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**STABILIZATION OF AN UNCERTAIN SIMPLE FISHERY
MANAGEMENT GAME**

By

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Stabilization of an uncertain simple fishery management game*

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July, 2017

Abstract: This note analyzes in a fishery management problem the effects of relaxing one of the usual assumptions in the literature of dynamic games. Specifically, the assumption that players restrict to strategies that stabilize the system. Previous works in the literature have shown that feedback Nash equilibria can exist in which a player can improve unilaterally by choosing a feedback control for which the closed-loop system is unstable. This paper considers in some more detail the implication this setting has in the framework of a simple fishery management.

It is shown that if the fishermen are not short-sighted in their valuation of future profits, the considered approach implies a division of them into two groups. One group going for maximal profits and the other group taking care of the imposed stability constraint.

To see how noise might impact these results we additionally consider a framework where fishermen take into account the possibility that the fish growth might be corrupted by external factors. We consider a deterministic approach of dealing with noise in this set-up.

The model predicts that the more people are involved in the group taking care of the stabilization constraint, the less active they get. Furthermore it predicts that the natural reaction of any of these persons is to increase his activities if he expects more noise. But that this activity is reduced, and partly replaced by more active control policies by group members, if the size of the group increases. Activity of fishermen going for maximal profits is not affected by noise expectations.

Keywords: fishery management, fishing strategies, linear quadratic differential games, feedback information structure, soft-constrained Nash equilibrium, infinite planning horizon.

Jel-codes: Q22, Q28, Q56, C61, C72, C73.

1 Introduction

Dynamic game theory brings together three features that are key to many situations in economy, ecology, and elsewhere: optimizing behavior, presence of multiple agents, and enduring consequences of decisions. For that reason this framework is often used to analyze various policy problems in these areas (see e.g. [11], [21] and [25]).

In this note we will use this framework to analyze a fishery management problem. Growth of the fish stock is mathematically modeled using a (set of) differential equation(s) including so-called input functions, modeling the fishing that occurs by the various fishermen over time. We assume

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the fishermen like to maximize total discounted profits over an infinite time period (their so-called performance function).

Since unlimited growth of the fish stock is unrealistic, it is usually assumed that the fishermen (players in the game) use fishing strategies that are such that the controlled fish stock converges to a steady state. This automatically implies that the performance functions take finite values, so that the involved optimization problems are well-posed. If a linear description of the fish stock growth is assumed, this mostly leads to the consideration of fishing strategies that depend (linearly) on the current stock of the population and which enforce the stock to converge to a steady state. As shown by [23] it may happen under such assumptions Nash equilibria exist where in fact players can improve still unilaterally by choosing an input function outside the presumed class of admissible input functions. This shows that the assumption that players stick to this kind of stabilizing control functions is really restrictive, and the question arises whether consideration of a larger set of admissible input functions may lead to different Nash equilibria for this fishery management game. In literature, theory has been developed how one might deal with cases where the assumption that the performance functions assume finite values is not made, see, e.g., [6] or [7]. Within the current context such an extension seems however too general. We will restrict here to the consideration of a class of input functions which yield a finite value for the performance but which not, beforehand, necessarily imply that the fish stock converges. We will see that under this assumption, under most parameter constellations, still Nash equilibria for the fishery management game result, with corresponding strategies that yield a steady state value of the fish stock. On the other hand, we will see that by considering this broader class of potential fishing strategies the game has a more rich structure, leading to interpretations which may make sense in practice. Two main conclusions are that, first, always a class of fishermen exists that takes care of a good management of the fish stock over time and, second, it is in fact from a stabilization efficiency point of view best that there exists just one fisherman taking this job.

In the past already various fishery management problems have been studied using the framework of dynamic games. One well-known early reference in this area is [8]. In that paper a very general model of fishing is introduced and next describes four sets of conditions under each of which there is a socially optimal open-loop Nash equilibrium path. For two simplified, though still non-linear, versions of this model [24] considered the existence of Markov perfect Nash equilibria. Except for the usual cooperative and non-cooperative information structures (see, e.g., [29] for a general survey) also problems have been addressed where one fisherman has a first mover advantage (see, e.g., [2]). Furthermore, incentive strategies that establish the efficient solution as an incentive equilibrium have been developed (see, e.g., [13]). And, more recently, also nonlinear incentive strategies have been proposed by [12] which are credible in a large region of the cooperative trajectory.

Clearly, most dynamic models that are considered are just an approximation of reality. If an accurate model can be formed at all, it will in general be complicated and difficult to handle. Moreover it may be unwise to optimize on the basis of a too detailed model, in view of possible changes in dynamics that may take place in the course of time and that may be hard to predict. It makes more sense for agents to work on the basis of a relatively simple model and to look for strategies that are robust with respect to deviations between the model and reality. Therefore, as a second item in this study, we consider the impact the introduction of uncertainty into the model has on above mentioned results.

In an economic context, the importance of incorporating aversion to specification uncertainty has been stressed for instance in [17]. Two strands of literature emerged to deal with uncertainty.

The first, and eldest, is to model uncertainty using a stochastic framework. Examples of fishery

management problems where the effect of uncertainty is modeled using a stochastic framework are, e.g., [10], [18], [20] and [30]¹. A disadvantage of the stochastic approach is that it requires an accurate specification of the statistics and information players have about the game.

Therefore in the early 1980's a second approach was introduced to deal with uncertainty (see also [27] for an early contribution in this area in economic science). This approach boils down to the design of a worst-case controller. That is, to design a control that performs well even when "nature" chooses an outcome, from a predefined set of disturbances, that is maximally harmful to the agent. In control theory, this theory of robust design is well documented in, e.g., [1]. We use this background here to arrive at a simple "uncertain" fishery management model.

Following this second approach, a malevolent disturbance input is introduced which is used in the modeling of aversion to specification uncertainty. That is, it is assumed that the growth of the fish stock is corrupted by a deterministic noise component, and that each fisherman has his own expectation about this noise. This is modeled by adapting for each fisherman his performance function accordingly. The fishermen cope with this uncertainty by considering a worst-case scenario. Consequently in this approach the equilibria of the game, in general, depend on the worst-case scenario expectations about the noise of each fisherman. We will see that the introduction of this deterministic noise does not impact the previous mentioned main results. Furthermore, we will see that in general fishermen intensify their stabilization efforts in case they expect more noise.

In this note we will assume that all fishermen act non-cooperative and have complete information on the fish stock at each point in time (that is a feedback information structure is considered). For results dealing with an open-loop information structure in a deterministic uncertain environment see e.g. [14, Chapter 7.4], [1], [22], [19] or [15]. This note builds on a preliminary two-person version of this game studied by the author in [14][Section 9.6]. We extend here the analysis to a general N -player setting. As a consequence a number of interpretations that were not clear for the 2-player context result.

The outline of the paper is as follows. Next section formalizes the problem statement and summarizes the main theoretical results. In section three we discuss in some more detail the interpretation of theoretical results for the fishery management problem. Section four discusses a number of shortcomings of the model studied here, and how they might be tackled in future studies. The last section summarizes and concludes. The paper has a number of appendices, mainly containing theoretical material. Appendix A summarizes a number of theoretical results that can be found elsewhere in literature and which are used to derive the main results presented from this paper, in Appendix B. Appendix C contains the proof of Theorem 2.5, whereas Appendix D substantiates a number of statements concerning the closed-loop dynamics of the fish stock.

