpha)$ will be formulated as the composition of two separate mappings, so that the mapping φ is actually written as a composition of three mappings. The required fixed-point theorem will be obtained as a consequence of an eigenvalue theorem in nonlinear Perron-Frobenius theory due to Oshime (1983). To make application of this theorem possible, a number of properties of the mapping φ must be established; among other things, it needs to be shown that φ can be extended to a continuous mapping defined on the closed unit simplex, and that positive eigenvectors of the mapping φ can only occur with eigenvalue 1, so that a positive eigenvector of φ gives rise to a fixed point. These properties are shown in a series of lemmas which are relegated to the appendix.

The reformulation of the problem of finding a PEFF solution as a fixed-point problem naturally suggests the use of iteration as a numerical procedure. The mapping φ as defined above does not map the unit simplex into itself. However, the mapping always produces a strictly positive vector, so that we can construct a related mapping ψ that does map the unit simplex into itself by

$$\psi(\alpha) = \frac{\varphi(\alpha)}{\sum_{i=1}^{n}(\varphi(\alpha))_i}\,. \tag{3.17}$$

A vector α in the open unit simplex is a fixed point of ψ if and only if it is an eigenvector of the mapping φ.

Theorem 3.8 *In the same setting as in Theorem 3.7, if Assumption 3.4 holds, then the sequence $(\alpha^k)_{k=1,2,\dots}$ defined by $\alpha^{k+1} = \psi(\alpha^k)$ converges to the unique fixed point in the open unit simplex of the mapping φ defined in (3.16), for any choice of the initial point $\alpha^0 \in \mathbb{R}^n_{++}$.*

The theorem can be proved in the same way as Corollary 2.23, on the basis of the properties of the mapping φ as established in the appendix.

3.4 Special cases

This section is devoted to some special cases in which explicit solutions exist to the problem of finding a Pareto efficient and financially fair risk sharing scheme.

3.4.1 Identical preferences and identical contributions

One situation in which existence and uniqueness of the PEFF solution can be proved directly, without a monotonicity assumption, is the case of agents with identical preferences and identical contributions.

Proposition 3.9 *If a collective consists of n agents who all employ the same utility function $u_0(x)$ and whose contributions to the collective all have the same economic value w_0, then the risk sharing problem allows a unique Pareto efficient and financially fair solution. This solution is given by $(\hat{X}; y_1, \dots, y_n)$ with*

$$y_i(x) = x/n \quad (i = 1, \dots, n) \tag{3.18a}$$

$$\hat{X} = \arg\max_{X \in \mathcal{X}} E[u_0(X/n)], \tag{3.18b}$$

where $\mathcal{X}$ represents the set of positions available to the collective.

Proof. The parameter vector $\alpha = [1, \ldots, 1]$ leads to the solution (3.18) which is financially fair. To prove uniqueness, suppose that a parameter vector $(\alpha_1, \ldots, \alpha_n) > 0$ gives rise to a risk sharing scheme $(X; y)$ that is Pareto efficient and financially fair. Write $Z = J(X; \alpha)$, and let $I_0(\cdot)$ denote the inverse marginal utility corresponding to the utility function $u_0(x)$. The Pareto optimal allocation functions are given by (3.6). From the financial fairness constraint, we then obtain that $E^Q[I_0(J(Z)/\alpha_i)] = w_0$ for all $i = 1, \ldots, n$, so that in particular for all i and j we have

$$E^Q[I_0(J(Z)/\alpha_i)] = E^Q[I_0(J(Z)/\alpha_j)].$$

Since the inverse marginal utility function, $I_0(\cdot)$, is strictly decreasing, the function $\alpha \mapsto E^Q[I_0(J(Z)/\alpha)]$ is strictly monotonic. The equality above therefore implies that $\alpha_i = \alpha_j$, and we find that all entries of α must be equal. □

In the situation of the proposition above, the agents may still differ in their initial endowments; also, they might face different trading constraints. As a consequence, the utility gains (or losses) of the agents in a Pareto efficient and financially fair risk sharing scheme may be different as well. When agents enter a PEFF risk sharing scheme under the conditions of equal preferences and equal contributions, they effectively replace their initial endowment X_i^{in} by the average initial endowment $\frac{1}{n}\sum_{i=1}^n X_i^{\text{in}}$. This replacement can be beneficial to all participants when the collective is of suitable composition, as illustrated in the example in Section 3.5. When agents are heterogeneous in terms of preferences and/or contributions as well, the benefits of cooperation in a collective may even be larger.

3.4.2 Complete market

In case the collective faces a complete market, the PEFF solution can be implemented as follows. All agents turn over their wealth v_i to the collective and communicate their individual optimal investment plans; the collective then implements these plans and returns to the agents the proceeds that they would have received if they would have implemented their plans on their own. The following proposition shows that this is indeed the only PEFF solution. We use X_i^* to indicate the solution of the individual optimization problem of agent i in a complete market:

$$E[u_i(X)] \to \max \quad \text{subject to } E^Q X = v_i. \tag{3.19}$$

Due to the assumptions we have imposed on the utility functions u_i, the above problem indeed has a unique solution $X_i^* > 0$. Moreover, the Lagrange multiplier associated to the budget constraint is positive, so that the optimal utility $E[u_i(X_i^*)]$ is a strictly increasing function of the available wealth v_i.

Proposition 3.10 *In a complete market, there is a unique Pareto efficient and financially fair risk sharing scheme, which is given by the collective investment decision*

$$X^* = \sum_{i=1}^{n} X_i^*, \quad \textit{with } X_i^* = \operatorname*{arg\,max}_{X:\, E^Q X = v_i} E[u_i(X)], \tag{3.20}$$

and by the agent shares

$$Y_i = X_i^* \qquad (i = 1, \ldots, n). \tag{3.21}$$

Proof. The position X^* given by (3.20) satisfies $E^Q X^* = \sum_{i=1}^n E^Q X_i^* = \sum_{i=1}^n v_i = v$, and therefore, since the market is complete, this is a feasible position. The allocation rules are also obviously financially fair. The Pareto efficiency of the scheme can be proved as in Xia (2004). To show the uniqueness of the PEFF solution, consider any risk sharing scheme $(X'; y')$ that is Pareto efficient and financially fair, and write $Y_i' = y_i'(X')$. Due to the financial fairness constraint, we must have $E[u_i(Y_i')] \leq E[u_i(X_i^*)]$ for all i. Because the scheme $(X'; y')$ is assumed to be Pareto efficient, in fact equality must hold. This implies that $Y_i' = X_i^*$ for all i, since the maximization problem in (3.20) has a unique solution. □

If there are no access restrictions for agents, so that the agents operate in the same complete market as the collective does, then the role of the collective is merely administrative. The collective can provide gains to the participants only by effects that are not modeled in this chapter, such as economies of scale.

Remark 3.11 The rule (3.21) does not specify the allocation functions explicitly. For such functions to be obtainable from (3.21), it is required that, for all $i = 1, \ldots, n$, the equality $X_i^*(\omega) = X_i^*(\omega')$ holds whenever the outcomes ω and ω' are such that $X^*(\omega) = X^*(\omega')$. Satisfaction of this requirement follows from the comonotonicity property of optimal portfolios. The optimal position of agent i is given by $X_{i,j}^* = I_i(\mu_i \theta_j)$ $(j = 1, \ldots, m)$, where $\mu_i > 0$ is a Lagrange multiplier chosen such that the individual budget constraint of agent i is satisfied, and θ_j is defined by $\theta_j = q_j / p_j$. If we define a function $\tilde{I} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ by $\tilde{I}(\theta) = \sum_{i=1}^n I_i(\mu_i \theta)$, then $\tilde{I}$ is strictly decreasing, and the allocation functions are given by $y_i(x) = I_i(\mu_i \tilde{I}^{-1}(x))$ $(i = 1, \ldots, n)$.

3.4.3 Equicautious HARA collectives

It is well known that equicautious HARA collectives enjoy special properties (see, for instance, Section 2.6).

Lemma 3.12 *Suppose that $u_1, \dots, u_n$ are utility functions of the HARA class, and write $-u_i'(x)/u_i''(x) = \sigma_i x + \tau_i$. If all coefficients σ_i are the same, say $\sigma_i = \sigma$ for all i, then, for any set of weights $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{++}$, the group utility function $u(x)$ defined in (3.10) is in the HARA class as well. Specifically, we have $-u'(x)/u''(x) = \sigma x + \tau$ where $\tau = \sum_{i=1}^n \tau_i$.*

In particular the lemma shows that, in the case of an equicautious HARA collective, the group risk tolerance does not depend on the weights α_i. Since the group risk tolerance defines the group utility up to equivalence of utility functions, it follows that the optimal decision $\hat{X}$ given by (3.12b) does not depend on $(\alpha_1, \dots, \alpha_n)$. From the point of view of allocation, the situation is then the same as when the risk is given exogenously. In this situation, the mapping $(\alpha_1, \dots, \alpha_n) \mapsto J(\hat{X})$ is strictly increasing, as already noted in the paragraph preceding the statement of Assumption 3.4. It follows that Assumption 3.4 is satisfied for equicautious HARA collectives, so that uniqueness holds in this case.

A utility function that is defined on the positive halfline, as required in this chapter, belongs to the HARA class if and only if it belongs to the CRRA class (constant relative risk aversion); these are the power utilities and logarithmic utility. An equicautious CRRA collective consists of agents who all have the same constant coefficient of risk aversion, so essentially all members have the same utility function. The group utility is also the same. The allocation rule in the unique Pareto efficient and financially fair risk sharing scheme for an equicautious CRRA collective is

$$y_i(x) = \frac{v_i}{E^Q X}\, x \qquad (i = 1, \dots, n).$$

Therefore this is a case in which the combination of Pareto efficiency and financial fairness leads to the simple ex-post proportional division rule. Moreover, the investment decision that is taken by the collective is the same as the one that would be taken by an individual who has the same coefficient of relative risk aversion as all participants, and whose capital is equal to the sum of the contributions of all participants.

3.5 Example

We present a simple example, which illustrates the notions of Pareto efficiency and financially fair risk sharing in complete and incomplete markets in the context of a coin tossing experiment. We compare the levels of utility that can be reached by the agents when they operate on their own and when they form a collective, under various assumptions on the restrictions to which their bets are subject. The example may be viewed as a highly stylized representation of the situation of two generations that take part in a collective pension fund.

Consider a situation in which two Bernoulli trials will take place, and in which there are two agents who are allowed to place bets. The two possible outcomes of the trials are denoted by $\mathtt{H}$ (heads) and $\mathtt{T}$ (tails), so that the outcome space is $\{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$. It will be assumed that all outcomes have equal Q-probabilities, or in other words, the state prices of all outcomes are the same. We shall consider a number of different settings, which are distinguished by the constraints that are placed on the positions that can be taken by the agents. First consider the "autarky" settings, in which the agents operate on their own.

A1 Agent A can only bet on the outcome of Coin 1, and agent B can only bet on the outcome of Coin 2. In this setting, both agents have only one degree of freedom. Specifically, agent A has access to the zero-price payoff given by the vector $[1,\ 1,\ -1,\ -1]$, and agent B has access to the zero-price payoff given by $[1,\ -1,\ 1,\ -1]$.

A2 Both agents can bet on both coins. The agents now each have access to the two-dimensional space of zero-price payoff vectors generated by $[1,\ 1,\ -1,\ -1]$ and $[1,\ -1,\ 1,\ -1]$.

A3 Both agents have access to the full (three-dimensional) set of payoffs that have zero price. For instance, this means that they may place bets on "first coin heads and second coin tails" with payoff vector $[-1,\ 3,\ -1,\ -1]$, which would not be possible in setting A2.

Next we proceed to the settings in which the agents may form a collective. We consider situations in which the agents are subject to financial fairness, as well as situations in which this constraint is not imposed.

C2F The collective has access to both coins, as in setting A2 for the individual agents, and can implement any financially fair division rule.

C2U The collective has access to both coins and can use any division rule.

C3F The collective has access to the full space of zero-price payoffs, as in setting A3, and can use any financially fair division rule.

C3U The collective has access to the full space of zero-price payoffs and can use any division rule.

All of these settings correspond to certain subspaces of the eight-dimensional space of vectors along which the initial endowments of the two agents can be modified. In the autarky settings, these subspaces are just the products of the corresponding subspaces for the agents individually, and consequently the subspaces corresponding to settings A1, A2, and A3 have dimensions 2, 4, and 6, respectively. The subspaces associated to the setting A3 and the setting C3F coincide; this reflects the fact that when agents individually have access to a complete market, they can by themselves construct any position that can be obtained from a collective implementing financially fair risk sharing (see Section 3.4.2). The subspace associated to setting C2F however has dimension 5 and is strictly larger than the space associated to A2. The subspace associated to C2U has dimension 6 just like the subspace associated to A3 and C3F, but these subspaces do not coincide. Finally, the subspace associated to C3U is subject only to the overall budget constraint; it has dimension 7 and contains all lower-dimensional subspaces mentioned above.

Given objective functions of the agents, any point in the product of the payoff spaces of the two agents gives rise to a point in the two-dimensional space of preference levels of the two agents. The collection of these points for any of the settings specified above generates a Pareto set (set of points for which no strict Pareto improvement exists within the specified setting). The inclusion relations discussed above give rise to certain relations among these sets which are indicated schematically in the left panel of Figure 3.1. If the agents' preferences are of the form as assumed in Section 3.2.2, it follows from our main result and Example 3.6 that there exist unique PEFF solutions both in the incomplete-market setting C2 and in the complete-market setting C3. In the diagram, these solutions are indicated by circles marked PEFF2 and PEFF3. The autarky settings give rise to solutions in which both agents select their optimal policy individually; these solutions appear as corner points of the corresponding Pareto sets. The autarky solution A3 coincides with PEFF3.

In order to get an impression of the actual location of various Pareto curves in a specific case, we assume the following. The real-world probability of outcome `H` is taken to be equal to 0.6 in both experiments, so that there is an incentive for both agents to place bets even when they are risk averse, since the risk-neutral probability of outcome `H` is 0.5. The agents' preferences are described by power utility $u_i(x) = x^{1-\gamma_i}/(1-\gamma_i)$, $i = 1, 2$, with relative risk aversion coefficients $\gamma_1 = \gamma_2 = 2$. The initial allocations of the agents are given by $X_1^{\text{in}} = [2, 1, 0.5, 1]$ and $X_2^{\text{in}} = [0.5, 1, 2, 1]$, so that the values of the contributions of the

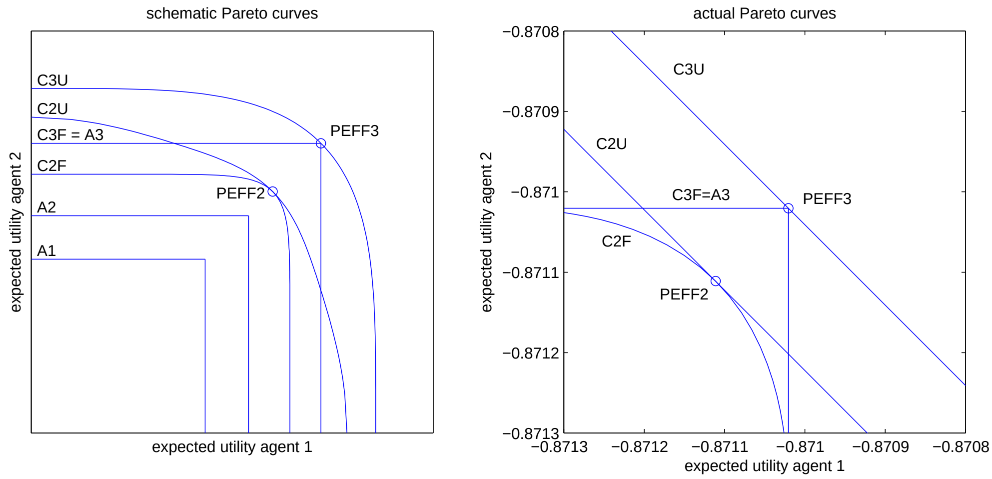

Figure 3.1: Schematic and actual Pareto curves for the coin toss example. PEFF solutions are indicated by circles. In the right panel, the scale is chosen such that the distinctions between the C2U and C3U curves and between the C2F and C3F/A3 curves are visible; as a consequence, only small parts of these curves are shown, and the Pareto curves A1 and A2 are outside the domain of the plot.

two agents are the same: $v_1 = v_2 = 1.125$. Table 3.1 shows the optimal allocations for both agents in the three autarky settings as well as the two PEFF solutions corresponding to the settings C2F and C3F. In both PEFF solutions, the proceeds are divided equally between the two agents in all states of the world, as it should be the case according to Proposition 3.9. In the example, it turns out that both the allocations and the utility gains for both agents in the PEFF solution under incomplete accessibility are only slightly below the ones obtained from the individually optimal complete-market solutions. The proximity of the incomplete-market PEFF solution to the complete-market autarky solution is also demonstrated in the right panel of Figure 3.1. By taking part in a collective that follows the PEFF rule, both agents in the C2 setting realize a utility gain that is almost the same as the one they would realize in a complete-market situation.

3.6 Conclusion

In this chapter, we have used the notions of Pareto efficiency and financially fairness to arrive at a particular solution of the joint investment and allocation problem. We have given conditions under which the solution is ensured to be unique. An iterative algorithm has been provided, which reduces the computation of the PEFF solution to a series of

	agent	HH	HT	TH	TT	premium
initial	1	2.0000	1.0000	0.5000	1.0000	–
	2	0.5000	1.0000	2.0000	1.0000	–
A1	1	1.7559	0.7559	0.7442	1.2441	6.52
	2	0.7435	0.7565	2.2435	0.7565	6.87
A2	1	1.8000	0.7500	0.7500	1.2000	6.53
	2	0.9000	1.5000	1.5000	0.6000	15.78
PEFF2	1	1.3500	1.1250	1.1250	0.9000	17.74
	2	1.3500	1.1250	1.1250	0.9000	29.48
PEFF3/A3	1	1.3638	1.1135	1.1135	0.9092	17.75
	2	1.3638	1.1135	1.1135	0.9092	29.49

Table 3.1: The final holdings of the agents in the coin toss example with different spaces of accessible trades. The term “premium” refers to the percentage of initial wealth that an agent would be maximally willing to give up in order to take part in a given risk sharing scheme. The solutions given in the autarky settings are the ones that are individually optimal for both agents.

relatively simple steps. Even when there is no guarantee of uniqueness, the algorithm can still be used to compute candidate solutions. It has been shown in an example that, in an incomplete market, agents may achieve substantial utility improvements by taking part in a collective, even while strictly adhering to financial fairness.

