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On interactive sequencing situations with exponential cost functions

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Abstract

This paper addresses interactive one-machine sequencing situations in which the costs of processing a job are given by an exponential function of its completion time. The main difference with the standard linear case is that the gain of switching two neighbors in a queue is time-dependent and depends on their exact position. We illustrate that finding an optimal order is complicated in general and we identify specific subclasses, which are tractable from an optimization perspective. More specifically, we show that in these subclasses, all neighbor switches in any path from the initial order to an optimal order lead to a non-negative gain. Moreover, we derive conditions on the time-dependent neighbor switching gains in a general interactive sequencing situation to guarantee convexity of the corresponding cooperative game. These conditions are satisfied within our specific subclasses of exponential interactive sequencing situations.

Keywords: interactive sequencing situation, initial order, exponential cost function, sequencing games, convexity
JEL classification: C44, C71

1 Introduction

In an interactive one-machine sequencing situation, several jobs have to be processed on a single machine. Each job is associated to a player with a specific cost function which is defined in terms of the completion time, that is, the time this job spends in the system. Furthermore, it is assumed that there is an initial order on the players that prescribes the rights to be processed by the machine. From an optimization perspective, the objective is to determine an optimal processing order that minimizes the total aggregate costs.

In the standard setting, the cost functions are assumed to be linear, specified by linear cost coefficients. Smith (1956) shows that, in order to minimize the total costs, the players should be processed in a non-increasing order of their urgency indices. Here, the urgency index is defined as the ratio of the linear cost coefficient and the processing time.

To obtain reasonable allocations of the total cost savings reached by reordering the players from the initial order to an optimal order, Curiel, Pederzoli, and Tijs (1989) defined for each standard sequencing situation an associated transferable utility cooperative game. It was shown that standard sequencing games allow for coalitionally stable cost allocations and in particular, allow for a coalitionally stable cost allocation rule, only based on repairs of neighbor misplacements. This Equal Gain Splitting rule (EGS-rule) is analyzed and characterized by Curiel et al. (1989) and defined by recursively splitting the corresponding neighbor switching gains equally in every step in a path from the initial order to an optimal order that repairs all neighbor misplacements. Moreover, standard sequencing games are shown to be convex, which provide a definite drive to indeed cooperate.

The analysis of interactive standard sequencing situations has been extended in many directions. We mention Hamers, Borm, and Tijs (1995) by imposing ready times, Borm, Fiestras-Janeiro, Hamers, Sánchez, and Voorneveld (2002) by imposing due dates, Rustogi and Strusevich (2012) by studying positional effects, Lohmann, Borm, and Slikker (2014) by analyzing just-in-time arrivals and Musegaas, Borm, and Quant (2015) by considering step out-step in sequencing games. Moreover, Grundel, Çiftçi, Borm, and Hamers (2013), Gerichhausen and Hamers (2009) and Çiftçi, Borm, Hamers, and Slikker (2013) studied the grouping of players in families or batches. Finally, we mention Klijn and Sánchez (2006), who studied uncertainty sequencing games.

In this paper, we primarily deal with interactive sequencing situations with exponential cost functions, specified by exponential cost coefficients. This type of cost functions models the fact that players could have increasing marginal costs, which is a standard modeling choice in several fields, in particular economics. We will show that finding an optimal order in exponential interactive sequencing situations, in general, is difficult. Therefore, we will carefully explore the differences between the standard linear model and the exponential model by investigating the underlying neighbor switching gains. Contrary to the standard linear model, neighbor switching gains are time-dependent and depend on the exact position in the queue. Moreover, it is seen that an optimal order can not always be reached from the initial order by recursively repairing neighbor misplacements such that every neighbor switching gain is non-negative.

We identify several subclasses that are tractable from an optimization perspective, that is, no difficulties arise with respect to the above-mentioned features. More specifically, for these subclasses, there exists a comparison index between the jobs to determine all optimal orders (similar to the Smith urgency index). These subclasses involve special cases of the exponential cost coefficients and the processing times. In particular, two subclasses consist of exponential interactive sequencing situations with identical exponential cost coefficients and identical processing times, respectively. The third subclass we study, consists of exponential interactive sequencing situations in which both the cost coefficients and the processing times can only take two possible values, respectively. This subclass is referred to as the high-low model.

To analyze allocations of the cost savings, we introduce exponential sequencing games, similar to the standard sequencing games as considered in Curiel et al. (1989). Again, using the component-additive structure, exponential sequencing games allow for coalitionally stable allocations. However, it is seen that convexity, in general, does not hold. Interestingly, convexity of the games corresponding to the specific subclasses of exponential interactive sequencing situations can be established on the basis of a general result for interactive sequencing situations. This result is obtained by imposing specific conditions on the time-dependent neighbor switching gains of misplacements and non-misplacements. In particular, these conditions will guarantee that an optimal order for subcoalitions can be obtained from an optimal order for the grand coalition.

Although the specific subclasses of exponential interactive sequencing situations lead to convex games and every path from the initial order to an optimal order recursively repairs neighbor misplacements, all with non-negative gains, we will see that however no direct extension of the EGS-rule will always lead to coalitionally stable cost allocations.

The structure of the paper is as follows. Section 2 describes interactive sequencing situations. Section 3 contains the analysis of exponential interactive sequencing situations, including the specific subclasses. Finally, Section 4 focuses on sequencing games and in particular, on the convexity of these games.

2 Interactive sequencing situations

An *interactive sequencing situation* is represented by a tuple (N, σ_0, p, c), where $N = \{1, 2, \ldots, n\}$ is the *set of players* that each have a job that needs to be processed on a single machine. The order σ_0 is the *initial order* which prescribes the initial rights to be processed by the machine. Formally, an order is described by a bijective function $\sigma : N \to \{1, 2, \ldots, n\}$ in which $\sigma(i) = k$ means that the job of player i is in position k. We denote by $\Pi(N)$ the *set of all orders* of N. The vector $p = (p_i)_{i \in N} \in \mathbb{R}^N_{++}$ is the vector that contains the *processing times* of the jobs of the different players. Finally, the vector $c = (c_i)_{i \in N}$ specifies the *cost functions* for the players, for all $i \in N$ given by $c_i : [0, \infty) \to \mathbb{R}$, where $t \in [0, \infty)$ represents the number of time units spend in the system. It is assumed that the machine starts processing at time $t = 0$ and that all players are present in the system at that time. The set of all interactive sequencing situations with player set N is denoted by SEQ^N.

A *standard sequencing situation* is an interactive sequencing situation $(N, \sigma_0, p, c) \in SEQ^N$ in which each cost function c_i is given by a linear function: $c_i : [0, \infty) \to \mathbb{R}$, with $c_i(t) = \alpha_i t$. Here, $\alpha_i \in \mathbb{R}_{++}$ is referred to as the *linear cost coefficient* of player $i \in N$. The set of all standard sequencing situations with player set N is denoted by $SSEQ^N$.

Let $(N, \sigma_0, p, c) \in SEQ^N$ be an interactive sequencing situation. The *completion time* of the job of player $i \in N$ with respect to an order $\sigma \in \Pi(N)$ is denoted by $C_i(\sigma)$ and given by $C_i(\sigma) = \sum_{k \in N : \sigma(k) \leq \sigma(i)} p_k$. Moreover, the *starting time* of the job of player $i \in N$ with respect to an order $\sigma \in \Pi(N)$ is denoted by $t_i(\sigma)$ and given by $t_i(\sigma) = \sum_{k \in N : \sigma(k) < \sigma(i)} p_k$. For a given order $\sigma \in \Pi(N)$, the *total costs* of this order are denoted by $TC(\sigma)$ and are given by

$$TC(\sigma) = \sum_{i \in N} c_i(C_i(\sigma)).$$

An order is *optimal* if the total costs of this order are minimal among all orders. Formally, an optimal order $\hat{\sigma}$ satisfies $TC(\hat{\sigma}) \leq TC(\sigma)$ for all $\sigma \in \Pi(N)$. Accordingly, the *set of misplacements* with respect to an optimal order $\hat{\sigma}$ is defined as the set of pairs of players that are ordered differently in the initial order σ_0 and the optimal order $\hat{\sigma}$:

$$MP(\sigma_0, \hat{\sigma}) = \{(i, j) \in N \times N \,:\, \sigma_0(i) < \sigma_0(j) \text{ and } \hat{\sigma}(i) > \hat{\sigma}(j)\}.$$

In every interactive sequencing situation, each optimal order can be obtained from the initial order by recursively switching two consecutive players, i.e., by recursively repairing neighbor misplacements. This is due to a basic result for permutations: every permutation is the product of transpositions, where a transposition is a permutation that only interchanges two elements. Hence, it is possible to obtain any optimal order from the initial one by recursively interchanging a pair of consecutive players that are misplaced.

For notational convenience, a sequence of orders $(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m)$ with $\sigma_m = \hat{\sigma}$ corresponding to transpositions $(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$ is called a *path* from the initial order σ_0 to an optimal order $\hat{\sigma}$ if, for every $k \in \{1, 2, \ldots, m\}$, $\sigma_{k-1}(j_k) = \sigma_{k-1}(i_k) + 1$ and $\sigma_k(i_k) = \sigma_{k-1}(j_k)$, $\sigma_k(j_k) = \sigma_{k-1}(i_k)$ and $\sigma_k(\ell) = \sigma_{k-1}(\ell)$ for every $\ell \in \{1, \ldots, n\}$ with $\ell \neq i_k, j_k$. In other words, a path from the initial order σ_0 to an optimal order $\hat{\sigma}$ repairs all misplacements in $MP(\sigma_0, \hat{\sigma})$ one by one, by only switching two misplaced neighbors in each step. Note that $m = |MP(\sigma_0, \hat{\sigma})|$.

In every step of a path from the initial order to an optimal order, the total costs change. Since this change in total costs is only caused by the pair of consecutive players that are switched in that step, the change in total costs is called the *(neighbor switching) gain* of these two players. Formally, the neighbor switching gain of players $i, j \in N$ with i directly in front of player j at time $t \in [0, \infty)$ (see also Figure 1) is defined by

$$g_{ij}(t) = c_i(t + p_i) + c_j(t + p_i + p_j) - c_i(t + p_i + p_j) - c_j(t + p_j). \tag{1}$$

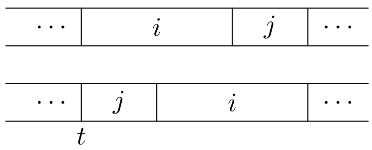

Figure 1: Interchanging players i and j, leading to the neighbor gain $g_{ij}(t)$.