2 A fishery management game

Consider N fishermen who fish a lake. Let $s(t)$ be the fish stock in the lake at time t . Assume that the price $p(t)$ the fishermen get for their fish is fixed, i.e.,

$$p(t) = p.$$

¹Note that this stochastic approach in a multi-player setting requires a good specification about what information players have on the game at each point in time. This may be a tricky point, see e.g. [26].

The growth of the fish stock in the lake is described by next linear differential equation

$$\dot{s}(t) = \beta s(t) - \sum_{i=1}^N u_i(t) - w(t), \quad s(0) = s_0 > 0. \quad (2.0.1)$$

Here $\beta > 0$ is the natural growth rate of the fish stock; u_i is the amount of fish caught by fisherman i ; and w is the aggregate of uncertain factors (like water pollution, weather, birds, local fishermen etc.) that impact the growth of the fish stock. We assume that the fishing strategy of every fisherman consists of fishing an amount depending on the available stock, $f_i s(t)$, additional to some fixed amount of fish, g_i . We do not assume upfront that all fishermen will stick to a policy that aims at stabilizing the fish stock. We allow for strategies resulting in some limited exponential growth of the fish stock in the lake. That is, the fishing strategies $u_i, i \in \mathbf{N}^2$, belong to the set:

$$\mathcal{F}^{aff} := \{(u_1, \dots, u_N) \mid u_i(t) = f_i s(t) + g_i, \text{ with } \beta - \sum_{i=1}^N f_i < \frac{1}{2}r\}. \quad (2.0.2)$$

Here r denotes the discount factor, the fishermen use in the valuation of future profits. This assumption implies we consider a very broad class of potential realizations of the future fish stock³.

Every fisherman has his own expectations about the consequences uncertain factors have on the fish growth, which is incorporated in his performance criterion. Given his expectations about the uncertainty, for every fishing strategy he uses (and his colleagues), there will be a realization of the uncertain variable that provides the worst outcome for his performance. We assume that ideally every fisherman is looking for that fishing strategy that gives him the best performance under all these worst-case outcomes.

In case no noise corrupts the fish growth (i.e. $w(t) = 0$ in (2.0.1)), we assume that every fisherman wants to maximize total discounted profits. Or, equivalently, determine the solution of next minimization problem.

$$J_i := \min_{u_i \in \mathcal{F}^{aff}} \int_0^\infty e^{-rt} \{-pu_i(t) + \gamma_i u_i^2(t)\} dt, \quad i \in \mathbf{N}. \quad (2.0.3)$$

Note that in this set-up the fishermen do not care about the fish stock. This, on the one hand may be a quite strong assumption. On the other hand we will see that it permits to solve the problem analytically and to clearly demonstrate the most important point of this paper. In Section 4 we will discuss this assumption in some more depth.

In case noise might corrupt the fish growth, we assume every fisherman wants to determine

$$J_i^{WC} := \min_{u_i \in \mathcal{F}^{aff}} \sup_{w \in L_2} \int_0^\infty e^{-rt} \{-pu_i(t) + \gamma_i u_i^2(t) - v_i w^2(t)\} dt, \quad i \in \mathbf{N}. \quad (2.0.4)$$

In this formulation all constants, r, γ_i and v_i , are positive. In above performance functions the term pu_i models the revenues fisherman i obtains from fishing and the term $\gamma_i u_i^2$ models the cost involved

² \mathbf{N} denotes the set $\{1, \dots, N\}$

³In principle this might lead to equilibria where the fish stock grows forever. However, we will see that usually this does not occur.

for him in catching an amount u_i of fish⁴. We will assume that in case the fishermen expect that noise might corrupt the fish growth, $v_i > \gamma_i$, $i \in \mathbf{N}$. That is, each fisherman does not expect that a situation will occur where the cost implied by unforeseen disturbances, measured by $v_i w^2$, will be larger than his normal cost of operation, measured by $\gamma_i u^2$. If this inequality does not hold, one might argue that the model does not mimic reality enough, and a better modeling of the involved noise should be looked for.

Notice that, since in this formulation the involved cost for the fishermen depends quadratically on the amount of fish they catch, catching large amounts of fish is not profitable for them. This observation might model the fact that catching a large amount of fish is, from a practical point of view, impossible for them. This might be due to either technical restrictions and/or the fact that there is not an abundant amount of fish in the lake. That is, catching much more fish requires much more advanced technology which costs rise quadratically. Furthermore in this formulation, the more external noise the fisherman expects, the lower he should choose the parameter v (observe, e.g., that with choosing v very large, his corresponding worst-case noise signal is approximately $w(\cdot) = 0$. See also the Appendix for some additional explanation.).

Since every fisherman affects by his fishing the fish stock and, therefore implicitly, the performance of all other fishermen, the actual amount every fisherman will catch is unclear yet. A well-known assumption, implying also a situation where it is clear what every fisherman will catch, is that every fisherman will fish such an amount that if he would change unilaterally this amount, his performance will degenerate. So, it is rational for every fisherman to stick to this policy. Strategies $(u_1, \dots, u_N) \in \mathcal{F}^{aff}$ satisfying this assumption are called Nash equilibria. In case the system is corrupted by noise we assume every fisherman first determines, for fixed strategies of his colleagues, the for him worst-case outcome that might result with respect to noise. This leads for every choice of strategies $(u_1, \dots, u_N) \in \mathcal{F}^{aff}$ for every fisherman to a corresponding worst-case performance. In case, based on these worst-case performances, strategies are such that no fisherman has an incentive to deviate unilaterally from his strategy we call them a soft-constrained Nash equilibria (SNE). The phrase "soft-constrained" is chosen here, because no hard constraints (bounds) are imposed in this formulation on the noise.

In Appendix A we included a formal definition of this equilibrium concept together with some theoretical results how to determine these equilibria in case the system is described by a linear differential equation and the performance criteria are such that they quadratically penalize both state and control variables. The results are provided both for a noise-free context and for the case players take noise into consideration. Results for this general setting are then used in Appendix B to arrive at the results presented below for this specific fishery management problem.

Both Theorem 2.2 and Theorem 2.3, below, show that the equilibrium outcome depends on whether the fishermen are short-sighted or not concerning their future instantaneous profits. Next definition formalizes this notion of short-sightedness. The fishermen are called short-sighted in case the discount factor they use in valuating future profits is larger than twice the natural growth rate of the fish. That is, in case future profits play only a minor role in their decision making⁵

Definition 2.1 *The fishermen are called short-sighted if and only if $r > 2\beta$.*

⁴One can also interpret it as that every fisherman maximizes total discounted profits on an own market where the inverse demand function, $D_i(u)$, is given by $D_i(u) := p - \gamma_i u$.

⁵In economic literature one often introduces the notion of "myopic players" under such conditions. These are players that disregard the dynamics of the state variable when solving their optimization problem. Note that this is not assumed here.

Next two theorems are proved in Appendix B. In these theorems the index set $\mathbf{N}_1 \subset \mathbf{N}$ contains all indices of the fisherman that catch a constant amount of fish over time, and $\bar{\mathbf{N}}_1 \neq \emptyset$ the set of the remaining fishermen.

Theorem 2.2 Consider the case no noise corrupts the fish stock growth. Let $\alpha := \frac{p}{2} \sum_{i=1}^N \frac{1}{\gamma_i}$.

