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Contents

Chapter 0

Introduction

Mechanism design is a field in economics and game theory that takes an engineering approach to design economic mechanisms or incentives, toward desired welfare objectives, where players have decentralized information and act in a Bayesian rational way. It has been studied since 1970s and applied extensively in practice, for example, in designing auctions as FCC spectrum auctions and Google AdWords auctions. The interface of mechanism design and computation also promotes innovations in electronic commerce. In this thesis, I further investigate mechanism design theory for general social choice problems.

The contents are organized as follows: In chapter 1, I provide an axiomatic characterization of the probability-weighted minimal norm solution for social choice problems with reference points. In chapter 2, I investigate the problem of characterizing feasibility conditions for general social choice problems. The examples include voting, auctions with externalities, combinatorial auctions and exchanges with complementary objects. In chapter 3, I consider the problem of selecting among ex post efficient solutions for a two-person bargaining problem, when multiple ex post efficient solutions exist. In chapters 4 and 5, I investigate two specific problems of designing trading mechanisms with monetary transfers to achieve certain welfare objectives. In chapter 4, I discuss the choice of information partitions together with a trading mechanism for the seller and the buyer in order to enlarge the trading surplus. In chapter 5, I consider a revenue-maximizing problem for a seller who wants to sell two complementary objects, in the presence of inter-buyer resale.

Chapter 1 provides a characterization of the probability-weighted minimal norm solution in a Bayesian social choice environment as introduced by Myerson (1979). In such a problem, incentive compatibility restricts the set of feasible utility allocations. I assume that there is some socially desirable reference utility allocation, but that this allocation is unattainable. For example, the allocation in the past may become infeasible when players receive more information. A problem is: If such a reference allocation is relevant for the compromise, which allocation will players finally agree on? The proposed solution prescribes the incentive feasible utility allocation that minimizes a probability-weighted norm to the interim reference point. The solution is uniquely determined by eight axioms: an IIA axiom, a Pareto optimality axiom, a symmetry axiom, a fairness axiom, a splitting types axiom, a scaling axiom, a translation invariance axiom, and a continuity axiom.

Chapter 2 considers the implementability of reduced form allocation rules for social choice problems with general utility functions and finite types. This class of problems is motivated by Maskin and Riley (1984), which discusses the optimal auction with risk averse buyers. Due to general utility functions, the optimal auction problem raises a reduced form implementation problem: Can a system of interim expected winning probabilities be generated by a feasible allocation rule? Border (1991) characterizes the implementability condition for single unit auctions. I consider general social choice problems and obtain a necessary and sufficient condition for implementability, as well as a necessary condition with finitely many inequalities. The results in this chapter can be used to study voting problems, package exchanges with complementary valuations, and package auctions with risk averse players.

Chapter 3 studies disagreement point monotone bargaining solutions. I consider a two-person bargaining problem where players' disagreement payoffs are correlated and it is common knowledge that players must agree. I investigate the existence of any ex post efficient utility allocation such that each player's interim utility is non-constant and weakly responsive to his disagreement payoff. I establish some impossibility results for such monotone solutions.

Chapter 4 revisits the bilateral trade problem of Myerson and Satterthwaite (1983). It investigates how the information structure (i.e. how information is distributed among players) influences the attainability of ex post efficient allocations. I construct coarser partitions together with a feasible trading procedure that induces more efficient trade than the constrained efficient solution of Myerson and Satterthwaite.

Chapter 5 considers the optimal sale of two complementary objects to two buyers in the presence of resale. It is assumed that it is common knowledge that one buyer obtaining the bundle is efficient. Assuming full transparency of the seller's auction outcome, I show that if buyers use a mediator to maximize the resale surplus in a sequentially optimal way, then the optimal revenue, as in Myerson (1981), is unattainable. I introduce a modified Myerson auction (MMA) that requires selling the bundle with some personalized reserve prices and the seller withholding one object in case these prices are not met. The revenue from MMA when resale is not allowed serves as an upper bound of the seller's revenue.

Chapter 1

Characterization of the Minimal Norm Solution with Incomplete Information

1.1. Introduction

In this chapter, we investigate the problem of an arbitrator trying to select a decision from a finite set of social alternatives for a group of players, when the arbitrator does not have information about their preferences except some prior estimate. On the one hand, the arbitrator has to respect incentive compatibility constraints: Each player must be incentivized to reveal his true preferences. That the social choice must be made at the interim stage (i.e. the players know their types but the arbitrator does not) restricts the set of feasible utility allocations. On the other hand, the players may agree that some incentive infeasible allocation is a relevant aspiration point. Such a reference point may be generated by some feasible allocation in the past or in the future. To determine a fair compromise, the arbitrator has to also respect the players' aspirations of what they are entitled to receive.

Consider a bilateral trade example. The seller has value zero for the object and the buyer's value can be either 1 or $v > 1$, with probabilities $p = (\underline{p}, \overline{p})$. In case of no trade, both players receive zero. In any incentive compatible, individual rational, and ex post efficient trading mechanism, the object is always being transferred and the buyer pays the seller a constant price $d \in [0, 1]$. The set of interim utility vectors for the seller and types 1 and v of the buyer is given by $X = \{(d, 1 - d, v - d) : d \in [0, 1]\}$. A natural question is: What would be the fair trading price at the interim stage?

A mediator trying to answer this question, could consider two possible benchmarks:

(i) Taking an ex ante point of view, i.e. the buyer not knowing his value, Nash's bargaining solution requires an equal split of $V = \underline{p} + \overline{p}v$ between the two players.

(ii) Taking an ex post point of view, i.e. both players knowing the buyer's value, Nash's solution requires an equal split of 1 or v, depending on the state.