Note that the neighbor switching gain can be either positive, negative or zero. Moreover, every path from the initial order to an optimal order leads to the same total cost savings. That is, for every path from the initial order to an optimal order, $(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m)$ with $\sigma_m = \hat{\sigma}$, corresponding to transpositions $(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$, it holds that

$$TC(\sigma_0) - TC(\hat{\sigma}) = \sum_{k=1}^{m} g_{i_k j_k}(t_{i_k}(\sigma_{k-1})).$$

For a standard sequencing situation $(N, \sigma_0, p, c) \in SSEQ^N$, Smith (1956) showed that the total costs are minimized if the players are ordered in non-increasing order of their urgency indices, where the *urgency index* u_i is defined by $u_i = \frac{\alpha_i}{p_i}$ for all $i \in N$. Consequently, using the urgency index as a comparison index, there exists an optimal order σ^* such that

$$MP(\sigma_0, \sigma^*) = \{(i, j) \in N \times N \ : \ \sigma_0(i) < \sigma_0(j) \text{ and } u_i < u_j\}. \tag{2}$$

Note that for this particular optimal order, the number of misplacements (i.e. the number of pairs of players that are ordered in the initial order in a different way than in the optimal order) is the smallest with respect to all optimal orders. In that sense, σ^* is the optimal order that is closest to the initial order.

Using Equation (1), the neighbor switching gain of players $i, j \in N$ at time $t \in [0, \infty)$ of a standard sequencing situation $(N, \sigma_0, p, c) \in SSEQ^N$ becomes independent of the time:

$$\begin{aligned} g_{ij}(t) &= c_i(t + p_i) + c_j(t + p_i + p_j) - c_i(t + p_i + p_j) - c_j(t + p_j) \\ &= \alpha_i t + \alpha_i p_i + \alpha_j t + \alpha_j p_i + \alpha_j p_j - \alpha_i t - \alpha_i p_i - \alpha_i p_j - \alpha_j t - \alpha_j p_j \\ &= \alpha_j p_i - \alpha_i p_j. \end{aligned} \tag{3}$$

Consequently, the total cost savings are equal to the sum of the neighbor switching gains corresponding to misplacements:

$$TC(\sigma_0) - TC(\hat{\sigma}) = \sum_{(i,j) \in MP(\sigma_0, \hat{\sigma})} (\alpha_j p_i - \alpha_i p_j).$$

For the remainder of this paper, we primarily focus on neighbor switching gains. Due to Smith (1956), we have the following lemma, which will be the baseline for the corresponding results in the upcoming sections.

Lemma 2.1. *Let $(N, \sigma_0, p, c) \in SSEQ^N$ be a standard sequencing situation. Let $i, j \in N$ be two players such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma}$ be an optimal order. Then,*

1) $g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$, if $(i, j) \in MP(\sigma_0, \hat{\sigma})$;

2) $g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$, if $(i, j) \notin MP(\sigma_0, \hat{\sigma})$.

Moreover, $g'_{ij}(t) = 0$ for all $t \in [0, \infty)$.

Proof. Note that $g_{ij}(t) = \alpha_j p_i - \alpha_i p_j$, according to Equation (3).

1) If $(i, j) \in MP(\sigma_0, \hat{\sigma})$, then, $u_i \leq u_j$. Hence, $\alpha_i p_j \leq \alpha_j p_i$ and thus $g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$.

2) If $(i, j) \notin MP(\sigma_0, \hat{\sigma})$, then, $u_i \geq u_j$. Hence, $\alpha_i p_j \geq \alpha_j p_i$ and thus $g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$.

Obviously, $g'_{ij}(t) = 0$ for all $t \in [0, \infty)$. □

Summarizing, we stress the following important features for a standard sequencing situation:

- The neighbor switching gains are constant in time and do not depend on the exact position in the queue;
- An optimal order can be reached from the initial order by consecutively repairing all neighbor misplacements, where the corresponding neighbor switching gains in each step are non-negative.

3 Exponential sequencing situations

An *exponential sequencing situation* is represented by an interactive sequencing situation $(N, \sigma_0, p, c) \in SEQ^N$, where the cost function of player $i \in N$ is given by an exponential function, i.e., $c_i : [0, \infty) \to \mathbb{R}$, with $c_i(t) = e^{\alpha_i t}$ and $\alpha_i \in \mathbb{R}_{++}$ the *exponential cost coefficient* of player $i \in N$. The set of all exponential sequencing situations with player set N is denoted by $ESEQ^N$.

For an exponential sequencing situation $(N, \sigma_0, p, c) \in ESEQ^N$, the neighbor switching gain of two consecutive players $i, j \in N$ at time $t \in [0, \infty)$ is given by

$$\begin{aligned} g_{ij}(t) &= c_i(t + p_i) + c_j(t + p_i + p_j) - c_i(t + p_i + p_j) - c_j(t + p_j) \\ &= e^{\alpha_i(t+p_i)} + e^{\alpha_j(t+p_i+p_j)} - e^{\alpha_i(t+p_j+p_i)} - e^{\alpha_j(t+p_j)}. \end{aligned} \tag{4}$$

Contrary to the standard sequencing situations, it immediately follows from Equation (4) that the neighbor switching gains for exponential sequencing situations are time-dependent. Moreover, the following example shows that it is possible that there are negative neighbor switching gains in every path from the initial order to an optimal order.

Example 3.1. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation, where $N = \{1, 2, 3, 4\}$, $\sigma_0 = (1, 2, 3, 4)$ and the exponential cost coefficients and processing times as specified by the table below.*

	player 1	*player 2*	*player 3*	*player 4*
α_i	*1.880*	*1.904*	*1.902*	*1.549*
p_i	*1.205*	*1.940*	*1.976*	*1.357*

The total costs for all orders are computed and shown below.

σ	$TC(\sigma)$	σ	$TC(\sigma)$	σ	$TC(\sigma)$
$(1, 2, 3, 4)$	40192.0978	$(2, 3, 1, 4)$	39731.0259	$(3, 4, 1, 2)$	232608.8238
$(1, 2, 4, 3)$	225869.7331	$(2, 3, 4, 1)$	199870.9139	$(3, 4, 2, 1)$	217726.1909
$(1, 3, 2, 4)$	40392.5376	$(2, 4, 1, 3)$	229338.6158	$(4, 1, 2, 3)$	229805.7814
$(1, 3, 4, 2)$	228882.7154	$(2, 4, 3, 1)$	217473.5852	$(4, 1, 3, 2)$	233055.4929
$(1, 4, 2, 3)$	229736.6016	$(3, 1, 2, 4)$	40397.1073	$(4, 2, 1, 3)$	229673.8692
$(1, 4, 3, 2)$	232986.3132	$(3, 1, 4, 2)$	228887.2851	$(4, 2, 3, 1)$	217808.8386
$(2, 1, 3, 4)$	40193.6761	$(3, 2, 1, 4)$	39747.2069	$(4, 3, 1, 2)$	232965.9098
$(2, 1, 4, 3)$	225871.3114	$(3, 2, 4, 1)$	199887.0949	$(4, 3, 2, 1)$	218083.2769

Note that $TC((2, 3, 1, 4)) = 39731.0259 < TC(\sigma)$ for all $\sigma \in \Pi(N)$. So there is a unique optimal order, $\hat{\sigma} = (2, 3, 1, 4)$. Then, $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3)\}$. Hence, there is only one path from the initial order to the optimal order:

$$(1, 2, 3, 4) \to (2, 1, 3, 4) \to (2, 3, 1, 4),$$

that is, the path $(\sigma_0, \sigma_1, \sigma_2)$ with $\sigma_0 = (1, 2, 3, 4), \sigma_1 = (2, 1, 3, 4)$ and $\sigma_2 = \hat{\sigma} = (2, 3, 1, 4)$ corresponding to transpositions $(1, 2)$ and $(1, 3)$, respectively. Notice that the neighbor switching gains in the first step of this path are negative, that is,

$$g_{12}(0) = TC(\sigma_0) - TC(\sigma_1) = 40192.0978 - 40193.6761 < 0.$$

In fact, every path from the initial order to any other order (optimal or not) has a negative neighbor switching gain in the first step: $g_{23}(p_1) = TC(\sigma_0) - TC((1,3,2,4)) < 0$ *and* $g_{34}(p_1 + p_2) = TC(\sigma_0) - TC((1,2,4,3)) < 0$.

Example 3.1 shows that finding an optimal order, in general, is complicated. One should allow for neighbor switches with a negative gain. Thus, bilateral considerations are not sufficient in finding an optimal order. Additionally, it is seen that, in general, the two important features as discussed at the end of Section 2 no longer hold if the cost functions are exponential.

3.1 A special case: identical exponential cost coefficients

This section is the first of three sections that describe specific subclasses of exponential sequencing situations. For this section, it is assumed that the exponential cost coefficients are common for all jobs, that is, there exists an $\alpha \in \mathbb{R}_{++}$ such that all individual exponential cost coefficients α_i equal α. For this special case, the following theorem characterizes an optimal order. It shows that this subclass is tractable from an optimization perspective.

Theorem 3.2. *Let* $(N, \sigma_0, p, c) \in ESEQ^N$ *be an exponential sequencing situation such that, for all* $i \in N$ *and all* $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$ *with* $\alpha \in \mathbb{R}_{++}$. *Let* $\hat{\sigma} \in \Pi(N)$ *be an order. Then it holds that* $\hat{\sigma}$ *is optimal if and only if*

$$p_{\hat{\sigma}^{-1}(1)} \leq \cdots \leq p_{\hat{\sigma}^{-1}(k-1)} \leq p_{\hat{\sigma}^{-1}(k)} \leq p_{\hat{\sigma}^{-1}(k+1)} \leq \cdots \leq p_{\hat{\sigma}^{-1}(n)}. \tag{5}$$

Proof. The proof consists of two parts: we first show the 'only if' part. Secondly, we show that any two orders that satisfy Equation (5) yield the same total costs. Together, this implies that an order is optimal if and only if it satisfies Equation (5).

For the first claim, assume that $\hat{\sigma}$ is optimal and suppose for the sake of contradiction that it does not satisfy Equation (5). Then there exists an $\ell \in \{1, 2, \ldots, n-1\}$ such that $p_{\hat{\sigma}^{-1}(\ell)} > p_{\hat{\sigma}^{-1}(\ell+1)}$. Denote $i = \hat{\sigma}^{-1}(\ell)$ and $j = \hat{\sigma}^{-1}(\ell + 1)$, such that we have that $p_i > p_j$ and i and j are consecutive players in the order $\hat{\sigma}$.

