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Proefschrift

ter verkrijging van de graad van doctor aan Tilburg University op gezag van de rector magnificus, prof. dr. E.H.L. Aarts, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op vrijdag 3 november 2017 om 10.00 uur door

Ferenc Horváth

geboren te Mátészalka, Hongarije.

PROMOTORES:	Prof. dr. F.C.J.M. de Jong Prof. dr. B.J.M. Werker
OVERIGE COMMISSIELEDEN:	Prof. dr. J.J.A.G. Driessen Prof. dr. N. Branger Prof. dr. A. Lucas Dr. A.G. Balter

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To this date, I clearly remember the moment when I first crossed the threshold of the Koopmans Building as an M.Sc. student on my Ph.D. fly-out in 2013. By the end of that day, after meeting with faculty members and Ph.D. students, my unconscious mind already knew: this city would be my home and the people I had just talked to hours ago my second family for the upcoming years. Choosing Tilburg University as the place to earn my Ph.D. degree was one of the best decisions of my life.

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Ferenc Horvath

Hong Kong, September 2017

Introduction

When investors do not know the outcome of future asset returns, but they know the probability distribution of these returns, they face *risk*. If investors do not even know the probability distribution of asset returns, they also face *uncertainty*. In reality, just as they are risk-averse, investors are also averse of uncertainty. If during their investment decisions they take into account the fact that they do not know the true distribution of asset returns, investors make *robust investment decisions*.

Robustness is the central concept of my doctoral dissertation. In the respective chapters I analyze how model and parameter uncertainty affect financial decisions of investors and fund managers, and what their equilibrium consequences are. Chapter 1 gives an overview of the most important concepts and methodologies used in the robust asset allocation and robust asset pricing literature, and it also reviews the most recent advances thereof. Chapter 2 provides a resolution to the bond premium puzzle by featuring robust investors, and – as a technical contribution – it develops a novel technique to solve robust dynamic asset allocation problems: the robust version of the martingale method. Chapter 3 contributes to the resolution of the liquidity premium puzzle by demonstrating that parameter uncertainty generates an additional, positive liquidity premium component, the liquidity uncertainty premium. Chapter 4 examines the effects of model uncertainty on optimal Asset Liability Management decisions.

In Chapter 1 we survey the literature on robust dynamic asset allocation with an emphasis on the Asset Liability Management of pension funds. After demonstrating the difference between risk and uncertainty, we introduce two levels of uncertainty: parameter uncertainty and model uncertainty. We describe four of the most widely used approaches in robust dynamic asset allocation problems: the penalty approach, the constraint approach, the Bayesian approach, and the approach of smooth recursive preferences. We then demonstrate the importance of uncertainty for investors (including pension funds) from both a normative and a positive aspect, then we review the literature on robust asset management and on robust Asset Liability Management.

In Chapter 2 we analyze a dynamic investment problem with interest rate risk and ambiguity. After deriving the optimal terminal wealth and investment policy, we expand our model into a robust general equilibrium model and calibrate it to U.S. data. We confirm the bond premium puzzle, i.e., we need an unreasonably high relative risk-aversion parameter to explain excess returns on long-term bonds. Our model with robust investors reduces this risk-aversion parameter substantially: a relative risk aversion of less than four suffices to match market data. Additionally, we provide a novel formulation of robust dynamic investment problems together with an alternative solution technique: the robust version of the martingale method.

In Chapter 3 I analyze a dynamic investment problem with stochastic transaction cost and parameter uncertainty. I solve the problem numerically, and I obtain the optimal consumption and investment policy and the least-favorable transaction cost process. Using reasonable parameter values, I confirm the liquidity premium puzzle, i.e., the representative agent model (without robustness) produces a liquidity premium which is by a magnitude lower than the empirically observed value. I show that my model with robust investors generates an additional liquidity premium component of 0.05%-0.10% (depending on the level of robustness) for the first 1% proportional transaction cost, and thus it provides a partial explanation to the liquidity premium puzzle. Additionally, I provide a novel non-recursive representation of discrete-time robust dynamic asset allocation problems with transaction cost, and I develop a numerical technique to efficiently solve such investment problems.

In Chapter 4 we analyze a dynamic Asset Liability Management problem with model uncertainty in a complete market. The fund manager acts in the best interest of the pension holders by maximizing the expected utility derived from the terminal funding ratio. We solve the robust multi-period Asset Liability Management problem in closed form, and identify two constituents of the optimal portfolio: the myopic demand, and the liability hedge demand. We find that even though the investment opportunity set is stochastic, the investor does not have intertemporal hedging demand. We also find that model uncertainty induces a more conservative investment policy regardless of the risk attitude of the fund manager, i.e., a robust investment strategy corresponds to risk exposures which provide a much stronger liability hedge.

Contents

Chapter 1

Robustness for Asset Liability Management of Pension Funds

Co-authors: Frank de Jong and Bas J.M. Werker

1.1 Introduction

Pension funds (and investors in general) face both risk and uncertainty during their everyday operation. The importance of distinguishing between risk and uncertainty was first emphasized in the seminal work of Knight (1921), and it has been an active research topic in the finance literature ever since. Risk means that the investor does not know what future returns will be, but she does know the probability distribution of the returns. On the other hand, uncertainty means that the investor does not know precisely the probability distribution that the returns follow. As a simple example, let us assume that the one-year return of a particular stock follows a normal distribution with 9% expected value and 20% standard deviation. A pension fund who knows that the return of the stock follows this particular distribution, faces risk, but it does not face uncertainty. Another pension fund only knows that the return of this stock follows a normal distribution, that its expected value lies between 8% and 12%, and that its standard deviation is 20%. This pension fund faces not only risk, but also uncertainty: not only does it not know the exact return in one year, it also does not know the precise probability distribution that the return follows.

A risk-averse investor is averse of the risk with known distribution, while an uncertainty-averse investor is averse of uncertainty.[1] Decisions which take into account the fact that the investor faces uncertainty, are called *robust* decisions. These decisions are robust to uncertainty because they protect the investor against uncertain outcomes ("bad events").

1.2 Classification of robustness

1.2.1 Parameter uncertainty and model uncertainty

We can distinguish two levels of uncertainty: parameter uncertainty and model uncertainty. If the investor knows the form of the underlying model but she is uncertain about the exact value of one or several parameters, the investor then faces parameter uncertainty. On the other hand, if the investor does not even know the form of the underlying model, she faces model uncertainty.

For example, if the investor knows that the return of a particular stock follows the geometric Brownian motion

$$\frac{\mathrm{d}S_t}{S_t} = \mu_S \mathrm{d}t + \sigma_S \mathrm{d}W_t^{\mathbb{P}} \tag{1.1}$$

with constant drift μ_S and constant volatility σ_S, but she does not know the exact value of these two parameters, she faces parameter uncertainty. But if she does not even know whether the stock return follows a geometric Brownian motion or any other type of stochastic process, she faces model uncertainty.

The distinction between parameter uncertainty and model uncertainty is in many cases not clear-cut. The most important example of this from the point-of-view of pension funds is the uncertainty about the drift parameters. If we take the simple example of the stock return in (1.1), then being uncertain about the drift can be translated into being uncertain about the probability measure $\mathbb{P}$, assuming that the investor considers only equivalent probability measures.[2] Uncertainty about the probability measure is considered model uncertainty according to the vast majority of

[1] In the behavioral finance and in the operations research literature, a distinction is made between *ambiguity* and *uncertainty*. Ambiguity refers to missing information that could be known, while uncertainty means that the information does not exist. For a detailed treatment of the difference between ambiguity and uncertainty, we refer to Dequech (2000). In the robust asset allocation literature the two terms are used interchangeably, and we also follow this practice in this paper.

[2] Two probability measures are said to be equivalent if and only if each is absolutely continuous with respect to the other. That is, the investor may be uncertain about the exact probability of events, but she is certain about which events happen for almost sure (i.e., with probability one) or with probability zero. The assumption that the investor considers only equivalent probability measures is quite common in the robustness literature.

the literature. This is the reason why, e.g., Maenhout (2004) and Munk and Rubtsov (2014) discuss model uncertainty, even though in their model the investor is uncertain only about the drift parameters.[3]

The assumption that the investor is uncertain only about the drift, but not about the volatility, is not unrealistic. If constant volatility is assumed, then the volatility parameter can be estimated to any arbitrary level of precision, as long as the investor can increase the observation frequency as much as she wants. Given that in today's world return data are available for every second (or even more frequently), the assumption that the investor is able to observe return data in continuous time is indeed justifiable. For the reasons why expected returns (i.e., the drift parameters) are notoriously hard to estimate, we refer to Merton (1980), Blanchard, Shiller, and Siegel (1993) and Cochrane (1998). Since pension funds' uncertainty mostly concerns uncertainty about the drift, and since it is common practice in the literature to assume that investors only consider equivalent probability measures, whenever we talk about uncertainty in the rest of this paper, we mean uncertainty about the drift term, unless we indicate otherwise.

1.2.2 A generic investment problem

Before discussing robustness in details, we formulate a generic non-robust dynamic asset allocation problem. Later in the paper we extend this model to formulate a robust framework.

Let us assume that the investor derives utility from consumption and terminal wealth. Her goal is to maximize her total expected utility. She has an initial wealth x, and her investment horizon is T. At the end of every period (i.e., at the end of the 0^{th} year, at the end of the 1^{st} year, ..., at the end of the (T-1)th year) she has to make a decision: how much of her wealth to consume and how to allocate her remaining wealth among the assets available on the financial market. For the sake of simplicity, we assume that the financial market consists of a risk-free asset, which pays a constant return r_f, and a stock, which follows the geometric Brownian motion in (1.1). Then we can formulate the investor's optimization problem as follows.

Problem 1. *Given initial wealth x, find an optimal pair $\{C_t, \pi_t\}\, \forall t \in [0, ..., T-1]$ for the utility*

[3] If two probability measures are equivalent, then the standard Wiener processes under the two measures differ only in their drift terms (if expressed under the same probability measure), and their volatility term is the same. For a detailed treatment of this topic we refer to Karatzas and Shreve (1991).

maximization problem

$$V_0(x) = \sup_{\{C_t,\pi_t\}\forall t\in[0,\dots,T-1]} \mathrm{E}^{\mathbb{P}}\left[\sum_{t=1}^{T} U_C(C_t) + U_T(X_T)\right] \tag{1.2}$$

subject to the budget constraint

$$\frac{\mathrm{d}X_t}{X_t} = \left[r_f + \pi_t(\mu_S - r_f) - \frac{C_t}{X_t}\right]\Delta t + \pi_t \sigma_S \Delta W_t^{\mathbb{P}}. \tag{1.3}$$

In Problem 1 X_t denotes the investor's wealth at time t, C_t is her consumption at time t, π_t is the ratio of her wealth invested in the stock, and $U_C(\cdot)$ and $U_T(\cdot)$ are her utility functions.

If the investor can consume and reallocate her wealth continuously, the continuous counterpart of Problem 1 can be formulated.

Problem 2. *Given initial wealth x, find an optimal pair* $\{C_t, \pi_t\}, t \in [0, T]$ *for the utility maximization problem*

$$V_0(x) = \sup_{\{C_t,\pi_t\} t\in[0,T]} \mathrm{E}^{\mathbb{P}}\left[\int_{t=0}^{T} U_C(C_t)\,\mathrm{d}t + U_T(X_T)\right] \tag{1.4}$$

subject to the budget constraint

$$\frac{\mathrm{d}X_t}{X_t} = \left[r_f + \pi_t\left(\mu_S^{\mathbb{B}} - r_f\right) - \frac{C_t}{X_t}\right]\mathrm{d}t + \pi_t \sigma_S \mathrm{d}W_t. \tag{1.5}$$

There are two main methods that can be used to solve optimization problems like Problem (1) and Problem (2): relying on the principle of dynamic programming (which makes use of the Bellman difference equation in discrete time optimization problems and of the Hamilton-Jacobi-Bellman (HJB) differential equation in continuous time optimization problems) and the martingale method of Cox and Huang (1989).[4]

We now briefly explain the intuition behind the principle of dynamic programming. When the investor is making a decision about how much of her wealth to consume and how to allocate the rest, she is working backwards. In the discrete setup of Problem 1 this means that first she solves the optimization problem as if she were at time $T-1$, assuming her wealth before making the decision is X_{T-1}. This way she solves a one-period optimization problem by maximizing the

[4] For a detailed treatment of the principle of dynamic programming we refer to Bertsekas (2005) and Bertsekas (2012), while the martingale method is treated in detail in Karatzas and Shreve (1998).

sum of her immediate utility from consumption and her expected[5] utility from terminal wealth with respect to C_{T-1} and π_{T-1}. This maximized sum is the investor's value function at time $T-1$, and we denote it by V_{T-1}. According to the principle of dynamic programming, the C_{T-1} and π_{T-1} values which the investor has just obtained, are also optimal solutions to the original optimization problem (Problem 1). Then she moves to time $T-2$. She wants to maximize the sum of her utility from immediate consumption C_{T-2}, her expected utility from C_{T-1}, and her expected utility from X_T,[6] with respect to C_{T-2}, π_{T-2}, C_{T-1} and π_{T-1}. But according to the principle of dynamic programming, she has already found the optimal values of C_{T-1} and π_{T-1} before. Thus her optimization problem at time $T-2$ eventually boils down to maximizing the sum of her utility from immediate consumption C_{T-2} and the expected value of her value function V_{T-1}, the expectation being conditional on the information available up to time $T-2$. Then she moves to time $T-3$, and continues solving the optimization problem in the same way, until she obtains the optimal C_t and π_t values for all t between 0 and $T-1$. The intuition of solving Problem 2 is the same, but mathematically it means that the investor first obtains the optimal $\{C_t\}$ and $\{\pi_t\}$ processes[7] in terms of the value function, then she solves a partial differential equation (the HJB equation) with terminal condition $V_T = U_T(X_T)$, to obtain the value function. Knowing the value function, she can substitute it back into the previously obtained optimal $\{C_t\}$ and $\{\pi_t\}$ processes.

Cox and Huang (1989) approached Problem 1 and Problem 2 from a different angle and were the first to use the martingale method to solve dynamic asset allocation problems. The basic idea of the martingale method is that first the investor obtains the optimal terminal wealth as a random variable and the optimal consumption process as a stochastic process. Then, making use of the martingale representation theorem (see, e.g., Karatzas and Shreve (1991), pp. 182, Theorem 3.4.15), she obtains the unique $\{\pi_t\}$ process that enables her to achieve the previously derived optimal terminal wealth and optimal consumption process. In many optimization problems the martingale method has not only mathematical advantages (one does not have to solve higher-order partial differential equations), but it also provides economic intuition and insights into the decision-making of the investor. For such an example, we refer to the optimization problem in Horvath,

[5] Conditionally on the information available up to time $T-1$.

[6] Both of these expectations are conditional on the information available up to time $T-2$.

[7] An indexed random variable (the index being t) in brackets denotes a stochastic process, e.g., $\{C_t\}$ is the consumption process.

de Jong, and Werker (2016).

1.2.3 Robust dynamic asset allocation models

In Problem 1 and Problem 2 we assumed that the investor knows the underlying model properly. In reality, however, investors face uncertainty: they do not know the precise distribution of returns. By incorporating this uncertainty into their optimization problem, they make robust consumption and investment decisions. In the remainder of the paper we assume, for the sake of simplicity, that the investor makes decisions in continuous time, but the intuition can always be carried over to the discrete counterpart of the model. In this subsection we also assume that the financial market consists of a risk-free asset with constant rate of return and a stock, the return of which follows a geometric Brownian motion with constant drift and volatility parameters. The investor knows the volatility parameter of the stock return process, but she is uncertain about the drift parameter. The approaches to robust asset allocation that we introduce in this subsection can straightforwardly be extended to more complex financial markets, e.g., one accommodating several stocks, long-term bonds, a stochastic risk-free rate, etc.

There are several ways to introduce robustness into Problem 2. In this subsection we describe four of the most common approaches in the literature: the penalty approach, the constraint approach, the Bayesian approach and the approach of smooth recursive preferences. The basic idea of all of these approaches is the same: the investor is uncertain about μ_S, thus she considers several μ_S-values that she thinks might be the true one. The differences between these four approaches are twofold: how the investor chooses which μ_S-values she considers possible, and how she incorporates these several possible μ_S-values into her optimization problem (e.g., she selects the worst case scenario, or she takes a weighted average of them, etc.).

1.2.4 The penalty approach

The penalty approach was introduced into the literature in Anderson, Hansen, and Sargent (2003). The investor has a μ_S-value in mind which she considers to be the most likely. We call this the *base parameter*, and denote it by $\mu_S^{\mathbb{B}}$. She is uncertain about the true μ_S, so she considers other μ_S-values as well. These are called *alternative parameters*, and denoted by $\mu_S^{\mathbb{U}}$. The relationship between $\mu_S^{\mathbb{B}}$ and $\mu_S^{\mathbb{U}}$ is expressed by

$$\mu_S^{\mathbb{B}} = \mu_S^{\mathbb{U}} + u_S \sigma_S \qquad \forall t \in [0, T]\,. \tag{1.6}$$

u_S is multiplied by the volatility parameter for scaling purposes.[8] Following the penalty approach, the investor adds a penalty term to her goal function, concretely

$$\int_0^T \Upsilon_t \frac{u_S^2}{2} \mathrm{d}t. \tag{1.7}$$

The parameter Υ_t expresses how uncertainty-averse the investor is, and one might assume that it is a constant, a deterministic function of time, or even a stochastic function of time. $\frac{u_S^2}{2}$ expresses the distance between the base parameter and the alternative parameter.[9]

The investor considers all possible $\mu_S^{\mathbb{U}}$ parameters and she chooses the one which results in the lowest possible value function. Putting it differently, she considers the worst case scenario. We now formalize the robust counterpart of Problem 2, using the penalty approach.

Problem 3. *Given initial wealth x, find an optimal triplet* $\{u_S, C_t, \pi_t\}, t \in [0, T]$ *for the robust utility maximization problem*

$$V_0(x) = \inf_{u_S} \sup_{\{C_t, \pi_t\}} \mathrm{E}\left\{ \int_{t=0}^T \left[U_C(C_t) + \Upsilon_t \frac{u_S^2}{2} \right] \mathrm{d}t \right. \\ \left. + U_T(X_T) \right\} \tag{1.8}$$

subject to the budget constraint

[8] The reason behind u_S being multiplied by σ_S lies in the fact that the investor being uncertain about the drift parameter is equivalent to her being uncertain about the physical probability measure, as long as she only considers probability measures that are equivalent to the base measure that she considers to be the most likely. Changing from her base measure to an alternative measure thus means that the base drift $\mu_S^{\mathbb{B}}$ changes to $\mu_S^{\mathbb{U}} + u_S \sigma_S$, and the stochastic process that under the base measure was a standard Wiener process changes to another stochastic process, namely one which is a standard Wiener process under the alternative measure.

[9] Mathematically, $u_S^2/2$ is the time-derivative of the Kullback-Leibler divergence, also known as the relative entropy. The reason why the Kullback-Leibler divergence is often used in the literature of robustness as the penalty function lies not only in its mathematical tractability, but also in its intuitive interpretation. Actually, the relative entropy of measure $\mathbb{U}$ with respect to measure $\mathbb{B}$ is the amount of information lost if one uses measure $\mathbb{B}$ to approximate measure $\mathbb{U}$. If in the definition of relative entropy the logarithm is of base 2, the amount of information is measured in bits (i.e., how many yes-no questions have to be answered in order to tell $\mathbb{U}$ and $\mathbb{B}$ apart). If the logarithm is of base e, the amount of information is measured in nats. For a detailed treatment of the Kullback-Leibler divergence we refer to Cover and Thomas (2006), Chapter 2).

$$\frac{\mathrm{d}X_t}{X_t} = \left[r_f + \pi_t \left(\mu_S^{\mathbb{B}} + u_S \sigma_S - r_f\right) - \frac{C_t}{X_t}\right]\mathrm{d}t + \pi_t \sigma_S \mathrm{d}W_t. \tag{1.9}$$

A robust investor with Problem 3 then solves her optimization problem either by making use of the principle of dynamic programming or by the martingale method, the same way as we described in Section 1.2.2 for the case of a non-robust investor. The only difference is that instead of only maximizing with respect to $\{C_t, \pi_t\}$ she first maximizes with respect to these two variables, then she minimizes with respect to u_S.[10]

The penalty approach is widely used in the literature to study the effects of robustness on dynamic asset allocation and asset prices. Maenhout (2004) finds that if one accounts for uncertainty aversion based on the penalty approach, it is actually possible to explain a substantial part of the "too high" equity risk premium that is termed the "equity premium puzzle" in the literature. Concretely, a robust Duffie-Epstein-Zin representative investor with reasonable risk-aversion and uncertainty-aversion parameters generates a 4% to 6% equity premium. To achieve this result, it is essential that Maenhout (2004) parameterized the uncertainty-aversion parameter Υ_t to be a function of the "value"[11] at the respective time.[12] As he points out, if the uncertainty-aversion parameter is constant (which is actually the case in Anderson et al. (2003)), it is not possible to give a closed form solution to the robust version of Mertons problem. Furthermore, Maenhout (2006) and Horvath et al. (2016) find that if the investment opportunity set is stochastic, robustness increases the importance of intertemporal hedging compared to the non-robust case.

Trojani and Vanini (2002) examine the asset pricing implications of robustness, comparing their results with those of Merton (1969). The penalty approach is extended by Cagetti, Hansen, Sargent, and Williams (2002) to allow the state variables to follow not only pure diffusion processes but mixed jump processes as well. Uppal and Wang (2003) use the penalty approach and explore a potential source of underdiversification. They find that if the investor is allowed to have different levels of ambiguity regarding the marginal distribution of any subsets of the return of the investment

[10]In most robust dynamic investment problems the supinf and infsup preferences lead to the same solution, because the order of maximization and minimization can be interchanged due to Sions maximin theorem (Sion (1958)). So it does not matter whether the investor first maximizes with respect to $\{C_t, \pi_t\}$ and then minimizes with respect to u_S, or if she interchanges the order of maximization and minimization.

[11]By "value" we mean the concept known in dynamic optimization theory: the value function at time t shows the highest possible expected value of utility at time t that the investor can achieve by properly allocating her resources among the available assets between time t and the end of her investment horizon.

[12]This approach is criticized by, e.g., Pathak (2002) for its recursive nature.

assets, there are circumstances when the optimal portfolio is significantly underdiversified compared to the usual mean-variance optimal portfolio. Liu, Pan, and Wang (2005) study the asset pricing implications of ambiguity about rare events using the penalty approach. Routledge and Zin (2009) study the connection between uncertainty and liquidity in the penalty framework.

1.2.5 The constraint approach

The penalty approach determined the set of alternative $\mu_S^{\mathbb{U}}$ parameters by adding a penalty term to the goal function and choosing the least favorable drift parameter. Another way to determine the set of alternative $\mu_S^{\mathbb{U}}$ parameters is to explicitly specify a constraint on u_S. Since the investor considers both positive and negative u_S values,[13] it is a straightforward choice to set a higher constraint on u_S^2, concretely

$$\frac{u_S^2}{2} \leq \eta. \tag{1.10}$$

Using a reasonable value for η, the model assures that the investor considers scenarios which are pessimistic and reasonable at the same time. So, for example, if $\mu_S B = 10\%$, then the investor will not consider $\mu_S^{\mathbb{U}} = -200\%$ as the drift parameter of the stock return, but she might consider $\mu_S^{\mathbb{U}} = 7\%$.

Now we formalize the robust optimization problem, using the constraint approach.

Problem 4. *Given initial wealth x, find an optimal triplet $\{u_S, C_t, \pi_t\}, t \in [0, T]$ for the robust utility maximization problem*

$$V_0(x) = \inf_{u_S} \sup_{\{C_t, \pi_t\}} \mathrm{E}\left\{\int_{t=0}^{T} U_C(C_t)\,\mathrm{d}t + U_T(X_T)\right\} \tag{1.11}$$

subject to

$$\frac{u_S^2}{2} \leq \eta. \tag{1.12}$$

and subject to the budget constraint (1.9).

The form of (1.12) is very similar to the penalty term in (1.7). This is not a coincidence: if and

[13] She is allowed to take short positions, so for her only the magnitude of the expected excess return matters, but not the sign.

only if Υ_t is a nonnegative constant, then there exists such an η, that the solution to Problem 3 and the solution to Problem 4 are the same. For the proof of this statement and a detailed comparison of the penalty approach and the constraint approach we refer to Appendix B in Lei (2001).

The constraint approach is used by, among others, Gagliardini, Porchia, and Trojani (2009) to study the implications of ambiguity-aversion to the yield curve and to characterize the market equilibrium if ambiguity-aversion is also accounted for; and by Leippold, Trojani, and Vanini (2008) to study equilibrium asset prices under ambiguity. Garlappi, Uppal, and Wang (2007) use a closely related approach to build their model for portfolio allocation, but contrary to the majority of literature in this field they examine a one-period (static) setup instead of a dynamic one. Peijnenburg (2014) extends the framework of the constraint-approach and considers, besides the maximin setup, the case of recursive smooth preferences. Moreover, she introduces the concept of learning into the model: the more time elapses, the less uncertain the investor is about the risk premium. We also refer to Cochrane and Saa-Requejo (1996), who use a model similar to the constraint approach to derive bounds on asset pricing in incomplete markets by ruling out "good deals".

1.2.6 The Bayesian approach

In both the penalty approach and the constraint approach the investor had a set of possible $\mu_S^{\mathbb{U}}$ parameters in mind, and she chose the one which minimized her value function. These two approaches did not make it possible to directly incorporate the investor's view on how likely the different $\mu_S^{\mathbb{U}}$ parameters are. The Bayesian approach builds around this exact idea: the investor has a set of possible $\mu_S^{\mathbb{U}}$ parameters in mind, and she renders likelihoods to all of these values that she considers possible. Putting it differently, she can construct a probability distribution on all $\mu_S^{\mathbb{U}}$ parameters.[14] This probability distribution on all $\mu_S^{\mathbb{U}}$ parameters reflects the view of the investor on how likely the various $\mu_S^{\mathbb{U}}$ values are to be the true parameter value. Now we formulate the optimization problem of a robust investor who uses the Bayesian approach.

Problem 5. *Given initial wealth x, find an optimal pair* $\{C_t, \pi_t\}, t \in [0, T]$ *for the robust utility*

[14]The wording is important here: she renders a likelihood value to all $\mu_S^{\mathbb{U}}$, regardless of whether she considers it possible or not. If she considers that a particular $\mu_S^{\mathbb{U}}$ cannot be the true parameter value, she renders a likelihood of zero to it.

maximization problem

$$V_0\left(x\right)= \sup_{\{C_t,\pi_t\}} \mathrm{E}\left\{\int_{t=0}^{T} U_C\left(C_t\right)\mathrm{d}t + U_T\left(X_T\right)\right\} \tag{1.13}$$

subject to the budget constraint (1.9), and assuming that u_s follows a particular probability distribution reflecting the investor's view on how likely different $\mu_S^{\mathbb{U}}$ parameters are.

In the recent literature on robustness, Hoevenaars, Molenaar, Schotman, and Steenkamp (2014) use the Bayesian approach to study the effects of parameter uncertainty on different asset classes, namely stocks, long-term bonds and short-term bonds (bills). They find that uncertainty raises the long-run volatilities of all three asset classes proportionally with the same vector, compared to the volatilities that are obtained using Maximum Likelihood. The consequence of this is that in the optimal asset allocation the horizon effect is much smaller compared to the case of using Maximum Likelihood. Pástor (2000) analyzes the effects of model uncertainty on asset allocation using the Bayesian approach. When calibrating his model to U.S. data, he finds that investors' belief in the domestic CAPM has to be very strong to reconcile the implications of his model with market data.

1.2.7 Smooth recursive preferences

The approach of smooth recursive preferences makes it possible to separate uncertainty (which reflects the investor's beliefs) and uncertainty-aversion (which reflects the investor's taste). The starting point of the smooth recursive preferences approach is the Bayesian framework in Problem 5. The investor does not know the exact value of the expected excess stock return, but she can construct a probability distribution on it. This probability distribution reflects her beliefs: it shows how likely she considers particular $\mu_S^{\mathbb{U}}$ values.

To incorporate her attitude towards uncertainty, she constructs a "distorted" probability distribution from the original distribution that reflects her beliefs on $\mu_S^{\mathbb{U}}$. Intuitively this means that she gives higher weight to "unfavorable events", i.e., to $\mu_S^{\mathbb{U}}$ values that result in low expected utility, and lower weight to "favorable events". Technically, distorting the probability distribution is achieved by applying a concave function (and later its inverse) on the original probabilities to change their relative importance to the investor. If the uncertainty-aversion of an investor with smooth recursive preferences is infinity, and she has a bounded set of priors for $\mu_S^{\mathbb{U}}$, her optimal solution will be the same as an investor who uses the infsup (minimax) setup in either the constraint approach or the penalty approach with constant Υ_t.

The approach of smooth recursive preferences was developed in Klibanoff, Marinacci, and Mukerji (2005), and it was axiomatized in Klibanoff, Marinacci, and Mukerji (2009). Hayashi and Wada (2010) analyzed the asset pricing implications of this approach. Chen, Ju, and Miao (2014) and Ju and Miao (2012) calibrate the uncertainty-aversion parameter within a framework of smooth recursive preferences.

1.3 The role of robustness in ALM of pension funds

1.3.1 Robust Asset Management

Several papers document the relevance of robustness in asset management. Garlappi et al. (2007) use international equity indices to demonstrate the importance of robustness in portfolio allocation. They assume that the investor is uncertain about the expected return of assets, and she makes robust decisions following the constraint approach described in Section 1.2.5. Their analysis suggests that robust portfolios deliver higher out-of-sample Sharpe ratios than their non-robust counterparts. Moreover, the robust portfolios are not only more balanced, but they also fluctuate much less over time - which is a desirable property due to attracting less transaction costs in total.[15]

Another paper emphasizing the importance of robustness in asset management is Glasserman and Xu (2013). They use daily commodity futures data to extract spot price changes. The investor is assumed to have a mean-variance utility function. The model parameters are estimated based on futures price data of the past 6 months, and they are re-estimated every week. The investor is uncertain about the expected return, and she makes robust decisions following the penalty approach (Section 1.2.4). The authors find that the portfolio that is based on robust investment decisions significantly outperforms the non-robust portfolio both in terms of the goal-function value and the Sharpe ratio. The difference in performance between the robust and non-robust portfolio is both statistically and economically significant. Moreover, they conclude that the improvement in performance comes mainly from the reduction of risk, rather than from the increase of return.

[15] Robust portfolios being more balanced and fluctuating less is a "side effect", i.e., they were not designed to have these properties.

Liu (2011) assumes a stochastic investment opportunity set and uncertainty about the expected return within the penalty framework, and demonstrates the superior out-of-sample performance of the robust portfolio. Hedegaard (2014) shows that the robust portfolio outperforms its non-robust counterpart also in the case when the investor knows the expected return, but she is uncertain about the alpha-decay of the predicting factors. Cartea, Donnelly, and Jaimungal (2014) analyze the optimal portfolio of a market maker who is uncertain about the drift of the midprice dynamics, about the arrival rate of market orders, and about the fill probability of limit orders. Using the penalty approach, they demonstrate that the robust strategy delivers a significantly higher out-of-sample Sharpe ratio.

Koziol, Proelss, and Schweizer (2011) show that robust portfolios achieving a significantly higher out-of-sample Sharpe ratio is not only a normative result, but that institutional investors are indeed highly uncertainty-averse and make robust decisions. They find that the average in-sample Sharpe ratio of their asset side is only 60% of the in-sample Sharpe ratio of the corresponding non-robust (i.e. unambiguous) asset portfolio. According to their argument, this result is not due to poor diversification, because institutional investors have the cognitive ability and financial knowledge to optimally diversify their portfolio; moreover, fund managers' compensation is in most of the cases somehow linked to the performance of the managed portfolio. So, a higher in-sample Sharpe ratio means higher compensation for them. The lower in-sample Sharpe ratio thus, as they argue, is a result of uncertainty-aversion. Besides institutional investors being uncertainty averse, they also find that robustness plays a more important role for alternative asset classes (e.g. real estate, private equity, derivatives, etc.) than for stocks and bonds.

As Garlappi et al. (2007), Glasserman and Xu (2013) and Liu (2011) point out, making robust investment decisions on the asset side ensures that the investment decisions will provide a better out-of-sample Sharpe-ratio on average not only if the pension fund manager knows the exact distribution of asset returns, but also if the model of asset returns that the pension fund manager had in mind turns out to be misspecified. That is, using robust investment decisions helps decrease the investment risk of pension funds.

The main reasons why it is wise for pension funds to make robust investment decisions is the difficulty of obtaining reliable estimates for the risk premiums (the best-known example of which in practice is the equity risk premium), for long term interest rates and for correlations (especially for large portfolios). Maenhout (2004) solves the robust version of the dynamic asset allocation problem of Merton (1971) using the penalty approach: the investor maximizes her expected utility from consumption plus a penalty term, her utility function is of CRRA type, and the financial market consists of a money market account (MMA) with constant risk-free rate and a stock market index. The investment horizon is finite. The investor is uncertain about the expected excess return of the stock market index. The penalty term is quadratic in the difference between the drift term according to the base model and the drift term according to the alternative model. Instead of multiplying the penalty term by a constant (as Anderson et al. (2003) did), Maenhout (2004) assumes that the uncertainty-aversion parameter is stochastic, concretely it is linear in the inverse of the value function itself.[16] This particular form of the penalty term makes it possible to obtain a closed form solution for the optimal consumption and investment policy. Moreover, the optimal investment policy is homothetic, i.e., the optimal ratio of wealth to be invested in the stock market index is independent of the wealth itself. Denoting the relative risk-aversion parameter by γ and the uncertainty-aversion parameter by θ, the optimal investment ratio is the same as in the problem of Merton (1971), the only difference being that instead of γ there is $\gamma + \theta$ in the denominator, i.e., $\frac{1}{\gamma+\theta}\frac{\mu_S - r_f}{\sigma_S^2}$. Thus what Maenhout (2004) finds, effectively, is that a robust investor has a lower portion of her wealth invested in the risky asset than a non-robust investor. Or, as sometimes stated in the literature: a robust investor is more conservative in her investment decision.

In another paper Maenhout (2006) analyzes a similar problem, but he assumes that the investment opportunity set is stochastic. To be more precise, the expected excess return of the stock market index follows a mean-reverting Ornstein-Uhlenbeck process. Robustness again decreases the optimal ratio of wealth to be invested in the stock market index, but it increases the intertemporal hedging demand. Thus robustness leads to more conservative decision in two aspects: on one hand the investor invests less in the risky asset by decreasing the myopic (speculative) demand, on the other hand she invests more in the risky asset by increasing the intertemporal hedging demand. The total effect of robustness is thus not straightforward. It can happen, for example, that a robust and a non-robust investor have the same optimal investment ratio, but their motives are different: a non-robust investor lays more emphasis on the speculative nature of the stock market index than the robust investor, while the robust investor lays more emphasis on its hedging nature than the

[16] This parameterization has been criticized by some researchers due to its recursive nature (e.g. Pathak (2002)).

non-robust investor.

Flor and Larsen (2014) find that it is more important to take uncertainty about stock dynamics into account than uncertainty about long-term bond dynamics. They find that the higher the Sharpe ratio of an asset, the more important the role of uncertainty about the price of that asset. Since historically the stock market has a slightly higher Sharpe ratio than the bond market, uncertainty about stock dynamics plays a more important role than uncertainty about bond dynamics.