1. If the fishermen are short-sighted the fishery management game (2.0.1, 2.0.3) has a unique Nash equilibrium within the control action set \mathcal{F}^{aff} (2.0.2).
All fishermen catch a constant amount of fish in this equilibrium: $u_i(t) = \frac{p}{2\gamma_i}$, $t \geq 0$. Total profits of fisherman i are: $J_i = \frac{p^2}{4r\gamma_i}$.
2. If the fishermen are not short-sighted the fishery management game has $2^N - 1$ equilibria within the control action set \mathcal{F}^{aff} (2.0.2).
All equilibria are characterized by N_1 fishermen j , $j \in \mathbf{N}_1$, fishing a constant amount of fish and the rest, $N - N_1$, $j \in \bar{\mathbf{N}}_1$, catching an amount that additionally depends on the fish stock. The amount of fish caught by fisherman $i \in \mathbf{N}_1$ is: $u_i(t) = \frac{p}{2\gamma_i}$, $t \geq 0$. The amount of fish caught by fisherman $j \in \bar{\mathbf{N}}_1$ is: $u_j(t) = \bar{x}_j s(t) + \frac{p}{2\gamma_j} - \frac{\alpha}{\beta} \bar{x}_j$, $t \geq 0$, where $\bar{x}_j = \frac{2\beta - r}{-1 + 2(N - N_1)}$.
Total profits fisherman $i \in \mathbf{N}_1$ gets are: $J_i = \frac{p^2}{4r\gamma_i}$;
total profits fisherman $j \in \bar{\mathbf{N}}_1$ gets are: $J_j = -(s_0 + \frac{-\alpha}{\beta})^2 \bar{x}_j \gamma_j + \frac{p^2}{4r\gamma_j}$. □

Theorem 2.3 Let $\alpha := \frac{p}{2} \sum_{i=1}^N \frac{1}{\gamma_i}$.

1. If the fishermen are short-sighted the fishery management game has a unique soft-constrained Nash equilibrium.
All fishermen catch a constant amount of fish in this equilibrium: $u_i(t) = \frac{p}{2\gamma_i}$, $t \geq 0$. The minimal profits fisherman i expects are: $J_i^{WC} = \frac{p^2}{4r\gamma_i}$.
2. If the fishermen are not short-sighted the fishery management game has $2^N - 1$ equilibria. All equilibria are characterized by N_1 fishermen j , $j \in \mathbf{N}_1$, fishing a constant amount of fish and the rest, $N - N_1$, $j \in \bar{\mathbf{N}}_1$, catching an amount that additionally depends on the fish stock. The amount of fish caught by fisherman $i \in \mathbf{N}_1$ is: $u_i(t) = \frac{p}{2\gamma_i}$, $t \geq 0$. The amount of fish caught by fisherman $j \in \bar{\mathbf{N}}_1$ is: $u_j(t) = x_j s(t) + \frac{p}{2\gamma_j} - \frac{\alpha}{\beta} x_j$, $t \geq 0$, where $x_j = \frac{2\beta - r}{-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i}} \frac{v_j}{\gamma_j + v_j}$.
The minimal profits fisherman $i \in \mathbf{N}_1$ expects are: $J_i^{WC} = \frac{p^2}{4r\gamma_i}$;
minimal profits fisherman $j \in \bar{\mathbf{N}}_1$ expects are: $J_j^{WC} = -(s_0 + \frac{-\alpha}{\beta})^2 x_j \gamma_j + \frac{p^2}{4r\gamma_j}$. □

One easily verifies that Theorem 2.2 coincides with Theorem 2.3, if one considers in this last-mentioned theorem v_i going off to infinity.

With respect to the case that fishermen do take noise into consideration we like to stress next observations.

Remark 2.4

1. Note that in the short-sighted equilibrium the equilibrium actions do not depend on the expected noise.

2. Note that the actual cost the fishermen are confronted with depends on the realization of the exogenous noise. In particular this implies that for almost every fisherman the expected worst-case cost will not realize. That is, profits will exceed his minimal expected value for them.
3. In case the fishermen are not short-sighted it follows from our parameter assumptions ($v_i > \gamma_i$ and $2\beta - r > 0$) that $x_i > 0$. Obviously, the worst-case cost for the fishermen who catch a fixed amount do not depend on these expectations since they catch the same fixed amount whatever the noise expectations are. That is, even under the worst-case disturbance that may occur to them, given their expectations, total profits remain the same for them. On the other hand we see that the fishermen whose catch depends on the fish stock are less optimistic under this scenario, as they expect a total profit that is lower than in case they would get when catching a fixed amount of fish over time. Note, in case the number of these fishermen is very large, their minimal expected profit almost coincides with the profit they would get when fishing a constant amount.

Furthermore, instantaneous profits (measured by $pu_i(t) - \gamma_i u_i^2(t)$) for the fishermen catching a constant amount of fish are $\frac{p^2}{4\gamma_i}$ whereas for the other fishermen they are $\frac{p^2}{4\gamma_j} - \frac{\gamma_j x_j^2}{\beta^2} (\beta s(t) - \alpha)^2$.

Before discussing in some more detail implications of the above results, we consider some additional results for the case fishermen are not short-sighted and take noise into consideration. For that purpose note that for a given stock of fish, $s(t)$, the parameter x_j determines how active the fishermen will respond to a change in their expectations about disturbances affecting this fish stock. In case this parameter increases (decreases) we will say fisherman j uses a more (less) active stabilization policy. Next result is shown in Appendix C.

Theorem 2.5 *Consider the case fishermen are not short-sighted and expect noise might impact the fish stock. Then for fishermen not catching a constant amount of fish we have next results.*

1. *Assume there is just one fisherman not catching a constant amount of fish. Then, the more this fisherman expects the fish stock will be disturbed by external factors,*
 - a. *the more active stabilization policies he will use.*
 - b. *his expected returns under the worst-case scenario decrease.*
2. *Assume there is more than one fisherman who is not catching a constant amount of fish. Then, the more fisherman j expects the fish stock will be disturbed by external factors*
 - a. *this fisherman j will use less active stabilization policies.*
 - b. *all other fishermen $k \in \bar{\mathbf{N}}_1$ pursue a more active stabilization policy.*
 - c. *the expected returns of this fisherman j under the worst-case scenario increase, whereas they decrease for all other fishermen $k \in \bar{\mathbf{N}}_1$.*

3 Discussion of Results

From the previous section we conclude that when fishermen are short-sighted, all fishermen will fish a fixed amount of fish, independent of the fact whether they expect the fish stock could be disturbed

by external factors or not. In particular the amount of fish they catch does not depend on these expectations.

In case fishermen are not short-sighted, we see fishing the lake can only survive in case there are "fishermen" who care about the level of the fish stock. Both Theorem 2.2 and Theorem 2.3 show that there must exist "altruistic" fishermen, i.e. fishermen that do not go for a maximum total profit, in order to realize a sustainable equilibrium. Since these fishermen can be viewed as (local) authorities/governments taking care of the fish stock in the lake, we will call them for ease of notation "governments" from now on. The amount of fish caught by these governments at any point in time, t , consists of a constant catch and a catch depending on the fish stock at time t . We already named the last mentioned catch the stabilization policy of a government. This, since this catch determines the growth of the fish stock in the long run (see Appendix D). Depending on whether $\frac{p}{2\gamma_j} > \frac{\alpha}{\beta}x_j$ (or, $\frac{p}{2\gamma_j} > \frac{\alpha}{\beta}\bar{x}_j$ in the noise free case) or, the reverse inequality holds, we see that the constant catch by the governments can be either positive or negative. All parameters in the model impact this sign. In case this catch is negative, we interpret this as that the government will plant fish. So, a government's fishing policy consists of on the one hand pursuing an active stabilization policy and, additional to that, on the other hand a policy consisting of either planting or catching a constant amount of fish at any point in time. In particular we see from above inequality that the lower the price of fish is, the higher is the chance that a government will pursue a planting policy. This, to create enough future revenues.

