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Publication date:
2017

Document Version
Early version, also known as pre-print

CentER

Discussion Paper

No. 2017-005

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1 February 2017

ISSN 0924-7815
ISSN 2213-9532

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February 1, 2017

Abstract

This paper analyzes bankruptcy games with nontransferable utility as a generalization of bankruptcy games with monetary payoffs. Following the game theoretic approach to NTU-bankruptcy problems, we study some appropriate properties and the core of NTU-bankruptcy games. Generalizing the core cover and the reasonable set to the class of NTU-games, we show that NTU-bankruptcy games are compromise stable and reasonable stable. Moreover, we derive a necessary and sufficient condition for an NTU-bankruptcy rule to be game theoretic.

Keywords: NTU-bankruptcy problem, NTU-bankruptcy game, compromise stability, reasonable stability, game theoretic bankruptcy rule
JEL classification: C71

1 Introduction

A bankruptcy problem is an elementary allocation problem in which claimants have individual claims on an estate which cannot be satisfied together. Bankruptcy theory studies allocations of the estate among the claimants, taking into account the corresponding claims. In a bankruptcy problem with transferable utility (cf. O'Neill (1982)), the estate and claims are of a monetary nature. These problems are well-studied, both from an axiomatic perspective and a game theoretic perspective. We refer to Thomson (2003) for an extensive survey, Thomson (2013) for recent advances, and Thomson (2015) for an update.

Dietzenbacher, Estévez-Fernández, Borm, and Hendrickx (2016) generalized monetary bankruptcy problems to bankruptcy problems with nontransferable utility in which individual utility is represented in incompatible measures. The estate can take a more general shape and corresponds to a set of feasible utility allocations. Dietzenbacher et al. (2016) analyzed these NTU-bankruptcy problems from an axiomatic perspective by formulating appropriate properties for bankruptcy rules and studying their implications. In particular, they focused on proportionality, equality, and duality in bankruptcy problems with nontransferable utility, which resulted in the proportional rule and the constrained relative equal awards rule.

Orshan, Valenciano, and Zarzuelo (2003) analyzed NTU-bankruptcy problems from a game theoretic perspective by introducing an associated NTU-bankruptcy game. As pointed out by Estévez-Fernández, Borm, and Fiestras-Janeiro (2014), coalitions can attain payoff allocations outside the estate in this game, which contradicts the original idea of O'Neill (1982). They redefined NTU-bankruptcy games to stay in line with this original idea about

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TU-bankruptcy games, while focusing on convexity and compromise stability. However, it turns out that their NTU-bankruptcy game does not straightforwardly generalize the original TU-bankruptcy game, since the attainable payoff allocations of subcoalitions are explicitly bounded by individual claims.

This paper introduces a new model for bankruptcy games with nontransferable utility which both generalizes the model for TU-bankruptcy games and stays in line with the idea of O'Neill (1982). Focusing on the structure of the core, we analyze NTU-bankruptcy games along the lines of Curiel, Maschler, and Tijs (1987). Their results imply that TU-bankruptcy games are compromise stable, i.e. the core equals the core cover, and reasonable stable, i.e. the core equals the reasonable set. Generalizing the core, the core cover, and the reasonable set to the class of NTU-games, we show that NTU-bankruptcy games are compromise stable and reasonable stable as well.

Curiel et al. (1987) also showed that a TU-bankruptcy rule is game theoretic if and only if it satisfies truncation invariance. This means that there exists a solution for TU-games which coincides on the class of bankruptcy games with a certain bankruptcy rule if and only if this bankruptcy rule satisfies truncation invariance. We generalize this characterization to rules for bankruptcy problems with nontransferable utility.

This paper is organized in the following way. Section 2 provides a formal overview of notions for bankruptcy games with transferable utility and bankruptcy problems with nontransferable utility. Section 3 introduces the class of nonnegative games with nontransferable utility and generalizes some notions from TU-games to NTU-games. Section 4 introduces and analyzes a new model for NTU-bankruptcy games. In Section 5, we formulate some concluding remarks and point out some suggestions for future research.

2 Preliminaries

2.1 Bankruptcy Games with Transferable Utility

Let N be a nonempty and finite set of *players*. An *order* of N is a bijection $\sigma : \{1, \ldots, |N|\} \to N$. The set of all orders of N is denoted by $\Pi(N)$ and the set of all *coalitions* is denoted by $2^N = \{S \mid S \subseteq N\}$. A *transferable utility game* is a pair (N, v) in which $v : 2^N \to \mathbb{R}$ assigns to each coalition $S \in 2^N$ its *worth* $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. Let TU^N denote the class of all transferable utility games with player set N. For convenience, we denote a TU-game by $v \in \mathrm{TU}^N$.

Let $v \in \mathrm{TU}^N$. The *marginal vector* $M^\sigma(v) \in \mathbb{R}^N$ corresponding to $\sigma \in \Pi(N)$ is for all $n \in \{1, \ldots, |N|\}$ given by

$$M^\sigma_{\sigma(n)}(v) = v(\{\sigma(1), \ldots, \sigma(n)\}) - v(\{\sigma(1), \ldots, \sigma(n-1)\}).$$

Let $K(v) \in \mathbb{R}^N$ for all $i \in N$ be given by

$$K_i(v) = v(N) - v(N \setminus \{i\}),$$

and let $k(v) \in \mathbb{R}^N$ for all $i \in N$ be given by

$$k_i(v) = \max_{S \in 2^N : i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} K_j(v) \right\}.$$