A basic assumption in this chapter is that a pricing functional is fixed, by negotiation or by authority. We have also taken the composition of the collective as fixed; we do not consider, for instance, different collectives that might be formed within a group of agents. Our focus has been on the existence and uniqueness of the PEFF solution for a given collective that uses a given price functional, and on the computation of solutions. It may turn out that the PEFF solution is not utility improving for some agents, or that a subgroup from the collective can do better by forming a collective of its own. Issues of such a nature have been discussed extensively in cooperative game theory. However, typically no financial fairness constraint has been applied in game-theoretic studies, so that there may be room for further investigation.

Several additional possible directions of future research can be mentioned. Technically, the analysis in this chapter is restricted to discrete probability spaces. A further step would be to formulate and solve problems combining Pareto efficiency and financial fairness for continuous and possibly unbounded random variables. A multiperiod version of the PEFF problem for given risks has been studied in Bao et al. (2017). It would be of interest to add the investment decision also in this context. We have worked in this chapter with agents who satisfy the von Neumann-Morgenstern axioms, so that their preferences can be described in terms of expected utility with respect to an objective probability measure. Of course this is a limitation, and one might consider collectives of agents whose preferences are not necessarily of this type. Another topic of possible interest is to investigate in what sense, if any, the investment decision of a collective following the PEFF rule can be viewed as representing preferences that are an aggregate of the preferences of the participants. Finally, Assumption 3.4 (monotonicity) presents a challenge. The assumption seems quite far from being necessary for existence and uniqueness of PEFF solutions, but on the other hand it appears that a rather different mathematical approach would be needed to avoid this assumption altogether.

3.7 Appendix

We start by introducing a notation for what might be called the "weighted inverse marginal group utility", and noting a few elementary properties of this function. Let $\mathcal{L}'$ denote the set of all continuous and strictly decreasing functions $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that

$$\lim_{x \downarrow 0} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = 0.$$

The notation J used below is in line with the definition in (3.7).

Lemma 3.13 *For any $\alpha \in \mathbb{R}^n_+ \setminus \{0\}$, the function I defined on $\mathbb{R}_{++}$ by*

$$I(z) := \sum_{i:\alpha_i \neq 0} I_i(z/\alpha_i) \qquad (z > 0) \tag{3.22}$$

is invertible. The inverse $J = I^{-1}$ belongs to $\mathcal{L}'$.

Proof. For given $\alpha \in \mathbb{R}^n_+ \setminus \{0\}$, the functions $I_i(\cdot/\alpha_i)$ in (3.22) are strictly decreasing and continuous on $(0, \infty)$. Therefore, their sum I is strictly decreasing and continuous as well, so that its inverse J exists. Clearly, J is strictly decreasing and continuous. The Inada conditions (3.1) imply that $\lim_{z\to\infty} I(z) = 0$ and $\lim_{z\downarrow 0} I(z) = \infty$. It follows that $\lim_{x\downarrow 0} J(x) = \infty$ and $\lim_{x\to\infty} J(x) = 0$, i.e., $J \in \mathcal{L}'$. □

Instead of $I(z)$ and $J(x)$ we sometimes also write $I(z; \alpha)$ and $J(x; \alpha)$ in order to stress the dependence on the parameter vector α.

3.7.1 Proof of Proposition 3.5

Since the random variable G is defined on a finite probability space, it can only take finitely many values, say $g_1, \dots, g_\ell$. For $k = 1, \dots, \ell$, write $S_k = \{j \mid G(\omega_j) = g_k\}$. The sets $S_1, \dots, S_\ell$ form a partition of the index set $\{1, \dots, m\}$. The Lagrangian function associated to the constrained optimization problem (3.9) can be written as

$$L(X_1, \dots, X_m, \lambda_1, \dots, \lambda_\ell) = \sum_{j=1}^{m} p_j u(X_j; \alpha) + \sum_{k=1}^{\ell} \lambda_k \sum_{j \in S_k} q_j (X_j^{\text{in}} - X_j)$$

where the function $u(x; \alpha)$ is defined in (3.10), and where X_j ($j = 1, \dots, m$) represents the outcome of total risk in future state j. Differentiation with respect to X_j leads to the conditions $p_j u'(X_j; \alpha) = \lambda_k q_j$ ($j \in S_k$, $k = 1, \dots, \ell$). It follows that the location of the optimum is given by

$$\hat{X}_j = I(\lambda_k q_j / p_j; \alpha) \qquad (j \in S_k,\ k = 1, \dots, \ell) \tag{3.23}$$

where $I(x; \alpha)$ is defined in (3.22). The Lagrange multipliers λ_k are determined by the constraints $\sum_{j \in S_k} q_j (X_j^{\text{in}} - \hat{X}_j) = 0$ ($k = 1, \dots, \ell$). These constraints may be written as

$$\sum_{j \in S_k} q_j I(\lambda_k q_j / p_j\,; \alpha) = \sum_{j \in S_k} q_j X_j^{\text{in}} \qquad (k = 1, \dots, \ell). \tag{3.24}$$

The expression that appears on the left-hand side of (3.24) is strictly decreasing as a function of λ_k and strictly increasing as a function of any of the weights α_i. Since the right-hand side of (3.24) does not depend on either the variable λ_k or the variables α_i, it follows that an increase of any of the variables α_i must be accompanied by an increase of λ_k in order to maintain equality in (3.24). In other words, the function $\alpha \mapsto \lambda_k$ defined implicitly by (3.24) is strictly increasing. Since $(J(\hat{X}))_j = J(\hat{X}_j) = \lambda_k q_j / p_j$ by (3.23), the statement of the proposition follows.

3.7.2 Mapping used in fixed-point theorem

The proof of our main result, Theorem 3.7, is based on showing existence and uniqueness of the fixed point of a particular mapping φ from the nonnegative cone into itself. This mapping is constructed as the composition of three mappings, as follows:

$$\mathbb{R}_+^n \xrightarrow{\varphi_1} \mathcal{L} \xrightarrow{\varphi_2} \mathbb{R}_{++}^m \cup \{0\} \xrightarrow{\varphi_3} \mathbb{R}_+^n.$$

Here, $\mathcal{L}$ is the cone defined by $\mathcal{L} = \mathcal{L}' \cup \{0\}$. Below we make use of $\mathcal{L}$ equipped with the topology of pointwise convergence.

Our aim is to show that the composite mapping φ satisfies the conditions for the existence of a unique positive eigenvector as formulated by Oshime (1983), and that the corresponding eigenvalue is equal to 1. In the following three subsections, the three mappings are defined and it is proved that they satisfy certain continuity and monotonicity properties. On the basis of this, we then show that the mapping φ indeed has the required properties.

First component φ_1

On the basis of Lemma 3.13, we can define a mapping φ_1 from $\mathbb{R}^n_+$ to $\mathcal{L}$ by

$$\varphi_1(\alpha) = \begin{cases} J \text{ as defined in Lemma 3.13} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0. \end{cases} \tag{3.25}$$

Given that J is defined as the inverse function of the mapping I which appears in (3.22), and also given that $\lim_{\alpha \downarrow 0} I(z; \alpha) = 0$ for all $z > 0$, the definition $\varphi_1(0) = 0$ in (3.25) can be viewed as corresponding to the convention that the inverse function of 0 is 0. While this might not seem intuitive at first, the convention becomes natural if we think of functions in $\mathcal{L}'$ in terms of their graphs as subsets of the extended nonnegative quadrant $[0, \infty]^2$. In this point of view, the function denoted by 0 in fact represents the union of the two nonnegative semi-axes $\big(\{0\} \times [0, \infty]\big) \cup \big([0, \infty] \times \{0\}\big)$, and the customary generalization of the notion of inverse function to multivalued mappings leads to the statement $0^{-1} = 0$. We now note some properties of the mapping φ_1 that will be needed below.

Lemma 3.14 *The mapping φ_1 is a continuous and homogeneous mapping from $\mathbb{R}^n_+$ to $\mathcal{L}$.*

Proof. For the first statement, see Lemma 2.15. Homogeneity is immediate. □

Lemma 3.15 *If $(\alpha^k)_k$ is a sequence in $\mathbb{R}^n_+$ such that there is an index i' with $\alpha^k_{i'} \to \infty$, then $\varphi_1(\alpha^k) \to \infty$.*

Proof. Let $(\alpha^k)_k$ be a sequence as in the statement of the lemma, and write $J^k = \varphi_1(\alpha^k)$. Take $x > 0$. By definition, we have

$$\sum_{i=1}^{n} I_i(J^k(x)/\alpha^k_i) = x. \tag{3.26}$$

If the sequence $(J^k(x))_k$ would have a bounded subsequence, then the corresponding subsequence of $(I_{i'}(J^k(x)/\alpha^k_{i'}))_k$ would tend to infinity. Since all terms on the left-hand side of (3.26) are positive, this leads to a contradiction. It follows that $J^k(x) \to \infty$, and since this holds for all $x > 0$, we obtain $J^k \to \infty$. □

Second component φ_2

When $u : (0, \infty) \to \mathbb{R}$ is a utility function of the type described in Section 3.2.2, then its derivative u' belongs to $\mathcal{L}'$. Given also the subspace $\mathcal{X}_0$ of accessible zero-cost trades and an initial endowment X^{in} such that $\big(\{X^{\mathrm{in}}\} + \mathcal{X}_0\big) \cap \mathbb{R}^m_{++} \neq \emptyset$, we can formulate the optimization problem

$$E[u(X)] \to \max \quad \text{subject to } X \in \mathcal{X} \cap \mathbb{R}^m_{++}, \tag{3.27}$$

where $\mathcal{X} := \{X^{\mathrm{in}}\} + \mathcal{X}_0$. A vector $\hat{X} \in \mathbb{R}^m_{++}$ is a solution of this optimization problem if and only if the conditions

$$\hat{X} \in \mathcal{X}, \quad J(\hat{X}) \in \mathcal{X}_0^{\perp} \tag{3.28}$$

hold, where $J := u'$ is the marginal utility and where orthogonality is taken in the sense of the inner product defined by expectation under P, i.e.

$$\mathcal{X}_0^{\perp} = \{Z \mid E[X_0 Z] = 0 \text{ for all } X_0 \in \mathcal{X}_0\}.$$

Lemma 3.16 *For $J \in \mathcal{L}'$, the equations (3.28) admit a unique solution $\hat{X} \in \mathbb{R}^m_{++}$.*

Proof. Let $u : (0, \infty) \to \mathbb{R}$ be such that $u' = J$. The function u is increasing, strictly concave, and can be extended continuously to a function from $[0, \infty)$ to $\mathbb{R} \cup \{-\infty\}$ which will still be denoted by u. The function $X \mapsto E[u(X)]$ is continuous, bounded from above on the compact set $\mathcal{X} \cap \mathbb{R}^m_+$, and strictly concave. Therefore there exists a unique point $\hat{X}$ in $\mathcal{X} \cap \mathbb{R}^m_+$ at which the function takes its maximum. The set $\mathcal{X} \cap \mathbb{R}^m_+$ is convex and has nonempty relative interior by assumption. Take $X^1 \in \mathcal{X} \cap \mathbb{R}^m_{++}$, and define the function $\tilde{u} : [0, 1] \to [-\infty, \infty)$ by

$$\tilde{u}(\lambda) = E[u\big((1 - \lambda)\hat{X} + \lambda X^1\big)].$$

Given that the function $E[u(X)]$ is maximized at $X = \hat{X}$, the function $\tilde{u}(\lambda)$ takes its maximum at $\lambda = 0$. The derivative of $\tilde{u}(\lambda)$ on the open interval $(0, 1)$ is given by

$$\tilde{u}'(\lambda) = \sum_{j=1}^{m} p_j J\big((1 - \lambda)\hat{X}_j + \lambda X^1_j\big)(X^1_j - \hat{X}_j).$$

For indices j such that $\hat{X}_j = 0$, we have $X_j^1 - \hat{X}_j = X_j^1 > 0$, so that the corresponding term in the summation at the right-hand side of the equation above tends to ∞ as λ tends to 0. In the case of indices j such that $\hat{X}_j \neq 0$, the corresponding term in the summation tends to a finite limit. Therefore, if there are any indices j such that $\hat{X}_j = 0$, then $\lim_{\lambda \downarrow 0} \tilde{u}'(\lambda) = \infty$, which is not possible given that $\tilde{u}$ takes its maximum at 0. It follows that all entries of $\hat{X}$ are strictly positive, and consequently the vector $\hat{X}$ satisfies the necessary conditions (3.28). From the strict concavity of the mapping $X \mapsto E[u(X)]$ it follows that these conditions are sufficient as well, so that the solution is unique. □

On the basis of the lemma above, we can define a mapping $\varphi_2 : \mathcal{L} \to \mathbb{R}^m_{++} \cup \{0\}$ as follows:

$$\varphi_2(J) = \begin{cases} J(\hat{X}) \text{ where } \hat{X} \text{ is defined by (3.28)} & \text{if } J \in \mathcal{L}' \\ 0 & \text{if } J = 0. \end{cases} \tag{3.29}$$

The question of continuity of the mapping φ_2 leads to the problem of showing that the solution of a portfolio optimization problem depends continuously on the preference specification. This problem has been studied in the literature in various contexts; a discrete-time formulation has been used by Carassus and Rasonyi (2007), while Jouini and Napp (2004) have obtained results in continuous time. Here we cannot follow the work of Carassus and Rasonyi entirely since they work with utility functions that are defined on the whole real line, but part of the proof of Lemma 3.18 below makes use of ideas in their paper. First we show an auxiliary result.

Lemma 3.17 *If $(J^k)_k$ is a sequence of functions in $\mathcal{L}'$ that converges pointwise to a function $J \in \mathcal{L}$, then the sequence $(J^k(\hat{X}^k))_k$ is bounded, where $\hat{X}^k$ denotes the element of $\mathbb{R}^m_{++}$ associated to J^k via Lemma 3.16.*

Proof. All vectors $\hat{X}^k$ belong to the bounded set $\mathcal{X}_0 \cap \mathbb{R}^m_{++}$, so that it is sufficient to conduct the proof under the assumption that the sequence $(\hat{X}^k)_k$ converges to a limit, say $\hat{X}^\infty$. Then $\hat{X}^\infty \in \mathcal{X}_0 \cap \mathbb{R}^m_+$. Because $J^k(\hat{X}^k) \in \mathcal{X}_0^\perp$ for all k, we have in particular

$$\sum_{j=1}^m J^k(\hat{X}_j^k)(\hat{X}_j^k - X_j^{\text{in}}) = 0 \tag{3.30}$$

for all k. Moreover, we may assume (shifting X^{in} by an element of $\mathcal{X}_0$ if necessary) that all components of X^{in} are positive. Define index sets S_+ and S_- by

$$S_+ = \{j \mid \hat{X}_j^\infty \geq X_j^{\text{in}}\}, \quad S_- = \{j \mid \hat{X}_j^\infty < X_j^{\text{in}}\}.$$

From (3.30) we have

$$\sum_{j\in S_+} J^k(\hat{X}_j^k)(\hat{X}_j^k - X_j^{\text{in}}) = \sum_{j\in S_-} J^k(\hat{X}_j^k)(X_j^{\text{in}} - \hat{X}_j^k). \tag{3.31}$$

For $j \in S_+$, the inequality $\hat{X}_j^k \geq \frac{1}{2}X_j^{\text{in}}$ holds for all sufficiently large k, so that

$$J^k(\hat{X}_j^k) \leq J^k(\tfrac{1}{2}X_j^{\text{in}}) \overset{k\to\infty}{\longrightarrow} J(\tfrac{1}{2}X_j^{\text{in}}).$$

This shows that the sequences $\big(J^k(\hat{X}_j^k)\big)_k$ are bounded for $j \in S_+$. Given that $\lim_{k\to\infty} \hat{X}_j^k = \hat{X}_j^\infty$, it follows that the left-hand side of (3.31) is bounded as k tends to infinity. Consequently the right-hand side is bounded as well. Since the sequences $\big(X_j^{\text{in}} - \hat{X}_j^k\big)_k$ for $j \in S_-$ converge to limits that are strictly positive, this implies the boundedness of $\big(J^k(\hat{X}_j^k)\big)_k$ for $j \in S_-$. □

Lemma 3.18 *Let $(J^k)_k$ be a sequence of functions in $\mathcal{L}'$ that converges pointwise to a function $J \in \mathcal{L}'$, and let $\hat{X}^k$ ($k = 1, 2, \dots$) and $\hat{X}$ be the elements of $\mathbb{R}^m_{++}$ that are associated to J^k and J, respectively, via Lemma 3.16. Under these conditions, we have $\lim_{k\to\infty} J^k(\hat{X}^k) = J(\hat{X})$.*

Proof. We first show that all limit points of the sequence $(\hat{X}^k)_k$ belong to the positive orthant $\mathbb{R}^m_{++}$. Let $\hat{X}^\infty$ denote such a limit point. Then there is a subsequence that converges to $\hat{X}^\infty$, which for convenience of notation will still be denoted by $(\hat{X}^k)_k$. All vectors $\hat{X}^k$ belong to $\mathbb{R}^m_{++}$, and therefore $\hat{X}^\infty \in \mathbb{R}^m_+$. Suppose that j' is such that $\hat{X}_{j'}^\infty = 0$. It follows from the previous lemma that the sequence $(J^k(\hat{X}_{j'}^k))_k$ must remain bounded as k tends to infinity. Let M be such that $J^k(\hat{X}_{j'}^k) \leq M$ for all k. Define $\bar{x} > 0$ by $J(2\bar{x}) = M$. Because the function J is strictly decreasing, we have $J(\bar{x}) > M$. The inequality $\hat{X}_{j'}^k < \bar{x}$ holds for all sufficiently large k, so that $J^k(\bar{x}) < J^k(X_{j'}^k) \leq M$ for all such k. But this implies that $\lim_{k\to\infty} J^k(\bar{x})$ is strictly less than $J(\bar{x})$, which contradicts the assumption that the sequence $(J^k)_k$ converges pointwise to J. Therefore the limit point $\hat{X}^\infty$ must be strictly positive.