Define $\tau \in \Pi(N)$ as follows: $\tau(k) = \hat{\sigma}(k)$ for all $k \in N \setminus \{i, j\}$, $\tau(i) = \hat{\sigma}(j)$ and $\tau(j) = \hat{\sigma}(i)$. In other words, τ is the order where players i and j are interchanged. Then,

$$\begin{aligned}
TC(\hat{\sigma}) - TC(\tau) &= \sum_{p \in N} e^{\alpha_p C_p(\hat{\sigma})} - \sum_{p \in N} e^{\alpha_p C_p(\tau)} \\
&= e^{\alpha_i C_i(\hat{\sigma})} + e^{\alpha_j C_j(\hat{\sigma})} - e^{\alpha_i C_i(\tau)} - e^{\alpha_j C_j(\tau)} \\
&= e^{\alpha t_i(\hat{\sigma})}(e^{\alpha p_i} + e^{\alpha(p_i+p_j)} - e^{\alpha(p_i+p_j)} - e^{\alpha p_j}) \\
&= e^{\alpha t_i(\hat{\sigma})}(e^{\alpha p_i} - e^{\alpha p_j}) > 0,
\end{aligned}$$

where the inequality follows from the fact that $e^{\alpha t} > 0$ for all $t \in [0, \infty)$ and $e^{\alpha p_i} - e^{\alpha p_j} > 0$, since $p_i > p_j$. Moreover, the second equality is due to the fact that $C_p(\hat{\sigma}) = C_p(\tau)$ for all $p \in N \setminus \{i, j\}$. Consequently, $TC(\hat{\sigma}) > TC(\tau)$, contradicting the optimality of $\hat{\sigma}$. This proves the first claim.

For the second claim, consider two different orders $\sigma, \sigma' \in \Pi(N), \sigma \neq \sigma'$ satisfying Equation (5). Since both orders satisfy Equation (5), the only differences can be within a block of players with identical processing times. For these players (within a certain block) it holds that both the processing times and the cost coefficients are identical, respectively. Hence, the combined costs of all players in one block is the same for both orders. Then it follows that $TC(\sigma) = TC(\sigma')$. □

In particular, Theorem 3.2 provides a comparison index to determine the set of misplacements beforehand. This is shown by the following corollary.

Corollary 3.3. *Let* $(N, \sigma_0, p, c) \in ESEQ^N$ *be an exponential sequencing situation such that, for all* $i \in N$ *and all* $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$ *with* $\alpha \in \mathbb{R}_{++}$. *Then, there exists an optimal order* $\sigma^* \in \Pi(N)$ *such that*

$$MP(\sigma_0, \sigma^*) = \{(i, j) \in N \times N \,:\, \sigma_0(i) < \sigma_0(j) \text{ and } p_i > p_j\}.$$

This comparison index, based on processing times only, provides all pairs of players that needs to be interchanged in order to reach a particular optimal order σ^* from the initial order. Similar as in Equation (2), σ^* corresponds to the optimal order that is closest to the initial order. Obviously, other optimal orders can be obtained from σ^* by interchanging two consecutive players that have identical comparison indices, i.e. identical processing times. By doing so, the total costs do not change.

Similar to Lemma 2.1, we show that all neighbor switching gains of misplacements are non-negative, while the neighbor switching gains are non-positive for non-misplacements.

Lemma 3.4. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that, for all $i \in N$ and all $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$ with $\alpha \in \mathbb{R}_{++}$. Let $i, j \in N$ be two players such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then,*

1) $g_{ij}(t) \geq 0$ *and* $g'_{ij}(t) \geq 0$ *for all* $t \in [0, \infty)$*, if* $(i, j) \in MP(\sigma_0, \hat{\sigma})$*;*
2) $g_{ij}(t) \leq 0$ *and* $g'_{ij}(t) \leq 0$ *for all* $t \in [0, \infty)$*, if* $(i, j) \notin MP(\sigma_0, \hat{\sigma})$*.*

Proof. Using Equation (4), we have that $g_{ij}(t) = e^{\alpha t}(e^{\alpha p_i} - e^{\alpha p_j})$ for all $t \in [0, \infty)$.

1) If $(i, j) \in MP(\sigma_0, \hat{\sigma})$, then, $p_i \geq p_j$. Hence, $e^{\alpha p_i} - e^{\alpha p_j} \geq 0$. Since $e^{\alpha t} > 0$ for all $t \in [0, \infty)$, $g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$. Similarly, we can conclude that $g'_{ij}(t) = \alpha e^{\alpha t}(e^{\alpha p_i} - e^{\alpha p_j}) \geq 0$.
2) If $(i, j) \notin MP(\sigma_0, \hat{\sigma})$, then, $p_i \leq p_j$. Hence, $e^{\alpha p_i} - e^{\alpha p_j} \leq 0$ and thus $g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$. Analogously, it can be readily checked that $g'_{ij}(t) = \alpha e^{\alpha t}(e^{\alpha p_i} - e^{\alpha p_j}) \leq 0$.

This concludes the proof. □

From the first statement of Lemma 3.4, we notice that every step in every path from the initial order to an optimal order comes with a non-negative neighbor switching gain.

3.2 A special case: identical processing times

This section describes another specific subclass of exponential sequencing situations. Here, it is assumed that the processing times are identical for all jobs, that is, there exists a $p \in \mathbb{R}_{++}$ such that $p_i = p$ for all $i \in N$. For this special case, the following theorem shows that this particular subclass is optimizationally tractable.

Theorem 3.5. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that, for all $i \in N$, $p_i = p$ with $p \in \mathbb{R}_{++}$. Let $\hat{\sigma} \in \Pi(N)$ be an order. Then it holds that $\hat{\sigma}$ is optimal if and only if*

$$\alpha_{\hat{\sigma}^{-1}(1)} \geq \cdots \geq \alpha_{\hat{\sigma}^{-1}(k-1)} \geq \alpha_{\hat{\sigma}^{-1}(k)} \geq \alpha_{\hat{\sigma}^{-1}(k+1)} \geq \cdots \geq \alpha_{\hat{\sigma}^{-1}(n)}. \tag{6}$$

Proof. Similar to the proof of Theorem 3.2, this proof is also based on two steps. First, we show that any optimal order should satisfy Equation (6). Secondly, we provide an argument to show that any two orders that satisfy Equation (6) have equal total costs. Together, this implies that an order is optimal if and only if Equation (6) is satisfied.

For the first step, assume that $\hat{\sigma}$ is optimal and suppose for the sake of contradiction that Equation (6) is not satisfied. Then there exists an $\ell \in \{1, 2, \ldots, n-1\}$ such that $\alpha_{\hat{\sigma}^{-1}(\ell)} < \alpha_{\hat{\sigma}^{-1}(\ell+1)}$. Denote $i = \hat{\sigma}^{-1}(\ell)$ and $j = \hat{\sigma}^{-1}(\ell+1)$, such that $\alpha_i < \alpha_j$ and i and j are consecutive players in the order $\hat{\sigma}$.

Define $\tau \in \Pi(N)$ as the order where players i and j are interchanged, i.e. $\tau(k) = \hat{\sigma}(k)$ for all $k \in N \setminus \{i, j\}$ and $\tau(i) = \hat{\sigma}(j)$ and $\tau(j) = \hat{\sigma}(i)$. Then,

$$\begin{aligned}
TC(\hat{\sigma}) - TC(\tau) &= \sum_{p \in N} e^{\alpha_p C_p(\hat{\sigma})} - \sum_{p \in N} e^{\alpha_p C_p(\tau)} \\
&= e^{\alpha_i C_i(\hat{\sigma})} + e^{\alpha_j C_j(\hat{\sigma})} - e^{\alpha_i C_i(\tau)} - e^{\alpha_j C_j(\tau)} \\
&= e^{\alpha_i (t_i(\hat{\sigma})+p)} + e^{\alpha_j (t_i(\hat{\sigma})+2p)} - e^{\alpha_i (t_i(\hat{\sigma})+2p)} - e^{\alpha_j (t_i(\hat{\sigma})+p)} \\
&= e^{\alpha_i (t_i(\hat{\sigma})+p)} \left(1 - e^{\alpha_i p}\right) + e^{\alpha_j (t_i(\hat{\sigma})+p)} \left(e^{\alpha_j p} - 1\right) \\
&> e^{\alpha_i (t_i(\hat{\sigma})+p)} \left(1 - e^{\alpha_i p}\right) + e^{\alpha_i (t_i(\hat{\sigma})+p)} \left(e^{\alpha_j p} - 1\right)
\end{aligned}$$

$$
\begin{aligned}
&> e^{\alpha_i(t_i(\hat{\sigma})+p)}\left(1-e^{\alpha_i p}\right)+e^{\alpha_i(t_i(\hat{\sigma})+p)}\left(e^{\alpha_i p}-1\right) \\
&= 0,
\end{aligned}
$$

where the first inequality is due to the fact that $e^{\alpha_j(t_i(\hat{\sigma})+p)} > e^{\alpha_i(t_i(\hat{\sigma})+p)}$, since $\alpha_i < \alpha_j$, together with $e^{\alpha_j p} - 1 > 0$. Moreover, the second inequality is due to the fact that $e^{\alpha_j p} > e^{\alpha_i p}$ together with $e^{\alpha_i(t_i(\hat{\sigma})+p)} > 0$. Hence, $TC(\hat{\sigma}) > TC(\tau)$, which is a contradiction.

For the second step, notice that the only differences between two orders $\sigma, \sigma' \in \Pi(N), \sigma \neq \sigma'$ satisfying Equation (6) occur within a block of players with identical cost parameters. Hence, all these players within a certain block have the identical processing times and cost parameters, such that the combined costs for every block of players is the same for both orders. Consequently, $TC(\sigma) = TC(\sigma')$. □

Analogously to the previous section, Theorem 3.5 provides a comparison index to determine the set of misplacements.

Corollary 3.6. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that, for all $i \in N$, $p_i = p$ with $p \in \mathbb{R}_{++}$. Then, there exists an optimal order $\sigma^* \in \Pi(N)$ such that*

$$MP(\sigma_0, \sigma^*) = \{(i,j) \in N \times N \, : \, \sigma_0(i) < \sigma_0(j) \text{ and } \alpha_i < \alpha_j\}.$$

Note again that due to this comparison index, based on the exponential cost coefficients only, it is possible to determine which pairs of players need to be interchanged in order to reach the specific optimal order σ^* from the initial order.

To show that also in this subclass we have that every step in every path from the initial order to an optimal order comes with a non-negative neighbor switching gain, we provide the following lemma.