Let $v \in \mathrm{TU}^N$. The *core* is given by

$$\mathcal{C}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall_{S \in 2^N} : \sum_{i \in S} x_i \geq v(S) \right\},$$

the *Weber set* (cf. Weber (1988)) is given by

$$\mathcal{W}(v) = \mathrm{Conv}\left\{ M^\sigma(v) \mid \sigma \in \Pi(N) \right\},$$

the *core cover* (cf. Tijs and Lipperts (1982)) is given by

$$\mathcal{CC}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), k(v) \leq x \leq K(v) \right\},$$

and the *reasonable set* (cf. Gerard-Varet and Zamir (1987)) is given by

$$\mathcal{R}(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in N} x_i = v(N), \forall_{i \in N} : \min_{\sigma \in \Pi(N)} M_i^\sigma(v) \leq x_i \leq \max_{\sigma \in \Pi(N)} M_i^\sigma(v) \right\}.$$

We have $\mathcal{C}(v) \subseteq \mathcal{W}(v) \subseteq \mathcal{R}(v)$ and $\mathcal{C}(v) \subseteq \mathcal{CC}(v) \subseteq \mathcal{R}(v)$. A TU-game $v \in \mathrm{TU}^N$ is called *convex* (cf. Shapley (1971) and Ichiishi (1981)) if $\mathcal{C}(v) = \mathcal{W}(v)$, and *compromise stable* (cf. Quant, Borm, Reijnierse, and Van Velzen (2005)) if $\mathcal{C}(v) = \mathcal{CC}(v)$ and $\mathcal{CC}(v) \neq \emptyset$. We state the following result.

Proposition 2.1.
A TU-game $v \in \mathrm{TU}^N$ is convex and compromise stable if and only if $\mathcal{C}(v) = \mathcal{R}(v)$.

Proof. Assume that $v \in \mathrm{TU}^N$ is convex and compromise stable. Then we have $\mathcal{C}(v) = \mathcal{CC}(v)$. Moreover, from convexity we know that $\min_{\sigma \in \Pi(N)} M_i^\sigma(v) = v(\{i\})$ and $\max_{\sigma \in \Pi(N)} M_i^\sigma(v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$, and we know that $k_i(v) = v(\{i\})$ for all $i \in N$. This means that $\min_{\sigma \in \Pi(N)} M_i^\sigma(v) = k_i(v)$ and $\max_{\sigma \in \Pi(N)} M_i^\sigma(v) = K_i(v)$ for all $i \in N$. Hence, $\mathcal{C}(v) = \mathcal{CC}(v) = \mathcal{R}(v)$.

Assume that $\mathcal{C}(v) = \mathcal{R}(v)$. Since $\mathcal{C}(v) \subseteq \mathcal{W}(v) \subseteq \mathcal{R}(v)$, this means that $\mathcal{C}(v) = \mathcal{W}(v)$, so $v \in \mathrm{TU}^N$ is convex. Since $\mathcal{C}(v) \subseteq \mathcal{CC}(v) \subseteq \mathcal{R}(v)$, this means that $\mathcal{C}(v) = \mathcal{CC}(v)$, so $v \in \mathrm{TU}^N$ is compromise stable. □

A *bankruptcy problem with transferable utility* (cf. O'Neill (1982)) is a triple (N, E, c) in which N is a nonempty and finite set of *claimants*, $E \in \mathbb{R}_+$ is the *estate*, and $c \in \mathbb{R}_+^N$ is the vector of *claims* of N on E for which $\sum_{i \in N} c_i \geq E$. Let TUBR^N denote the class of all bankruptcy problems with transferable utility with claimant set N. For convenience, we denote a TU-bankruptcy problem by $(E, c) \in \mathrm{TUBR}^N$.

Let $(E, c) \in \mathrm{TUBR}^N$. The corresponding *bankruptcy game with transferable utility* $v^{E,c} \in \mathrm{TU}^N$ is given by $v^{E,c}(S) = \max\{E - \sum_{i \in N \setminus S} c_i, 0\}$ for all $S \in 2^N$. Curiel et al. (1987) showed that TU-bankruptcy games are convex and compromise stable. Quant et al. (2005) showed that a convex and compromise stable TU-game is strategically equivalent to a bankruptcy game.

2.2 Bankruptcy Problems with Nontransferable Utility

Let N be a nonempty and finite set of claimants. For any $x, y \in \mathbb{R}_+^N$, we denote $x \leq y$ if $x_i \leq y_i$ for all $i \in N$, and $x < y$ if $x_i < y_i$ for all $i \in N$. The zero-vector $x \in \mathbb{R}_+^N$ with $x_i = 0$ for all $i \in N$ is denoted by 0^N. For any $E \subseteq \mathbb{R}_+^N$,

– the (nonnegative) *comprehensive hull* is given by $\text{comp}(E) = \{x \in \mathbb{R}^N_+ \mid \exists_{y \in E} : y \geq x\}$;

– the *upper contour set* is given by $\text{UC}(E) = \{x \in \mathbb{R}^N_+ \mid \neg\exists_{y \in E, y \neq x} : y \geq x\}$;

– the *strong Pareto set* is given by $\text{SP}(E) = \{x \in E \mid \neg\exists_{y \in E, y \neq x} : y \geq x\}$;

– the *weak Pareto set* is given by $\text{WP}(E) = \{x \in E \mid \neg\exists_{y \in E} : y > x\}$.

Note that $\text{SP}(E) = \text{WP}(E) \cap \text{UC}(E)$.

A *bankruptcy problem with nontransferable utility* (cf. Dietzenbacher et al. (2016)) is a triple (N, E, c) in which $E \subset \mathbb{R}^N_+$ is the estate satisfying the following conditions:

– E is nonempty, closed, and bounded;

– E is comprehensive, i.e., $E = \text{comp}(E)$;

– E is non-leveled, i.e., $\text{SP}(E) = \text{WP}(E)$,

and $c \in \text{UC}(E)$ is the vector of claims. Let BR^N denote the class of all bankruptcy problems with nontransferable utility with claimant set N. For convenience, we denote an NTU-bankruptcy problem by $(E, c) \in \text{BR}^N$. Note that $0^N \in E$ for all $(E, c) \in \text{BR}^N$. Moreover, any TU-bankruptcy problem $(E^*, c) \in \text{TUBR}^N$ can be written as an NTU-bankruptcy problem $(E, c) \in \text{BR}^N$ with $E = \{x \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \leq E^*\}$.