We next show that the sequence $(\hat{X}^k)_k$ converges to $\hat{X}$. Since the set $\mathcal{X} \cap \mathbb{R}^m_{++}$ is bounded, it is enough to show that any limit point of the sequence $(\hat{X}^k)_k$ is equal to $\hat{X}$. Let $\hat{X}^\infty$ denote such a limit point. Passing to a subsequence if necessary, we can assume that $\lim_{k\to\infty} \hat{X}^k = \hat{X}^\infty$. A classical theorem of analysis (Buchanan and Hildebrandt, 1908) states that pointwise convergence of the monotonically decreasing functions J^k to J implies uniform convergence of the sequence $(J^k)_k$ to J on compacts. Let u and u^k be primitive functions of J and J^k, respectively; we may and will assume that $u(1) = u^k(1) = 0$ for all k. Under this condition, it is an easy exercise to show that the uniform convergence of $(J^k)_k$ to J on compacts implies the uniform convergence of $(u^k)_k$ to u on compacts. By definition, we have

$$E[u^k(\hat{X}^k)] \geq E[u^k(\hat{X})] \tag{3.32}$$

for all k. The estimate

$$\left|E[u^k(\hat{X}^k)] - E[u(\hat{X}^\infty)]\right| \leq \left|E[u^k(\hat{X}^k)] - E[u(\hat{X}^k)]\right| + \left|E[u(\hat{X}^k)] - E[u(\hat{X}^\infty)]\right|$$

shows, in combination with the fact that $\hat{X}^\infty \in \mathbb{R}^m_{++}$ and the uniform convergence of $(u^k)_k$ on compacts, that $\lim_{k\to\infty} E[u^k(\hat{X}^k)] = E[u(\hat{X}^\infty)]$. Since the right-hand side of the inequality (3.32) converges to $E[u(\hat{X})]$ as k tends to infinity, it follows that $E[u(\hat{X}^\infty)] = E[u(\hat{X})]$ and hence that $\hat{X}^\infty = \hat{X}$, by the uniqueness of the maximizer in the optimization problem defined by J.

Given that the sequence $(\hat{X}^k)_k$ converges to $\hat{X} \in \mathbb{R}^m_{++}$, the convergence of $(J^k(\hat{X}^k))_k$ to $J(\hat{X})$ is now immediate from the uniform convergence of $(J^k)_k$ to J on compacts. □

To complete the proof of sequential continuity of φ_2 on $\mathcal{L}$, we show the continuity at 0.

Lemma 3.19 *Let $(J^k)_k$ be a sequence of functions in $\mathcal{L}'$ converging pointwise to 0, and let $\hat{X}^k$ ($k = 1, 2, \dots$) be the elements of $\mathbb{R}^m_{++}$ that are associated to J^k via Lemma 3.16. Then $J^k(\hat{X}^k) \to 0$.*

Proof. Write $Z^k := J^k(\hat{X}^k)$. It follows from Lemma 3.17 that the sequence $(Z^k)_k$ is bounded, so that without loss of generality we may assume that the sequence tends to a limit. By restricting to a further subsequence if necessary, we may furthermore assume that the sequence $(\hat{X}^k)_k$ tends to a limit as well; let the limit be denoted by $\hat{X}^\infty$. Since $Z^k \in \mathcal{X}_0^\perp$ for all k, we have in particular

$$\sum_{j=1}^m p_j Z_j^k X_j^{\text{in}} = \sum_{j=1}^m p_j Z_j^k \hat{X}_j^k \qquad (k = 1, 2, \dots) \tag{3.33}$$

where, as before, we may assume that $X_j^{\text{in}} > 0$ for all j. For j such that $\hat{X}_j^\infty > 0$, we have $0 \leq Z_j^k = J^k(\hat{X}_j^k) \leq J^k(\frac{1}{2}\hat{X}_j^\infty)$ for all sufficiently large k, and since $\lim_{k\to\infty} J^k(\frac{1}{2}\hat{X}_j^\infty) = 0$ it follows that the term $Z_j^k \hat{X}_j^k$ tends to 0 as k tends to infinity. For j such that $\hat{X}_j^\infty = 0$, the term $Z_j^k \hat{X}_j^k$ tends to 0 as well, since the sequence $\left(Z^k\right)_k$ is bounded as shown in Lemma 3.17. Therefore the right-hand side of (3.33) tends to 0 as k tends to infinity. The left-hand side therefore converges to 0 as well; given that $p_j > 0$ and $X_j^{\text{in}} > 0$ for all j, this implies that $Z^k \to 0$. □

Finally we establish two simple properties of the mapping φ_2 that will be required below.

Lemma 3.20 *The mapping φ_2 is homogeneous.*

Proof. Multiplication of $J \in \mathcal{L}'$ by a positive scalar has no effect on the vector $\hat{X}$ defined by (3.28). Hence the mapping $J \mapsto J(\hat{X})$ is homogeneous for all $J \in \mathcal{L}'$. For $J = 0$, the statement holds trivially. □

Lemma 3.21 *Let $\left(J^k\right)_k$ be a sequence of functions in $\mathcal{L}$ with vectors $\hat{X}^k$ $(k = 1, 2, \dots)$ associated via Lemma 3.16. If $J^k(x) \to \infty$ for all $x > 0$, then $J^k(\hat{X}^k) \to \infty$.*

Proof. Let M be such that $\hat{X}_j^k \leq M$ for all k and all j. Then, for all j, $J^k(\hat{X}_j^k) \geq J^k(M) \to \infty$ as $k \to \infty$. □

Third component φ_3

Lemma 3.22 *Let $Z \in \mathbb{R}^m_{++}$ and let $v_i > 0$, for $i = 1, \dots, n$, be given. Then there exists a unique vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{++}$ such that*

$$E^Q\left[I_i(Z/\alpha_i)\right] = v_i \qquad (i = 1, \dots, n). \tag{3.34}$$

Proof. For each i, the mapping $\alpha_i \mapsto E^Q I_i(Z/\alpha_i)$ is strictly monotonic and continuous. From the Inada conditions (3.1), we also have $\lim_{\alpha_i \to \infty} E^Q[I_i(Z/\alpha_i)] = \infty$ and $\lim_{\alpha_i \downarrow 0} E^Q I_i(Z/\alpha_i) = 0$. Therefore, there exists a unique positive vector $\alpha = (\alpha_1, \dots, \alpha_n)$ that solves (3.34). □

The above lemma enables us to define a mapping φ_3 from $\mathbb{R}^m_{++} \cup \{0\}$ to $\mathbb{R}^n_{+}$ in the following way:

$$\varphi_3(Z) = \begin{cases} \alpha \text{ as defined by (3.34)} & \text{if } Z > 0 \\ 0 \in \mathbb{R}^n & \text{if } Z = 0. \end{cases} \tag{3.35}$$

Lemma 3.23 *The mapping φ_3 defined by (3.35) is a continuous mapping from $\mathbb{R}^m_{++} \cup \{0\}$ to R^n_+.*

Proof. Let $(Z^k)_k$ be a sequence in $\mathbb{R}^m_{++} \cup \{0\}$ converging to $\bar{Z} \in \mathbb{R}^m_{++} \cup \{0\}$. Write $\alpha^k = (\alpha^k_1, \ldots, \alpha^k_n) = \varphi_3(Z^k)$, and $\bar{\alpha} = (\bar{\alpha}_1, \ldots, \bar{\alpha}_n) = \varphi_3(\bar{Z})$. First consider the case in which $\bar{Z} = 0$, and take $i \in \{1, \ldots, n\}$. To avoid trivialities, we can assume that $Z^k > 0$ for all k. If there would be a subsequence of $(\alpha^k_i)_k$ that would be bounded away from 0, then the corresponding quantities $E^Q[I_i(Z^k/\alpha^k_i)]$ would tend to infinity given that $Z^k \to 0$, contradicting the requirement $E^Q[I_i(Z^k/\alpha^k_i)] = v_i$. It follows that $\lim_{k\to\infty} \alpha^k_i = 0$ for all i.

Now, consider the case in which $\bar{Z} > 0$. There is no loss in generality if we assume that $Z^k > 0$ for all k. Fix $i \in \{1, \ldots, n\}$. The numbers α^k_i and $\bar{\alpha}_i$ are defined uniquely by the equations $f_i(\alpha^k_i, Z^k) = v_i$ and $f_i(\bar{\alpha}_i, \bar{Z}) = v_i$, respectively, where f_i is the mapping from $\mathbb{R}_{++} \times \mathbb{R}^m_{++}$ defined by $f_i(\alpha, Z) = E^Q[I_i(Z/\alpha)]$. Each mapping f_i is continuous. Moreover, because the sequence $(Z^k)_k$ is bounded, the sequence $(\alpha^k_i)_k$ is bounded as well. Indeed, if there would be a subsequence converging to infinity, then the corresponding quantities $E^Q[I_i(Z^k/\alpha^k_i)]$ would converge to infinity also, which is a contradiction. We can therefore conclude, following Lemma 2.14, that $\lim_{k\to\infty} \alpha^k_i = \bar{\alpha}_i$. Since this holds for each i, it follows that $\lim_{k\to\infty} \alpha^k = \bar{\alpha}$. □

Lemma 3.24 *The mapping φ_3 is strongly increasing.*

Proof. Take Z^1 and Z^2 in $\mathbb{R}^m_{++} \cup \{0\}$ with $Z^2 \gneq Z^1$, and write $\alpha^i = \varphi_3(Z^i)$ $(i = 1, 2)$. If $Z^1 = 0$, then $\alpha^1 = 0 < \alpha^2$. Assume now that $Z^1 > 0$, and take $i \in \{1, \ldots, n\}$. Since the mapping $Z \mapsto E^Q[I_i(Z)]$ is strongly decreasing, it would follow from the supposition $\alpha^2_i \leq \alpha^1_i$ that $Z^2/\alpha^2_i \gneq Z^1/\alpha^1_i$ and hence

$$v_i = E^Q\big[I_i(Z^2/\alpha^2_i)\big] < E^Q\big[I_i(Z^1/\alpha^1_i)\big] = v_i,$$

which is a contradiction. It follows that $\alpha^2_i > \alpha^1_i$ for all i. □

Lemma 3.25 *If the sequence $(Z^k)_k$ in $\mathbb{R}^m_{++} \cup \{0\}$ tends to infinity, then the sequence $(\alpha^k)_k$ defined by $\alpha^k = \varphi_3(Z^k)$ tends to infinity as well.*

Proof. Suppose to the contrary that for some $i \in \{1, \ldots, n\}$ there are a subsequence $(\alpha^{k_j}_i)_j$ and a constant M such that $\alpha^{k_j}_i \leq M$ for all j. Then

$$v_i = E^Q[I_i(Z^{k_j}/\alpha^{k_j}_i)] \leq E^Q[I_i(Z^{k_j}/M)].$$

The right-hand side tends to 0 because Z^{k_j} tends to infinity, and so we arrive at a contradiction since $v_i > 0$. □

Lemma 3.26 *The mapping φ_3 is homogeneous.*

Proof. This is immediate from the definition. □

3.7.3 Proof of Theorem 3.7

We now define a mapping φ from the nonnegative cone to itself by

$$\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1, \tag{3.36}$$

where the components φ_1, φ_2, and φ_3 are given by (3.25), (3.29), and (3.35), respectively.

Lemma 3.27 *The mapping φ defined by (3.36) is continuous and homogeneous. Under Assumption 3.4, the mapping φ is strongly increasing on $\mathbb{R}^n_{++}$*

Proof. Continuity follows from Lemmas 3.14, 3.18, 3.19, and 3.23; homogeneity results from Lemmas 3.14, 3.20, and 3.26. From Assumption 3.4 and Lemma 3.24 it follows that φ is strongly increasing on $\mathbb{R}^n_{++}$. □

To allow the application of an appropriate fixed-point theorem, we verify an additional condition that acts as a substitute for the indecomposability condition of linear Perron-Frobenius theory. The definition below follows Oshime (1983).

Definition 3.28 A mapping φ from $\mathbb{R}^n_+$ into itself is *nonsectional* if, for every partition of the index set $\{1, \ldots, n\}$ into two nonempty subsets R and S, there exists $s \in S$ such that

(i) $(\varphi(x))_s > (\varphi(y))_s$ for all x, $y \in \mathbb{R}^n_+$ such that $x_R > y_R$ and $x_S = y_S > 0$;

(ii) $(\varphi(x^k))_s \to \infty$ for all sequences $(x^k)_k$ in $\mathbb{R}^n_+$ such that $x^k_R \to \infty$ while x^k_S is fixed and positive.

Lemma 3.29 *Under Assumption 3.4, the mapping φ defined by (3.36) is nonsectional.*

Proof. Let a nontrivial partition of the index set $\{1, \dots, n\}$ into complementary subsets R and S be given, and take any $s \in S$. Item (i) follows from the fact that φ is strongly increasing (Lemma 3.27). Item (ii) follows from Lemmas 3.15, 3.21, and 3.25. □

We will use the following nonlinear Perron-Frobenius theorem, due to Oshime (1983).

Theorem 3.30 (Oshime, 1983) *If a mapping φ from $\mathbb{R}^n_+$ into itself is continuous, monotonic, homogeneous of degree 1, and nonsectional, then the mapping φ has a positive eigenvector, which is unique up to scalar multiplication. In other words, there exist a constant $\lambda^* > 0$ and a vector $x^* \in \mathbb{R}^n_{++}$ such that $\varphi(x^*) = \lambda^* x^*$, and if $\lambda > 0$ and $x \in \mathbb{R}^n_{++}$ are such that $\varphi(x) = \lambda x$, then x is a scalar multiple of x^*.*

The theorem by itself does not prove that the mapping φ has a fixed point. For this, it needs to be shown that the corresponding eigenvalue is equal to 1. This is established in the lemma below.

Lemma 3.31 *Let φ be defined by (3.36). If $\lambda \in \mathbb{R}_{++}$ is such that $\varphi(\alpha) = \lambda\alpha$ for some $\alpha \in \mathbb{R}^n_{++}$, then $\lambda = 1$.*

Proof. Let $\alpha \in \mathbb{R}^n_{++}$ and $\lambda > 0$ be such that $\varphi(\alpha) = \lambda\alpha$. Write $J = \varphi_1(\alpha)$. Because $\varphi_3(J(\hat{X})) = \lambda\alpha$, we have

$$\sum_{i=1}^{n} E^Q[I_i(J(\hat{X})/(\lambda\alpha_i))] = \sum_{i=1}^{n} v_i =: v. \tag{3.37}$$

On the other hand, from the relation $\sum_{i=1}^n I_i(J(x)/\alpha_i) = x$ it follows that

$$\sum_{i=1}^{n} E^Q[I_i(J(\hat{X})/\alpha_i)] = E^Q\hat{X} = v. \tag{3.38}$$

Since the mapping $\lambda \mapsto E^Q[I_i(J(\hat{X})/(\lambda\alpha_i))]$ is strictly increasing, a comparison of (3.37) and (3.38) shows that $\lambda = 1$. □

This leads to the proof of our main result.

Proof of Theorem 3.7. By Thm. 3.1 in Xia (2004), the collection of all Pareto efficient risk sharing schemes $\big(\hat{X}; y_1(\cdot), \dots, y_n(\cdot)\big)$ is parametrized completely by the open unit simplex in $\mathbb{R}^n$ through the prescription (3.12). It was already argued in the main text, following the statement of Theorem 3.7, that PEFF schemes correspond exactly to those parameter vectors that are fixed points under the mapping φ. By Lemma 3.27 and Lemma 3.29, if Assumption 3.4 holds, then the mapping φ satisfies all conditions of Oshime's theorem, and consequently a unique positive eigenvector exists. Lemma 3.31 shows that the corresponding eigenvalue is equal to 1, and consequently the eigenvector gives rise to a fixed point of the mapping φ on the open unit simplex. Since conversely any fixed point of φ on the open simplex gives rise to an eigenvector, the uniqueness of the eigenvector implies the uniqueness of the fixed point. The statement of the theorem follows. □

The method of proof that we have followed naturally suggests a computational algorithm. Numerical implementation of the mappings φ_1 and φ_3 requires a routine to determine the root of a scalar monotonic and continuous function. The mapping φ_2 calls for the solution of an optimization problem, which can be more difficult; however, the problem is of convex type, so that there are no local optima. To solve the equations (3.28), the primary set of unknowns could be taken as X or as $J(X)$; the former is likely to be more convenient when the dimension of the space $\mathcal{X}_0$ is low (few degrees of freedom in the investment decision), whereas the latter may be preferable when the dimension of that space is high (close-to-complete market).

3.7.4 On the necessity of the monotonicity assumption

The following example demonstrates that the condition of Assumption 3.4 is not always satisfied. Let two agents be given with identical marginal utilities defined by

$$u_i'(x) = \begin{cases} \frac{1}{x} & \text{for } x \in (0, z_1), \\ \frac{z_2 - x}{z_2 - z_1}\frac{1}{z_1} + \frac{x - z_1}{z_2 - z_1}\frac{1}{z_1 + d} & \text{for } x \in [z_1, z_2), \\ \frac{1}{x - z_2 + z_1 + d} & \text{for } x \in [z_2, \infty) \end{cases} \tag{3.39}$$

($i = 1, 2$), with $z_1 = 0.05$, $z_2 = 2$, and $d = 0.01$. (Actually a small modification of the definition above would be needed to ensure continuity of the second derivative as required by our assumptions; we choose here to ignore this issue.) A plot of the marginal utility defined by (3.39) is shown in Figure 3.3. The agents also have identical initial endowments given by [1, 1, 1]. The objective probability measure is given by $[p_1,\ p_2,\ p_3] = [0.1,\ 0.1,\ 0.8]$, whereas the pricing measure is defined by $[q_1,\ q_2,\ q_3] = [0.5,\ 0.125,\ 0.375]$.

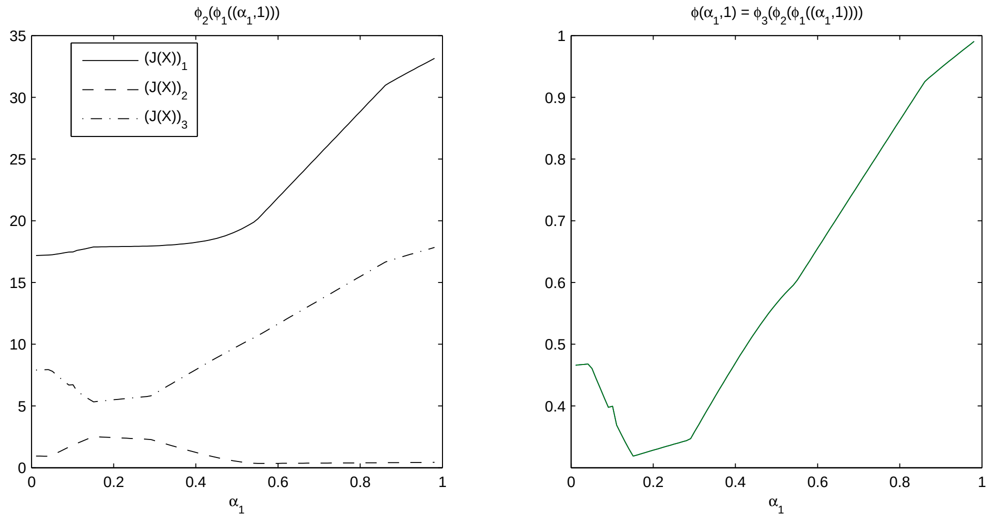

Figure 3.2: The panels show the three components of $J(\hat{X};\alpha) = \varphi_2(\varphi_1(\alpha))$ (left) and the two components of $\varphi(\alpha)$ (right) as a function of α_1, while α_2 is kept fixed at the value 1. The two components of $\varphi(\alpha)$ are identical to each other.

