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Publication date:
2018

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Compernolle, T., Huisman, K., Kort, P., Lavrutich, M., Nunes, C., & Thijssen, J. J. J. (2018). *Investment Decisions with Two-Factor Uncertainty*. (CentER Discussion Paper; Vol. 2018-003). CentER, Center for Economic Research.

CentER

Discussion Paper

No. 2018-003

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6 February 2018

ISSN 0924-7815
ISSN 2213-9532

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Investment decisions with two-factor uncertainty

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October 6, 2017

Abstract

This paper considers investment problems in real options with non-homogeneous two-factor uncertainty. It shows that, despite claims made in the literature, the method used to derive an analytical solution in one dimensional problems cannot be straightforwardly extended to problems with two stochastic processes. To illustrate, we analyze an investment problem with two stochastic revenue streams and a constant sunk cost. We show that a semi-analytical approach leads to a sub-optimal investment policy. The main message of our paper is that non-homogeneous investment problems can only be solved numerically.

Keywords: Investment analysis; Optimal stopping time problem; Two-factor uncertainty

1 Introduction

The first real options model with two-factor uncertainty occurs in McDonald and Siegel (1986). Their value function is homogenous of degree one, and the two stochastic processes are the output price and the investment cost. In such cases, the investment threshold level can be determined for the price-to-cost ratio. This allows to reformulate the problem in terms of the relative price, and to reduce the number of stochastic variables to one. In this way a standard one-factor real options model is obtained for which a closed-form solution exists. The result of this analysis is, however, not a threshold point but a threshold boundary at which it is optimal to invest (Nunes and Pimentel (2017)). Hu and Øksendal (1998) generalize this solution to the n-dimensional case. Armada *et al.* (2013) consider a problem where the output price and quantity are stochastic. Here the dimension of the state space can be reduced to a one-dimensional space, because the only relevant payoff variable is revenue (price times quantity). The problem then reduces to finding an optimal revenue threshold that makes investment optimal.

Several authors have tried to use this dimension-reduction approach to cases characterized by multiple stochastic processes and a constant sunk cost. Huisman *et al.* (2013) and Compernolle *et al.* (2017) consider price and cost uncertainty and determine the investment threshold level for the price-to-cost ratio. However, there are some problems with this approach. Due to the presence of a constant sunk cost investment, homogeneity of degree one does not hold. For this reason, the state space cannot be reduced to a one-dimensional one. This is also revealed in these papers, because two processes (price/cost and cost) remain present in the equations.

For problems of this kind, Adkins and Paxson (2011b) propose a quasi-analytical approach that results in a set of equations to determine the optimal investment boundary. They solve this set of equations simultaneously while keeping one of the stochastic threshold variables fixed. The present paper shows that this methodology can lead to sub-optimal solutions. In fact, the results of the quasi-analytical approach will generically-speaking be incorrect. To put it succinctly, Adkins and Paxson (2011b) use a "local" approach to solve a "global" problem. In addition, Adkins and Paxson (2011b) do not derive the value function explicitly. This is problematic, because the value function and the optimal exercise boundary need to be solved for simultaneously: they are two sides of the same coin. In order to solve multi-dimensional problems that are not homogeneous of degree one, several numerical schemes are available. Here we propose a finite differencing scheme that is straightforward to implement.

The methodology developed in Adkins and Paxson (2011b) has been applied in several contributions. In the context of replacement options, Adkins and Paxson (2013a) examine premature and postponed replacement in the presence of technological progress, where revenue and operating costs are treated as stochastic. Adkins and Paxson (2017) use a general replacement model to investigate when it is optimal to replace an asset whose operating cost and salvage value deteriorate stochastically. In addition, the quasi-analytical approach is applied to investment problems in the energy sector. Adkins and Paxson (2011a) solve a switching model for two alternative energy inputs with fixed switching costs. Boomsma and Linnerud (2015) examine investment in a renewable energy project under both market and policy uncertainty. Fleten *et al.* (2016) study investment decisions in the renewable energy sector, where the revenue comes from selling electricity and from receiving subsidies. Both revenue flows are considered stochastic. Adkins and Paxson (2016) consider the optimal investment policy for an energy facility with price and quantity uncertainty under different subsidy schemes. Støre *et al.* (2016) determine the optimal timing to switch from oil to gas production. Both the oil price and the gas price are considered stochastic. Heydari *et al.* (2012) extend the quasi-analytical approach to a three-factor model, which is employed to value the choice between investing in full or partial CCS retrofits given uncertainty in electricity, CO_2 and fuel prices.

To illustrate the mathematical pitfalls of the quasi-analytical approach, we apply it to a model with two stochastic revenue streams. We determine the optimal timing of investment in the presence of a constant sunk investment cost. We provide numerical examples where the quasi-analytical approach approach violates certain properties of the optimal boundary, which implies that the quasi-analytical solution gives, generically, a sub-optimal investment policy. In a nutshell, the problem is that in solving the partial differential equation that governs the value function, two power parameters are assumed to be constant, whereas in the numerical implementation they are treated as variable.

One may still wonder whether the obtained solution by the quasi-analytical approach results in a useful approximation. However, we show that for the considered model this is not the case. In particular, it holds that in some situations the threshold is lower in a more uncertain environment. This does not comply with a, if not *the*, major result from real options theory: "an increase in uncertainty leads to an increase in project value". We formally prove that this major result should also hold for the model under consideration. In contrast, our finite differencing scheme does exhibit the expected behavior in relation to an increase in uncertainty.

The remainder of this paper is organized as follows. Section 2 introduces a non-homogeneous investment problem characterized by two uncertain revenue flows. Section 3 applies the methodological approach in Adkins and Paxson (2011b) to solve the model presented in Section 2, and highlights the mathematical problems with the solution. Section 4 proposes an alternative numerical approach to solve the model. Section 5 concludes.