Let $(E, c) \in \text{BR}^N$. The vector of *utopia values* $u^E \in \mathbb{R}^N_+$ is for all $i \in N$ given by

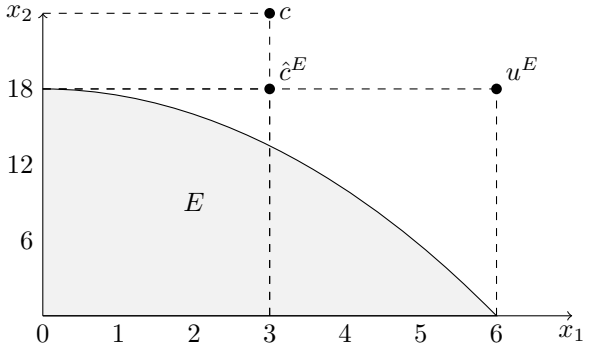
$$u^E_i = \max\{x_i \mid x \in E\}.$$

The vector of *truncated claims* $\hat{c}^E \in \mathbb{R}^N_+$ is for all $i \in N$ given by

$$\hat{c}^E_i = \min\left\{c_i, u^E_i\right\}.$$

Note that $(E, \hat{c}^E) \in \text{BR}^N$ for all $(E, c) \in \text{BR}^N$.

Example 1.
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$. We have $u^E = (6, 18)$, which means that $\hat{c}^E_1 = \min\{c_1, u^E_1\} = \min\{3, 6\} = 3$ and $\hat{c}^E_2 = \min\{c_2, u^E_2\} = \min\{24, 18\} = 18$.



△

A *bankruptcy rule* $f : \text{BR}^N \to \mathbb{R}^N_+$ assigns to any $(E, c) \in \text{BR}^N$ a payoff allocation $f(E, c) \in \text{SP}(E)$ for which $f(E, c) \leq c$. A bankruptcy rule $f : \text{BR}^N \to \mathbb{R}^N_+$ satisfies *truncation invariance* if $f(E, c) = f(E, \hat{c}^E)$ for all $(E, c) \in \text{BR}^N$.

3 Nonnegative Games with Nontransferable Utility

This section introduces nonnegative games with nontransferable utility and generalizes some notions for TU-games to this new model. Many classes of TU-games are nonnegative, e.g. cost savings games such as sequencing games and other operations research games, airport games, simple games, glove games, and bankruptcy games. This lower bound for the worth of coalitions arises naturally from the assumption that allocating nothing to a player corresponds to a payoff of zero utility, which implies for some allocation problems that negative utility payoffs do not have any interpretation. Following the same lines of reasoning, this lower bound can also be applied in the context of NTU-games.

Definition 3.1 (Nonnegative Game with Nontransferable Utility)**.**
A *nonnegative game with nontransferable utility* is a pair (N, V) in which N is a nonempty and finite set of players, and V assigns to each nonempty coalition $S \in 2^N \setminus \{\emptyset\}$ a set of payoff allocations $V(S) \subset \mathbb{R}_+^S$ satisfying the following conditions:

- $V(S)$ is nonempty, closed, and bounded;
- $V(S)$ is comprehensive, i.e., $V(S) = \text{comp}(V(S))$.

A nonnegative NTU-game (N, V) is called *monotonic* if $V(S) \subseteq \{x_S \mid x \in V(T)\}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$. Let NTU_+^N denote the class of all monotonic nonnegative NTU-games with player set N. For convenience, we denote such an NTU-game by $V \in \text{NTU}_+^N$. Note that a nonnegative TU-game $v \in \text{TU}^N$ gives rise to the nonnegative NTU-game $V \in \text{NTU}_+^N$ with $V(S) = \{x \in \mathbb{R}_+^S \mid \sum_{i \in S} x_i \leq v(S)\}$ for all $S \in 2^N \setminus \{\emptyset\}$. A *solution* for monotonic nonnegative NTU-games $F : \text{NTU}_+^N \to \mathbb{R}_+^N$ assigns to any $V \in \text{NTU}_+^N$ a payoff allocation $F(V) \in \text{WP}(V(N))$.

Let $V \in \text{NTU}_+^N$. Similar to Otten, Borm, Peleg, and Tijs (1998), we define the marginal vector $M^\sigma(V) \in \mathbb{R}^N$ corresponding to $\sigma \in \Pi(N)$ for all $n \in \{1, \ldots, |N|\}$ by

$$M_{\sigma(n)}^\sigma(V) = \max \left\{ x \in \mathbb{R}_+ \;\middle|\; (M_{\sigma(1)}^\sigma(V), \ldots, M_{\sigma(n-1)}^\sigma(V), x) \in V(\{\sigma(1), \ldots, \sigma(n)\}) \right\}.$$

Note that the conditions on V imply that this maximum exists. As in the context of TU-games, the marginal contribution of a player in a certain order can be interpreted as its maximal payoff when joining its predecessors, which have already been allocated their marginal contributions. Inspired by Borm, Keiding, McLean, Oortwijn, and Tijs (1992), we define $K(V) \in \mathbb{R}^N$ for all $i \in N$ by

$$K_i(V) = \max \left\{ x_i \;\middle|\; x \in V(N), x_{N \setminus \{i\}} \in \text{UC}(V(N \setminus \{i\})) \right\},$$

and $k(V) \in \mathbb{R}^N$ for all $i \in N$ by

$$k_i(V) = \max_{S \in 2^N : i \in S} \sup \left\{ x \in \mathbb{R}_+ \;\middle|\; (x, K_{S \setminus \{i\}}(V)) \in V(S) \right\}.$$

Note that the conditions on V imply that these maxima exist. As in the context of TU-games, $K_i(V)$ can be interpreted as the maximal payoff of player $i \in N$ within an allocation of $V(N)$ which is stable against a coalitional deviation of the other players together. Moreover, $k_i(V)$ can be interpreted as the minimal right of player $i \in N$, the maximal payoff which can be obtained within some coalition $S \in 2^N$, with $i \in S$, when each other member $j \in S \setminus \{i\}$ is allocated $K_j(V)$.