The space of accessible zero-cost trades is defined by

$$\mathcal{X}_0 = \ker \begin{bmatrix} 4 & 1 & 3 \\ 10 & 2 & 15 \end{bmatrix}.$$

The left panel of Figure 3.2 shows a plot of the three components of the vector $J(\hat{X};\alpha)$ as a function of α_1, when α_2 is kept constant at the value 1. Monotonicity does not hold. The right panel shows the behavior of the two components of $\varphi(\alpha_1, 1)$; since the agents have identical utility functions and identical wealths, these components are equal to each other. It is seen that the mapping φ is in this case also not monotonic. However, the symmetry between the agents implies that there is a unique Pareto efficient and financially fair solution, as shown in Proposition 3.9. It is therefore apparent that the monotonicity in Assumption 3.4 is only a sufficient and not a necessary condition.

The example shown above was obtained by experimentation, and seems too complicated for a clear economic interpretation. The particular shape of the utility is likely to play a role in the violation of monotonicity.

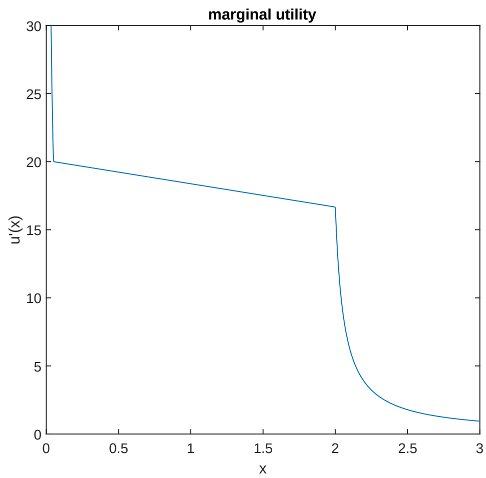

Figure 3.3: Marginal utility as defined by (3.39).

Chapter 4

Bid-ask spreads and intra-group trading

Joint work with J.M. Schumacher and B.J.M. Werker

4.1 Introduction

We consider a setting where a group of agents can conclude both trades within the group (intra-group or internal trades) and trades with an external market (extra-group or external trades). We assume that the extra-group trades take place within exogenously given bid-ask spreads, whereas the intra-group trades are concluded at endogenously determined prices with zero bid-ask spreads. We assume that the bid-ask spreads on an external market are the sole group's incentive to trade internally. In this chapter, we provide a framework to explore which bid-ask spreads on the external market allow the group of agents to reach an equilibrium position purely by extra-group trades, purely by intra-group trades, or by both means of trades.

To define an external market, we build on the work of Jouini and Kallal (1999) and Cherny and Madan (2010). We consider an external market with bid-ask spreads that can be used by agents as a counterparty for trading. In such trades the agents are assumed to act as price takers in the sense that both the bid and the ask prices on the external market are exogenously specified to the agents. We choose to model the bid-ask spreads as a convex cone. This convex cone represents the tradeable assets and can be referred to as a cone of acceptable risks or, as we will call it in this chapter, a cone of marketed assets. We can then refer to such external market as a 'conic market' and hence the name 'conic economy' which we will use throughout the chapter. We assume that the external market, can accept unlimited quantities of marketed assets and is not affected by them. Some basic properties of the conic external market are studied in Jouini and Kallal (1999) and in Schachermayer (2004), where the conditions for no arbitrage and the fundamental theorem of asset pricing are stated, respectively.

An indication of the way in which observed bid-ask spreads can be related to convex cone is given in Example 4.34. When there are only zero bid-ask spreads on the external market, the cone of marketed assets is a half-space and such external market is said to be complete. On the contrary, when there are non-zero bid-ask spreads on the external market, the cone of marketed assets is strictly smaller than a half space and the external market is said to be incomplete. The 'incompleteness' here could be understood in the context of financial mathematics as non-existence of a unique pricing functional, rather than in the context of standard Arrow-Debreu definition, because all underlying assets could be in fact tradeable.

The reason why agents perceive bid-ask spreads on the external market can vary. For instance, the bid-ask spreads can be caused by transaction costs such as market fees or management fees. We do not investigate the origin of bid-ask spreads on external market and we simply take bid-ask spreads on external market as a primitive of our model.

In addition to an external market, we consider a group of finitely many agents who act as price takers with respect to the external market and as price makers with respect to the internal (intra-group) market, which has to satisfy a market clearing condition. Intra-group trades may be in general motivated by minimization of a counterparty risk or avoidance of transaction costs such as market fees and bid-ask spreads. In this chapter, we consider the bid-ask spreads to be the only incentive for intra-group trades. In this setting, we use a concept of equilibrium which is formally identical to competitive equilibrium in an economy where all agents have access to the same production set, and this production set is given by a closed convex cone. For brevity, and in line with the terminology already used above, we will refer to this equilibrium concept as "conic equilibrium", although Madan and Schoutens (2012) use the same term for a somewhat different notion of equilibrium. The conic equilibrium determines prices that are used by agents on internal market and, when required, also indicates whether bid or ask price is used on the external market.

The conic equilibrium presented here is, in fact, a type of Walrasian equilibrium in a production economy, as we show in Proposition 4.9. However, we will work mainly in the context of conic equilibria, where both external and internal trades are explicitly expressed, and where we can use sets of nested cones to measure the group's tendency to trade internally. In general, a conic equilibrium can be reached using external trades or using intra-group trades or it may be necessary to use both types of trades. The first two ways of reaching the equilibrium are of our particular interest as one type of trade, internal or external, is superfluous.

Our framework can be used to model, for instance, a situation of a pension fund with several generations or a general mutual fund with several sub-funds. The generations or the sub-funds typically cover various needs of clients by offering different strategies ranging from aggressive to conservative. Each pension fund generation or sub-fund can act on its own and trade assets for bid or ask prices on external markets. However, the pension fund or a mutual fund can choose to trade some assets in-house. Our result provides conditions on the external market bid-ask spreads under which building an internal trading desk is superfluous or, on the contrary, is favorable.

We state necessary and sufficient conditions under which the cone of marketed assets induces purely internal equilibrium trades in Section 4.4. We show that the cones of marketed assets that are compatible with purely intra-group equilibrium trades have a nesting property, i.e., if a cone is compatible with purely intra-group equilibrium trades, then so are all embedded cones of marketed assets. Later, in Section 4.6, we build on this result and define a measure of tendency to trade internally. Our proposed measure is expressed by the size of the maximal cone of marketed assets that is compatible with only internal equilibrium trades. Further, the measure of tendency to trade internally can be linked to the term "heterogeneity measure of a group" in the following way. If a group consists of identical agents, such group can be considered homogeneous, and one expects that agents would prefer to conclude only trades with the external conic market. Hence, their tendency to trade internally will be relatively low. On the other hand, if the group consists of agents with complementary needs, i.e., the group exhibits some level of heterogeneity, one would expect that the demand for internal trades would be relatively higher. Hence, the group's tendency to trade internally will be also higher. In terms of bid-ask spreads, the group heterogeneity can be linked to the tendency to trade internally measured by the smallest bid-ask spreads under which a specific group is still trading purely internally.

The description of market situations in which a group concludes only external equilibrium trades allows us to address situations in which agents are "on the same side of the market". This term intuitively describes situations in which no agent benefits from any internal trade. Being on the same side of the market is easily understood in case of two agents and two assets as equally signed demand for each asset. But this intuition may fail in higher dimensions. By analyzing cones of marketed assets compatible with purely external equilibrium trades in Section 4.5, we provide a geometrical interpretation of the term "being on the same side of market" for a more general case. We observe that this term can be understood as "being in the interior of the same face of the cone of marketed assets". We support this by showing that when the individual demands of all agents, computed as if each agent would act on her own on the external conic market, are in the interior of the same face of the cone of marketed assets, then intra-group trades are superfluous.

Section 4.2 details the settings and assumptions on a group of agents and a conic market. Section 4.3 links concepts of conic and competitive equilibria and provides conditions for existence of conic equilibria. In Section 4.4, we analyze the conic economies in which the conic equilibrium can be achieved using intra-group trades only. We provide a necessary and sufficient condition on the cone of marketed assets that admit purely intra-group equilibrium trades. In Section 4.5, we focus on situations in which agents can achieve the conic equilibrium using only external trades. Also, we describe such situations by providing sufficient and necessary conditions on the cone of marketed assets. Because these conditions might be difficult to check, we also provide a sufficient condition that is easier to verify. In Section 4.6, we propose and formalize a measure to this effect and we investigate how this measure is influenced by a correlation of the initial risks, the risk aversion of agents, and by the number of agents within the group. We compare these results with the intuition behind the term "group heterogeneity". In the last section, we summarize and conclude this chapter. Proofs are mainly presented in the appendix.

4.2 Setting

We consider an economy in which assets are represented by m-dimensional vectors, i.e., let $X = \mathbb{R}^m$ denote the linear space of available assets. For convenience, we define $X_+ = \{x \in X \mid x \geq 0\}$ as the non-negative orthant and $X_{++} = \{x \in X \mid x > 0\}$ as the strictly positive orthant, where the vector inequalities are used in the elementwise sense. Let X' denote the space of linear functionals on X. Given $p \in X'$, the value or price assigned by p to an asset $x \in X$ is denoted as px. On this market we consider a finite group of n agents. Each agent i, $i = 1, \ldots, n$, is endowed with an objective function $U_i : X_+ \to \mathbb{R} \cup \{-\infty\}$, and initial endowment $x_i^0 \in X_{++}$.

Assumption 4.1 The functions $U_i : X_+ \to \mathbb{R} \cup \{-\infty\}$, $i = 1, \ldots, n$, are on X_{++} strictly concave, twice continuously differentiable, and strictly increasing in all of their arguments.

We assume that all agents have access to an external market where they can trade certain assets. More precisely, we introduce a closed convex cone $C \subset X$ which denotes a set of vectors tradeable at zero price on the external market. We call C the *cone of marketed assets*. The interpretation is that any agent can freely dispose of any asset $x \in C$ in the external market. A natural requirement on C is free disposal of a non-negative asset, i.e., $X_+ \subseteq C$.

Definition 4.2 A *cone of marketed assets* is a closed convex cone $C \subset X$ that contains the non-negative cone X_+. A *pricing functional* is a non-negative and non-zero element of X'.

Let C^* denote the set of pricing functionals that assign a non-negative value to all elements of the cone of marketed assets C,

$$C^* = \{q \in X' \mid qx \geq 0 \text{ for all } x \in C\}.$$

Because C is convex, closed and nonempty, from the bipolar theorem, we get that the cone of marketed assets C can be obtained from C^*, its dual cone, as

$$C = \{x \in X \mid qx \geq 0 \text{ for all } q \in C^*\}.$$

Therefore agents cannot dispose of assets in the external market if and only if there exists a pricing functional in C^* for which the assets have a negative value.

By definition, the cone of marketed assets is convex and includes the non-negative orthant. Let us have a look at the extreme cases. The smallest cone of marketed assets is the non-negative orthant X_+ itself. If $C = X_+$, agents can only dispose of non-negative assets. Because by Assumption 4.1 we consider agents' objective functions to be increasing, any trading on such external market would decrease the agent's objective function and, hence, would not be rational. The other extreme case is where the cone of marketed assets is a half-space containing X_+. In that case, the set of pricing functionals is a singleton, thus, bid and ask prices are equal for each asset. The most interesting cases, for our purposes, occur in between these two extremes. Such a situation represents an external market with different bid and ask prices for individual assets.

In addition to disposing of assets in the external market, we assume that agents also have a possibility to exchange assets among each other. We call this exchange of assets internal or intra-group trades and we say that it takes place on the internal or intra-group market. We assume that the internal trades happen at prices that are determined endogenously, which is contrary to the situation on the external market where we assume that agents act as price takers with prices given by a cone of marketed assets.

A general market situation in which a group of agents has access to both internal and external markets will be called a conic economy.

Definition 4.3 An *economy* is specified by objective functions U_i, $i = 1, \ldots, n$, fulfilling Assumption 4.1, and initial endowments $x_i^0 \in X_{++}$, $i = 1, \ldots, n$. An economy together with a cone of marketed assets C is a *conic economy*.

Now we can formulate the notion of equilibrium for a conic economy.

Definition 4.4 In a conic economy specified by $\big(C, (U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$, a pricing functional p achieves a *conic equilibrium (CE)* if there exists a collection of triples $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, such that the following conditions hold:

(i) (budget constraint) $p\, x_i^{*\text{in}} = 0$, $x_i^{*\text{ex}} \in C$, and $x_i^* = x_i^0 + x_i^{*\text{in}} - x_i^{*\text{ex}}$, for all $i = 1, \ldots, n$;

(ii) (individual optimality) for any given $i \in \{1, \ldots, n\}$ and for any triple $(x_i, x_i^{\text{in}}, x_i^{\text{ex}})$ such that $p\, x_i^{\text{in}} = 0$, $x_i^{\text{ex}} \in C$, and $x_i = x_i^0 + x_i^{\text{in}} - x_i^{\text{ex}}$, the inequality $U_i(x_i) \leq U_i(x_i^*)$ holds;

(iii) (internal market clearing)

$$\sum_{i=1}^{n} x_i^{*\text{in}} = 0.$$

The triplet $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$ in the above definition of conic equilibrium describes the equilibrium allocation of agent i, where x_i^*, $x_i^{*\text{in}}$, and $x_i^{*\text{ex}}$ denote the agent's *total allocation*, the *internal trade*, and the *external trade*, respectively. We call $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$ a triplet associated to the *equilibrium pricing functional* p.

Remark 4.5 When a pricing functional p achieves a conic equilibrium then $p \in C^*$. In fact, for $p \in C^*$ it is sufficient that p fulfills only Conditions (i) and (ii) in Definition 4.4 and only for one agent. To prove this, assume the contrary; $p \notin C^*$. Then from the definition of C^* there exist $y \in C$ and $x \in X_{++}$ such that $py < 0$ and $px + py = 0$. Let $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$ be a triplet satisfying Conditions (i) and (ii) in Definition 4.4. Then the triplet $(x_i^* + x, x_i^{*\text{in}} + x + y, x_i^{*\text{ex}} + y)$ is such that $p\,(x_i^{*\text{in}} + x + y) = 0$, $x_i^{*\text{ex}} + y \in C$ and $U_i(x_i^* + x) > U_i(x_i^*)$, which contradicts the individual optimality of $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$.

The following proposition asserts that the pricing functional, for which the conic equilibrium is achieved, must assign zero value to all external trades, i.e., external equilibrium trades have zero value. This follows solely from the individual optimality condition and the budget constraint. The proof is given in the appendix.

Proposition 4.6 *Consider a conic economy given by* $\big(C, (U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$ *and let* $p \in X'$ *achieve a conic equilibrium with a collection of triplets* $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i = 1, \ldots, n$. *Then* $px_i^{*\text{ex}} = 0$, *for all* $i = 1, \ldots, n$.

Note that when a conic equilibrium is achieved for some pricing functional p, then the internal and external equilibrium trades of agents are not necessarily uniquely determined. For instance, without any influence on total allocations any non-zero external equilibrium trade of one agent can be substituted with a non-zero external equilibrium trade of a different agent followed by internal reselling.

4.3 On the existence of conic equilibria

The concept of conic equilibria can be linked to the concept of Walrasian (competitive) equilibria in a production economy with a constant returns to scale technology. First we recall the notion of production economy in our setting.

Definition 4.7 A *production economy* is an economy $\big((U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$ together with production sets $Y_j \subset X$, $j = 1, \ldots, k$, and ownership shares $\theta_{i,j} \geq 0$, $\sum_{i=1}^n \theta_{i,j} = 1$, for $i = 1, \ldots, n$ and $j = 1, \ldots, k$.

For a production economy (a private ownership economy) the standard notion of equilibrium is defined in textbooks, e.g. Mas-Colell et al. (1995).

Definition 4.8 *Walrasian equilibrium (Mas-Colell et al., 1995, Def. 17.B.1)* In a production economy given by $\big((U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$ and $\big((Y_1, \ldots, Y_k), (\theta_{1,1}, \ldots, \theta_{n,k})\big)$, an allocation $(x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_k^*)$, where $x_i^* \in X$ for all i and $y_j^* \in Y_j$ for all j, and a pricing functional $p \in X'$ constitute a Walrasian equilibrium if

(i) for any $j \in \{1, \ldots, k\}$ and any $y_j \in Y_j$, the inequality $py_j \leq py_j^*$ holds,

(ii) for any $i \in \{1, \ldots, n\}$, $px_i^* \leq px_i^0 + \sum_{j=1}^k \theta_{i,j} py_j^*$, and for any $x_i \in X$ such that $px_i \leq px_i^0 + \sum_{j=1}^k \theta_{i,j} py_i^*$, the inequality $U_i(x_i) \leq U_i(x_i^*)$ holds,

(iii) $\sum_{i=1}^n x_i^* = \sum_{i=1}^n x_i^0 + \sum_{j=1}^k y_k^*$.

The conic equilibrium is, in fact, a special case of the Walrasian equilibrium where the production sets are represented by the opposite cone of marketed assets, i.e., by $-C$, and where each agent owns one company (or one production unit, or accesses one production set). The access of each agent to one production set mimics the access to the external market.