From Appendix D, formula (5.0.22) it follows that, assuming normal operating conditions hold (that is, when time proceeds the fish stock converges to some fixed amount), combined fishing policies by all fishermen result in an active stabilization policy complemented with a positive fixed catch at any point in time. The converged fish stock always equals the same number: $\frac{\alpha}{\beta}$. In particular, this stock does not depend on how many of the fishermen act as governments. Also in case fishermen take noise into consideration fish stock always converges to this same number (assuming the realization of noise is actually zero, i.e. $w(\cdot) = 0$ occurs, and the normal operating conditions apply). This steady-state value indicates that a higher price for the fish and/or more fishermen fishing the lake requires maintaining a higher steady-state of the fish stock.

Another point the model predicts is that the stabilization efforts by a government declines the larger the number of involved governments gets (see Theorem 2.2, item 2), whereas his constant catch increases with this number (or, planting decreases). We observe the same effects w.r.t. the cumulative response of all pursued policies if the number of governments increases (see Appendix D formulae (5.0.22,5.0.23)). In fact, in this model, even an unstable growth of fish stock may result if the number of governments increases. That is, the normal operation condition gets violated when the number of governments increases too much. Or, stated differently, all the more governments are involved, the more difficult it is to meet the stabilization constraint. Therefore, from a stabilization point of view it seems best to have just a single authority taking care of the fish stock.

If fishermen expect external disturbances might impact the fish growth in the lake, Theorem 2.5 shows that, except for the governments, the amount of fish caught by all fishermen is not affected by expectations concerning these disturbances. Governments, however, do respond in this model on such expectations.

More detailed, we see that there is a difference in reaction between the cases that just one government is taking care of the fish stock in the lake, or more than one government is concerned about it. Theorem 2.5 shows that in case just one government takes care about the fish stock, this government will use a more active stabilization policy when he expects more serious disturbances might impact

the growth of the fish stock in the lake. This is accompanied by a reduction of his constant catch (or, increase of the number of fish that is planted) in such a way that, when actually no disturbances occur, the steady-state fish stock coincides with that observed in the noise free case (provided the fish stock is stabilized).

In case more than one government takes care about the fish stock in the lake, the model predicts a reversal of controls by that same government. In that case he will respond with a less active stabilization policy and increase of his fixed catch (or, decrease of planting fish). Other governments react on his increased expectation of noise. They, on the contrary, will respond with using more active stabilization policies and a decrease of their constant catch (or, plant more fish). So, it seems other governments anticipate this behavior of the government with changed noise expectations and compensate for the anticipated change in policy. From Appendix D, (5.0.22,5.0.24), it follows that the total impact of a unilateral expected increase of noise on fishing is that total stabilization efforts are increased and the constant catch is decreased (or, planting of fish is increased).

Also in this noisy environment the effect of having more governments active in stabilization efforts will be that stabilization activities by these government will lower (cf. Theorem 2.3) and, moreover, also total stabilization activity declines (cf. (5.0.24)). So, also in the noisy case, from a policy point of view it seems preferable to have just one government taking care of the stabilization policy.

Finally, we notice that total stabilization activity of governments is always higher in the noisy case than in the corresponding noise free equilibrium (see Appendix D).

4 Modeling Issues

Linear-quadratic games can be viewed as a natural approximation to more realistic models. [4], e.g., shows that under some general conditions in a two player scalar setting, when nonlinear perturbations are added to the dynamics of the system and to the cost functions, the perturbed game still has a Nash equilibrium in feedback form, close to the original one. Within that context one should also view the current contribution. The model, clearly, has a number of shortcomings. And a major question is, in how far observations we made will apply for a more realistic context too. Note that observations we made in the previous section are mainly based on the consideration of parameter constellations where the actual closed-loop system is stabilized. That is, for that part of the model that makes sense in a fishing context.

A first issue to be addressed concerns the assumption that the fish stock grows linearly. A great number of papers dealing with fishery management in a dynamic framework assume a logistic specification for this growth function. But this brings on a non-linearity which is analytically much more difficult to handle. As a compromise, e.g., [3] considers a piecewise linear approximation of the logistic specification. Following [3], one might replace the dynamics in (2.0.1) by next dynamics

$$\dot{s}(t) = \begin{cases} \beta s(t) - \sum_{i=1}^N u_i(t) - w(t), & \text{if } s(t) \leq \bar{s} \\ \beta \frac{\bar{s}}{s_{max} - \bar{s}} (\bar{s} - s(t)) - \sum_{i=1}^N u_i(t) - w(t), & \text{if } s(t) > \bar{s}. \end{cases} \quad (4.0.5)$$

where \bar{s} is the stock of fish beyond which the fish stock declines and s_{max} represents the habitat carrying capacity beyond which the growth of the fish stock is negative due to the limited availability of food and space. Optimal control of piecewise linear systems is a subject that has been studied a lot in literature. Probably one of the first references in this area is [9]. [9] already showed that in this kind of systems an important issue is to determine the switching times. That is the times,

when the state of the system enters a region where the description of the dynamics change. Usually it is only possible to calculate these switching points numerically. And a lot of algorithms have been designed in literature to calculate them. See, e.g. [31]. In a multi-player linear-quadratic framework context some preliminary results with discontinuous linear dynamics are reported in [28]. There it is shown that, under the assumption that a symmetric equilibrium occurs and at most a finite number of switches take place, multiple switchings indeed may occur. Therefore, under similar assumptions, we expect a similar behaviour for the model with dynamics given by (4.0.5) and performance (2.0.3). However, since even in case we have a symmetric model, our analysis shows we can expect qualitatively different equilibria, probably one has to analyze numerically an $N + 1$ -dimensional set of involved differential equations instead of a 2-dimensional one. Maybe the continuity of the dynamics at the point where the switching occurs here simplifies the analysis, but this remains a topic for future research.

A second point which needs further attention is that the current model does not take into account that the fish stock always remains positive. This point could be addressed by including this as a hard bound constraint into the model. But, again, this makes the model nonlinear and, therefore, complicates the analysis. It remains a challenge to define the set of admissible control strategies in such a way that these restrictions can be dealt with in an analytically tractable way too. Against the backdrop of our results, one outcome of such a modeling could be we see the introduction of a third type of fishermen, taking care of this constraint.

A third issue is that the current model does not take into account that fishing gets more complicated the less fish there is in the lake. This point could be addressed by replacing $u_j^2(t)$ by $(u_j(t) - u^*)^2$ in the performance (2.0.3) of fisherman j , where u^* is the from the fishermen's point of view optimal operating point (or even $(u_j(t) - u_j^*)^2$ if fishermen have different perceptions on this). This adaptation does not bring on any fundamental changes in results. From a technical point of view all that changes is that in all results the price p has to be increased to $p + \gamma u^*$.