Using these notions, we can generalize the core, the core cover, and the reasonable set to the context of NTU-games. Let $V \in \mathrm{NTU}_+^N$. The (strong) core is defined by

$$\mathcal{C}(V) = \left\{ x \in V(N) \;\middle|\; \forall_{S \in 2^N \setminus \{\emptyset\}} : x_S \in \mathrm{UC}(V(S)) \right\},$$

the core cover is defined by

$$\mathcal{CC}(V) = \{ x \in \mathrm{SP}(V(N)) \mid k(V) \leq x \leq K(V) \},$$

and the reasonable set is defined by

$$\mathcal{R}(V) = \left\{ x \in \mathrm{SP}(V(N)) \;\middle|\; \forall_{i \in N} : \min_{\sigma \in \Pi(N)} M_i^\sigma(V) \leq x_i \leq \max_{\sigma \in \Pi(N)} M_i^\sigma(V) \right\}.$$

Lemma 3.1.
Let $V \in \mathrm{NTU}_+^N$. Then $\mathcal{C}(V) \subseteq \mathcal{CC}(V)$.

Proof. Let $x \in \mathcal{C}(V)$. For all $i \in N$, we can write

$$\begin{aligned} x_i &\leq \max\{x_i \mid x \in \mathcal{C}(V)\} = \max\{x_i \mid x \in V(N), \forall_{S \in 2^N \setminus \{\emptyset\}} : x_S \in \mathrm{UC}(V(S))\} \\ &\leq \max\{x_i \mid x \in V(N), x_{N \setminus \{i\}} \in \mathrm{UC}(V(N \setminus \{i\}))\} = K_i(V). \end{aligned}$$

Suppose that there exists an $i \in N$ for which $x_i < k_i(V)$. Let $S \in 2^N$ with $i \in S$ be such that $(k_i(V), K_{S \setminus \{i\}}(V)) \in V(S)$. Then $x_S \leq (k_i(V), K_{S \setminus \{i\}}(V))$ and $x_S \neq (k_i(V), K_{S \setminus \{i\}}(V))$. Since $V(S)$ is comprehensive, this means that $x_S \notin \mathrm{UC}(V(S))$. This contradicts that $x \in \mathcal{C}(V)$, so $k(V) \leq x \leq K(V)$. Hence, $x \in \mathcal{CC}(V)$. □

Definition 3.2 (Compromise Stability)**.**
An NTU-game $V \in \mathrm{NTU}_+^N$ is called *compromise stable* if $\mathcal{C}(V) = \mathcal{CC}(V)$ and $\mathcal{CC}(V) \neq \emptyset$.

Contrary to TU-games, the following example shows that the generalized reasonable set does not necessarily contain the core of an NTU-game.

Example 2.
Let $N = \{1, 2, 3\}$ and consider the game $V \in \mathrm{NTU}_+^N$ which is for all $S \in 2^N \setminus \{\emptyset\}$ given by

$$V(S) = \begin{cases} \left\{ x \in \mathbb{R}_+^S \;\middle|\; x_1^2 + x_2^2 \leq (9 - x_3)^2, x_3 \leq 9 \right\} & \text{if } S = N; \\ \left\{ x \in \mathbb{R}_+^S \;\middle|\; x_1 + x_2 \leq 4 \right\} & \text{if } S = \{1, 2\}; \\ 0^S & \text{otherwise.} \end{cases}$$

All marginal vectors are presented below.

σ	$M_1^\sigma(V)$	$M_2^\sigma(V)$	$M_3^\sigma(V)$
$(1,2,3)$	0	4	5
$(1,3,2)$	0	9	0
$(2,1,3)$	4	0	5
$(2,3,1)$	9	0	0
$(3,1,2)$	0	9	0
$(3,2,1)$	9	0	0

This means that the reasonable set is given by

$$\mathcal{R}(V) = \{ x \in \mathrm{SP}(V(N)) \mid 0 \leq x_1 \leq 9, 0 \leq x_2 \leq 9, 0 \leq x_3 \leq 5 \}.$$

One can verify that $(2, 2, 9 - 2\sqrt{2}) \in \mathcal{C}(V) \setminus \mathcal{R}(V)$. Hence, $\mathcal{C}(V) \not\subseteq \mathcal{R}(V)$. △

Although the reasonable set does not necessarily contain the core, the minimal and maximal marginal contributions can still be considered as reasonable bounds for payoff allocations. For that reason, we introduce the notion of reasonable stability to describe games for which the core and the reasonable set coincide.

Definition 3.3 (Reasonable Stability)**.**
An NTU-game $V \in \mathrm{NTU}_+^N$ is called *reasonable stable* if $\mathcal{C}(V) = \mathcal{R}(V)$.

Note that reasonable stability is stronger than *marginal convexity* (cf. Hendrickx, Borm, and Timmer (2002)), which requires that $M^\sigma(V) \in \mathcal{C}(V)$ for all $\sigma \in \Pi(N)$. Moreover, in view of Proposition 2.1, reasonable stability is equivalent to the combination of convexity and compromise stability on the class of TU-games.

4 Bankruptcy Games with Nontransferable Utility

This section introduces and analyzes a new model for bankruptcy games with nontransferable utility. Orshan et al. (2003) introduced a first model for NTU-bankruptcy games. As pointed out by Estévez-Fernández et al. (2014), coalitions can attain payoff allocations outside the estate in this game, which contradicts the original idea of O'Neill (1982). They redefined NTU-bankruptcy games to stay in line with this original idea about TU-bankruptcy games. However, the following example shows that their NTU-bankruptcy game does not straightforwardly generalize the original TU-bankruptcy game.