Proposition 4.9 *Consider a conic economy given by* $\big(C, (U_1, \dots, U_n), (x_1^0, \dots, x_n^0)\big)$ *and a production economy given by* $\big((U_1, \dots, U_n), (x_1^0, \dots, x_n^0)\big)$ *and* $\big((Y_1, \dots, Y_k), (\theta_{1,1}, \dots, \theta_{n,k})\big)$*. Let* $k = n$*,* $Y_i = -C$*, for* $i = 1, \dots, n$*, and let* $\theta_{i,j} = 1$*, for* $i = j$*, and* $\theta_{i,j} = 0$ *else.*

(i) If a pricing functional p *achieves a conic equilibrium with a collection of triplets* $(x_i^*, x_i^{*\mathrm{in}}, x_i^{*\mathrm{ex}})$*,* $i = 1, \dots, n$*, then the set of allocations* $(x_1^*, \dots, x_n^*, -x_1^{*\mathrm{ex}}, \dots, -x_n^{*\mathrm{ex}})$ *and* p *constitute a Walrasian equilibrium.*

(ii) If a pricing functional p *and an allocation* $(x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*)$ *constitute a Walrasian equilibrium, then* p *achieves a conic equilibrium with the collection of triplets* $(x_i^*, x_i^* - y_i^* - x_i^0, -y_i^*)$*,* $i = 1, \dots, n$*.*

We express equilibria in this chapter in the context of a conic economy rather than in the context of a production economy because this allows us to observe directly whether external or internal equilibrium trades occur. This might be advantageous for instance in the context of markets where different risks may be associated with intra-group and extra-group trading.

There are some convenient properties of the conic equilibria that follow instantly from their equivalence with Walrasian equilibria. For instance, we get the validity of the Fundamental Theorems of Welfare Economics for conic equilibria. Under assumption of non-zero bid-ask spreads, we obtain also the existence of equilibrium.

Proposition 4.10 *For any conic economy, where* $-C \cap C = \{0\}$*, there exists a pricing functional that achieves a conic equilibrium.*

Proof. From Proposition 4.9, we get the equivalence between the existence of a conic equilibrium and a Walrasian equilibrium. The existence of the latter is a classic result. For instance, we can use Debreu (1959, p. 83) where we verify Assumptions (a)–(d.4). Here, the assumption $-C \cap C = \{0\}$ is directly Assumption (d.3). The other assumptions are direct consequences of our settings. □

The conic equilibrium depends both on the characteristics of the agents and on the cone of marketed assets. In the following sections we investigate the dependency on the cone of marketed assets while keeping the characteristics of agents fixed. Namely, we investigate how the cone of marketed assets influences the intra-group and extra-group equilibrium trades of these agents. In general, three statements about equilibrium trades can be made. Firstly, a conic equilibrium can be achieved by using intra-group trades only; secondly, a conic equilibrium can be achieved by using extra-group trades only; thirdly, a conic equilibrium can be achieved using both internal and external trades. Note that two or even all three statements can be true simultaneously, Example 4.31.

4.4 Intra-group trades only

In this section, we describe conic economies in which a given group of agents can reach their conic equilibrium allocations using intra-group trades only. This can be used to identify external markets which agents do not have to use to reach equilibrium allocations.

The following definition formalizes the notion of cones of marketed assets that admit conic equilibria using internal trading only.

Definition 4.11 In an economy given by $\big((U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$, a cone of marketed assets C is *compatible with internal trading only (CIO)* if there exists a pricing functional p and triplets $(x_i^*, x_i^{*\text{in}}, 0)$, $i = 1, \ldots, n$, such that p achieves a conic equilibrium with these associated triplets.

The CIO property of cones naturally depends on the objective functions of the agents as well as on their initial endowments. From the definition above, we also see that we can relate the CIO property of a cone to a pricing functional that would arise in a pure exchange market equilibrium. For our purposes we call such equilibrium Internal-Market-Only equilibrium to emphasise the link with an internal market.

Definition 4.12 Assume an economy given by $\big((U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$. A pricing functional p achieves an *internal-market-only (IMO) equilibrium* if there exists a collection of pairs $(x_i^*, x_i^{*\text{in}})$, $i = 1, \ldots, n$, such that:

(i) (budget constraint) $p\, x_i^{*\text{in}} = 0$ and $x_i^* = x_i^0 + x_i^{*\text{in}}$ for all $i = 1, \ldots, n$;

(ii) (individual optimality) for all $i \in \{1, \ldots, n\}$ and for any pair (x_i, x_i^{in}) such that $p\, x_i^{\text{in}} = 0$ and $x_i = x_i^0 + x_i^{\text{in}}$, we have $U_i(x_i) \leq U_i(x_i^*)$;

(iii) (internal market clearing) $\sum_{i=1}^{n} x_i^{*\text{in}} = 0$.

We can characterize CIO cones using the just defined IMO equilibria. Intuitively, a cone of marketed assets is a CIO cone if and only if there exists an IMO equilibrium with the same allocations as a conic equilibrium. This can occur only in cases when the sum of all external equilibrium trades equals zero. A formal characterization of a CIO cone using IMO equilibria is provided in the following proposition.

Proposition 4.13 *Consider an economy $\big((U_1,\dots,U_n),(x_1^0,\dots,x_n^0)\big)$ and a cone of marketed assets C. The cone of marketed assets C is a CIO cone if and only if there exists a pricing functional p and triplets $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i=1,\dots,n$, such that p achieves a conic equilibrium with these triplets $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i=1,\dots,n$, and an IMO equilibrium with pairs $(x_i^*, x_i^{*\text{in}} - x_i^{*\text{ex}})$, $i=1,\dots,n$.*

Remark 4.14 Note that if a conic and an IMO equilibrium are achieved by the same pricing functional, then there exist total equilibrium allocations that are equal for both equilibria. This can be justified by the following argument. Firstly, for a given pricing functional we note that the optimization problems in Conditions (ii) in both Definition 4.4 and Definition 4.12 have unique solutions because of Assumption 4.1. Secondly, the optimal solutions of the problem in Definition 4.4(ii) are always feasible for the problem in Definition 4.12(ii) (see Lemma 4.33). Thirdly, the feasible solutions of the problem in Definition 4.12(ii) are included in the feasible solutions of the problem in Definition 4.4(ii). Therefore, these two optimization problems for the same pricing functionals will have the same solutions which are the total equilibrium allocations.

In view of this remark, we can state a necessary and sufficient condition for a cone of marketed assets to be CIO in terms of IMO equilibrium prices.

Proposition 4.15 *Assume an economy given by $\big((U_1,\dots,U_n),(x_1^0,\dots,x_n^0)\big)$. Let $\mathcal{P}$ be the set of all functionals in which IMO equilibria are achieved. Then a cone of marketed assets C is a CIO cone if and only if $p \in C^*$ for some $p \in \mathcal{P}$.*

Using this proposition, if we compute the IMO equilibria pricing functionals and compare them with the dual of the cone of marketed assets, we observe directly whether the cone of marketed assets is CIO. Let us now return to our two extreme examples: a cone of marketed assets being a half-space and the positive orthant.

Remark 4.16 From Proposition 4.15, it follows that a half-space is a CIO cone if and only if an IMO equilibrium is achieved in the pricing functional orthogonal to the half-space.

Remark 4.17 The non-negative orthant X_+ is a CIO cone. This immediately follows from Propositions 4.10 and 4.15.

An advantageous property of CIO cones is that they remain CIO when reduced.

Proposition 4.18 *Assume a conic economy. Any cone of marketed assets that is included in a CIO cone is a CIO cone.*

Proof. Let K be a CIO cone and let C, $C \subset K$, be a cone of marketed assets. Let $\mathcal{P}$ be the set of functionals in which an IMO equilibrium is achieved. From Proposition 4.15, we know that there exists $p \in \mathcal{P}$ such that $p \in K^*$. Since $C \subset K$, we have $K^* \subset C^*$ and, hence, $p \in C^*$. Using Proposition 4.15 once more, we see that C is also a CIO cone. □

Due to this nesting property of CIO cones, we have a simple rule for identifying a whole set of market situations in which no external market is needed for a given group of agents.

From the theory of general equilibrium, we know that uniqueness of equilibria is not granted without additional conditions. The conditions that would guarantee uniqueness are not within the scope of this chapter; we refer to Mas-Colell (1991) for a thorough discussion. However, we can express the uniqueness of IMO equilibria using the notion of CIO cones.

Proposition 4.19 *In a conic economy, a pricing functional that achieves an IMO equilibrium is unique up to scalar multiplication if and only if all CIO cones are in a single half-space.*

Proof. Consider pricing functionals on a unit sphere. Firstly, assume that there exists a unique pricing functional, say p, that achieves an IMO equilibrium. Then, by Proposition 4.15, only the cones of marketed assets whose dual cones contain p are CIO cones. All these cones are contained in the half-space that is obtained as the dual cone of p, by Proposition 4.18 and Remark 4.16. Secondly, assume that there would exist two different pricing functionals $p \neq q$ that achieve IMO equilibrium. Then there exist two different half-spaces that contain the positive orthant and that are dual to p or q. According to Remark 4.16, these half-spaces are CIO cones. Hence, there are two CIO cones that do not belong to the same half-space. □

Remark 4.20 The set of cones that are CIO has a unique maximal cone if and only if there exists a unique ray of pricing functional that achieves an IMO equilibrium. This maximal cone is a half-space which is the dual cone of the IMO equilibrium pricing functional. This is a direct consequence of Propositions 4.15, 4.18, and 4.19.

In the next section, a similar analysis that was done for CIO cones is done for the cones of marketed assets that allow solely external equilibrium trades.

4.5 External trades only

In this section, we describe conic economies in which a given group of agents can reach equilibrium allocations by using external trades only. By this, we identify market situations where an internal market provides no additional utility improvement in reaching equilibrium allocation for the agents. Let us start by formalizing the notion of cones of marketed assets that admit conic equilibria using external trading only.

Definition 4.21 Consider an economy given by $\big((U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$ and a cone of marketed assets C. The cone of marketed assets C is *compatible with external trading only (CEO)* if there exists a pricing functional p and triplets of the form $(x_i^*, 0, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, such that p achieves a conic equilibrium with these associated triplets.

Let us list at least one example of a cone of marketed assets with the CEO property.

Remark 4.22 Let a cone of marketed assets be a half-space and let a conic equilibrium exist, then the cone of marketed assets is a CEO cone. This can be justified as follows. Let C be a half-space; then C^* contains only one pricing functional, say p. Since the conic equilibrium exists, there exists a collection of triplets $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, fulfilling Conditions (i)–(iii) in Definition 4.4. Given that $p \in C^*$ and $px_i^{*\text{in}} = 0$, we have $x_i^{*\text{in}} \in -C$ for $i = 1, \ldots, n$. Because $-x_i^{*\text{in}} + x_i^{*\text{ex}} \in C$, the collection of triplets $(x_i^*, 0, -x_i^{*\text{in}} + x_i^{*\text{ex}})$, $i = 1, \ldots, n$, fulfills Conditions (i)–(iii) in Definition 4.4. Therefore, C is a CEO cone.

In the previous section, Proposition 4.15 relates the concept of CIO cones to equilibrium pricing functionals that arise in an economy which allows internal trades only. Similarly, the CEO property can be linked to market situations where each agent can access only the external market. We, therefore, introduce the following notion of an optimal position. We use the term optimal position rather than equilibrium as no interaction among agents is required.

Definition 4.23 Let $\big(C, (U_1, \dots, U_n), (x_1^0, \dots, x_n^0)\big)$ denote a conic economy. Then a collection of pairs $(x_i^*, x_i^{*\mathrm{ex}})$, $i = 1, \dots, n$, achieves an optimal *external-market-only (EMO)* position if the following conditions hold:

(i) (budget constraint) the relations $x_i^{*\mathrm{ex}} \in C$ and $x_i^* = x_i^0 - x_i^{*\mathrm{ex}}$ are satisfied for all $i = 1, \dots, n$;

(ii) (individual optimality) for any given $i \in \{1, \dots, n\}$ and for any pair (x_i, x_i^{ex}) such that $x_i^{\mathrm{ex}} \in C$ and $x_i = x_i^0 - x_i^{\mathrm{ex}}$, the inequality $U_i(x_i) \leq U_i(x_i^*)$ holds.

The relation between CEO cones and optimal EMO positions is similar to that between CIO cones and IMO equilibria, i.e., a cone is CEO if and only if an optimal EMO position and conic equilibrium have the same total allocations.

Proposition 4.24 *Assume an economy given by $\big((U_1, \dots, U_n), (x_1^0, \dots, x_n^0)\big)$ and a cone of marketed assets C. The cone of marketed assets C is CEO if and only if there exists a pricing functional p and triplets $(x_i^*, x_i^{*\mathrm{in}}, x_i^{*\mathrm{ex}})$, $i = 1, \dots, n$, such that p achieves a conic equilibrium with these triplets $(x_i^*, x_i^{*\mathrm{in}}, x_i^{*\mathrm{ex}})$, $i = 1, \dots, n$, and an optimal EMO position is achieved with the collection of pairs $(x_i^*, x_i^{*\mathrm{ex}} - x_i^{*\mathrm{in}})$, $i = 1, \dots, n$.*

For a given group of agents, we have shown in the previous section that if there exists a unique IMO equilibrium then the set of CIO cones has a unique maximal element. One could ask if a similar result, i.e., the existence of a unique minimal element, holds for the set of CEO cones. But unfortunately, CEO cones do not have a nesting property like CIO cones and hence a unique minimal CEO cone does not necessarily exist. This lack of a nesting property is shown in Example 4.30 and sets of CEO cones are illustrated later in Example 4.31.

Identifying the CEO property using the previous proposition would involve computing both the optimal EMO positions and all conic equilibria, and hence it would be inconvenient. The identification of a CEO cone can be based solely on conic equilibria, as hinted by the definition of the CEO property. Namely, if there exists a conic equilibrium in which all individual excess demands are inside the opposite of the cone of marketed assets (inside $-C$), then the agents can satisfy their demands by using the external market only, and hence such a cone of marketed assets would be CEO. This is formalized in the following proposition.

Proposition 4.25 *Let* $\big((U_1,\dots,U_n),(x_1^0,\dots,x_n^0)\big)$ *denote an economy and* C *a cone of marketed assets. The cone* C *is a CEO cone if and only if there exists a pricing functional* p *and* $d_i \in -C$*, for all* $i=1,\dots,n$*, such that a conic equilibrium is achieved in* p *and*

$$d_i = \underset{\{d|pd=0\}}{\arg\max}\, U_i(x_i^0 + d), \qquad \textit{for } i = 1,\dots,n.$$

The above proposition provides a necessary and sufficient condition for a cone of marketed assets to be a CEO cone. However, this condition is still not easily verifiable as it involves computing all pricing functionals in which the conic equilibrium is achieved. Therefore, one may ask if we can recognize a CEO cone without the computation of equilibrium pricing functionals. The rest of this section will be devoted to several less computationally demanding tests that can help us to identify CEO cones.

Proposition 4.25 can be interpreted as that there is no need for internal trade if all individual demands in conic equilibria are in one hyperplane which is orthogonal to a conic equilibrium pricing functional. Hence, using Proposition 4.24, we can see that a necessary condition for a cone to be a CEO cone is that all external trades in the EMO situation must be in the same hyperplane. The verification of this condition is much simpler, as we do not need to consider any equilibrium pricing functional and we simply solve n separate optimizations. We formalize this necessary condition in the following proposition.

Proposition 4.26 *Let* $\big((U_1,\dots,U_n),(x_1^0,\dots,x_n^0)\big)$ *denote an economy and* C *a cone of marketed assets. Let a collection of pairs* $(x_i^*, x_i^{*\mathrm{ex}})$*,* $i=1,\dots,n$ *achieve an optimal EMO position. If the cone* C *is a CEO cone, then there exists* $p \in C^*$ *such that* $px_i^{*\mathrm{ex}} = 0$ *for all* $i=1,\dots,n$*.*

Proof. Assumption 4.1 guarantees that $(x_i^*, x_i^{*\mathrm{ex}})$, $i=1,\dots,n$, are uniquely determined. From Proposition 4.24, it follows that there exists a pricing functional p with associated triplets $(x_i^*, 0, x_i^{*\mathrm{ex}})$, $i=1,\dots,n$, achieving a conic equilibrium. From Proposition 4.6 and Remark 4.5, we find that $p \in C^*$ and $px_i^{*\mathrm{ex}} = 0$, for $i=1,\dots,n$. □

Proposition 4.26 states only a necessary condition for the CEO property. Naturally, this condition does not have to be sufficient because the allocations $(x_i^*, 0, x_i^{*\mathrm{ex}})$, where $px_i^{*\mathrm{ex}} = 0$, $i=1,\dots,n$, do not have to constitute a conic equilibrium. We provide a simple example showing this in Example 4.30. A set of sufficient and necessary conditions can be derived by comparing local optimality conditions of the optimal EMO allocations and the conic equilibrium allocations. To illustrate this approach we describe a cone of marketed assets using a finite number of convex functions.

Proposition 4.27 *Let $\big(C, (U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$ denote a conic economy and let g^j, for $j = 1, \ldots, k$, be twice differentiable convex functions such that $C = \{x \in X \mid g^j(x) \le 0, \forall j \in \{1, \ldots, k\}\}$. Let a collection of pairs $(x_i^*, x_i^{*\mathrm{ex}})$, $i = 1, \ldots, n$, be an optimal EMO position. The cone C is a CEO cone if and only if there exists $p \in C^*$ such that $\frac{\partial}{\partial x} U_i(x_i^0 - x_i^{*\mathrm{ex}}) \propto p$ for all $i = 1, \ldots, n$.*

Up to now, we have mainly considered a cone of marketed assets to be a general convex cone. In the rest of this section we concentrate on polyhedral cones, i.e., cones given by an intersection of finitely many half-spaces which may arise from specification of the cone of marketed assets in terms of bid-ask spreads, see, for instance, Example 4.34. Polyhedral cones of marketed assets and Proposition 4.27 will help us to derive a simple sufficiency condition on the CEO property. But firstly, with this assumption let us revisit Proposition 4.26 from which it follows that a cone can be a CEO cone only if all trades in the EMO situation are in the same hyperplane. For polyhedral cones, this is possible only if all trades in the EMO situation are in the same face of this cone, i.e. if a polyhedral cone of marketed assets has CEO property then all trades in the EMO situation are in the same face. If all trades in an EMO situation are in the interior of the same face of a polyhedral cone of marketed assets then the polyhedral cone of marketed assets has CEO property.

Remark 4.28 To some extent, the market settings, where all trades in an EMO situation are in the interior of the same face of the cone of marketed assets, can be perceived as a formal definition and generalization of the term "being on the same side of the market". The natural intuition behind this term is that agents that are on the same side of the market have no incentive to trade internally, even if such trades are allowed. This is exactly what happens when the conic economy is described by a polyhedral cone of marketed assets and all EMO trades are in the interior of the same face of the cone.