Vardas and Xepapadeas (2015) show that robustness does not necessarily induce more conservative investment behavior: if there are two risky assets and a risk-free asset (with constant risk-free rate) in the market and the investor is uncertain about the price processes of the risky assets, it might be the case that the total holding of risky assets is higher than in the case of no uncertainty. Moreover the authors find that if the levels of uncertainty about the price processes of the two risky assets are different, then the investor will decrease her investment in the asset about the price process of which she is more uncertain and she will increase her investment in the asset about the price process of which she is less uncertain. If one of the risky assets represents home equity and the other represents foreign equity, and the investor is more uncertain about the foreign assets, this finding provides an explanation for the home-bias.

Uppal and Wang (2003) also assume a financial market with several risky assets. The investor is uncertain not only about the joint distribution of the asset returns, but she is also uncertain – to various degrees – about the marginal distribution of any subset of the asset returns. The authors find that under specific circumstances[17] the optimal portfolio is significantly underdiversified compared to the optimal mean-variance portfolio.

1.3.2 Robust liability management and ALM

Introducing robustness into the liability side is less straightforward. If the liability side is given as a one-dimensional stochastic process (in practice this usually means a one-dimensional geometric Brownian Motion), one can add a perturbation term just like one did on the asset side. More sophisticated models allow several state variables to influence the liability side, some of which

[17] Concretely: if the uncertainty about the joint return distribution is high and the level of uncertainty about the marginal return distributions is different from each other – even if these differences are small in magnitude.

might influence the asset side as well. Such state variables often used by pension funds include interest rates, wage growth and inflation.

Since inflation can also influence the asset side by, e.g., holding inflation-indexed bonds or by influencing the discount rate used to value bonds, its robust treatment requires a joint ALM framework. The most important papers on robustness about inflation are Ulrich (2013) and Munk and Rubtsov (2014). Ulrich (2013) uses empirical data from the 1970s to the 2010s and concludes that the term premium of U.S. government bonds can be explained by a model with a representative investor with log-utility and uncertainty about the inflation process. Horvath et al. (2016) find similar results (using a two-factor Vašiček-model, without specifying inflation as a factor): if the investment horizon is assumed to be 30 years, they find that a relative risk-aversion parameter of 1.73 is needed to explain the term premium (the log-utility case corresponds to a relative risk aversion of 1), assuming model uncertainty. Without model uncertainty a relative risk-aversion of 6.54 is needed to explain market data.

Munk and Rubtsov (2014) also solve a robust dynamic investment problem with stochastic investment opportunity set. But contrary to Maenhout (2006), the stochasticity of the investment opportunity set comes from the short rate being stochastic, and the financial market also includes a long-term nominal bond. Inflation is also explicitly included in the model, and the investor is uncertain not only about the drift of the financial assets (a stock and a long-term nominal bond), but also about the drift of the inflation process. The optimal portfolio weight for both the stock and the bond is the sum of one speculative and three hedging components. The latter three components hedge against adverse changes in the realized inflation, the short rate and the expected inflation. Contrary to Maenhout (2004) and Maenhout (2006), in the optimal solution of the investment problem the uncertainty-aversion parameter is not simply added to the risk-aversion parameter, but it is multiplied by several combinations of the correlation between the inflation process and asset price processes. Intuitively this means that there is a spill-over effect: uncertainty about the inflation process induces uncertainty about the asset price processes. Both the variable nature of the investment opportunity set and the correlation of inflation with asset prices lead to the total effect of uncertainty being not straightforward: whether it increases or decreases the holding of a particular asset depends on which component of the demand (myopic component and three hedging components) of that asset is influenced by a higher degree by uncertainty.

Wage growth can be treated separately as a state variable (where, if inflation is included, wage should be measured in real terms), and the pension fund managers robustness with respect to this state variable can be expressed by adding an additional penalty term. This is done by Shen (2014). If pensions are indexed to the wage level and/or inflation, higher wage growths and higher inflation leads to higher liabilities. At the same time – assuming that the contribution rate is not changed – the value of the asset side will increase as well. If the pension fund manager makes robust investment decisions and she is ambiguous about the model describing inflation and wage growth, she will effectively base her decision on higher or lower drifts of the wage growth and inflation processes. Whether robustness means higher or lower drifts, depends – among others – on the specification of the financial market (i.e. on other state variables) and on the funding ratio of the pension fund.

Once both the asset and liability sides are described as stochastic processes (which can be functions of several underlying stochastic processes), the objective function can be formulated. The objective function is an expectation of two terms: the utility function and a penalty term. Choosing the exact form of the utility function is a core step in robust optimization for the pension fund, since it determines what exactly the pension fund wants to hedge against. A simple approach, which is used by Shen, Pelsser, and Schotman (2014), is to take the utility function as $-\left[L_T - A_T\right]^+$, where L_T is the value of liabilities at time T and A_T is the value of assets at time T. Then the pension fund managers goal is to make decisions regarding the state variables (e.g. investment policy, contribution rate, etc.) such that the value function (which contains the above utility function and a penalty term) is maximized (i.e. the manager hedges against the shortfall risk as much as possible) but under the worst case scenario. In mathematical terms this means solving the following optimization problem:

$$\min_{\mathbb{U}} \max_{\Theta} \mathrm{E}^{\mathbb{U}} \Big\{ -\left[L_T - A_T\right]^+ + \int_0^T \Upsilon_s \frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_s\right]}{\partial s} \mathrm{d}s \Big\}, \tag{1.14}$$

where $\mathbb{B}$ is the base measure, $\mathbb{U}$ is the alternative measure, Υ_s is the (deterministic and time-dependent) uncertainty-aversion parameter and Θ is the set of decision variables (which can be stochastic). The pension fund manager solves the above optimization problem such that the budget constraint holds. The authors find that a robust pension-fund manager follows a more conservative hedging policy. But the effect of robustness heavily depends on the instantaneous funding ratio, precisely: robustness has a significant effect on the hedging policy only if the instantaneous funding ratio is low. Intuitively: if the pension fund is strong enough to hedge against future malevolent events, the robust and the non-robust hedging strategies will be practically identical. This follows from the particular form of the goal function: according to (1.14), the pension fund manager's goal is to avoid being underfunded, but once there is no significant threat of becoming underfunded (i.e., the funding ratio is high), she does not have any other objectives based on which to optimize. This is reflected in the kink in the goal function. Moreover: the robust hedging policy differs from the non-robust hedging policy only if the drift terms of the state variables are overestimated.[18]

1.4 Conclusion

Accounting for uncertainty is of crucial importance for proper asset-liability management of pension funds. As we demonstrated in Section 1.3, robust investment decisions outperform non-robust investment decisions in terms of both expected utility and the Sharpe ratio. The difference in performance between robust and non-robust portfolios is both statistically and economically significant.

Pension funds can use several approaches to make robust investment decisions. The ones most commonly used are the penalty approach, the constraint approach, the Bayesian approach and the smooth recursive preferences approach. These approaches differ from each other in the assumptions they use and in how they formulate the robust optimization problem. Once this optimization problem is formulated, one can either use the principle of dynamic programming or the martingale method to obtain the optimal investment policy.

Although the vast majority of the robustness literature focuses on the implications of uncertainty on asset management, for the prudent functioning of pension funds it is at least as important to properly account for uncertainty regarding the liability side. As we demonstrated in Section 1.3.2, there are several factors that influence both the asset and the liability side of pension funds (the most important of which are wage growth and inflation), and accounting for ambiguity about these

[18] According to the model, the pension fund manager is only uncertain about the drift of the state variables.

factors is the basis for robust asset-liability management.

Chapter 2

Robust Pricing of Fixed Income Securities

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2.1 Introduction

Similarly to the equity premium puzzle, there exists a bond premium puzzle. The risk premium on long-term bonds is higher than predicted by mainstream models using reasonable parameter values. This phenomenon was first described in Backus, Gregory, and Zin (1989), who show that in bond markets "*... the representative agent model with additively separable preferences fails to account for the sign or the magnitude of risk premiums*". Although the bond premium puzzle has received much less attention in the literature than the equity premium puzzle, it is by no means of less importance. As Rudebusch and Swanson (2008) remark, the value of outstanding long-term bonds in the U.S. is much larger than the value of outstanding equity. The present literature about equilibrium bond pricing estimates an unreasonably high relative risk aversion parameter. For instance Piazzesi and Schneider (2007) estimate the relative risk-aversion parameter to be 57; van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez (2012) estimate a value of relative risk-aversion in their model around 80; while Rudebusch and Swanson (2012) estimate it to be 110. Although several potential explanations can be found in the literature, none of the provided solutions are generally accepted – just as in the case of the equity premium puzzle.

In this paper, we approach the bond premium puzzle from a new angle. A key parameter in any investment allocation model is the risk premium earned on investing in bonds or, equivalently, the prices of risk in a factor model. However, given a limited history of data, the investor faces substantial uncertainty about the magnitude of these risk premiums. We build on the literature on robust decision making and asset pricing and formulate a dynamic investment problem under robustness in a market for bonds and stocks. Our model features stochastic interest rates driven by a two-factor Gaussian affine term structure model. The investor chooses optimal portfolios of bonds and stocks taking into account the uncertainty about the bond and stock risk premiums. We solve the representative investor's optimization problem and give an explicit solution to the optimal terminal wealth, the least-favorable physical probability measure, and the optimal investment policy.

We calibrate the risk aversion and robustness parameters by equating the optimal portfolio weights implied by the model to weights observed in actual aggregate portfolio holdings. This is different from the existing literature, which typically calibrates first-order conditions of a consumption-based asset pricing model to the observed expected returns. Given the very low volatility of consumption, and the low correlation of stock and bond returns with consumption growth, that approach requires high levels of risk aversion to fit the observed risk premiums. Our approach only uses the optimal asset demands, which do not directly involve the volatility of or correlations with consumption; only the volatilities of the returns and intertemporal hedging demands for the assets are required. We then use the concept of detection error probabilities to disentangle the risk- and uncertainty-aversion parameters.

We estimate the parameters of our model using 42 years of U.S. market data by Maximum Likelihood. We find that matching the optimal equity demand to market data gives reasonable values for the risk-aversion parameter. If we assume that the financial market consists of only a stock market index and a money market account, the calibrated risk-aversion parameter in the non-robust version of our terminal wealth utility model is reasonable, 1.92 (see Table 2.5). For the bond demand the situation is very different: reasonable values of the risk-aversion parameter imply far too low bond risk premiums. The reason for this are the high bond risk premiums relative to the bond price volatilities. Only with robustness we can find reasonable values for the risk-aversion parameter to match observed demands. When calibrating the non-robust version of our model to

U.S. market data, we need a value between 7.9 and 69.1 (depending on the investment horizon of the representative investor) for the risk-aversion parameter to explain the market data. After accounting for robustness, these values decrease to 2.9 and 25.5, respectively (see Table 2.6). Thus, our model can to a large extent resolve the bond premium puzzle.

Apart from our results on the bond premium puzzle, another contribution of our paper is of a more technical nature. We develop a novel method to solve the robust portfolio problem, the robust martingale method. We show in Theorem 1 that the robust dynamic investment problem can be interpreted as a non-robust dynamic investment problem with so-called least-favorable risk premiums, which are time-dependent but deterministic. This means that, based on our Theorem 1, we can formulate the objective function of the investor at time zero *non-recursively.* The importance of this contribution is stressed by Maenhout (2004), who writes: "*Using the value function V itself to scale θ may make it difficult to formulate a time-zero problem, as V is only known once the problem is solved.*" In Theorem 1 we prove that the time-zero robust dynamic investment problem can be formulated ex ante, before solving for the value function itself, and we also provide this alternative (but equivalent) formulation of the investment problem in closed form. Since without this non-recursive formulation the problem can only be solved recursively, the literature so far had to rely exclusively on the Hamilton-Jacobi-Bellman differential equation to solve robust dynamic investment problems. Our alternative formulation of the problem makes it possible to apply an alternative technique to solve robust dynamic investment problems, namely a robust version of the martingale method. This method is likely also applicable in other settings.

Our paper relates to the literature on the bond premium puzzle. Backus et al. (1989) use a consumption-based endowment economy to study the behavior of risk premiums. They conclude that in order for their model to match the risk premium observed in market data, the coefficient of relative risk aversion must be at least around 8-10. This value for the relative risk-aversion parameter is considered too high by the majority of the literature to reconcile with both economic intuition and economic experiments.[1] Further early discussion on the bond premium puzzle can

[1]The majority of the literature considers relative risk aversion parameter values lower than five reasonable. Several studies attempt to estimate the relative risk aversion, usually by using consumption data or by conducting experiments. Friend and Blume (1975) estimate the relative risk aversion parameter to be around 2; Weber (1975) and Szpiro (1986) estimate it to be between about 1.3 and 1.8; the estimates of Hansen and Singleton (1982) and Hansen and Singleton (1983) are 0.68–0.97 and 0.26–2.7, respectively; using nondurable consumption data, Mankiw (1985) estimates the relative risk aversion to be 2.44–5.26, and using durable goods consumption data it to be 1.79–3.21; Barsky, Juster, Kimball, and Shapiro (1997) use an experimental survey to estimate the relative risk aversion

be found in Donaldson, Johnsen, and Mehra (1990) and Den Haan (1995). They demonstrate that the bond premium puzzle is not a peculiarity of the consumption-based endowment economy, but it is also present in real business-cycle models. This remains true even if one allows for variable labor and capital or for nominal rigidities. Rudebusch and Swanson (2008) examine whether the bond premium puzzle is still present if they use a more sophisticated macroeconomic model instead of either the consumption-based endowment economy or the real business-cycle model. They use several DSGE setups and find that the bond premium puzzle remains even if they extend their model to incorporate large and persistent habits and real wage bargaining rigidities. However, Wachter (2006) provides a resolution to the puzzle by incorporating habit-formation into an endowment economy. Piazzesi, Schneider et al. (2006) use Epstein-Zin preferences, but to match market data, they still need a relative risk aversion of 59.

Our paper obviously relates to the literature on robust dynamic asset allocation. Investors are uncertain about the parameters of the distributions that describe returns. In robust decision making, the investor makes decisions that "*not only work well when the underlying model for the state variables holds exactly, but also perform reasonably well if there is some form of model misspecification*" (Maenhout (2004)). We use the minimax approach to robust decision making. A comparison of the minimax approach with other approaches, such as the recursive smooth ambiguity preferences approach, can be found in Peijnenburg (2014). The minimax approach assumes that the investor considers a set of possible investment paths regarding the parameters she is uncertain about. She chooses the worst case scenario, and then she makes her investment decision using this worst case scenario to maximize her value function. To determine the set of possible parameters we use the penalty approach. This means that we do not set an explicit constraint on the parameters about which the investor is uncertain, but we introduce a *penalty term* for these parameters. Deviations of the parameters from a so-called base model are penalized by this function. Then the investor solves her unconstrained optimization problem using this new goal function. The penalty approach was introduced into the literature first by Anderson et al. (2003), and it was applied by Maenhout (2004) and Maenhout (2006) to analyze equilibrium equity prices. Maenhout (2004) finds that in the case of a constant investment opportunity set robustness increases the equilibrium equity

parameter of the subjects, the mean of which turns out to be 4.17; while in the study of Halek and Eisenhauer (2001) the mean relative risk aversion is 3.7.

premium[2] and it decreases the risk-free rate. Concretely, a robust Duffie-Epstein-Zin representative investor with reasonable risk-aversion and uncertainty-aversion parameters generate a 4% to 6% equity premium. Furthermore, Maenhout (2006) finds that, if the investment opportunity set is stochastic, robustness increases the importance of intertemporal hedging compared to the non-robust case. We confirm this result in our setting (see Corollary 1). While our paper sheds light on the importance of parameter uncertainty for asset prices, several papers analyzed the effects of parameter uncertainty on asset allocation. Branger, Larsen, and Munk (2013) solve a stock-cash allocation problem with a constant risk-free rate and uncertainty aversion. The model of Flor and Larsen (2014) features a stock-bond-cash allocation problem with stochastic interest rates and ambiguity, while Munk and Rubtsov (2014) also account for inflation ambiguity. Feldhütter, Larsen, Munk, and Trolle (2012) investigate the importance of parameter uncertainty for bond investors empirically.

The paper is organized as follows. Section 2.2 introduces our model, i.e., the financial market and the robust dynamic optimization problem. Section 2.2 also provides the solution to the robust investment problem, using the martingale method. In Section 2.3 we calibrate our model to our data. In Section 2.4 we solve for the equilibrium prices and in Section 2.5 we disentangle the risk aversion from the uncertainty aversion using detection error probabilities. Section 2.6 concludes.

2.2 Robust Investment Problem

We consider agents that have access to an arbitrage-free complete financial market consisting of a money market account, constant maturity bond funds, and a stock market index. The short rate r_t is assumed to be affine in an N-dimensional factor $\boldsymbol{F}_t$, i.e.,

$$r_t = A_0 + \boldsymbol{\iota}'\boldsymbol{F}_t, \tag{2.1}$$

[2] The intuition is as follows. Assuming a constant investment opportunity set, uncertainty *decreases* the optimal weight of the risky asset, i.e., uncertainty leads to a *more conservative* investment decision. If the market is in general equilibrium, due to market clearing the demand of the risky asset is equal to the supply, regardless of whether the investor is robust or not. In other words: in general equilibrium a robust and a non-robust investor holds the same amount of the risky asset, concretely, the amount equal to the supply. The uncertainty-averse investor will hold the same amount of the risky asset as the non-uncertainty-averse investor only if she is compensated for it, i.e., if the equity premium is higher than it were for a non-uncertainty-averse investor. Hence, uncertainty leads to a *more conservative* investment decision, and in general equilibrium it leads to a *higher* equity premium.

where $\boldsymbol{\iota}$ denotes a column vector of ones. The factors $\boldsymbol{F}_t$ follow an N-dimensional Ornstein-Uhlenbeck process, i.e.,

$$\mathrm{d}\boldsymbol{F}_t = -\boldsymbol{\kappa}(\boldsymbol{F}_t - \boldsymbol{\mu}_F)\mathrm{d}t + \boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}^{\mathbb{Q}}_{F,t}. \tag{2.2}$$

Here $\boldsymbol{\mu}_F$ is an N-dimensional column vector of long-term averages, $\boldsymbol{\kappa}$ is an $N \times N$ diagonal mean-reversion matrix, $\boldsymbol{\sigma}_F$ is an $N \times N$ lower triangular matrix with strictly positive elements in its diagonal, and $\boldsymbol{W}^{\mathbb{Q}}_{F,t}$ is an N-dimensional column vector of independent standard Wiener processes under the risk-neutral measure $\mathbb{Q}$. The value of the available stock market index is denoted by S_t and satisfies

$$\mathrm{d}S_t = S_t r_t \mathrm{d}t + S_t \left(\boldsymbol{\sigma}'_{FS} \mathrm{d}\boldsymbol{W}^{\mathbb{Q}}_{F,t} + \sigma_{N+1} \mathrm{d}W^{\mathbb{Q}}_{N+1,t} \right), \tag{2.3}$$

where σ_{N+1} is strictly positive, $\boldsymbol{\sigma}_{FS}$ is an N-dimensional column vector governing the covariance between stock and bond returns, and $W^{\mathbb{Q}}_{N+1,t}$ is a standard Wiener process (still under the risk-neutral measure $\mathbb{Q}$) that is independent of $\boldsymbol{W}^{\mathbb{Q}}_{F,t}$. As our financial market is arbitrage-free and complete, such a risk-neutral measure $\mathbb{Q}$ indeed exists and is unique.

Although we will study the effect of ambiguity on investment decisions and equilibrium prices below, it is important to note that, due to the market completeness, the risk-neutral measure $\mathbb{Q}$ is unique and agents cannot be ambiguous about it. Indeed, the risk-neutral measure $\mathbb{Q}$ is uniquely determined by market prices and, thus, if all investors accept that there is no arbitrage opportunity on the market and they observe the same market prices, then they all have to agree on the risk-neutral measure $\mathbb{Q}$ as well. Investors will be ambiguous in our model about the physical probability measure or, equivalently, about the prices of risk of the Wiener processes. We denote $\boldsymbol{W}^{\mathbb{Q}}_{F,t}$ and $W^{\mathbb{Q}}_{N+1,t}$ jointly as

$$\boldsymbol{W}^{\mathbb{Q}}_t = \begin{bmatrix} \boldsymbol{W}^{\mathbb{Q}}_{F,t} \\ W^{\mathbb{Q}}_{N+1,t} \end{bmatrix}. \tag{2.4}$$

Now consider an investor with investment horizon T. She derives utility from terminal wealth. This investor is ambiguous about the physical probability measure. She has a physical probability measure $\mathbb{B}$ in mind which she considers the most probable, but she is uncertain about whether this is the true physical probability measure or not. This measure $\mathbb{B}$ is called the *base measure*. As the investor is not certain that the measure $\mathbb{B}$ is the true physical probability measure, she considers other possible physical probability measures as well. These measures are called *alternative (physical) measures* and denoted by $\mathbb{U}$. We formalize the relationship between $\mathbb{Q}$, $\mathbb{B}$, and $\mathbb{U}$ as

$$\mathrm{d}\boldsymbol{W}_t^{\mathbb{B}} = \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} - \boldsymbol{\lambda}\mathrm{d}t, \tag{2.5}$$

$$\mathrm{d}\boldsymbol{W}_t^{\mathbb{U}} = \mathrm{d}\boldsymbol{W}_t^{\mathbb{B}} - \boldsymbol{u}(t)\mathrm{d}t, \tag{2.6}$$

where $\boldsymbol{W}_t^{\mathbb{B}}$ and $\boldsymbol{W}_t^{\mathbb{U}}$ are $(N+1)$-dimensional standard Wiener processes under the measures $\mathbb{B}$ and $\mathbb{U}$, respectively. Thus, $\boldsymbol{\lambda}$ can be identified as the $(N+1)$-dimensional vector of prices of risk of the base measure $\mathbb{B}$, while $\boldsymbol{u}(t)$ denotes the $(N+1)$-dimensional vector of prices of risk of $\mathbb{U}$.[3] It is important to emphasize that the investor assumes $\boldsymbol{u}(t)$ to be a deterministic function of time, i.e., $\boldsymbol{u}(t)$ is assumed to be non-stochastic.[4]

We can now formalize the investor's optimization problem, given a CRRA utility function with risk aversion $\gamma > 1$,[5] time-preference parameter $\delta > 0$, and a stochastic and non-negative parameter Υ_t, which expresses the investor's attitude towards uncertainty, and which we will describe in more details later in this section.

Problem 6. *Given initial wealth x, find an optimal pair $(X_T, \mathbb{U})$ for the robust utility maximization problem*

$$V_0(x) = \inf_{\mathbb{U}} \sup_{X_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma} + \int_0^T \Upsilon_s \exp(-\delta s) \frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_s\right]}{\partial s}\mathrm{d}s \right\}, \tag{2.7}$$

subject to the budget constraint

$$\mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right) X_T\right] = x. \tag{2.8}$$

The investor's optimization problem as it is formulated here follows the so-called martingale

[3] In order for $\mathbb{U}$ to be well defined, we assume throughout $\int_0^T \|\boldsymbol{u}(s)\|^2\mathrm{d}s < \infty$.

[4] A natural question here is why we allow $\boldsymbol{u}$ to be a deterministic function of time, but assume $\boldsymbol{\lambda}$ to be constant. Allowing $\boldsymbol{\lambda}$ to be a deterministic function of time would not change our conclusions, but it would result in more complicated expressions due to time-integrals involving $\boldsymbol{\lambda}(t)$. Moreover, we could not calibrate our model to market data without assuming some functional form for $\boldsymbol{\lambda}(t)$. Thus since for our purposes a constant $\boldsymbol{\lambda}$ suffices and it allows straightforward model calibration, we throughout take $\boldsymbol{\lambda}$ to be constant.

[5] In the case of $\gamma = 1$ the investor has log-utility. All of our results can be shown to hold in this case as well.

method. Given that our financial market is complete, the martingale method maximization is over terminal wealth X_T only. It is not necessary, mathematically, to consider optimization of the portfolio strategy as the optimal strategy will simply be that one that achieves the optimal terminal wealth X_T. For (mathematical) details we refer to Karatzas and Shreve (1998).

The outer inf in Problem 6 adds robustness to the investment problem as the investor considers the worst case scenario, i.e., she chooses the measure $\mathbb{U}$ which minimizes the value function (evaluated at the optimal terminal wealth). The investor considers all alternative probability measures $\mathbb{U}$ which are equivalent[6] to the base measure $\mathbb{B}$.

The first part of the expression in brackets in (2.7) expresses that the investor cares about her discounted power utility from terminal wealth X_T. The second term represents a penalty: if the investor calculates her value function using a measure $\mathbb{U}$ which is very different from $\mathbb{B}$, then the penalty term will be high. We will be more explicit about what we mean by two probability measures being *very different* from each other in the next paragraph. The fact that the investor considers a worst-case scenario, including the penalty term, ensures that she considers "pessimistic" probability measures (which result in low expected utility), but at the same time she only considers "reasonable" probability measures (that are not too different from the base measure).

Following Anderson et al. (2003), we quantify how different probability measures are by their *Kullback-Leibler* divergence, which is also known as the relative entropy. The reason why we use the Kullback-Leibler divergence as the penalty function lies not only in its intuitive interpretation (see, e.g., Cover and Thomas (2006), Chapter 2), but also in its mathematical tractability.

We now rewrite Problem 6 and, following Maenhout (2004), introduce a concrete specification for Υ_t. In view of (2.6) and Girsanov's theorem, we obtain

$$\frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_t\right]}{\partial t} = \frac{\partial}{\partial t}\mathrm{E}^{\mathbb{U}}\left[\frac{1}{2}\int_0^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s - \int_0^t \boldsymbol{u}(s)\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}\right] \tag{2.9}$$

$$= \frac{1}{2}\|\boldsymbol{u}(t)\|^2, \tag{2.10}$$

where $\|\boldsymbol{u}(t)\|$ denotes the Euclidean norm of $\boldsymbol{u}(t)$. Furthermore, in order to ensure homotheticity

[6] Two probability measures are said to be equivalent if and only if each is absolutely continuous with respect to the other. That is, the investor may be uncertain about the exact probability of events, but she is certain about which events happen for sure (i.e., with probability 1) or with probability zero. This is a common, sometimes implicit, assumption in this literature.

of the investment rule,[7] we use the following specification of Υ_t, introduced in Maenhout (2004).[8]

$$\Upsilon_t = \exp(\delta t)\,\frac{1-\gamma}{\theta} V_t(X_t), \tag{2.11}$$

where X_t denotes optimal wealth at time t. Substituting (2.10) and (2.11) into (2.7), the value function becomes

$$V_0(x) = \inf_{\mathbb{U}} \sup_{X_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma} + \int_0^T \frac{(1-\gamma)\,\|\boldsymbol{u}(t)\|^2}{2\theta} V_t(X_t)\mathrm{d}t \right\}. \tag{2.12}$$

This expression of the value function is recursive, in the sense that the right-hand side contains future values of the same value function. The following theorem gives a non-recursive expression. All proofs are in the appendix.

Theorem 1. *The solution to (2.12) with initial wealth x is given by*

$$V_0(x) = \inf_{\mathbb{U}} \sup_{X_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t\right) \frac{X_T^{1-\gamma}}{1-\gamma} \right\}. \tag{2.13}$$

subject to the budget constraint

$$\mathrm{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s \mathrm{d}s\right) X_T\right] = x. \tag{2.14}$$

Theorem 1 gives an alternative interpretation to the robust investment Problem 6, with parameterization (2.11). Effectively, the investor maximizes her expected discounted utility of terminal wealth, under the least-favorable physical measure $\mathbb{U}$, using an adapted subjective discount factor

$$\delta - \frac{1-\gamma}{2\theta}\frac{1}{T}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t. \tag{2.15}$$

As $\theta > 0$, we obtain, for $\gamma > 1$, that the subjective discount rate increases in the time-average

[7] Homotheticity of the investment rule means that the optimal ratio of wealth to be invested in a particular asset at time t does not depend on the wealth at time t itself.

[8] This specification has been criticized in, e.g., Pathak (2002) for its recursive nature. Alternatively we could have specified the robust investment problem directly as in Theorem 1.

of $\|\boldsymbol{u}(t)\|^2$, i.e., in deviations of the least-favorable physical measure $\mathbb{U}$ from the base measure $\mathbb{B}$. The investor thus becomes effectively more impatient. However, this is not the reason why the robustness affects the asset allocation, as the subjective time preference does not affect the asset allocation in this standard terminal wealth problem. Rather, the effect of robustness is that the prices of risk that the robust investor uses are affected by the least-favorable measure $\mathbb{U}$ in the value function in (2.13). We show this formally in the following section.

2.2.1 Optimal terminal wealth

We now solve Problem 6 using the reformulation in Theorem 1. In the present literature, dynamic robust investment problems are mostly solved by making use of a "twisted" Hamilton-Jacobi-Bellman (HJB) differential equation. Solving this HJB differential equation determines both the optimal investment allocation and the optimal final wealth. However, we propose to use the so-called martingale method. This approach has not only mathematical advantages (one does not have to solve higher-order partial differential equations), but it also provides economic intuition and insights into the decision-making of the investor. We provide this intuition at the end of this section, directly after Theorem 2. The martingale method was developed by Cox and Huang (1989) for complete markets, and a detailed description can be found in Karatzas and Shreve (1998). The basic idea is to first determine the optimal terminal wealth X_T (Theorem 2) and to subsequently determine the asset allocation that the investor has to choose in order to achieve that optimal terminal wealth (Corollary 1).

In our setting of robust portfolio choice, we can still follow this logic. It is important to note that the budget constraint (2.8) is, obviously, not subject to uncertainty, i.e., $\mathbb{Q}$ is given. The value function in Theorem 1 contains an inner (concerning X_T) and an outer (concerning $\mathbb{U}$) optimization. Solving these in turn leads to the following result, whose proof is again in the appendix.

Theorem 2. *The solution to the robust investment Problem 6 under (2.11) is given by*

$$\hat{X}_T = x \frac{\exp\left(\frac{1}{\gamma}\int_0^T (\hat{\boldsymbol{u}}(t)+\boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} + \frac{1}{\gamma}\int_0^T r_t \mathrm{d}t\right)}{\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1}{\gamma}\int_0^T (\hat{\boldsymbol{u}}(t)+\boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} + \frac{1-\gamma}{\gamma}\int_0^T r_t \mathrm{d}t\right)}, \tag{2.16}$$

with the least-favorable distortions

$$\hat{\boldsymbol{u}}(t)_F = -\frac{\theta}{\gamma+\theta}\left[\boldsymbol{\lambda}_F + \boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota}\right], \tag{2.17}$$

$$\hat{\boldsymbol{u}}(t)_{N+1} = -\frac{\theta}{\gamma+\theta}\boldsymbol{\lambda}_{N+1}, \tag{2.18}$$

where $\boldsymbol{B}(\cdot)$ is defined in (2.54).

Equation (2.16) shows the stochastic nature of the optimal terminal wealth. The denominator is a scaling factor, and the numerator can be interpreted as the exponential of a (stochastic) yield on the investment horizon T. The investor achieves this yield on her initial wealth x if she invests optimally throughout her life-cycle.

The absolute value of the least-favorable distortions (2.17) and (2.18) increase as θ increases. This is in line with the intuition that a more uncertainty-averse investor considers alternative measures that are "more different" from the base measure. If the investor is not uncertainty-averse, i.e., $\theta = 0$, the least-favorable distortions are all zero. This means that the investor considers only the base measure and she makes her investment decision based on that measure. On the other hand, if the investor is infinitely uncertainty-averse, i.e., $\theta = \infty$, the least-favorable distortions are

$$\tilde{\boldsymbol{u}}(t)_F = -\left[\boldsymbol{\lambda}_F + \boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota}\right], \tag{2.19}$$

$$\tilde{\boldsymbol{u}}(t)_{N+1} = -\boldsymbol{\lambda}_{N+1}. \tag{2.20}$$

An infinitely uncertainty-averse investor thus uses $-\boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota}$ as the market price of risk induced by $\boldsymbol{W}_{F,t}^{\mathbb{U}}$ and 0 as the market price of risk induced by $W_{N+1,t}^{\mathbb{U}}$.

2.2.2 Optimal portfolio strategy

The final step in our theoretical analysis is to derive the investment strategy that leads to the optimal final wealth $\hat{X}_T$ derived in Theorem 2. The following theorem gives the optimal exposures to the driving Brownian motion $\boldsymbol{W}_t^{\mathbb{Q}}$. The proof can be found in the appendix.

Theorem 3. *Under the conditions of Theorem 2, the optimal final wealth $\hat{X}_T$ is achievable using* *the wealth evolution*

$$\mathrm{d}\hat{X}_t = \ldots \mathrm{d}t + \left[\frac{\boldsymbol{\lambda}_F'}{\gamma+\theta} + \frac{1-(\gamma+\theta)}{\gamma+\theta}\boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F;\ \frac{\lambda_{N+1}}{\gamma+\theta}\right]\hat{X}_t\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}, \tag{2.21}$$

starting from $\hat{X}_0 = x$.

Introducing the notation

$$\boldsymbol{\mathcal{B}}(\tau) = \left[\boldsymbol{B}(\tau_1)\boldsymbol{\iota}; \ldots; \boldsymbol{B}(\tau_N)\boldsymbol{\iota}\right], \tag{2.22}$$

where τ_j denotes the maturity of bond fund j, we can state the following corollary (proved in the appendix).

Corollary 1. *Under the conditions of Theorem 2, the optimal investment is a continuous rebalancing strategy where the fraction of wealth invested in the constant maturity bond funds is*

$$\begin{aligned}\hat{\boldsymbol{\pi}}_{B,t} = &-\frac{1}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\left(\boldsymbol{\sigma}_F'\right)^{-1}\left(\boldsymbol{\lambda}_F - \frac{\lambda_{N+1}}{\sigma_{N+1}}\boldsymbol{\sigma}_{FS}\right)\\ &-\frac{1-\gamma-\theta}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\boldsymbol{B}(T-t)\end{aligned} \tag{2.23}$$

and the fraction of wealth invested in the stock market index is

$$\hat{\pi}_{S,t} = \frac{\lambda_{N+1}}{(\gamma+\theta)\,\sigma_{N+1}}. \tag{2.24}$$

Equations (2.23) and (2.24) provide closed-form solutions for the optimal fractions of wealth to be invested in the bond and stock markets. For the latter one, this fraction is time independent and it is equal to the market price of the idiosyncratic risk of the stock market (i.e., λ_{N+1}) divided by $(\gamma+\theta)$ and by the volatility of the unspanned stock market risk σ_{N+1}. So even though the return on the stock market is influenced by all of the $N+1$ sources of risk, only the stock market specific risk matters when the investor decides how much to invest in the stock market. This investment policy closely resembles the solution to Merton's problem (Merton (1969)), the main difference being that γ is replaced by $\gamma+\theta$ in the denominator.

The optimal fraction to be invested in the bonds has two components (similarly to Merton's intertemporal consumption model (Merton (1973)). The first component is the myopic demand, and its form is similar to that of (2.24). The second component represents the intertemporal hedging demand, and it is present due to the stochastic nature of the investment opportunity set: the investor holds this component in order to protect herself against unfavorable changes of the N factors. The main difference to the solution to Merton's problem is again γ being replaced by $\gamma+\theta$. This means that if we compare a robust and a non-robust investor's optimal investment policy (with the same level of relative risk-aversion), the only difference between them is that in the robust investor's case the risk-aversion parameter γ is replaced by the sum of the risk-aversion and the uncertainty aversion parameters, $\gamma+\theta$. In (2.23) we find that robustness increases the intertemporal hedging demand of the investor. This finding is in accordance with Maenhout (2006).