2 Investment decision given two uncertain revenue flows

Consider a profit-maximizing, risk-neutral firm that has the opportunity to invest in a production plant by paying a constant investment cost, I. The plant can produce two different products, the prices of which are stochastic and follow correlated geometric Brownian motions X and Y, i.e.,

$$dX_t = \alpha_1 X_t dt + \sigma_1 X_t dW_{X,t}, \quad dY_t = \alpha_2 Y_t dt + \sigma_2 Y_t dW_{Y,t} \tag{1}$$

with

$$X_0 = x, Y_0 = y, \tag{2}$$

and $E[dW_{X,t} dW_{Y,t}] = \rho dt$.

At any instant, if the prices of each product are x and y, the instantaneous profit of the firm is given by:

$$\pi(x, y) = Q_1 x + Q_2 y, \tag{3}$$

where Q_1 and Q_2 denote the quantities of the products produced. Moreover, at that instant the firm's value is equal to the perpetual revenue flow from selling two products:[1]

$$F(x, y) = \mathbb{E}_{(x,y)} \left[\int_0^\infty \mathrm{e}^{-rt} \pi(X_t, Y_t) dt - I \right] = \frac{Q_1 x}{\delta_1} + \frac{Q_2 y}{\delta_2} - I, \tag{4}$$

with $\delta_i := r - \alpha_i, i \in \{1, 2\}$. To ensure finite integrals, we assume that $r > \alpha_1 \vee \alpha_2$.

The firm needs to determine the optimal time to undertake the investment, and, thus, solves the following optimal stopping problem

$$V(X_t, Y_t) = \sup_{\tau \geq t} \mathbb{E}_{(X_t, Y_t)} \left[\mathrm{e}^{-r(\tau - t)} F(X_\tau, Y_\tau) \right] \tag{5}$$

where τ is a stopping time with respect to the filtration generated by the joint process (W_X, W_Y).

Using standard calculations from optimal stopping theory (see, e.g., Øksendal and Sulem (2007)), we derive the following Hamilton-Jacobi-Bellman (HJB) equation for this problem:

$$\min\{rV(x, y) - \mathcal{L}V(x, y), V(x, y) - F(x, y)\} = 0, \quad \forall\, (x, y) \in \Re_+ \times \Re_+. \tag{6}$$

Here $\mathcal{L}$ denotes the infinitesimal generator of the process (X, Y), which is given by (Øksendal and Sulem (2007)):

$$\begin{aligned} \mathcal{L}V(x, y) &= \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 V(x, y)}{\partial x} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 V(x, y)}{\partial y^2} + \rho\sigma_1\sigma_2 xy \frac{\partial^2 V(x, y)}{\partial x \partial y} \\ &+ \alpha_1 x \frac{\partial V(x, y)}{\partial x} + \alpha_2 y \frac{\partial V(x, y)}{\partial y}. \end{aligned} \tag{7}$$

[1] $\mathbb{E}_{(x,y)}$ denotes the expectation conditional on $(X_0, Y_0) = (x, y)$.

Moreover, we let the set $D := \{(x, y) \in \Re^2_+ | V(x, y) > F(x, y)\}$ denote the *continuation region*, and $S := \Re^2_+ \setminus D = \{(x, y) \in \Re^2_+ | V(x, y) = F(x, y)\}$ denote the *stopping region.* Following the general theory of optimal stopping, it then follows that τ^*, the time at which the investment should be undertaken, is given by the first exit time of the continuation region, i.e.,

$$\tau^* = \inf\{t > 0; (X_t, Y_t) \notin D\}. \tag{8}$$

In view of the equation (6), it follows that

$$rV(x, y) - \mathcal{L}V(x, y) \geq 0 \ \wedge \ V(x, y) \geq F(x, y), \ \forall \ (x, y) \in \Re^2_+.$$

Moreover,

$$rV(x, y) - \mathcal{L}V(x, y) = 0 \ \wedge \ V(x, y) \geq F(x, y), \ \forall \ (x, y) \in D, \tag{9}$$

whereas

$$rF(x, y) - \mathcal{L}F(x, y) \geq 0 \ \wedge \ V(x, y) = F(x, y), \ \forall \ (x, y) \in S. \tag{10}$$

The solution of the HJB equation, V, must satisfy the following boundary condition:

$$V(0_+, 0_+) = 0, \tag{11}$$

which reflects the fact that the value of the firm will be zero if the prices are zero. Also the following value-matching and smooth-fit conditions should hold (see Pham (1997), Tankov (2003) and Larbi and Kyprianou (2005)):

$$V(x, y) = F(x, y) \quad \text{and} \quad \nabla V(x, y) = \nabla F(x, y), \quad \text{for } (x, y) \in \partial D. \tag{12}$$

Here ∂D denotes the boundary of D, which we call *critical boundary*, and ∇ is the gradient operator. Therefore, the solution of the problem is continuous at the critical boundary, not only for itself but also for its derivatives. The resulting threshold is a curve separating the two regions (the continuation and the stopping regions).

Note that $x = 0$ and $y = 0$ are absorbing barriers. Consequently, at these boundaries, the firm only receives revenues from one product and only one stochastic process is in use. Therefore, the threshold at these points corresponds to the standard solution for a one-dimensional problem. It follows that the investment triggers at the y and x axis are

$$x^* = \frac{\beta_1}{\beta_1 - 1}\delta_1 I, \quad \text{and} \quad y^* = \frac{\eta_1}{\eta_1 - 1}\delta_2 I,$$

respectively. The parameters $\beta_1 > 1$ and $\eta_1 > 1$ are the positive roots of the quadratic equations

$$\frac{1}{2}\sigma_1\beta(\beta - 1) + \alpha_1\beta - r = 0, \quad \text{and} \quad \frac{1}{2}\sigma_2\eta(\eta - 1) + \alpha_2\eta - r = 0,$$

respectively. The next theorem derives some qualitative features of the threshold boundary for the problem defined in (5).