A polyhedral cone C can be represented by a matrix $A \in \mathbb{R}^{m \times k}$ as follows: $-C = \{x \in X \mid \exists y \ge 0 : Ay = x\}$. In the context of general equilibrium theory, the matrix A is often called the activity, technology or production matrix. This is because the matrix A can be read in the following way. The columns are production options for the agents, and negative entries denote production inputs, whereas positive entries denote production output. The vector y is then called an activity vector or a production vector. Note that the external trades of agents can be fully described by a particular activity vector. Based on activity vectors, we can derive a sufficient condition for a cone to be a CEO cone. This condition involves computing only the trades in an EMO situation and, hence, it is relatively easy to verify as it leads to n independent optimization problems. The main idea of this condition is that if all trades in an EMO situation are in the interior of one face, then $\frac{d}{dx}U_i(x_i^0 - x_i^{*\text{ex}}) \propto p$, for $i = 1, \ldots, n$, and hence, from Proposition 4.27, the cone of marketed assets is a CEO cone. When expressed in terms of activity vectors of each agent y_i, this condition reduces to counting non-zero elements of y_i and comparing their positions.

Proposition 4.29 *Let $\big(C, (U_1, \ldots, U_n), (x_1^0, \ldots, x_n^0)\big)$ denote a conic economy. Let $-C = \{x \in X \mid \exists y \geq 0 : Ay = x\}$, where $A \in \mathbb{R}^{m \times k}$, $m \leq k$, and where no column of A can be expressed as a non-negative linear combination of the remaining columns. Let a collection of pairs $(x_i^*, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, be an optimal EMO position. Let $J_1 = \{j_1, \ldots, j_{m-1}\}$ and $J_0 = \{j_m \ldots, j_k\}$ be index sets denoting columns of A such that $J_1 \cup J_0 = \{1, \ldots, k\}$. Let vectors $y_i \in \mathbb{R}_+^k$, $i = 1, \ldots, n$, be such that $Ay_i = -x_i^{*\text{ex}}$ and let $(y_i)_j > 0$ for $j \in J_1$ and $(y_i)_j = 0$ for $j \in J_0$. Let the columns of the matrix A indexed by J_1 be linearly independent. Then the cone C is a CEO cone.*

This proposition states that if there are no redundant columns in the activity matrix A, then a sufficient condition for the cone of marketed assets to be a CEO cone is relatively simple. It is enough to compute activity vectors in the EMO situation for each agent separately, y_i's, and compare whether the positions of the non-zero elements that are indexed by set J_1 are the same across the agents. If each of these activity vectors has exactly $m - 1$ non-zero elements on the same positions J_1, then the cone of marketed assets is a CEO cone. For instance, in case of two dimensional asset space it is sufficient to verify that each agent has one non-zero element on the same position in its activity vector. We illustrate the use of this proposition in the following example. The following example can also be used to demonstrate that the CEO cones do not possess the nesting property.

Example 4.30 Assume two agents, expected utility maximizers, with utility functions $u_1(x) = -x^{-1}$, $u_2(x) = \log(x)$. Consider a three-dimensional asset space, where assets express a position in Arrow-Debreu securities with the probability measure $P = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ which is also equal to the agents' beliefs. Let the agents' initial endowments be $x_1^0 = (1, 4, 1)$ and $x_2^0 = (2, 3, 2)$. We will consider two conic economies given by cones of marketed assets

Conic economy given by C_1			
pricing functional	0.4828	0.2759	0.2414
total excess demand	0.1084	−2.2167	2.3165
agent 1 demand	0.5370	−1.9667	1.1737
agent 2 demand	−0.4286	−0.2500	1.1429

Conic economy given by C_2			
pricing functional	0.2857	0.5714	0.1429
total excess demand	3.0436	−3.3479	7.3043
agent 1 demand	2.0436	−1.8479	3.3043
agent 2 demand	1.0000	−1.5000	4.0000

Table 4.1: A conic equilibrium pricing functional and corresponding demands for the settings described in Example 4.30.

C_1 and C_2 such that $-C_i = \{x \in X \mid \exists y \geq 0 : A_i y = x\}$ with production matrices

$$A_i = \begin{pmatrix} -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & a_i & 0 \\ 0 & 0 & -1 & 2 & 2 \end{pmatrix},$$

for $i = 1, 2$, $a_1 = -3.5$, and $a_2 = -1$. Note that $-C_1 = \operatorname{pos}(A_1)$, $-C_2 = \operatorname{pos}(A_2)$, where $\operatorname{pos}(\cdot)$ is the set of nonnegative linear combinations of the columns of a matrix. Also note that the fourth column of matrix A_1 can be expressed as a positive linear combination of the second and fourth column of matrix A_2, namely $(1, -3.5, 2)' = 2.5 \cdot (0, -1, 0)' + 1 \cdot (1, -1, 2)'$. Because the fourth column of matrix A_2 cannot be expressed as a positive linear combination of columns in A_1, $C_1 \subsetneq C_2$. For illustration we present the conic equilibrium pricing functionals and the corresponding demands in Table 4.1. In general, if the individual demands are in the cone of marketed assets, then the agents can satisfy their needs by external trading only. From Proposition 4.25, we know that only in such a case the cone of marketed assets is a CEO cone. To find out whether the individual demands are inside the cones, we can use the fact that the cones C_1 and C_2 are defined by activity matrices. Hence, the individual demands are inside the cone of marketed assets if the corresponding activity vector y exists. The activity vectors are listed in Table 4.2. One individual demand in the conic economy given by C_2 cannot be expressed using an

Case		Activity vectors				
Cone C_1 (matrix A_1)	agent 1 demand	0.0000	0.0000	0.0000	0.5619	0.0498
	agent 2 demand	0.0000	0.0000	0.0000	0.0714	1.0000
Cone C_2 (matrix A_2)	agent 1 demand	does not exist				
	agent 2 demand	0.0000	0.0000	0.0000	1.5000	1.0000

Table 4.2: The corresponding activity vectors to individual demands listed in Table 4.1. Activity matrices A_1 and A_2 are specified in Example 4.30. In the conic economy described by activity matrix A_2, the equilibrium demand of agent 1 cannot be described by an activity vector and, thus, is not in the cone of marketed asset C_2. Therefore the demand of agent 1 cannot be satisfied in external-market-only situation and C_2 is not CEO cone.

activity vector. Therefore, by Proposition 4.25, the cone C_1 is a CEO cone, but C_2 is not. Because $C_1 \subset C_2$, we observe that CEO cones are not nested in general. In fact, we could show that C_1 is a CEO cone without computing the conic equilibrium pricing functional, by using the sufficiency condition defined in Proposition 4.29. This can be done by computing the optimal EMO trades and the corresponding activity vectors individually for each agent. For the cone C_1, these activity vectors are the same as presented above. Note that these activity vectors have exactly 2 positive entries in the same positions, and the dimension of asset space, m, equals 3, we have verified the condition in Proposition 4.29 (here $J_0 = \{1, 2, 3\}$ and $J_1 = \{4, 5\}$), and hence C_1 is a CEO cone.

This example is visualized in Figure 4.1 where we can see individual demands in both conic equilibrium and EMO situations for both cones of marketed assets. In this figure, we observe that the optimal EMO trades are in the same face for both cone of marketed assets. However, the optimal EMO trades are in the interior of face only in the case of C_1. This is in line with our Remark 4.28 that the term "being on the same side of the market" can be understood as "being in the interior of the same face" in higher dimensional asset spaces. We can also observe that the necessary condition for a cone to be a CEO cone from Proposition 4.26, i.e., being in the same hyperplane, is satisfied in both cases. Because C_2 is not a CEO cone, we see that this condition is indeed not sufficient.

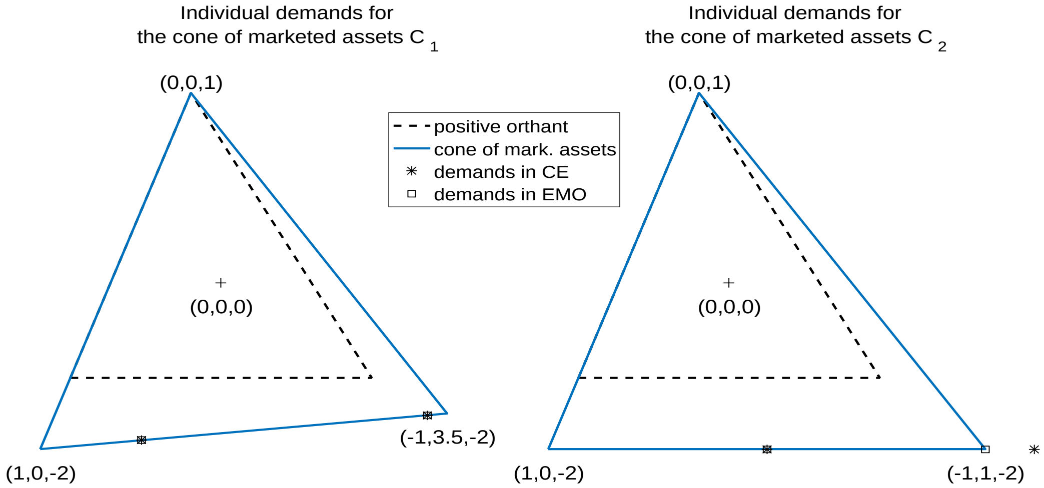

Figure 4.1: A diagram of individual demands in two conic economies (CE) with the agents as described in Example 4.30. For the cone of marketed assets C_2, we observe that even though external-market-only (EMO) individual trades are in the same face, the cone is not compatible with external trading only, because one CE individual demand is outside the face.

4.6 Measures of group's tendency to trade internally

The focus of this section is to quantify the group's tendency to trade internally. We propose a measure of group's tendency to trade internally based on bid-ask spreads and we illustrate how various factors, such as agent's risk aversion, correlation of risks, and the number of agents, influence this measure. Since we will rely heavily on the nesting property of CIO cones, we illustrate how CIO cones are nested also in case of multiple equilibria and we show that the CEO cones do not have the same favorable nested structure.

If the group consists of identical agents in terms of preference as well as in terms of initial endowment, one can say that such a group is homogeneous. In that case, one would expect that agents in a conic economy conclude only external trades, if any. On the other hand, if the group consists of agents with complementary needs, one can say that the group is heterogeneous and one would expect that the need for external trades would be low because agents can satisfy their needs by internal trading. This suggests that the measure of group's tendency to trade internally can indicate the heterogeneity of the group. To test this idea, we investigate the way in which the tendency to trade internally depends on parameters which relate to different forms of heterogeneity.

To formalize a higher or lower tendency to trade internally, we use the nesting property of CIO cones, Proposition 4.18. With a suitable parametrization of cones of marketed assets we can use this nesting property to find the maximal CIO cone. Then we can say that one group of agents tends to trade internally more than a second group if the maximal CIO cone of the first group includes the maximal CIO cone of the second group. Given the equivalence between cones of marketed assets and bid-ask spreads, Example 4.34, we can express this relation more naturally in terms of bid-ask spreads. Assume two groups of agents that trade only internally for the given bid-ask spreads. If we decrease the bid-ask spreads and the first group still trades only internally while the second group does not, we say that the first group tends to trade internally more than the second group. To summarize the arguments above, we will say that one group tends to trade internally more than the other group if it has a bigger maximal CIO cone than the other group.

For the proposed measure of tendency to trade internally, we need to be able to compare the CIO cones. In general, this can be complicated as there is no simple method to compare two general cones which possibly intersect each other. To avoid these problems, we parametrize sets of nested cones of marketed assets and for their generating we use families of distortion functions (see (4.1) below). We consider a distortion function g to be a non-decreasing concave function such that $g : [0, 1] \to [0, 1]$, $g(0) = 0$ and $g(1) = 1$. Let $\{g_\alpha\}_{\alpha\in[0,1]}$ denote a set of distortion functions parametrized by α such that $g_{\alpha_1} \geq g_{\alpha_2}$ if $\alpha_1 \leq \alpha_2$; and let $g_0(u) = 1$ and $g_1(u) = u$ for $u \in (0, 1]$. For economic and statistical interpretation of distortion functions, see for instance Wang (2000).

In the rest of this section, we assume that the assets in a conic economy are random cashflows. Any element of the asset space $X = \mathbb{R}^m$ identifies a specific holding of m Arrow-Debreu securities, i.e. $x \in X$ denotes a random variable on a finite probability space $(P, \mathcal{F}, \Omega)$. This allows us to define the set of cones of marketed assets $\{C_\alpha\}_{\alpha\in[0,1]}$ as follows

$$C_\alpha = \{x \in X \mid E^Q x \geq 0, \forall \text{ prob. measures } Q, \text{ s.t. } Q(A) \leq g_\alpha(P(A)) \; \forall A \subset \Omega\}, \quad (4.1)$$

where g_α is a distortion function. The chosen parametrization of cones follows recent works in the field of conic finance where tradeable assets are linked to distortion functions, e.g., Eberlein et al. (2012) and Madan and Schoutens (2012).

Note some properties of $\{C_\alpha\}_{\alpha\in[0,1]}$. For $\alpha_1 \leq \alpha_2$ we have that $C_{\alpha_1} \subset C_{\alpha_2}$, because $g_{\alpha_1} \geq g_{\alpha_2}$, and hence elements in C_{α_1} have to pass the positivity test against a larger set of test measures Q than elements in C_{α_2}. Therefore, the cones in the set $\{C_\alpha\}_{\alpha\in[0,1]}$ are nested. Because we assumed that $g_0(u) = 1$ and $g_1(u) = u$ for $u \in (0, 1]$, the boundary cones C_0 and C_1 are the positive orthant X_+ and a half-space given by the reference measure P, respectively.

We can use any family of distortion functions to generate the sets of nested cones of

Wang's distortion function	$g^{w}_{\alpha}(u) = \Phi\Big(\Phi^{-1}(u) - \Phi^{-1}(\alpha/2)\Big)$
proportional hazard distortion function	$g^{p}_{\alpha}(u) = u^{\alpha}$
minvar distortion function	$g^{mv}_{\alpha}(u) = 1 - (1-u)^{\frac{1}{\alpha}}$
minmaxvar distortion function	$g^{mmv}_{\alpha}(u) = 1 - (1-u^{\alpha})^{\frac{1}{\alpha}}$
CVaR	$g^{c}_{\alpha}(u) = \min\{u/\alpha, 1\}$

Table 4.3: The list of families of distortion functions and their reparametrizations that are considered in later examples. We define all listed functions for $u \in [0,1]$, $\alpha \in (0,1)$, and Φ denotes the cumulative distribution function of the standard normal distribution.

marketed assets. To investigate how the proposed measure of tendency to trade internally depends on a particular choice of a family of distortion functions, we use several distortion functions that are often used in the literature. To adhere to the above assumptions, we use an elementary reparametrization of these distortion functions. In later examples we consider families of proportional hazard, minvar, minmaxvar, Wang's, and CVaR distortion functions; see Table 4.3 for their reparametrizations. The proportional hazard distortion function is sometimes also called a power or maxvar distortion function in this reparametrization, see Wang (1995) and Cherny and Madan (2009). For motivation and more information about minvar and minmaxvar distortion functions we refer to Cherny and Madan (2009) and for Wang's distortion function to Wang (2000). The CVaR distortion function is a function corresponding to the conditional value-at-risk, a measure often used in risk measurement. For illustration, these selected distortion functions are displayed in Figure 4.2.

IMO equilibria are not unique in general, therefore, according to Remark 4.20, there does not have to exist a unique maximal CIO cone. However, because CIO cones are nested according to Proposition 4.18, any set of nested cones of marketed assets will have a unique maximal CIO cone regardless of the number of IMO equilibria. Therefore, we can define a maximal CIO cone for each of our proposed sets of nested cones of marketed assets. For a given parametrization, we identify the maximal CIO cone by a parameter $\alpha^{\text{CIO}} \in [0,1]$, where $\alpha^{\text{CIO}} = \sup\{\alpha | C_{\alpha} \text{ is CIO cone}\}$. The parameter α^{CIO} denotes a critical bid-ask spread for only internal trading and can be used to express the above discussed tendency to trade internally as follows. Assume two groups of agents and a given set of nested cones. If the value of α^{CIO} is larger for one group than for another, then the first group tends to trade internally under a smaller bid-ask spread. Therefore, larger values of α^{CIO} correspond to a stronger tendency to trade internally.

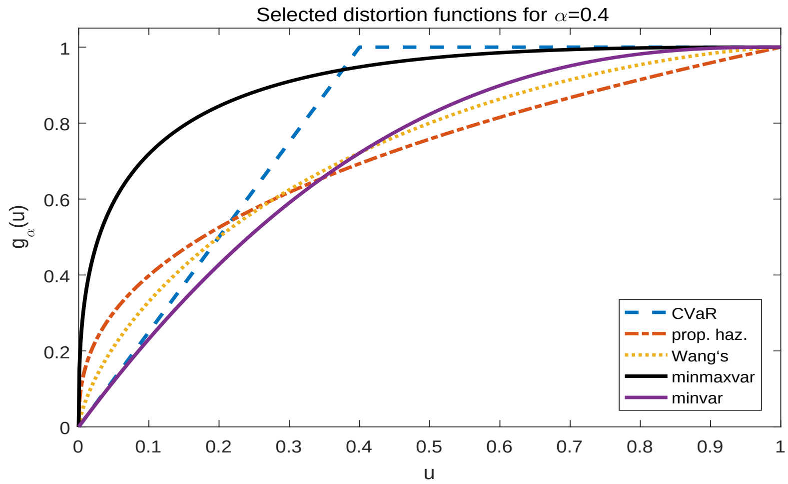

Figure 4.2: Selected distortion functions that can be used for generating nested cones.

In the following example we demonstrate the concept of a maximal CIO cone in case of multiple equilibria. For illustration we use a simple two-dimensional asset space and a Shapley-Shubik symmetric economy. In this example, for the sake of simplicity, the utility functions do not fulfill Assumption 4.1 on the entire X_+. However, as the considered utility functions could be approximated by strictly concave, increasing and differentiable functions on the concerned domain, the breach of the strict concavity assumption is not crucial.