2.3 Model calibration

In this section we calibrate our model to market data. We use weekly observations from 5 January 1973 to 29 January 2016. We use continuously compounded zero-coupon yields with maturities of 3 months, 1 year, 5 years, and 10 years. The zero-coupon yields for maturities of 1 year, 5 years, and 10 years were obtained from the US Federal Reserve Data Releases, while the yields for maturity of 3 months are the 3-month T-bill secondary market rates from the St. Louis Fed Fred Economic Data.[9] The spot rates of 1 and 2 months of maturity sometimes show extremely large changes within one period, so the shortest maturity that we used is 3 months. Regarding the sample period we did not go back further than 1973, because yield curve estimations from earlier years contain relatively high standard errors due to the many missing values. Moreover, the monetary policy before the 1970s was very different from the period afterwards. As stock market index we use the continuously compounded total return index of Datastream's US-DS Market.[10]

To utilize both cross-sectional and time-series bond market data, we follow the estimation methodology of de Jong (2000) based on the Kalman filter and Maximum Likelihood. The error terms in the observation equation are allowed to be cross-sectionally correlated, but they are assumed to be serially uncorrelated. We initialize the Kalman filter recursions by conditioning on the first observations and the initial MSE matrix equals the covariance matrix of the observation errors. The initial factor values are set equal to their long-term mean. For identification purposes $\boldsymbol{\mu}_F$ is

[9]These 3-month yields assumed quarterly compounding, so we manually transformed them into continuously compounded yields.

[10]Datastream code: TOTMKUS(RI).

assumed to be a zero vector. The standard errors of the estimated parameters are obtained as the square roots of the diagonal elements of the inverted Hessian matrix. Table 2.1 contains our estimates.[11]

Table 2.1. Parameter estimates and standard errors
Estimated parameters and standard errors using Maximum Likelihood with weekly observations. At each time we observed four points on the U.S. zero-coupon, continuously compounded yield curve, corresponding to maturities of 3 months, 1 year, 5 years and 10 years; and the total return index of Datastream's US-DS Market. The observation period is from 5 January 1973 to 29 January 2016.

	Estimated parameter	*Standard error*
$\hat{\kappa}_1$	0.0763***	*0.0024*
$\hat{\kappa}_2$	0.3070***	*0.0108*
$\hat{A}_0$	0.0862***	*0.0013*
$\hat{\lambda}_{F,1}$	−0.1708	*0.1528*
$\hat{\lambda}_{F,2}$	−0.5899***	*0.1528*
$\hat{\lambda}_{N+1}$	0.3180**	*0.1528*
$\hat{\sigma}_{F,11}$	0.0208***	*0.0009*
$\hat{\sigma}_{F,21}$	−0.0204***	*0.0012*
$\hat{\sigma}_{F,22}$	0.0155***	*0.0003*
$\hat{\sigma}_{FS,1}$	−0.0035	*0.0038*
$\hat{\sigma}_{FS,2}$	−0.0121***	*0.0035*
$\hat{\sigma}_{N+1}$	0.1659***	*0.0025*

One of the two factors exhibits reasonably strong mean reversion as $\hat{\kappa}_2$ is higher than 0.3, while the mean-reversion parameter of the other factor is quite small (around 0.08), however still statistically significantly different from zero. The long-term mean of the short rate, under the risk-neutral measure, A_0 is estimated to be around 9%. The negative sign of $\hat{\boldsymbol{\sigma}}_{F,21}$ shows that there is a negative correlation between the two factors. The two elements of $\hat{\boldsymbol{\sigma}}_{FS}$ are economically not significant and only one of them is statistically significantly different from zero. $\hat{\sigma}_{N+1}$, on the other hand, is both statistically and economically significant: it is around 17% per annum. The market prices of the two risk sources that influence the factors, $\boldsymbol{\lambda}_F$, are both negative.

Using these estimates, we calculate the model-implied instantaneous expected excess returns, volatilities and Sharpe ratios for the constant maturity bond funds with maturities of 1 year, 5 years, and 10 years and for the stock market index fund (Table 2.2). The stock market index fund has the highest expected excess return (6.05%), but also the highest volatility (16.64%). Regarding

[11] As a robustness check, we also estimated the parameters using monthly and quarterly data. Our estimates are very similar to the ones we obtained using weekly data, which verifies that our model is indeed a good description of the behaviour of returns. All our observations are on the last trading day of the particular period.

Table 2.2. Model implied instantaneous excess returns, volatilities and Sharpe ratios The model implied instantaneous expected excess returns, volatilities of returns and Sharpe ratios using the estimates in Table 2.1.

	Expected excess return	Volatility of return	Sharpe ratio
Constant maturity (1 y.) bond fund	0.83%	1.36%	0.61
Constant maturity (5 y.) bond fund	2.93%	5.25%	0.56
Constant maturity (10 y.) bond fund	4.25%	9.53%	0.45
Stock market index	6.05%	16.64%	0.36

the constant maturity bond funds, the expected excess returns are lower than for the stock market index fund: for 10 years of maturity, the expected excess return is slightly higher than 4%. At the same time, the lower expected excess returns come with lower volatilities, which are 1.36%, 5.25% and 9.53% for the three bond funds, respectively. The highest Sharpe ratio is produced by the constant maturity bond fund with 1 year of maturity: its volatility is relatively low (only 1.36%) given its expected excess return of 0.83%.

For comparison purposes, we also estimated the expected excess returns, the volatilities and the Sharpe ratios directly using excess return data. We used the 3 month T-bill rate as a proxy for the risk-free rate.[12] We used the same stock market index data that we used when estimating the parameters, with weekly observations. To directly estimate the returns, the volatilities and the Sharpe ratios of constant maturity bond funds, we assumed that the investor buys the bond fund at the beginning of the quarter and sells it at the end of the quarter, when its maturity is already 3 months less. To this end we used the same continuously compounded, zero-coupon yield curve observations that before, now with quarterly frequency, plus we observed three additional points on the yield-curve, namely for maturities of 6 months, 4 years and 9 years. These three additional maturities were used – together with the maturities of 1 year, 5 years and 10 years – to obtain our estimates by linear interpolation for the continuously compounded, zero coupon yields for 9 months, 4.75 years and 9.75 years. Our direct estimates of the expected excess returns, the volatilities, and the Sharpe ratios can be found in Table 2.3.

The directly estimated Sharpe ratio of the stock market index fund is 0.37, which is practically the same as the model-implied value. The estimated expected excess returns of the constant

[12]We compared the model implied risk-free rates and the 3 month T-bill rates for the entire estimation period, and they are very close to each other. So the 3 month T-bill rate is a good proxy for the risk-free rate.

Table 2.3. Directly estimated excess returns, volatilities and Sharpe ratios
The directly estimated excess returns, volatilities and Sharpe ratios, using the same observation period as for estimating the parameters in Table 2.1, assuming 3 months holding period.

	Expected excess return	Volatility	Sharpe ratio
Constant maturity (1 y.) bond fund	1.53%	1.64%	0.94
Constant maturity (5 y.) bond fund	2.85%	7.08%	0.40
Constant maturity (10 y.) bond fund	4.40%	12.86%	0.34
Stock market index	6.07%	16.58%	0.37

maturity bond funds with maturities of 5 and 10 years are very close to their model implied counterparts: they are 2.85% and 4.40%. The directly estimated expected excess return for the constant maturity bond fund with 1 year of maturity is somewhat higher than its model-implied counterpart. The directly estimated volatilities of the bond funds are slightly higher than what the model implies, which leads to lower values of the directly estimated Sharpe ratios for maturities of 5 years and 10 years.

We can clearly recognize the bond premium puzzle in Tables 2.2 and 2.3. The Sharpe ratios of all of the constant maturity bond funds are higher than the Sharpe ratio of the stock market index, the only exception being the directly estimated Sharpe ratio of the constant maturity bond fund with 10 years of maturity. Even without considering the supply side, these Sharpe ratios of long-term bonds seem very high given a reasonable level of risk aversion. Interestingly, the puzzle is stronger in the case of the shorter maturity bond funds: the Sharpe ratio of the 1-year constant maturity bond fund is 0.61, while that of the 10-year constant maturity bond fund is 0.45. We now proceed to show that ambiguity aversion can explain this bond-premium puzzle to a large extent.

2.4 Robust general equilibrium

In Section 2.2 we obtained the optimal investment policy of the representative investor: if she would like to maximize her value function (2.7) subject to her budget constraint (2.8), she has to invest according to (2.23) and (2.24). In this section we introduce the concept of robust general equilibrium and we estimate the sum of the risk-aversion and uncertainty-aversion parameters $\gamma+\theta$. Section 2.5 will be devoted to separately identifying γ and θ using the concept of detection error probabilities.

Definition 1. *The market is in robust general equilibrium if the following conditions are satisfied:*

1. *The representative investor solves Problem 6 with parameterization (2.11).*
2. *All of the security markets (the bond markets, the stock market and – as a consequence – the money market) clear continuously, i.e., for all* $t \in [0, T]$*,*

$$\hat{\boldsymbol{\pi}}_{B,t} = \boldsymbol{\pi}_B^* \qquad \forall\, t \in [0,T]\,, \tag{2.25}$$

$$\hat{\pi}_{S,t} = \pi_S^* \qquad \forall\, t \in [0,T]\,, \tag{2.26}$$

where $\boldsymbol{\pi}_B^*$ and π_S^* denote the exogenously given supply of the N constant maturity bond funds and of the stock market index as a fraction of the total wealth of the economy.

If we substitute the optimal investment ratios for the bonds and the stock into the market clearing equations (2.25)-(2.26), we obtain that the market is in robust equilibrium if, for all $t \in [0, T]$, we have

$$\begin{aligned}
\boldsymbol{\pi}_B^* =& -\frac{1}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\left(\boldsymbol{\sigma}_F'\right)^{-1}\left(\boldsymbol{\lambda}_F - \frac{\lambda_{N+1}}{\sigma_{N+1}}\sigma_{FS}\right) \\
&-\frac{1-\gamma-\theta}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\boldsymbol{B}(T-t)\,\boldsymbol{\iota}, \qquad (2.27)\\
\pi_S^* =& \frac{\lambda_{N+1}}{(\gamma+\theta)\,\sigma_{N+1}}. \qquad (2.28)
\end{aligned}$$

We perform the calibration assuming different exogenous supply sides and several different sets of market clearing conditions, which will be subsets of the market clearing conditions (2.25)-(2.26). To calibrate $\gamma + \theta$, we use our estimates from Table 2.1. In (2.27)-(2.28), γ and θ appear only as a sum, so they cannot be separately identified. Moreover, since the system (2.27)-(2.28) is overidentified, an exact solution of $\gamma + \theta$ does not exist, hence we minimize the sum of squared differences to estimate $\gamma+\theta$. I.e., we calculate $\|\boldsymbol{\pi}_B^* - \hat{\boldsymbol{\pi}}_B\|^2 + (\pi_S^* - \hat{\pi}_S)^2$ for several $T-t$ and $\gamma+\theta$ combinations, and then, for each $T - t$, we select that value of $\gamma + \theta$ that minimizes this sum of squared differences.

Since our model has two factors driving the short rate, we need two constant maturity bond funds with different maturities to assure market completeness. To determine the maturities of these two bond funds, we use the data of U.S. government debt (Table 2.4). Since we have data

Table 2.4. U.S. government debt by maturities
The government debt of the U.S. by maturities as of 1 September 2015 and the average maturities of the maturity-clusters. When calculating the average maturities, we assumed that the distribution of the debts within the clusters is uniform. Moreover, we assumed that the average maturity of debts with more than 20 years of maturity is 25 years. Source: Datastream.

Maturity	Debt outstanding (million USD)	Ratio of total debt outstanding	Average maturity
<1 year	2,890,796	0.2824	-
1-5 years	4,335,287	0.4235	3 years
5-10 years	2,035,095	0.1988	7.5 years
10-20 years	187,318	0.0183	15 years
>20 years	789,260	0.0771	25 years

available only for the clustered maturities as shown in Column 1, we assume that the distribution of maturities within the clusters is uniform. This way we can calculate the average maturities for each cluster (Column 4). For bonds with more than 20 years of maturity we assume that their average maturity is 25 years. Because the shortest maturity cluster is less than one year, we treat this cluster of bonds as the money market account. Besides the money market account our model needs two zero-coupon bonds with different maturities in order to have a complete market. Since bonds with 1-5 years of maturity make up more then 42% of all government debt, we assume in the calibration that one of the available zero-coupon bonds has a maturity of 3 years. To determine the maturity of the other zero-coupon bond, we calculate the weighted average of the maturities of bonds with maturities of more than 5 years, where the weights are determined by their amount outstanding. This way we obtain an average 12.55 years maturity for the coupon bonds. As the duration of these coupon bonds is shorter than their maturity, in the calibration we assume that a zero-coupon bond with a maturity of 10 years is available.

Table 2.5 shows the calibrated $\gamma+\theta$ values for different investment horizons and market clearing conditions. As a first experiment, we only impose that the stock market is in equilibrium, $\pi_S^* = 1$. In this case, the required $\gamma + \theta$ parameter is reasonable, 1.92, regardless of the investment horizon. Second, we only impose the restrictions on the bond markets, assuming it is in zero net supply, $\boldsymbol{\pi}_B^* = \mathbf{0}$. The calibrated $\gamma + \theta$ value now depends on the assumed investment horizon $T - t$. The values for $\gamma + \theta$ are more than 150 for $T - t = 10$ years and gradually decrease to 6.67 as $T - t$ approaches infinity.

In the next experiment, we combine both restrictions on the stock and bond market and assume

Table 2.5. Calibrated values of the sum of the risk and uncertainty aversion parameters Calibrated $\gamma + \theta$ values for several investment horizons and market clearing conditions. Estimates for all of the other parameters are from Table 2.1 and the maturities of the available zero-coupon bonds are $\tau_1 = 3$ years and $\tau_2 = 10$ years. The market is assumed to be in equilibrium according to Definition 1.

$T - t$	$\gamma + \theta$			
	$\pi_S = 1$	$\boldsymbol{\pi}_{B_{\tau_1}} = 0$ $\boldsymbol{\pi}_{B_{\tau_2}} = 0$	$\pi_S = 1$ $\boldsymbol{\pi}_{B_{\tau_1}} = 0$ $\boldsymbol{\pi}_{B_{\tau_2}} = 0$	$\pi_S = 0.67$ $\boldsymbol{\pi}_{B_{\tau_1}} = 0.14$ $\boldsymbol{\pi}_{B_{\tau_2}} = 0.10$
10	1.92	154.20	97.78	69.14
15	1.92	19.17	18.09	16.90
20	1.92	11.85	11.48	11.00
25	1.92	9.44	9.23	8.92
30	1.92	8.32	8.16	7.92
50	1.92	6.96	6.86	6.70
100	1.92	6.68	6.59	6.44
∞	1.92	6.67	6.58	6.43

the following exogenous stock and bond supply

$$\pi_S^* = 1, \tag{2.29}$$

$$\boldsymbol{\pi}_B^* = \mathbf{0}. \tag{2.30}$$

The reasoning behind (2.29)-(2.30) is that if an agent in the economy borrows money either by buying a long-term bond or by investing in the money-market account, another player has to lend that money to her. As a result, in general equilibrium the bond holdings of the representative agent will be zero and she keeps all of her wealth invested in the stock market. Thus in general equilibrium the representative investor's holdings of long-term bonds is zero, and all of her wealth is invested in the stock market. The results of this experiment are shown in Column 4 of Table 2.5. If the investor's investment horizon is 10 years, she needs a risk-aversion plus uncertainty-aversion parameter of nearly 100 in order to match market data. As the investment horizon increases, the calibrated $\gamma + \theta$ value decreases to 6.58. The first paper which described the bond premium puzzle, Backus et al. (1989), calibrates the risk-aversion parameter to be around 8, which is what we find using a reasonable investment horizon of 30 years. Hence, our results confirm the bond premium puzzle: if we calibrate our our model in the absence of ambiguity aversion (i.e., setting $\theta = 0$), we need an unreasonably high risk-aversion parameter.

Although the assumption that long-term government bonds are in zero net supply is customary in the literature,[13] it is by no means innocuous, as noted in Donaldson and Mehra (2008). One may argue that the bond supply is exogenously determined by the government (since we only consider government bonds). Then the market clearing conditions are (2.25)-(2.26) with $\boldsymbol{\pi}_B^*$ and π_S^* equal to the weights in a value-weighted market portfolio of stocks and bonds. The U.S. stock market capitalization on 1 September 2015 was $20,885,920$ million USD (using Datastream's TOTMKUS index). Adding up the amounts of debt outstanding in Table 2.4, we find that the total amount of government debt of the U.S. is $10,237,756$ million USD. According to our model the financial market constitutes of the stock market, the bond market, and the money market (assumed to be in zero net supply), thus the total financial market capitalization is $31,123,676$ million USD. This leads to actual aggregate supply portfolio weights equal to

$$\pi_S^* = \frac{20,885,920}{31,123,676} = 0.67, \tag{2.31}$$

$$\pi_{B_{\tau_1}}^* = \frac{4,335,287}{31,123,676} = 0.14, \tag{2.32}$$

$$\pi_{B_{\tau_2}}^* = \frac{3,011,673}{31,123,676} = 0.10. \tag{2.33}$$

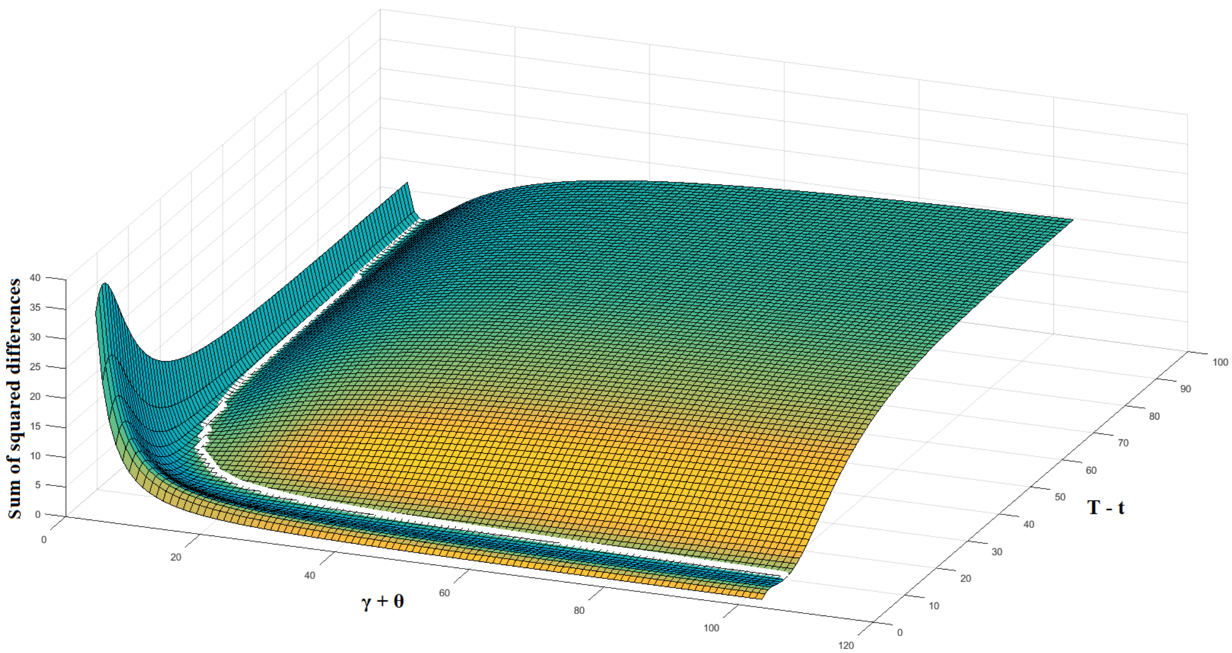
Using these market clearing conditions, the calibrated values of $\gamma + \theta$ are very close to the previous calibration. For $T - t = 10$, the calibrated value of $\gamma + \theta$ is 69.14, and it decreases gradually for longer investment horizons. At $T - t = 30$ it is 7.92 and as the investment horizon increases to infinity, $\gamma + \theta$ converges to 6.43. Hence, also if we use the more realistic market clearing conditions of (2.32)-(2.31) instead of (2.30)-(2.29), the bond-premium puzzle remains with similar magnitude.

Figure 2.1 illustrates the behavior of the calibrated $\gamma + \theta$ values as the time horizon changes. We assume the market clearing condition (2.31)–(2.33). The white line shows the minimized sum of squared differences for given investment horizons up to 100 years. As $T - t$ approaches infinity, the minimizing $\gamma + \theta$ converges to 6.43. If we decrease the investment horizon, the minimizing $\gamma + \theta$ is increasing. So the presence of the bond premium puzzle is robust to the investment horizon: given any reasonable investment horizon for our representative investor, if we calibrate our model in the absence of uncertainty aversion (i.e., setting $\theta = 0$), the risk-aversion parameter γ is large.

[13] Note that several authors, e.g., Gomes and Michaelides (2008), assume that long-term government bonds are in strictly positive net supply.

For example, for a 15 years investment horizon, the value of γ is 16.90 and for a 25 years horizon it is 8.92.

Figure 2.1. Sum of squared differences between supply and demand of the securities Sum of squared differences between supply and demand of the two long-term bonds and the stock for different investment horizons $T - t$. The maturities of the two long-term bonds are 3 years and 10 years. The white line shows the minimized sum of squared differences for given investment horizons $T - t$ up to 100 years.



We have illustrated above that the bond risk-premium puzzle is also present in the sample period we consider, in the absence of uncertainty aversion, i.e., for $\theta = 0$. In the next section, we will identify the risk-aversion parameter γ separately from the uncertainty-aversion parameter θ.

2.5 Separating risk and uncertainty aversion

In Section 2.4 we calibrated the sum of the risk and uncertainty aversion parameters $(\gamma + \theta)$ to market data, but without further assumptions we cannot identify them separately. In this section we recall the concept of detection error probability (Anderson et al. (2003)) in order to identify the risk-aversion parameter γ and the uncertainty-aversion parameter θ separately.

We assume that the investor observes the prices of N constant maturity bond funds and of the stock market index. As assumed in Section 2.2, the investor can observe these stock and bond prices continuously. The observation period is from $t - H$ to t, where t is the moment of observation and $H > 0$. We assume that she then performs a likelihood-ratio test to decide whether the true physical probability measure is $\mathbb{B}$ or the least-favorable $\mathbb{U}$ as derived in Theorem 2. To be more precise: we assume that she calculates the ratio of the likelihoods of the physical probability measure being $\mathbb{B}$ and $\mathbb{U}$, respectively, and accepts the measure with the larger likelihood. If the true measure is $\mathbb{B}$, then the probability that the investor will be wrong is

$$P^{\mathbb{B}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}} < 0\right). \tag{2.34}$$

Similarly, if the true measure is $\mathbb{U}$, the probability that the investor will be wrong when determining the probability measure given a sample of data is

$$P^{\mathbb{U}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}} > 0\right). \tag{2.35}$$

We now define the detection error probability, following Anderson et al. (2003).

Definition 2. *The detection error probability (DEP) is defined as*

$$DEP = \frac{1}{2}P^{\mathbb{B}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}} < 0\right) + \frac{1}{2}P^{\mathbb{U}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}} > 0\right). \tag{2.36}$$

In the following theorem we give the detection error probability for any arbitrary $\mathbb{U}$. Then, in Corollary 2 we give the detection error probability for the least favorable $\mathbb{U}$ as derived in Theorem 2. Both proofs again are in the appendix.

Theorem 4. *Assume that the investor (continuously) observes the prices of N constant maturity bond funds and of the stock market index. The observation period lasts from $t-H$ to the moment of observation, t. Then, the detection error probability of the investor for given $\mathbb{U}$ is*

$$DEP = 1 - \Phi\left(\frac{1}{2}\sqrt{\int_{t-H}^{t} \|\boldsymbol{u}(s)\|^2 \mathrm{d}s}\right), \tag{2.37}$$

where $\boldsymbol{u}(\cdot)$ is defined in (2.6).

Plugging in the least-favorable $\mathbb{U}$ as derived in Theorem 2 leads to the following corollary.

Corollary 2. *Assume that the conditions of Theorem 4 hold. Then, the detection error probability of the investor for the least-favorable* $\mathbb{U}$ *is*

$$DEP = 1 - \Phi\left(\frac{\theta}{2(\gamma+\theta)}\sqrt{H\left(\lambda_{N+1}^2 + \|\boldsymbol{\lambda}_F\|^2\right) + \Delta_1 + \Delta_2}\right), \tag{2.38}$$

with

$$\begin{aligned}\Delta_1 &= 2\boldsymbol{\lambda}_F'\boldsymbol{\sigma}_F'\left[\left(\exp\left(-\boldsymbol{\kappa}\left(T-t+H\right)\right) - \exp\left(-\boldsymbol{\kappa}\left(T-t\right)\right)\right)\boldsymbol{\kappa}^{-2}\right.\\ &\qquad \left. + H\boldsymbol{\kappa}^{-1}\right]\boldsymbol{\iota}, \qquad (2.39)\\ \Delta_2 &= \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\left(\boldsymbol{\sigma}_F\boldsymbol{\sigma}_F'\right)_{ij}}{\kappa_i\kappa_j}\left[\boldsymbol{B}_{ii}\left(T-t\right) + \boldsymbol{B}_{jj}\left(T-t\right) - \boldsymbol{B}_{ii}\left(T-t+H\right)\right.\\ &\qquad \left. - \boldsymbol{B}_{jj}\left(T-t+H\right) - \boldsymbol{C}_{ij}\left(T-t\right) + \boldsymbol{C}_{ij}\left(T-t+H\right) + H\right] \qquad (2.40)\end{aligned}$$

If the investor is not uncertainty-averse at all, her θ parameter is 0, thus her detection error probability is $\frac{1}{2}$. This means that when solving Problem 6, she chooses the alternative measure $\mathbb{U}$ which is exactly the same as the base measure $\mathbb{B}$, hence it is impossible to distinguish between them based on a sample of data. This is reflected in her detection error probability as well: in 50% of the cases she will be wrong when basing her decision on the likelihood-ratio test described above, so she might as well just flip a coin instead of basing her decision on a sample of data.

The more uncertainty-averse the investor is, that is, the larger θ, the larger the distance between the chosen least-favorable $\mathbb{U}$ measure and the base $\mathbb{B}$ measure. And the larger this distance between $\mathbb{U}$ and $\mathbb{B}$, the easier it is to determine (based on a sample of data) which of the two measures is the true one. This is reflected in the detection error probability of the investor: if θ is larger, $\Phi(\cdot)$ is larger, and her detection error probability is lower. That is, she will make the correct decision when distinguishing between the two measures in a higher proportion of the cases.

The detection error probability also depends on the observation span H: the longer time period the investor observes, the higher H will be and, thus, ceteris paribus $\Phi(\cdot)$ will be higher and the detection error probability will be lower. If the observation span goes to infinity, the detection error probability goes to zero.

Anderson et al. (2003) argue that the largest amount of uncertainty that should be entertained as reasonable corresponds to a detection error probability of approximately 10%. Since we have 42 years of data available to estimate our model, it is reasonable to assume that our representative investor also has access to 42 years of data to distinguish between the base model and the alternative model. Given that her observation period of 42 years is relatively long and that she can observe the data in continuous time, we can reasonably assume that the detection error probability equals 10%. Table 2.6 presents the induced calibrated values for the γ and θ separately, based on the estimates in Table 2.5. Besides the γ and θ values, Column 5 of Table 2.6 shows by how much the risk-aversion parameter decreases by the introduction of an uncertainty-averse representative agent.

Table 2.6. Disentangled risk- and uncertainty-aversion parameters
Calibrated $\gamma + \theta$ values and their disentangled γ and θ components, based on the estimates in Table 2.5. The detection error probability is 10%. The observation period is 42 years. The last column shows by how much robustness decreases the risk-aversion parameter that we need to match market data compared to the non-robust model.

$T-t$	$\gamma+\theta$	γ	θ	Decrease in γ due to robustness
10	69.1	25.5	43.6	63%
15	16.9	6.2	10.7	63%
20	11.0	4.1	6.9	63%
25	8.9	3.3	5.6	63%
30	7.9	2.9	5.0	63%

As Table 2.6 shows, regardless of the investment horizon of the representative investor, the required risk-aversion parameter to match market data is decreased by about 63% as a result of the introduction of robustness in our model. The relative risk aversion of the robust investor is much more reasonable than that of a non-robust investor: for investment periods between 15 and 30 years, it is between about 2.9 and 6.2, compared to it being between 7.9 and 16.9 in the case of a non-robust investor.

As a robustness check of our results, we also disentangle the sum of risk- and uncertainty aversion parameters using higher (i.e., more conservative) detection error probability values, concretely 15% and 20%. The results are shown in Table 2.7 and Table 2.8, respectively. We see that assuming a detection error probability of 15% still reduces the originally required risk-aversion parameter by more than half of its original value, and even a rather conservative assumption of 20% detection error probability results in a decrease of 41.5% in the required risk-aversion parameter. This reinforces the robustness of our findings that parameter uncertainty can explain a large fraction of the bond premium puzzle.

Table 2.7. Disentangled risk- and uncertainty-aversion parameters
Calibrated $\gamma + \theta$ values and their disentangled γ and θ components, based on the estimates in Table 2.5. The detection error probability is 15%. The observation period is 42 years. The last column shows by how much robustness decreases the risk-aversion parameter that we need to match market data compared to the non-robust model.

$T-t$	$\gamma+\theta$	γ	θ	Decrease in γ due to robustness
10	69.1	33.9	35.2	51%
15	16.9	8.3	8.6	51%
20	11.0	5.4	5.6	51%
25	8.9	4.4	4.5	51%
30	7.9	3.9	4.0	51%

Table 2.8. Disentangled risk- and uncertainty-aversion parameters
Calibrated $\gamma + \theta$ values and their disentangled γ and θ components, based on the estimates in Table 2.5. The detection error probability is 20%. The observation period is 42 years. The last column shows by how much robustness decreases the risk-aversion parameter that we need to match market data compared to the non-robust model.

$T-t$	$\gamma+\theta$	γ	θ	Decrease in γ due to robustness
10	69.1	40.5	28.6	41%
15	16.9	9.9	7.0	41%
20	11.0	6.4	4.6	41%
25	8.9	5.2	3.7	41%
30	7.9	4.6	3.3	41%

To further justify that our model with robust investors explains a substantial portion of the bond premium puzzle, we calculate the model-implied expected instantaneous excess returns on the constant maturity bond funds and on the stock market index with and without robustness, for different investment horizons. To this end, we use the market clearing conditions in the last column of Table 2.5. We further assume that the investor has access to 42 years of continuous market data, and that her Detection Error Probability is 10%. Our results can be found in Table 2.9.

We find again that without robustness the bond premium puzzle arises: as Backus et al. (1989) documents, the model with non-robust agents cannot match either the sign or the magnitude of the expected excess bond returns.[14] Intuitively, the reason of the negative sign of the model-implied

[14]Note that we use the market clearing conditions in the last column of Table 2.5 to calculate the model-implied bond premiums. These market clearing conditions allow for both positive and negative model-implied bond premiums. E.g., if the investor has a logarithmic utility function, and her investment horizon is 10 years, in general equilibrium

bond premiums is that even though they have the attractive feature of providing intertemporal hedging, in general equilibrium the representative investor is assumed to invest only a small fraction of her wealth or nothing at all in them (depending on the set of market clearing conditions used). However, this investment decision is optimal for the investor only if – besides providing intertemporal heding – some other properties of the constant maturity bond funds are unattractive to her, namely, they offer a negative premium above the risk-free rate.

Introducing uncertainty aversion into our model alleviates the discrepancy between the model implied and the empirically observed bond premiums significantly: although the model-implied bond premiums are still negative, their magnitude is substantially smaller, hence they are much closer to their empirically observed, positive counterpart. Moreover, the model-implied stock market index premiums are also much closer to the empirically observed value.

Table 2.9. Model-implied excess returns.
Model-implied expected instantaneous excess returns on constant maturity bond funds with 3 and 10 years of maturity and on the stock market index, with and without robustness, for different investment horizons. The market clearing conditions in the last column of Table 2.5 hold. The investor is assumed to have access to 42 years of continuous market data, and her Detection Error Probability is assumed to be 10%.

$T-t$	Model-implied expected instantaneous excess return					
	Bond ($\tau = 3$)		Bond ($\tau = 10$)		Stock market index	
	Non-robust	Robust	Non-robust	Robust	Non-robust	Robust
15	-4.2%	-1.3%	-16.4%	-5.2%	29.9%	11.1%
20	-3.1%	-0.9%	-12.5%	-3.7%	19.4%	7.3%
25	-2.6%	-0.7%	-11%	-3.1%	15.7%	5.9%
30	-2.4%	-0.6%	-10.3%	-2.7%	13.9%	5.2%

2.6 Conclusion

We have shown that the introduction of uncertainty aversion in a standard financial market offers a potential solution to the bond-premium puzzle. In the presence of uncertainty aversion, the risk aversion of the representative agent decreases to levels consistent with both economic intuition and experiments. At the same time, our paper offers the methodological contribution to formulate and solve the robust investment problem an uncertainty-averse investor faces using the martingale

the model-implied premiums on bonds with 3 and 10 years of maturity are 0.07% and 0.19%, respectively. However, if we used the market clearing conditions in the fourth column of Table 2.5, the model-implied bond premiums would always be negative.

method.

To disentangle risk aversion from uncertainty aversion, we assumed a detection error probability of 10%, and we used 15% and 20% detection error probability values as robustness checks of our findings. Assuming detection error probability values between 10% and 20% is common in the literature, but there is little research on what determines the level of the detection error probability. This can be a fruitful line of future research. We also assumed that the investor is not uncertain about the volatility. Relaxing this assumption leads us out of the realm of the framework of Anderson et al. (2003). Extending the penalty approach of dynamic robust asset allocation in a direction that allows for uncertainty about the volatility is another potential area of future research.

Appendix

Proof of Theorem 1. From (2.12) the value function at time t satisfies

$$\begin{aligned} V_t(X_t) &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma} + \int_t^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta} V_s(X_s)\mathrm{d}s\right\} \\ &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma}\right\} + \mathrm{E}_t^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta} V_s(X_s)\mathrm{d}s\right\} \\ &\quad - \int_0^t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2 V_s(X_s)}{2\theta}\mathrm{d}s, \end{aligned} \tag{2.41}$$

where X_T and $\mathbb{U}$ denote the optimal terminal wealth and least-favorable physical measure, respectively. Introduce the square-integrable martingales, under $\mathbb{U}$,

$$M_{1,t} = \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma}\right\}, \tag{2.42}$$

$$M_{2,t} = \mathrm{E}_t^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta} V_s(X_s)\mathrm{d}s\right\}. \tag{2.43}$$

According to the martingale representation theorem (see, e.g., Karatzas and Shreve (1991), pp. 182, Theorem 3.4.15), there exist square-integrable stochastic processes $\boldsymbol{Z}_{1,t}$ and $\boldsymbol{Z}_{2,t}$ such that

$$M_{1,t} = \mathrm{E}_0^{\mathbb{U}}\left\{\exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma}\right\} + \int_0^t \boldsymbol{Z}'_{1,s}\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}, \tag{2.44}$$

$$M_{2,t} = \mathrm{E}_0^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta} V_s(X_s)\mathrm{d}s\right\} + \int_0^t \boldsymbol{Z}'_{2,s}\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}. \tag{2.45}$$

Substituting in (2.41), we can express the dynamics of the value function as

$$\mathrm{d}V_t(X_t) = -\frac{(1-\gamma)\,\|\boldsymbol{u}\,(t)\,\|^2}{2\theta} V_t(X_t)\mathrm{d}t + (Z_{1,t} + Z_{2,t})'\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{U}}. \tag{2.46}$$

This is a linear backward stochastic differential equation, that (together with the terminal condition $V_T(X_T) = \exp(-\delta T)X_T^{1-\gamma}/(1-\gamma)$) can be solved explicitly; see Pham (2009), pp. 141-142. The unique solution to (2.46) is given by

$$\Gamma_t V_t(X_t) = \mathrm{E}_t^{\mathbb{U}}\left\{\Gamma_T \exp(-\delta T)\frac{X_T^{1-\gamma}}{1-\gamma}\right\}, \tag{2.47}$$

where Γ_t solves the linear differential equation

$$\mathrm{d}\Gamma_t = \Gamma_t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(t)\,\|^2}{2\theta}\mathrm{d}t; \qquad \Gamma_0 = 1, \tag{2.48}$$

i.e.,

$$\Gamma_t = \exp\left(\int_0^t \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}\mathrm{d}s\right). \tag{2.49}$$

Substituting into (2.47), we obtain the value function in closed form as

$$V_t(X_t) = \mathrm{E}_t^{\mathbb{U}}\left\{\exp\left(\int_t^T \frac{(1-\gamma)\,\|\boldsymbol{u}\,(s)\,\|^2}{2\theta}\mathrm{d}s - \delta T\right)\frac{X_T^{1-\gamma}}{1-\gamma}\right\}. \tag{2.50}$$

Recall that X_t and $\mathbb{U}$ represent the optimal wealth and least-favorable physical measure. As a result, we obtain (2.13). □

Before continuing with the proofs of the remaining theorems, we recall some well-known results.