Finally, one might also consider a case where (some) fishermen are less individualistic in the sense that they do take into consideration that emptying the lake is not really what they want, and, therefore maintaining a certain stock of fish, s^* , is desirable. Also this point can be addressed within the current framework. That is, by adding the term $(s(t) - s^*)^2$ (or even $(s(t) - s_j^*)^2$, if fishermen have different set points for this stock) in (2.0.3). Probably this adaptation will obscure the basic result that a number of fishermen will take care of the stabilization constraint. This, since now all fishermen care about the fish stock. Consequently, all fishermen will use a strategy in which their catch depends also on the current fish stock. However, again we expect to see in that case one or more fishermen that act like a "government" in the fishing game. That is, to see fishermen that take more care about the fish stock than others. This is motivated by the consideration of the full symmetric case with $p = 0$. This case was analyzed in detail in [16][Theorem 4.3]. In that theorem it was observed that in most cases no full symmetric equilibrium exists for that specific game. That is, not all players act the same way. Apparently this has to be attributed to the imposed stabilization constraint.

5 Summary and Concluding Remarks

In this note we introduced a very stylized dynamic fishing model, which differs from the usual ones studied in literature in the sense that we allow for a broader class of potential fishing strategies that can be pursued by the fishermen. Using equilibrium concepts developed within the framework of

dynamic differential games we studied the predicted behaviour. Basically, the framework used is that of linear-quadratic non-cooperative differential games played over an infinite time horizon, where all fishermen have complete knowledge about the fish stock at any point in time.

Due to its simple structure we could do this analytically. We considered both a complete deterministic setting and an uncertain setting of the fish stock growth. That is, the case that the fishing model perfectly describes reality and the case that the fishing might be corrupted by (modest) external factors, and the fishermen take this uncertainty into account in their fishing strategy. In the last mentioned case the strategy is designed such that it performs well when nature chooses an outcome that is maximally harmful to the fisherman. That is, the fishing strategies depend on the worst-case scenario expectations about the noise of every fisherman.

The model predicts that basically two scenarios may occur. Depending whether the fishermen are short-sighted or have a longer time span concerning their objectives.

In case fishermen don't care too much about future profits, the model predicts every fisherman will catch a constant amount of fish over time and this amount will not be affected by potential disturbances they expect that might corrupt the fish growth. Basically, this result is due to the fact that future profits are negligible for them. So they have no direct interest in the development of the fish stock.

In case the fishermen are not short-sighted with respect to future returns, the model predicts that two types of fishermen exist in the model. Additional to fishermen catching a constant amount of fish, one or more "governments" must exist that control the fish stock of the lake. This, in order for the fishing to be sustainable. The governments' catch consists of some fixed amount and, additional to that, an amount that depends on the current state of the fish stock. Their so-called stabilization activity. Depending on, in particular the price of fish, it may be that the constant catch by the governments becomes a negative number. In those cases the governments' policy can be interpreted as, on the one hand catching an amount of fish that depends on the fish stock, and on the other hand planting a fixed amount of fish at any point in time.

The model predicts that the stabilization activity of the involved government depends on whether there is a single authority controlling the fish stock or whether there will be more than one. The more governments there are, the less active stabilization policies will be pursued. Both individually and collectively. Therefore, from a stabilization point of view it seems best to have just a single authority taking care of the fish stock. Assuming the fish stock is stabilized, the steady-state value turns out to be the same, whatever the number of governments is taking care of the stabilization, and, irrespective of whether governments consider noise expectations or not (in the last case, under the assumption that actual realization of noise is zero at any point in time). A more detailed analysis shows this is achieved by a fine-tuning of stabilization activity and constant catch by every government.

The response of a single government when he expects more noise is to use a more active stabilization policy and to lower his fixed catch (or, increase the planting). In general, we see that total stabilization activity of governments is higher in the noisy case than in the corresponding noise free equilibrium.

Basically, these results are due to the fact that in the non short-sighted model future profits are important for the fishermen too. So there is a need for fishermen who take care of the future development of the fish stock. In this case one might view the fishermen who fish a constant amount as free-riders in the game.

Clearly, the model used in this note is very stylized and motivated by analytic tractability. A basic question is in how far results obtained within this setting generalize for more realistic models.

On the other hand, the considered model is an appropriate framework for other application areas too. And, therefore, the obtained preliminary conclusions may be relevant there too.

In the previous section we discussed a number of issues that need to be addressed within the context of this model and potential lines for future research. Finally, an interesting relevant question can be which fishermen should act as "government". Any fishermen acting as a "government" will loose by taking that role. So, if there is no "natural" government the question remains whether there are some additional motives to "select" the government(s) or whether side payments (taxes) should be enforced in order to compensate the fishermen taking this role.

Appendix A

Here we recall some general results concerning the existence of so-called soft-constrained feedback Nash equilibria. Proofs and more details can be found in, e.g., [5] or [14].

Consider next dynamic model:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^N B_i u_i(t) + Ew(t), \quad x(0) = x_0. \quad (5.0.6)$$

Here $x \in \mathbb{R}^n$ is the state of the system, $u_i \in \mathbb{R}^{m_i}$ the control efforts used by agent i to control the system and $w \in L_2^q(0, \infty)$ is a q -dimensional disturbance vector affecting the system and A , B_i and E are constant matrices of appropriate dimensions.

In case above system (5.0.6) is void of noise (i.e., $w = 0$), it is assumed that every agent likes to minimize his performance criterion

$$J_i(x_0, u_i) = \lim_{T \rightarrow \infty} J_i(x_0, u_i, T), \quad i \in \mathbf{N}. \quad (5.0.7)$$

Here

$$J_i(x_0, u_i, T) = \int_0^T \{x^T(t)Q_i x(t) + u_i^T(t)R_i u_i(t)\} dt, \quad (5.0.8)$$

where matrices Q_i and R_i , $i \in \mathbf{N}$, are symmetric and R_i , $i \in \mathbf{N}$, are positive definite.

Assuming agents have full access to the current state of the system and they use constant linear feedback strategies, i.e.,

$$u_i(t) = F_i x(t), \quad \text{with } F_i \in \mathbb{R}^{m_i \times n}, \quad i \in \mathbf{N},$$

where (F_1, \dots, F_N) belongs to the set⁶

$$\mathcal{F} := \{F = (F_1, \dots, F_N) \mid A + \sum_{i=1}^N B_i F_i \text{ is stable}\},$$

the concept of a linear feedback Nash equilibrium on an infinite-planning horizon is then defined as follows.

⁶A necessary and sufficient condition for this set to be not empty is that the matrix pair $(A, [B_1, \dots, B_N])$ is stabilizable.

Definition 5.1 $F^* := (F_1^*, \dots, F_N^*) \in \mathcal{F}$ is called a *stationary linear feedback Nash equilibrium* if the following inequalities hold:

$$J_i(x_0, F^*) \leq J_i(x_0, F_{-i}^*(F_i))$$

for each x_0 and for each state feedback matrix F_i such that $(F_{-i}^*(F_i)) \in \mathcal{F}$. \square

The stabilization constraint is imposed to ensure the finiteness of the infinite-horizon cost integrals that we will consider.

Next, with $S_i := B_i R_i^{-1} B_i^T$, consider the set of coupled algebraic Riccati equations

$$0 = (A - \sum_{i=1}^N S_i K_i)^T K_j + K_j (A - \sum_{i=1}^N S_i K_i) + K_j S_j K_j + Q_j, \quad j \in \mathbf{N}. \quad (5.0.9)$$

Theorem 5.2 below states that feedback Nash equilibria are completely characterized by *stabilizing solutions* of (5.0.9). That is, by solutions (K_1, \dots, K_N) for which the closed-loop system matrix $A - \sum_{i=1}^N S_i K_i$ is stable.