Theorem 1 *The boundary between D and S can be described by a mapping $x \mapsto b(x)$, where:*

1. $b(x) = \sup \{ y \in \Re_+ \mid V(x,y) > F(x,y) \}$ *for all* $x \in (0, x^*)$*;*
2. b *is non-increasing on* $(0, x^*)$*;*
3. b *is convex on* $(0, x^*)$*;*
4. b *is continuous;*
5. $b(x) < y^*$ *on* $(0, x^*)$*, and* $b(x) = 0$ *on* $[x^*, \infty)$*.*

In addition, the stopping set S is:

1. *closed;*
2. *convex.*

Finally, the value function V satisfies:

1. $V > 0$ *on* $\Re^2_{++}$*;*
2. V *is convex;*
3. V *is continuous;*
4. V *is increasing in x and y.*

Remark 1 *Theorem 1 leads to the following observations.*

1. *We can write*

$$D = \left\{ (x,y) \in \Re^2_+ \;\middle|\; y < b(x) \right\}, \quad \textit{and} \quad S = \left\{ (x,y) \in \Re^2_+ \;\middle|\; y \geq b(x) \right\}.$$

2. *The optimal stopping boundary can never lie below the Net Present Value boundary $\bar{b}$, i.e.*

$$b(x) > \bar{b}(x) := \delta_2(I - x/\delta_1), \quad \textit{all } x \in (0, \delta_1 I).$$

3 The quasi-analytical approach

Following the approach in Adkins and Paxson (2011b), we start by postulating a solution to equation (7) of the following form:

$$v(x,y) = Ax^{\beta}y^{\eta}, \tag{13}$$

where A, β, and η are constants. Simple calculations lead to the conclusion that for (13) to be a solution to (7) it must hold that β and η are the roots of the characteristic root equation:

$$\mathcal{Q}(\beta,\eta) = \frac{1}{2}\sigma_2^2\eta(\eta-1) + \frac{1}{2}\sigma_1^2\beta(\beta-1) + \rho\sigma_1\sigma_2\beta\eta + \alpha_1\beta + \alpha_2\eta - r = 0. \tag{14}$$

The set of solutions to (14) defines an ellipse that intersects all quadrants of $\Re^2$, with β (η) on the horizontal (vertical) axis.

Adkins and Paxson (2011b) hypothesize that the boundary between the continuation and stopping regions is of the form $x \mapsto b(x)$. As Theorem 1 shows, this is correct. In order to find this boundary, Adkins and Paxson (2011b) try to extend the standard value-matching and smooth-pasting conditions to a two-dimensional setting. The way this is done is as follows: on the boundary it should hold for every $\hat{x} \in (0, x^*)$ that

$$v(\hat{x}, b(\hat{x})) = \frac{Q_1\hat{x}}{r-\alpha_1} + \frac{Q_2 b(\hat{x})}{r-\alpha_2} - I \quad \text{(value matching)} \tag{15}$$

$$\frac{\partial v(x,y)}{\partial x}_{|x=\hat{x},y=b(\hat{x})} = \frac{Q_1}{r-\alpha_1} \quad \text{(smooth pasting in } x\text{-direction)} \tag{16}$$

$$\frac{\partial v(x,y)}{\partial y}_{|x=\hat{x},y=b(\hat{x})} = \frac{Q_2}{r-\alpha_2} \quad \text{(smooth pasting in } y\text{-direction).} \tag{17}$$

Now, *if* the value function is of the form[2]

$$v(x,y) = Ax^{\beta}y^{\eta},$$

then it should hold that $\beta, \eta > 0$ since the boundary conditions $\lim_{x\downarrow 0} v(x,y) = \lim_{y\downarrow 0} v(x,y) = 0$ should be satisfied. Therefore, for every $\hat{x} \in (0, x^*)$ we can solve the system of non-linear equations

$$A\hat{x}^{\beta}b(\hat{x})^{\eta} = \frac{Q_1\hat{x}}{r-\alpha_1} + \frac{Q_2 b(\hat{x})}{r-\alpha_2} - I \tag{18}$$

$$\beta A\hat{x}^{\beta-1}b(\hat{x})^{\eta} = \frac{Q_1}{r-\alpha_1} \tag{19}$$

$$\eta A\hat{x}^{\beta}b(\hat{x})^{\eta-1} = \frac{Q_2}{r-\alpha_2} \tag{20}$$

$$\mathcal{Q}(\beta,\eta) = 0, \tag{21}$$

[2]Note that Adkins and Paxson (2011b) assume this is the case.

in b, A, β, and η, under the condition that $\beta, \eta > 0$.

Using the approach presented in Støre *et al.* (2016) to solve this system, we find the explicit solution for the boundary[3]:

$$\hat{b}(x) = x\frac{\eta(x)(r-\alpha_2)Q_1}{\beta(x)(r-\alpha_1)Q_2}, \tag{22}$$

where

$$\beta(x) = \tfrac{\sigma_1^2-2\alpha_1+C^*(x)(2\alpha_2+\sigma_2^2)}{2(\sigma_1^2+\sigma_2^2C^*(x))} + \sqrt{\left(\tfrac{\sigma_1^2-2\alpha_1+C^*(x)(2\alpha_2+\sigma_2^2)}{2(\sigma_1^2+\sigma_2^2C^*(x))}\right)^2 + 2\tfrac{r-\alpha_2}{\sigma_1^2+\sigma_2^2C^*(x)}}, \tag{23}$$

$$\eta(x) = 1-\beta(x)C^*(x), \tag{24}$$

$$C^*(x) = 1-\tfrac{(r-\alpha_1)I}{xQ_1}. \tag{25}$$

Therefore, solving (19)-(21) leads to values of β and η that *do* depend on the value of x and, thus, cannot be treated as fixed parameters. This is also the case for the problem in Adkins and Paxson (2011b), as illustrated by their Figure 3.[4] The same holds for A and b.