Lemma 1. *Under (2.1) and (2.2), we have*

$$\begin{aligned} r_t =& A_0 + \boldsymbol{\iota}'\left[\boldsymbol{\mu}_F + \exp\left(-\boldsymbol{\kappa}t\right)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right)\right] \\ &+ \int_0^t \boldsymbol{\iota}' \exp\left(-\boldsymbol{\kappa}(t-s)\right)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,s}^{\mathbb{Q}}. \end{aligned} \tag{2.51}$$

Proof. The proof directly follows from the solution of the stochastic differential equation of the Ornstein-Uhlenbeck process, for which see, e.g., Chin, Nel, and Ólafsson (2014), pp. 132–133. □

Lemma 2. *Under (2.1) and (2.2), we have*

$$\int_0^t r_s \mathrm{d}s = \left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right]t + \boldsymbol{\iota}'\boldsymbol{B}(t)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right) + \int_0^t \boldsymbol{\iota}'\boldsymbol{B}(t-v)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,v}^{\mathbb{Q}}, \tag{2.52}$$

with $\boldsymbol{B}$ defined as the matrix integral

$$\boldsymbol{B}(t) = \int_0^t \exp\left(-\boldsymbol{\kappa}s\right)\mathrm{d}s. \tag{2.53}$$

Since $\boldsymbol{\kappa}$ is a matrix whose elements are constants, $\boldsymbol{B}(t)$ can be expressed in the more compact form

$$\boldsymbol{B}(t) = \left(\boldsymbol{I} - \exp\left\{-\boldsymbol{\kappa} t\right\}\right)\boldsymbol{\kappa}^{-1}. \tag{2.54}$$

Proof. From Lemma 1 we find

$$\begin{aligned}
\int_0^t r_s \mathrm{d}s &= \left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] t + \boldsymbol{\iota}'\boldsymbol{B}(t)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right) + \int_0^t \int_0^s \boldsymbol{\iota}' \exp\left(-\boldsymbol{\kappa}(s-v)\right)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,v}^{\mathbb{Q}} \mathrm{d}s \\
&= \left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] t + \boldsymbol{\iota}'\boldsymbol{B}(t)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right) + \int_0^t \int_v^t \boldsymbol{\iota}' \exp\left(-\boldsymbol{\kappa}(s-v)\right)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,v}^{\mathbb{Q}} \mathrm{d}s \\
&= \left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] t + \boldsymbol{\iota}'\boldsymbol{B}(t)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right) + \int_0^t \boldsymbol{\iota}'\boldsymbol{B}(t-v)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,v}^{\mathbb{Q}}.
\end{aligned}$$

This completes the proof. □

Lemma 2 immediately leads to the price of bonds in our financial market. We briefly recall this for completeness. The price at time t of a nominal bond with remaining maturity τ is given by

$$\begin{aligned}
P_t(\tau) &= \mathrm{E}_t^{\mathbb{Q}} \exp\left(-\int_t^{t+\tau} r_s \mathrm{d}s\right) \\
&= \mathrm{E}_t^{\mathbb{Q}} \exp\left(-\left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right]\tau - \boldsymbol{\iota}'\boldsymbol{B}(\tau)\left(\boldsymbol{F}_t - \boldsymbol{\mu}_F\right) - \int_t^{t+\tau} \boldsymbol{\iota}'\boldsymbol{B}(t+\tau-s)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,s}^{\mathbb{Q}}\right) \\
&= \exp\left(-\left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right]\tau - \boldsymbol{\iota}'\boldsymbol{B}(\tau)\left(\boldsymbol{F}_t - \boldsymbol{\mu}_F\right) + \frac{1}{2}\int_0^{\tau} \|\boldsymbol{\iota}'\boldsymbol{B}(\tau-s)\boldsymbol{\sigma}_F\|^2 \mathrm{d}s\right).
\end{aligned} \tag{2.55}$$

As a result, the exposure of a constant τ-maturity bond fund to the factors $\boldsymbol{F}_t$ is given by $-\boldsymbol{\iota}'\boldsymbol{B}(\tau)$.

We now continue with the proofs of Theorem 2, 3, and 4.

Proof of Theorem 2. The first step of the optimization is to determine the optimal terminal wealth, given the budget constraint. In order to determine the optimal terminal wealth, we form the

Lagrangian from (2.13) and (2.14). This Lagrangian is given by

$$\begin{aligned}
L(x) &= \inf_{\mathbb{U}} \sup_{X_T} \Bigg\{ \mathrm{E}^{\mathbb{U}} \exp\left(\frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T \right) \frac{X_T^{1-\gamma}}{1-\gamma} \\
&\quad - y \left[\mathrm{E}^{\mathbb{Q}} \exp\left(- \int_0^T r_t \mathrm{d}t \right) X_T - x \right] \Bigg\} \\
&= \inf_{\mathbb{U}} \sup_{X_T} \Bigg\{ \mathrm{E}^{\mathbb{Q}} \exp\Bigg(\int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} - \frac{1}{2} \int_0^T \|\boldsymbol{u}(t) + \boldsymbol{\lambda}\|^2 \mathrm{d}t \\
&\qquad + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T \Bigg) \frac{X_T^{1-\gamma}}{1-\gamma} \\
&\quad - y \left[\mathrm{E}^{\mathbb{Q}} \exp\left(- \int_0^T r_t \mathrm{d}t \right) X_T - x \right] \Bigg\}, \qquad (2.56)
\end{aligned}$$

where y is the Lagrange-multiplier and the second equality uses the Girsanov transformation (2.6).

We first consider the inner optimization, i.e., the optimal choice of the final wealth X_T given $\mathbb{U}$. The first order condition for optimal final wealth, denoted by $\hat{X}_T$, is

$$\begin{aligned}
&\exp\left(\int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} - \frac{1}{2} \int_0^T \|\boldsymbol{u}(t) + \boldsymbol{\lambda}\|^2 \mathrm{d}t + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T \right) \hat{X}_T^{-\gamma} \\
&= \; y \exp\left(- \int_0^T r_t \mathrm{d}t \right). \qquad (2.57)
\end{aligned}$$

As the Lagrange multiplier y is still to be determined by the budget constraint, we can subsume all deterministic terms in a new Lagrange multiplier y_1 and solve

$$\exp\left(\int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \hat{X}_T^{-\gamma} = y_1 \exp\left(- \int_0^T r_t \mathrm{d}t \right), \qquad (2.58)$$

i.e.,

$$\hat{X}_T = y_1^{-1/\gamma} \exp\left(\frac{1}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right), \qquad (2.59)$$

We can now substitute this optimal terminal wealth into the budget constraint (2.14) in order to obtain the optimal value of the Lagrange multiplier (denoted by $\hat{y}_1$).

$$\hat{y}_1^{-1/\gamma} = \frac{x}{\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right)}. \qquad (2.60)$$

Substituting (2.60) into (2.59), the optimal terminal wealth given $\mathbb{Q}$ is obtained explicitly. This yields (2.16).

We proceed by solving the outer optimization in (2.13), i.e., we find the least-favorable distortions $\boldsymbol{u}$. Substituting the optimal final wealth (2.59) into the value function leads to

$$\begin{aligned}
V_0(x) &= \inf_{\mathbb{U}} \mathrm{E}^{\mathbb{U}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t \right) \frac{\hat{X}_T^{1-\gamma}}{1-\gamma} \right\} \\
&= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \mathrm{E}^{\mathbb{U}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t + \frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t \right.\right. \\
&\qquad \left.\left. + \frac{1-\gamma}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right\} \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right]^{\gamma-1}. \qquad (2.61)
\end{aligned}$$

We now use the Girsanov transforms (2.5)–(2.6) to obtain

$$\begin{aligned}
V_0(x) &= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \mathrm{E}^{\mathbb{Q}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t + \frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t \right.\right. \\
&\qquad \left.\left. + \frac{1-\gamma}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{Q}} \right\} \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right]^{\gamma-1} \\
&= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \mathrm{E}^{\mathbb{Q}} \left\{ \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t + \frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t \right.\right. \\
&\qquad \left.\left. + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} - \frac{1}{2} \int_0^T \|\boldsymbol{u}(t) + \boldsymbol{\lambda}\|^2 \mathrm{d}t \right) \right\} \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right]^{\gamma-1} \\
&= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \exp\left(-\delta T + \frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \frac{1}{2} \int_0^T \|\boldsymbol{u}(t) + \boldsymbol{\lambda}\|^2 \mathrm{d}t \right) \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma} \int_0^T r_t \mathrm{d}t + \frac{1}{\gamma} \int_0^T (\boldsymbol{u}(t) + \boldsymbol{\lambda})' \, \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} \right) \right]^{\gamma}. \qquad (2.62)
\end{aligned}$$

Using Lemma 2 we find

$$\begin{aligned}
V_0(x) &= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \exp\left(-\delta T + \frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \frac{1}{2}\int_0^T \|\boldsymbol{u}(t)+\boldsymbol{\lambda}\|^2 \mathrm{d}t\right) \\
&\quad \times \exp\left((1-\gamma)\left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] T + (1-\gamma)\boldsymbol{\iota}'\boldsymbol{B}(T)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right)\right) \\
&\quad \times \left[\mathrm{E}^{\mathbb{Q}} \exp\left(\frac{1-\gamma}{\gamma}\int_0^T \boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}_{F,t}^{\mathbb{Q}} \mathrm{d}t + \frac{1}{\gamma}\int_0^T (\boldsymbol{u}(t)+\boldsymbol{\lambda})' \mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right)\right]^{\gamma} \\
&= \frac{x^{1-\gamma}}{1-\gamma} \inf_{\mathbb{U}} \exp\left(-\delta T + \frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \frac{1}{2}\int_0^T \|\boldsymbol{u}(t)+\boldsymbol{\lambda}\|^2 \mathrm{d}t\right) \\
&\quad \times \exp\left((1-\gamma)\left[A_0 + \boldsymbol{\iota}'\boldsymbol{\mu}_F\right] T + (1-\gamma)\boldsymbol{\iota}'\boldsymbol{B}(T)\left(\boldsymbol{F}_0 - \boldsymbol{\mu}_F\right)\right) \\
&\quad \times \exp\left(\frac{1}{2\gamma}\int_0^T \|(1-\gamma)\boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F + (\boldsymbol{u}_F(t)+\boldsymbol{\lambda}_F)'\|^2 \mathrm{d}t \right. \\
&\qquad \left. + \frac{1}{2\gamma}\int_0^T \left(u_{N+1}(t) + \lambda_{N+1}\right)^2 \mathrm{d}t\right).
\end{aligned} \tag{2.63}$$

Recall that $\boldsymbol{u}_F(t)$ is an N-dimensional column vector containing the first N elements of $\boldsymbol{u}(t)$ and $u_{N+1}(t)$ is its last element. Similarly, $\boldsymbol{\lambda}_F$ is the N-dimensional column vector containing the first N elements of $\boldsymbol{\lambda}$ and λ_{N+1} denotes its last element.

This expression of the value function allows to perform the outer minimization with respect to $\mathbb{U}$, i.e., with respect to $\boldsymbol{u}$. The first-order condition of this minimization reads, with respect to $\boldsymbol{u}_F(t)$,

$$\begin{aligned}
0 &= \frac{1-\gamma}{\theta}\boldsymbol{u}_F(t) - \boldsymbol{u}_F(t) - \boldsymbol{\lambda}_F + \frac{1}{\gamma}\left((1-\gamma)\boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota} + \boldsymbol{u}_F(t) + \boldsymbol{\lambda}_F\right) \\
&= \frac{1-\gamma}{\theta}\boldsymbol{u}_F(t) + \frac{1-\gamma}{\gamma}\left(\boldsymbol{u}_F(t) + \boldsymbol{\lambda}_F\right) + \frac{1-\gamma}{\gamma}\boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota} \\
&= (1-\gamma)\frac{\gamma+\theta}{\gamma\theta}\boldsymbol{u}_F(t) + \frac{1-\gamma}{\gamma}\left[\boldsymbol{\sigma}_F'\boldsymbol{B}(T-t)'\boldsymbol{\iota} + \boldsymbol{\lambda}_F\right].
\end{aligned} \tag{2.64}$$

This proves (2.17). Finally, minimizing the value function $V_0(x)$ with respect to $u_{N+1}(t)$ we find the first-order condition

$$\begin{aligned}
0 &= \frac{1-\gamma}{\theta}u_{N+1}(t) - u_{N+1}(t) - \lambda_{N+1} + \frac{1}{\gamma}\left(u_{N+1}(t) + \lambda_{N+1}\right) \\
&= \left[\frac{1-\gamma}{\theta} + \frac{1-\gamma}{\gamma}\right] u_{N+1}(t) + \frac{1-\gamma}{\gamma}\lambda_{N+1}.
\end{aligned} \tag{2.65}$$

This proves (2.18) and completes the proof. □

Proof of Theorem 3. Theorem 2 has actually a fairly simple form in the driving Brownian motion $\boldsymbol{W}_t^{\mathbb{Q}}$. Using Lemma 2 we find

$$\begin{aligned}
&\exp\left(-\int_0^T r_t \mathrm{d}t\right)\hat{X}_T \\
\propto\ &\exp\left(\int_0^T \frac{\hat{\boldsymbol{u}}(t)' + \boldsymbol{\lambda}'}{\gamma}\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}} + \frac{1-\gamma}{\gamma}\int_0^T \left[\boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F; 0\right]\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right) \\
=\ &\exp\left(\int_0^T \left[\frac{\boldsymbol{\lambda}_F'}{\gamma+\theta} + \frac{1-(\gamma+\theta)}{\gamma+\theta}\boldsymbol{\iota}'\boldsymbol{B}(T-t)\boldsymbol{\sigma}_F; \frac{\boldsymbol{\lambda}_{N+1}}{\gamma+\theta}\right]\mathrm{d}\boldsymbol{W}_t^{\mathbb{Q}}\right).
\end{aligned} \tag{2.66}$$

As $\exp\left(-\int_0^t r_s \mathrm{d}s\right)\hat{X}_t = \mathrm{E}_t^{\mathbb{Q}}\exp\left(-\int_0^T r_t \mathrm{d}t\right)\hat{X}_T$, the wealth evolution (2.21) is easily obtained. □

Proof of Corollary 1. This is a direct consequence of Theorem 3 and the martingale representation theorem (see, e.g., Karatzas and Shreve (1991), pp. 182, Theorem 3.4.15). □

Proof of Theorem 4. The investor observes the prices of N constant maturity bond funds and of the stock market index in continuous time from $t-H$ to t. The moment of observation is t. Using (2.6) we have

$$\begin{aligned}
\log\left(\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}\right)_H &= -\frac{1}{2}\int_{t-H}^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s + \int_{t-H}^t \boldsymbol{u}(s)\,\mathrm{d}W_s^{\mathbb{U}} \\
&\overset{\mathbb{U}}{\sim}\mathcal{N}\left(-\frac{1}{2}\int_{t-H}^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s, \int_{t-H}^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s\right).
\end{aligned} \tag{2.67}$$

This implies

$$P^{\mathbb{U}}\left(\log\left(\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}\right)_H > 0\right) = 1 - \Phi\left(\frac{1}{2}\sqrt{\int_{t-H}^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s}\right). \tag{2.68}$$

Due to symmetry, we have

$$\begin{aligned}
P^{\mathbb{B}}\left(\log\left(\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}\right)_H < 0\right) &= P^{\mathbb{B}}\left(\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_H > 0\right) \\
&= P^{\mathbb{U}}\left(\log\left(\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}}\right)_H > 0\right).
\end{aligned} \tag{2.69}$$

Thus, using Definition 2, the detection error probability is

$$DEP = 1 - \Phi\left(\frac{1}{2}\sqrt{\int_{t-H}^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s}\right). \tag{2.70}$$

□

Proof of Corollary 2. Substituting (2.19) and (2.20) into (2.70) and working out the integrals, (2.38) follows directly. □

Chapter 3

Parameter Uncertainty: The Missing Piece of the Liquidity Premium Puzzle?

3.1 Introduction

One of the many asset pricing puzzles in the finance literature is the liquidity premium puzzle. Theoretical work finds that transaction cost only has a second-order effect on expected returns, while empirical research documents a first-order effect. For example, the seminal work of Constantinides (1986) finds that a 1% proportional transaction cost on stocks increases the expected stock return only by 0.2%. On the other hand, Amihud and Mendelson (1986) empirically find that for a 1% proportional transaction cost, the liquidity premium is about 2%, based on NYSE data. Even though in the past four decades stock liquidity increased substantially and at the same time the liquidity premium decreased (Ben-Rephael, Kadan, and Wohl (2015)), the difference between the empirically documented and the theoretically implied liquidity premium is still significant.

In this paper, I approach the liquidity premium puzzle from a new angle. A key feature of dynamic asset allocation problems with transaction cost is the set of assumptions about the transaction cost. In my model I not only allow the transaction cost to be stochastic, but I also relax the assumption that the investor knows the precise probability distribution of future transaction costs. She takes this uncertainty about the transaction cost process into account when she solves a dynamic investment problem, and she makes robust decisions with respect to consumption and asset allocation.

A robust investor is uncertain about one or several parameters of the underlying model. Acknowledging this uncertainty, she makes investment decisions that work well not only if the model that she had in mind is correct, but which work reasonably well even if her model turns out to be misspecified. In this paper I use the minimax approach of Anderson et al. (2003). The investor is uncertain about the (future) transaction costs in the underlying model, but she has a *base transaction cost level* in mind, which she considers to be the most reasonable. Since she is uncertain about the true future transaction costs, she considers other *alternative transaction cost levels* as well. But instead of setting an explicit constraint on the transaction costs to be considered, I introduce a penalty term in the goal function. If the transaction cost level under consideration is very different from the base transaction cost level, the investor is penalized through this function. Since she wants to make robust decisions, she prepares for the worst-case scenario, i.e., she uses the alternative transaction cost level which results in the lowest value function (including the penalty term). The investor's tolerance towards uncertainty is expressed by the uncertainty-tolerance parameter, which multiplies the penalty term.

I solve the robust dynamic optimization problem numerically, using parameter estimates of the existing literature. I compare the optimal consumption and asset allocation decision of a robust investor – using several different levels of uncertainty aversion – to the decisions of an otherwise equivalent non-robust investor, and I determine the effects of robustness on asset allocation. Uncertainty-aversion affects the investor's decision about asset allocation through two channels, and these influence the investor's decision in opposite directions. On the one hand, a robust investor is induced to buy more of the risky asset now, so that she will not have to buy it later at an unknown (and potentially high) transaction cost. At the same time, she is also induced to buy less risky asset now, to avoid having to sell these asset in the future for consumption purposes – again, at an unknown and potentially high transaction cost. The argument is the same if the investor wants to sell some of her risky asset, instead of buying. Besides the parameter values of the model, the current value of the transaction cost and the inherited investment ratio jointly determine which of the two effects will dominate.

After showing the effects of robustness on the optimal asset allocation, I introduce the definitions of liquidity premium and its several components, and I study the asset pricing implications of robustness in my model. Concretely, I show that even a moderate level of robustness can generate a modest additional liquidity premium (0.05% for the first 1% proportional transaction cost), and this additional liquidity premium can be even higher if I assume higher levels of robustness. Thus, parameter uncertainty can provide a partial resolution to the liquidity premium puzzle.

Besides these economic contributions, my paper also provides two technical contributions. First, I provide a novel, non-recursive representation of a robust, discrete-time, dynamic asset allocation problem. In its original form, this problem is of non-standard recursive nature, and practically impossible to solve numerically. My novel, non-recursive representation of the problem not only gives more economic insight into the effects of robustness on the investor's decisions, but it makes it possible to derive the Bellman equation, which is already in standard recursive form, and thus it is possible to solve the problem within a reasonable amount of time.[1]

As a second technical contribution, I provide a novel approach to efficiently solve discrete-time robust dynamic asset allocation problems involving transaction cost. Given the robust nature of the problem, one cannot make use of the standard optimization techniques which work well for maximization and minimization problems. Moreover, the value function is of class C^0, i.e., its first partial derivative is non-continuous, due to the presence of a no-trade region, as documented in Constantinides (1986). To circumvent these difficulties, I develop a semi-analytical approach to solve the optimization problem. First I derive the first-order conditions analytically, then I solve the resulting non-linear equation system numerically. Since these equations are the partial derivatives of the value function, they contain non-continuous functions. These non-continuities prevent me from making use of the Gaussian quadrature rule to evaluate numerically the arising double integrals.[2] So I determine the no-trade boundaries explicitly, and I calculate the double integrals part-wise numerically, applying the Gaussian quadrature rule.

My paper relates to the literature on the liquidity premium puzzle. The first paper documenting the puzzle is by Constantinides (1986). Constantinides repeats the calculation including quasi-fixed

[1]Using a computer with 16 threads and an Intel Xeon E5-2650 v2 Processor, it takes approximately 4 hours to solve a robust dynamic optimization problem with an investment horizon of 10 years and annual discretization.

[2]These double integrals are a result of having to evaluate a double expectation due to the two sources of risk: a stochastic stock return and a stochastic transaction cost.

transaction cost,[3] and he concludes that its effect is still of second order. Liu (2004) not only extends the model with a fixed transaction cost, but he also proposes a different definition of the liquidity premium: instead of matching the expected utility, he matches the average holding of the risky asset. The generated liquidity premium is, however, lower than in Constantinides (1986). Vayanos (1998) proposes a general equilibrium model instead of the partial equilibrium model of Constantinides (1986), but he finds that general equilibrium models can generate even lower liquidity premiums than partial equilibrium models. Buss, Uppal, and Vilkov (2014) propose a model with an Epstein-Zin utility function instead of the CRRA utility function of Constantinides (1986), but they find that there are no substantial differences in the generated liquidity premiums. Jang, Koo, Liu, and Loewenstein (2007) introduce a stochastic investment opportunity set into the model, and they find that this more than doubles the generated liquidity premium to 0.45% for the first 1% proportional transaction cost from the original 0.20%. The model of Lynch and Tan (2011) features three novel elements, which can all generate additional reasonable liquidity premium, but still not enough to match empirical facts. Their model features unhedgeable labor income, return predictability, and countercyclical stochastic transaction cost. Besides Lynch and Tan (2011) and the present paper, to the best of my knowledge only three other papers assume stochastic transaction cost: Driessen and Xing (2015) focus on quantifying the liquidity risk premium, while Garleanu and Pedersen (2013) and Glasserman and Xu (2013) analyze the effects of stochastic transaction cost on portfolio allocation, but due to the unconventional utility function which they use, they cannot derive conclusions regarding the level of the liquidity premium.

My paper also relates to the literature on robust dynamic asset allocation. The seminal paper introducing the penalty approach into the robust dynamic asset allocation and asset pricing literature is Anderson, Hansen, and Sargent (2003). Then Maenhout (2004) applies this approach to analyze equilibrium stock prices. In a follow-up paper, Maenhout (2006) allows for a stochastic investment opportunity set, and he finds that robustness increases the relative importance of the intertemporal hedging demand, compared to the non-robust case. Other papers analyzing different aspects of the effects of robustness on dynamic asset allocation include Branger et al. (2013), Flor and Larsen (2014), and Munk and Rubtsov (2014).

[3] In the case of quasi-fixed transaction cost, each trade incurs a transaction cost that is proportional to the investor's current wealth, regardless of the value of the trade.

This paper is organized as follows. Section 3.2 introduces my model, i.e., the financial market and the robust dynamic asset allocation problem. Section 3.2 also provides the least-favorable transaction cost, and the optimal consumption and investment policy. Section 3.3 introduces the definitions of liquidity premium and its components, and it provides the model-implied liquidity premiums. Section 3.4 concludes. Appendix A provides the numerical procedure that I used to solve the robust optimization problem. Appendix B contains the proofs of theorems and lemmas.

3.2 Robust Investment Problem

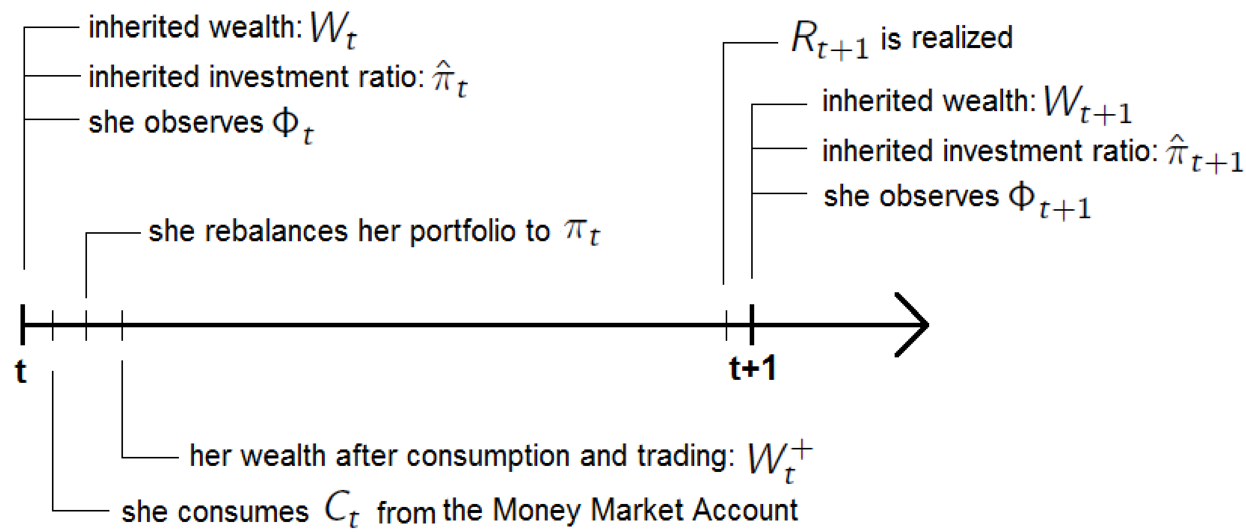
The financial market consists of a money-market account with constant continuously compounded risk-free rate r_f and a risky security (a stock) with continuously compounded return r_t, which follows an i.i.d. normal distribution with mean μ_r and variance σ_r^2. Buying and selling the money-market account is free. On the other hand, when the investor buys or sells the stock, she encounters a transaction cost, which is proportional to the traded dollar amount. This transaction cost is denoted by Φ_t, and it follows an i.i.d. log-normal distribution with parameters μ_ϕ and σ_ϕ^2. I assume that Φ_t and r_t are uncorrelated.[4]

The order of decisions is shown in Figure 3.1. At each time t the investor inherits wealth W_t. A proportion of this wealth, $\hat{\pi}_t$ is inherited in the risky asset. This proportion I call the *inherited investment ratio*. The investor first learns about the proportional transaction cost Φ_t, then she consumes C_t. I assume that she finances her consumption from the riskless asset. This assumption is common in the literature, and it is in line with economic intuition, allowing one to think about the riskless asset as a money market account, which can be directly used to buy any goods for consumption purposes.[5] After consumption, she decides about her portfolio weight π_t, and then she trades in the securities (and pays the transaction cost) to obtain this portfolio weight.

[4]Empirically, transaction costs are countercyclical, i.e., there is a negative correlation between current transaction costs and expected future returns. Driessen and Xing (2015) show that this results in a negative liquidity risk premium, however, its magnitude is very small, approximately 0.03%. Thus, assuming zero correlation between the current transaction cost and both current and future stock returns in my model does not influence my results significantly.

[5]In contrast to this assumption, Lynch and Tan (2011) assume that the investor finances her consumption by costlessly liquidating her risky and riskless assets in proportions $\hat{\pi}_t$ and $1 - \hat{\pi}_t$. They justify this by the fact that equities pay dividend, and this justification is reasonable in an infinite-horizon setup. However, in the finite-horizon setup of my model the optimal consumption ratio becomes higher and higher as the investor approaches the end of her investment horizon (and eventually becomes equal to one when $t = T$), making my assumption about financing consumption from the riskless asset more realistic.

Figure 3.1. Order of the representative investor's decisions
At time t the investor inherits wealth W_t. A proportion of this wealth, $\hat{\pi}_t$ is inherited in the risky asset. The investor learns about the proportional transaction cost Φ_t, then she consumes C_t. After consumption, she decides about her portfolio weight π_t, and then she trades in the securities (and pays the transaction cost) to obtain this portfolio weight. Then one time period elapses, and at time $t+1$ the investor observes the return R_{t+1}, and the series of decisions starts over again.



The investor's wealth process is thus

$$W_{t+1} = W_t^+ \left[R_f + \pi_t \left(R_{t+1} - R_f\right)\right], \tag{3.1}$$

where $R_f = \exp(r_f)$, $R_{t+1} = \exp(r_{t+1})$, and W_t^+ denotes the investor's wealth at time t, after she has consumed, and rebalanced her portfolio (and paid the transaction cost). I.e.,

$$W_t^+ = W_t - C_t - \Phi_t |W_t^+ \pi_t - W_t \hat{\pi}_t|. \tag{3.2}$$

Since W_t^+ appears on both sides of equation (3.2), it is convenient to express it in a form which does not contain W_t^+ on the right-hand side. I.e.,

$$W_t^+ = \frac{W_t \left(1 + I_t \Phi_t \hat{\pi}_t\right) - C_t}{1 + I_t \Phi_t \pi_t}, \tag{3.3}$$

where I_t is an indicator function, which is equal to 1 if the investor is buying additional risky assets when rebalancing her portfolio (i.e., if $W_t^+ \pi_t > W_t \hat{\pi}_t$), -1 if the investor is selling part of her risky assets when rebalancing her portfolio (i.e., if $W_t^+\pi_t < W_t\hat{\pi}_t$), and 0 if the investor is not trading to rebalance her portfolio (i.e., if $W_t^+\pi_t = W_t\hat{\pi}_t$). At time $t+1$ the investor learns about the outcome of R_{t+1}, and her investment in the risky security as a fraction of her total wealth becomes

$$\hat{\pi}_{t+1} = \frac{\pi_t R_{t+1}}{R_f + \pi_t\left(R_{t+1} - R_f\right)}. \tag{3.4}$$

This $\hat{\pi}_{t+1}$ I call the *inherited investment ratio at time t+1*.

Now I consider an investor with a finite investment horizon T. She derives utility from consumption, and she has a CRRA utility function with relative risk aversion γ. Her goal is to maximize her expected utility, but she is uncertain about the mean of the stochastic log-transaction cost process, μ_ϕ. She has a base parameter value in mind, which she considers to be the most likely. This parameter value is denoted by $\mu_\phi^{\mathbb{B}}$. But she is uncertain about the true value of μ_ϕ, so she considers other (alternative) parameter values as well. These alternative parameter values are denoted by $\mu_\phi^{\mathbb{U}}$. I formalize the relationship between $\mu_\phi^{\mathbb{B}}$ and $\mu_\phi^{\mathbb{U}}$ as

$$\mu_\phi^{\mathbb{U}} = \mu_\phi^{\mathbb{B}} + u_t, \tag{3.5}$$

where u_t is a stochastic decision variable, just as C_t and π_t are.

The investor wants to protect herself against unfavorable outcomes, so she makes robust investment decisions. Now I formalize the investor's robust optimization problem.

Problem 7. *Given $W_0 > 0$, $\hat{\pi}_0$, $0 \geq \Phi_0 \leq 1$, $\gamma > 1$, and $\delta \geq 0$, find an optimal triple $\{C_t, \pi_t, u_t\}\,\forall t \in [0, T-1]$ for the robust utility maximization problem*

$$V_0\left(W_0, \hat{\pi}_0, \Phi_0\right) = \inf_{u_t} \sup_{\{C_t, \pi_t\}} \mathrm{E}_0^{\mathbb{U}} \sum_{t=0}^{T} \left[\exp\left(-\delta t\right) \frac{C_t^{1-\gamma}}{1-\gamma} + \Upsilon_t \frac{u_t^2}{2}\right], \tag{3.6}$$

subject to the budget constraints (3.1) and (3.2), to the terminal condition $C_T = W_T\left(1 - \hat{\pi}_T\Phi_T\right)$, and where $\mathrm{E}_0^{\mathbb{U}}$ means that the expectation is calculated assuming $\mu_\phi^{\mathbb{U}}$, conditional on all information available up to time 0.

The penalty term, $\sum_{t=0}^{T} \Upsilon_t u_t^2/2$ corresponds to the Kullback-Leibler divergence from the base probability measure $\mathbb{B}$ to the alternative measure $\mathbb{U}$, where the only source of difference between the two probability measures is the Wiener term of the transaction cost process. Intuitively, if the base measure and the alternative measure are very different from each other (i.e., the alternative transaction cost drift is very different from the base transaction cost drift), the penalty term is large, while if the alternative transaction cost drift is very close to the base transaction cost drift, the penalty term is small. For the only difference between the base and the alternative measures is the drift distortion of the transaction cost process, i.e., u_t, the Kullback-Leibler divergence from $\mathbb{B}$ to $\mathbb{U}$ will be essentially quadratic in u_t.

To ensure homotheticity of the optimal consumption ratio, the optimal investment policy, and the least-favorable distortion,[6] I scale the uncertainty tolerance parameter Υ_t (following Maenhout (2004)) as

$$\Upsilon_t = \frac{(1-\gamma) V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})}{\theta}, \tag{3.7}$$

where $\theta > 0$. Substituting the parameterization of the uncertainty tolerance (3.7) into the value function (3.6), the value function becomes

$$V_0(W_0, \hat{\pi}_0, \Phi_0) = \inf_{u_t} \sup_{\{C_t, \pi_t\}} \mathrm{E}_0^{\mathbb{U}} \sum_{t=0}^{T} \left[\exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma) u_t^2 V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})}{2\theta} \right]. \tag{3.8}$$

This representation of the value function is recursive, i.e., the right-hand side of (3.8) contains future values of the value function itself. The following theorem (which I prove in Appendix B) gives a representation of the value function which is not recursive.

Theorem 5. *The solution to (3.8) with initial wealth x is given by*

$$V_0 = \inf_{u_t} \sup_{\{C_t, \pi_t\}} \mathrm{E}_0^{\mathbb{U}} \sum_{t=0}^{T} \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} \prod_{s=0}^{t-1} \left[1 + \frac{u_s^2 (1-\gamma)}{2\theta} \right] \right\}, \tag{3.9}$$

subject to the budget constraint (3.1) and (3.2), and to the terminal condition $C_T = W_T(1 - \hat{\pi}_T \Phi_T)$.

[6] Homotheticity means that the optimal consumption ratio, the optimal investment policy, and the least-favorable distortion will all be independent of the wealth level.