Now consider the interim stage. There are two alternatives to modify Nash's solution:

(i) The ex ante Nash's solution used for the seller and both types of the buyer is given by $(V/2, V/2, V/2)$. This utility vector is infeasible.

(ii) The interim expectation of the ex post Nash's solution for the seller and each type of the buyer is given by $(V/2, 1/2, v/2)$. This utility vector is again infeasible.

While such a utility allocation in either (i) or (ii) is infeasible, the players may agree that it is qualified as a relevant interim reference point, hence, that it should influence the fair compromise. It turns out that with each of these perspectives in either (i) or (ii), the solution provided in this chapter (the minimal norm solution) prescribe a price $(\underline{p} + \overline{p}v)/2$ and the solutions of the two problems are the same. In this case, the seller and both types of the buyer are indifferent between these reference points. Notice that an increase in $\overline{p}$ leads to a higher price and both types of the buyer are worse off.

Apart from this bargaining example, Bayesian social choice problems with reference points arise more generally in economic environments. An example is the bankruptcy problem with complete information as in Aumann and Maschler (1985). A man dies and leaves debts $r_1, ..., r_n$ totalling more than his estate E. The authors investigate the rules about how should the estate be divided among n creditors. Now suppose the estate is indivisible and the creditors may value it differently. $\tilde{E}_1, ..., \tilde{E}_n$ are random variables and each creditor privately observes his value of the estate. A question is: How should the estate be divided among the creditors under incomplete information?

For social choice problems with complete information, Yu (1973) was the first to propose a class of Euclidean compromise solutions. Such a solution minimizes the Euclidean distance between the feasible set and the utopia point of that set.[1] It reflects that players must reach a compromise based on an endogenously determined, but generally infeasible, ideal point whose coordinates correspond to the maximum feasible payoffs attainable by the players. Thus, the solution minimizes a measure of the group regret. Voorneveld, van den Nouweland and McLean (2011) and Conley, McLean and Wilkie (2014) obtain two characterizations of Yu's solution. In this chapter, we consider (and axiomatize) a generalization of Yu's solution for social choice problems with reference points under incomplete information.

Each problem (p, X, r) specifies a system of marginal probabilities $p = (p_i)_{i \in N}$ with p_i supported by the individual type set T_i for each player $i \in N$. The set of incentive feasible interim utility allocations X is a convex compact subset in the interim utility space, and a reference point r is an interim utility allocation outside X, which is further required to strictly dominate one of the strong Pareto optimal allocations. Then, the minimal norm solution F selects the unique vector in X that minimizes the total quadratic utility losses from r weighted by the marginal probabilities p of different types of players. An increase in the marginal probability of a certain type lowers the utility loss that this type has to bear.

We characterize the minimal norm solution by eight axioms: independence of irrelevant alternatives (IIA), weak Pareto optimality (WPO), symmetry for TU problems (TU), individual fairness (IF), splitting types (ST), scaling (SCA), translation invariance

[1] In the utopia point, each coordinate corresponds to the maximal utility that a player can get in the feasible set.

(T.INV), and feasible set continuity (F.CONT).

The first axiom is similar to Nash's IIA axiom, except that a feasible set is defined differently and the disagreement point is replaced by the reference point. Weak Pareto optimality requires that the solution belongs to the weak Pareto set of the feasible set. The axiom of symmetry for TU problems requires that if the feasible set induces an ex ante transfer hyperplane and the interim reference point induces a symmetric reference point ex ante, then in the solution, the ex ante utilities of the players must be equal. It reduces to Nash's symmetry axiom in case of complete information. Individual fairness requires that all types of the players bear some losses in a TU problem. The axiom of splitting types, modified from Harsanyi and Selten (1975) and Weidner (1992), requires that if one problem is derived from another by splitting a type of a player into a twin, then the new solution is derived from the previous solution by splitting this type. We use symmetry for TU problems and splitting types to establish a consistency across problems with different systems of marginal probabilities. Scaling, translation invariance and feasible set continuity are used by Voorneveld et al. (2011) and Conley et al. (2014) to characterize the Yu solution.

The remainder of this chapter is organized as follows. Sections 1.2 and 1.3 introduce social choice problems with reference points and the axioms. Section 1.4 provides the characterization theorem. Section 1.5 investigates a minimal norm duality and independence of the axioms. Section 1.6, discusses the generation of reference points in specific economic contexts. Section 1.7 reviews the related literature and investigates whether the minimal norm solution satisfies the axioms in the literature. Section 1.8 concludes.

1.2. The Problem

1.2.1. Bayesian Social Choice Problems with Reference Points

We introduce Bayesian social choice problems of Myerson (1979), in which privately informed players select some social alternative from a finite set and there are no monetary transfers. Let N be a finite set of players. For each player $i \in N$, there is a finite type set T_i. Let $T = \times_{i\in N} T_i$ be the product type set, with the common prior $\pi \in \Delta(T)$. We assume $\pi(t) > 0$ for all $t \in T$. Denote $T_{-i} = \times_{j\neq i} T_j$. The conditional belief of player i with type t_i on the set of other players' types is given by $\pi_i(t_{-i}|t_i)$ for all $t_{-i} \in T_{-i}$. Denote p_{i,t_i} the marginal probability of t_i, or $p_{i,t_i} = \sum_{t_{-i}\in T_{-i}} \pi(t)$, and denote p the system of the marginals of π. Let $\mathring{T} = \bigcup_{i\in N} T_i$.