Theorem 5.2 Let (K_1, \dots, K_N) be a stabilizing solution of (5.0.9) and define $F_i^* := -R_i^{-1} B_i^T K_i$, $i \in \mathbf{N}$. Then (F_1^*, \dots, F_N^*) is a feedback Nash equilibrium (FNE). Moreover, the cost incurred by player i by playing this equilibrium action is $x_0^T K_i x_0$, $i = 1, 2$.

Conversely, if (F_1^*, \dots, F_N^*) is a feedback Nash equilibrium, there exists a stabilizing solution (K_1, \dots, K_N) of (5.0.9) such that $F_i^* = -R_i^{-1} B_i^T K_i$. \square

In case system (5.0.6) does depend on noise $w(\cdot)$, the description of the agents' objectives given above needs to be modified in order to express a desire for robustness. To that end, criterion (5.0.8) is modified to

$$\bar{J}_i^{SC}(x_0, F_i) := \sup_{w \in L_2^q(0, \infty)} J_i(x_0, F_i, w) \quad (5.0.10)$$

where

$$J_i(x_0, F_i, w) := \int_0^\infty \{x^T(Q_i + F_i^T R_i F_i)x(t) - w^T V_i w\} dt. \quad (5.0.11)$$

The weighting matrix V_i is symmetric and positive definite. Because it occurs with a minus sign in (5.0.11), this matrix constrains the disturbance vector w in an indirect way. Therefore, matrix V_i can be interpreted as modeling the risk aversion of player i . Specifically, if the quantity $w^T V_i w$ is large for a vector $w \in \mathbb{R}^q$, this means that agent i does not expect large deviations of the nominal dynamics in the direction of Ew . So, the larger he chooses V_i , the closer the worst case signal he can be confronted with in this model will approach the zero input signal (that is: $w(\cdot) = 0$). So, choosing V_i large models the case agent i does not expect external disturbances will have a serious impact on the system.

This formulation is called the "soft-constrained" one (opposed to having strict constraints on the disturbance).

The equilibrium concept that we use in this deterministic noise setting is based on the adjusted cost functions (5.0.10).

⁷Here $F_{-i}(\alpha) := (F_1, \dots, F_{i-1}, \alpha, F_{i+1}, \dots, F_N)$.

Definition 5.3 $\bar{F} = (\bar{F}_1, \dots, \bar{F}_N) \in \mathcal{F}$ is called a *soft-constrained Nash equilibrium* if for all $i \in \mathbf{N}$ the following inequality holds:

$$\bar{J}_i(x_0, \bar{F}) \leq \bar{J}_i(x_0, \bar{F}_{-i}(F_i))$$

for all initial states x_0 and for all $F_i \in \mathbb{R}^{m_i \times n}$ such that $\bar{F}_{-i}(F_i) \in \mathcal{F}$. \square

Using the shorthand notation

$$M_i := EV_i^{-1}E^T,$$

we have next analogue of the noise-free case.

Theorem 5.4 Consider the differential game defined by (5.0.6–5.0.10). Assume there exist real symmetric $n \times n$ matrices X_i , $i = 1, 2$, and real symmetric $n \times n$ matrices Y_i , $i = 1, 2$, such that

$$\left(A - \sum_{i=1}^N S_i X_i\right)^T X_j + X_j \left(A - \sum_{i=1}^N S_i X_s\right) + X_j (S_j + M_j) X_j + Q_j = 0, \quad j \in \mathbf{N} \quad (5.0.12)$$

$$A - \sum_{i=1}^N S_i X_i + M_j X_j, \quad j \in \mathbf{N} \text{ are stable,} \quad (5.0.13)$$

$$A - \sum_{i=1}^N S_i X_i \text{ is stable} \quad (5.0.14)$$

$$-\left(A - \sum_{i \neq j} S_i X_i\right)^T Y_j - Y_j \left(A - \sum_{i \neq j} S_i X_i\right) + Y_j S_j Y_j - Q_j \leq 0, \quad j \in \mathbf{N}. \quad (5.0.15)$$

Define $\bar{F} = (\bar{F}_1, \dots, \bar{F}_N)$ by

$$\bar{F}_i := -R_i^{-1} B_i^T X_i, \quad i \in \mathbf{N}.$$

Then $\bar{F} \in \mathcal{F}$, and \bar{F} is a soft-constrained Nash equilibrium. Furthermore, the worst-case signal \bar{w}_i from player i 's perspective is

$$\bar{w}(t) = V_i^{-1} E^T X_i e^{(A - \sum_{j=1}^N S_j X_j + M_i X_i)t} x_0.$$

Moreover the cost for player i under the realization of his worst-case expectations are

$$\bar{J}_i^{SC}(\bar{F}, x_0) = x_0^T X_i x_0, \quad i \in \mathbf{N}.$$

Conversely, if $(\bar{F}_1, \dots, \bar{F}_N)$ is a soft-constrained Nash equilibrium, the equations (5.0.12–5.0.14) have a set of real symmetric solutions (X_1, \dots, X_N) .

Corollary 5.5 If $Q_i \geq 0$, $i \in \mathbf{N}$, the matrix inequalities (5.0.15) are trivially satisfied with $Y_i = 0$, $i \in \mathbf{N}$. So, under these conditions the differential game defined by (5.0.6–5.0.10) has a soft-constrained Nash equilibrium if and only if the equations (5.0.12–5.0.14) have a set of real symmetric $n \times n$ matrices X_i , $i \in \mathbf{N}$. \square

Appendix B: Proof of Theorems 2.2 and 2.3

Since the proof of Theorem 2.2 can be given along the lines of that from Theorem 2.3, we will just present the proof of this last-mentioned theorem here.

Introducing $x^T(t) := [s(t) \ 1]$, the optimization problem can be rewritten as

$$\min_{u_i \in \mathcal{F}^{aff}} \sup_{w \in L_2(0, \infty)} \int_0^\infty e^{-rt} \{ [x^T(t) \ u_i^T(t)] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}p \\ 0 & -\frac{1}{2}p & \gamma_i \end{bmatrix} \begin{bmatrix} x(t) \\ u_i(t) \end{bmatrix} - v_i w^2(t) \} dt, \quad i = 1, 2,$$

subject to the dynamics

$$\dot{x}(t) = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} x(t) + \sum_{i=1}^N \begin{bmatrix} -1 \\ 0 \end{bmatrix} u_i(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t), \quad x(0) = \begin{bmatrix} s_0 \\ 1 \end{bmatrix}.$$

Using the transformation

$$\tilde{u}_i := u_i - \frac{p}{2\gamma_i}, \quad i \in \mathbf{N},$$

and introducing the short-hand notation $\alpha := \frac{p}{2} \sum_{i=1}^N \frac{1}{\gamma_i}$, the optimization problem can be rewritten as

$$\min_{\tilde{u}_i \in \mathcal{F}^{aff}} \sup_{w \in L_2(0, \infty)} \int_0^\infty e^{-rt} \{ x^T(t) \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4\gamma_i} \end{bmatrix} x(t) + \gamma_i \tilde{u}_i^2(t) - v_i w^2(t) \} dt, \quad i \in \mathbf{N}$$

subject to the dynamics

$$\dot{x}(t) = \begin{bmatrix} \beta & -\alpha \\ 0 & 0 \end{bmatrix} x(t) + \sum_{i=1}^N \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tilde{u}_i(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(t), \quad \tilde{x}(0) = \begin{bmatrix} s_0 \\ 1 \end{bmatrix}.$$