Lemma 3.7. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that, for all $i \in N$, $p_i = p$ with $p \in \mathbb{R}_{++}$. Let $i, j \in N$ be two players such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then,*

1) *$g_{ij}(t) \geq 0$ and $g'_{ij}(t) \geq 0$ for all $t \in [0, \infty)$, if $(i,j) \in MP(\sigma_0, \hat{\sigma})$;*
2) *$g_{ij}(t) \leq 0$ and $g'_{ij}(t) \leq 0$ for all $t \in [0, \infty)$, if $(i,j) \notin MP(\sigma_0, \hat{\sigma})$.*

Proof. Using Equation (4), we have that $g_{ij}(t) = e^{\alpha_i(t+p)}(1 - e^{\alpha_i p}) - e^{\alpha_j(t+p)}(1 - e^{\alpha_j p})$ for all $t \in [0, \infty)$. In order to prove 1), assume that $(i,j) \in MP(\sigma_0, \hat{\sigma})$. Then, $\alpha_i \leq \alpha_j$ and, for all $t \in [0, \infty)$,

$$
\begin{aligned}
g_{ij}(t) = e^{\alpha_i(t+p)}(1-e^{\alpha_i p}) - e^{\alpha_j(t+p)}(1-e^{\alpha_j p}) &= e^{\alpha_i(t+p)}(1-e^{\alpha_i p}) + e^{\alpha_j(t+p)}(e^{\alpha_j p}-1) \\
&\geq e^{\alpha_i(t+p)}(1-e^{\alpha_i p}) + e^{\alpha_i(t+p)}(e^{\alpha_j p}-1) \\
&\geq e^{\alpha_i(t+p)}(1-e^{\alpha_i p}) + e^{\alpha_i(t+p)}(e^{\alpha_i p}-1) \\
&= 0,
\end{aligned}
$$

where the first inequality follows from the fact that $\alpha_i \leq \alpha_j$ implies that, for all $t \in [0, \infty)$, $e^{\alpha_j(t+p)} \geq e^{\alpha_i(t+p)}$ and $e^{\alpha_j p} - 1 > 0$. The second inequality is due to the fact that $e^{\alpha_j p} \geq e^{\alpha_i p}$. Moreover,

$$
\begin{aligned}
g'_{ij}(t) = \alpha_i e^{\alpha_i(t+p)}(1-e^{\alpha_i p}) - \alpha_j e^{\alpha_j(t+p)}(1-e^{\alpha_j p}) &= \alpha_i e^{\alpha_i(t+p)}(1-e^{\alpha_i p}) + \alpha_j e^{\alpha_j(t+p)}(e^{\alpha_j p}-1) \\
&\geq \alpha_i e^{\alpha_i(t+p)}(1-e^{\alpha_i p}) + \alpha_i e^{\alpha_i(t+p)}(e^{\alpha_i p}-1) \\
&= 0,
\end{aligned}
$$

according to a similar reasoning as before. The proof of 2) can be obtained analogously. □

3.3 A special case: the high-low model

Finally, we describe a third subclass of exponential sequencing situations based on both the exponential cost coefficients and the processing times. We study exponential sequencing situations in which each exponential cost coefficient can only take two possible values, either a high value or a low value. Moreover, also each processing time can only take either a high value or a low value. Consequently, there are only four possible different types of players: a player can either have a high or a low exponential cost coefficient and either have a high or a low processing time. These four groups are denoted by G_{LL}, G_{LH}, G_{HL} and G_{HH}. Formally, for an exponential sequencing situation $(N, \sigma_0, p, c) \in ESEQ^N$ such that, for every player $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}$ and $p_i \in \{p_L, p_H\}$ with $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ satisfying $\alpha_L < \alpha_H$ and $p_L < p_H$:

- $G_{LL} = \{i \in N \,:\, \alpha_i = \alpha_L \text{ and } p_i = p_L\}$;
- $G_{LH} = \{i \in N \,:\, \alpha_i = \alpha_L \text{ and } p_i = p_H\}$;
- $G_{HL} = \{i \in N \,:\, \alpha_i = \alpha_H \text{ and } p_i = p_L\}$;
- $G_{HH} = \{i \in N \,:\, \alpha_i = \alpha_H \text{ and } p_i = p_H\}$.

For convenience, for all $i \in N$, G_i denotes the group G_{LL}, G_{LH}, G_{HL} or G_{HH} to which player i belongs. In addition, we introduce, for any order $\sigma \in \Pi(N)$, an order relation $\prec_\sigma$ on the set of players: for all $i, j \in N$, $i \prec_\sigma j$ if $\sigma(i) < \sigma(j)$. With a slight abuse of notation, we extend this order relation to be used for groups of players, that is, for any order $\sigma \in \Pi(N)$ and any two groups of players $G, G' \in \{G_{LL}, G_{LH}, G_{HL}, G_{HH}\}$, $G \prec_\sigma G'$ if for all $i \in G$ and all $j \in G'$ we have that $i \prec_\sigma j$.

For an exponential sequencing situation in this particular subclass, there are only four types of players, such that it follows that, if we found an optimal order, we can make other optimal orders by interchanging players from the same group with each other.

The following theorem shows that any optimal order starts with the players in G_{HL}, i.e. players having a high exponential cost coefficient and low processing times and ends with G_{LH}, i.e. players with a low exponential cost coefficient and a high processing time.

Theorem 3.8. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}$ and $p_i \in \{p_L, p_H\}$ with $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ satisfying $\alpha_L < \alpha_H$ and $p_L < p_H$. Let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then,*

1) $G_{HL} \prec_{\hat{\sigma}} G$ for all $G \in \{G_{LL}, G_{LH}, G_{HH}\}$;

2) $G \prec_{\hat{\sigma}} G_{LH}$ for all $G \in \{G_{LL}, G_{HL}, G_{HH}\}$.

Proof. Suppose for the sake of contradiction that $\hat{\sigma}$ does not satisfy 1) or 2). Then there are two neighbors that are not ordered according to 1) or 2). That is, there exists an $\ell \in \{1, 2, \ldots, n-1\}$ such that for $i = \hat{\sigma}^{-1}(\ell)$ and $j = \hat{\sigma}^{-1}(\ell + 1)$ one of the following five cases is satisfied:

i) $i \in G_{LL}$ and $j \in G_{HL}$.

ii) $i \in G_{LH}$ and $j \in G_{HL}$;

iii) $i \in G_{HH}$ and $j \in G_{HL}$;

iv) $i \in G_{LH}$ and $j \in G_{LL}$;

v) $i \in G_{LH}$ and $j \in G_{HH}$.

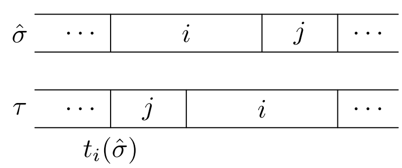

Figure 2: Interchanging i and j in orders $\hat{\sigma}$ and τ.

Define $\tau \in \Pi(N)$ such that players i and j are interchanged. That is, $\tau(i) = \hat{\sigma}(j), \tau(j) = \hat{\sigma}(i)$ and $\tau(k) = \hat{\sigma}(k)$ for all $k \in N \setminus \{i, j\}$. Figure 2 is a schematic representation of both orders. Note that it is also possible that $p_i = p_j$. We deal with the above five cases separately.

i) For the first case, we have that $\alpha_i = \alpha_L, p_i = p_L, \alpha_j = \alpha_H$ and $p_j = p_L$. Then,

$$\begin{aligned}
TC(\hat{\sigma}) - TC(\tau) &= \sum_{p\in N} e^{\alpha_p C_p(\hat{\sigma})} - \sum_{p\in N} e^{\alpha_p C_p(\tau)} \\
&= e^{\alpha_i C_i(\hat{\sigma})} + e^{\alpha_j C_j(\hat{\sigma})} - e^{\alpha_i C_i(\tau)} - e^{\alpha_j C_j(\tau)} \\
&= e^{\alpha_L (t_i(\hat{\sigma})+p_L)} + e^{\alpha_H (t_i(\hat{\sigma})+p_L+p_L)} - e^{\alpha_L (t_i(\hat{\sigma})+p_L+p_L)} - e^{\alpha_H (t_i(\hat{\sigma})+p_L)} \\
&= e^{\alpha_H (t_i(\hat{\sigma})+p_L)}(e^{\alpha_H p_L} - 1) - e^{\alpha_L (t_i(\hat{\sigma})+p_L)}(e^{\alpha_L p_L} - 1) \\
&> 0,
\end{aligned}$$

where the last inequality follows from the fact that $e^{\alpha_H p_L} - 1 > e^{\alpha_L p_L} - 1 > 0$ and $e^{\alpha_H (t_i(\hat{\sigma})+p_L)} > e^{\alpha_L (t_i(\hat{\sigma})+p_L)} > 0$, since $\alpha_H > \alpha_L$.

ii) Now, we have that $\alpha_i = \alpha_L, p_i = p_H, \alpha_j = \alpha_H$ and $p_j = p_L$. Then,

$$\begin{aligned}
TC(\hat{\sigma}) - TC(\tau) &= \sum_{p\in N} e^{\alpha_p C_p(\hat{\sigma})} - \sum_{p\in N} e^{\alpha_p C_p(\tau)} \\
&= e^{\alpha_i C_i(\hat{\sigma})} + e^{\alpha_j C_j(\hat{\sigma})} - e^{\alpha_i C_i(\tau)} - e^{\alpha_j C_j(\tau)} \\
&= e^{\alpha_L (t_i(\hat{\sigma})+p_H)} + e^{\alpha_H (t_i(\hat{\sigma})+p_H+p_L)} - e^{\alpha_L (t_i(\hat{\sigma})+p_L+p_H)} - e^{\alpha_H (t_i(\hat{\sigma})+p_L)} \\
&> e^{\alpha_L (t_i(\hat{\sigma})+p_H)} + e^{\alpha_H (t_i(\hat{\sigma})+p_H+p_L)} - e^{\alpha_L (t_i(\hat{\sigma})+p_L+p_H)} - e^{\alpha_H (t_i(\hat{\sigma})+p_H)} \\
&= e^{\alpha_H (t_i(\hat{\sigma})+p_H)}(e^{\alpha_H p_L} - 1) - e^{\alpha_L (t_i(\hat{\sigma})+p_H)}(e^{\alpha_L p_L} - 1) \\
&> 0,
\end{aligned}$$

where the first inequality follows from the fact that $e^{\alpha_H (t_i(\hat{\sigma})+p_L)} < e^{\alpha_H (t_i(\hat{\sigma})+p_H)}$, since $p_L < p_H$, and the last inequality from a similar reasoning as in the first case, now using both $\alpha_L < \alpha_H$ and $p_L < p_H$.

iii) In this case, we have that $\alpha_i = \alpha_H, p_i = p_H, \alpha_j = \alpha_H$ and $p_j = p_L$. Then,

$$\begin{aligned}
TC(\hat{\sigma}) - TC(\tau) &= \sum_{p\in N} e^{\alpha_p C_p(\hat{\sigma})} - \sum_{p\in N} e^{\alpha_p C_p(\tau)} \\
&= e^{\alpha_i C_i(\hat{\sigma})} + e^{\alpha_j C_j(\hat{\sigma})} - e^{\alpha_i C_i(\tau)} - e^{\alpha_j C_j(\tau)} \\
&= e^{\alpha_H (t_i(\hat{\sigma})+p_H)} + e^{\alpha_H (t_i(\hat{\sigma})+p_H+p_L)} - e^{\alpha_H (t_i(\hat{\sigma})+p_H+p_L)} - e^{\alpha_H (t_i(\hat{\sigma})+p_L)} \\
&= e^{\alpha_H (t_i(\hat{\sigma})+p_H)} - e^{\alpha_H (t_i(\hat{\sigma})+p_L)} \\
&> 0,
\end{aligned}$$

where the inequality is ensured by $p_L < p_H$.