Let D be a nonempty finite set of decisions. The utility function of player i is given by $u_i : D \times T \to \mathbb{R}$.[2] A social choice problem S is given by $(\pi, D, (u_i)_{i\in N})$. Denote $\mathcal{S}(\pi)$

[2]As we mention later, the minimal norm solution is not invariant to positive affine transformations of the utility functions, which implies the domain of a mechanism cannot be measurable with respect

the set of all such problems by fixing π (and hence N and T) and varying $(D,(u_i)_{i\in N})$.

Let $S \in \mathcal{S}(\pi)$. A mechanism is a function $\mu : D\times T \to [0,1]$ satisfying $\sum_{d\in D}\mu(d|t) = 1$ for all $t \in T$. Let $\mathcal{M}(S)$ be the set of all mechanisms for S. For any $\mu \in \mathcal{M}(S)$, the expected utility for type $t_i \in T_i$ from reporting $\hat{t}_i \in T_i$, while the other players report honestly is given by

$$U_i(\mu,\hat{t}_i|t_i) = \sum_{t_{-i}\in T_{-i}}\sum_{d\in D}\mu(d|\hat{t}_i,t_{-i})u_i(d,t)\pi_i(t_{-i}|t_i). \tag{1.2.1}$$

A mechanism μ is incentive compatible (IC) if $U_i(\mu,t_i|t_i) \geq U_i(\mu,\hat{t}_i|t_i)$, for all $\hat{t}_i, t_i \in T_i$, and $i \in N$. Denote $\mathcal{B}(S)$ the set of all IC mechanisms for S. For $\mu \in \mathcal{B}(S)$, denote the interim utility of player i by $U_i(\mu|t_i) = U_i(\mu,t_i|t_i)$.

There are situations where no information transmission is available among the social planner and the players and decisions must be made under the veil of ignorance. With such communication constraints, we define a "simple lottery problem" from S, in which the social planner is constrained to choose from the set of constant mechanisms,

$$\mathcal{M}_c(S) = \{\mu \in \mathcal{M}(S) : \mu(\cdot|t) = \delta \text{ for all } t \in T, \quad \text{for some } \delta \in \Delta(D)\}. \tag{1.2.2}$$

Any mechanism in $\mathcal{M}_c(S)$ is IC, hence, $\mathcal{M}_c(S) \subset \mathcal{B}(S)$.

Every $\mu \in \mathcal{B}(S)$ defines an *incentive feasible* interim utility vector $U(\mu) \in \times_{i\in N}\mathbb{R}^{T_i}$. Denote n the dimension of the interim utility space, i.e. $n = \sum_{i\in N}|T_i|$. Denote $\mathcal{U}(S) \subset \mathbb{R}^n$ the set of all incentive feasible utility vectors, and by $\mathcal{U}_c(S) \subset \mathbb{R}^n$ the set of all incentive feasible allocations from all constant mechanisms of S.

For any $X \subset \mathbb{R}^n$, the strong (interim) Pareto boundary is given by[3]

$$PO(X) = \{x \in X : \text{ for all } y \in X, y \geq x \text{ implies } y = x\}. \tag{1.2.3}$$

Similarly, we define the weak Pareto boundary by

$$WPO(X) = \{x \in X : \nexists y \in X \text{ such that } y > x\}. \tag{1.2.4}$$

In some contexts, the players might agree that some infeasible utility allocation is relevant for a compromise. A *reference point* is an interim utility allocation $r \in \mathbb{R}^n$ that strictly Pareto dominates some strong Pareto allocation. We write $\mathcal{R}(X)$ for the set of all such reference points with respect to X,

$$\mathcal{R}(X) = \{r \in \mathbb{R}^n : \exists x \in PO(X) \text{ such that } r > x\}. \tag{1.2.5}$$

to the equivalent classes of vNM utility functions.

[3]For $x, y \in \mathbb{R}^n$, $y \geq x$ means $y_k \geq x_k$, for all $k = 1, ..., n$, and $y > x$ means $y_k > x_k$, for all $k = 1, ..., n$.

That is, a reference point yields strictly higher utilities for all coordinates than some point on the strong Pareto boundary.

By varying $(D, (u_i)_{i \in N})$, we obtain two classes of interim utility sets generated either by all social choice problems or by all simple lottery problems, given by[4]

$$\mathcal{X}(\pi) = \{X : \exists S \in \mathcal{S}(\pi) \text{ such that } X = \mathcal{U}(S)\},$$
$$\mathcal{X}_c(\pi) = \{X : \exists S \in \mathcal{S}(\pi) \text{ such that } X = \mathcal{U}_c(S)\}.$$

By fixing (N, T) and varying π, we obtain the classes of interim utility sets generated by different priors on a common type set. By fixing N and varying (T, π), we obtain the classes of interim utility sets across different type sets. Our axioms used for the characterization result allow for this consistency over different type sets.[5]

1.2.2. The Reduced Problems

Denote Π the set of all prior probabilities by varying (N, T, π). Define

$$\mathcal{X} = \bigcup_{\pi \in \Pi} \mathcal{X}(\pi), \quad \mathcal{X}_c = \bigcup_{\pi \in \Pi} \mathcal{X}_c(\pi), \text{ and}$$
$$\mathcal{X}_0 = \{X : X \text{ is a polytope in } \mathbb{R}^n \text{ for some finite } n\}.$$

In this chapter, we consider $\mathcal{X}_0$ as the domain of the feasible sets. On the other hand, Myerson (1984) requires the domain of the feasible sets to be $\mathcal{X}$. While $\mathcal{X}$ is the most natural domain, the next result shows that $\mathcal{X}$ is "coarse" since it is a subset of all polytopes. As we will mention in Section 1.6, a characterization on $\mathcal{X}$ is more difficult than that on $\mathcal{X}_0$.