Example 3.
Let $N = \{1, 2, 3\}$ and consider the TU-bankruptcy problem $(E, c) \in \mathrm{TUBR}^N$ in which $E = 4$ and $c = (1, 2, 3)$. The corresponding TU-bankruptcy game $v^{E,c} \in \mathrm{TU}^N$ is presented below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v^{E,c}(S)$	0	0	1	1	2	3	4

Consider coalition $\{2, 3\}$. A straightforward generalization to an NTU-bankruptcy game $V^{E,c} \in \mathrm{NTU}_+^N$ would prescribe $V^{E,c}(\{2,3\}) = \{x \in \mathbb{R}_+^{\{2,3\}} \mid x_2 + x_3 \leq 3\}$. However, the NTU-bankruptcy game $(N, W^{E,c})$ introduced by Estévez-Fernández et al. (2014) assigns the set of payoff allocations $W^{E,c}(\{2,3\}) = \{x \in \mathbb{R}^N \mid x_2 + x_3 \leq 3, x_1 = 1, x_2 \leq 2, x_3 \leq 3\}$, which is essentially different due to the upper bound on the payoff of player 2. △

Next, we introduce a model for NTU-bankruptcy games which generalizes TU-bankruptcy games and simultaneously stays in line with the original idea of O'Neill (1982).

Definition 4.1 (Bankruptcy Game with Nontransferable Utility)**.**
Let $(E, c) \in \mathrm{BR}^N$ be a bankruptcy problem with nontransferable utility. The corresponding *bankruptcy game with nontransferable utility* $V^{E,c} \in \mathrm{NTU}_+^N$ is for all $S \in 2^N \setminus \{\emptyset\}$ defined by

$$V^{E,c}(S) = \begin{cases} \{x \in \mathbb{R}_+^S \mid (x, c_{N\setminus S}) \in E\} & \text{if } (0^S, c_{N\setminus S}) \in E; \\ 0^S & \text{if } (0^S, c_{N\setminus S}) \notin E. \end{cases}$$

Note that $V^{E,c}(S)$ is indeed nonempty, closed, bounded and comprehensive for all $S \in 2^N \setminus \{\emptyset\}$, since E is nonempty, closed, bounded and comprehensive. Moreover, $V^{E,c}$ is monotonic and $V^{E,c}(N) = E$. As in TU-bankruptcy games, coalitions can attain the payoff allocations within the estate in which the other players are allocated their claims.

Example 4.
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$ as in Example 1. We have $V^{E,c}(\{1\}) = 0$, $V^{E,c}(\{2\}) = [0, 13\frac{1}{2}]$, and $V^{E,c}(N) = E$. △

Contrary to the models of Orshan et al. (2003) and Estévez-Fernández et al. (2014), every subgame of the new bankruptcy game is a bankruptcy game too, as is the case for TU-bankruptcy games. For any NTU-game $V \in \mathrm{NTU}^N_+$, the *subgame* $V_S \in \mathrm{NTU}^S_+$ on coalition $S \in 2^N \setminus \{\emptyset\}$ is defined by $V_S(R) = V(R)$ for all $R \in 2^S \setminus \{\emptyset\}$.

Proposition 4.1.
Each subgame of a bankruptcy game is a bankruptcy game as well.

Proof. Let $(E, c) \in \mathrm{BR}^N$ and let $S \in 2^N \setminus \{\emptyset\}$. Then $V^{E,c}(S)$ is nonempty, closed, bounded and comprehensive. Moreover, $V^{E,c}(S)$ is non-leveled since E is non-leveled and comprehensive, and $c_S \in \mathrm{UC}(V^{E,c}(S))$ since $c \in \mathrm{UC}(E)$. This means that $(V^{E,c}(S), c_S) \in \mathrm{BR}^S$. For all $R \in 2^S \setminus \{\emptyset\}$, we can write

$$\begin{aligned}
V^{V^{E,c}(S),c_S}(R) &= \begin{cases} \{x \in \mathbb{R}^R_+ \mid (x, c_{S\setminus R}) \in V^{E,c}(S)\} & \text{if } (0^R, c_{S\setminus R}) \in V^{E,c}(S); \\ 0^R & \text{if } (0^R, c_{S\setminus R}) \notin V^{E,c}(S) \end{cases} \\
&= \begin{cases} \{x \in \mathbb{R}^R_+ \mid (x, c_{S\setminus R}, c_{N\setminus S}) \in E\} & \text{if } (0^R, c_{S\setminus R}, c_{N\setminus S}) \in E; \\ 0^R & \text{if } (0^R, c_{S\setminus R}, c_{N\setminus S}) \notin E \end{cases} \\
&= \begin{cases} \{x \in \mathbb{R}^R_+ \mid (x, c_{N\setminus R}) \in E\} & \text{if } (0^R, c_{N\setminus R}) \in E; \\ 0^R & \text{if } (0^R, c_{N\setminus R}) \notin E \end{cases} \\
&= V^{E,c}(R) \\
&= V^{E,c}_S(R).
\end{aligned}$$

Hence, $V^{E,c}_S \in \mathrm{NTU}^S_+$ is a bankruptcy game. □

The remainder of this section studies the relationship between the core, the core cover, and the reasonable set of NTU-bankruptcy games. For this, we need to find expressions for the upper and lower bounds of the core cover and the reasonable set. A useful observation for this analysis is that bankruptcy games are invariant under claim truncation.

Lemma 4.2.
Let $(E, c) \in \mathrm{BR}^N$. Then $V^{E,c} = V^{E,\hat{c}^E}$.