iv) Here, $\alpha_i = \alpha_L, p_i = p_H, \alpha_j = \alpha_L$ and $p_j = p_L$. Then,

$$\begin{aligned}
TC(\hat{\sigma}) - TC(\tau) &= \sum_{p\in N} e^{\alpha_p C_p(\hat{\sigma})} - \sum_{p\in N} e^{\alpha_p C_p(\tau)} \\
&= e^{\alpha_i C_i(\hat{\sigma})} + e^{\alpha_j C_j(\hat{\sigma})} - e^{\alpha_i C_i(\tau)} - e^{\alpha_j C_j(\tau)} \\
&= e^{\alpha_L (t_i(\hat{\sigma})+p_H)} + e^{\alpha_L (t_i(\hat{\sigma})+p_H+p_L)} - e^{\alpha_L (t_i(\hat{\sigma})+p_L+p_H)} - e^{\alpha_L (t_i(\hat{\sigma})+p_L)} \\
&= e^{\alpha_L (t_i(\hat{\sigma})+p_H)} - e^{\alpha_L (t_i(\hat{\sigma})+p_L)} \\
&> 0,
\end{aligned}$$

since $p_L < p_H$.

v) In the last case, we have that $\alpha_i = \alpha_L, p_i = p_H, \alpha_j = \alpha_H$ and $p_j = p_H$. Then,

$$\begin{aligned}
TC(\hat{\sigma}) - TC(\tau) &= \sum_{p\in N} e^{\alpha_p C_p(\hat{\sigma})} - \sum_{p\in N} e^{\alpha_p C_p(\tau)} \\
&= e^{\alpha_i C_i(\hat{\sigma})} + e^{\alpha_j C_j(\hat{\sigma})} - e^{\alpha_i C_i(\tau)} - e^{\alpha_j C_j(\tau)} \\
&= e^{\alpha_L (t_i(\hat{\sigma})+p_H)} + e^{\alpha_H (t_i(\hat{\sigma})+p_H+p_H)} - e^{\alpha_L (t_i(\hat{\sigma})+p_H+p_H)} - e^{\alpha_H (t_i(\hat{\sigma})+p_H)}
\end{aligned}$$

$$= e^{\alpha_H(t_i(\hat{\sigma})+p_H)}(e^{\alpha_H p_H} - 1) - e^{\alpha_L(t_i(\hat{\sigma})+p_H)}(e^{\alpha_L p_H} - 1)$$
$$> 0,$$

according to a similar reasoning as in the first two cases.

In all cases, $TC(\hat{\sigma}) - TC(\tau) > 0$, contradicting the fact that $\hat{\sigma}$ is an optimal order. $\square$

The following example shows that ordering the players in G_{LL} and G_{HH} optimally is complicated in general. In particular, it shows that these players need not be ordered in groups.

Example 3.9. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation, where $N = \{1,2,3,4,5\}$, $\sigma_0 = (1,2,3,4,5)$ and, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}$ with $\alpha_L = 1.748$ and $\alpha_H = 1.781$ and $p_i \in \{p_L, p_H\}$ with $p_L = 0.342$ and $p_H = 0.368$. The processing times and the exponential cost coefficients of the players are specified by the following table:*

	player 1	*player 2*	*player 3*	*player 4*	*player 5*
α_i	α_H	α_H	α_L	α_L	α_L
p_i	p_L	p_H	p_H	p_L	p_L

Note that $G_{LL} = \{4,5\}, G_{LH} = \{3\}, G_{HL} = \{1\}$ and $G_{HH} = \{2\}$. According to Theorem 3.8, player 1 is the first player and player 3 the last player in any optimal order. That leaves only six orders, for which the total costs are computed and shown below.

σ	$TC(\sigma)$
$(1,2,4,5,3)$	44.8630
$(1,2,5,4,3)$	44.8630
$(1,4,2,5,3)$	44.8495
$(1,4,5,2,3)$	44.8862
$(1,5,2,4,3)$	44.8495
$(1,5,4,2,3)$	44.8862

Consequently, it is easily seen that both $(1,4,2,5,3)$ and $(1,5,2,4,3)$ are optimal. Note that the players in G_{LL} are not ordered as a group in both optimal orders.

The following theorem provides a condition on the four high-low parameters for which the corresponding exponential sequencing situation is tractable from an optimization perspective.

Theorem 3.10. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}$ and $p_i \in \{p_L, p_H\}$ with $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ satisfying $\alpha_L < \alpha_H$ and $p_L < p_H$. Let $\hat{\sigma} \in \Pi(N)$ be an optimal order. If it holds that*

$$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)}, \tag{7}$$

then $G_{HH} \prec_{\hat{\sigma}} G_{LL}$.

Proof. Assume that $\alpha_L, \alpha_H, p_L, p_H$ satisfy Equation (7) and suppose for the sake of contradiction that $G_{HH} \not\prec_{\hat{\sigma}} G_{LL}$. According to Theorem 3.8, the players from the group G_{HL} are all in the beginning of $\hat{\sigma}$ and the players from the group G_{LH} at the end of $\hat{\sigma}$. Therefore, there exists an $\ell \in \{1,2,\ldots,n-1\}$ such that for $i = \hat{\sigma}^{-1}(\ell)$ and $j = \hat{\sigma}^{-1}(\ell+1)$ it holds that $i \in G_{LL}$ and $j \in G_{HH}$. Then, $\alpha_i = \alpha_L, p_i = p_L, \alpha_j = \alpha_H$ and $p_j = p_H$. Similar as in the proof of Theorem 3.8, define $\tau \in \Pi(N)$ such that players i and j are interchanged, i.e. $\tau(i) = \hat{\sigma}(j), \tau(j) = \hat{\sigma}(i)$ and $\tau(k) = \hat{\sigma}(k)$ for all $k \in N \setminus \{i,j\}$ (see also Figure 2). Consequently,

$$\begin{aligned}
TC(\hat{\sigma}) - TC(\tau) &= \sum_{p \in N} e^{\alpha_p C_p(\hat{\sigma})} - \sum_{p \in N} e^{\alpha_p C_p(\tau)} \\
&= e^{\alpha_i C_i(\hat{\sigma})} + e^{\alpha_j C_j(\hat{\sigma})} - e^{\alpha_i C_i(\tau)} - e^{\alpha_j C_j(\tau)} \\
&= e^{\alpha_L(t_i(\hat{\sigma})+p_L)} + e^{\alpha_H(t_i(\hat{\sigma})+p_L+p_H)} - e^{\alpha_L(t_i(\hat{\sigma})+p_L+p_H)} - e^{\alpha_H(t_i(\hat{\sigma})+p_H)}
\end{aligned}$$

$$
\begin{aligned}
&= e^{\alpha_L t_i(\hat{\sigma})}\left(e^{\alpha_L p_L} - e^{\alpha_L(p_L+p_H)}\right) + e^{\alpha_H t_i(\hat{\sigma})}\left(e^{\alpha_H(p_L+p_H)} - e^{\alpha_H p_H}\right) \\
&> e^{\alpha_L t_i(\hat{\sigma})}\left(e^{\alpha_L p_L} - e^{\alpha_H p_H} + e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)}\right) \\
&\geq 0,
\end{aligned}
$$

where the first inequality follows from the fact that $e^{\alpha_H t_i(\hat{\sigma})} > e^{\alpha_L t_i(\hat{\sigma})}$, since $\alpha_H > \alpha_L$. The second inequality follows from the fact that $e^{\alpha_L t_i(\hat{\sigma})} > 0$, by definition, and $e^{\alpha_L p_L} - e^{\alpha_H p_H} + e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)} \geq 0$, by assumption.

Subsequently, $TC(\tau) < TC(\hat{\sigma})$, which yields a contradiction. □

Note that, if $\frac{\alpha_L}{p_L} \leq \frac{\alpha_H}{p_H}$, then Equation (7) is satisfied. Interestingly, combining Theorem 3.8 and Theorem 3.10 leads to a characterization of an optimal order for an exponential sequencing situation within the high-low model with the high-low parameters satisfying Equation (7). An optimal order $\hat{\sigma}$ should order players from the same group consecutively and order the four groups in the following way:

$$G_{HL} \prec_{\hat{\sigma}} G_{HH} \prec_{\hat{\sigma}} G_{LL} \prec_{\hat{\sigma}} G_{LH}.$$

This leads to a comparison index based on the exponential cost coefficients and processing times only, which is shown by the following corollary.

Corollary 3.11. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}$ and $p_i \in \{p_L, p_H\}$ with $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ satisfying $\alpha_L < \alpha_H$ and $p_L < p_H$. If it holds that*

$$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)}, \tag{8}$$

then there exists an optimal order $\sigma^* \in \Pi(N)$ such that

$$
\begin{aligned}
MP(\sigma_0, \sigma^*) = \Big\{(i,j) \in N \times N \ : \ \sigma_0(i) < \sigma_0(j) \text{ and either } & i \in G_{LH}, j \in G_{HL} \cup G_{HH} \cup G_{LL} \\
\text{or } & i \in G_{LL}, j \in G_{HL} \cup G_{HH} \\
\text{or } & i \in G_{HH}, j \in G_{HL}\Big\}.
\end{aligned}
$$

Consequently, the pairs of players that need to be interchanged in order to reach an optimal order from the initial order can be determined beforehand.

Similar as Lemma 3.4 and Lemma 3.7, we have that every step in every path from the initial order to an optimal order comes with a non-negative neighbor switching gain for all exponential sequencing situations within the high-low model such that the corresponding parameters satisfy Equation (7).

Lemma 3.12. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}$ and $p_i \in \{p_L, p_H\}$ with $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ satisfying $\alpha_L < \alpha_H$ and $p_L < p_H$. Let $i, j \in N$ be two players such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. If it holds that*

$$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)}, \tag{9}$$

then,

1) $g_{ij}(t) \geq 0$ and $g'_{ij}(t) \geq 0$ for all $t \in [0, \infty)$, if $(i,j) \in MP(\sigma_0, \hat{\sigma})$;

2) $g_{ij}(t) \leq 0$ and $g'_{ij}(t) \leq 0$ for all $t \in [0, \infty)$, if $(i,j) \notin MP(\sigma_0, \hat{\sigma})$.

Proof. According to Equation (4), the neighbor switching gain of players i and j is, for all $t \in [0, \infty)$ given by

$$g_{ij}(t) = e^{\alpha_i(t+p_i)} + e^{\alpha_j(t+p_i+p_j)} - e^{\alpha_i(t+p_j+p_i)} - e^{\alpha_j(t+p_j)}.$$

First, remark that both players i and j could be in the same group, i.e. $G_i = G_j$. In that case, $g_{ij}(t) = 0$ and $g'_{ij}(t) = 0$ for all $t \in [0, \infty)$, since $\alpha_i = \alpha_j$ and $p_i = p_j$. Secondly, assume that players i and j are not in the same group. We only prove 2), since the proof of 1) can be obtained analogously and follows the same reasoning as in the proofs of Theorems 3.8 and 3.10. If $(i,j) \notin MP(\sigma_0, \hat{\sigma})$, then one of the following six cases holds:

i) $i \in G_{HL}$ and $j \in G_{HH}$;

ii) $i \in G_{HL}$ and $j \in G_{LL}$;

iii) $i \in G_{HL}$ and $j \in G_{LH}$;

iv) $i \in G_{HH}$ and $j \in G_{LL}$;

v) $i \in G_{HH}$ and $j \in G_{LH}$;

vi) $i \in G_{LL}$ and $j \in G_{LH}$.