Let $\mathbf{u} = [\,\beta(x) \quad \eta(x) \quad A(x) \quad \hat{b}(x)\,]^T$ denote the vector of solutions resulting from (19)-(21). Then at $\hat{b}(x)$, the value of the firm can be written as

$$v(x,\hat{b}(x)) = A(x)x^{\beta(x)}\hat{b}^{\eta(x)}(x). \tag{26}$$

Note that from (9) the partial differential equation $rV(x,y) - \mathcal{L}V(x,y) = 0$ must also hold along the threshold boundary, implying that

$$\begin{aligned}\frac{1}{2}\sigma_1^2x^2\left(\frac{\partial^2 v(x,\hat{b})}{\partial x^2} + \frac{\partial}{\partial \mathbf{u}}\left(\frac{\partial v(x,\hat{b})}{\partial x}\right)\frac{\partial \mathbf{u}}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial v(x,\hat{b})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial x}\right)\right) + \frac{1}{2}\sigma_2^2\hat{b}^2\frac{\partial^2 v(x,\hat{b})}{\partial \hat{b}^2} \\ +\alpha_1 x\left(\frac{\partial v(x,\hat{b})}{\partial x} + \frac{\partial v(x,\hat{b})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial x}\right) + \alpha_2\hat{b}\frac{\partial v(x,\hat{b})}{\partial \hat{b}} - rv(x,\hat{b}) = 0.\end{aligned} \tag{27}$$

Using (26) we can rewrite (27) as

$$\begin{aligned}\frac{1}{2}\sigma_1^2x^2\left(\frac{\partial}{\partial \mathbf{u}}\left(\frac{\partial v(x,\hat{b})}{\partial x}\right)\frac{\partial \mathbf{u}}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial v(x,\hat{b})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial x}\right)\right) + \alpha_1 x\left(\frac{\partial v(x,\hat{b})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial x}\right) \\ +Ax^\beta\hat{b}^\eta\left(\frac{1}{2}\sigma_2^2\eta(\eta-1) + \frac{1}{2}\sigma_1^2\beta(\beta-1) + \alpha_1\beta + \alpha_2\eta - r\right) = 0,\end{aligned} \tag{28}$$

[3]For simplicity, henceforth we assume that $\rho = 0$.

[4]The same holds for (Adkins and Paxson, 2011a), see Table 2; Heydari *et al.* (2012), see equation (19); Adkins and Paxson (2013a), see equation (9); Adkins and Paxson (2013b), see Figure 2; Fleten *et al.* (2016), see equation (17); Støre *et al.* (2016), see equation (18); and Adkins and Paxson (2017), see Table 3.

The first two terms in (28) represent the contributions of the partial derivatives of $\hat{b}$, A, β and η with respect to x, whereas the last term is equal to $Ax^{\beta}\hat{b}^{\eta}\mathcal{Q}(\beta,\eta)$. If the solution proposed in (13) is correct, then the latter should be equal to zero, and we can still use the system (18)-(21) to determine the threshold boundary. In what follows we verify whether the contribution of the partial derivatives is negligible for the numerical example in Table 1.

$\hat{x}$	Contribution of partial derivatives
10	-10841.14
20	-54 856.60
30	-9040.40

Table 1: The value of the first two terms of equation (28) for the following set of the parameter values: $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, and $I = 2000$.

For $\hat{x} = 10$, and the set of parameter values in Table 1, $F(\hat{x}, \hat{b}(\hat{x})) = 7281.23$. Therefore, we conclude that the contribution of the partial derivatives cannot be neglected. As a result, substitution of the solution (26) in (7) leads to the conclusion that the condition for η and β is no longer (14). In fact, (14) needs to be modified to incorporate terms involving $\beta'(x)$, $\eta'(x)$, $\beta''(x)$, $\eta''(x)$, $A'(x)$, $A''(x)$, $\hat{b}'(x)$ and $\hat{b}''(x)$. The implication is that solving the system (18)-(21) for different values of $\hat{x}$ does not result in a correct threshold boundary.

3.1 Results of the quasi-analytical approach

After having shown that the boundary $\hat{b}$, as determined by the quasi-analytical approach, is not the true boundary b, it could still be the case that $\hat{b}$ is a good approximation of b. This section, however, provides an argument that this is not the case, at least for the problem in Section 2.

We start out by presenting the following proposition.

Proposition 2 *The value function, V, is monotonically increasing in both σ_1 and σ_2.*

Proof. The result follows from Proposition 3 in Olsen and Stensland (1992) using the fact that the optimal value function is convex, as stated in our Theorem 1. ■

Corollary 3 *Let $b(\sigma_1, \sigma_2; x)$ denote the optimal investment threshold boundary for a given level of x. Then it holds that $b(\sigma_1, \sigma_2; x)$ is increasing in both σ_1 and σ_2.*

Proof. We prove the result by contradiction. Without loss of generality we only consider a change in σ_1. Consider two different values of σ_1, such that $\hat{\sigma}_1 > \bar{\sigma}_1$, and $b(\hat{\sigma}_1, \sigma_2; x) < b(\bar{\sigma}_1, \sigma_2; x)$ for some x.

Let $V(\sigma_1, \sigma_2; x, y)$ denote the optimal value function for a given level of x. Then,

$$V(\hat{\sigma}_1, \sigma_2; x, y) = \begin{cases} > F(x,y) & \text{for } y < b(\hat{\sigma}_1, \sigma_2; x), \\ = F(x,y) & \text{for } y \geq b(\hat{\sigma}_1, \sigma_2; x), \end{cases} \tag{29}$$

$$V(\bar{\sigma}_1, \sigma_2; x, y) = \begin{cases} > F(x,y) & \text{for } y < b(\bar{\sigma}_1, \sigma_2; x), \\ = F(x,y) & \text{for } y \geq b(\bar{\sigma}_1, \sigma_2; x). \end{cases} \tag{30}$$

If $b(\hat{\sigma}_1, \sigma_2; x) < b(\bar{\sigma}_1, \sigma_2; x)$, then for $y \in (b(\hat{\sigma}_1, \sigma_2; x), b(\bar{\sigma}_1, \sigma_2; x))$, it holds that $V(\bar{\sigma}_1, \sigma_2; x, y) > F(x, y) = V(\hat{\sigma}_1, \sigma_2; x, y)$, which contradicts Proposition 1. ■

Figure 1 illustrates the quasi-analytical threshold boundaries for different values of σ_2.