Proof. Let $S \in 2^N \setminus \{\emptyset\}$. If $\hat{c}^E_{N\setminus S} = c_{N\setminus S}$, then clearly $V^{E,c}(S) = V^{E,\hat{c}^E}(S)$. Suppose that $\hat{c}^E_{N\setminus S} \neq c_{N\setminus S}$. Then there exists an $i \in N \setminus S$ for which $\hat{c}^E_i = u^E_i < c_i$. This means that $(0^S, c_{N\setminus S}) \notin E$, so $V^{E,c}(S) = 0^S$. Since E is non-leveled, we have $V^{E,\hat{c}^E}(S) = \{x \in \mathbb{R}^S_+ \mid (x, \hat{c}^E_{N\setminus S}) \in E\} = 0^S$ if $(0^S, \hat{c}^E_{N\setminus S}) \in E$. Hence, $V^{E,c}(S) = V^{E,\hat{c}^E}(S)$. □

For any bankruptcy game $V^{E,c} \in \mathrm{NTU}^N_+$, we define the vector $m(E, c) \in \mathbb{R}^N_+$ by $m_i(E, c) = \max\{x \in V^{E,c}(\{i\})\}$ for all $i \in N$. Together with the vector of truncated claims, this vector appears to play a central role in the bounds of the core cover of bankruptcy games.

Lemma 4.3.
Let $(E,c) \in \mathrm{BR}^N$. Then

(i) $K(V^{E,c}) = \hat{c}^E$;

(ii) $k(V^{E,c}) = m(E,c)$.

Proof. (i) From Lemma 4.2 we know that $V^{E,c} = V^{E,\hat{c}^E}$, so $K(V^{E,c}) = K(V^{E,\hat{c}^E})$. Let $i \in N$. We have $(\hat{c}_i^E, x) \in E$ for all $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$, so $(\hat{c}_i^E, x) \in V^{E,\hat{c}^E}(N)$ for all $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. This implies that $K_i(V^{E,\hat{c}^E}) \geq \hat{c}_i^E$. Suppose that $K_i(V^{E,\hat{c}^E}) > \hat{c}_i^E$. Let $x \in \mathrm{UC}(V^{E,\hat{c}^E}(N \setminus \{i\}))$ be such that $(K_i(V^{E,\hat{c}^E}), x) \in V^{E,\hat{c}^E}(N)$. Since $V^{E,\hat{c}^E}(N)$ is comprehensive, we have $(\hat{c}_i^E, x) \in V^{E,\hat{c}^E}(N)$, so $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$. This means that $(\hat{c}_i^E, x) \notin \mathrm{SP}(V^{E,\hat{c}^E}(N))$ and $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. Since $V^{E,\hat{c}^E}(N)$ is non-leveled, we have $(\hat{c}_i^E, x) \notin \mathrm{WP}(V^{E,\hat{c}^E}(N))$. This means that there exists a $y \in V^{E,\hat{c}^E}(N)$ for which $y > (\hat{c}_i^E, x)$. Since $V^{E,\hat{c}^E}(N)$ is comprehensive, we have $(\hat{c}_i^E, y_{N\setminus\{i\}}) \in V^{E,\hat{c}^E}(N)$. This means that $y_{N\setminus\{i\}} \in V^{E,\hat{c}^E}(N \setminus \{i\})$, which contradicts that $x \in \mathrm{SP}(V^{E,\hat{c}^E}(N \setminus \{i\}))$. Hence, $K_i(V^{E,c}) = K_i(V^{E,\hat{c}^E}) = \hat{c}_i^E$.

(ii) Let $i \in N$. We can write

$$k_i(V^{E,c}) \geq \sup\left\{x \in \mathbb{R}_+ \mid x \in V^{E,c}(\{i\})\right\} = \max\left\{x \in V^{E,c}(\{i\})\right\} = m_i(E,c).$$

Suppose that we have $k_i(V^{E,c}) > m_i(E,c) = \max\{x \in V^{E,c}(\{i\})\}$. Let $S \in 2^N$ with $i \in S$ be such that $(k_i(V^{E,c}), K_{S\setminus\{i\}}(V^{E,c})) \in V^{E,c}(S)$. Then we know from Lemma 4.2 and (i) that $(k_i(V^{E,\hat{c}^E}), \hat{c}^E_{S\setminus\{i\}}) \in V^{E,\hat{c}^E}(S)$. This means that $(k_i(V^{E,\hat{c}^E}), \hat{c}^E_{S\setminus\{i\}}, \hat{c}^E_{N\setminus S}) \in E$, which implies that $k_i(V^{E,\hat{c}^E}) \in V^{E,\hat{c}^E}(\{i\})$. This contradicts that $k_i(V^{E,\hat{c}^E}) > m_i(E,c)$. Hence, $k_i(V^{E,c}) = k_i(V^{E,\hat{c}^E}) = m_i(E,c)$. □

From Lemma 3.1 and Example 2 we know that, contrary to TU-games, the core cover is not necessarily contained in the reasonable set of an NTU-game. Surprisingly, for the reasonable set of an NTU-bankruptcy game we find the same upper bound and lower bound as for its core cover, which means that the core cover and the reasonable set of an NTU-bankruptcy game still coincide.

Lemma 4.4.
Let $(E,c) \in \mathrm{BR}^N$ and let $i \in N$. Then

(i) $\max_{\sigma\in\Pi(N)} M_i^\sigma(V^{E,c}) = \hat{c}_i^E$;

(ii) $\min_{\sigma\in\Pi(N)} M_i^\sigma(V^{E,c}) = m_i(E,c)$.