LEMMA 1.1: *(i) For every $\pi \in \Pi$, $\mathcal{X}_c(\pi)$ contains all polytopes in $\mathbb{R}^n$. That is, $\mathcal{X}_c = \mathcal{X}_0$. (ii) $\mathcal{X} \subsetneq \mathcal{X}_0$.*

We define a Bayesian social choice problem with the reference point, or a *reduced problem* $\Gamma = (p, X, r)$ by

i. An interim utility space $\mathbb{R}^n$.

ii. $p \in \mathbb{R}^n_{++}, \sum_{j \in T_i} p_{ij} = 1$, for all $i \in N$.[6]

iii. $X \subset \mathbb{R}^n$ is a polytope.

iv. $r \in \mathbb{R}^n$ with $r > x$ for some $x \in PO(X)$.

[4]These classes of interim utility sets implicitly assume a welfarism: Two social choice problems with the same players, types sets and priors, but with different decision sets and utility functions can generate the same utility sets. Notice that we do distinguish two utility sets in the same space but with different priors.

[5]By fixing (N, T) and varying π, we may investigate alternative axioms and characterizations.

[6]We write $\mathbb{R}^n_+$,$\mathbb{R}^n_-$ and $\mathbb{R}^n_{++}$ for the vectors with all coordinates nonnegative, nonpositive, and strictly positive in $\mathbb{R}^n$.

This definition implicitly assumes that if two problems have the same marginals and differ only in the priors, then their solutions are the same. We emphasize that this reduction is only for simplicity.[7] We can instead use some prior $\pi \in \Pi$ as input of Γ. Finally, p and X are consistent in the definition, since every polytope X can be generated by $\mathcal{U}_c(S)$ for some $S \in \mathcal{S}(\pi)$.

Denote Σ the set of all (reduced) problems. A *solution* f assigns a unique feasible utility allocation for each problem,

$$f : \Sigma \to \mathcal{X}_0 \text{ such that } f(p, X, r) \in X, \text{ for all } (p, X, r) \in \Sigma. \tag{1.2.6}$$

Notation. Before the formal analysis, we introduce some notation. For any $x, y \in \mathbb{R}^n$, define the vector from coordinatewise multiplication $x * y \in \mathbb{R}^n$ by $(x * y)_k = x_k y_k$ for all $k = 1, ..., n$. If $x_k \neq 0$ for all $k = 1, ..., n$, define the inverse x^{-1} by $(x^{-1})_k = 1/x_k$, for all $k = 1, ..., n$. For $X \subset \mathbb{R}^n$ and $h \in \mathbb{R}^n$, define $h * X = \{h * x : x \in X\}$. For $y \in \mathbb{R}^n$, define $X + y = \{x + y : x \in X\}$. For $x, y \in \mathbb{R}^n$, the line through x with direction y is given by $l(x, y) = \{x + \alpha y : \alpha \in \mathbb{R}\}$.

For any $q \in \mathbb{R}^n_{++}$, the q-inner product $\langle ., . \rangle_q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$\langle x, y \rangle_q = \sum_i \sum_j x_{ij} q_{ij} y_{ij}. \tag{1.2.7}$$

The q-inner product induces the q-norm $\|x\|_q = \sqrt{\langle x, x \rangle_q}$. For any closed convex set $X \subset \mathbb{R}^n$ and vector $r \in \mathbb{R}^n$, define the q-projection of r onto X by

$$\phi(q, X, r) = \arg\min_{x \in X} \|r - x\|_q. \tag{1.2.8}$$

The q-projection is well defined: By the projection theorem for closed convex subsets in an Euclidean space, the q-projection of r onto X exists and is unique. For $r \in \mathbb{R}^n$ and $m > 0$, the q-normed ball $B_q(r, m)$ is given by $\{x \in \mathbb{R}^n : \|x - r\|_q \leq m\}$.

1.3. The Axioms

A solution F is the minimal norm solution, if for every problem $\Gamma = (p, X, r)$ in Σ, it is the p-projection of r onto X, or

$$F(\Gamma) = \phi(p, X, r). \tag{1.3.1}$$

[7] The information of marginals is sufficient for our characterization result. One consequence of this definition is that it requires to introduce an axiom of splitting types based on marginal probabilities, which must be modified from those in Harsanyi and Selten (1975) and Weidner (1992).

In this section, we provide eight axioms that characterize the minimal norm solution: An IIA axiom, a weak Pareto optimality axiom, a symmetry for TU problems axiom, an individual fairness axiom, a splitting types axiom, a scaling axiom, a translation invariance axiom, and a feasible set continuity axiom.

Axiom 1.1: Independence of Irrelevant Alternatives (IIA). Let $\Gamma = (p, X, r)$ and $\Gamma' = (p, X', r)$ in Σ. If $X \subseteq X'$ and $f(\Gamma') \in X$, then $f(\Gamma) = f(\Gamma')$.

The first axiom requires that if a feasible set becomes smaller and the solution for the larger set remains feasible, then it must be chosen in the smaller set. It resembles Nash's IIA axiom where the disagreement point is replaced by the reference point. With our domain, every larger feasible set can be extended trivially from a smaller feasible set in the interim utility space. The underlying economic environments or incentives do not restrict the existence of such an extension. This is in contrast to the extension axiom of Myerson (1984), which is defined on the space of social choice problems. We discuss this issue in Section 1.7.

The second axiom requires that for each problem, the solution belongs to the weak Pareto boundary of that problem.

Axiom 1.2: Weak Pareto Optimality (WPO). Let $\Gamma = (p, X, r)$ in Σ. Then $f(\Gamma) \in WPO(X)$.