In line with Horvath et al. (2016), equation (3.9) shows that introducing robustness effectively increases the subjective discount rate δ. I.e., a robust investor is effectively more impatient than an otherwise equal non-robust investor. However, this is not the only place where robustness plays a role in (3.9), but it also affects the expectation operator by changing the probability density function of the transaction cost $\Phi_t \forall t \in \{1, ..., T\}$. Actually, as it is shown in Merton (1969) and in Merton (1971), the change in the subjective discount factor affects only the optimal consumption policy, but not the optimal investment policy. The effect of robustness on the optimal investment policy is due to the change from $\mathrm{E}_0^{\mathbb{B}}$ to $\mathrm{E}_0^{\mathbb{U}}$.

As I mentioned previously, the optimal consumption ratio, the optimal investment rule, and the least-favorable distortion are homothetic, i.e., they do not depend on the wealth level. This is formally stated in the following theorem, which I again prove in Appendix B.

Theorem 6. *Denoting the consumption ratio by $c_t = C_t/W_t$, the optimal solution $\{c_t^*, \pi_t^*, u_t^*\}$ to the robust optimization problem (7) with parameterization (3.7) is independent of the wealth level W_t, $\forall t \in \{0, 1, ..., T\}$. Moreover, the value function can be expressed in the form*

$$V_0\left(W_0, \hat{\pi}_0, \Phi_0\right) = \frac{W_0^{1-\gamma}}{1-\gamma} v_0\left(\hat{\pi}_0, \Phi_0\right). \tag{3.10}$$

To solve the optimization problem, I apply the principle of dynamic programming (Bellman (1957)). As a first step of this approach, I formulate the one-period optimization problem at time $T-1$ as

$$\begin{aligned} v_{T-1} = \inf_{u_{T-1}} \sup_{\{c_{T-1}, \pi_{T-1}\}} & \Big\{ \exp\left(-\delta\left(T-1\right)\right) c_{T-1}^{1-\gamma} \\ & + \exp\left(-\delta T\right) \left(1 + \frac{u_{T-1}^2\left(1-\gamma\right)}{2\theta}\right) \frac{\left(1 - c_{T-1} + I_{T-1}\Phi_{T-1}\hat{\pi}_{T-1}\right)^{1-\gamma}}{\left(1 + I_{T-1}\Phi_{T-1}\pi_{T-1}\right)^{1-\gamma}} \\ & \times \mathrm{E}_{T-1}^{\mathbb{U}}\left[\left(1 - \hat{\pi}_T\Phi_T\right)^{1-\gamma}\left(R_f + \pi_{T-1}\left(R_T - R_f\right)\right)^{1-\gamma}\right]\Big\}, \end{aligned} \tag{3.11}$$

the detailed derivation of which can be found in Appendix B, in the proof of Lemma 3. After obtaining the optimal $\{c_{T-1}, \pi_{T-1}, u_{T-1}\}$ triple, I apply backward induction to find the optimal $\{c_t, \pi_t, u_t\}$ triples for $t \in \{0, 1, ..., T-2\}$. To this end, I formulate the Bellman equation (the derivation of which can be found in Appendix B, in the proof of Theorem 6).

$$v_t(\hat{\pi}_t, \Phi_t) = \inf_{u_t} \sup_{\{c_t, \pi_t\}} \left\{ \exp(-\delta t) \, {c_t}^{1-\gamma} + \left[1 + \frac{u_t^2 (1-\gamma)}{2\theta} \right] \left(\frac{1 + I_t \Phi_t \hat{\pi}_t - c_t}{1 + I_t \Phi_t \pi_t} \right)^{1-\gamma} \right.$$
$$\left. \times \mathrm{E}_t^{\mathbb{U}} \left[(R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} \, v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}) \right] \right\}. \tag{3.12}$$

A closed form solution to the robust utility maximization Problem 7 does not exist, therefore I solve the problem numerically. Still, obtaining the optimal consumption and investment policies and the least-favorable transaction cost $\mu_\phi^{\mathbb{U}}$ parameter is computationally challenging for several reasons. First, the robust nature of the problem (i.e., the minimax setup) results in a saddle-point solution, which prevents one from making use of standard numerical optimization techniques, that are otherwise well suited for maximization and minimization problems. Second, the value function is of class C^0, since its first partial derivative with respect to the investment ratio, π_t, is not continuous due to the presence of a no-trade region. This implies that when I calculate the expected value of the first partial derivative of the value function with respect to π_t numerically, I cannot directly apply the Gaussian quadrature rule. Third, since my model contains two sources of risk (R_t and Φ_t), calculating the first partial derivative of the value function with respect to π_t involves the numerical approximation of a definite double-integral instead of the integral of a function of only one variable.

To circumvent these difficulties, I provide a semi-analytical approach: first I derive the first-order conditions on the Bellman equation (3.12) in closed form,[7] then I solve the resulting non-linear equation system numerically. To efficiently approximate the involved double integrals numerically, I first determine the boundaries of the no-trade region, then apply the Gaussian quadrature rule separately on the sell, no-trade, and buy regions, in which the first partial derivative of the value function with respect to the investment ratio is continuous. Using this approach, I can solve the robust optimization Problem 7 numerically in a feasible time.

[7]Since the first-order condition on the Bellman equation (3.12) contains the partial derivative of the value function at time $t+1$ with respect to π_t, I apply the Benveniste-Scheinkman Condition (Envelope Theorem) to transform this first-order condition into a closed-form equation. The details can be found in Appendix A.

3.2.1 Model parameterization

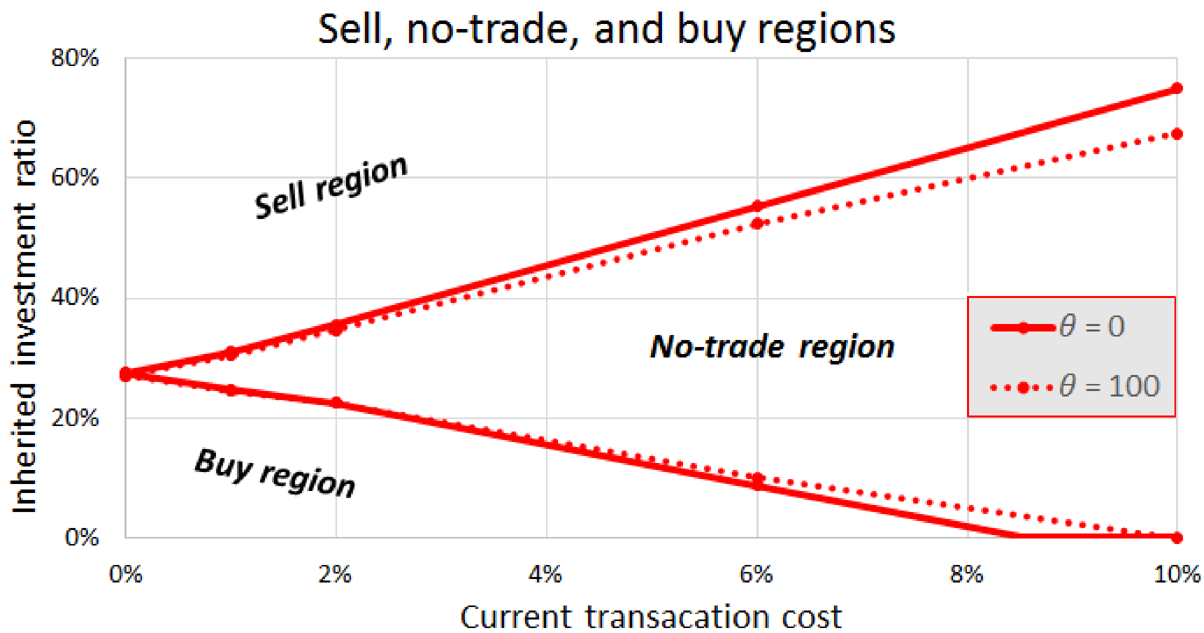
Regarding the choice of the parameter values, I follow the literature to make my findings comparable to existing results. I assume that the investment horizon is 9 years, and the discretization frequency is annual. The effective annual risk-free rate is 3%. The gross stock returns, R_t, are i.i.d. log-normally distributed with parameters $\mu_r = 8\%$ and $\sigma_r = 20\%$. Following Constantinides (1986) and the estimates of Lesmond, Ogden, and Trzcinka (1999), I assume that the expected future transaction cost is 1%. About the standard deviation of the transaction cost, I assume it to be 0.5%. In their paper, Lynch and Tan (2011) use 0.76% standard deviation and 2% expected value for the transaction cost process. Since higher standard deviation of the transaction cost produces higher negative liquidity risk premium, I use 0.50% standard deviation, which is higher than half of the value used by Lynch and Tan (2011) for 2% transaction cost. This way I underestimate the total liquidity premium, which makes my results stronger.[8] The representative investor's relative risk-aversion parameter is $\gamma = 5$, her subjective discount factor is $\delta = 5\%$, and I vary her uncertainty-aversion parameter θ between 0 and 100, 0 corresponding to a non-robust investor, and 100 corresponding to a highly robust investor.

Determining a reasonable level of robustness is an important aspect of the model. Unfortunately, the uncertainty-aversion parameter θ does not have such a universal interpretation as the relative risk-aversion parameter γ, the reasonable value of which is between 1 and 5 according to the literature. The uncertainty-aversion parameter θ is always model specific.

To still give a general measure of uncertainty aversion, the dynamic asset allocation literature provides two approaches. A statistical approach is to make an additional assumption on the Detection Error Probability of the representative investor, as in, e.g., Anderson, Hansen, and Sargent (2003). The other approach relies more on economic intuition: it suggests deriving the worst-case scenario (e.g., worst-case mean transaction cost parameter in my model) explicitly, and the difference between the base parameter and the worst-case parameter gives an economic meaning to the uncertainty-aversion level of the particular investor. This approach is also followed by Maenhout (2004). In this paper, since calculating the exact detection-error probabilities to the various

[8] I model the logarithm of the transaction cost, instead of the transaction cost itself, to ensure that the realized transaction cost values are non-negative. Although the log-normal distribution theoretically allows the transaction cost to take values that are above 100%, given my parameter choices the probability of such an event is negligible.

Figure 3.2. Sell, no-transaction, and buy zones
The investment horizon is 9 years, the discretization frequency is annual, the effective annual risk-free rate is 3%, the gross stock returns are IID and log-normally distributed with parameters $\mu_r = 8\%$ and $\sigma_r = 20\%$. The proportional transaction costs are IID and log-normally distributed with parameters $\mu_\phi^{\mathbb{B}} = -4.7167$ and $\sigma_\phi = 0.4724$, which correspond to a base expected transaction cost of 1% and standard deviation of 0.50%. The relative risk aversion is $\gamma = 5$, and the subjective discount factor is $\delta = 5\%$.



uncertainty-aversion parameter values would require computationally intensive recursive calculations, I use the second approach and provide the least-favorable expected future transaction cost levels for for three – otherwise identical – investors with different levels of uncertainty aversion: $\theta = 0$ (no uncertainty aversion), $\theta = 50$ (moderate uncertainty aversion), and $\theta = 100$ (high uncertainty aversion). These are shown – as a function of the current transaction cost – in Figure 3.3, for three different inherited investment ratios.

As Constantinides (1986) showed, the optimal investment decision is determined by a single state variable, the inherited investment ratio. If this ratio is within given boundaries (determined by the model parameters), the optimal decision is to not trade. If the ratio is above the upper boundary, the investor optimally sells a fraction of her risky assets, while if it is below the lower boundary, she buys additional risky assets. The sell, no-trade, and buy regions in my calibrated model are shown in Figure 3.2.

The solid line represents the boundaries of the no-trade region for the non-robust investor. If her inherited investment ratio is above the upper boundary, she has to sell part of her risky assets, while if it is below the lower boundary, she has to buy additional risky assets. Allowing the investor to be uncertainty-averse shrinks the no-trade region. If she is in the sell region, an uncertainty-averse investor is supposed to sell more (so that she will have to sell less later at the anticipated higher expected transaction cost). On the other hand, if she is in the buy region, an uncertainty-averse investor is supposed to buy more, so that she will have to buy less in the future at the anticipated higher expected transaction cost.

Let me emphasize that this conclusion about the shrinkage of the no-trade boundaries depends on the level of the investor's uncertainty aversion. An investor with a different uncertainty-aversion parameter might buys less (if she is in the buy region), so that she has to sell less later at the anticipated higher expected transaction cost. This is so because in the case of this other robust investor the "buy less, so that you have to sell less later" motive dominates, while in the case of the uncertainty-averse investor shown in Figure 3.2 the dominant motive is the "buy more, so that you will have to buy less later at the anticipated higher expected transaction cost".

The effect of uncertainty on the consumption policy is quantitatively negligible, less than 0.04% on average. Moreover, the optimal consumption ratio is also very stable among the different states: it is around 11.85%.

As a running example, let us consider an investor whose investment horizon is 9 years, and who inherited all of her wealth in cash, i.e., in the risk-free secuitiy. She does not know the true expected future transaction costs, and she considers 1% as her "best educated guess" (i.e., the mean of future transaction costs under her base measure is 1%). Her uncertainty aversion parameter is $\theta = 100$. At time zero she decides to consume slightly less than 12% of her wealth, and after consumption she still has her entire wealth invested in the risk-free security.

3.2.2 Least-favorable transaction cost

The least-favorable future expected transaction cost is state-dependent, i.e., it varies among different combinations of the current values of the two state variables of my model: the inherited investment ratio and the current transaction cost. The top panel of Figure 3.3 shows the investor in the running example, who inherited everything in the risk-free security. If the current transaction cost for her is 1%, she considers a future mean transaction cost of 1.2%, instead of her base 1% value. The investor in the middle panel inherited approximately 31% of her wealth in the risky asset, while the bottom panel features an investor who inherited everything in the risky asset. Regardless of the current state of the system, higher robustness always means a higher expected future transaction cost. This is intuitive: a robust investor prepares for the worst-case scenario, thus she considers a higher expected transaction cost in the future. The investor in the running example considers slightly less than 1.2% as her least-favorable expected future transaction cost.

On the other hand, it is not straightforward how the least-favorable expected future transaction cost changes if – ceteris paribus – we change one of the current state variables. If the investor has inherited everything in the riskless asset (top panel of Figure 3.3), then increasing the current transaction cost induces a higher least-favorable expected future transaction cost. The intuition behind this is that a higher current transaction cost will result in a lower optimal investment ratio, i.e., the investor will have less of her wealth invested in the risky security. Since uncertainty about the transaction cost shows up via having to sell the risky security in the future to finance consumption,[9] having a lower optimal investment ratio now means that a bad outcome will hurt the investor less. Thus, given the same level of uncertainty aversion, to prepare for the worst-case scenario, she will consider a higher future expected transaction cost.

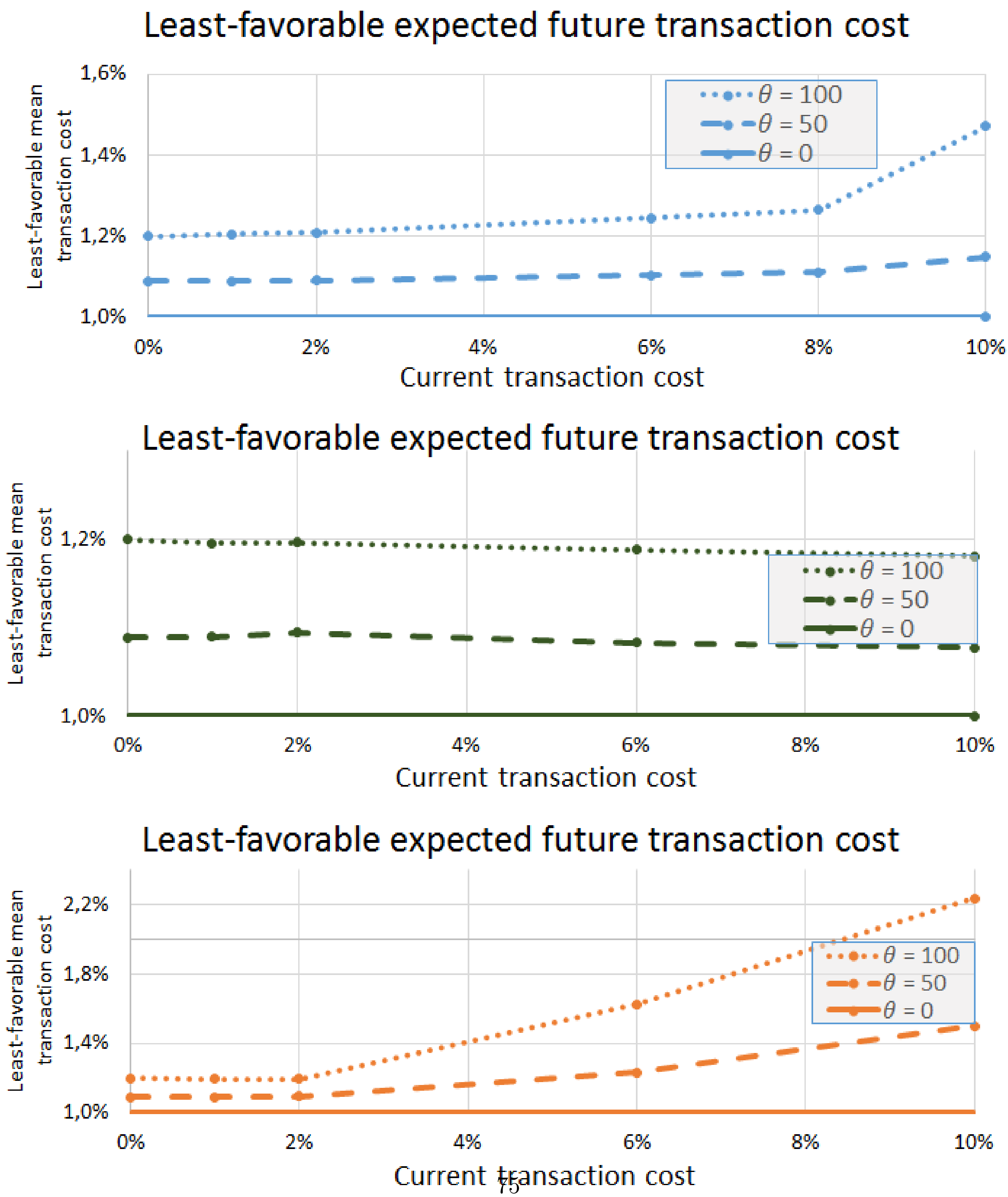
If the investor inherits part of her portfolio in the risky asset (middle panel of Figure 3.3), then the effect of increasing the current transaction cost is the opposite: a higher transaction cost induces a lower least-favorable expected future transaction cost. The intuition is that a higher current transaction cost will result in a higher optimal investment ratio (since the inherited investment ratio is above the no-transaction-cost optimum), thus the investor is exposed to more uncertainty. To compensate for this, she will consider a lower least-favorable transaction cost.

Regarding the comparative statics changing the other state variable, the inherited investment ratio, we can observe that the intercepts in Figure 3.3 are the same. This means that if the current transaction cost is zero, the least-favorable expected future transaction cost is the same, regardless of the inherited investment ratio. Moreover, if the inherited investment ratio is the same as the zero-current-transaction-cost optimal investment ratio (post-consumption), then regardless of the

[9] There is a second effect of a higher current transaction cost, which has the opposite direction: since the investor buys less risky asset now, she will have to buy more in the future, and if she considers a higher expected future transaction cost now, this additional demand will decrease her value function now. This effect is, however, of second order, and quantitatively it is dominated by the effect described in the main text.

Figure 3.3. Least-favorable expected future transaction costs, for different inherited investment ratios.

The investment horizon is 9 years, the discretization frequency is annual, the effective annual risk-free rate is 3%, the gross stock returns are IID and log-normally distributed with parameters $\mu_r = 8\%$ and $\sigma_r = 20\%$. The proportional transaction costs are IID and log-normally distributed with parameters $\mu_\phi^{\mathbb{B}} = -4.7167$ and $\sigma_\phi = 0.4724$, which correspond to a base expected transaction cost of 1% and standard deviation of 0.50%. The relative risk aversion is $\gamma = 5$, and the subjective discount factor is $\delta = 5\%$. The inherited investment ratios in the three graphs are 0%, 31%, and 100%, respectively.



current transaction cost, the investor does not have to trade, and the least-favorable transaction cost will be constant at the same level as the intercepts in Figure 3.3. This is intuitive, since in this particular case the current transaction cost does not have any effect on the probabilities of the future states of the system. If the inherited investment ratio is between zero and the zero-current-transaction-cost optimal investment ratio, then the least-favorable transaction cost will be an increasing function of the current transaction cost, and the higher the inherited investment ratio, the lower the slope of this function. The same is true if the inherited investment ratio is between the zero-current-transaction-cost optimal investment ratio and one: the least-favorable expected future transaction cost is an increasing function of the current transaction cost, and the higher the inherited investment ratio, the higher the slope of this function.

3.2.3 Optimal investment policy

Similarly to the least-favorable expected future transaction cost, the optimal investment ratio is also state-dependent, as it is shown in Figure 3.4. If the investor inherits everything in the risk-free asset, and she is non-robust, then she will buy the risky security to have an investment ratio of 31.21%. This is the intercept of the solid line in the top panel of Figure 3.4. If the current transaction cost is higher, the non-robust investor buys less of the risky asset, and at a current transaction cost of just above 8% she will not trade at all, but leave her entire wealth invested in the riskless asset.

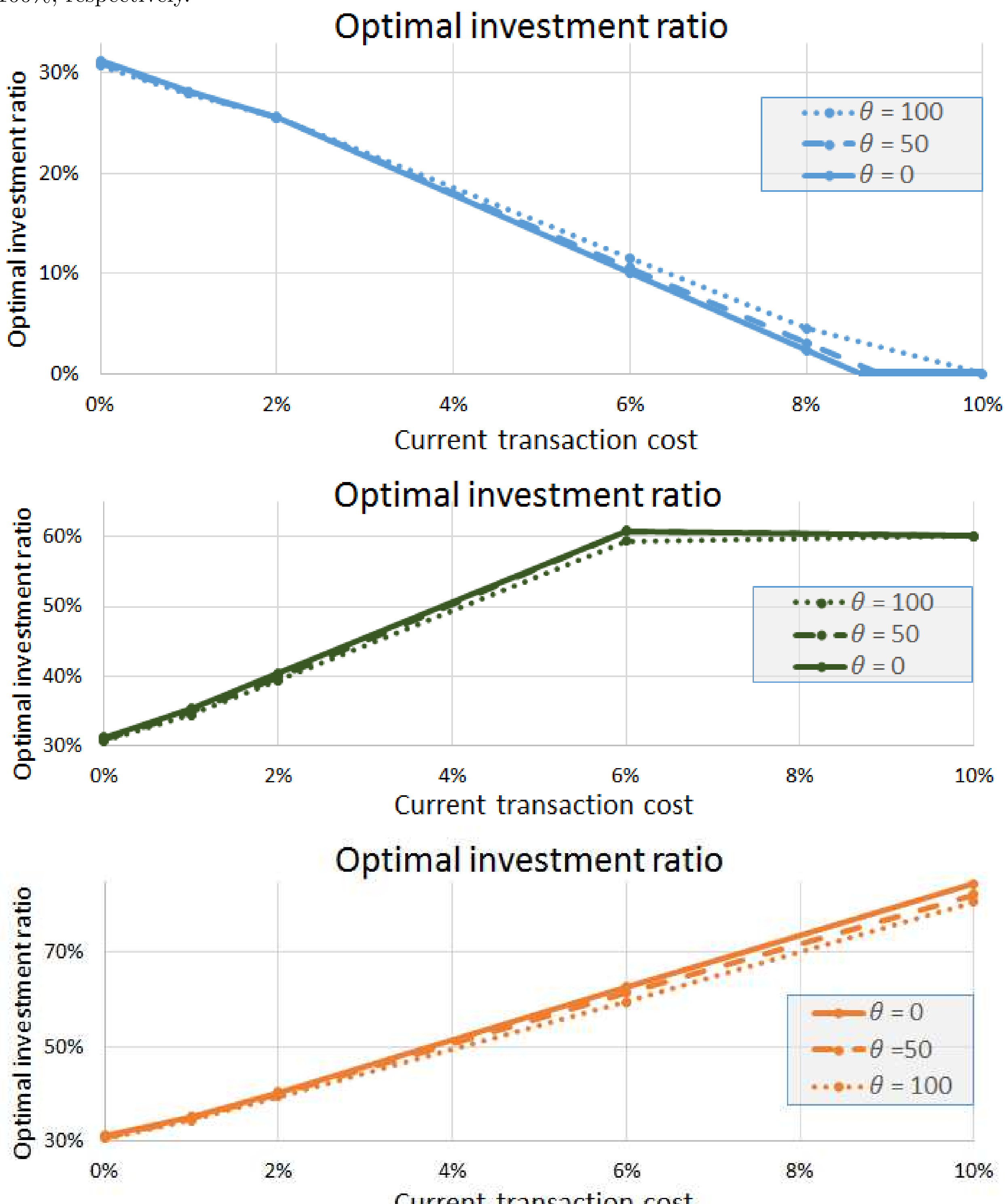
If the investor is uncertainty-averse, she will anticipate a higher expected transaction cost than a non-robust investor (see Figure 3.3). In the case of such an uncertainty-averse investor, there are two effects working in the opposite directions. On the one hand, the investor wants to buy less risky asset than her non-robust counterpart, so that later she will have to sell less risky asset at the higher anticipated expected transaction cost. Instead, she will hold more of her wealth now in the riskless asset, which later she can use for consumption purposes without having to pay transaction cost to sell it. On the other hand, she wants to buy more risky asset now at the known transaction cost, so that later she will have to buy less risky asset at the higher anticipated expected transaction cost. As we can see in the top panel of Figure 3.4, in the case of an uncertainty-averse investor who inherited everything in the risk-free asset the first effect is the dominant: she will buy less risky security now than her non-robust counterpart. Given the 1% current transaction cost, the optimal investment ratio for the investor in the running example is around 27%.

If the investor inherits everything in the risky asset (bottom panel of Figure 3.4), she will have to sell part of her portfolio to achieve the optimal investment ratio. If the investor is non-robust, the zero-current-transaction-cost optimal investment ratio for her is 31.21% – the same is for the non-robust investor who inherited everything in the risk-free asset. This is intuitive: if the current transaction cost is zero, then the investor can rebalance her portfolio for free, hence the inherited portfolio allocation does not matter for her. But if the current transaction cost is non-zero, the optimal investment ratio becomes higher – i.e., she will sell less of her risky asset to save on the transaction cost. And as the current transaction cost is higher and higher, she will sell less and less, until she achieves the point where she will not trade any more and rather keep all of her wealth in the risky asset.

If this investor is uncertainty-averse, then again there will be two effects working in the opposite directions. On the one hand, the robust investor wants to sell more of the risky security so that she will have to sell less later at the higher anticipated expected transaction cost. On the other hand, she wants to sell less of the risky security to avoid having to buy additional risky security in the future for rebalance purposes. In the bottom panel of Figure 3.4, we can see that regardless of the current transaction cost, the second effect will be the dominant one, i.e., a robust investor will always sell more of the risky security now than a non-robust investor. This is due to the fact that her zero-current-transaction-cost optimal portfolio is very different from her inherited portfolio. If these two portfolios were more similar in terms of asset allocation to each other – as it is the case in the top panel of Figure 3.4 –, then the first effect might be the dominant. But even now we can observe the fact that we encountered in the case of the investor who inherited everything in the risk-free asset that as the investor becomes more and more uncertainty-averse, the dominance of the second effect is less and less strong, especially if the current transaction cost is not too high (less than 2%).

If the investor inherited part of her wealth in the risky security and the other part invested in the money market account, a higher current transaction cost will reduce her trading. This is shown in the middle panel of Figure 3.4, where the investor inherited 31% of her wealth in the risky asset and 69% in the risk-free asset. First she consumes part of her wealth from the money market account, and after consumption she has about 60% of her wealth invested in the risky asset (since

Figure 3.4. Optimal investment ratio, for different inherited investment ratios. The investment horizon is 9 years, the discretization frequency is annual, the effective annual risk-free rate is 3%, the gross stock returns are IID and log-normally distributed with parameters $\mu_r = 8\%$ and $\sigma_r = 20\%$. The proportional transaction costs are IID and log-normally distributed with parameters $\mu_\phi^{\mathbb{B}} = -4.7167$ and $\sigma_\phi = 0.4724$, which correspond to a base expected transaction cost of 1% and standard deviation of 0.50%. The relative risk aversion is $\gamma = 5$, and the subjective discount factor is $\delta = 5\%$. The inherited investment ratios in the three graphs are 0%, 31%, and 100%, respectively.



she consumed part of her risk-free assets). If the transaction cost is zero, she will sell almost half of her risky assets to achieve the optimal 31.25% investment ratio. If the current transaction cost is higher than 0%, she will sell less, and if the current transaction cost is 6% or higher, the optimal decision for her is to not trade, hence the optimal investment ratio will be around 60%. The effect of robustness is also in accordance with the previous two cases: if the investor is uncertainty-averse, she will trade more than an otherwise identical, but non-uncertainty-averse investor.

3.3 Liquidity Premium

After showing the effects of uncertainty about future transaction cost on asset allocation in Section 3.2, now I go one step further and I analyze the effects of uncertainty about future transaction costs on asset pricing. To be more precise, I show that my model with a robust representative investor generates an additional liquidity premium of 0.05%-0.10% for the first 1% proportional transaction cost, depending on the level of uncertainty aversion of the investor.

To separate the effects of the level of the transaction cost, the volatility of the transaction cost, and the parameter uncertainty about the transaction cost process, I introduce the definition of the *liquidity-uncertainty premium*, the *liquidity-risk premium*, and the *liquidity-level premium.*[10]

Definition 3. *Let us consider a representative agent with uncertainty-aversion parameter θ solving the robust optimization Problem 7, and obtaining V_0. If we impose the restriction $u_t = 0 \; \forall \; t \in \{0, 1, ..., T-1\}$, the continuously compounded expected return on the risky security, μ_r, has to be decreased to $\mu_r^{\theta=0}$ so that the investor achieves the same level of value V_0 as without this restriction. The difference $\mu_r - \mu_r^{\theta=0}$ is called the liquidity-uncertainty premium.*

Definition 4. *Let us consider the representative investor in Definition 3. If we impose the restriction $u_t = 0 \; \forall \; t \in \{0, 1, ..., T-1\}$, and we change the volatility parameter of the transaction cost process from σ_ϕ to 0, the continuously compounded expected return on the risky security, μ_r, has to be decreased to $\mu_r^{\theta=0,\sigma_\phi=0}$ so that the investor achieves the same level of value V_0 as originally. The difference $\mu_r^{\theta=0} - \mu_r^{\theta=0,\sigma_\phi=0}$ is called the liquidity-risk premium.*

[10] My liquidity premium definitions are in line with the majority of the literature, though some researchers use alternative definitions, e.g., Liu (2004).

Definition 5. *Let us consider the representative investor in Definition 3. If we impose the restriction that* $u_t = 0 \ \forall \ t \in \{0, 1, ..., T-1\}$*, we change the volatility parameter of the transaction cost process from* σ_ϕ *to* 0*, and we also change the base mean parameter of the transaction cost from* $\mu_\phi^{\mathbb{B}}$ *to* $\tilde{\mu}_\phi^{\mathbb{B}}$ *so that* $\mathrm{E}_t^{\mathbb{B}}\{\Phi_{t+1}\} = 0 \ \forall \ t \in \{0, 1, ..., T-1\}$*, the continuously compounded expected return on the risky security,* μ_r*, has to be decreased to* $\mu_r^{\theta=0,\sigma_\phi=0,\mu_\Phi=0}$ *so that the investor achieves the same level of value* V_0 *as originally. The difference* $\mu_r^{\theta=0,\sigma_\phi=0} - \mu_r^{\theta=0,\sigma_\phi=0,\mu_\Phi=0}$ *is called the liquidity-level premium.*

Using Definitions 3-5 and the parameter values described in Section 3.2, my model generates the liquidity-level premium, liquidity-risk premium, and liquidity-uncertainty premium values described in Table 3.1, and also shown in Figure 3.5 and Figure 3.6.

Table 3.1. Model implied liquidity premiums
Model implied liquidity-level premiums, liquidity-risk premiums, and liquidity-uncertainty premiums for different levels of uncertainty aversion. The investment horizon is T=10 years, the discretization is annual. The continuously compounded expected stock return is $\mu_r = 8\%$, the volatility of the return is $\sigma_r = 20\%$, the risk-free rate is 3%, the base transaction cost mean parameter is $\mu_\phi^{\mathbb{B}} = 1\%$, the transaction cost volatility parameter is $\sigma_\phi = 0.5\%$. The investor's risk aversion parameter is $\gamma = 5$, and her subjective discount rate is $\delta = 5\%$.

	$\theta = 0$	$\theta = 50$	$\theta = 100$
Liquidity uncertainty premium	0.00%	0.05%	0.10%
Liquidity risk premium	−0.03%	−0.02%	−0.04%
Liquidity level premium	0.83%	0.71%	0.72%
Total liquidity premium	0.80%	0.74%	0.78%

In the non-robust version of my model, the total liquidity premium generated is 0.80% for a proportional liquidity premium with expected value of 1% and standard deviation of 0.50%. The majority of this premium is due to the level of the transaction cost: the liquidity level premium is 0.83%. The liquidity-risk premium is negative, it is -0.03%. At first this might seem counterintuitive, but it is actually very logical. A rigorous demonstration of why the liquidity premium is negative if the correlation between the stock return and the transaction cost is zero can be found in Driessen and Xing (2015). The intuition is that the stochastic nature of the transaction cost gives additional opportunity to the investor to choose when to rebalance her portfolio. If market conditions are bad (i.e., the transaction cost is high), she can postpone rebalancing, while if market conditions are good (i.e., the transaction cost is low), she can trade more. Driessen and Xing (2015) call this the *Choice Effect*.

Introducing a moderate level of robustness decreases the total liquidity premium by 0.06%. Though the liquidity-level premium decreased, a liquidity-uncertainty premium showed up, which is 0.05%. The decrease in the liquidity level premium is due to the definition of the three components of the total liquidity premium: instead of taking $\mu_r = 8\%$ as the baseline non-robust value and increasing it when introducing robustness to calculate the liquidity-uncertainty premium, I take $\mu_r = 8\%$ as the robust baseline value. I do this because the μ_r value is observed/estimated independently of the model specification, and if the representative investor is assumed to be robust, then this representative investor generates $\mu_r = 8\%$ expected return, that we can observe on the market, and not a return which is higher than this by the liquidity uncertainty premium. Thus, the main message of the third column of Table 3.1 is: even a moderate level of uncertainty aversion can generate a liquidity uncertainty premium of 0.05%, which is 25% of the total liquidity premium generated in Constantinides (1986). Increasing the uncertainty aversion to a higher level, will produce a liquidity-uncertainty premium of 0.10%.

3.4 Conclusion

I have shown that uncertainty-aversion can explain a reasonable portion of the liquidity-premium puzzle. With a moderate level of uncertainty aversion my model can generate an additional 0.05% liquidity premium, which is even higher if I allow for a higher level of uncertainty aversion. I have also shown that uncertainty-aversion has two different channels through which it affects the optimal investment behavior, and the total effect depends on the current transaction cost and the inherited investment ratio. Besides these two economic contributions, I provided two technical contributions to efficiently solve robust dynamic asset allocation problems involving transaction cost.

When I introduced uncertainty aversion into my model, I relied on the magnitude of the least-favorable transaction cost values to assess the level of uncertainty aversion. This is an intuitive and appealing approach, however, more rigorous definitions of the level of uncertainty aversion, e.g., by using simulation-based detection error probabilities can be a fruitful line of further research.

Moreover, developing a continuous-time version of the robust model and using the relative entropy as a penalty term might provide opportunities for intuitive interpretation of the uncertainty-aversion parameter in the context of ambiguity about the future transaction costs, and it can also

Figure 3.5. Decomposition of the liquidity premium.
The three components of the liquidity premium: the liquidity-level premium, the (negative) liquidity-risk premium, and the liquidity-uncertainty premium.