Proof. (i) From Lemma 4.2 we know that $V^{E,c} = V^{E,\hat{c}^E}$, so $\max_{\sigma\in\Pi(N)} M_i^\sigma(V^{E,c}) = \max_{\sigma\in\Pi(N)} M_i^\sigma(V^{E,\hat{c}^E})$. Let $\hat{\sigma} \in \Pi(N)$ be such that $\hat{\sigma}(|N|) = i$. We have $(x, \hat{c}_i^E) \in E$ for all $x \in V^{E,\hat{c}^E}(N \setminus \{i\})$, so $(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(|N|-1)}(V^{E,\hat{c}^E}), \hat{c}_i^E) \in V^{E,\hat{c}^E}(N)$. This implies that $\max_{\sigma\in\Pi(N)} M_i^\sigma(V^{E,\hat{c}^E}) \geq \hat{c}_i^E$. Suppose that $\max_{\sigma\in\Pi(N)} M_i^\sigma(V^{E,\hat{c}^E}) > \hat{c}_i^E$. Let $\hat{\sigma} \in \Pi(N)$ be such that $M_i^{\hat{\sigma}}(V^{E,\hat{c}^E}) = \max_{\sigma\in\Pi(N)} M_i^\sigma(V^{E,\hat{c}^E})$. Let $n \in \{2, \ldots, |N|\}$ be such that $\hat{\sigma}(n) = i$. Then we have $(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(n)}(V^{E,\hat{c}^E})) \in V^{E,\hat{c}^E}(\{\hat{\sigma}(1), \ldots, \hat{\sigma}(n)\})$, which means that

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}), \ldots, M^{\hat{\sigma}}_{\hat{\sigma}(n)}(V^{E,\hat{c}^E}), \hat{c}^E_{\hat{\sigma}(n+1)}, \ldots, \hat{c}^E_{\hat{\sigma}(|N|)}\right) \in E.$$

Since E is comprehensive, we have

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}),\ldots,M^{\hat{\sigma}}_{\hat{\sigma}(n-1)}(V^{E,\hat{c}^E}),\hat{c}^E_{\hat{\sigma}(n)},\ldots,\hat{c}^E_{\hat{\sigma}(|N|)}\right)\in E\setminus \mathrm{SP}(E).$$

Since E is non-leveled, we have

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}),\ldots,M^{\hat{\sigma}}_{\hat{\sigma}(n-1)}(V^{E,\hat{c}^E}),\hat{c}^E_{\hat{\sigma}(n)},\ldots,\hat{c}^E_{\hat{\sigma}(|N|)}\right)\in E\setminus \mathrm{WP}(E).$$

This means that there exists a $y\in E$ for which

$$y>\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}),\ldots,M^{\hat{\sigma}}_{\hat{\sigma}(n-1)}(V^{E,\hat{c}^E}),\hat{c}^E_{\hat{\sigma}(n)},\ldots,\hat{c}^E_{\hat{\sigma}(|N|)}\right).$$

Since E is comprehensive, we have

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}),\ldots,M^{\hat{\sigma}}_{\hat{\sigma}(n-2)}(V^{E,\hat{c}^E}),y_{\hat{\sigma}(n-1)},\hat{c}^E_{\hat{\sigma}(n)},\ldots,\hat{c}^E_{\hat{\sigma}(|N|)}\right)\in E.$$

This means that

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}),\ldots,M^{\hat{\sigma}}_{\hat{\sigma}(n-2)}(V^{E,\hat{c}^E}),y_{\hat{\sigma}(n-1)}\right)\in V^{E,\hat{c}^E}(\{\hat{\sigma}(1),\ldots,\hat{\sigma}(n-1)\}),$$

which contradicts that $M^{\hat{\sigma}}_{\hat{\sigma}(n-1)}(V^{E,\hat{c}^E})$ equals

$$\max\left\{x\in\mathbb{R}_+\;\middle|\;(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}),\ldots,M^{\hat{\sigma}}_{\hat{\sigma}(n-2)}(V^{E,\hat{c}^E}),x)\in V^{E,\hat{c}^E}(\{\hat{\sigma}(1),\ldots,\hat{\sigma}(n-1)\})\right\}.$$

Hence, $\max_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,c})=\max_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,\hat{c}^E})=\hat{c}^E_i$.

(ii) Let $\hat{\sigma}\in\Pi(N)$ be such that $\hat{\sigma}(1)=i$. We can write

$$M^{\hat{\sigma}}_i(V^{E,c})=\max\left\{x\in\mathbb{R}_+\mid x\in V^{E,c}(\{i\})\right\}=\max\left\{x\in V^{E,c}(\{i\})\right\}=m_i(E,c).$$

This implies that $\min_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,c})\leq m_i(E,c)$. Suppose that $\min_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,c})<m_i(E,c)$. From Lemma 4.2 we know that $V^{E,c}=V^{E,\hat{c}^E}$, so $\min_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,c})=\min_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,\hat{c}^E})$. Let $\hat{\sigma}\in\Pi(N)$ be such that $M^{\hat{\sigma}}_i(V^{E,\hat{c}^E})=\min_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,\hat{c}^E})$. Let $n\in\{2,\ldots,|N|\}$ be such that $\hat{\sigma}(n)=i$. Then we have

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}),\ldots,M^{\hat{\sigma}}_{\hat{\sigma}(n-1)}(V^{E,\hat{c}^E}),m_i(E,c)\right)\notin V^{E,\hat{c}^E}(\{\hat{\sigma}(1),\ldots,\hat{\sigma}(n)\}),$$

which means that

$$\left(M^{\hat{\sigma}}_{\hat{\sigma}(1)}(V^{E,\hat{c}^E}),\ldots,M^{\hat{\sigma}}_{\hat{\sigma}(n-1)}(V^{E,\hat{c}^E}),m_i(E,c),\hat{c}^E_{\hat{\sigma}(n+1)},\ldots,\hat{c}^E_{\hat{\sigma}(|N|)}\right)\notin E.$$

Since E is comprehensive, we know from (i) that $(m_i(E,c),\hat{c}^E_{N\setminus\{i\}})\notin E$, which contradicts that $m_i(E,c)\in V^{E,\hat{c}^E}(\{i\})$. Hence, $\min_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,c})=\min_{\sigma\in\Pi(N)}M^{\sigma}_i(V^{E,\hat{c}^E})=m_i(E,c)$. □

From Lemma 4.3 and Lemma 4.4 we know that the core cover and the reasonable set of a bankruptcy game coincide. From Lemma 3.1 we know that the core is contained in the core cover, which means that the core of a bankruptcy game is also contained in its reasonable set. Next, we generalize the result for TU-bankruptcy games which states that the core of a bankruptcy game coincides with its core cover and its reasonable set.