The third axiom is a new symmetry axiom introduced in this chapter. First, consider a complete information problem (e.g. each player's type set is a singleton) with transferable utility, defined by the reference point $r^0 = 0 \in \mathbb{R}^N$ and the feasible set

$$X^0_{w,\kappa} = \{x \in \mathbb{R}^N : \sum_i x_i \leq w \text{ and } x_i \geq \kappa \quad \text{for all } i \in N\}, \tag{1.3.2}$$

for some $\kappa < w < 0$. Then, Nash's symmetry axiom requires that each player obtains the same utility.

To apply Nash's symmetry to incomplete information problems, the third axiom introduces a linear map from a class of interim hyperplane problems to a class of ex ante transferable utility problems. We call these hyperplane problems (probability-weighted) TU problems. A problem $\Gamma = (p, X, r)$ is a p-TU problem, if $r = 0 \in \mathbb{R}^n$ and X is given by

$$X_{w,\kappa} = \{x \in \mathbb{R}^n : \sum_i \sum_j p_{ij} x_{ij} \leq w \text{ and } x_{ij} \geq \kappa, \text{ for all } (i,j) \in \mathring{T}\}, \tag{1.3.3}$$

for some $\kappa \leq \min_{(i,j)} w/p_{ij} < w < 0$. By varying (w, κ), we have all p-TU problems, and by varying p, we define the class of TU problems.

To see that Γ is well-defined, notice that $X_{w,\kappa}$ is a polytope and $0 \in \mathcal{R}(X_{w,\kappa})$. As shown below, requiring κ being uniform for all $(i,j) \in \mathring{T}$ is such that the ex ante transferable utility problems are symmetric. Requiring $\kappa < 0$ small enough relates to a larger class of linear problems derived from p-TU problems.[8]

Axiom 1.3: Symmetry for TU Problems (TU). Let $\Gamma = (p, X, 0)$ be a p-TU problem. Then,

$$\sum_j p_{ij} f_{ij}(\Gamma) = \sum_m p_{km} f_{km}(\Gamma), \text{ for all } i, k \in N. \tag{1.3.4}$$

Intuitively, the axiom requires that if we define a linear transformation $\varphi : \mathbb{R}^n \to \mathbb{R}^N$ by

$$(\varphi x)_i = \sum_j p_{ij} x_{ij}, \text{ for all } i \in N, \tag{1.3.5}$$

then $\varphi X_{w,\kappa} = X^0_{w,\kappa}$ and $\varphi r = r^0$. Then the ex ante utility allocation is symmetric among the players.

The fourth axiom is a modified form of the individual fairness axiom of Conley et al. (2014). It requires all types of players bearing some losses in TU problems. As the hyperplane in a p-TU problem intercepts all axes with strictly negative values, each type of a player should be given some weight in determining the final allocation. Then, every type of a player must bear some strictly positive loss.

Axiom 1.4: Individual Fairness (IF). Let $\Gamma = (p, X, 0)$ be a p-TU problem. Then, $f(\Gamma) < 0$.

The fifth axiom on irrelevant splitting of types was first introduced by Harsanyi and Selten (1972), which considers an inessential way of transforming a problem. Since our problems are defined by marginal probabilities rather than priors, we use a modified version which is implied by their definition.

Definition 1.1: Let $\Gamma = (p, X, r)$ and $\Gamma' = (p', X', r')$ in Σ, with p, X, r in $\mathbb{R}^n$ and p', X', r' in $\mathbb{R}^{n+1}$. Γ' is obtained from Γ by splitting a type s of player 1 with probability $\alpha \in (0,1)$, if

i. $N' = N$, $T'_i = T_i$ for all $i \neq 1$, and $T'_1 = (T_1 \setminus \{s\}) \cup \{a, b\}$.

ii. $p'_{ij} = p_{ij}$ for all $j \in T'_i$, $i \neq 1$, and $p'_{1j} = p_{1j}$ for all $j \in T'_1 \setminus \{a, b\}$, and $p'_{1a} = \alpha p_{1s}$, $p'_{1b} = (1-\alpha) p_{1s}$.

iii. $r'_{ij} = r_{ij}$ for all $j \in T'_i$, $i \neq 1$ and, $r'_{1j} = r_{1j}$ for all $j \in T'_1 \setminus \{a, b\}$, and $r'_{1a} = r'_{1b} = r_{1s}$.

iv. $x' \in X'$ if and only if there exists $x \in X$ such that $x'_{ij} = x_{ij}$ for all $j \in T'_i$, $i \neq 1$ and, $x'_{1j} = x_{1j}$ for all $j \in T'_1 \setminus \{a, b\}$, and $x'_{1a} = x'_{1b} = x_{1s}$.

[8] This requirement is for technical convenience and becomes clear in the proof of Lemma 1.2 (ii).

This definition is a reduced version derived from a more primitive definition by comparing two social choice problems with different priors, as in Weidner (1992). To see it intuitively, suppose the system of marginals p is derived from a prior π, then Definition 1.1 (ii) is implied by the following operation on the prior,

$$\pi'(j,\cdot) = \begin{cases} \pi(j,\cdot) & \text{if } j \in T_1' \setminus \{a,b\}, \\ \alpha\pi(j,\cdot) & \text{if } j = a, \\ (1-\alpha)\pi(j,\cdot) & \text{if } j = b. \end{cases}$$

This operation does not affect the utility functions and the conditional beliefs of types a and b. Hence, both types a and b of player 1 have the same decision problems. Also, the operation does not affect the utility functions, the conditional beliefs, the decisions of other types of player 1 and of types of other players. So, these two social choice problems must generate the same incentive compatible utility allocations, except that the interim utilities that types a and b receive from x' will be the same as type s receives from x. Now the axiom of splitting types is self-explanatory.