With, $\hat{x}(t) := e^{-\frac{1}{2}rt} x(t)$, $\hat{u}_i(t) := e^{-\frac{1}{2}rt} \tilde{u}_i(t)$, $i \in \mathbf{N}$ and $\hat{w}(t) := e^{-\frac{1}{2}rt} w(t)$, we obtain the standard formulation.

$$\min_{\hat{u}_i \in \mathcal{F}} \sup_{\hat{w} \in L_2(0, \infty)} \int_0^\infty \{ \hat{x}^T(t) \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4\gamma_i} \end{bmatrix} \hat{x}(t) + \gamma_i \hat{u}_i^2(t) - v_i \hat{w}^2(t) \} dt, \quad i \in \mathbf{N},$$

subject to the dynamics

$$\dot{\hat{x}}(t) = \begin{bmatrix} \beta - \frac{1}{2}r & -\alpha \\ 0 & -\frac{1}{2}r \end{bmatrix} \hat{x}(t) + \sum_{i=1}^N \begin{bmatrix} -1 \\ 0 \end{bmatrix} \hat{u}_i(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \hat{w}(t), \quad \hat{x}(0) = \begin{bmatrix} s_0 \\ 1 \end{bmatrix}.$$

According Theorem 5.4 the soft-constrained Nash equilibria for this game are obtained as

$$\hat{u}_i(t) = \begin{bmatrix} \frac{1}{\gamma_i} & 0 \end{bmatrix} X_i \hat{x}(t),$$

where with

$$A := \begin{bmatrix} \beta - \frac{1}{2}r & -\alpha \\ 0 & -\frac{1}{2}r \end{bmatrix}; S_i := \begin{bmatrix} \frac{1}{\gamma_i} & 0 \\ 0 & 0 \end{bmatrix}; M_i := \begin{bmatrix} \frac{1}{v_i} & 0 \\ 0 & 0 \end{bmatrix}; Q_i := \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4\gamma_i} \end{bmatrix};$$

(X_1, \dots, X_N) solve (5.0.12) and satisfy the conditions (5.0.13–5.0.14). Notice that with

$$Y_i := \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4r\gamma_i} \end{bmatrix}, \quad i \in \mathbf{N},$$

the inequalities (5.0.15) are satisfied.

In case the fishermen are short-sighted, $r > 2\beta$, we see by straightforward substitution that

$$X_i := \begin{bmatrix} 0 & 0 \\ 0 & \frac{-p^2}{4r\gamma_i} \end{bmatrix}, \quad i \in \mathbf{N}, \quad (5.0.16)$$

satisfy the equations (5.0.12–5.0.14). So one soft-constrained Nash equilibrium is in that case provided by

$$u_i^*(t) = \frac{p}{2\gamma_i}. \quad (5.0.17)$$

That is, irrespective of the growth of the fish population, the fishermen catch a constant amount of fish each time. This amount is completely determined by their cost function and the price of the fish.

To see whether there exist still different equilibria, we elaborate the equations (5.0.12). To that end introduce

$$X_i =: \begin{bmatrix} x_{i1} & x_{i2} \\ x_{i2} & x_{i3} \end{bmatrix}, \quad i \in \mathbf{N}.$$

Then, the equations (5.0.12) can be rewritten as

$$(2\beta - r - 2 \sum_{i=1}^N \frac{x_{i1}}{\gamma_i})x_{j1} + (\frac{1}{\gamma_j} + \frac{1}{v_j})x_{j1}^2 = 0, \quad j \in \mathbf{N}, \quad (5.0.18)$$

$$(-\alpha - \sum_{i=1}^N \frac{x_{i2}}{\gamma_i})x_{j1} + (\beta - r - \sum_{i=1}^N \frac{x_{i1}}{\gamma_i})x_{j2} + (\frac{1}{\gamma_j} + \frac{1}{v_j})x_{j1}x_{j2} = 0, \quad j \in \mathbf{N}, \quad (5.0.19)$$

$$\frac{p^2}{4\gamma_j} + 2(\alpha + \sum_{i=1}^N \frac{x_{i2}}{\gamma_i})x_{j2} - (\frac{1}{\gamma_j} + \frac{1}{v_j})x_{j2}^2 + rx_{j3} = 0, \quad j \in \mathbf{N}. \quad (5.0.20)$$

The first equation (5.0.18) completely determines x_{j1} , $j \in \mathbf{N}$. Given these values, the variables x_{j2} , $j \in \mathbf{N}$, are then determined by the linear equations (5.0.19). Finally, by (5.0.20), it follows then that $x_{j3} = -\frac{1}{r}(\frac{p^2}{4\gamma_j} + 2(\alpha + \sum_{i=1}^N \frac{x_{i2}}{\gamma_i})x_{j2} - (\frac{1}{\gamma_j} + \frac{1}{v_j})x_{j2}^2)$.

Now, consider in some more detail (5.0.18). From this equation it follows that either

$$(i) \ x_{j1} = 0 \text{ or } (ii) \ 2 \sum_{i \neq j} \frac{x_{i1}}{\gamma_i} + (\frac{1}{\gamma_j} - \frac{1}{v_j})x_{j1} = 2\beta - r, \quad j \in \mathbf{N}.$$

In case (i), $x_{j1} = 0$, (5.0.19) yields that $x_{j2} = 0$ (under the assumptions that $\beta - r - \sum_{i=1}^N \frac{x_{i1}}{\gamma_i} \neq 0$). Equation (5.0.20) shows then that necessarily $x_{j3} = -\frac{p^2}{4r\gamma_j}$. That is, $X_j = \frac{1}{r}Q_j$ (as already reported in (5.0.16)).

In case $x_{j1} = 0$, $j \in \mathbf{N}_1$, and x_{j1} differs from zero, $j \in \bar{\mathbf{N}}_1$, it follows that for all these last-mentioned fishermen, x_{j1} is given by (ii). By straightforward substitution one verifies that X_j given by (5.0.16), $j \in \mathbf{N}_1$, and X_j given by next matrices (5.0.21), $j \in \bar{\mathbf{N}}_1$, satisfy (5.0.12):

$$X_j = x_j \gamma_j \begin{bmatrix} 1 \\ \frac{-\alpha}{\beta} \end{bmatrix} \begin{bmatrix} 1 & \frac{-\alpha}{\beta} \end{bmatrix} + \frac{1}{r}Q_j, \text{ where } x_j = \frac{2\beta - r}{-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i}} \frac{v_j}{\gamma_j + v_j}. \quad (5.0.21)$$

Without going into details, we notice here that it can be shown that this set of linear equations (5.0.19) has a unique solution, given our parametric assumptions. So, with this we determined all solutions of (5.0.12).

Next consider (5.0.14). Simple calculations show that with X_i as outlined above,

$$\begin{aligned} A - \sum_{i=1}^N S_i X_i &= A - \sum_{i \in \bar{\mathbf{N}}_1} S_i X_i = A - \sum_{i \in \bar{\mathbf{N}}_1} x_i \begin{bmatrix} 1 \\ \frac{-\alpha}{\beta} \end{bmatrix} \begin{bmatrix} 1 & \frac{-\alpha}{\beta} \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & -\frac{1}{2}r \end{bmatrix}, \end{aligned}$$

where $\alpha_1 = \beta - \frac{1}{2}r - \sum_{i \in \bar{\mathbf{N}}_1} x_i = (2\beta - r) \left(\frac{1}{2} - \frac{1}{-1 + 2 \sum_{j \in \bar{\mathbf{N}}_1} \frac{v_j}{\gamma_j + v_j}} \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i} \right) = -\frac{2\beta - r}{2} \frac{1}{-1 + 2 \sum_{j \in \bar{\mathbf{N}}_1} \frac{v_j}{\gamma_j + v_j}}$.