Theorem 4.5.
Every bankruptcy game is compromise stable and reasonable stable.

Proof. Let $(E,c) \in \mathrm{BR}^N$. From Lemma 4.3 and Lemma 4.4 we know that $\mathcal{CC}(V^{E,c}) = \mathcal{R}(V^{E,c})$, so it suffices to show that $\mathcal{CC}(V^{E,c}) \neq \emptyset$ and $\mathcal{C}(V^{E,c}) = \mathcal{CC}(V^{E,c})$. From Lemma 4.3 we know that $\mathcal{CC}(V^{E,c}) = \{x \in \mathrm{SP}(E) \mid m(E,c) \leq x \leq \hat{c}^E\}$. Since $c \in \mathrm{UC}(E)$, we have $m(E,c) \in E$, $\hat{c}^E \in \mathrm{UC}(E)$, and $m(E,c) \leq \hat{c}^E$. This means that there exists an $x \in \mathrm{SP}(E)$ for which $m(E,c) \leq x \leq \hat{c}^E$, so $\mathcal{CC}(V^{E,c}) \neq \emptyset$.

Let $x \in \mathcal{CC}(V^{E,c})$. Then we have $x \leq \hat{c}^E \leq c$. Suppose that $x \notin \mathcal{C}(V^{E,c})$. Then there exists an $S \in 2^N \setminus \{\emptyset\}$ for which $x_S \notin \mathrm{UC}(V^{E,c}(S))$. This means that there exists a $y \in V^{E,c}(S)$ for which $y \geq x_S$ and $y \neq x_S$. Then we have $(y, c_{N\setminus S}) \in E$. Since $x \leq (y, c_{N\setminus S})$, this means that $x \notin \mathrm{SP}(E)$, which contradicts that $x \in \mathcal{CC}(V^{E,c})$. Hence, $x \in \mathcal{C}(V^{E,c})$ and we have $\mathcal{CC}(V^{E,c}) \subseteq \mathcal{C}(V^{E,c})$. From Lemma 3.1 we know that $\mathcal{C}(V^{E,c}) \subseteq \mathcal{CC}(V^{E,c})$, so $\mathcal{C}(V^{E,c}) = \mathcal{CC}(V^{E,c})$. □

Using Lemma 4.3, Lemma 4.4, and Theorem 4.5, we derive a compact expression for the core of a bankruptcy game.

Corollary 4.6.
Let $(E,c) \in \mathrm{BR}^N$. *Then* $\mathcal{C}(V^{E,c}) = \{x \in \mathrm{SP}(E) \mid x \leq c\}$.

In other words, all bankruptcy rules assign to each bankruptcy problem a core element of the corresponding bankruptcy game. This means that a solution for NTU-games corresponds on the class of bankruptcy games to a bankruptcy rule if and only if it assigns to any bankruptcy game a core element. The other way around, the question arises under which conditions a bankruptcy rule corresponds to a solution for NTU-games on the class of bankruptcy games. Such a bankruptcy rule is called game theoretic.

Definition 4.2 (Game Theoretic Bankruptcy Rule)**.**
A bankruptcy rule f is called *game theoretic* if there exists a solution $F : \mathrm{NTU}_+^N \to \mathbb{R}_+^N$ for which $f(E,c) = F(V^{E,c})$ for all $(E,c) \in \mathrm{BR}^N$.

Similar to bankruptcy rules for TU-bankruptcy problems, a necessary and sufficient condition for an NTU-bankruptcy rule to be game theoretic is to satisfy truncation invariance.

Theorem 4.7.
A bankruptcy rule is game theoretic if and only if it satisfies truncation invariance.

Proof. Let f be a game theoretic bankruptcy rule. Let $(E,c) \in \mathrm{BR}^N$. From Lemma 4.2 we know that $V^{E,c} = V^{E,\hat{c}^E}$. We can write

$$f(E,c) = F(V^{E,c}) = F(V^{E,\hat{c}^E}) = f(E,\hat{c}^E).$$

Hence, f satisfies truncation invariance.

Let f be a bankruptcy rule satisfying truncation invariance. Let $F : \mathrm{NTU}_+^N \to \mathbb{R}_+^N$ be a solution such that $F(V^{E,c}) = f(V^{E,c}(N), K(V^{E,c}))$ for any bankruptcy game $V^{E,c} \in \mathrm{NTU}_+^N$. Let $(E,c) \in \mathrm{BR}^N$. From Lemma 4.3 we know that $K(V^{E,c}) = \hat{c}^E$. We can write

$$f(E,c) = f(E,\hat{c}^E) = f(V^{E,c}(N), K(V^{E,c})) = F(V^{E,c}).$$

Hence, f is game theoretic. □

5 Concluding Remarks

A solution for NTU-games corresponds on the class of bankruptcy games to a bankruptcy rule if and only if it assigns to any bankruptcy game a core element. The compromise value for NTU-games (cf. Borm et al. (1992)) and the MC-value for monotonic NTU-games (cf. Otten et al. (1998)) are such solutions. Future research could study the interpretation and axiomatic significance of these and other corresponding bankruptcy rules in more detail.

The other way around, a bankruptcy rule is game theoretic if and only if it satisfies truncation invariance. The constrained relative equal awards rule (cf. Dietzenbacher et al. (2016)) is such a bankruptcy rule. Future research could study the corresponding solutions for NTU-games in order to further extend the relation between NTU-bankruptcy problems and NTU-games.

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