We prove that $g_{ij}(t) < 0$ for all $t \in [0, \infty)$ for the above six cases separately.

i) In the first case, we have that $\alpha_i = \alpha_H, p_i = p_L, \alpha_j = \alpha_H$ and $p_j = p_H$. Then, for all $t \in [0, \infty)$,

$$\begin{aligned}
g_{ij}(t) &= e^{\alpha_H(t+p_L)} + e^{\alpha_H(t+p_L+p_H)} - e^{\alpha_H(t+p_H+p_L)} - e^{\alpha_H(t+p_H)} \\
&= e^{\alpha_H(t+p_L)} - e^{\alpha_H(t+p_H)} \\
&= e^{\alpha_H t}(e^{\alpha_H p_L} - e^{\alpha_H p_H}) \\
&< 0,
\end{aligned}$$

where the inequality follows from the fact that $e^{\alpha_H t} > 0$ and $e^{\alpha_H p_L} - e^{\alpha_H p_H} < 0$ (since $p_L < p_H$). With regard to the derivative, we have that, for all $t \in [0, \infty)$,

$$\begin{aligned}
g'_{ij}(t) &= \alpha_H e^{\alpha_H(t+p_L)} + \alpha_H e^{\alpha_H(t+p_L+p_H)} - \alpha_H e^{\alpha_H(t+p_H+p_L)} - \alpha_H e^{\alpha_H(t+p_H)} \\
&= \alpha_H e^{\alpha_H(t+p_L)} - e^{\alpha_H(t+p_H)} \\
&= \alpha_H e^{\alpha_H t}(e^{\alpha_H p_L} - e^{\alpha_H p_H}) \\
&< 0,
\end{aligned}$$

according to a similar reasoning as above.

ii) Secondly, $\alpha_i = \alpha_H, p_i = p_L, \alpha_j = \alpha_L$ and $p_j = p_L$. Then, for all $t \in [0, \infty)$,

$$\begin{aligned}
g_{ij}(t) &= e^{\alpha_H(t+p_L)} + e^{\alpha_L(t+p_L+p_L)} - e^{\alpha_H(t+p_L+p_L)} - e^{\alpha_L(t+p_L)} \\
&= e^{\alpha_H(t+p_L)}(1 - e^{\alpha_H p_L}) + e^{\alpha_L(t+p_L)}(e^{\alpha_L p_L} - 1) \\
&< e^{\alpha_H(t+p_L)}(1 - e^{\alpha_H p_L}) + e^{\alpha_H(t+p_L)}(e^{\alpha_L p_L} - 1) \\
&= e^{\alpha_H(t+p_L)}(e^{\alpha_L p_L} - e^{\alpha_H p_L}) \\
&< 0,
\end{aligned}$$

where the first inequality follows from $\alpha_L < \alpha_H$ and $e^{\alpha_L p_L} - 1 > 0$ and the second inequality follows from $\alpha_L p_L < \alpha_H p_L$. In a similar way,

$$\begin{aligned}
g'_{ij}(t) &= \alpha_H e^{\alpha_H(t+p_L)} + \alpha_L e^{\alpha_L(t+p_L+p_L)} - \alpha_H e^{\alpha_H(t+p_L+p_L)} - \alpha_L e^{\alpha_L(t+p_L)} \\
&= \alpha_H e^{\alpha_H(t+p_L)}(1 - e^{\alpha_H p_L}) + \alpha_L e^{\alpha_L(t+p_L)}(e^{\alpha_L p_L} - 1) \\
&< \alpha_H e^{\alpha_H(t+p_L)}(1 - e^{\alpha_H p_L}) + \alpha_H e^{\alpha_H(t+p_L)}(e^{\alpha_L p_L} - 1) \\
&= \alpha_H e^{\alpha_H(t+p_L)}(e^{\alpha_L p_L} - e^{\alpha_H p_L}) \\
&< 0.
\end{aligned}$$

iii) In this case, $\alpha_i = \alpha_H, p_i = p_L, \alpha_j = \alpha_L$ and $p_j = p_H$. Then, for all $t \in [0, \infty)$,

$$\begin{aligned}
g_{ij}(t) &= e^{\alpha_H(t+p_L)} + e^{\alpha_L(t+p_L+p_H)} - e^{\alpha_H(t+p_H+p_L)} - e^{\alpha_L(t+p_H)} \\
&< e^{\alpha_H(t+p_H)} + e^{\alpha_L(t+p_L+p_H)} - e^{\alpha_H(t+p_H+p_L)} - e^{\alpha_L(t+p_H)} \\
&= e^{\alpha_H(t+p_H)}(1 - e^{\alpha_H p_L}) + e^{\alpha_L(t+p_H)}(e^{\alpha_L p_L} - 1)
\end{aligned}$$

$$
\begin{aligned}
&< e^{\alpha_H(t+p_H)}(1-e^{\alpha_H p_L}) + e^{\alpha_H(t+p_H)}(e^{\alpha_L p_L}-1) \\
&= e^{\alpha_H(t+p_H)}(e^{\alpha_L p_L}-e^{\alpha_H p_L}) \\
&< 0,
\end{aligned}
$$

using the fact that $p_L < p_H$ in the first inequality, $\alpha_L < \alpha_H$ together with $e^{\alpha_L p_L} - 1 > 0$ in the second inequality and finally, $\alpha_L p_L < \alpha_H p_L$ in the last one. Similarly, $g'_{ij}(t) < 0$ for all $t \in [0, \infty)$.

iv) Here, $\alpha_i = \alpha_H, p_i = p_H, \alpha_j = \alpha_L$ and $p_j = p_L$. Then, for all $t \in [0, \infty)$,

$$
\begin{aligned}
g_{ij}(t) &= e^{\alpha_H(t+p_H)} + e^{\alpha_L(t+p_H+p_L)} - e^{\alpha_H(t+p_L+p_H)} - e^{\alpha_L(t+p_L)} \\
&= e^{\alpha_H t}(e^{\alpha_H p_H} - e^{\alpha_H(p_L+p_H)}) + e^{\alpha_L t}(e^{\alpha_L(p_L+p_H)} - e^{\alpha_L p_L}) \\
&< e^{\alpha_H t}(e^{\alpha_H p_H} - e^{\alpha_H(p_L+p_H)}) + e^{\alpha_H t}(e^{\alpha_L(p_L+p_H)} - e^{\alpha_L p_L}) \\
&= e^{\alpha_H t}(e^{\alpha_H p_H} - e^{\alpha_H(p_L+p_H)} + e^{\alpha_L(p_L+p_H)} - e^{\alpha_L p_L}) \\
&\leq 0,
\end{aligned}
$$

where we used $\alpha_L < \alpha_H$ and $e^{\alpha_L(p_L+p_H)} - e^{\alpha_L p_L} > 0$ for the strict inequality and the assumption of Equation (9) for the last inequality. Similarly, $g'_{ij}(t) < 0$ for all $t \in [0, \infty)$.

v) For the fifth case, $\alpha_i = \alpha_H, p_i = p_H, \alpha_j = \alpha_L$ and $p_j = p_H$. Then, for all $t \in [0, \infty)$,

$$
\begin{aligned}
g_{ij}(t) &= e^{\alpha_H(t+p_H)} + e^{\alpha_L(t+p_H+p_H)} - e^{\alpha_H(t+p_H+p_H)} - e^{\alpha_L(t+p_H)} \\
&= e^{\alpha_H(t+p_H)}(1-e^{\alpha_H p_H}) + e^{\alpha_L(t+p_H)}(e^{\alpha_L p_H}-1) \\
&< e^{\alpha_H(t+p_H)}(e^{\alpha_L p_H}-e^{\alpha_H p_H}) \\
&< 0,
\end{aligned}
$$

using $\alpha_L < \alpha_H$ twice. Similarly, $g'_{ij}(t) < 0$ for all $t \in [0, \infty)$.

vi) Finally, $\alpha_i = \alpha_L, p_i = p_L, \alpha_j = \alpha_L$ and $p_j = p_H$. Then, for all $t \in [0, \infty)$,

$$
\begin{aligned}
g_{ij}(t) &= e^{\alpha_L(t+p_L)} + e^{\alpha_L(t+p_L+p_H)} - e^{\alpha_L(t+p_H+p_L)} - e^{\alpha_L(t+p_H)} \\
&= e^{\alpha_L(t+p_L)} - e^{\alpha_L(t+p_H)} \\
&= e^{\alpha_L t}(e^{\alpha_L p_L} - e^{\alpha_L p_H}) \\
&< 0,
\end{aligned}
$$

using $p_L < p_H$. Similarly, $g'_{ij}(t) < 0$ for all $t \in [0, \infty)$.

Together, this proves 2). □

4 Sequencing games

This section analyzes interactive sequencing situations from a game-theoretic perspective. A *TU-game* is a pair (N, v), where N is the *set of players* and v is the *characteristic function*, that is, a map from 2^N to $\mathbb{R}$, satisfying that $v(\emptyset) = 0$. In general, for all $S \in 2^N$, $v(S)$ is the *worth* of the coalition, representing the joint pay-off obtained in case of cooperation by players in S.

A TU-game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for every pair of coalitions $S, T \in 2^N$ with $S \cap T = \emptyset$. Moreover, a game (N, v) is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for every pair $S, T \in 2^N$.

Let $(N, \sigma_0, c, p) \in SEQ^N$ be an interactive sequencing situation. The associated *sequencing game* is defined as a game (N, v), where $v(S)$ corresponds to the maximal cost savings the coalition can achieve by admissible rearrangements with respect to σ_0. Formally, for all $S \in 2^N \setminus \{\emptyset\}$,

$$v(S) = \max_{\sigma \in \mathcal{A}(\sigma_0, S)} \left\{ \sum_{i \in S} c_i(C_i(\sigma_0)) - \sum_{i \in S} c_i(C_i(\sigma)) \right\},$$

where $\mathcal{A}(\sigma_0, S)$ is defined in the standard way as the set of admissible orders for S with respect to $\sigma_0 \in \Pi(N)$. An order $\sigma \in \Pi(N)$ is *admissible* for a coalition $S \in 2^N \setminus \{\emptyset\}$ with respect to $\sigma_0 \in \Pi(N)$ if, for all $i \in N \setminus S$, $\{k \in N : \sigma(k) < \sigma(i)\} = \{k \in N : \sigma_0(k) < \sigma_0(i)\}$.

Moreover, an order $\hat{\sigma}_S \in \Pi(N)$ is *optimal for* S if $TC(\hat{\sigma}_S) \leq TC(\sigma)$ for all $\sigma \in \mathcal{A}(\sigma_0, S)$, that is, the admissible order for S which minimizes the total costs. Note that, for the coalition N, this reduces to the definition of an optimal order, since $\mathcal{A}(\sigma_0, N) = \Pi(N)$.