Axiom 1.5: Splitting Types (ST). Let $\Gamma = (p, X, r)$ and $\Gamma' = (p', X', r')$ in Σ. Suppose Γ' is obtained from Γ by splitting a type s of player 1 with probability $\alpha \in (0,1)$. Then

$$f_{1j}(\Gamma') = \begin{cases} f_{1j}(\Gamma) & \text{if } j \in T_1' \setminus \{a,b\}, \\ f_{1s}(\Gamma) & \text{if } j \in \{a,b\}, \end{cases}$$

and

$$f_{ij}(\Gamma') = f_{ij}(\Gamma), \text{ for all } j \in T_i,\ i \in N \setminus \{1\}.$$

The following scaling axiom is modified from Voorneveld et al. (2011) and Conley et al. (2014). It establishes a link between the class of TU problems and a larger class of linear problems. If a linear problem is derived by scaling a p-TU problem, in which type j of player i has twice the relative weight of type m of player k as in the original problem, then the utility loss to type j relative to that of type m should be half the utility loss to type j relative to that of type m in the original problem.

Axiom 1.6: Scaling (SCA). Let $\Gamma = (p, X, 0)$ be a p-TU problem. For any $h \in \mathbb{R}^n_{++}$, define a linear problem $h * \Gamma = (p, h * X, 0)$. Then

$$h_{ij} f_{km}(\Gamma) f_{ij}(h * \Gamma) = h_{km} f_{ij}(\Gamma) f_{km}(h * \Gamma), \tag{1.3.6}$$

for all $(i,j),(k,m) \in \mathring{T}$.

To see $h * \Gamma$ is well defined, notice that $h * X$ is a polytope and $0 \in \mathcal{R}(h * X)$. If each player's type set is a singleton, this axiom reduces to the scaling axiom of Voorneveld et al. (2011), except that they define a hyperplane problem differently and the utopia point is replaced by the reference point.

Axiom 1.7: Translation Invariance (T.INV). Let $\Gamma = (p, X, r)$ in Σ. For any $z \in \mathbb{R}^n$, define $\Gamma + z = (p, X + z, r + z)$. Then $f(\Gamma + z) = f(\Gamma) + z$.

The last axiom is a mild regularity condition. It states that a small change in the feasible set does not lead to a drastic change in the solution outcome.

Axiom 1.8: Feasible Set Continuity (F.CONT). Let $\Gamma = (p, X, r)$ and $\Gamma_k = (p, X_k, r)$, $k = 1, 2, ...$, be a sequence of problems in Σ, and $X_k \to X$ in the Hausdorff metric[9]. Then, $f(\Gamma_k) \to f(\Gamma)$.

We now provide some final remarks on these axioms.

IIA is often used in individual and social choice theory, but a solution that satisfies IIA may violates an axiom of individual monotonicity introduced by Kalai and Smorodinsky (1975), which requires that if players have more resources to share, all of the players must be weakly better off. Also, IIA is too strong to be satisfied by any voting rule in some environment.

WPO is a weak interim welfare criterion. Holmstrom and Myerson (1983) shows that various concepts of Pareto optimality under uncertainty can be equivalently represented through measurability restrictions on individual weights in a social welfare function. At the interim stage, it is natural to require the welfare weights depending only on one's own types.

To interpret TU, notice that the arbitrator as an outsider has no private information. A hyperplane problem and the corresponding ex ante transferable utility problem are observable equivalent to him, and a symmetry on the set of players applies. ST is a fairly weak axiom. Since the operation of splitting types is an inessential transformation of a problem by dividing a type, the arbitrator should not distinguish two problems before and after the splitting. IF implicitly requires players being treated fairly in a TU problem, which is probability-weighted symmetric. For the uniform marginal probabilities, IF requires the solution not to favor a certain type of a player. IF is weaker than symmetry across all types of the players. SCA and T.INV impose certain ways of comparison of utilities across types and players. Finally, F.CONT is for technical reasons.

1.4. Characterization

In this section, we provide a characterization of the minimal norm solution. The following theorem is the main result of the chapter.

[9]Let X, Y be two nonempty closed subsets in $\mathbb{R}^n$. The Hausdorff metric is given by

$$h(X, Y) = \max\{\sup_{x \in X} d_E(x, Y), \sup_{y \in Y} d_E(y, X)\}, \tag{1.3.7}$$

where d_E is the Euclidean metric on $\mathbb{R}^n$ and $d_E(x, Y) = \min_{y \in Y} d_E(x, y)$.

Theorem 1.1: *A solution f satisfies Axioms 1.1-1.8 if and only if $f = F$.*

It is easy to verify that F satisfies IIA, T.INV, F.CONT, and WPO. We use Lemma 1.2 to show that F satisfies the other axioms. The proofs of the lemmas are provided in Appendix 1.A.

Lemma 1.2: *F satisfies TU, IF, SCA, and ST.*

The proof that F is the unique solution that satisfies all axioms is divided into several steps. Denote the *normalized* p-TU problem $\Gamma^p = (p, X^p, 0)$ by $X^p = X_{w,\kappa}$ where $(w, \kappa) = (-|N|, \min_{(i,j)} -|N|/p_{ij})$. Denote the solution to the normalized p-TU problem by

$$e = f(\Gamma^p). \tag{1.4.1}$$

f being IF implies $e < 0$. Lemma 1.3 states that if f is IIA, WPO, and SCA, then for any linear problem $\Gamma = (p, h * X, 0)$ scaled from a p-TU problem $\tilde{\Gamma} = (p, X, 0)$ for some $h \in \mathbb{R}^n_{++}$,

$$f(\Gamma) = \phi(p^e, h * X, 0), \tag{1.4.2}$$

where

$$p^e = p * (-e)^{-1}. \tag{1.4.3}$$

Then by TU, ST, and Lemma 1.4, we have for the normalized p-TU problem, $e = (-1, ..., -1)$. Finally, by IIA and F.CONT, we establish $f = F$.