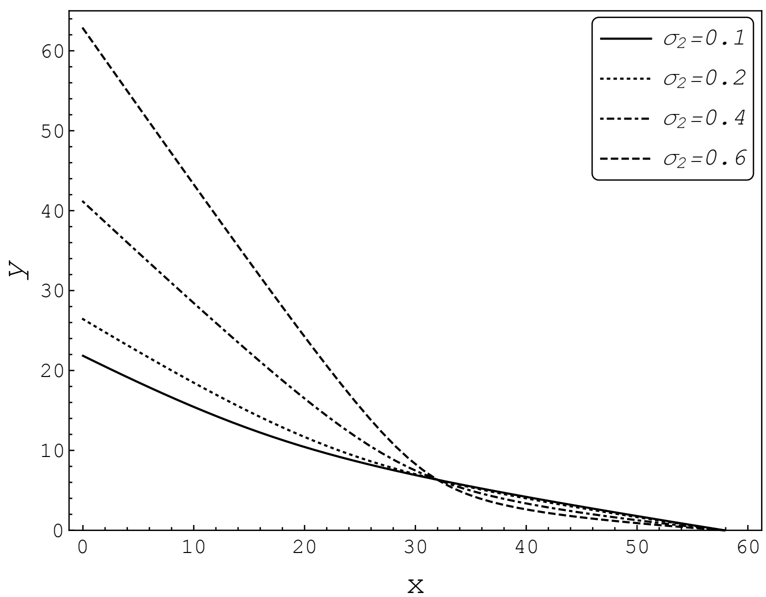

Figure 1: The threshold boundaries, $\hat{b}$, for the following set of parameter values: $\sigma_1 = 0.2$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$, and different values of σ_2.

Evidently, the numerical example violates Corollary 3, since the threshold boundaries intersect. Moreover, this result does not correspond to what we would expect from real options theory, i.e. that the firm invests for a larger threshold level in a more uncertain environment. In fact, for $x > 32$, the quasi-analytical approach suggests that the firm should invest for a lower threshold level when σ_2 is larger. This clearly leads to a sub-optimal decision, so the quasi-analytical solution falls short in being a useful approximation to the optimal solution in this case.

4 Numerical Solution

This section develops a finite difference algorithm to solve the optimal stopping problem in (5). The results of the numerical approach are different from the results obtained by the analytical approach *and* in line with Theorem 1 and Proposition 2.

We start by generating a discrete grid over the domain of the partial differential equation in (7):

$$x_i = ih \qquad i = 0, 1, 2, .., N_x, \qquad h = \frac{x^*}{N_x}, \tag{31}$$

$$y_j = jg \qquad i = 0, 1, 2, .., N_y, \qquad g = \frac{y^*}{N_y}. \tag{32}$$

Recall that x^* and y^* are the optimal investment triggers in case the other state variable is zero and, thus, are the natural end points of the grid. Moreover, we consider the following notation. Let $V_{i,j}$ denote the value of the firm at the grid points (x_i, y_j). Then we are able to derive a linear system of equations that allows to solve for the discrete grid points simultaneously.

We discretize the partial differential equation using a weighted sum of the function values at the neighboring point approximations to the partial derivatives. This yields

$$\begin{aligned}\frac{1}{2}\sigma_1^2 p_1^2 \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h^2} + \frac{1}{2}\sigma_2^2 p_2^2 \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{g^2} + \alpha_1 x \frac{V_{i+1,j} - V_{i,j}}{h} + \alpha_2 y \frac{V_{i,j+1} - V_{i,j}}{g} \\ + \rho\sigma_1\sigma_2 xy \frac{V_{i+1,j+1} - V_{i+1,j-1} - V_{i-1,j+1} + V_{i-1,j-1}}{4hg} - rV_{i,j} = 0.\end{aligned} \tag{33}$$

Rearranging the terms, gives

$$\begin{aligned}V_{i,j}\left(-\sigma_1^2 i^2 - \alpha_1 ih - \sigma_2^2 \frac{j^2h^2}{g^2} - \alpha_2 \frac{jh^2}{g} - rh^2\right) + V_{i,j+1}\left(\frac{1}{2}\sigma_2^2 \frac{j^2h^2}{g^2} + \alpha_2 \frac{jh^2}{g}\right) + V_{i+1,j}\left(\frac{1}{2}\sigma_1^2 i^2 + \alpha_1 ih\right) + \\ V_{i,j-1}\left(\frac{1}{2}\sigma_2^2 \frac{j^2h^2}{g^2}\right) + V_{i-1,j}\left(\frac{1}{2}\sigma_1^2 i^2\right) + \rho\sigma_1\sigma_2 \frac{ijh}{4g}(V_{i+1,j+1} - V_{i+1,j-1} - V_{i-1,j+1} + V_{i-1,j-1}) = 0.\end{aligned} \tag{34}$$

The vector of unknown grid points, $\mathbf{v}$, can be ordered in the following way

$$\mathbf{v} = \begin{bmatrix} V_{0,0} \\ V_{0,1} \\ \vdots \\ V_{0,N_j} \\ \vdots \\ V_{N_i,N_j} \end{bmatrix} \tag{35}$$

Then the partial differential equation (33) can be represented as a system of linear equations

$$B\mathbf{v} = 0, \tag{36}$$

where B is the matrix of coefficients resulting from (34).

This system can be solved by applying appropriate boundary conditions. We use the fact that the value at $(x*, 0)$ and $(0, y*)$ must equalthe value of the immediate investment. In addition, if either x_i or y_j is equal to zero the problem is reduced to one-dimension, and the grid points together with the threshold boundary can be found analytically. Given a candidate threshold function, the system (36) in combination with the boundary conditions in zero and final nodes, yield a solution for the unknown grid points. To determine the optimal threshold we implement the following procedure. First, we propose a shape of the exercise boundary. For example, the results that we present in Figure 2 are based on the quadratic function, i.e. $y = a + bx + cx^2$. The unknown parameters, a and b can be determined using the analytical threshold boundaries when either x_i or y_j is zero. In order to find c, we compute the derivative of the option value at the candidate threshold boundary at each node, and compare it with the derivatives resulting from the smooth pasting conditions. Next, we compute the sum squared error of the differences and minimize it with respect to unknown parameter c, which allows to determine the optimal threshold in such a way that the smooth pasting condition is satisfied.

We now replicate Figure 1 using our finite-differencing scheme and depict it in Figure 2.