A coalition $S \in 2^N \setminus \{\emptyset\}$ is called *connected* with respect to an order $\sigma \in \Pi(N)$ if for all $i, j \in S$ and $k \in N$ such that $\sigma(i) < \sigma(k) < \sigma(j)$, it holds that $k \in S$. Moreover, for any $S \in 2^N \setminus \emptyset$ and $\sigma \in \Pi(N)$, a *component* of S with respect to σ is a maximally connected subset of S with respect to σ. Let $S|_\sigma$ denote the set of all components of a coalition $S \in 2^N \setminus \emptyset$ and an order $\sigma \in \Pi(N)$. We introduce the following types of connected coalitions:

$$\begin{aligned}
(i, j]_\sigma &= \{k \in N : \sigma(i) < \sigma(k) \leq \sigma(j)\};\\
[i, j)_\sigma &= \{k \in N : \sigma(i) \leq \sigma(k) < \sigma(j)\};\\
(i, j)_\sigma &= \{k \in N : \sigma(i) < \sigma(k) < \sigma(j)\};\\
[i, j]_\sigma &= \{k \in N : \sigma(i) \leq \sigma(k) \leq \sigma(j)\},
\end{aligned}$$

where $i, j \in N$ are two players such that $\sigma(i) < \sigma(j)$, for an order $\sigma \in \Pi(N)$.

Finally, Curiel, Potters, Prasad, Tijs, and Veltman (1993) introduced the class of σ_0-*component additive games*. Given an order $\sigma_0 \in \Pi(N)$, a game (N, v) is called a σ_0-component additive game if the following three conditions hold:

1) $v(\{i\}) = 0$, for all $i \in N$;
2) (N, v) is superadditive;
3) $v(S) = \sum_{T \in S|_{\sigma_0}} v(T)$ for all $S \in 2^N \setminus \{\emptyset\}$.

Note that every sequencing game is a σ_0-component additive game. In what follows, if $(N, \sigma_0, p, c) \in SSEQ^N$, the associated game (N, v) is called a *standard sequencing game* and if $(N, \sigma_0, p, c) \in ESEQ^N$, the associated game is called an *exponential sequencing game*. In particular, standard sequencing games and exponential sequencing games are σ_0-component additive games.

4.1 Convexity of exponential sequencing games

In this section, we study the convexity of the sequencing games that arise from exponential sequencing situations. First, Example 4.1 illustrates that exponential sequencing games need not be convex in general.

Example 4.1. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation, where $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, and the exponential cost coefficients and processing times as specified by the table below.*

	player 1	*player 2*	*player 3*
α_i	*1.65*	*1.6*	*1.65*
p_i	*2*	*1*	*2*

The total costs for all orders are computed and shown below.

σ	$TC(\sigma)$
$(1,2,3)$	3976.2489
$(1,3,2)$	3743.1658
$(2,1,3)$	3973.7538
$(2,3,1)$	3973.7538
$(3,1,2)$	3743.1658
$(3,2,1)$	3976.2489

Both $(1,3,2)$ and $(3,1,2)$ are optimal orders. To compute the associated exponential sequencing game (N,v), note that $TC((1,2,3)) - TC((2,1,3)) = 2.4951$ and $TC((1,2,3)) - TC((1,3,2)) = 233.0831$. This exponential sequencing game is shown below.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	*0*	*0*	*0*	*0*	*2.4951*	*0*	*233.0831*	*233.0831*

This exponential sequencing game (N,v) is not convex, since

$$v(N) + v(\{2\}) = 233.0831 < 2.4951 + 233.0831 = v(\{1,2\}) + v(\{2,3\}).$$

Interestingly, for the three subclasses of exponential sequencing situations as discussed in Sections 3.1, 3.2 and 3.3, we show that the associated exponential sequencing games are convex. This is shown by imposing conditions on the time-dependent neighbor switching gains of misplacements and non-misplacements in a general interactive sequencing situation to establish convexity of the associated sequencing game.

Let $(N,\sigma_0,p,c) \in SEQ^N$ be a (general) interactive sequencing situation and let $\sigma \in \Pi(N)$ be an order. We say that σ *induces* an order $\sigma_S \in \Pi(N)$ for a given $S \in 2^N \setminus \{\emptyset\}$ if $\sigma_S \in \mathcal{A}(\sigma_0,S)$ and $\{k \in S : \sigma(k) < \sigma(i)\} = \{k \in S : \sigma_S(k) < \sigma_S(i)\}$ for all $i \in S$. This means that players outside S are in the same positions as in the initial order, while players inside S are ordered within the components of S according to σ.

Lemma 4.2 below ensures, by imposing a condition on the time-dependent neighbor switching gains of misplacements and non-misplacements, that we can deduct optimal orders for coalitions from an optimal order for the grand coalition.

Lemma 4.2. *Let $(N,\sigma_0,p,c) \in SEQ^N$ be an interactive sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. If, for all $t \in [0,\infty)$,*

1) *$g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0,\hat{\sigma})$;*
2) *$g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0,\hat{\sigma})$,*

then, for every $S \in 2^N \setminus \{\emptyset\}$, the induced order $\hat{\sigma}_S$ is optimal for S.

Proof. Assume that, for all $t \in [0,\infty)$, $g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0,\hat{\sigma})$ and $g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0,\hat{\sigma})$. Let $S \in 2^N \setminus \{\emptyset\}$. First, $\hat{\sigma}$ induces the order $\hat{\sigma}_S$. Then, there exists a path $(\sigma_0,\ldots,\hat{\sigma}_S,\ldots,\hat{\sigma})$ from σ_0 to $\hat{\sigma}$ corresponding to the misplacements $MP(\sigma_0,\hat{\sigma})$ such that $\hat{\sigma}_S$ is on this path. This implies that the first part of this path, from σ_0 to $\hat{\sigma}_S$, only consists of admissible orders for S with respect to σ_0. In order to prove that $\hat{\sigma}_S$ is optimal for S, let $\sigma'_S \in \mathcal{A}(\sigma_0,S)$. We will prove that $TC(\hat{\sigma}_S) \leq TC(\sigma'_S)$.

If σ'_S is on the path $(\sigma_0,\ldots,\hat{\sigma}_S,\ldots,\hat{\sigma})$, then σ'_S is on the first part of this path, i.e. the part from σ_0 to $\hat{\sigma}_S$, since $\sigma'_S \in \mathcal{A}(\sigma_0,S)$. Then we can go from σ'_S to $\hat{\sigma}_S$ along this path, that is, by repairing specific misplacements of $MP(\sigma_0,\hat{\sigma})$. This implies that $TC(\sigma'_S) \geq TC(\hat{\sigma}_S)$, since $g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0,\hat{\sigma})$.

If σ'_S is not on the path $(\sigma_0,\ldots,\hat{\sigma}_S,\ldots,\hat{\sigma})$, then consider a path from $\hat{\sigma}_S$ to σ'_S. Note that every step in this path corresponds to a non-misplacement, i.e. $(i,j) \notin MP(\sigma_0,\hat{\sigma})$ for every transposition (i,j), because the path starts in $\hat{\sigma}_S$ and can never redo a repair of a misplacement. Hence, $TC(\hat{\sigma}_S) \leq TC(\sigma'_S)$, since $g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0,\hat{\sigma})$.

Together, we have that $TC(\hat{\sigma}_S) \leq TC(\sigma'_S)$ for all $\sigma'_S \in \mathcal{A}(\sigma_0,S)$. Hence, $\hat{\sigma}_S$ is optimal for S. $\square$

In addition to the two conditions of Lemma 4.2, if we add an extra condition on the first derivative of the time-dependent neighbor switching gains of misplacements, then we can show that any sequencing game is convex. The proof of this main result is based on a result of Borm et al. (2002), which is stated below.

Proposition 4.3. *Let $\sigma_0 \in \Pi(N)$ be an order and let (N, v) be a σ_0-component additive game. Then, (N, v) is convex if and only if for all $i, j \in N$ with $\sigma_0(i) < \sigma_0(j)$,*

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) - v((i,j]_{\sigma_0}) + v((i,j)_{\sigma_0}) \geq 0.$$

Finally, we need some extra notation regarding the neighbor switching gains. Extending the definition as stated in Equation (1), the consecutively neighbor switching gains of player $i \in N$ and a group $J \subseteq N$ at time $t \in [0, \infty)$ with player i directly in front of the group $J = \{j_1, \dots, j_m\}$ as represented in Figure 3, is defined by

$$g_{iJ}(t) = g_{ij_1}(t) + g_{ij_2}(t + p_{j_1}) + g_{ij_3}(t + p_{j_1} + p_{j_2}) + \ldots + g_{ij_m}(t + p_{j_1} + \ldots + p_{j_{m-1}}).$$

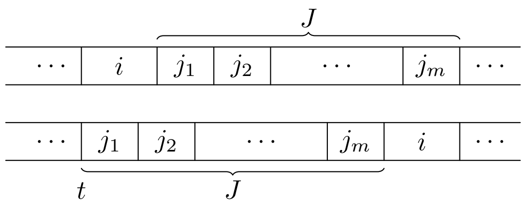

Figure 3: Interchanging player i with a group of players J, leading to the gain $g_{iJ}(t)$.

Theorem 4.4. *Let $(N, \sigma_0, p, c) \in SEQ^N$ be a sequencing situation and let $\hat{\sigma}$ be an optimal order. If, for all $t \in [0, \infty)$,*

1) $g_{ij}(t) \geq 0$ for all $(i, j) \in MP(\sigma_0, \hat{\sigma})$;

2) $g_{ij}(t) \leq 0$ for all $(i, j) \notin MP(\sigma_0, \hat{\sigma})$;

3) $g'_{ij}(t) \geq 0$ for all $(i, j) \in MP(\sigma_0, \hat{\sigma})$,

then, the associated sequencing game (N, v) is convex.

Proof. Let (N, v) be the associated sequencing game to (N, σ_0, p, α). Using Proposition 4.3, it suffices to check that $v([i,j]_{\sigma_0}) - v((i,j]_{\sigma_0}) \geq v([i,j)_{\sigma_0}) - v((i,j)_{\sigma_0})$ for all $i, j \in N$ with $\sigma_0(i) < \sigma_0(j)$. For convenience, we leave out the subscript σ_0 in the notation for the connected coalitions.

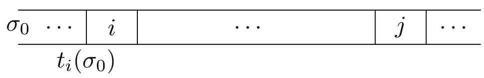

Figure 4: Players $i, j \in N$ in the initial order σ_0.