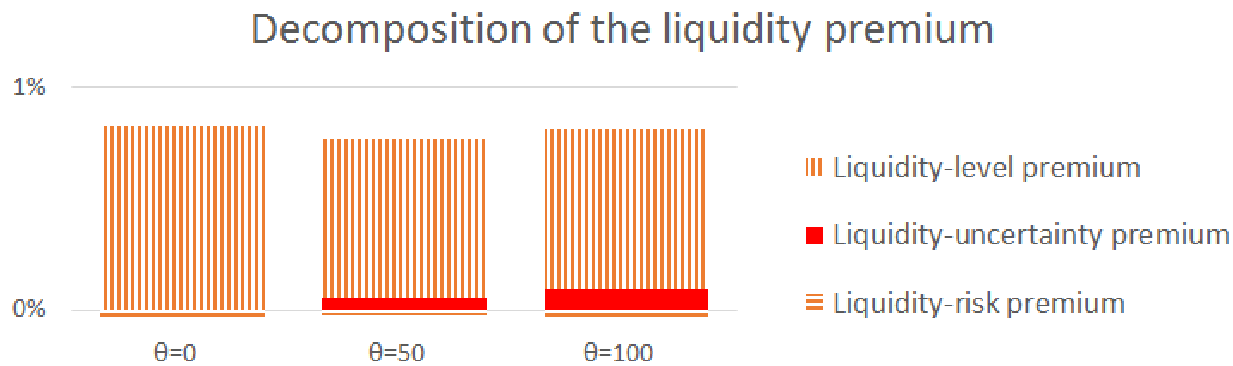
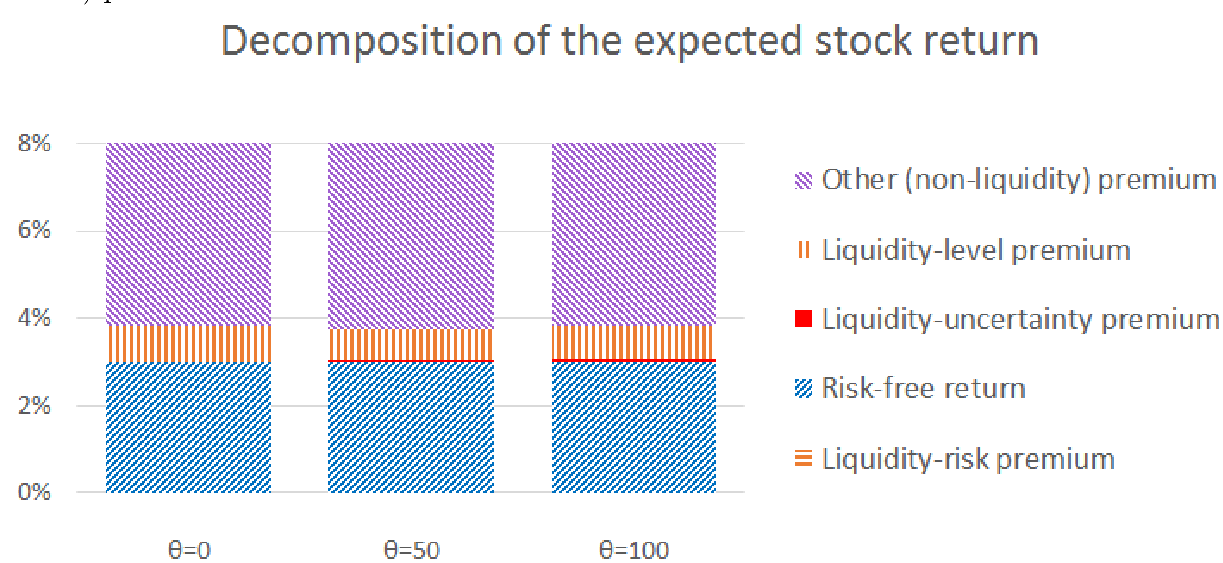


Figure 3.6. Decomposition of the expected stock return.
Decomposition of the expected stock return into the risk-free return, the (negative) liquidity-risk premium, the liquidity-uncertainty premium, the liquidity-level premium, and other (non-liquidity-related) premiums.



give additional insight into the working mechanism of the two channels through which robustness affects the investor's decision. This is also a promising area of future research.

Appendix A

Solving the robust dynamic optimization Problem 7 numerically is computationally challenging for several reasons. First, due to the minimax setup of the problem, the solution will be a saddle point, thus I cannot use the standard numerical optimization techniques which can be used to efficiently solve maximization and minimization problems. Second, the value function is of class C^0, i.e., its partial derivative with respect to the investment ratio is non-continuous due to the presence of the no-trade region. Third, my model contains two sources of uncertainty (both the stock return and the transaction cost are stochastic), thus calculating the expected value for the value function (and its partial derivatives) involves numerically approximating a double integral, where the function to be integrated is not necessarily continuous.

To solve the problem within a reasonable time, I take several measures. First, I show in Theorem 6 that the optimal consumption ratio, the optimal investment ratio and the least-favorable distortion at time t do not depend on the wealth level at time t. Since the wealth level is one of the state variables, this theorem substantially reduces the required computational time to solve the optimization problem. Second, I develop a novel technique to solve discrete-time robust dynamic asset allocation problems with transaction cost. This involves rewriting Problem 7 in a non-recursive formulation. I provide this reformulation in Theorem 5. Then I write down the first-order conditions in closed form, making use of the Benveniste-Scheinkman Condition (Envelope Theorem). This results in a non-linear equation system of three equations, with three variables. The equations themselves are the first partial derivatives of the value function, thus some of them are non-continuous. To evaluate the double expectations (which are equivalent to numerically evaluating double integrals), I explicitly determine the non-continuity lines, then I apply the Gaussian quadrature rule to calculate the numerical integral of the function part-wise. Moreover, I also parallelize the solution algorithm to distribute the computational workload among several working units.

The steps of the solution procedure are as follows.

1. First, I create grids for possible $\hat{\pi}_{T-1}$ and Φ_{T-1} values. The grids for $\hat{\pi}_{T-1}$ lie in $\{0, 0.5, 1\}$, while the grids for Φ_{T-1} lie in $\{0, 0.01, 0.02, 0.06, 0.1\}$. I obtain the optimal investment ratio

π_{T-1}, the optimal consumption ratio c_{T-1}, and the least-favorable transaction cost parameter u_{T-1} for each $\{\hat{\pi}_{T-1}, \Phi_{T-1}\}$ pairs by numerically solving the first order conditions with respect to c_{T-1}, π_{T-1}, and u_{T-1}. These first-order conditions are (3.42), (3.43), and (3.44).

2. Now I go one period backwards. Since the solution procedure will be the same for all $t \in \{1, 2, ..., T-2\}$, instead of $T-2$ I use the more general time period t in this step, keeping in mind that immediately after the above step I have $t = T-2$. Just as in the previous step, I create grids for possible $\hat{\pi}_t$ and Φ_t values. The grids for $\hat{\pi}_t$ lie in $\{0, 0.5, 1\}$, while the grids for Φ_t lie in $\{0, 0.01, 0.02, 0.06, 0.1\}$. The first order condition on the Bellman equation (3.12) with respect to c_t is

$$c_t = \left\{ \left[\exp(\delta t) \left[1 + \frac{u_t^2 (1-\gamma)}{2\theta} \right] \frac{(1 + I_t \Phi_t \hat{\pi}_t)^{-\gamma}}{(1 + I_t \Phi_t \pi_t)^{1-\gamma}} \right. \right.$$
$$\left. \left. \times \mathrm{E}_t^{u_t} \left[(R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1} (\hat{\pi}_{t+1}, \Phi_{t+1}) \right] \right]^{\frac{1}{\gamma}} + (1 + I_t \Phi_t \hat{\pi}_t)^{-1} \right\}^{-1}, \quad (3.13)$$

the first order condition on the Bellman equation (3.12) with respect to π_t is

$$\mathrm{E}_t^{u_t} \left\{ (1-\gamma) v_{t+1} (\hat{\pi}_{t+1}, \Phi_{t+1}) (R_f + \pi_t (R_{t+1} - R_f))^{-\gamma} \right.$$
$$\times \left[R_{t+1} - R_f - (R_f + \pi_t (R_{t+1} - R_f)) \frac{I_t \Phi_t}{1 + I_t \Phi_t \pi_t} \right]$$
$$\left. + R_f R_{t+1} (R_f + \pi_t (R_{t+1} - R_f))^{-1-\gamma} \frac{\partial v_{t+1} (\hat{\pi}_{t+1}, \Phi_{t+1})}{\partial \hat{\pi}_{t+1}} \right\} = 0, \quad (3.14)$$

and the first order condition on the Bellman equation (3.12) with respect to u_t is

$$\mathrm{E}_t^{u_t} \left\{ (R_f + \pi_t (R_{t+1} - R_f))^{1-\gamma} v_{t+1} (\hat{\pi}_{t+1}, \Phi_{t+1}) \right.$$
$$\left. \times \left[\left(1 + \frac{u_t^2 (1-\gamma)}{2\theta} \right) \frac{\phi_{t+1} - \mu_\phi - u_t}{\sigma_\phi^2} + \frac{u_t (1-\gamma)}{\theta} \right] \right\} = 0. \quad (3.15)$$

Condition (3.14) contains $\partial v_{t+1} (\hat{\pi}_{t+1}, \Phi_{t+1}) / \partial \hat{\pi}_{t+1}$, which not only makes the evaluation of the expected value in (3.14) computationally intensive, but it also decreases the numerical accuracy of the obtained results. To circumvent this, I make use of the Benveniste-Scheinkman

Condition (Envelope Theorem). Let us introduce the function

$$\begin{aligned}\bar{v}_t\left(c_t,\pi_t,u_t,\hat{\pi}_t,\Phi_t\right) = &\left(1+\frac{u_t^2\left(1-\gamma\right)}{2\theta}\right)\left(\frac{1+I_t\Phi_t\hat{\pi}_t-c_t}{1+I_t\Phi_t\pi_t}\right)^{1-\gamma}\\ &\times \mathrm{E}_t^{u_t}\left[\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)^{1-\gamma}v_{t+1}\left(\hat{\pi}_{t+1},\Phi_{t+1}\right)\right],\end{aligned} \tag{3.16}$$

the partial derivative of which with respect to $\hat{\pi}_t$ is

$$\begin{aligned}\frac{\partial\bar{v}_t\left(c_t,\pi_t,u_t,\hat{\pi}_t,\Phi_t\right)}{\partial\hat{\pi}_t} = &\left(1+\frac{u_t^2\left(1-\gamma\right)}{2\theta}\right)\left(1-\gamma\right)\left(\frac{1+I_t\Phi_t\hat{\pi}_t-c_t}{1+I_t\Phi_t\pi_t}\right)^{-\gamma}\\ \times\frac{I_t\Phi_t}{1+I_t\Phi_t\pi_t}&\mathrm{E}_t^{u_t}\left[\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)^{1-\gamma}v_{t+1}\left(\hat{\pi}_{t+1},\Phi_{t+1}\right)\right].\end{aligned} \tag{3.17}$$

Substituting (3.16) into (3.12), the Bellman equation becomes

$$v_t\left(\hat{\pi}_t,\Phi_t\right) = \inf_{u_t}\sup_{\{c_t,\pi_t\}}\left\{\exp\left(-\delta t\right)c_t^{1-\gamma}+\bar{v}_t\left(c_t,\pi_t,u_t,\hat{\pi}_t,\Phi_t\right)\right\}. \tag{3.18}$$

Then the following theorem (which I prove in Appendix B) holds.

Theorem 7 (Benveniste-Scheinkman Condition (Envelope Theorem))**.** *If* $c_t = c_t^*$*,* $\pi_t = \pi_t^*$*, and* $u_t = u_t^*$*, then*

$$\frac{\partial v_t\left(\hat{\pi}_t,\Phi_t\right)}{\partial\hat{\pi}_t} = \frac{\partial\bar{v}_t\left(c_t^*,\pi_t^*,u_t^*,\hat{\pi}_t,\Phi_t\right)}{\partial\hat{\pi}_t}. \tag{3.19}$$

Using Theorem 7 and equation (3.17), I rewrite the first order condition with respect to π_t, i.e., (3.14), as

$$\begin{aligned}0 =&\mathrm{E}_t^{u_t}\left\{\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)^{-\gamma}v_{t+1}\left(\hat{\pi}_{t+1},\Phi_{t+1}\right)\right.\\ &\times\left[R_{t+1}-R_f-\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)\frac{I_t\Phi_t}{1+I_t\Phi_t\pi_t}\right]\\ &+R_fR_{t+1}\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)^{-1-\gamma}\left(1+\frac{\left(u_{t+1}^*\right)^2\left(1-\gamma\right)}{2\theta}\right)\\ &\times\frac{\left(1+I_{t+1}\Phi_{t+1}\hat{\pi}_{t+1}-c_{t+1}^*\right)^{-\gamma}}{\left(1+I_{t+1}\Phi_{t+1}\pi_{t+1}^*\right)^{1-\gamma}}I_{t+1}\Phi_{t+1}\\ &\left.\times\mathrm{E}_{t+1}^{u_{t+1}^*}\left[\left(R_f+\pi_{t+1}^*\left(R_{t+2}-R_f\right)\right)^{1-\gamma}v_{t+2}\left(\hat{\pi}_{t+2},\Phi_{t+2}\right)\right]\right\}.\end{aligned} \tag{3.20}$$

Appendix B

Proof of Theorem 5. I will prove the theorem using backward induction. Throughout the proof, I assume that the choice variables, i.e., C_t, π_t, and u_t are always optimally chosen, thus I omit the inf and sup operators. Moreover, since it will not cause any confusion, instead of $V_t\left(W_t, \hat{\pi}_t, \Phi_t\right)$ I simply write V_t. At time T the value function is

$$\begin{aligned} V_T &= \exp\left(-\delta T\right) \frac{C_T^{1-\gamma}}{1-\gamma} + \frac{u_T^2\left(1-\gamma\right)}{2\theta} V_{T+1} \\ &= \exp\left(-\delta T\right) \frac{C_T^{1-\gamma}}{1-\gamma}, \end{aligned} \tag{3.21}$$

since $V_{T+1} = 0$. Going one step backwards, the value function at time $T-1$ is by definition

$$\begin{aligned} V_{T-1} &= \left\{ \exp\left\{-\delta\left(T-1\right)\right\} \frac{C_{T-1}^{1-\gamma}}{1-\gamma} + \frac{u_{T-1}^2\left(1-\gamma\right)}{2\theta} V_T + \exp\left(-\delta T\right) \frac{C_T^{1-\gamma}}{1-\gamma} \right\} \\ &= \mathrm{E}_{T-1} \left\{ \exp\left\{-\delta\left(T-1\right)\right\} \frac{C_{T-1}^{1-\gamma}}{1-\gamma} + \exp\left(-\delta T\right) \frac{C_T^{1-\gamma}}{1-\gamma} \left(1 + \frac{u_{T-1}^2\left(1-\gamma\right)}{2\theta}\right) \right\} \\ &= \mathrm{E}_{T-1} \sum_{t=T-1}^{T} \left\{ \exp\left(-\delta t\right) \frac{C_t^{1-\gamma}}{1-\gamma} \prod_{s=T-1}^{t-1} \left[1 + \frac{u_s^2\left(1-\gamma\right)}{2\theta}\right] \right\}. \end{aligned} \tag{3.22}$$

Going one more step backwards, I can write down the value function V_{T-2} in the same way:

$$\begin{aligned} V_{T-2} =& E_{T-2} \left\{ \exp\left(-\delta\left(T-2\right)\right) \frac{C_{T-2}^{1-\gamma}}{1-\gamma} + \exp\left(-\delta\left(T-1\right)\right) \frac{C_{T-1}^{1-\gamma}}{1-\gamma} \left(1 + \frac{u_{T-2}^2\left(1-\gamma\right)}{2\theta}\right) \right. \\ & \left. + \exp\left(-\delta T\right) \frac{C_T^{1-\gamma}}{1-\gamma} \left(1 + \frac{u_{T-1}^2\left(1-\gamma\right)}{2\theta}\right) \left(1 + \frac{u_{T-2}^2\left(1-\gamma\right)}{2\theta}\right) \right\} \\ =& \mathrm{E}_{T-2} \sum_{t=T-2}^{T} \left\{ \exp\left(-\delta t\right) \frac{C_t^{1-\gamma}}{1-\gamma} \prod_{s=T-2}^{t-1} \left[1 + \frac{u_s^2\left(1-\gamma\right)}{2\theta}\right] \right\}. \end{aligned} \tag{3.23}$$

Progressing backwards in the same way, I can write the value function at time t as

$$V_t = \mathrm{E}_t \sum_{s=t}^{T} \left\{ \exp\left(-\delta s\right) \frac{C_s^{1-\gamma}}{1-\gamma} \prod_{m=t}^{s-1} \left[1 + \frac{u_m^2\left(1-\gamma\right)}{2\theta}\right] \right\}, \tag{3.24}$$

and the value function at time 0 as

$$V_0 = \mathrm{E}_0 \sum_{t=0}^{T} \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} \prod_{s=0}^{t-1} \left[1 + \frac{u_s^2 (1-\gamma)}{2\theta} \right] \right\}. \tag{3.25}$$

This is the same as in Theorem 5, and it completes the proof. □

Proof of Theorem 6. I will prove the theorem using mathematical induction. Following Theorem 5, the value function at time t is

$$V_t (W_t, \hat{\pi}_t, \Phi_t) = \inf_{u_s} \sup_{\{C_s, \pi_s\}} \mathrm{E}_t \sum_{s=t}^{T} \left\{ \exp(-\delta s) \frac{C_s^{1-\gamma}}{1-\gamma} \prod_{m=t}^{s-1} \left[1 + \frac{u_m^2 (1-\gamma)}{2\theta} \right] \right\}. \tag{3.26}$$

From (3.26) follows that the Bellman equation is[11]

$$\begin{aligned} V_t (W_t, \hat{\pi}_t, \Phi_t) = \inf_{u_t} \sup_{\{C_t, \pi_t\}} & \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} + \left[1 + \frac{u_t^2 (1-\gamma)}{2\theta} \right] \right. \\ & \left. \times \mathrm{E}_t^{u_t} \left[V_{t+1} (W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1}) \right] \right\}, \end{aligned} \tag{3.27}$$

where $\mathrm{E}_t^{u_t}$ means that the expectation is taken assuming μ_ϕ^U.

The first order conditions on the right-hand side of the Bellman equation in (3.27) with respect to C_t, π_t, and u_t are

$$\begin{aligned} 0 &= \exp(-\delta t) C_t^{-\gamma} \\ &\quad + \left[1 + \frac{u_t^2 (1-\gamma)}{2\theta} \right] \mathrm{E}_t^{u_t} \left[\frac{\partial V_{t+1} (W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})}{\partial C_t} \right], \end{aligned} \tag{3.28}$$

$$0 = \mathrm{E}_t^{u_t} \left[\frac{\partial V_{t+1} (W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})}{\partial \pi_t} \right], \tag{3.29}$$

and

$$\begin{aligned} & -\frac{u_t (1-\gamma)}{\theta} \mathrm{E}_t^{u_t} \left[V_{t+1} (W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1}) \right] \\ & = \left[1 + \frac{u_t^2 (1-\gamma)}{2\theta} \right] \mathrm{E}_t^{u_t} \left\{ \frac{\phi_{t+1} - \mu_\phi - u_t}{\sigma_\phi^2} V_{t+1} (W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1}) \right\}, \end{aligned} \tag{3.30}$$

respectively, where $\phi_{t+1} = \log(\Phi_{t+1})$.

[11] The Bellman equation can be also directly derived from the definition of the value function by rewriting (3.24).

Now let us assume (I will prove this in Lemma 3) that the value function $V_{t+1}\left(W_{t+1},\hat{\pi}_{t+1},\Phi_{t+1}\right)$ can be expressed in the form

$$V_{t+1}\left(W_{t+1},\hat{\pi}_{t+1},\Phi_{t+1}\right) = \frac{W_{t+1}^{1-\gamma}}{1-\gamma}v_{t+1}\left(\hat{\pi}_{t+1},\Phi_{t+1}\right). \tag{3.31}$$

Then the first-order condition with respect to C_t, i.e., (3.28) becomes

$$\begin{aligned} c_t = \Bigg\{ \Bigg[& \exp\left(\delta t\right)\left[1+\frac{u_t^2\left(1-\gamma\right)}{2\theta}\right]\frac{\left(1+I_t\Phi_t\hat{\pi}_t\right)^{-\gamma}}{\left(1+I_t\Phi_t\pi_t\right)^{1-\gamma}} \\ & \times \mathrm{E}_t^{u_t}\Big[\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)^{1-\gamma}v_{t+1}\left(\hat{\pi}_{t+1},\Phi_{t+1}\right)\Big]\Bigg]^{\frac{1}{\gamma}} + \left(1+I_t\Phi_t\hat{\pi}_t\right)^{-1}\Bigg\}^{-1}, \end{aligned} \tag{3.32}$$

the first-order condition with respect to π_t, i.e., (3.29) becomes

$$\begin{aligned} & \mathrm{E}_t^{u_t}\Big\{\left(1-\gamma\right)v_{t+1}\left(\hat{\pi}_{t+1},\Phi_{t+1}\right)\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)^{-\gamma} \\ & \quad \times \left[R_{t+1}-R_f-\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)\frac{I_t\Phi_t}{1+I_t\Phi_t\pi_t}\right] \\ & \quad + R_f R_{t+1}\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)^{-1-\gamma}\frac{\partial v_{t+1}\left(\hat{\pi}_{t+1},\Phi_{t+1}\right)}{\partial\hat{\pi}_{t+1}}\Big\} = 0, \end{aligned} \tag{3.33}$$

and the first-order condition with respect to u_t, i.e., (3.30) becomes

$$\begin{aligned} & \mathrm{E}_t^{u_t}\Big\{\left(R_f+\pi_t\left(R_{t+1}-R_f\right)\right)^{1-\gamma}v_{t+1}\left(\hat{\pi}_{t+1},\Phi_{t+1}\right) \\ & \quad \times\left[\left(1+\frac{u_t^2\left(1-\gamma\right)}{2\theta}\right)\frac{\phi_{t+1}-\mu_\phi-u_t}{\sigma_\phi^2}+\frac{u_t\left(1-\gamma\right)}{\theta}\right]\Big\} = 0 \end{aligned} \tag{3.34}$$

Since none of the three first-order conditions contain W_t apart from the consumption ratio c_t (which is constant regardless of the level of W_t as (3.32) shows), I can conclude that if the value function at time $t+1$ can be expressed in the form (3.31), then the optimal consumption ratio c_t^*, the optimal investment ratio π_t^*, and the least-favorable distortion u_t^* will all be independent of the wealth level, W_t.

Now I prove the second part of Theorem 6, which states that the value function at time t can be expressed in the form

$$V_t\left(W_t,\hat{\pi}_t,\Phi_t\right) = \frac{W_t^{1-\gamma}}{1-\gamma}v_t\left(\hat{\pi}_t,\Phi_t\right). \tag{3.35}$$

Let us now again assume (I will prove this in Lemma 3) that the value function $V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1})$ can be expressed in the form

$$V_{t+1}(W_{t+1}, \hat{\pi}_{t+1}, \Phi_{t+1}) = \frac{W_{t+1}^{1-\gamma}}{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1}). \tag{3.36}$$

Substituting this form of the value function (the right-hand side of (3.36)) into the value function at time t, i.e., into (3.27), I obtain

$$\begin{aligned} V_t(W_t, \hat{\pi}_t, \Phi_t) = & \inf_{u_t} \sup_{\{C_t, \pi_t\}} \left\{ \exp(-\delta t) \frac{C_t^{1-\gamma}}{1-\gamma} \right. \\ & \left. + \left[1 + \frac{u_t^2(1-\gamma)}{2\theta}\right] \mathrm{E}_t^{u_t}\left[\frac{W_{t+1}^{1-\gamma}}{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1})\right]\right\} \\ = & \frac{W_t^{1-\gamma}}{1-\gamma} \inf_{u_t} \sup_{\pi_t} \left\{ \exp(-\delta t)\, c_t^{*1-\gamma} + \left[1 + \frac{u_t^2(1-\gamma)}{2\theta}\right] \right. \\ & \times \left(\frac{1 + I_t \Phi_t \hat{\pi}_t - c_t^*}{1 + I_t \Phi_t \pi_t}\right)^{1-\gamma} \\ & \left. \times \mathrm{E}_t^{u_t}\left[(R_f + \pi_t(R_{t+1} - R_f))^{1-\gamma} v_{t+1}(\hat{\pi}_{t+1}, \Phi_{t+1})\right]\right\}. \end{aligned} \tag{3.37}$$

In the last equation I already used the optimal consumption, C_t^*, thus I have omitted C_t below the sup operator; I substituted $c_t^* W_t$ in the place of C_t^*, since I showed in (3.32) that the optimal consumption ratio does not depend on the wealth level; and I also substituted the wealth dynamics (3.1) in the place of W_{t+1}. As we can see in (3.37), the value function at time t depends on the wealth level at time t only through the multiplicative factor $W_t^{1-\gamma}/(1-\gamma)$. Thus I can conclude that if the value function at time $t+1$ can be expressed in the form (3.36), then the value function at time t can be expressed in the form (3.35).

Following the logic of mathematical induction, the only thing left to prove is that the value function at time $T-1$ can be expressed in the form

$$V_{T-1}(W_{T-1}, \hat{\pi}_{T-1}, \Phi_{T-1}) = \frac{W_{T-1}^{1-\gamma}}{1-\gamma} v_{T-1}(\hat{\pi}_{T-1}, \Phi_{T-1}), \tag{3.38}$$

which is exactly what Lemma 3 contains. This completes the proof. □

Lemma 3. *The value function at time $T-1$ can be expressed in the form*

$$V_{T-1}\left(W_{T-1}, \hat{\pi}_{T-1}, \Phi_{T-1}\right) = \frac{W_{T-1}^{1-\gamma}}{1-\gamma} v_{T-1}\left(\hat{\pi}_{T-1}, \Phi_{T-1}\right). \tag{3.39}$$

Proof of Lemma 3. First I show that both the optimal consumption ratio c_{T-1} and the optimal investment ratio π_{T-1} are independent of the wealth level W_{T-1}. Then, using the fact of these independences, I show that the value function at time $T-1$ can indeed be written in the form (3.39).

At time T, the investor consumes all of her wealth. The proportion of her wealth that she inherited in the risky asset, $\hat{\pi}_T$, she liquidates encountering a transaction cost of Φ_T, while the proportion which she inherited in the riskless asset she liquidates for free. Thus her consumption at time T is

$$C_T = W_T\left(1 - \hat{\pi}_T \Phi_T\right), \tag{3.40}$$

and her value function at time $T-1$ is

$$\begin{aligned} V_{T-1} = \inf_{u_{T-1}} \sup_{\{C_{T-1}, \pi_{T-1}\}} &\left\{ \exp\left(-\delta\left(T-1\right)\right) \frac{C_{T-1}^{1-\gamma}}{1-\gamma} \right. \\ &\left. + \exp\left(-\delta T\right)\left(1 + \frac{u_{T-1}^2\left(1-\gamma\right)}{2\theta}\right) \mathrm{E}_{T-1}^{u_{T-1}}\left[\frac{W_T^{1-\gamma}\left(1-\hat{\pi}_T\Phi_T\right)^{1-\gamma}}{1-\gamma}\right]\right\}. \end{aligned} \tag{3.41}$$

To obtain the optimal consumption, the optimal investment ratio, and the least-favorable distortion at time $T-1$, I write down the first-order conditions on the expression within the brackets on the right-hand side of (3.41) with respect to C_{T-1}, π_{T-1}, and u_{T-1}. The first-order condition with respect to C_{T-1} is

$$\begin{aligned} c_{T-1} = \left(1 + I_{T-1}\Phi_{T-1}\hat{\pi}_{T-1}\right) &\left\{ \exp\left(-\frac{\delta}{\gamma}\right)\left(1 + \frac{u_{T-1}^2\left(1-\gamma\right)}{2\theta}\right)^{\frac{1}{\gamma}} \right. \\ &\times \left\{\mathrm{E}_{T-1}^{u_{T-1}}\left[\left(1-\hat{\pi}_T\Phi_T\right)^{1-\gamma}\left(R_f + \pi_{T-1}\left(R_T - R_f\right)\right)^{1-\gamma}\right]\right\}^{\frac{1}{\gamma}} \\ &\left. \times \left(1 + I_{T-1}\Phi_{T-1}\pi_{T-1}\right)^{\frac{\gamma-1}{\gamma}} + 1\right\}^{-1}, \end{aligned} \tag{3.42}$$

the first-order condition with respect to π_{T-1} is

$$\begin{aligned}
0 = \mathrm{E}_{T-1}^{u_{T-1}} \Big\{ & \left(R_f + \pi_{T-1}\left(R_T - R_f\right)\right)^{-\gamma}\left(1 - \hat{\pi}_T \Phi_T\right)^{1-\gamma} \\
& \times \left[R_T - R_f - \frac{I_{T-1}\Phi_{T-1}\left(R_f + \pi_{T-1}\left(R_T - R_f\right)\right)}{1 + I_{T-1}\Phi_{T-1}\pi_{T-1}} \right] \\
& - \left(R_f + \pi_{T-1}\left(R_T - R_f\right)\right)^{-1-\gamma}\left(1 - \hat{\pi}_T \Phi_T\right)^{-\gamma} \Phi_T R_f R_T \Big\},
\end{aligned} \tag{3.43}$$

and the first-order condition with respect to u_{T-1} is

$$\begin{aligned}
0 = & \mathrm{E}_{T-1}^{u_{T-1}} \Big\{ \left(R_f + \pi_{T-1}\left(R_T - R_f\right)\right)^{1-\gamma}\left(1 - \hat{\pi}_T \Phi_T\right)^{1-\gamma} \\
& \times \left[\frac{u_{T-1}\left(1-\gamma\right)}{\theta} + \left(1 + \frac{u_{T-1}^2\left(1-\gamma\right)}{2\theta}\right) \frac{\phi_T - \mu_\phi - u_{T-1}}{\sigma_\phi^2} \right] \Big\}.
\end{aligned} \tag{3.44}$$

This proves that at time $T-1$ each of the optimal consumption ratio, the optimal investment ratio, and the least-favorable distortion is independent of the wealth level W_{T-1}.

To show that the value function at $T-1$ can be expressed in the form (3.39), I substitute the wealth dynamics (3.1) into the original form of the value function at time $T-1$ (i.e., into (3.41)), and make use of the fact that none of the optimal consumption ratio c_{T-1}^*, the optimal investment ratio π_{T-1}^*, and the least-favorable distortion u_{T-1}^* depends on the wealth level W_{T-1}.

$$\begin{aligned}
V_{T-1} = & \frac{W_{T-1}^{1-\gamma}}{1-\gamma} \inf_{u_{T-1}} \sup_{\{c_{T-1}, \pi_{T-1}\}} \Big\{ \exp\left(-\delta\left(T-1\right)\right) c_{T-1}^{1-\gamma} \\
& + \exp\left(-\delta T\right) \left(1 + \frac{u_{T-1}^2\left(1-\gamma\right)}{2\theta}\right) \frac{\left(1 - c_{T-1} + I_{T-1}\Phi_{T-1}\hat{\pi}_{T-1}\right)^{1-\gamma}}{\left(1 + I_{T-1}\Phi_{T-1}\pi_{T-1}\right)^{1-\gamma}} \\
& \times \mathrm{E}_{T-1}^{u_{T-1}} \left[\left(1 - \hat{\pi}_T \Phi_T\right)^{1-\gamma} \left(R_f + \pi_{T-1}\left(R_T - R_f\right)\right)^{1-\gamma} \right] \Big\}
\end{aligned} \tag{3.45}$$

Remember, in the previous paragraph I showed that c_{T-1}^*, π_{T-1}^*, and u_{T-1}^* are all independent of W_{T-1}. Thus the right-hand side of (3.45) depends on the wealth level only through the multiplicative term $W_{T-1}^{1-\gamma}/\left(1-\gamma\right)$. This completes the proof. □

Proof of Theorem 7. For the proof of the generalized version of the Envelope Theorem for saddle-point problems, see Milgrom and Segal (2002). □

Chapter 4

Dynamic Asset Liability Management under Model Uncertainty

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4.1 Introduction

Financial institutions, such as pension funds and insurance companies, are exposed to several sources of risk through their assets and their liabilities. During their decision-making process, they simultaneously consider the potential effects of their decisions on both the asset and the liability side of their balance sheet, hence the term Asset Liability Management. However, unless the decision maker (the financial institution)[1] knows the true model (the data-generating process driving the asset and liability values) precisely, it faces not only risk, but also uncertainty. Disregarding uncertainty can lead to suboptimal investment and Asset Liability Management decisions, thus financial institutions want to make decisions that *"not only work well when the underlying model for the state variables holds exactly, but also perform reasonably well if there is some form of model misspecification"* (Maenhout (2004)). In the literature these decisions are called *robust decisions*. Although the utility loss resulting from model misspecification can be substantial (Branger and Hansis (2012)), the majority of the literature still assumes perfect knowledge of the underlying

[1] In this paper we focus on pension funds, but our results can be interpreted in a more general sense, and they are valid for any financial institution which has to make Asset Liability Management decisions.

model on the decision maker's side. Our aim with this paper is to fill this gap in the dynamic Asset Liability Management literature.

Our model features a complete financial market with stochastic interest rates governed by an N-factor Gaussian affine term structure model. The fund manager solves a dynamic Asset Liability Management problem under model uncertainty. Using the martingale method of Cox and Huang (1989), we provide the optimal terminal wealth, the least-favorable physical probability measure, and the optimal investment policy in closed form. We find that the optimal portfolio weights consist of two components: the myopic demand and the liability hedge demand, but notwithstanding the stochastic investment opportunity set, the fund manager does not have an intertemporal hedging demand component. We then use 42 years of U.S. data to calibrate our model. We show that robustness induces a more conservative investment policy: a robust fund manager's optimal risk exposures are closer to the liability risk exposures, hence reducing the speculative demand and increasing the liability-hedging demand. In parallel with this, a robust fund manager invests less in the constant maturity bond fund with a relatively short maturity (in our numerical example 1 year) and also in the stock market index than an otherwise identical non-robust fund manager. The portfolio weight of the constant maturity bond fund with maturity equal to the investment horizon increases due to its strong liability hedge effect, and thus effectively reduces the exposure to the factor-specific risk sources.

Our paper relates to the literature on Asset Liability Management. The modern Asset Liability Management literature dates back to Leibowitz (1987), who introduces the concept of the *surplus function* (the excess of the plan's asset value over the value of its liabilities). Based on this notion, Sharpe and Tint (1990) extend the Mean-Variance portfolio allocation model of Markowitz (1952) to an Asset Liability Management model. The basic idea is that instead of asset returns, the investor cares about the surplus returns, where surplus means the value of assets minus the value of liabilities. Sharpe and Tint (1990) find that the optimal portfolio consists of two components: a speculative portfolio and a liability-hedge portfolio. Moreover, only the speculative portfolio depends on the investor's preferences, the liability-hedge portfolio is the same for each investor. More recently, Hoevenaars, Molenaar, Schotman, and Steenkamp (2008) extend the multi-period portfolio selection model of Campbell and Viceira (2005) into a multi-period Asset Liability Management model. They confirm the finding of Sharpe and Tint (1990) that the optimal portfolio consists of two parts: a speculative portfolio and a liability hedge portfolio. Instead of maximizing a subjective utility function, Shen et al. (2014) assume that the fund manager minimizes the expected shortfall (i.e., the expected amount by which the value of liabilities exceeds the value of assets) at the terminal date. This assumption emphasizes that the fund manager acts in the best interest of the sponsoring firm, but does not consider the interest of the pension holders. In contrast to this, van Binsbergen and Brandt (2016) assume that the objective function is a sum of two parts: the first part expresses the (positive) utility of the pension holders, while the second part represents the (negative) utility of the sponsoring firm. To be more concrete, pension holders derive (positive) utility from a high funding ratio at the terminal date, while the sponsoring firm derives (negative) utility from having to provide additional contributions to the fund in order to keep the funding ratio above one throughout its life cycle. This model of van Binsbergen and Brandt (2016) nests the non-robust version of the model of Shen, Pelsser, and Schotman (2014), if the weight of the utility function of the pension holders is set to zero.