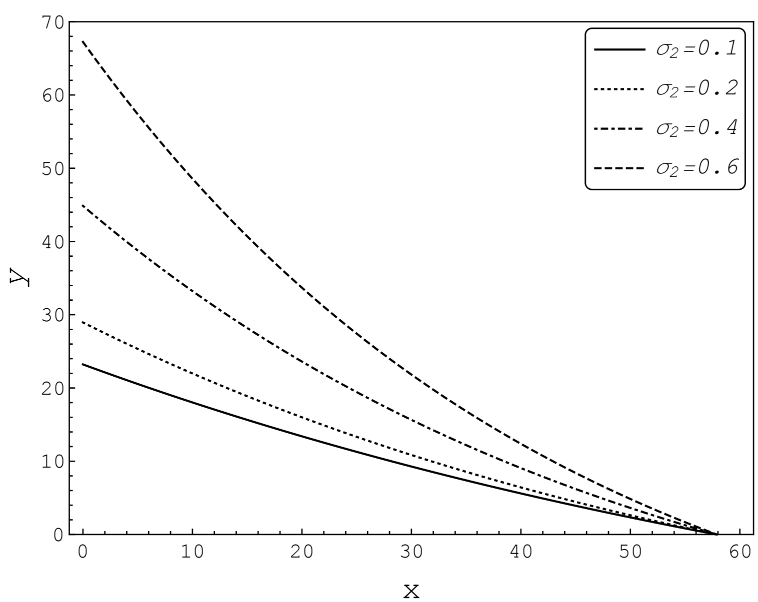

Figure 2: The numerical threshold boundary for the following set of parameter values: $\sigma_1 = 0.2$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$, and different values of σ_2.

This numerical example results in a more intuitive shape of thresholds boundaries and represent a standard result from the real options theory. Namely, an increase in volatility leads to an increase of the optimal investment threshold.

In addition, finite differencing also allows for the calculation of an approximation to the value function that is implied by the quasi-analytical boundary $\hat{b}$. This can be done by solving (36) for the boundary in (22). Figure 3 illustrates the comparison between the implied value function and the numerical solution represented by quadratic boundary for a fixed level of x and different values of y.

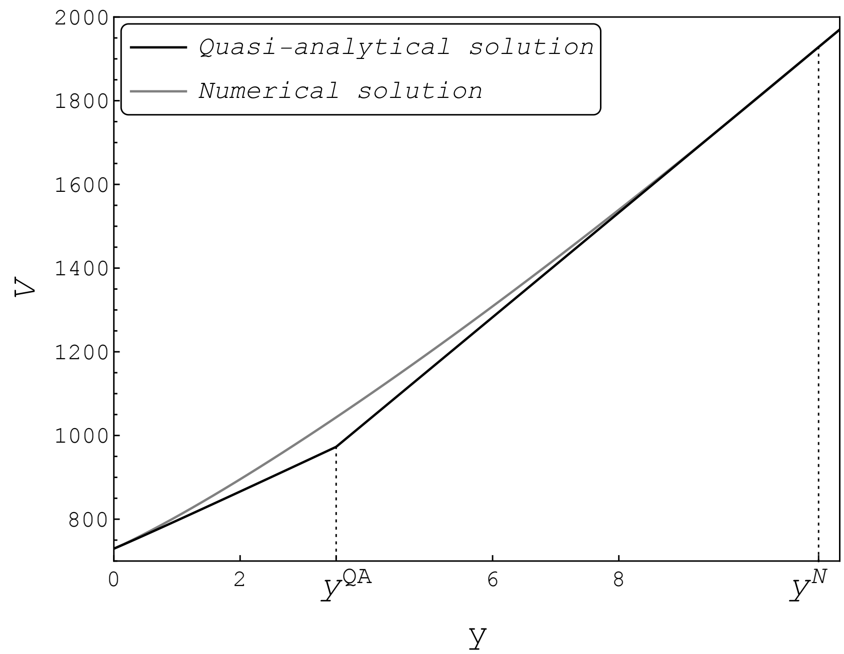

Figure 3: The numerical threshold boundary for the following set of parameter values: $x = 40.52$, $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$, and different values of y.

From Figure 3 it is evident that the value function implied by the quasi-analytical solution has a kink at the boundary point $y^{QA} = \hat{b}(40.52) = 2.49$, violating the smooth-pasting condition. Consequently, the quasi-analytical approach underestimates the true value function, which leads to a sub-optimal investment decision rule for large values of x. Note that the quasi-analytical approach suggests a much lower trigger than our finite-differencing scheme. For $x = 40.52$, the numerical procedure based on the finite-difference algorithm gives the boundary point $y^{QA} = 10.26$, such that the smooth-pasting condition holds. Figure 4 illustrates the value function for different values of x and y, as well as the threshold boundary.

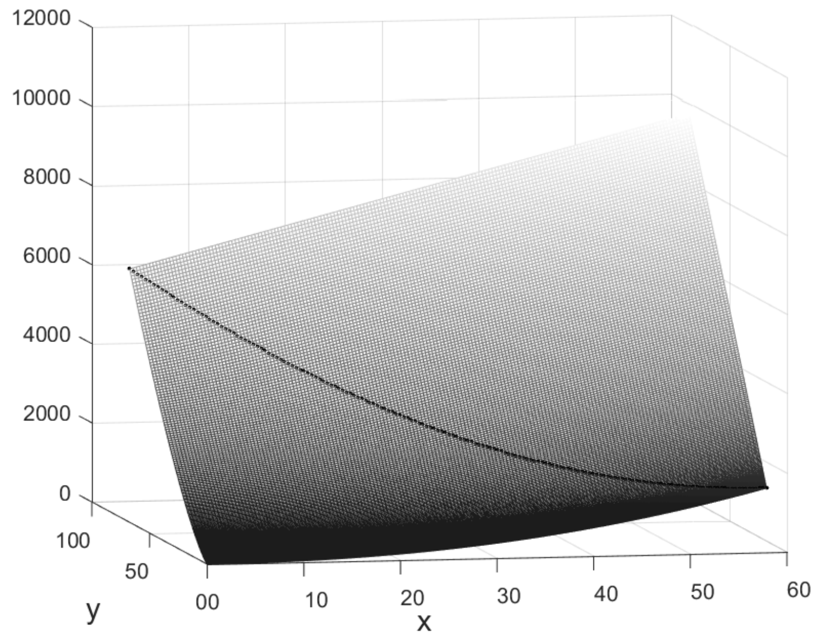

Figure 4: The numerical value function and threshold boundary (solid black curve) for the following set of parameter values: $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$, and different values of x and y.