Example 4.31 *Shapley-Shubik economy equilibria, Bergstrom et al. (2009)* Assume two agents with objective functions given by

$$U_1(x, y) = x + \frac{7}{2}y - \frac{1}{2}y^2, \qquad U_2(x, y) = \frac{7}{2}x - \frac{1}{2}x^2 + y.$$

Assume that the initial holdings of agents are $x_1^0 = (4, 0)$ and $x_2^0 = (0, 4)$. By computation, we can verify that the agents achieve IMO equilibrium for three pricing functionals: $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, \frac{1}{3})$. The three equilibria imply that three half-spaces, dual to the pricing functionals, are both CIO and CEO cones. Therefore, from Proposition 4.19 it follows that the set of CIO cones does not have a unique maximal element. Any two-dimensional cone of marketed assets can be described by two angles that give its distance from the positive quadrant. In Figure 4.3 we plot all combination of these angles that determine the cones of marketed assets in this setting and we display whether they have the CIO or CEO property. Any parametrized set of nested cones of marketed assets is displayed in this plot as a non-decreasing curve connecting the origin (the cone of marketed assets

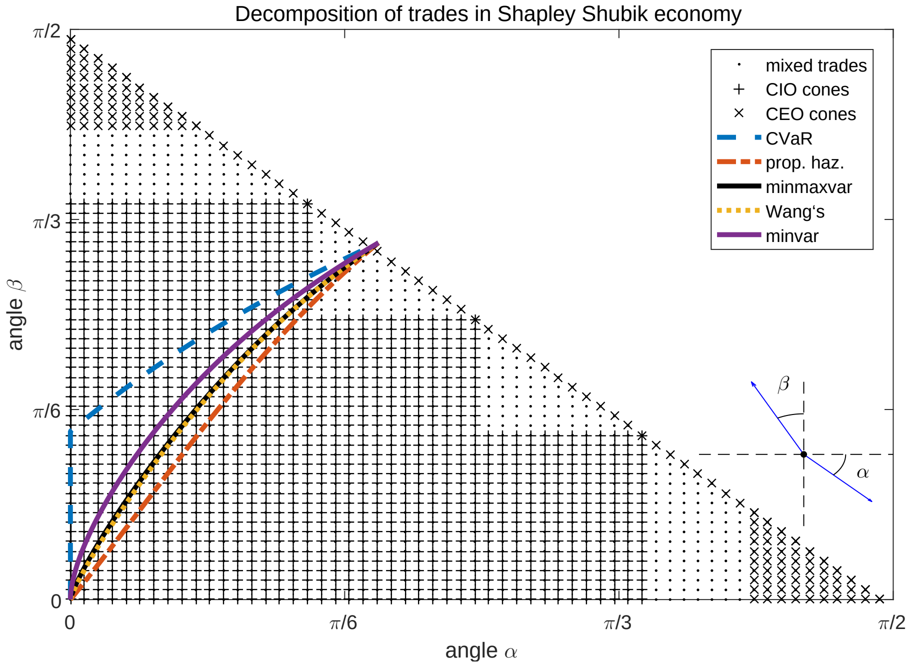

Figure 4.3: A diagram representing a set of all possible cones of marketed assets in the two-agents-two-assets case for Shapley-Shubik economy given in Example 4.31. The cones are represented by angles from the positive quadrant. These angles are plotted on the axes. The triangular shape is caused by considering only convex cones. In the diagram, we can observe what type of equilibrium trades can be realized for a given cone of marketed assets.

being the positive quadrant) and a point on diagonal line (the cone of marketed assets being a half-space). The point on the diagonal line is, in case of the considered distortion functions, given by the agents' beliefs and expressed by a probability measure P in (4.1); here we set $P = (0.4, 0.6)$. The maximal CIO cone for a given parametrization would be displayed as a point where the corresponding curve leaves the set of CIO cones.

As we have motivated and argued above, as a measure of group's tendency to trade internally we use the CIO cone that is maximal from a set of nested cones. Such measure depends on a particular parametrization of a set of nested cones. To test this dependency, we compute α^{CIO} for each of the proposed parametrizations for various groups of agents. We vary the group characteristics to investigate to which extent the measure of group's tendency to trade internally can capture the group heterogeneity. Namely, we investigate how correlation of initial risks, risk aversion of agents, and the number of agents in the group influence this measure.

Example 4.32 Assume n agents, expected utility maximizers with power utility functions $u_i(x) = \frac{x^{1-\gamma}}{1-\gamma}$, for $\gamma > 0$, $\gamma \neq 1$, and $u_i(x) = \log(x)$, for $\gamma = 1$, $i = 1, \ldots, n$. Assume that the initial holdings of agents are given by random variables X_i^0, $i = 1, \ldots, n$, such that

$$X_i^0 = \begin{cases} 2 & \text{if } \sqrt{1-\rho^2}Y_i + \rho Y_0 > 0, \\ 1 & \text{if } \sqrt{1-\rho^2}Y_i + \rho Y_0 \leq 0, \end{cases}$$

where $Y_i \sim N(0,1)$, $i = 0, 1, \ldots, n$, are independent random variables and $\rho \in [0,1]$ is a given correlation coefficient. Assume that agents' beliefs, expressed by P in (4.1), are equal to a reference probability measure of agents initial holdings.

In the setting of Example 4.32, the groups of agents are identified by three parameters: by the number of agents n, by the correlation of assets ρ, and by the risk aversion parameter γ. By varying these parameters, we get groups with different characteristics. For these different groups, we will observe the behaviour of the measure α^{CIO} and whether it is in line with a common perception of group heterogeneity and tendency to trade internally.

Example 4.32 *(continued) (correlation)* Assume $\gamma = 2$ and a given number of agents n. Then by varying ρ over $[0,1]$, we get groups of agents with initial positions ranging from uncorrelated risks to fully correlated ones. In Figure 4.4, we have plotted α^{CIO} depending on the correlation of initial holdings, i.e. $\text{corr}(X_1^0, X_2^0)$, for differently parametrized sets of nested cones and for groups of 2 and 4 agents. Except for the parametrizations by CVaR and minvar distortion functions, we can observe that, with higher correlation, the proposed measure of tendency to trade internally α^{CIO} decreases. The decrease corresponds to general intuition behind group heterogeneity with respect to correlation, as an increase in correlation of the agents' initial holdings leads to a decrease in the group heterogeneity.

In the same manner, we investigate how risk aversion of individual agents in the group influences the proposed measure of group's tendency to trade internally.

Example 4.32 *(continued) (risk aversion)* Assume $n = 2$ and a given risk correlation ρ. Then by varying the coefficient of relative risk aversion γ, we get groups of agents with relatively low risk aversion and groups of agents with relatively high risk aversion. For these groups, we investigate the behavior of our proposed measure, α^{CIO}. In Figure 4.5, we

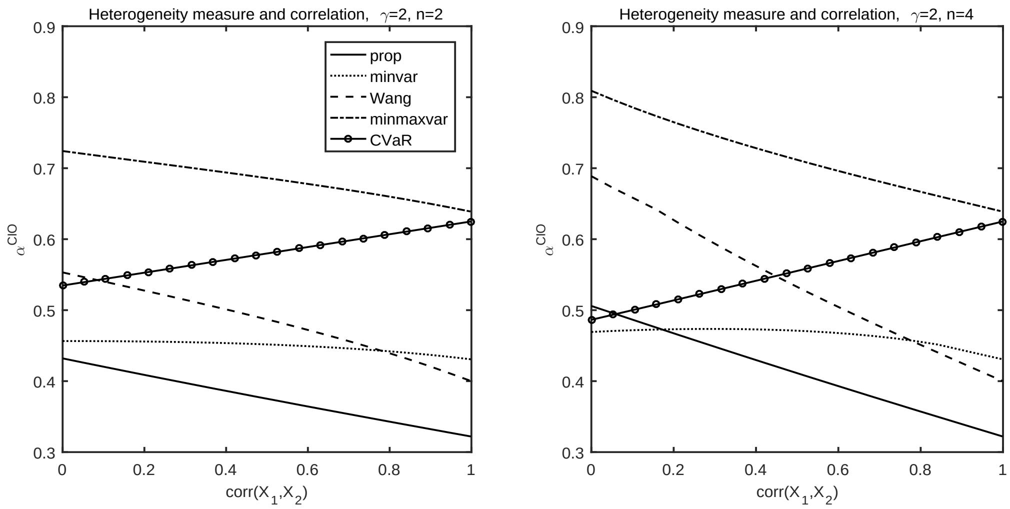

Figure 4.4: The dependency of α^{CIO} on the correlation of initial risks of agents $\text{corr}(X_1^0, X_2^0)$ given by $\frac{2}{\pi}\arcsin\rho^2$. The plots display the dependency for groups of 2 and 4 agents, respectively.

have plotted α^{CIO} using differently parametrized sets of nested cones. We can observe that, with higher risk aversion, the proposed measure is decreasing. The more risk-averse agents are, the more they require access to all assets to diversify their risks and to obtain a less variable outcome. Because the group can provide them only with certain assets, the more risk-averse agents will seek more external than internal trades. By this argumentation, we can say that our result is in line with the general notion of tendency to trade internally.

As a final example, we investigate how the number of agents in a group influences the proposed measure of group's tendency to trade internally.

Example 4.32 *(continued) (number of agents)* Assume n agents with a given relative risk aversion coefficient γ and risk correlation ρ. By varying the number of agents n, we investigate the behavior of α^{CIO}. In Figure 4.6, we have plotted α^{CIO} using differently parametrized sets of nested cones. Except for the parametrizations by CVaR and minvar distortion functions, we can observe that as the number of agents increases, the proposed measure of group's tendency to trade internally is also increasing. Our proposed measure can be linked with group heterogeneity in this example as follows. Each agent brings his own specific source of risk. Hence, a large group can be perceived as more heterogeneous. Therefore, except for the CVaR and minvar parametrizations, we can say that the proposed

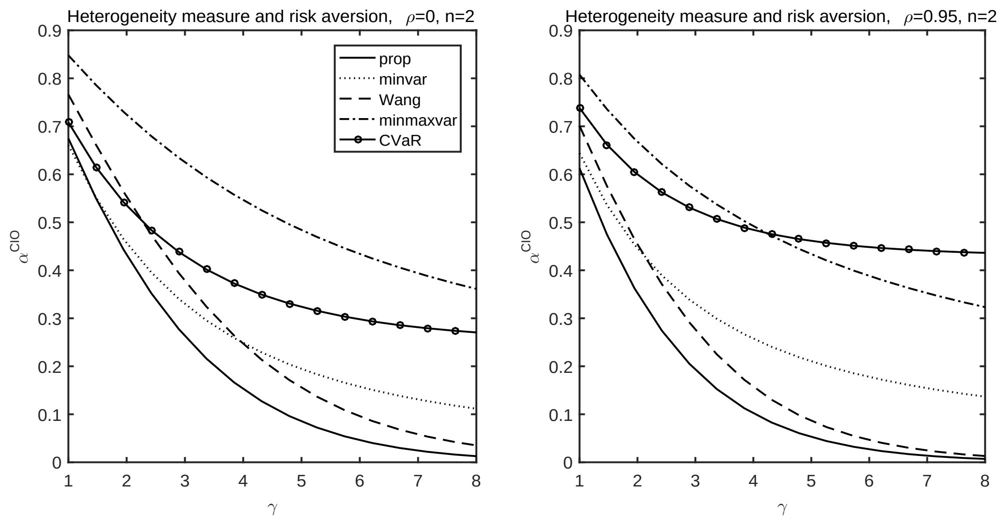

Figure 4.5: The dependency of the proposed measure of tendency to trade internally on the relative risk aversion parameter of agents, γ. The plots display the dependency for groups with correlation parameter $\rho = 0$ and $\rho = 0.95$, respectively.

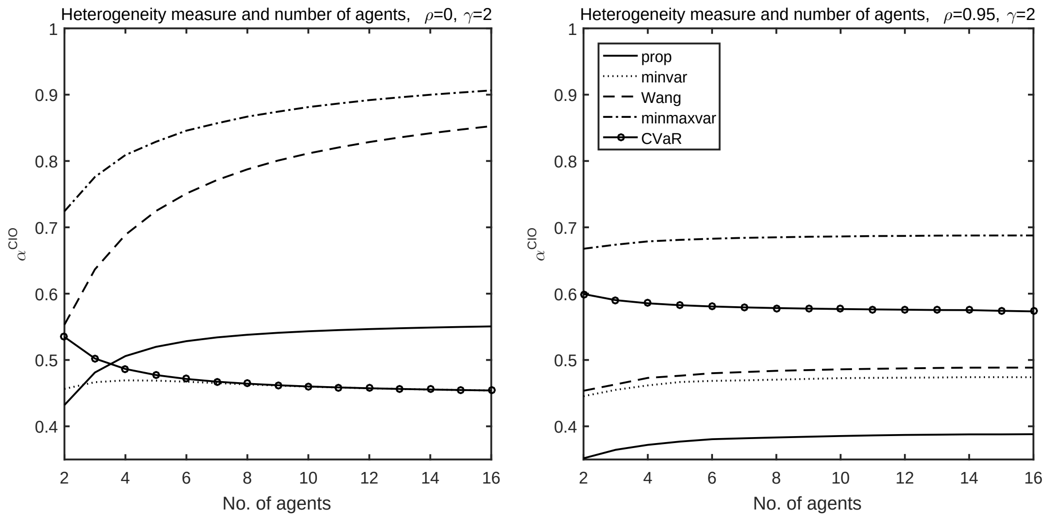

Figure 4.6: The dependency of the proposed measure of tendency to trade internally on the number of agents, n. The plots display the dependency for groups with correlation parameter $\rho = 0$ and $\rho = 0.95$, respectively.

measure of tendency to trade internally corresponds to general intuition behind the group heterogeneity with respect to the number of agents.

Based on the examples, it is confirmed that the proposed measure of tendency to trade internally is dependent on the chosen parametrization. While the measure of tendency to trade internally based on proportional hazard, minmaxvar and Wang's distortion functions is in line with an intuition behind the term measure of heterogeneity of group, the behavior of the measure for CVaR and minvar parametrizations is not. Within the settings of our example, we found that minvar parametrization can lead to nonmonotonic behavior of the measure of tendency to trade internally. Therefore, for practical usage of the proposed measure a suitable choice of distortion function must be done first.

4.7 Conclusions

In this chapter, we have analyzed different types of equilibrium trades in a conic economy. We considered a group of agents that is allowed to conclude intra-group trades as well as external trades with a conic market described by inelastic bid and ask prices. We have shown the equivalence between the conic equilibrium and a simplified Walrasian equilibrium. Firstly, we have described for which external conic markets the group can reach the conic equilibrium by internal trades only. We have described the structure of these external conic markets and their relation with the uniqueness of equilibria. Secondly, we have described the external conic markets for which the group can reach the conic equilibrium by external trades only. We have proposed a computationally simple rule on how to identify such external conic markets. Thirdly, we have proposed a measure of tendency to trade internally and its link with heterogeneity of the group. We have shown behavior of this measure with respect to chosen characteristics, and we concluded that for some parametrizations this behavior conforms with a general perception of measure of groups heterogeneity.

Our setting has several limitations, as we have considered only a finite asset space. However, this assumption is not crucial and could be relaxed. This shortfall could indicate the immediate direction of possible further research.

Another direction of possible further research regards the proposed measure of tendency to trade internally and its usage as a measure of heterogeneity. A more extensive comparison would provide a deeper insight into effects of different parametrizations of nested cones.

4.8 Appendix

This appendix gathers the proofs of various results in the main text of the chapter. We start by a simple observation that will be used several times.

Lemma 4.33 *Consider a conic economy given by* $\big(C, (U_1, \dots, U_n), (x_1^0, \dots, x_n^0)\big)$ *and let* p *be a pricing functional. Let, for some* i, x_i^*, $x_i^{*\mathrm{in}}$, $x_i^{*\mathrm{ex}}$ *be such that* $x_i^* = x_i^0 + x_i^{*\mathrm{in}} - x_i^{*\mathrm{ex}}$, $px_i^{*\mathrm{in}} = 0$, *and* $x_i^{*\mathrm{ex}} \in C$. *If* x_i^* *fulfills Condition (ii) in Definition 4.4, then* $px_i^{*\mathrm{ex}} = 0$.

Proof. By Remark 4.5, $p \in C^*$. From Condition (ii) in Definition 4.4, we find that $(x_i^{*\mathrm{in}}, x_i^{*\mathrm{ex}})$ solves

$$\max_{\{(x_i^{\mathrm{in}}, x_i^{\mathrm{ex}}) | x_i^0 + x_i^{\mathrm{in}} - x_i^{\mathrm{ex}} \in X_+,\, px_i^{\mathrm{in}} = 0,\, x_i^{\mathrm{ex}} \in C\}} U(x_i^0 + x_i^{\mathrm{in}} - x_i^{\mathrm{ex}}). \tag{4.2}$$

Because $x_i^{*\mathrm{ex}} \in C$ and $p \in C^*$, we have $px_i^{*\mathrm{ex}} \geq 0$ from the definition of C^*. If $px_i^{*\mathrm{ex}} = \alpha > 0$, then we can find $x \in X_{++}$ such that $px = \alpha$ and thus $p(x_i^{*\mathrm{ex}} - x) = 0$. Because $p(x_i^{*\mathrm{in}} - x_i^{*\mathrm{ex}} + x) = 0$, the pair $(x_i^{*\mathrm{in}} - x_i^{*\mathrm{ex}} + x, 0)$ is a feasible solution of (4.2). However, $U(x_i^0 + x_i^{*\mathrm{in}} - x_i^{*\mathrm{ex}} + x) = U(x_i^* + x) > U(x_i^*)$, which contradicts the optimality of $(x_i^{*\mathrm{in}}, x_i^{*\mathrm{ex}})$. □

Proof of Proposition 4.6. The claim follows by applying Lemma 4.33 to each individual agent. □

Proof of Proposition 4.9. We start with claim (i). Let p be a pricing functional that achieves the conic equilibrium with allocations $(x_i^*, x_i^{*\mathrm{in}}, x_i^{*\mathrm{ex}})$, $i = 1, \dots, n$. We want to show that p, x_i^*, and $y_i^* := -x_i^{*\mathrm{ex}}$, $i = 1, \dots, n$ fulfill Conditions (i)–(iii) in Definition 4.8. By Remark 4.5, $p \in C^*$ and therefore $-px^{\mathrm{ex}} \leq 0$ for all $x^{\mathrm{ex}} \in C$. From Lemma 4.33, we find that $0 = -px_i^{*\mathrm{ex}}$ and, hence, $-px_i^{\mathrm{ex}} \leq 0 = -px_i^{*\mathrm{ex}}$ for all $x_i^{\mathrm{ex}} \in C$. By setting $y_i = -x_i^{\mathrm{ex}}$, we get that $py_i \leq py_i^*$ for all $y_i \in Y$. Hence Condition (i) in Definition 4.8 holds. To fulfill Condition (ii) in Definition 4.8, we have to show that, for each $i = 1, \dots, n$, the allocation x_i^* solves the optimization problems

$$\max_{\{x_i \in X_+ | px_i \leq px_i^0 + py_i^*\}} U_i(x_i). \tag{4.3}$$

Note that $py_i^* = 0$ and the function U_i is strictly increasing. Therefore, x_i^* solves problem (4.3) if and only if it solves

$$\max_{\{x_i \in X_+ | px_i = px_i^0\}} U_i(x_i). \tag{4.4}$$

From the individual optimality of the conic equilibrium, we find that $x_i^* = x_i^0 + x_i^{*\text{in}} - x_i^{*\text{ex}}$ solves

$$\max_{\{x_i \in X_+ | x_i = x_i^0 + x_i^{\text{in}} - x_i^{\text{ex}},\, px_i^{\text{in}} = 0,\, x_i^{\text{ex}} \in C\}} U_i(x_i). \tag{4.5}$$

By Proposition 4.6 and because $px_i^{*\text{in}} = 0$, we have $px_i^* = px_i^0 + px_i^{*\text{in}} - px_i^{*\text{ex}} = px_i^0$. Hence, x_i^* is a feasible solution of problem (4.4). In fact, by Lemma 4.33, any solution of problem (4.5) is a feasible solution of problem (4.4). And since trivially the set of feasible solutions of problem (4.4) is included in the set of feasible solutions of problem (4.5) and the objective functions of both problems are the same, we see that x_i^* solves problem (4.4) if and only if it solves problem (4.5). By equivalence of problems (4.3) and (4.4) and by equivalence of problems (4.5) and (4.4), x_i^* solves problem (4.3) and Condition (ii) in Definition 4.8 holds. Condition (iii) in Definition 4.8 is fulfilled immediately because $\sum_{i=1}^n x_i^* = \sum_{i=1}^n (x_i^0 + x_i^{\text{in}} - x_i^{*\text{ex}}) = \sum_{i=1}^n x_i^0 + \sum_{i=1}^n y_i^*$.