Lemma 1.3: *Suppose f satisfies IF, IIA, WPO, and SCA. For any p-TU problem $\Gamma = (p, X, 0)$ and $h \in \mathbb{R}^n_{++}$, $f(h * \Gamma) = \phi(p^e, h * X, 0)$.*

Lemma 1.4: *Suppose f satisfies IF, IIA, WPO, SCA, TU, and ST. Then for the normalized p-TU problem Γ^p, $f(\Gamma^p) = (-1, ..., -1)$.*

Proof of Theorem 1.1. Suppose f is a solution satisfying all the axioms. Fix $\Gamma = (p, X, r)$. By T.INV, we can translate the problem to $r = 0$. Denote $y = F(\Gamma)$.

Then X and the ball $B_p(0, \|y\|_p)$ has y as the unique point in common. Since both sets are convex and compact, by a hyperplane separation theorem, there exists a hyperplane

$$H_{\lambda,w} = \{x \in \mathbb{R}^n : \langle \lambda, x \rangle_p = w\} \tag{1.4.4}$$

that separates X and the ball $B_p(0, \|y\|_p)$, and supports the ball at y. Since $B_p(0, \|y\|_p)$ is smooth, $\lambda = -y$. Furthermore, $\phi(p, H_{\lambda,w}, 0) = y$.

Case 1. $y < 0$. Then $\lambda \in \mathbb{R}^n_{++}$ and $w < 0$. Now consider a linear problem $h * \tilde{\Gamma} = (p, h * \tilde{X}, 0)$, where $h = \lambda^{-1}$ and $\tilde{\Gamma} = (p, \tilde{X}, 0)$ is a p-TU problem with $\tilde{w} = w$ and $\tilde{\kappa} < 0$ small enough such that $X \subseteq h * \tilde{X}$. Because $h * \tilde{X} \subset \{x : \langle \lambda, x \rangle_p \leq w\}$, and $y \in (h * \tilde{X}) \cap H_{\lambda,w}$, it implies $\phi(p, h * \tilde{X}, 0) = \phi(p, H_{\lambda,w}, 0) = y$. By Lemma 1.4,

$e = (-1, ..., -1)$ and $p^e = p$. By Lemma 1.3, $f(h * \tilde{\Gamma}) = \phi(p^e, h * \tilde{X}, 0) = \phi(p, h * \tilde{X}, 0)$. Since f is IIA, $f(\Gamma) = f(h * \tilde{\Gamma}) = y = F(\Gamma)$.

Case 2. $y \leq 0$ and $y_{ij} = \lambda_{ij} = 0$ for some $(i, j) \in \mathring{T}$. Lemma 1.8 in Appendix 1.A shows that there exists a sequence $\Gamma_k = (p, X_k, 0)$, $k = 1, 2, ...$, such that $y_k = F(\Gamma_k) < 0$ for all k, $X_k \to X$ in the Hausdorff metric, and $y_k \to y$. Then apply the result in Case 1, $f(\Gamma_k) = y_k = F(\Gamma_k)$. By F.CONT, $X_k \to X$ implies that $f(\Gamma) = \lim_{k\to\infty} f(\Gamma_k) = \lim_{k\to\infty} F(\Gamma_k) = F(\Gamma)$. □

1.5. Discussion

1.5.1. Minimum Norm Duality

We now discuss a minimal norm duality between welfare weights and interim utility for the minimal norm solution. In a canonical two-person complete information bargaining problem (X, d) with X being a convex compact subset of $\mathbb{R}^2$ and $d \in X$, Harsanyi (1963) and Shapley (1969) characterize that a feasible allocation $x \in X$ is the Nash solution if and only if there exists a welfare weighting vector $\lambda \in \mathbb{R}^2_{++}$ such that x is both λ-utilitarian and λ-egalitarian, i.e.

$$\sum_i \lambda_i x_i = \max_{y \in X} \sum_i \lambda_i y_i \text{ and } \lambda_1(x_1 - d_1) = \lambda_2(x_2 - d_2). \tag{1.5.1}$$

For such a λ, the solution is desirable in terms of both efficiency and equity, and hence λ is a natural weighting vector. Myerson (1984) shows that the set of neutral bargaining solutions under incomplete information have a similar property. For our social choice problem with a reference point, a natural question is how to define λ-egalitarian allocations and whether the weighting vector in our solution has a similar characterization.

Definition 1.2: Let $\Gamma \in \Sigma$. (i) $x \in X$ is interim λ-utilitarian, if there exists $\lambda \in \mathbb{R}^n_{++}$ such that

$$\langle \lambda, x \rangle_p = \max_{y \in X} \langle \lambda, y \rangle_p. \tag{1.5.2}$$

(ii) $x \in X$ has interim λ-equal loss, if there exists $\lambda \in \mathbb{R}^n_{++}$ such that

$$\frac{\lambda_{ij}}{\lambda_{km}} = \frac{r_{ij} - x_{ij}}{r_{km} - x_{km}}, \tag{1.5.3}$$

for all $(i, j), (k, m) \in \mathring{T}$.

To see what a λ-equal loss allocation refers to, consider a feasible set $X \subset \mathbb{R}^2$ with a reference point $r \in \mathcal{R}(X)$. A λ-equal loss allocation requires that the players bear losses proportional to their social weights, i.e. $\lambda_1/\lambda_2 = (r_1 - x_1)/(r_2 - x_2)$. Then, the player with the higher social weight bears more losses.[10]

With these definitions, we have the following result that characterizes F by a natural welfare weighting vector.