Our paper also relates to the literature on robust dynamic asset allocation. Anderson, Hansen, and Sargent (2003) in their seminal paper develop a framework for dynamic asset allocation models, which allows the investor to account for being uncertain about the physical probability measure. Within this framework, Maenhout (2004) provides an analytical and homothetic solution to the robust version of the Merton problem. Maenhout (2006) extends this model to incorporate a stochastic investment opportunity set. Branger, Larsen, and Munk (2013) solve a robust dynamic stock-cash allocation problem including return predictability, while Munk and Rubtsov (2014) also allow for ambiguity about the inflation process. Horvath, de Jong, and Werker (2016) provide a non-recursive formulation of the problem of Maenhout (2004), and also extend it to models featuring interest rate risk.

The paper is organized as follows. Section 4.2 introduces our model, i.e., the financial market and the fund manager's objective function. Moreover, Section 4.2 also provides the analytical solution of the robust dynamic Asset Liability Management problem. In Section 4.3 we calibrate our model to 42 years of U.S. market data using Maximum Likelihood and the Kalman filter. In Section 4.4 we link the level of uncertainty aversion to the theory of Detection Error Probabilities. In Section 4.5 we quantitatively analyze the effects of model uncertainty on the optimal Asset Liability Management decision. Section 4.6 concludes.

4.2 Robust Asset Liability Management Problem

Our model features a complete financial market and a robust fund manager. By robustness we mean that the fund manager is uncertain[2] about the underlying model. To be more precise, we assume that she is uncertain about the physical probability measure. She has a *base measure* $\mathbb{B}$ in mind, which she thinks to be the most reasonable probability measure. But she is uncertain about whether the base measure is indeed the true measure or not, so she considers other probability measures as well. We call these *alternative measures* and denote them by $\mathbb{U}$. We provide the exact relationship between $\mathbb{B}$ and $\mathbb{U}$, and also the restrictions on the set of $\mathbb{U}$ measures under consideration in Section 4.2.3.

4.2.1 Financial market

We consider pension funds which have access to a complete, arbitrage-free financial market consisting of a money-market account, N constant maturity bond funds, and a stock market index. The short rate is assumed to be affine in an N-dimensional factor $\boldsymbol{F}_t$, i.e.,

$$r_t = A_0 + \boldsymbol{\iota}'\boldsymbol{F}_t, \tag{4.1}$$

where $\boldsymbol{\iota}$ denotes a column vector of ones. The factor $\boldsymbol{F}_t$ follows an Ornstein-Uhlenbeck process under the base measure, i.e.,

$$\mathrm{d}\boldsymbol{F}_t = -\boldsymbol{\kappa}(\boldsymbol{F}_t - \boldsymbol{\mu}_F)\mathrm{d}t + \boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}^{\mathbb{B}}_{F,t}. \tag{4.2}$$

Here $\boldsymbol{\kappa}$ is an $N \times N$ diagonal matrix with the mean reversion parameters in its diagonal; $\boldsymbol{\mu}_F$ is an N-dimensional column vector containing the long-term means of the factors under the base measure $\mathbb{B}$; $\boldsymbol{\sigma}_F$ is an $N \times N$ lower triangular matrix, with strictly positive elements in its diagonal; and $\boldsymbol{W}^{\mathbb{B}}_{F,t}$ is an N-dimensional column vector of independent standard Wiener processes under $\mathbb{B}$.

[2]In the behavioral finance and the operations research literature, there is a distinction made between the terms *uncertainty* and *ambiguity*. In the robust asset allocation literature, however, they are used interchangeably. Since our paper primarily belongs to this latter branch of the literature, we do not differentiate between the meaning of *uncertainty* and *ambiguity*, and use the two words interchangeably.

The stock market index can be correlated with the factor $\boldsymbol{F}_t$, i.e.,

$$\mathrm{d}S_t = S_t\left[r_t + \boldsymbol{\sigma}'_{F,S}\boldsymbol{\lambda}_F + \sigma_{N+1,S}\lambda_{N+1}\right]\mathrm{d}t + S_t\left(\boldsymbol{\sigma}'_{F,S}\mathrm{d}\boldsymbol{W}^{\mathbb{B}}_{F,t} + \sigma_{N+1,S}\mathrm{d}W^{\mathbb{B}}_{N+1,t}\right), \tag{4.3}$$

where $\boldsymbol{\lambda}_F$ and λ_{N+1} are the market prices of risk corresponding to the base measure $\mathbb{B}$, $\boldsymbol{\sigma}_{F,S}$ is an N-dimensional column vector governing the covariance between stock and bond returns, $\sigma_{N+1,S}$ is a strictly positive constant, and $W^{\mathbb{B}}_{N+1,t}$ is a standard Wiener process under the base measure $\mathbb{B}$, which is independent of $\boldsymbol{W}^{\mathbb{B}}_{F,t}$. The liability of the pension fund is assumed to evolve according to

$$\mathrm{d}L_t = L_t\left(r_t + \boldsymbol{\sigma}'_{F,L}\boldsymbol{\lambda}_F + \sigma_{N+1,L}\lambda_{N+1}\right)\mathrm{d}t + L_t\left(\boldsymbol{\sigma}'_{F,L}\mathrm{d}\boldsymbol{W}^{\mathbb{B}}_{F,t} + \sigma_{N+1,L}\mathrm{d}W^{\mathbb{B}}_{N+1,t}\right), \tag{4.4}$$

where $\boldsymbol{\sigma}_{F,L}$ is an N-dimensional column vector, and $\sigma_{N+1,L}$ is a scalar.

To simplify notation, we denote $\boldsymbol{W}^{\mathbb{B}}_{F,t}$ and $W^{\mathbb{B}}_{N+1,t}$ jointly as

$$\boldsymbol{W}^{\mathbb{B}}_t = \begin{bmatrix} \boldsymbol{W}^{\mathbb{B}}_{F,t} \\ W^{\mathbb{B}}_{N+1,t} \end{bmatrix}, \tag{4.5}$$

$\boldsymbol{\lambda}_F$ and λ_{N+1} jointly as

$$\boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_F \\ \lambda_{N+1} \end{bmatrix}, \tag{4.6}$$

$\boldsymbol{\sigma}_{F,S}$ and $\sigma_{N+1,S}$ jointly as

$$\boldsymbol{\sigma}_S = \begin{bmatrix} \boldsymbol{\sigma}_{F,S} \\ \sigma_{N+1,S} \end{bmatrix}, \tag{4.7}$$

and $\boldsymbol{\sigma}_{F,L}$ and $\sigma_{N+1,L}$ jointly as

$$\boldsymbol{\sigma}_L = \begin{bmatrix} \boldsymbol{\sigma}_{F,L} \\ \sigma_{N+1,L} \end{bmatrix}. \tag{4.8}$$

4.2.2 The liability-risk-neutral measure

Our fund manager, as we describe in more detail in Section 4.2.3, is optimizing over the terminal funding ratio. Therefore, to facilitate the problem solving process, throughout the paper we use the liability value as numeraire. Since the financial market is complete and free of arbitrage opportunities, there exists a unique probability measure under which the value of any traded asset scaled by the value of liability is a martingale.

Let X_t be the value of any traded asset, with

$$\mathrm{d}X_t = X_t \left[r_t + \boldsymbol{\sigma}_X' \boldsymbol{\lambda}\right] \mathrm{d}t + X_t \boldsymbol{\sigma}_X' \mathrm{d}\boldsymbol{W}_t^{\mathbb{B}}. \tag{4.9}$$

Applying Ito's lemma, we find the dynamics of the asset price scaled by the value of the liability as

$$\mathrm{d}\left(\frac{X}{L}\right)_t = \left(\frac{X}{L}\right)_t (\boldsymbol{\sigma}_X - \boldsymbol{\sigma}_L)' (\boldsymbol{\lambda} - \boldsymbol{\sigma}_L) \,\mathrm{d}t + \left(\frac{X}{L}\right)_t (\boldsymbol{\sigma}_X - \boldsymbol{\sigma}_L)' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{B}}. \tag{4.10}$$

Defining

$$\mathrm{d}\boldsymbol{W}_t^{\mathbb{B}} = \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}} - \boldsymbol{\lambda}^{\mathbb{L}} \mathrm{d}t, \tag{4.11}$$

with

$$\boldsymbol{\lambda}^{\mathbb{L}} = \boldsymbol{\lambda} - \boldsymbol{\sigma}_L, \tag{4.12}$$

the dynamics of the asset X_t scaled by the liability value can be rewritten as

$$\mathrm{d}\left(\frac{X}{L}\right)_t = \left(\frac{X}{L}\right)_t (\boldsymbol{\sigma}_X - \boldsymbol{\sigma}_L)' \,\mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}. \tag{4.13}$$

Then (4.12), together with (4.11), uniquely determines the relationship between the liability-risk-neutral measure $\mathbb{L}$ and the base measure $\mathbb{B}$.

4.2.3 Preferences, beliefs, and problem formulation

We consider a pension fund manager who acts in the best interest of the pension holders. She is risk-averse, and she has CRRA preferences over the terminal funding ratio. The fund manager wants to maximize her expected utility, but she is uncertain about the physical probability measure under which the expectation is supposed to be calculated. She has a base measure ($\mathbb{B}$) in mind, but she considers other, alternative probability measures ($\mathbb{U}$) as well. We assume that the investor knows which events will happen with probability one and with probability zero, i.e., she considers only alternative probability measures which are equivalent to the base measure. We now formalize the relationship between the base measure $\mathbb{B}$ and the alternative measure $\mathbb{U}$ as

$$\mathrm{d}\boldsymbol{W}_t^{\mathbb{U}} = \mathrm{d}\boldsymbol{W}_t^{\mathbb{B}} - \boldsymbol{u}(t)\mathrm{d}t, \tag{4.14}$$

where $\boldsymbol{W}_t^{\mathbb{B}}$ and $\boldsymbol{W}_t^{\mathbb{U}}$ are $(N+1)$-dimensional standard Wiener processes under the measures $\mathbb{B}$ and $\mathbb{U}$, respectively. Similarly to identifying $\boldsymbol{\lambda}$ as the $(N+1)$-dimensional vector of prices of risks of the base measure $\mathbb{B}$, we can identify $\boldsymbol{u}(t)$ as the $(N+1)$-dimensional vector of prices of risks of $\mathbb{U}$.[3] We assume that $\boldsymbol{\lambda}$ is constant, while $\boldsymbol{u}(t)$ is assumed to be a deterministic function of time.[4]

We now formalize the robust optimization problem of the fund manager. Her investment horizon is T, she has a utility function with a constant relative risk aversion of $\gamma > 1$ over the terminal funding ratio,[5] and a subjective discount rate of $\delta > 0$. Her uncertainty-tolerance is determined by the parameter Υ_t, which is allowed to be stochastic.

Problem 8. *Given initial funding ratio A_0/L_0, find an optimal pair $\{A_T, \mathbb{U}\}$ for the robust utility maximization problem*

$$V_0\left(\frac{A_0}{L_0}\right) = \inf_{\mathbb{U}} \sup_{A_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp(-\delta T) \frac{\left(\frac{A_T}{L_T}\right)^{1-\gamma}}{1-\gamma} + \int_0^T \Upsilon_s \exp(-\delta s) \frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_s\right]}{\partial s} \mathrm{d}s \right\}, \tag{4.15}$$

subject to the budget constraint

$$\mathrm{E}^{\mathbb{L}}\left(\frac{A_T}{L_T}\right) = \frac{A_0}{L_0}. \tag{4.16}$$

A natural question might arise why we formulate the utility function in terms of the terminal funding ratio A_T/L_T instead of the terminal surplus asset net of liabilities, i.e., $A_T - L_T$. There are two schools of thought regarding how to measure the risks faced by pension funds: one focuses on the surplus of assets over liabilities, i.e., $A_t - L_t$ (e.g., Sharpe and Tint (1990) and Leibowitz, Kogelman, and Bader (1992) in the early ALM literature, or more recently Waring and Whitney

[3]Throughout the paper we assume $\int_0^T \|\boldsymbol{u}(s)\|^2 \mathrm{d}s < \infty$.

[4]We could allow $\boldsymbol{\lambda}$ to be a deterministic function of time without much change in our conclusions, but it would result in more complex expressions due to time-integrals involving $\boldsymbol{\lambda}(t)$. Thus, since for our purposes a constant $\boldsymbol{\lambda}$ suffices, we throughout take $\boldsymbol{\lambda}$ to be constant.

[5]The case $\gamma = 1$ corresponds to the fund manager having log-utility. All of our results can be shown to hold for the log-utility case as well.

(2009) and Ang, Chen, and Sundaresan (2013)), while the other concentrates on the ratio of assets over liabilities, i.e., A_t/L_t, called the funding ratio (e.g., Leibowitz, Kogelman, and Bader (1994) and van Binsbergen and Brandt (2016)). Although at first it might seem more natural to formulate the utility function in terms of the surplus asset value, once we interpret A_t/L_t as the value of the fund's assets expressed in units of the fund's liabilities at any time t, using the funding ratio as the argument of the fund's utility function immediately becomes more intuitive.

The formulation of Problem 8 follows the logic of the Martingale Method of Cox and Huang (1989): the fund manager optimizes over the terminal wealth A_T.[6] The first part of the objective function in Problem 8 expresses that the fund manager derives utility from the terminal funding ratio. The second part is a penalty term, which assures that the investor will use a pessimistic, but reasonable physical probability measure to calculate her expected utility. This penalty term – in line with Anderson, Hansen, and Sargent (2003) – is the integral of the discounted time-derivative of the Kullback-Leibler divergence (also known as the relative entropy) between the base measure $\mathbb{B}$ and the alternative measure $\mathbb{U}$, multiplied by the fund manager's uncertainty-tolerance parameter Υ_s. Intuitively, this penalty term is high if the alternative measure $\mathbb{U}$ and the base measure $\mathbb{B}$ are very different from each other, and low if they are similar to each other. If $\mathbb{U}$ and $\mathbb{B}$ coincide, the penalty term is zero. Using Girsanov's theorem, we can express the Kullback-Leibler divergence as

$$\begin{aligned}\frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_t\right]}{\partial t} &= \frac{\partial}{\partial t}\mathrm{E}^{\mathbb{U}}\left[\frac{1}{2}\int_0^t \|\boldsymbol{u}(s)\|^2 \mathrm{d}s - \int_0^t \boldsymbol{u}(s)\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}\right] \\ &= \frac{1}{2}\|\boldsymbol{u}(t)\|^2. \end{aligned} \tag{4.17}$$

To insure homotheticity of the solution, i.e., that the optimal portfolio weights do not depend on the actual funding ratio, we – following Maenhout (2004) – express the manager's uncertainty-tolerance parameter as

$$\Upsilon_t = \exp(\delta t)\,\frac{1-\gamma}{\theta} V_t\left(\frac{A_t}{L_t}\right), \tag{4.18}$$

[6]Actually, the fund manager optimizes over the terminal funding ratio A_T/L_T. However, as we describe it in more detail in Section 4.2.1, the liability process L_t is assumed to be exogenous, hence choosing an optimal terminal funding ratio A_T/L_T is equivalent to choosing "only" an optimal terminal wealth A_T.

where $V_t\left(A_t/L_t\right)$ is the value function of the fund manager at time t, i.e.,

$$V_t\left(\frac{A_t}{L_t}\right) = \inf_{\mathbb{U}} \sup_{A_T} \mathrm{E}_t^{\mathbb{U}} \left\{ \exp(-\delta T) \frac{\left(\frac{A_T}{L_T}\right)^{1-\gamma}}{1-\gamma} \right. \\ \left. + \int_t^T \Upsilon_s \exp(-\delta s) \frac{\partial \mathrm{E}^{\mathbb{U}}\left[\log\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{B}}\right)_s\right]}{\partial s} \mathrm{d}s \right\}, \tag{4.19}$$

subject to the budget constraint

$$\mathrm{E}_t^{\mathbb{L}}\left(\frac{A_T}{L_T}\right) = \frac{A_t}{L_t}. \tag{4.20}$$

Substituting (4.18) into the value function (4.15), and also making use of (4.17), we can rewrite Problem 8 in a form which has a non-recursive goal function. This is stated in the following theorem, the proof of which is provided in the Appendix.

Theorem 8. *If the fund manager's uncertainty-tolerance parameter Υ_t takes the form (4.18), then the value function in Problem 8 is equivalent to*

$$V_0\left(\frac{A_0}{L_0}\right) = \inf_{\mathbb{U}} \sup_{A_T} \mathrm{E}^{\mathbb{U}} \left\{ \exp\left(\frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T\right) \frac{\left(\frac{A_T}{L_T}\right)^{1-\gamma}}{1-\gamma} \right\}, \tag{4.21}$$

subject to the budget constraint

$$\mathrm{E}^{\mathbb{L}}\left(\frac{A_T}{L_T}\right) = \frac{A_0}{L_0}. \tag{4.22}$$

As noted by Horvath, de Jong, and Werker (2016), the expression in (4.21) provides an alternative interpretation of robustness: the goal function of a robust fund manager is equivalent to the goal function of a more impatient[7] non-robust fund manager. Besides increasing the subjective discount rate, the other effect of robustness is a change in the physical probability measure from $\mathbb{B}$ to $\mathbb{U}$.

4.2.4 Optimal Terminal Funding Ratio

To solve the robust dynamic ALM problem, we apply the martingale method (developed by Cox and Huang (1989), and adapted to robust problems by Horvath, de Jong, and Werker (2016)). The

[7] By a fund manager being more impatient, we mean that her subjective discount rate is higher.

next theorem – which we prove in the Appendix – provides the optimal terminal wealth and the least-favorable distortions.

Theorem 9. *The solution to Problem 8 under (4.18) is given by*

$$\hat{A}_T = L_T \frac{A_0}{L_0} \frac{\exp\left[\frac{1}{\gamma}\int_0^T \left(\boldsymbol{\lambda}^{\mathbb{L}} + \hat{\boldsymbol{u}}(t)\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{\gamma}\int_0^T \left(\boldsymbol{\lambda}^{\mathbb{L}} + \hat{\boldsymbol{u}}(t)\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right]}{\mathrm{E}^{\mathbb{L}} \exp\left[\frac{1}{\gamma}\int_0^T \left(\boldsymbol{\lambda}^{\mathbb{L}} + \hat{\boldsymbol{u}}(t)\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{\gamma}\int_0^T \left(\boldsymbol{\lambda}^{\mathbb{L}} + \hat{\boldsymbol{u}}(t)\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right]}, \tag{4.23}$$

with the least-favorable distortion

$$\hat{\boldsymbol{u}}(t) = -\frac{\theta}{\gamma + \theta} \boldsymbol{\lambda}^{\mathbb{L}}. \tag{4.24}$$

Using the martingale method to solve the robust dynamic ALM problem has the advantage of providing insight into the optimization process of the fund manager. The form of the optimal terminal wealth (4.23) suggests that the decision process of the fund manager can be separated into two parts. First, as a starting point, she wants to obtain a perfect hedge for the liabilities at time T, i.e., she wants a terminal wealth equal to L_T. Then, she modifies this terminal wealth based on her preferences to achieve the optimal terminal wealth.

The least-favorable distortion of the fund manager differs in two important aspects from the least-favorable distortion of an otherwise identical investor who optimizes over her terminal wealth, instead of the terminal funding ratio (see Horvath, de Jong, and Werker (2016)). First, the least-favorable distortion of the fund manager is independent of time, while the least-favorable distortion of an investor deriving utility from terminal wealth contains a time-dependent component. This time-dependent component is present due to the intertemporal hedging potential of the constant maturity bond funds, and it results in the investor having a “less severe” distortion. However – as we show in Section 4.2.5 –, deriving utility from the terminal funding ratio instead of the terminal wealth, the fund manager does not have an intertemporal hedging demand for the constant maturity bond funds. So, intuitively, regardless of how far away from the end of her investment horizon the fund manager is, she will distort her base measure to the same extent.

Another aspect in which the least-favorable distortion of the fund manager differs from the least-favorable distortion of an otherwise identical investor deriving utility from terminal wealth is that the market price of risk in which the least-favorable distortion is affine corresponds to the liability value as numeraire, instead of to the money market account.[8] Intuitively, this means that the magnitude of the distortion is reduced due to the fund manager deriving utility from the terminal funding ratio instead of the terminal wealth. This is true for both $\hat{\boldsymbol{u}}_F$ and for $\hat{u}_{N+1}$. Because the liability process behaves very similarly to a zero-coupon bond with approximately 15 years of maturity, we expect the elements of $\boldsymbol{\sigma}_{F,L}$ to be negative. Since the first N elements of $\boldsymbol{\lambda}$ are also negative, and the difference between the market price of risk corresponding to to the money market account as numeraire and the market price of risk corresponding to the liability as numeraire is $\boldsymbol{\sigma}_L$, deriving utility from the terminal funding ratio instead of the terminal wealth reduces the (positive) elements of $\hat{\boldsymbol{u}}_F$. The same logic applies to $\hat{u}_{N+1}$. The market price of risk using the money market account as numeraire, i.e., λ_{N+1} is positive, and intuition suggests $\sigma_{L,N+1}$ is also positive, therefore, optimizing over the terminal funding ratio instead of the terminal wealth will reduce the magnitude of the (negative) $\hat{u}_{N+1}$.

If the fund manager is not uncertainty averse at all, her θ parameter is equal to zero and her least-favorable distortion reduces to zero as well. In other words, she will use her base probability measure $\mathbb{B}$ to evaluate her expected utility. At the other extreme, if her uncertainty aversion (i.e., her θ parameter) is infinity, she uses the globally-least-favorable distortion

$$\tilde{\boldsymbol{u}} = -\boldsymbol{\lambda}^{\mathbb{L}}. \tag{4.25}$$

Optimizing over the terminal wealth, the globally-least-favorable distortion would be equal to the market price of risk using the money market account as numeraire, and the investor would consider the scenario when she receives no compensation above the risk-free rate for bearing any risk. For a fund manager optimizing over the funding ratio, however, this scenario would still be "of value" in the sense that she would still be willing to bear some risk, due to its hedging potential.[9] To achieve the least-favorable distortion, the fund manager has to correct for this and hence her least-favorable distortion becomes the market price of risk of the $\mathbb{L}$ measure over the base measure $\mathbb{B}$.

[8] That is, the fund manager's least-favorable distortion is affine in $\boldsymbol{\lambda}^{\mathbb{L}}$, while the least-favorable distortion of an otherwise identical investor who derives utility from terminal wealth is affine in $\boldsymbol{\lambda}$, i.e., the market price of risk of the base measure $\mathbb{B}$ over the risk-neutral measure with the money market account as numeraire.

[9] We would like to emphasize here that this hedging potential refers to the liability hedge, i.e., by being exposed to some risk in the above-mentioned scenario the investor can achieve a lower volatility of her terminal funding ratio than by investing everything in the money market account.

4.2.5 Optimal Portfolio Strategy

Since our financial market is complete, there exists a unique investment process which enables the fund manager to achieve the optimal terminal wealth (4.23). We provide the optimal risk exposure process corresponding to this optimal investment policy in Corollary 3, and the optimal investment process itself in Corollary 4. Both proofs are provided in the Appendix.

Corollary 3. *Under the conditions of Theorem 2, the optimal investment is a continuous rebalancing strategy where the exposures to the N+1 risk sources – as a fraction of wealth – are*

$$\hat{\mathbf{\Pi}}_t = \frac{1}{\gamma+\theta}\boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{\sigma}_L \tag{4.26}$$

$$= \frac{1}{\gamma+\theta}\boldsymbol{\lambda} + \left(1 - \frac{1}{\gamma+\theta}\right)\boldsymbol{\sigma}_L. \tag{4.27}$$

The form of the optimal exposure to the risk sources in (4.26) also reflects the separation of the investment decision into two parts which we described in Section 4.2.4, i.e., the fund manager first achieves a perfect hedge of the liability (second part of (4.26)), then she modifies her exposure according to her preferences (first part of (4.26)). If the correlation between the asset returns and the liability return is zero (i.e., if $\boldsymbol{\sigma}_L = \mathbf{0}$), then the optimal exposure to the risk sources is equal to the scaled market price of risk, i.e., $\boldsymbol{\lambda}^{\mathbb{L}}/(\gamma+\theta)$, which in that case coincides with $\boldsymbol{\lambda}/(\gamma+\theta)$.[10]

In the next corollary we provide the unique optimal investment process, with notation

$$\boldsymbol{\mathcal{B}}(\tau) = [\boldsymbol{B}(\tau_1)\boldsymbol{\iota};\ldots;\boldsymbol{B}(\tau_N)\boldsymbol{\iota}], \tag{4.28}$$

where τ_j denotes the maturity of bond fund j, and $\boldsymbol{B}(t)$ is defined as

$$\boldsymbol{B}(t) = (\boldsymbol{I} - \exp\{-\boldsymbol{\kappa}t\})\boldsymbol{\kappa}^{-1}. \tag{4.29}$$

Corollary 4. *Under the conditions of Theorem 9, the optimal investment is a continuous re-*

[10] We want to stress that this does not mean that the optimal decision for an investor optimizing over terminal wealth only (instead of over the terminal funding ratio) is equal to $\boldsymbol{\lambda}^{\mathbb{L}}/(\gamma+\theta)$ or $\boldsymbol{\lambda}/(\gamma+\theta)$. The reason of the difference is that even if the liability process is a constant (i.e., $\boldsymbol{\sigma}_L = \mathbf{0}$), the fund manager still hedges against it, and the liability hedge demand is equal to the negative of the intertemporal hedge demand. Thus, the two latter demand components of the fund manager cancel out. In contrast with this, if the investor optimizes over her terminal wealth only, she still has a non-zero intertemporal hedging demand.

balancing strategy where the fraction of wealth invested in the constant maturity bond funds is

$$\begin{aligned}\hat{\boldsymbol{\pi}}_{B,t} = & -\frac{1}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\left(\boldsymbol{\sigma}_F'\right)^{-1}\left(\boldsymbol{\lambda}_F - \frac{\lambda_{N+1}}{\sigma_{N+1,S}}\boldsymbol{\sigma}_{F,S}\right) \\ & + \frac{1-\gamma-\theta}{\gamma+\theta}\boldsymbol{\mathcal{B}}(\tau)^{-1}\left(\boldsymbol{\sigma}_F'\right)^{-1}\left(\boldsymbol{\sigma}_{F,L} - \frac{\sigma_{N+1,L}}{\sigma_{N+1,S}}\boldsymbol{\sigma}_{F,S}\right),\end{aligned} \tag{4.30}$$

and the fraction of wealth invested in the stock market index is

$$\hat{\pi}_{S,t} = \frac{\lambda_{N+1}}{(\gamma+\theta)\,\sigma_{N+1,S}} - \frac{(1-\gamma-\theta)}{(\gamma+\theta)}\frac{\sigma_{N+1,L}}{\sigma_{N+1,S}}. \tag{4.31}$$

In line with Sharpe and Tint (1990) and Hoevenaars, Molenaar, Schotman, and Steenkamp (2008), we find that the optimal portfolio consists of two parts: a speculative portfolio (first line of (4.30) and first part of (4.31)), and a liability hedge portfolio (second line of (4.30) and second part of (4.31)). The source of the liability hedge demand is the covariance between the asset returns and the liability returns. The higher the covariance between the bond fund returns and the liability returns (i.e., the lower the elements of $\boldsymbol{\sigma}_{F,L}$), the higher the optimal portfolio weight of the constant maturity bond funds. Also: the higher the covariance between the stock market index return and the liability return (i.e., the higher $\sigma_{N+1,L}$), the higher the optimal portfolio weight of the stock market index. Intuitively, a higher covariance between the return of an asset and the liability induces a higher optimal investment in that particular asset, because a higher covariance provides a higher hedging potential and therefore makes the asset more desirable. The second terms within the brackets in both the first and the second line of (4.30) are correction terms to the speculative constant maturity bond fund demand and the liability hedge constant maturity bond fund demand, respectively. These two correction terms arise due to the covariance between the bond returns and the stock market index return. The higher this covariance (i.e., the lower the elements of $\boldsymbol{\sigma}_{F,S}$), the lower the correction term to both the speculative bond demand and the liability hedge bond demand. The intuition of this is that a higher covariance between the constant maturity bond fund returns and the stock market index return results in the same investment in the stock market index providing a higher exposure to the N factors, and hence to retain the optimal exposure to these factors, the constant maturity bond funds should have lower portfolio weights than with zero covariance.

Table 4.1. Parameter estimates and standard errors
Estimated parameters and standard errors using Maximum Likelihood. We observed four points weekly on the U.S. zero-coupon, continuously compounded yield curve, corresponding to maturities of 3 months, 1 year, 5 years and 10 years; and the total return index of Datastream's US-DS Market. The observation period is from 5 January 1973 to 29 January 2016.

	Estimated parameter	*Standard error*
$\hat{\kappa}_1$	0.0763^{***}	*0.0024*
$\hat{\kappa}_2$	0.3070^{***}	*0.0108*
$\hat{A}_0$	0.0862^{***}	*0.0013*
$\hat{\lambda}_{F,1}$	-0.1708	*0.1528*
$\hat{\lambda}_{F,2}$	-0.5899^{***}	*0.1528*
$\hat{\lambda}_{N+1}$	0.3180^{**}	*0.1528*
$\hat{\sigma}_{F,11}$	0.0208^{***}	*0.0009*
$\hat{\sigma}_{F,21}$	-0.0204^{***}	*0.0012*
$\hat{\sigma}_{F,22}$	0.0155^{***}	*0.0003*
$\hat{\sigma}_{FS,1}$	-0.0035	*0.0038*
$\hat{\sigma}_{FS,2}$	-0.0121^{***}	*0.0035*
$\hat{\sigma}_{N+1}$	0.1659^{***}	*0.0025*

We find that the optimal asset allocation is determined by the sum of the risk-aversion parameter and the uncertainty-aversion parameter, i.e., by $\gamma + \theta$. This is in line with, e.g., Maenhout (2004), Maenhout (2006), and Horvath, de Jong, and Werker (2016). Intuitively, a robust fund manager behaves the same way as a non-robust, but more risk-averse fund manager.

4.3 Model Calibration

The two-factor version of our model for the financial market is identical to the model of Horvath, de Jong, and Werker (2016). Hence, we directly adapt the estimates therein for our model parameters. For completeness, we briefly recall the estimation methodology followed by Horvath, de Jong, and Werker (2016). The model is calibrated to U.S. market data using the Kalman filter and Maximum Likelihood. The data consist of weekly observations of the 3-month, 1-year, 5-year, and 10-year points of the yield curve, and Datastream's U.S. Stock Market Index. The observation period is from 1 January 1973 to 29 January 2016. The starting values of the filtered factors are equal to their long-term means. The parameter estimates can be found in Table 4.1.

All model parameters are estimated with small standard errors, the only exception being the market price of risk. This confirms the validity of our model setup, namely, that the fund manager is uncertain about the physical probability measure, which – together with her considering only equivalent probability measures – is equivalent to saying that she is uncertain about the market price of risk.

As a proxy for the liability process, we follow van Binsbergen and Brandt (2016) and use the price of a zero-coupon bond. Intuitively, we think of the liability as a rolled-over asset with constant duration. As of the duration itself, we use 15 years, which is approximately the average duration of U.S. pension fund liabilities (van Binsbergen and Brandt (2016)). Then, the volatility parameters of the liabilities are

$$\boldsymbol{\sigma}_{F,L} = -\boldsymbol{\sigma}_F' \boldsymbol{B}\,(15)\,\boldsymbol{\iota} \tag{4.32}$$

and

$$\sigma_{N+1,L} = 0. \tag{4.33}$$

Using our parameter estimates in Table 4.1, the estimated volatility vector of the liability process is

$$\boldsymbol{\sigma}_L' = \begin{bmatrix} -0.1201 & -0.0501 & 0 \end{bmatrix}. \tag{4.34}$$

4.4 Detection Error Probabilities

In the previous section we estimated the model parameters related to the financial market, based on historical data. Calibrating the parameters related to the preferences, i.e., the risk-aversion parameter γ and the uncertainty-aversion parameter θ, is less straightforward.

There is no agreement in the literature about what the relative risk aversion of a representative investor precisely is, but the majority of the literature considers risk aversion parameters between 1 and 5 to be reasonable. Several studies attempt to estimate what a reasonable risk aversion value is, usually by using consumption data or by conducting experiments. Friend and Blume (1975) estimate the relative risk aversion parameter to be around 2; Weber (1975) and Szpiro (1986) estimate it to be between about 1.3 and 1.8; the estimates of Hansen and Singleton (1982) and Hansen and Singleton (1983) are 0.68–0.97 and 0.26-2.7, respectively; using nondurable consumption data, Mankiw (1985) estimates the relative risk aversion to be 2.44-5.26, and using durable goods consumption data it to be 1.79-3.21; Barsky et al. (1997) use an experimental survey to estimate the relative risk aversion parameter of the subjects, the mean of which turns out to be 4.17; while in the study of Halek and Eisenhauer (2001) the mean relative risk aversion is 3.7. Later in this section we vary the risk-aversion parameter between 1 and 5 to see its effect on the optimal investment decision.

Calibrating the uncertainty-aversion parameter θ is even more complicated than the calibration of the risk aversion. Ever since the seminal paper of Anderson, Hansen, and Sargent (2003), the most puzzling questions in the robust asset pricing literature are related to how to quantify uncertainty aversion, and how much uncertainty is reasonable. Anderson, Hansen, and Sargent (2003) propose a theory to address these problems based on the Detection Error Probabilities. They assume that the investor can observe a sample of historical data, and she performs a likelihood ratio test to decide whether these data are generated by a data-generating process corresponding to the base measure $\mathbb{B}$, or by a data-generating process corresponding to the alternative measure $\mathbb{U}$. Based on this test, the investor is assumed to be able to correctly guess the true physical probability measure in $p\%$ of the cases, i.e., she is wrong in $(1-p)\,\%$ of the cases. Making this $(1-p)\,\%$ equal to the probability of making an error based on the likelihood ratio test, we can disentangle the risk aversion and the uncertainty aversion. The question of what a reasonable level of $(1-p)\%$, i.e., the Detection Error Probability, is, is the subject of an active line of research. Anderson, Hansen, and Sargent (2003) suggest that Detection Error Probabilities between 10% and 30% are plausible. Now we give the formal definition of the Detection Error Probability.

Definition 6. *The Detection Error Probability (DEP) is defined as*

$$DEP = \frac{1}{2}P^{\mathbb{B}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}} < 0\right) + \frac{1}{2}P^{\mathbb{U}}\left(\log\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{U}} > 0\right). \tag{4.35}$$

Following the reasoning of Horvath, de Jong, and Werker (2016), we can express the Detection Error Probability in closed form. This is stated in Theorem 10 and in Corollary 2.

Theorem 10. *Assume that the fund manager continuously observes the prices of N constant maturity bond funds, and the level of the stock market index. The observation period lasts from $t-H$ to the moment of observation, t. Then, the detection error probability of the fund manager* *for given* $\mathbb{U}$ *is*

$$DEP = 1 - \Phi\left(\frac{1}{2}\sqrt{\int_{t-H}^{t} \|\boldsymbol{u}(s)\|^2 \mathrm{d}s}\right), \tag{4.36}$$

where $\boldsymbol{u}(\cdot)$ *is defined in (4.14).*

Substituting the least-favorable distortion (4.24) into (4.36), we obtain the closed-form expression in Corollary 2.