As can be seen, the value appears to be smooth for different values of x and y in the grid. The average squared error resulting from the numerical procedure is equal to 0.44, which corresponds to 0.17% of the true value of total derivative of the value function. Therefore, we conclude that the proposed numerical method is a good approximation for the true value function and optimal threshold.

Lastly, in order to give an indication how often a firm would make a poorly timed investment decision, we simulate the passage time for the processes X_t and Y_t to reach the quasi-analytical boundary. We then run the procedure 5000 times for a specific set of starting values (x_0, y_0), and calculate the percentage of cases of the threshold being reached within the next 5 years. We perform a similar procedure, to determine the investment probabilities for our numerical solution. The results for the different starting points are presented in Table 2.

(x_0, y_0)	5	10	15
10	10.06%	23.97%	39.03%
15	21.69%	42.87%	61.69%
20	40.32%	68.34%	90.21%

(a) Quasi-analytical boundary

(x_0, y_0)	5	10	15
10	5.40%	5.56%	5.51%
15	5.57%	5.41%	5.57%
20	5.43%	5.24%	5.59%

(b) Numerical boundary

Table 2: Percentage of cases when a firm undertakes an investment within the next 5 years for the set of parameter values: $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$.

Table 2b shows that, for example, for the starting values (15,10) the firm should invest in 5.41% of the cases. According to the quasi-analytical approach however, the firm invests in 42.87% of the cases, implying that many times the firm invests while it is in fact not optimal to do so.

5 Conclusion

Since the seminal works of Dixit and Pindyck (1994) and Trigeorgis (1996), it became clear that real investment problems must be solved using a real options approach. Instead, application of the standard net present value decision rule could lead to investment decisions that are “very wrong”, as is extensively demonstrated in the above books. Since firm investment decisions lie at the basis of economic growth, it is crucial to take these decisions in the right way. From this perspective it is clear that it is of main importance to work on the development of the theory of real options.

Where in the basic analysis the real options model consists of a single firm having the opportunity to invest in a project of given size, with revenue governed by a single geometric Brownian motion process, several authors have extended this framework. Smets (1991) is the first to consider a scenario where two firms can invest in the same market. The revenue in this market is still governed by one stochastic process, also after both firms have already invested and thus are active in this market. The assumption ”project of given size” is relaxed in Bar-Ilan and Strange (1999) and Dangl (1999), in which the firm not only needs to decide about the time, but also about the size of the investment. Huisman and Kort (2015) combine the two extensions by considering a duopoly market where both firms also have to determine the investment size.

The above mentioned contributions have in common that just one stochastic process governs the dynamics of the model. This can be a major shortcoming, especially when problems are considered with multiple firms or products. Therefore, also considering real options problems driven by multiple sources of uncertainty is an important extension to the basic real options analysis, especially from a practical perspective. Adkins and Paxson (2011b) develop a so-called quasi-analytical approach to attack such problems. This approach has already been adopted by several other authors, as the overview in Section 1 shows. Our paper argues, however, that this quasi-analytical method does not result in the correct investment decision rule. We get to our point by analyzing a particular real options model with two-factor uncertainty. It considers an investment project where after investment the firm is able to produce two different products. The output price of each of these products follows a different geometric Brownian motion process. The investment cost is constant and sunk.

From the analysis of this two-factor real options problem we obtain that the quasi-analytical investment decision rule also fails to be a reasonable approximation to the optimal decision. In particular, we find that the quasi-analytical solution does not comply with the (analytical) result that the investment threshold boundary must be monotonically increasing in the volatility parameters of both stochastic processes.

The ultimate conclusion is that non-homogenous real options problems with two-factor uncertainty should be solved using a different numerical procedure. We propose an easy-to-implement finite difference algorithm. Note, however, that if our two-factor uncertainty problem is homogenous, then a standard (cf. McDonald and Siegel (1986)) reduction in dimensionality can be obtained, leading to an analytical solution.

Appendix

Proof of Theorem 1

Throughout the proof, we will denote the unique solution to (1) for given starting point $(X_0, Y_0) \in \Re^2_+ \setminus \{0\}$ by (X^x, Y^y). Note that $(X^x, Y^y) = (xX^1, yY^1)$.

1. ($V > 0$ on $\Re^2_{++}$) On S the result is trivial. Let $(x, y) \in D \cap \Re^2_{++}$. Consider the stopping time

$$\tau = \inf\{t \geq 0 | F(X_\tau, Y_\tau) > 0\}.$$

Since $e^{-r\tau}F(X_\tau, Y_\tau) = 0$ on $\{\tau = \infty\}$ (since $r > \max\{\alpha_1, \alpha_2\}$) and $\mathsf{P}(\tau < \infty) > 0$, it holds that

$$V(x, y) \geq \mathsf{E}\left[e^{-r\tau}F(X_\tau, Y_\tau)\right] > 0.$$

2. (Convexity of V) On S the result is trivial. Take $(x', y'), (x'', y'') \in D$ and $\lambda \in (0, 1)$. Define