From which it is clear that this matrix is stable iff. $\alpha_1 < 0$. From above expression for α_1 it follows directly that this is the case iff. $2\beta - r > 0$.

Finally notice that

$$A - \sum_{i=1}^N S_i X_i + M_j X_j = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & -\frac{1}{2}r \end{bmatrix} + \frac{x_j \gamma_j}{v_j} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{-\alpha}{\beta} \end{bmatrix} = \begin{bmatrix} \tilde{\alpha}_1 & \tilde{\alpha}_2 \\ 0 & -\frac{1}{2}r \end{bmatrix},$$

$$\begin{aligned} \text{where } \tilde{\alpha}_1 &= \alpha_1 + \frac{2\beta - r}{-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i}} \frac{\gamma_j}{\gamma_j + v_j} = \frac{2\beta - r}{-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i}} \left(\frac{-1}{2} + \frac{\gamma_j}{\gamma_j + v_j} \right) \\ &= -\frac{2\beta - r}{2} \frac{1}{-1 + 2 \sum_{j \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i}} \frac{v_j - \gamma_j}{\gamma_j + v_j} < 0. \end{aligned}$$

Appendix C: Proof of Theorem 2.5

By definition the stabilization activity of fisherman j is measured by parameter x_j . So, the change of the impact of expected noise on the stabilization activity of fisherman j is measured by the partial derivative $\frac{\partial x_j}{\partial v_j}$. Recall that an increase of the variable v_j means that the fisherman j expects less disturbances will corrupt the fishing. Consequently, if $\frac{\partial x_j}{\partial v_j} < 0$, this implies that this fisherman gets more active when he expects more disturbances.

From Theorem 2.3, item 2., it follows that

$$\begin{aligned}
\frac{\partial x_j}{\partial v_j} &= (2\beta - r) \frac{\partial}{\partial v_j} \left(\frac{1}{-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i}} \frac{v_j}{\gamma_j + v_j} \right) \\
&= \frac{(2\beta - r) \gamma_j}{(\gamma_j + v_j)^2} \frac{1}{(-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i})^2} \left(\frac{-v_j}{\gamma_j + v_j} - 1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i} \right) \\
&= \frac{(2\beta - r) \gamma_j}{(\gamma_j + v_j)^2} \frac{1}{(-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i})^2} \left(-1 + \frac{v_j}{\gamma_j + v_j} + 2 \sum_{i \in \bar{\mathbf{N}}_1 \setminus j} \frac{v_i}{\gamma_i + v_i} \right)
\end{aligned}$$

From this some elementary calculations show that in case $\bar{\mathbf{N}}_1 = \{j\}$, the above expression is negative, whereas in case $\bar{\mathbf{N}}_1$ consists of more than one element, this expression is always positive (using the assumption $v_i > \gamma_i$ again).

All statements in this theorem about the effect of disturbance expectations on expected minimal profits follow directly from the observation that the sign of $\frac{\partial J^{WC}}{\partial v_i}$ coincides with $\frac{\partial x_j}{\partial v_i}$ (where $i \in \bar{\mathbf{N}}_1$).

Part 2.b It is easily verified that

$$\begin{aligned}
\frac{\partial x_j}{\partial v_i} &= (2\beta - r) \frac{\partial}{\partial v_j} \left(\frac{1}{-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i}} \frac{v_j}{\gamma_j + v_j} \right) \\
&= -\frac{(2\beta - r) v_j}{\gamma_j + v_j} \frac{1}{(-1 + 2 \sum_{i \in \bar{\mathbf{N}}_1} \frac{v_i}{\gamma_i + v_i})^2} \frac{\gamma_i}{(\gamma_i + v_i)^2} < 0.
\end{aligned}$$

Appendix D: Study of closed-loop dynamics

In case the fishermen are short-sighted and no noise actually occurs (i.e. $w(\cdot) = 0$) from item 1. Theorem 2.2 and 2.3 we get that the closed-loop dynamics of the system are described by:

$$\dot{s}(t) = \beta s(t) - \alpha, \quad s(0) = s_0.$$

That is, $s(t) = (s_0 - \frac{\alpha}{\beta})e^{\beta t} + \frac{\alpha}{\beta}$.

So, depending on the sign of $s_0 - \alpha$ the fish stock will either exterminate in finite time, exponentially grow or remain constant forever.

In case the fishermen are not short-sighted consider $\bar{a}_{cl} := \beta - \sum_{j \in \bar{\mathbf{N}}_1} \bar{x}_j$. From Theorem 2.2, item

2, the next closed-loop dynamics of the system result:

$$\dot{s}(t) = \bar{a}_{cl} s(t) - \frac{\alpha}{\beta} \bar{a}_{cl}, \quad s(0) = s_0. \quad (5.0.22)$$

That is, $s(t) = (s_0 - \frac{\alpha}{\beta})e^{\bar{a}_{cl} t} + \frac{\alpha}{\beta}$.

So, in case $\bar{a}_{cl} < 0$, the fish stock will converge to the steady-state $\frac{\alpha}{\beta}$. Note that in that case the total

fixed catch by all fishermen is positive, and that the steady-state does not depend on how many of the fishermen act as government in the stabilization process.

If $\bar{a}_{cl} > 0$, the fish stock will either exterminate in finite time or grow exponentially.

Some simple calculations show that $\bar{a}_{cl} = \frac{-\beta+r\bar{N}_1}{-1+2\bar{N}_1}$, where $\bar{N}_1 = N - N_1$. So,

$$\frac{d\bar{a}_{cl}}{d\bar{N}_1} = \frac{2\beta - r}{(-1 + 2\bar{N}_1)^2} > 0. \quad (5.0.23)$$

Therefore, the closed-loop dynamics of the system get less stable, if the number of "governments" increases.

Similar conclusions result if we consider the noisy case, assuming the realization of the noise signal is zero again, i.e. $w(\cdot) = 0$.

From Theorem 2.3, item 2, we obtain again the closed-loop dynamics (5.0.22), where \bar{x}_j is replaced by x_j and \bar{a}_{cl} is replaced by $a_{cl} := \beta - \sum_{j \in \bar{N}_1} x_j$. Again simple calculations show that a_{cl} can be rewritten as $a_{cl} = \frac{-\beta+r\Delta}{-1+2\Delta}$, where $\Delta := \sum_{j \in \bar{N}_1} \frac{v_j}{\gamma_j + v_j}$. So, similarly as above, we conclude that in this case

$$\frac{da_{cl}}{d\Delta} = \frac{2\beta - r}{(-1 + 2\Delta)^2} > 0. \quad (5.0.24)$$

Therefore, again we conclude the closed-loop dynamics of the system get less stable, in this case the more Δ increases. Note, in particular, this occurs if either the number of "governments" increases or v_j increases.

Furthermore, since $\frac{v_j}{\gamma_j + v_j} < 1, \forall j$, it follows by a straightforward comparison that $a_{cl} < \bar{a}_{cl}$. That is, under noisy expectations the closed-loop will be more stabilized.

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