We continue by showing the validity of the claim in (ii). We are given a pricing functional p and the set of allocations (x_i^*, y_i^*), $i = 1, \dots, n$, that constitute a Walrasian equilibrium, and we want to show that p achieves a conic equilibrium with the collection of triplets $(x_i^*, x_i^* - y_i^* - x_i^0, -y_i^*)$, $i = 1, \dots, n$. For $x_i^{*\text{in}} := x_i^* - y_i^* - x_i^0$, $x_i^{*\text{ex}} := -y_i^*$ we verify Conditions (i)–(iii) in the conic equilibrium definition, Definition 4.4.

Assume that $p \in C^*$. From the choice of $x_i^{*\text{in}}$ and $x_i^{*\text{ex}}$ it follows that $x_i^{*\text{ex}} \in C$ and $x_i^* = x_i^0 + x_i^{*\text{in}} - x_i^{*\text{ex}}$. To verify Condition (i) in Definition 4.4 it remains to show that $px_i^{*\text{in}} = 0$. Since $py_i \le py_i^*$ for all $y_i \in -C$, and $0 \in C$, we have that $py_i^* = 0 = px_i^{*\text{ex}}$. From the optimality of x_i^* in Condition (ii) in Definition 4.8 we get that $px_i^* = px_i^0$. By substitution we have that $px_i^{*\text{in}} = px_i^* + px_i^{*\text{ex}} - px_i^0 = 0$ and Condition (i) in Definition 4.4 holds. Condition (ii) in Definition 4.4 follows from a similar argumentation as above. Because x_i^* is a Walrasian equilibrium allocation, it solves problem (4.3). Because $px_i^{*\text{ex}} = -py_i^* = 0$, x_i^* also solves problem (4.4). Since the solution sets of problems (4.4) and (4.5) are the same, by the discussion above, x_i^* also solves (4.5). Conditions (iii) in Definition 4.4 follows immediately because

$$\sum_{i=1}^n x_i^{*\text{in}} = \sum_{i=1}^n (x_i^* - y_i^* - x_i^0) = \sum_{i=1}^n x_i^0 + \sum_{i=1}^n y_i^* + \sum_{i=1}^n (-y_i^* - x_i^0) = 0.$$

It remains to show that $p \in C^*$. Assume that $p \notin C^*$. Then $0 < py$ for some $y \in -C$. From the definition of the Walrasian equilibrium, we know that $py \le py_i^*$ for all $y \in -C$.

Therefore, $0 < py_i^*$. Because $py \leq py_i^*$ for all $y \in -C$, it also holds for $y := 2y_i^* \in -C$. However, then $0 < 2py_i^* \leq py_i^*$ and we have a contradiction. Therefore, $p \in C^*$. □

Proof of Proposition 4.13. If C is a CIO cone, then there exists a pricing functional p and a collection of triplets $(x_i^*, x_i^{*\text{in}}, 0)$, $i = 1, \dots, n$, such that they fulfill the conditions in Definition 4.4. Then the pricing functional p and the collection of pairs $(x_i^*, x_i^{*\text{in}})$, $i = 1, \dots, n$, also fulfill Conditions (i)–(iii) in Definition 4.12.

For the reverse implication, let a pricing functional p with a collection of triplets $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i = 1, \dots, n$, achieve conic equilibrium and with a collection of pairs $(x_i^*, x_i^{*\text{in}} - x_i^{*\text{ex}})$, $i = 1, \dots, n$, achieve IMO equilibrium. Then we show that p with the collection of triplets $(x_i^*, x_i^{*\text{in}} - x_i^{*\text{ex}}, 0)$, $i = 1, \dots, n$, achieves a conic equilibrium by verifying Conditions (i)–(iii) in Definition 4.4. Because the pairs $(x_i^*, x_i^{*\text{in}} - x_i^{*\text{ex}})$, $i = 1, \dots, n$, are positions in IMO equilibrium, we find that $p(x_i^{*\text{in}} - x_i^{*\text{ex}}) = 0$ so that Condition (i) holds. Since p with $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i = 1, \dots, n$, achieves a conic equilibrium, the total allocations of agents, x_i^*, fulfill Condition (ii). The remaining Condition (iii) is satisfied because $\sum_{i=1}^n (x_i^{*\text{in}} - x_i^{*\text{ex}}) = 0$ by the definition of the IMO equilibrium. Hence, the cone of marketed assets is a CIO cone. □

Proof of Proposition 4.15. Let C be a CIO cone. By Proposition 4.13 there exists $p \in \mathcal{P}$ which also achieves a conic equilibrium and hence $p \in C^*$, by Remark 4.5.

Let $p \in C^*$ achieve IMO equilibrium with a collection of pairs $(x_i^*, x_i^{*\text{in}})$, $i = 1, \dots, n$. We will show that p and the collection of triples $(x_i^*, x_i^{*\text{in}}, 0)$, $i = 1, \dots, n$, constitute a conic equilibrium. Conditions (i) and (iii) in Definition 4.4 are satisfied trivially. To satisfy Condition (ii), it remains to show that, for each $i = 1, \dots, n$, x_i^* solves

$$\max_{\{x_i \in X_+ | x_i = x_i^0 + x_i^{\text{in}} - x_i^{\text{ex}},\, px_i^{\text{in}} = 0,\, x_i^{\text{ex}} \in C\}} U_i(x_i). \tag{4.6}$$

Using Lemma 4.33, any solution of (4.6) also solves

$$\max_{\{x_i \in X_+ | x_i = x_i^0 + x_i^{\text{in}},\, px_i^{\text{in}} = 0\}} U_i(x_i). \tag{4.7}$$

Note also that all feasible solutions of (4.7) are included in the set of feasible solutions (4.6). Therefore, the sets of solutions for both problems are equal. Because $(x_i^*, x_i^{*\text{in}})$, $i = 1, \dots, n$, are IMO equilibrium allocations, each x_i^* solves problem (4.7) and, hence, x_i^*

solves also problem (4.6). Hence, p achieves a conic equilibrium with a collection of triples $(x_i^*, x_i^{*\text{in}}, 0)$ and by Proposition 4.13 the cone C is a CIO cone. □

Proof of Proposition 4.24. Let C be a CEO cone. From the definition, there exist a pricing functional p and a collection of triplets $(x_i^*, 0, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, such that the conditions in Definition 4.4 are fulfilled. Then the pricing functional p and the collection of pairs $(x_i^*, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, fulfill Conditions (i)–(ii) in Definition 4.23.

For the reverse implication, let there exist a pricing functional p which achieves a conic equilibrium with a collection of triplets $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, and an optimal EMO position with the collection of pairs $(x_i^*, x_i^{*\text{ex}} - x_i^{*\text{in}})$, $i = 1, \ldots, n$. To conclude the proof, we show that p with associated triplets $(x_i^*, 0, x_i^{*\text{ex}} - x_i^{*\text{in}})$, $i = 1, \ldots, n$, achieves a conic equilibrium. From the budget constraint in EMO position, we know that $x_i^{*\text{ex}} - x_i^{*\text{in}} \in C$ and thus Condition (i) of Definition 4.4 holds. Since p with $(x_i^*, x_i^{*\text{in}}, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, achieves a conic equilibrium, the total allocations of agents, x_i^*, fulfill Condition (ii). Condition (iii) holds trivially. Hence, C is a CEO cone. □

Proof of Proposition 4.25. Let C be a CEO cone; then there exist a pricing functional p and a collection of triplets $(x_i^*, 0, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, satisfying Condition (ii) in Definition 4.4. Hence, $(0, x_i^{*\text{ex}})$ solves

$$\max_{\{(x_i^{\text{in}}, x_i^{\text{ex}}) | x_i^0 + x_i^{\text{in}} - x_i^{\text{ex}} \in X_+,\, p x_i^{\text{in}} = 0,\, x_i^{\text{ex}} \in C\}} U_i(x_i^0 + x_i^{\text{in}} - x_i^{\text{ex}}). \tag{4.8}$$

From Proposition 4.6 we know that $p x_i^{*\text{ex}} = 0$; hence, also $(-x_i^{*\text{ex}}, 0)$ solves problem (4.8). Thus, $-x_i^{*\text{ex}}$ solves

$$\max_{\{s | x_i^0 + s \in X_+,\, ps = 0\}} U_i(x_i^0 + s). \tag{4.9}$$

The conditions are satisfied by $d_i := -x_i^{*\text{ex}}$.

For the reverse implication, we assume that p achieves a conic equilibrium and that $-C \ni d_i = \arg\max_{\{s|ps=0\}} U_i(x_i^0 + s)$, for all $i = 1, \ldots, n$. By Proposition 4.6, problems (4.8) and (4.9) have the same value of the objective function in the optimum. Hence, the triplets $(x_i^0 + d_i, 0, -d_i)$ comply with Condition (ii) in Definition 4.4. Conditions (i) and (iii) hold trivially for these triplets. Therefore, C is a CEO cone by definition. □

Proof of Proposition 4.27. We start by showing necessity. Let a cone C be a CEO cone. From Proposition 4.25 we know that there exist a pricing functional p and allocations d_i,

$i = 1, \ldots, n$, such that d_i solves

$$\max_{\{x|px=0\}} U_i(x_i^0 + x), \qquad \text{for } i = 1, \ldots, n. \tag{4.10}$$

From the first-order conditions of (4.10), it follows that $\left[\frac{\partial}{\partial x} U_i(x_i^0 + x)\right]_{x=d_i} \propto p$ for all i. From the same proposition it follows that a conic equilibrium is reached in p and that $d_i \in -C$, for $i = 1, \ldots, n$. Hence, by Propositions 4.24 and 4.6 the pair $(x_i^*, -d_i)$ is an optimal EMO position. The strict concavity of U_i, $i = 1, \ldots, n$, and the convexity of the constraint set guarantee the uniqueness of such position. Hence, $d_i = -x_i^{*\text{ex}}$ which concludes the necessity part.

To show the sufficiency, let there exist $p \in C^*$ such that $\left[\frac{\partial}{\partial x} U_i(x_i^0 + x)\right]_{x=-x_i^{*\text{ex}}} \propto p$ for $i = 1, \ldots, n$. Because $(x_i^*, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, are the optimal EMO positions, we have that $x_i^{*\text{ex}}$ solves

$$\max_{\{x|g^j(x)\leq 0\, j=1,\ldots,k\}} U_i(x_i^0 - x),$$

for each $i = 1, \ldots, n$. From the Kuhn-Tucker conditions, we know that there exist scalars $v_i^j \geq 0$, $j = 1, \ldots, k$, such that

$$\begin{aligned} 0 &= \left[\frac{\partial}{\partial x} U_i(x_i^0 - x)\right]_{x=x_i^{*\text{ex}}} - \sum_{j=1}^{k} v_i^j \left[\frac{\partial}{\partial x} g^j(x)\right]_{x=x_i^{*\text{ex}}}, \\ 0 &= v_i^j g^j(x_i^{*\text{ex}}), \\ 0 &\geq g^j(x_i^{*\text{ex}}). \end{aligned}$$

We want to show that the pricing functional p achieves the conic equilibrium with the collection of triplets $(x_i^*, 0, x_i^{*\text{ex}})$, $i = 1, \ldots, n$, because then C is a CEO cone. Conditions (i) and (iii) in the definition of conic economy are fulfilled trivially. It remains to be shown that $(0, x_i^{*\text{ex}})$ solves

$$\max_{\{(x^{\text{in}},x^{\text{ex}})|px^{\text{in}}=0,\, g^j(x^{\text{ex}})\leq 0\, j=1,\ldots,k\}} U_i(x_i^0 + x^{\text{in}} - x^{\text{ex}}). \tag{4.11}$$

From the Kuhn-Tucker conditions, if some $(\hat{x}_i^{\text{in}}, \hat{x}_i^{\text{ex}})$ solves problem (4.11), then scalars λ_i

and $w_i^j \geq 0$, $j = 1, \dots, k$, exist such that

$$0 = \left[\frac{\partial}{\partial x} U_i(x_i^0 + x - \hat{x}_i^{\text{ex}})\right]_{x=\hat{x}_i^{\text{in}}} - \lambda_i p,$$

$$0 = \left[\frac{\partial}{\partial x} U_i(x_i^0 + \hat{x}_i^{\text{in}} - x)\right]_{x=\hat{x}_i^{\text{ex}}} - \sum_{j=1}^{k} w_i^j \left[\frac{\partial}{\partial x} g^j(x)\right]_{x=\hat{x}_i^{\text{ex}}},$$

$$0 = p\hat{x}_i^{\text{in}},$$

$$0 = w_i^j g^j(\hat{x}_i^{\text{ex}}),$$

$$0 \geq g^j(\hat{x}_i^{\text{ex}}).$$

The second order condition is satisfied because we assume that $-U_i$ and g_i are twice differentiable convex functions. Therefore, if and only if there exist scalars λ_i and w_i^j, $j = 1, \dots, k$, fulfilling these local optimality conditions, then $(\hat{x}_i^{\text{in}}, \hat{x}_i^{\text{ex}})$ solves problem (4.11). Since $\left[\frac{\partial}{\partial x} U_i(x_i^0 + x)\right]_{x=-x_i^{*\text{ex}}} \propto p$, let us set λ_i such that $\lambda_i p = \left[\frac{\partial}{\partial x} U_i(x_i^0 + x)\right]_{x=-x_i^{*\text{ex}}}$. By setting $\hat{x}_i^{\text{in}} := 0$, $\hat{x}_i^{\text{ex}} := x_i^{*\text{ex}}$, and $w_i^j := v_i^j$, $j = 1, \dots, n$, the local optimality conditions are satisfied and hence $(0, x_i^{*\text{ex}})$ solves problem (4.11). Therefore, p achieves a conic equilibrium with the collection of allocations $(x_i^*, 0, x_i^{*\text{ex}})$, $i = 1, \dots, n$, and the cone C is a CEO cone. □

Proof of Proposition 4.29. Fix $i \in \{1, \dots, n\}$. Because $(x_i^*, x_i^{*\text{ex}})$ is the optimal EMO allocation and $Ay_i = -x_i^{*\text{ex}}$, y_i solves

$$\max_{\{y | y \geq 0\}} U_i(x_i^0 + Ay).$$

To simplify our notation, let $u_i(x) := \frac{\partial}{\partial x} U_i(x)$. From the Kuhn-Tucker conditions, there exists a vector $v_i \in \mathbb{R}_+^k$ such that

$$A^\top u_i(x_i^0 + Ay_i) - v_i = 0$$
$$y_i^\top v_i = 0.$$

Because $(y_i)_j > 0$, for $j \in J_1$, the last equality implies that $(v_i)_j = 0$, for $j \in J_1$. Let B denote the submatrix of $A^\top$ formed from the rows indexed by J_1. From the assumptions, the rows of $B \in \mathbb{R}^{(m-1)\times m}$ are linearly independent. Therefore, the dimension of the null space of B is 1. Note that the vector $Bu_i(x_i^0 + Ay_i) = 0 \in \mathbb{R}^{m-1}$, for all $i = 1, \dots, n$. Since the dimension of the null space of B is 1, we have that $u_{i_1}(x_{i_1}^0 + Ay_{i_1}) \propto u_{i_2}(x_{i_2}^0 + Ay_{i_2})$ for any $i_1, i_2 \in \{1, \ldots, n\}$. Now we will show that $u_i(x_i^0 + Ay_i) \in C^*$. Because the functions U_i are strictly increasing, we have that $u_i(x_i^0 + Ay_i) > 0$. We know that for any $x \in C$, there exists $y \geq 0$ such that $x = Ay$. Note that $q \in C^*$ if $A^\top q \geq 0$, because for any $x \in C$, we have $q^\top x = q^\top Ay \geq 0$. From the nonnegativity of v_i, we have that $A^\top u_i(x_i^0 + Ay_i) \geq 0$ and hence $u_i(x_i^0 - x_i^{*\mathrm{ex}}) \in C^*$. We have shown that there exists a pricing functional $p \in C^*$ such that $u_i(x_i^0 - x_i^{*\mathrm{ex}}) \propto p$ for all i. By Proposition 4.27, we conclude that C is a CEO cone. □

Example 4.34 *(Construction of a cone of marketed assets from bid-ask spreads)* Suppose that the space of assets is three-dimensional, and that it is generated by the assets a_0, a_1, and a_2. Assume that the external market allows anyone to buy from the market assets a_1 and a_2 for p_1 and p_2 units of asset a_0, respectively; and to sell these assets to the market for q_1 and q_2 units of a_0, respectively. To avoid degenerate cases assume that $p_i \geq q_i \geq 0$, for $i = 1, 2$. The external market also allows the free disposal of non-negative assets. All these allowed trades with external market can be organized as columns of a matrix, for instance,

$$A = \begin{pmatrix} -1 & 0 & 0 & -p_1 & q_1 & -p_2 & q_2 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The cone of marketed assets C is then defined as $C = -\operatorname{pos}(A)$, where $\operatorname{pos}(A)$ is the set of nonnegative linear combinations of the columns of the matrix A.

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