Let $i, j \in N$ such that $\sigma_0(i) < \sigma_0(j)$ (see also Figure 4). Moreover, let $\hat{\sigma} \in \Pi(N)$ be an optimal order. According to Lemma 4.2, the induced orders $\hat{\sigma}_{[i,j]}, \hat{\sigma}_{(i,j]}, \hat{\sigma}_{[i,j)}$ and $\hat{\sigma}_{(i,j)}$ are optimal for $[i,j], (i,j], [i,j)$ and (i,j), respectively. We distinguish two cases: either $\hat{\sigma}(i) < \hat{\sigma}(j)$ or $\hat{\sigma}(i) > \hat{\sigma}(j)$.

In the first case, $\hat{\sigma}(i) < \hat{\sigma}(j)$, player i appears before player j in the optimal order. Figure 5 provides a schematic overview of the order of the relevant players. Since player i is before player j in the optimal order, i is also before j in the induced order for $[i,j]$. The top two orders in Figure 5 show that only the players of $\mathcal{I}$, defined by

$$\mathcal{I} = \{\ell \in (i,j)_{\sigma_0} : (i, \ell) \in MP(\sigma_0, \hat{\sigma})\},$$

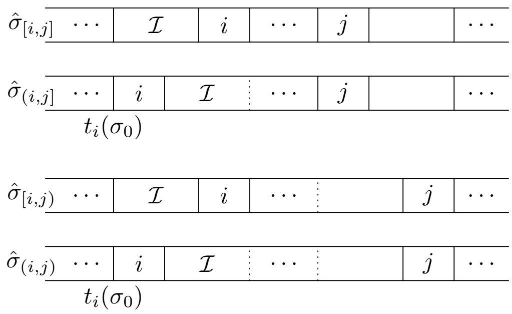

Figure 5: Schematic overview of the first case.

have to switch with player i. Note that this switch of i with the group of players $\mathcal{I}$ is the only difference between the induced optimal order $\hat{\sigma}_{(i,j]}$ and $\hat{\sigma}_{[i,j]}$. Moreover, this is also the only difference between the bottom two orders in Figure 5. Hence,

$$\begin{aligned} v([i,j]) - v((i,j]) &= g_{i\mathcal{I}}(t_i(\sigma_0)) \\ &= v([i,j)) - v((i,j)). \end{aligned}$$

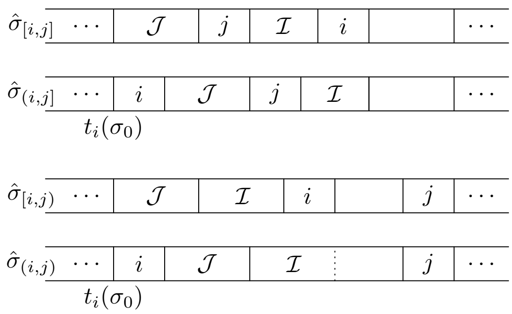

Figure 6: Schematic overview of the second case.

In the second case, $\hat{\sigma}(i) > \hat{\sigma}(j)$, player j appears before player i in the optimal order. Figure 6 provides a schematic overview of the order of the relevant players. The top two orders show that, in order to reach $\hat{\sigma}_{[i,j]}$ from $\hat{\sigma}_{(i,j]}$, player i has to switch with the group $\mathcal{J}$, with player j and with the group $\mathcal{I}$, where $\mathcal{J}$ and $\mathcal{I}$ are defined by

$$\mathcal{J} = \{k \in (i,j)_{\sigma_0} : (j,k) \in MP(\sigma_0, \hat{\sigma})\}$$

and

$$\mathcal{I} = \{\ell \in (i,j)_{\sigma_0} : (i,\ell) \in MP(\sigma_0, \hat{\sigma})\} \setminus \mathcal{J}.$$

Hence,

$$v([i,j]) - v((i,j]) = g_{i\mathcal{J}}(t_i(\sigma_0)) + g_{ij}(t_i(\sigma_0) + \sum_{k \in \mathcal{J}} p_k) + g_{i\mathcal{I}}(t_i(\sigma_0) + \sum_{k \in \mathcal{J}} p_k + p_j).$$

Furthermore, the bottom two orders show that the differences between $\hat{\sigma}_{(i,j)}$ and $\hat{\sigma}_{[i,j)}$ are the switches between i and the groups $\mathcal{J}$ and $\mathcal{I}$. Hence,

$$v([i,j)) - v((i,j)) = g_{i\mathcal{J}}(t_i(\sigma_0)) + g_{i\mathcal{I}}(t_i(\sigma_0) + \sum_{k \in \mathcal{J}} p_k).$$

Since $\sigma_0(i) < \sigma_0(j)$ and $\hat{\sigma}(i) > \hat{\sigma}(j)$, we have that $(i,j) \in MP(\sigma_0, \hat{\sigma})$, which implies that

$$g_{ij}(t_i(\sigma_0) + \sum_{k \in \mathcal{J}} p_k) \geq 0,$$

due to condition 1). Moreover,

$$g_{i\mathcal{I}}(t_i(\sigma_0) + \sum_{k \in \mathcal{J}} p_k + p_j) \geq g_{i\mathcal{I}}(t_i(\sigma_0) + \sum_{k \in \mathcal{J}} p_k),$$

due to condition 3). Together, this implies that

$$v([i,j]) - v((i,j]) \geq v([i,j)) - v((i,j)),$$

which finalizes the proof. □

Theorem 4.4 proves the convexity of any sequencing game associated to an interactive sequencing situation where the neighbor switching gains of misplacements are non-negative and non-decreasing and the gains of non-misplacements are non-positive. In particular, Theorem 4.4 can be used to prove convexity of sequencing games associated to particular subclasses of interactive sequencing situations. For example, it is known that any standard sequencing game is convex. This can now also be seen from Theorem 4.4, since the neighbor switching gains of misplacements are non-negative and constant, while the gains of non-misplacements are non-positive, according to Lemma 2.1.

Moreover, we can use Lemma 3.4, Lemma 3.7 and Lemma 3.12 to show that for the three subclasses of exponential sequencing situations as considered in Sections 3.1, 3.2 and 3.3, the three conditions of Theorem 4.4 are satisfied. Hence, the associated exponential sequencing games are convex.

Corollary 4.5. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that one of the following three cases holds:*

i) for all $i \in N$ and all $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$ with $\alpha \in \mathbb{R}_{++}$;

ii) for all $i \in N$, $p_i = p$ with $p \in \mathbb{R}_{++}$;

iii) for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}$ and $p_i \in \{p_L, p_H\}$ with $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ satisfying $\alpha_L < \alpha_H$, $p_L < p_H$ and

$$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H (p_L + p_H)} - e^{\alpha_L (p_L + p_H)}.$$

Then, the associated exponential sequencing game (N, v) is convex.

4.2 A path-based allocation rule

For standard sequencing situations, Curiel et al. (1989) introduced and analyzed the *Equal Gain Splitting rule* (EGS-rule). The EGS-rule is defined by recursively splitting the corresponding neighbor switching gains equally in every step in a path from the initial order to the optimal order that is closest to the initial order that repairs all neighbor misplacements. The EGS-rule thus assigns the cost savings obtained by a neighbor switch only to the two players that are involved and the assignment is made equally. Since the neighbor switching gains are constant in time for standard sequencing situations, all paths from the initial order to an optimal order lead to the same cost allocation. Interestingly, the EGS-rule will provide a coalitionally stable cost allocation for any standard sequencing game.

In this section, we show that no direct extension of the EGS-rule to the class of exponential sequencing situations can always lead to coalitionally stable cost allocations for any exponential sequencing game. In fact, this does not even hold for the convex cases like provided in Sections 3.1, 3.2 and 3.3, where it holds that every step in every path from the initial order to the optimal order that is closest to the initial one comes with a non-negative neighbor switching gain. On each such path, one can follow the idea of the EGS-rule. This however could lead to different cost allocations per path. A natural way to deal with this, is to average among all cost allocations per path.

Example 4.6. *Let $(N, \sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation, where $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, and the exponential cost coefficients and processing times are specified in the table below.*

	player 1	*player 2*	*player 3*
α_i	*1*	*1*	*1*
p_i	*1.5*	*1.45*	*1*

The total costs for all orders are computed and shown below.

σ	$TC(\sigma)$
$(1, 2, 3)$	75.5230
$(1, 3, 2)$	68.5995
$(2, 1, 3)$	75.3044
$(2, 3, 1)$	67.7868
$(3, 1, 2)$	66.8361
$(3, 2, 1)$	66.2420

Obviously, $\hat{\sigma} = (3, 2, 1)$ is the optimal order and $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$. Hence, there are two paths from the initial order to the optimal order:

$$(1, 2, 3) \to (2, 1, 3) \to (2, 3, 1) \to (3, 2, 1),$$

and

$$(1, 2, 3) \to (1, 3, 2) \to (3, 1, 2) \to (3, 2, 1).$$

Figure 7 provides a schematic overview of the two paths from the initial order to the optimal order with the corresponding total costs and neighbor switching gains.

$$\begin{tikzcd}[column sep=small]
& \begin{array}{c}\sigma_0=(1,2,3)\\ TC(\sigma_0)=75.5230\end{array} \arrow[dl, "{(1,2)}"'] \arrow[dr, "{(2,3)}"] & \\
\begin{array}{c}\sigma_1=(2,1,3)\\ TC(\sigma_1)=75.3044\end{array} \arrow[d, "{(1,3)}"'] & & \begin{array}{c}\sigma_1'=(1,3,2)\\ TC(\sigma_1')=68.5995\end{array} \arrow[d, "{(1,3)}"] \\
\begin{array}{c}\sigma_2=(2,3,1)\\ TC(\sigma_2)=67.7868\end{array} \arrow[dr, "{(2,3)}"'] & & \begin{array}{c}\sigma_2'=(3,1,2)\\ TC(\sigma_2')=66.8361\end{array} \arrow[dl, "{(1,2)}"] \\
& \begin{array}{c}\hat{\sigma}=(3,2,1)\\ TC(\hat{\sigma})=66.2420\end{array} &
\end{tikzcd}$$

Figure 7: Schematic overview of the two paths.

Dividing the neighbor switching gains in every step equally among the two involved neighbors, the left-side path of Figure 7 results in the path-EGS cost allocation vector

$$(3.8681, 0.8817, 4.5312),$$

since $g_{12}(t_1(\sigma_0)) = 0.2186, g_{13}(t_1(\sigma_1)) = 7.5176$ and $g_{23}(t_2(\sigma_2)) = 1.5448$, while the right-side path of

Figure 7 results in the path-EGS allocation vector

$$(1.1788, 3.7588, 4.3434),$$

since $g_{23}(t_2(\sigma_0)) = 6.9235, g_{13}(t_1(\sigma_1')) = 1.7634$ *and* $g_{12}(t_1(\sigma_2')) = 0.5941$. *Averaging we get the EGS allocation vector*

$$(2.5234, 2.3203, 4.4373).$$

Since,

$$v(\{2,3\}) = TC((1,2,3)) - TC((1,3,2)) = 75.5230 - 68.5995 = 6.9235,$$

the EGS cost allocation vector is not coalitionally stable, because

$$2.3203 + 4.4373 = 6.7576 < 6.9235 = v(\{2,3\}).$$

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