Corollary 5. *Assume that the conditions of Theorem 10 hold. Then, the detection error probability of the fund manager for the least-favorable* $\mathbb{U}$ *is*

$$DEP = 1 - \Phi\left(\frac{\theta}{2(\gamma+\theta)}\sqrt{H}\|\boldsymbol{\lambda}^{\mathbb{L}}\|\right). \tag{4.37}$$

The Detection Error Probability used by a fund manager who is not uncertainty-averse at all (i.e., whose θ parameter is zero) is 0.5. That is, she might as well flip a coin to distinguish between two probability measures instead of performing a likelihood-ratio test on a sample of data. On the other hand, a fund manager with an uncertainty aversion parameter of infinity uses the lowest possible Detection Error Probability, which is $1 - \Phi\left((1/2)\sqrt{H}\|\boldsymbol{\lambda}^{\mathbb{L}}\|\right)$.[11]

We assume that the observation period of the investor is 42 years,[12] and that her Detection Error Probability is 10%. Given that she has access to a relatively long sample of data, and that she can observe the prices continuously, our choice of 10% as the Detection Error Probability is justifiable.

4.5 Policy Evaluation

Now we use our parameter estimates from Section 4.3 to analyze the effects of robustness on the optimal ALM decision, if the fund manager has access to a money market account, to two constant maturity bond funds with 1 and 15 years of maturities, and to a stock market index. Her investment

[11] One might expect that the lowest possible Detection Error Probability is zero, which would mean that the fund manager knows the physical probability measure precisely. However, as (4.37) also shows, this is only the case if the length of her observation period is infinity, i.e., $H = \infty$. If her observation period is finite, the limitation of available data will always result in the fund manager not being able to correctly tell apart two probability measures in 100% of the cases.

[12] We use 42 years of market data to estimate our model parameters, thus it is a reasonable assumption that the fund manager has access to the same length of data. Even though our observation frequency is weekly, assuming that the fund manager can observe data continuously does not cause a significant difference.

horizon is 15 years. We show the quantitative relationship between the level of uncertainty aversion and the optimal exposure to the different sources of risk, and also the optimal portfolio weights. We find that regardless of the risk attitude of the fund manager, robustness substantially changes the magnitude of her optimal portfolio weights. Generally speaking, robustness translates into making more conservative ALM decisions. More concretely, while investing significantly less in the stock market index and the constant maturity bond fund with the shorter (1 year) maturity, the fund manager increases her investments in the constant maturity bond fund with the same maturity as her investment horizon (15 years) and in the money market account.

Figure 4.1 shows the optimal exposure of the fund manager to the three risk sources for different levels of risk aversion and uncertainty aversion.[13] Her uncertainty aversion is measured by the Detection Error Probability. If she uses a Detection Error Probability of 50%, then she is not uncertainty-averse at all, while if she uses the lowest possible level of Detection Error Probability (which in our case is 2.08%), her uncertainty aversion is infinitely high. Figure 4.2 shows the optimal portfolio weights, which enable the fund manager to achieve the optimal exposure to the risk sources. Due to the inherent nature of affine term structure models, our fund manager takes a high short position in the money market account, and she uses this money to obtain a highly leveraged long position in the 1-year constant maturity bond fund.[14] In Table 4.2 we provide the numerical values of the optimal exposures and the optimal portfolio weights for different levels of risk aversion of robust and non-robust fund managers. A non-robust fund manager applies a Detection Error Probability of 50%, while a robust fund manager assumes a Detection Error Probability of 10%.

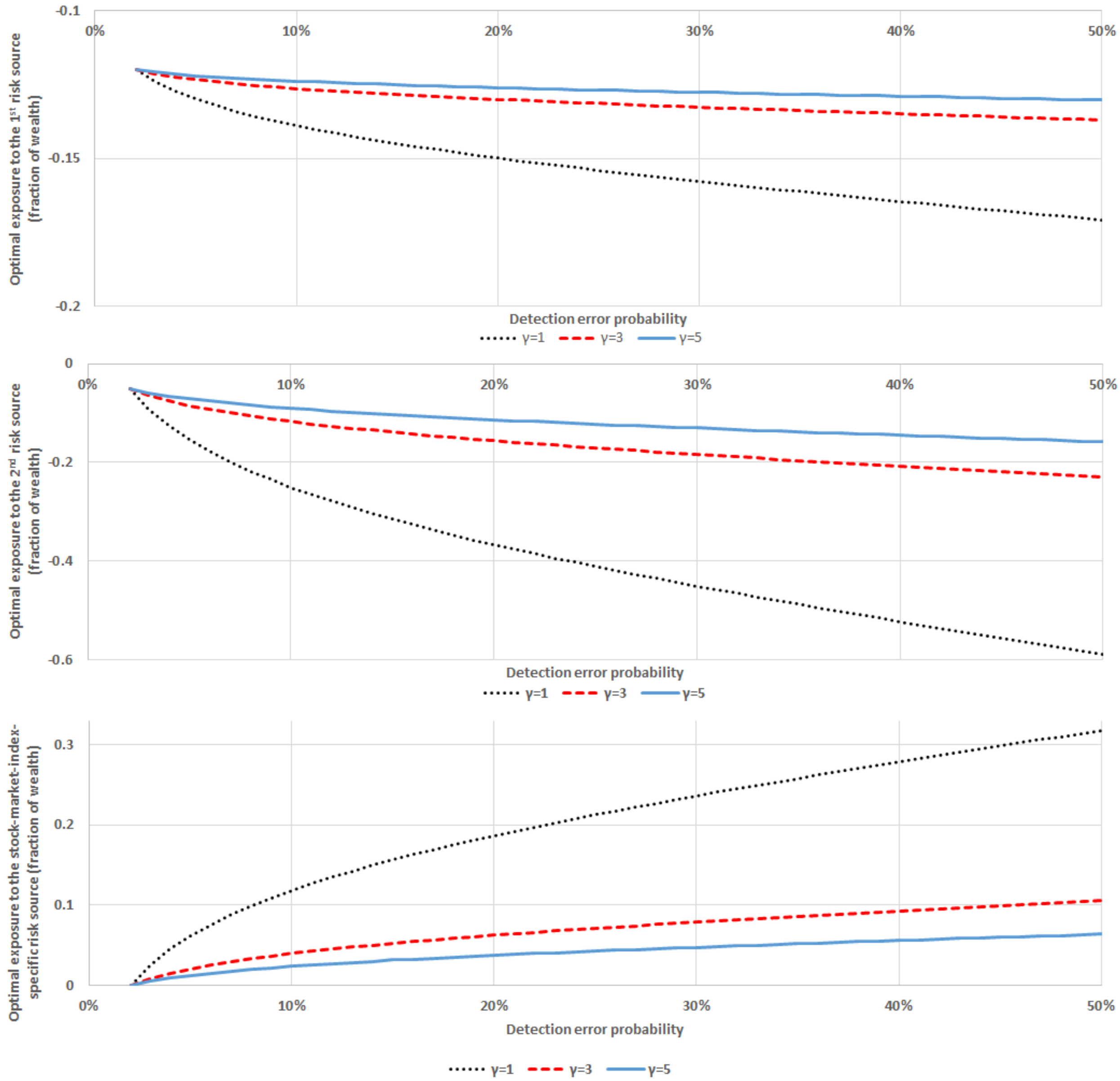
We find that the fund manager – regardless of her risk-aversion and uncertainty-aversion – always chooses a negative exposure to the first two risk sources. This is intuitive, since these two risk sources have negative market prices of risk; moreover, she has a positive exposure to these two risk sources due to her liabilities. But it is less straightforward why an infinitely risk-averse or an infinitely uncertainty-averse fund manager decides to take a strictly negative exposure to these risk sources, instead of opting for zero exposure. Given an infinitely high risk or uncertainty aversion,

[13]The exposure of the liability to the first two sources of risks is 0.12 and 0.05, respectively, while the liability exposure to the stock-market-index-specific risk source is zero.

[14]If there are no constraints on the position which the fund manager can take in the different assets, it is a common finding in the literature that she takes extremely large short and longe positions to achieve the optimal risk exposures. See, e.g., Brennan and Xia (2002), Figure 4 and Figure 6.

Figure 4.1. Optimal exposure to the risk sources

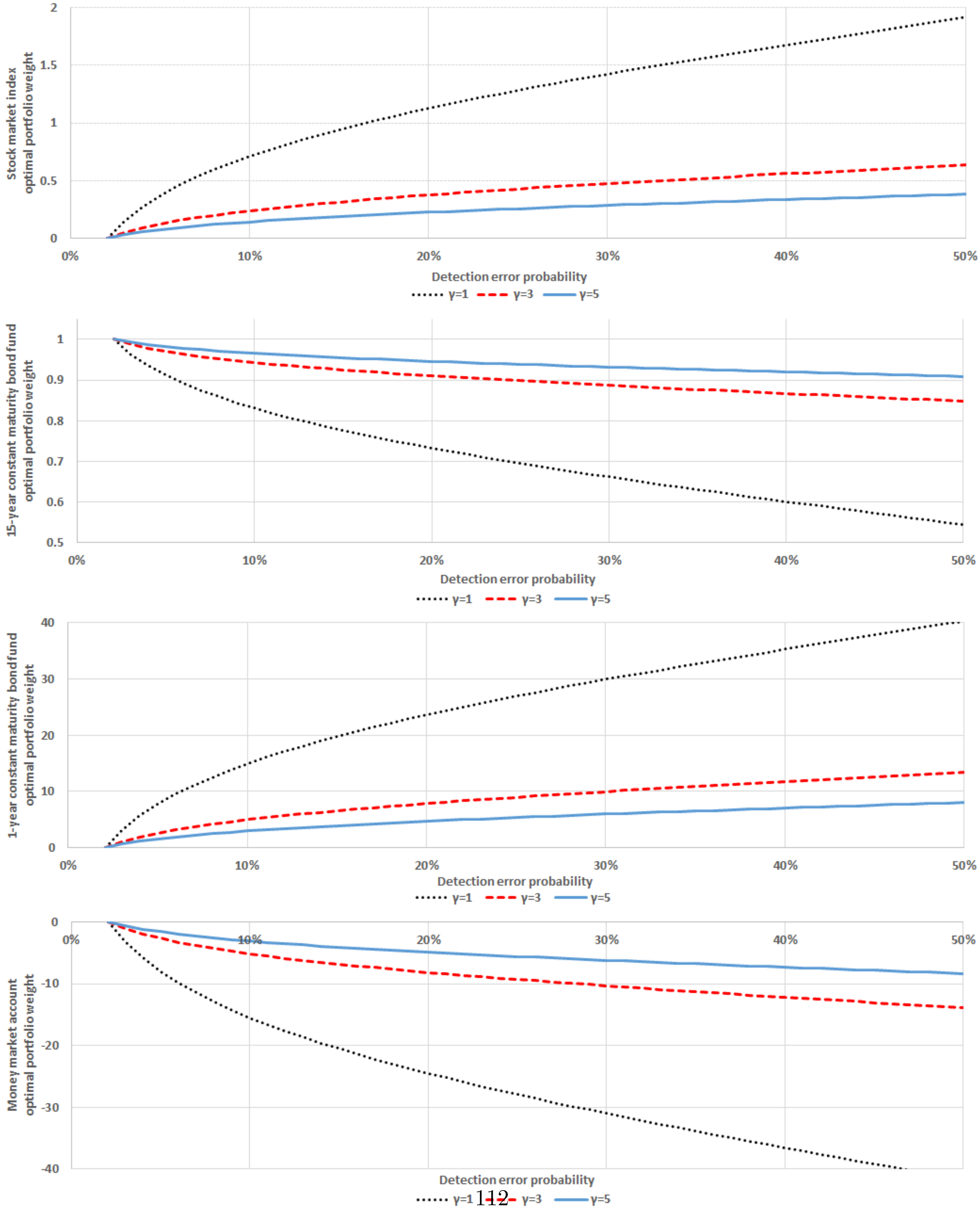
Optimal exposure of the fund manager to the risk sources as a function of the Detection Error Probability (DEP), for different levels of relative risk aversion. We use our parameter estimates in Table 4.1, and assume that the liability value is always equal to a zero-coupon bond with 15 years of maturity. The fund manager's investment horizon is 15 years. A DEP of 50% corresponds to a non-uncertainty-averse fund manager, while a DEP of 2.08% corresponds to a fund manager with infinitely high uncertainty aversion.



the fund manager's optimal ALM decision is to obtain a perfect hedge for the liabilities. Since the liabilities are a linear combination of the first two risk sources, she will expose herself to these risk sources to an extent which is equal to the exposure of the liabilities to them: 0.12 to the first

Figure 4.2. Optimal portfolio weights

Optimal portfolio weights as a function of the Detection Error Probability (DEP), for different levels of relative risk aversion. We use our parameter estimates in Table 4.1, and assume that the liability value is always equal to a zero-coupon bond with 15 years of maturity. The fund manager's investment horizon is 15 years. A DEP of 50% corresponds to a non-uncertainty-averse fund manager, while a DEP of 2.08% corresponds to a fund manager with infinitely high uncertainty aversion.



risk source, 0.05 to the second risk source, and 0 to the stock-market-index specific risk source. This is shown in Figure 4.2 and it can also be shown analytically by taking the limit of the right-hand-side of (4.26) as $\theta \to \infty$. Were the fund manager optimizing over the terminal wealth instead of the terminal funding ratio, her optimal exposure to the first two factors would be also strictly negative, but for a different reason: in this case her goal would be to achieve an exposure equal to that of a zero-coupon bond with the maturity of her investment horizon, thus eliminating risk and uncertainty totally, since she will receive the face value of the zero-coupon bond at the end of her investment horizon for sure. Looking at her decision from a different angle: her myopic demand for the constant maturity bond funds would be zero, and the entire total (strictly positive) demand would be due to the intertemporal hedging demand. In our case, when the fund manager optimizes over the terminal funding ratio, and she is either infinitely risk-averse or infinitely uncertainty-averse, her myopic demand for the constant maturity bond funds is zero, and her total demand is due to the liability hedge demand.

The exposure of an infinitely risk-averse or infinitely uncertainty-averse fund manager to the stock-market-index-specific source of risk is zero, because we assumed that the liability value is always equal to the value of a zero-coupon bond with 15 years of maturity, and the value of such a bond is not influenced by the stock-market-index-specific risk source. In absence of this assumption, the optimal exposure of an infinitely risk-averse or an infinitely uncertainty-averse fund manager would be $\sigma_{N+1,L}$, due to the reasoning in the previous paragraph.[15] If the fund manager were optimizing over the terminal wealth instead of the terminal funding ratio, her optimal exposure to the stock-market-index-specific source of risk would be zero even without our previous assumption about the liabilities, because her total demand would be equal to the intertemporal hedging demand, and the stock-market-specific risk source cannot be hedged intertemporally.

We also find that both a higher risk aversion and a higher uncertainty aversion result in a lower optimal exposure in absolute value to the risk sources, and in order to achieve this lower exposure the fund manager has lower optimal portfolio weight (again, in absolute value) in the constant maturity bond fund with 1 year of maturity and in the stock market index. Her optimal portfolio weight for the constant maturity bond fund with 15 years of maturity is, on the other hand, an

[15]The stock-market-index-specific risk source affects the liability of a pension fund if, e.g., the pension payout is linked to the industry wage level, and the industry wage level is affected by the stock-market-index-specific risk source via, e.g., performance-dependent wage schemes.

Table 4.2. Optimal risk exposures and portfolio weights
Optimal exposures and portfolio weights for different levels of risk aversion of robust and non-robust fund managers. The non-robust exposures and portfolio weights correspond to a Detection Error Probability of 50%, while their robust counterparts assume a Detection Error Probability of 10%. Optimal exposures and portfolio weights are calculated using the assumptions of Section 4.3. The exposure of the liability to the first two sources of risk is 0.12 and 0.05, respectively, while the liability exposure to the stock-market-index-specific risk source is zero.

	Optimal portfolio weights and exposures					
	$\gamma = 1$		$\gamma = 3$		$\gamma = 5$	
	Non-robust	Robust	Non-robust	Robust	Non-robust	Robust
Π_1	-0.17	-0.14	-0.14	-0.13	-0.13	-0.12
Π_2	-0.59	-0.25	-0.23	-0.12	-0.16	-0.09
Π_3	0.32	0.12	0.11	0.04	0.06	0.02
$\pi_{B(1)}$	4030%	1494%	1343%	498%	806%	299%
$\pi_{B(15)}$	54%	83%	85%	94%	91%	97%
π_S	192%	71%	64%	24%	38%	14%
π_{MMA}	-4176%	-1548%	-1392%	-516%	-835%	-310%

increasing function of both risk aversion and uncertainty aversion, due to its strong liability hedge potential.[16]

Accounting for uncertainty aversion has a substantial effect on both the optimal exposures to the risk sources and the optimal portfolio weights. We find that the optimal exposures (in absolute value) to each of the risk sources are a decreasing function of the level of robustness. I.e., the more uncertainty-averse the fund manager, the less exposure she finds optimal to each risk source. The decrease in the optimal exposure is especially substantial in the case of the stock-market-specific risk source (more than 60%) and the factor-specific risk source with a higher (absolute value of) market price of risk (more than 40%), while it is less significant in the case of the factor-specific risk source with a lower (absolute value of) market price of risk. The intuition behind this is that as the uncertainty aversion of the fund manager increases and approaches infinity, the optimal exposures approach the volatility loadings of the liability, concretely -0.1201, -0.0501, and 0. The optimal exposure of a non-robust fund manager with log-utility (i.e., $\gamma = 1$) to the first risk source is -0.17, hence there is not much scope for reduction in the magnitude of this exposure.[17] In contrast with this, the exposure of a non-robust fund manager with log-utility to the second and third risk

[16] The fact that the effects of risk aversion and uncertainty aversion have the same sign can directly be deduced from the risk-aversion parameter and the uncertainty-aversion parameter appearing only as a sum in the optimal portfolio weights in (4.30) and (4.31).

[17] I.e., even if her level of robustness is infinity, her optimal exposure would still be -0.1201, which is equal to the liability exposure to this risk source.

sources is relatively higher in magnitude (-0.59 and 0.32) due to their higher market price of risk (in absolute terms). Moreover, the magnitudes of liability exposures to the second and third risk sources are lower than that of the first risk source (-0.0501 and 0, respectively), therefore there is more scope for reduction in the optimal exposure as the uncertainty aversion increases. The lower exposure levels due to robustness translate to a lower demand for the constant maturity bond fund with 1 year of maturity (more than 62% decrease) and for the stock-market-index (also more than 62%). The demand for the constant maturity bond fund with a maturity equal to the investment horizon of the fund manager, however, increases with the level of robustness, due to its liability hedging potential.

4.6 Conclusion

We have shown that model uncertainty has significant effects on Asset Liability Management decisions. A fund manager who derives utility from the terminal funding ratio and who accounts for model uncertainty does not necessarily have an intertemporal hedging demand component, even though the investment opportunity set is stochastic. Robustness substantially changes the optimal exposures to the risk sources: as the level of uncertainty aversion increases, the optimal exposures approach the liability exposures to the respective risk sources. In the case of a two-factor affine term structure model and an additional, stock-market-specific risk source, optimal exposures can change by more than 60% due to robustness. These changes in the risk exposures translate into substantial changes in the optimal portfolio weights as well: while a robust fund manager invests less in the constant maturity bond fund with a relatively short maturity and also in the stock market index, she increases her investment in the constant maturity bond fund with a maturity equal to her investment horizon to make use of its liability hedge potential.

In our model we assume that the fund manager acts in the best interest of the pension holders, and she does not consider the interest of the pension fund sponsors. Extending the model to include the negative utility derived by the pension fund sponsors from having to contribute to the fund can provide further insight into the effects of model uncertainty on more complex Asset Liability Management decisions.

We also assume that the fund manager's uncertainty tolerance parameter is linear in the value function, hence the solution of our robust dynamic Asset Liability Management problem is homothetic, and it can be obtained in closed form. There is, however, an active and current debate in the literature whether this functional form of the uncertainty tolerance is justifiable. Solving our robust dynamic Asset Liability Managent problem with a differently formulated uncertainty tolerance parameter is another fruitful line of further research.

Appendix

Proof of Theorem 8. Substituting (4.17) and (4.18) into (4.15), the value function at time t satisfies

$$\begin{aligned} V_t\left[\left(\frac{A}{L}\right)_t\right] &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma} + \int_t^T \frac{(1-\gamma)\left\|\boldsymbol{u}(s)\right\|^2}{2\theta} V_s\left[\left(\frac{A}{L}\right)_s\right]\mathrm{d}s\right\} \\ &= \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma}\right\} + \mathrm{E}_t^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\left\|\boldsymbol{u}(s)\right\|^2}{2\theta} V_s\left[\left(\frac{A}{L}\right)_s\right]\mathrm{d}s\right\} \\ &\quad - \int_0^t \frac{(1-\gamma)\left\|\boldsymbol{u}(s)\right\|^2 V_s\left[\left(\frac{A}{L}\right)_s\right]}{2\theta}\mathrm{d}s, \end{aligned} \tag{4.38}$$

where $\left(\frac{A}{L}\right)_T$ and $\mathbb{U}$ denote the optimal terminal wealth and least-favorable physical measure, respectively. Introduce the square-integrable martingales, under $\mathbb{U}$,

$$M_{1,t} = \mathrm{E}_t^{\mathbb{U}}\left\{\exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma}\right\}, \tag{4.39}$$

$$M_{2,t} = \mathrm{E}_t^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\left\|\boldsymbol{u}(s)\right\|^2}{2\theta} V_s\left[\left(\frac{A}{L}\right)_s\right]\mathrm{d}s\right\}. \tag{4.40}$$

The martingale representation theorem (see, e.g., Karatzas and Shreve (1991), pp. 182, Theorem 3.4.15) states that there exist square-integrable stochastic processes $\boldsymbol{Z}_{1,t}$ and $\boldsymbol{Z}_{2,t}$ such that

$$M_{1,t} = \mathrm{E}_0^{\mathbb{U}}\left\{\exp(-\delta T)\frac{\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}}{1-\gamma}\right\} + \int_0^t \boldsymbol{Z}_{1,s}'\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}, \tag{4.41}$$

$$M_{2,t} = \mathrm{E}_0^{\mathbb{U}}\left\{\int_0^T \frac{(1-\gamma)\left\|\boldsymbol{u}(s)\right\|^2}{2\theta} V_s\left[\left(\frac{A}{L}\right)_s\right]\mathrm{d}s\right\} + \int_0^t \boldsymbol{Z}_{2,s}'\mathrm{d}\boldsymbol{W}_s^{\mathbb{U}}. \tag{4.42}$$

Substituting in (4.38), we can express the dynamics of the value function as

$$\mathrm{d}V_t\left[\left(\frac{A}{L}\right)_t\right] = -\frac{(1-\gamma)\left\|\boldsymbol{u}(t)\right\|^2}{2\theta} V_t\left[\left(\frac{A}{L}\right)_t\right]\mathrm{d}t + (Z_{1,t}+Z_{2,t})'\,\mathrm{d}\boldsymbol{W}_t^{\mathbb{U}}. \tag{4.43}$$

This linear backward stochastic differential equation with the terminal condition $V_T\left[\left(\frac{A}{L}\right)_T\right] = \exp(-\delta T)\left[\left(\frac{A}{L}\right)_T\right]^{1-\gamma}/(1-\gamma))$ has an explicit particular solution (see, e.g., Pham (2009), pp. 141-

142). The unique solution to (4.43) is given by

$$\Gamma_t V_t \left[\left(\frac{A}{L} \right)_t \right] = \mathrm{E}_t^{\mathbb{U}} \left\{ \Gamma_T \exp(-\delta T) \frac{\left[\left(\frac{A}{L} \right)_T \right]^{1-\gamma}}{1-\gamma} \right\}, \tag{4.44}$$

where Γ_t solves the linear differential equation

$$\mathrm{d}\Gamma_t = \Gamma_t \frac{(1-\gamma) \left\| \boldsymbol{u}(t) \right\|^2}{2\theta} \mathrm{d}t; \qquad \Gamma_0 = 1, \tag{4.45}$$

i.e.,

$$\Gamma_t = \exp \left(\int_0^t \frac{(1-\gamma) \left\| \boldsymbol{u}(s) \right\|^2}{2\theta} \mathrm{d}s \right). \tag{4.46}$$

Substituting into (4.44), we obtain the closed-form solution of the value function as

$$V_t(X_t) = \mathrm{E}_t^{\mathbb{U}} \left\{ \exp \left(\int_t^T \frac{(1-\gamma) \left\| \boldsymbol{u}(s) \right\|^2}{2\theta} \mathrm{d}s - \delta T \right) \frac{X_T^{1-\gamma}}{1-\gamma} \right\}, \tag{4.47}$$

with $\left[\left(\frac{A}{L} \right)_t \right]$ and $\mathbb{U}$ representing the optimal funding ratio and the least-favorable physical probability measure. As a result, we obtain (4.21). □

Proof of Theorem 9. The first step of the optimization is to determine the optimal terminal wealth, given the budget constraint. In order to determine the optimal terminal wealth, we form the Lagrangian from (4.21) and (4.22). This Lagrangian is

$$\begin{aligned} \mathcal{L}(A_0) &= \inf_{\mathbb{U}} \sup_{A_T} \left\{ \mathrm{E}^{\mathbb{U}} \exp \left(\frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T \right) \frac{\left(\frac{A_T}{L_T} \right)^{1-\gamma}}{1-\gamma} \right. \\ &\quad \left. - y \left[\mathrm{E}^{\mathbb{L}} \left(\frac{A_T}{L_T} \right) - \frac{A_0}{L_0} \right] \right\} \\ &= \inf_{\mathbb{U}} \sup_{A_T} \left\{ \mathrm{E}^{\mathbb{L}} \left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{L}} \right)_T \exp \left[\frac{1-\gamma}{2\theta} \int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T \right] \frac{\left(\frac{A_T}{L_T} \right)^{1-\gamma}}{1-\gamma} \right. \\ &\quad \left. - y \left[\mathrm{E}^{\mathbb{L}} \left(\frac{A_T}{L_T} \right) - \frac{A_0}{L_0} \right] \right\}, \end{aligned} \tag{4.48}$$

where y is the Lagrange-multiplier. Now we solve the inner optimization, taken $\mathbb{U}$ as given. The first-order condition for the optimal terminal funding ratio, denoted by $\hat{A}_T/L_T$, is

$$\frac{\hat{A}_T}{L_T} = \frac{y^{-\frac{1}{\gamma}}}{\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{L}}\right)_T^{-\frac{1}{\gamma}} \exp\left\{-\frac{1}{\gamma}\left[\frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T\right]\right\}}. \tag{4.49}$$

After substituting the optimal terminal funding ratio into the budget constraint, we obtain the Lagrangian as

$$y^{-\frac{1}{\gamma}} = \frac{A_0}{L_0 \mathrm{E}^{\mathbb{L}}\left\{\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{L}}\right)_T^{\frac{1}{\gamma}} \exp\left\{\frac{1}{\gamma}\left[\frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T\right]\right\}\right\}}. \tag{4.50}$$

Together with the Radon-Nikodym derivative

$$\begin{aligned}\left(\frac{\mathrm{d}\mathbb{U}}{\mathrm{d}\mathbb{L}}\right)_t =& \exp\left\{\int_0^t \left(\boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{u}(s)\right)' \mathrm{d}\boldsymbol{W}_s^{\mathbb{L}}\right. \\ &\left. + \int_0^t \left[\left(\boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{u}(s)\right)' \boldsymbol{\sigma}_L - \frac{1}{2}\left(\|\boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{\sigma}_L + \boldsymbol{u}(s)\|^2 - \|\boldsymbol{\sigma}_L\|^2\right)\right] \mathrm{d}s\right\},\end{aligned} \tag{4.51}$$

we substitute the Lagrangian back into (4.49) to determine the optimal terminal funding ratio as

$$\frac{\hat{A}_T}{L_T} = \frac{A_0}{L_0} \frac{\exp\left[\frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right]}{\mathrm{E}^{\mathbb{L}} \exp\left[\frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{\gamma}\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}\right]}. \tag{4.52}$$

Multiplying both sides by L_T, we obtain (4.23), and this proves the first part of Theorem 9. Now we solve the outer optimization problem. Substituting the optimal terminal wealth back into the value function, we obtain

$$\begin{aligned}V_0(A_0) =& \frac{\left(\frac{A_0}{L_0}\right)^{1-\gamma}}{1-\gamma} \exp\left(\frac{1-\gamma}{2\theta}\int_0^T \|\boldsymbol{u}(t)\|^2 \mathrm{d}t - \delta T - \frac{1}{2}\int_0^T \left[\|\boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{\sigma}_L + \boldsymbol{u}(t)\|^2 - \|\boldsymbol{\sigma}_L\|^2\right] \mathrm{d}t\right) \\ & \times \exp\left(\int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\right)' \boldsymbol{\sigma}_L \mathrm{d}t + \frac{1}{2\gamma}\int_0^T \|\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}}\|^2 \mathrm{d}t\right).\end{aligned} \tag{4.53}$$

Now we can write down the first-order condition for $\boldsymbol{u}(t)$ and we obtain

$$\boldsymbol{u}(t) = -\frac{\theta}{\gamma+\theta}\boldsymbol{\lambda}^{\mathbb{L}}, \tag{4.54}$$

which is indeed the same as (4.24). This completes the proof. □

Proof of Corollary 3. The optimal wealth process can be written as

$$\hat{A}_t = L_t \mathrm{E}_t^{\mathbb{L}} \left(\frac{\hat{A}_T}{L_T} \right). \tag{4.55}$$

Substituting the optimal terminal wealth (4.23) into (4.55), the optimal wealth at time t becomes

$$\begin{aligned} \hat{A}_t =& \frac{A_0}{L_0} L_t \exp\left[\frac{1}{\gamma} \int_0^t \left(\hat{\boldsymbol{u}}(s) + \boldsymbol{\lambda}^{\mathbb{L}} \right)' \mathrm{d}\boldsymbol{W}_s^{\mathbb{L}} + \frac{1}{\gamma} \int_0^T \left(\hat{\boldsymbol{u}}(s) + \boldsymbol{\lambda} \right)' \boldsymbol{\sigma}_L \mathrm{d}s \right] \\ & \times \frac{\exp\left(\frac{1}{2\gamma^2} \int_t^T \| \hat{\boldsymbol{u}}(s) + \boldsymbol{\lambda}^{\mathbb{L}} \|^2 \mathrm{d}s \right)}{\mathrm{E}^{\mathbb{L}} \exp\left[\frac{1}{\gamma} \int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}} \right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}} + \frac{1}{\gamma} \int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}} \right)' \boldsymbol{\sigma}_L \mathrm{d}t \right]}. \end{aligned} \tag{4.56}$$

Substituting the solution of the stochastic differential equation (4.4) for L_t, the optimal wealth at time t becomes

$$\begin{aligned} \hat{A}_t =& A_0 \exp\left\{ \int_0^t \left(\frac{1}{\gamma} \left(\hat{\boldsymbol{u}}_F + \boldsymbol{\lambda}_F^{\mathbb{L}} \right) + \boldsymbol{\sigma}_{F,L} + \boldsymbol{\sigma}_F' \boldsymbol{B}(t-s) \boldsymbol{\iota} \right)' \mathrm{d}\boldsymbol{W}_{F,s}^{\mathbb{L}} \right. \\ & + \int_0^t \left(\frac{1}{\gamma} \left(\hat{\boldsymbol{u}}_{N+1} + \boldsymbol{\lambda}_{N+1}^{\mathbb{L}} \right) + \boldsymbol{\sigma}_{N+1,L} \right)' \mathrm{d}\boldsymbol{W}_{N+1,s}^{\mathbb{L}} + \int_0^t \boldsymbol{\iota}' \boldsymbol{B}(t-s) \boldsymbol{\sigma}_F \boldsymbol{\sigma}_L \mathrm{d}s \\ & + \left(A_0 + \boldsymbol{\iota}' \left(\boldsymbol{\mu}_F^{\mathbb{L}} - \boldsymbol{\sigma}_F \boldsymbol{\sigma}_L \right) \right) t + \boldsymbol{\iota}' \boldsymbol{B}(t) \left(\boldsymbol{F}_0 - \left(\boldsymbol{\mu}_F^{\mathbb{L}} - \boldsymbol{\sigma}_F \boldsymbol{\sigma}_L \right) \right) + \frac{1}{2} \| \boldsymbol{\sigma}_L \|^2 t \\ & \left. + \frac{1}{\gamma} \int_0^T \left(\hat{\boldsymbol{u}}(s) + \boldsymbol{\lambda}^{\mathbb{L}} \right)' \boldsymbol{\sigma}_L \mathrm{d}s \right\} \times \frac{\exp\left(\frac{1}{2\gamma^2} \int_t^T \| \hat{\boldsymbol{u}}(s) + \boldsymbol{\lambda}^{\mathbb{L}} \|^2 \mathrm{d}s \right)}{\mathrm{E}^{\mathbb{L}} \exp\left[\frac{1}{\gamma} \int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}} \right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}} + \frac{1}{\gamma} \int_0^T \left(\hat{\boldsymbol{u}}(t) + \boldsymbol{\lambda}^{\mathbb{L}} \right)' \boldsymbol{\sigma}_L \mathrm{d}t \right]}. \end{aligned} \tag{4.57}$$

Applying Ito's lemma, the optimal wealth dynamics can be expressed as

$$\mathrm{d}\hat{A}_t = ...\mathrm{d}t + \frac{\hat{A}_t}{\gamma} \left(\hat{\boldsymbol{u}} + \boldsymbol{\lambda}^{\mathbb{L}} + \gamma \boldsymbol{\sigma}_L \right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}. \tag{4.58}$$

Substituting the least-favorable distortion for $\hat{\boldsymbol{u}}$, we obtain

$$\mathrm{d}\hat{A}_t = ...\mathrm{d}t + \hat{A}_t \left(\frac{1}{\gamma + \theta} \boldsymbol{\lambda}^{\mathbb{L}} + \boldsymbol{\sigma}_L \right)' \mathrm{d}\boldsymbol{W}_t^{\mathbb{L}}. \tag{4.59}$$

From (4.59) the optimal risk exposures follow directly. This completes the proof. □

Proof of Corollary 4. Using the portfolio weights $\boldsymbol{\pi}_{B,t}$ and $\boldsymbol{\pi}_{\boldsymbol{S},\boldsymbol{t}}$, the optimal wealth dynamics can be expressed as

$$\mathrm{d}\hat{A}_t = ...\mathrm{d}t + \hat{A}_t \left(-\boldsymbol{\pi}'_{B,t}\mathcal{B}'\boldsymbol{\sigma}_F \mathrm{d}\boldsymbol{W}^{\mathbb{L}}_{F,t} + \pi_{S,t} \left[\boldsymbol{\sigma}'_{F,S}; \sigma_{N+1,S} \right] \mathrm{d}\boldsymbol{W}^{\mathbb{L}}_t \right). \tag{4.60}$$

Moreover, the optimal wealth dynamics can equivalently be written as (4.59). Then, due to the martingale representation theorem we can write down a system of N+1 equations by making the exposures to the N+1 risk sources in (4.59) and (4.60) equal to each other. Solving this equation system, we indeed obtain the optimal portfolio weights (4.30) and (4.31). This completes the proof. □

Proof of Theorem 10. From (4.11) and (4.14) the Radon-Nikodym derivative $\mathrm{d}\mathbb{B}/\mathrm{d}\mathbb{U}$ follows directly. Substituting this into Definition 6 and evaluating the expectations, the closed-form solution in (4.36) is obtained. □

Proof of Corollary 5. Substituting the least-favorable distortions (4.24) into (4.36), (4.37) is immediately be obtained. □

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