$(x, y) := \lambda(x', y') + (1-\lambda)(x'', y'')$. It then holds that

$$\begin{aligned}
V(x,y) =& \sup_\tau \mathsf{E}\left[e^{-r\tau}F(x',y')\right]\\
=& \sup_\tau \mathsf{E}\left[e^{-r\tau}\left(\frac{xX^1_\tau}{\delta_1}+\frac{yY^1_\tau}{\delta_2}-I\right)\right]\\
=& \sup_\tau \mathsf{E}\left[e^{-r\tau}\left(\frac{(\lambda x'+(1-\lambda)x'')X^1_\tau}{\delta_1}+\frac{(\lambda y'+(1-\lambda)y'')Y^1_\tau}{\delta_2}-I\right)\right]\\
=& \sup_\tau \mathsf{E}\left[\lambda e^{-r\tau}\left(\frac{x'X^1_\tau}{\delta_1}+\frac{y'Y^1_\tau}{\delta_2}-I\right)+(1-\lambda)e^{-r\tau}\left(\frac{x''X^1_\tau}{\delta_1}+\frac{y''Y^1_\tau}{\delta_2}-I\right)\right]\\
\leq& \lambda\sup_\tau \mathsf{E}\left[e^{-r\tau}F(x',y')\right]+(1-\lambda)\sup_\tau \mathsf{E}\left[e^{-r\tau}F(x'',y'')\right]\\
=& \lambda V(x',y')+(1-\lambda)V(x'',y'').
\end{aligned}$$

3. (Continuity of V) This property follows from the general theory of stochastic processes, see, e.g., (Krylov, 1980, Theorem 3.1.5).

4. (Monotonicity of V) We prove that V is (strictly) increasing in x. Again, the result is trivial on S. Take $(x, y) \in D$ and let $\varepsilon > 0$ be such that $(x + \varepsilon, y) \in D$ (such ε exists since D is open; see below). Take any stopping time τ. It then holds that

$$\mathsf{E}\left[e^{-r\tau}\left(\frac{(x+\varepsilon)X^1_\tau}{\delta_1}+\frac{yY^1_\tau}{\delta_2}-I\right)\right] \geq \mathsf{E}\left[e^{-r\tau}\left(\frac{xX^1_\tau}{\delta_1}+\frac{yY^1_\tau}{\delta_2}-I\right)\right],$$

with equality only when $\{\tau = \infty\}$ a.s.. Note that τ with $\{\tau = \infty\}$ a.s. is never optimal. Take $\tau^* = \inf\{t \geq 0 | Y_t \geq \delta_1 I + 1\}$. Then $\mathsf{P}(\tau^* < \infty) > 0$ and, thus, we have that $\mathsf{E}\left[e^{-r\tau^*}F(X^x_{\tau^*}, Y^y_{\tau^*})\right] > 0.]$ Therefore, $V(x + \varepsilon, y) > V(x, y)$.

5. (Closedness of D) Take a sequence $(x^{(n)}, y^{(n)})_{n\in\aleph}$ in S with limit (x, y). Then $V(x^{(n)}, y^{(n)}) = F(x^{(n)}, y^{(n)})$ for all $n \in \aleph$. Since $\lim_{n\to\infty} F(x^{(n)}, y^{(n)}) = F(x, y)$ and V is continuous, it holds that $V(x, y) = F(x, y)$. This implies that $(x, y) \in S$.

6. (Convexity of D) Suppose there exists $(x', y'), (x'', y'') \in S$ and $\lambda \in (0, 1)$ such that $(x, y) := \lambda(x', y') + (1-\lambda)(x'', y'') \in D$. It then holds that

$$V(x,y) > F(x,y) = \lambda F(x',y') + (1-\lambda)F(x'',y'') = \lambda V(x',y') + (1-\lambda)V(x'',y'').$$

This contradicts convexity of V.

7. ($b(x)$ can be written as a sup) Take $(x, y) \in D$. The there exists a stopping time τ^* such that $(X_{\tau^*}, Y_{\tau^*}) \in D$, a.s.. Hence,

$$V(x,y) = \sup_\tau \mathsf{E}\left[e^{-r\tau}F(X_\tau,Y_\tau)\right] \geq \mathsf{E}\left[e^{-r\tau^*}F(X_{\tau^*},Y_{\tau^*})\right] > F(x,y).$$

Now take $\varepsilon \in (0, y)$. Then

$$\begin{aligned}
V(x, y-\varepsilon) \geq & \mathsf{E}\left[e^{-r\tau^*} F(X_{\tau^*}, Y_{\tau^*})\right] \\
= & \mathsf{E}\left[e^{-r\tau^*}\left(\frac{xX^1_{\tau^*}}{\delta_1} + \frac{(y-\varepsilon)Y^1_{\tau^*}}{\delta_2} - I\right)\right] \\
= & \mathsf{E}\left[e^{-r\tau^*}\left(\frac{xX^1_{\tau^*}}{\delta_1} + \frac{yY^1_{\tau^*}}{\delta_2} - I\right)\right] - \mathsf{E}\left[e^{-r\tau^*}\frac{\varepsilon Y^1_{\tau^*}}{\delta_2}\right] \\
\overset{(*)}{\geq} & \mathsf{E}\left[e^{-r\tau^*}\left(\frac{xX^1_{\tau^*}}{\delta_1} + \frac{yY^1_{\tau^*}}{\delta_2} - I\right)\right] - \frac{\varepsilon}{\delta_2} \\
> & \mathsf{E}\left[e^{-r\tau^*}\left(\frac{xX^1_{\tau^*}}{\delta_1} + \frac{yY^1_{\tau^*}}{\delta_2} - I\right)\right] > F(x, y) > F(x, y-\varepsilon),
\end{aligned}$$

where $(*)$ follows from the fact that $e^{-rt}Y_t$ is a supermartingale. Therefore, $(x, y-\varepsilon) \in D$.

8. (b is non-increasing) This follows from the fact that for all $(x, y) \in D$ and all $\varepsilon \in (0, x)$ it holds that $(x-\varepsilon, y) \in D$. This can be proved using a similar argument as above.

9. (b is convex) Convexity of b follows from the fact that its epigraph is the convex set S.

10. (b is continuous) Continuity of b on $(0, \infty)$ is immediate, because it is a convex function on an open convex set (see, for example, Berge, 1963, Theorem 8.5.7). Continuity at $x = 0$ follows from the fact that the stopping set is closed.

11. (boundedness of b) The boundedness properties follow from continuity and x^* and y^* being the solutions of the optimal stopping problem on $\Re_+ \times \{0\}$ and $\{0\} \times \Re_+$, respectively.

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