PROPOSITION 1.1: *Let $\Gamma \in \Sigma$ with $F(\Gamma) < r$. Then, $y = F(\Gamma)$ if and only if there exists $\lambda \in \mathbb{R}^n_{++}$ such that y is interim λ-utilitarian and has interim λ-equal loss.*

Proof of Proposition 1.1. (Only If) Among the supporting hyperplanes of X at $y = F(\Gamma)$, there exists one hyperplane $H_{\lambda,w} = \{x : \langle \lambda, x \rangle_p = w\}$ with $\lambda = r - y$ being the normal vector. From $F(\Gamma) < r$, $\lambda \in \mathbb{R}^n_{++}$. Then y is interim λ-utilitarian and λ-equal loss.

(If) Suppose there is $y' \in X$ and $\lambda \in \mathbb{R}^n_{++}$ such that y' is λ-utilitarian and λ-equal loss. Then with $\lambda' = r - y' = k\lambda$ where $k = \|r - y'\|_p / \|\lambda\|_p$, y' is also λ'-utilitarian and λ'-equal loss. Since y' is λ'-utilitarian, $\langle r - y', x - y' \rangle_p \leq 0$, for all $x \in X$. By the projection theorem, $y' = y$. □

To interpret this result, note that the minimum p-norm from r to X, or

$$\min_{x \in X} \|r - x\|_p \tag{P}$$

is equal to the maximum of p-norms from r to hyperplanes separating r and X. Hence, the dual problem of (P) is given by

$$\max_{\|\lambda\|_p = 1} \left(\langle \lambda, r \rangle_p - \max_{x \in X} \langle \lambda, x \rangle_p \right). \tag{D}$$

By a no duality gap theorem (Luenberger, 1969), when (P) has a solution $x^* = F(\Gamma)$, then the optimal solution λ^* to (D) is aligned with $r - x^*$. Define the linear social welfare function by

$$SW(\lambda, x, p) = \sum_i \sum_j \lambda_{ij} p_{ij} x_{ij} \tag{1.5.4}$$

for some welfare weights $\lambda \in \mathbb{R}^n_{++}$. We have

$$\|r - x^*\|_p = \langle \lambda^*, r \rangle_p - \langle \lambda^*, x^* \rangle_p = SW(\lambda^*, r, p) - SW(\lambda^*, x^*, p). \tag{1.5.5}$$

The duality pair (x^*, λ^*) allows the following interpretation. First, λ^* is the only weighting vector such that x^* is both interim λ^*-utilitarian and λ^*-equal loss. With such a natural weighting vector λ^*, the value of the primal problem has a "transferable utility"

[10] This may counter the intuition that the player with a higher social weight bears less loss. In this case, two players' weighted losses are equalized, i.e. $\lambda_1(r_1 - x_1) = \lambda_2(r_2 - x_2)$. However, for the λ-egalitarian criterion in a bargaining problem, a player with a higher weight receives a lower utility gain and thus is disadvantaged in the solution. Since λ is endogenously determined, the λ-equal loss criterion is more consistent with the λ-egalitarian criterion.

interpretation: It is equal to the utility gap between the λ^*-weighted social welfare from the reference point and the λ^*-weighted social welfare from the optimum.

1.5.2. Independence of Axioms

We investigate the logical independence of the axioms in Theorem 1.1 by considering that one of the axioms is violated while some other axioms are satisfied. We provide some counterexamples for some, but not all of the axioms.

Example 1. Let q be a function that associates each system of marginals p with a system of marginals $q(p)$ such that $q(p)$ and p have the same type sets but $q(p) \neq p$. For each $\Gamma \in \Sigma$, define

$$F^q(\Gamma) = \phi(q(p), X, r). \tag{1.5.6}$$

For all q, F^q satisfies IIA, WPO, IF, SCA, T.INV, and F.CONT, but F^q violates TU. Depending on q, F^q may satisfy ST. For example, for every $\Gamma = (p, X, r)$ and $\Gamma' = (p', X', r')$ derived from Γ by splitting type s of player 1, $q(p')$ is derived from $q(p)$ by splitting type s of player 1. On the other hand, if $q(p)$ associates with uniform distributions for all problems, we have the minimal Euclidean distance solution. Then F^q violates TU and ST.

Example 2. For each $\Gamma = (p, X, r)$, define the nadir point $m(X)$ by $m_{ij}(X) = \min_{x \in X} x_{ij}$, for all $(i,j) \in \mathring{T}$. For each $\Gamma \in \Sigma$, define

$$NB(\Gamma) = \arg\max_{x \in X} \sum_i \sum_j p_{ij}(x_{ij} - m_{ij}(X)). \tag{1.5.7}$$

Then, NB satisfies WPO, IF, T.INV, and F.CONT. Moreover, NB satisfies TU and ST. However, NB violates SCA. Voorneveld et al. (2011) observe that the scaling axioms are special to quadratic norms.

1.5.3. Other Axioms

It is worth noting that F also satisfies the following two properties. The axiom of prior continuity says that the solution is robust to small changes in priors. Here, N, T and hence n are fixed.

Axiom 1.9: Prior Continuity (P.CON). Let $\Gamma = (p, X, r)$ and $\Gamma_k = (p_k, X, r)$, $k = 1, 2...$, in Σ and $p_k \to p$. Then $f(\Gamma_k) \to f(\Gamma)$.

Finally, we introduce an axiom of reference point convexity. This axiom requires that for any problem, when a mixture of the original reference point and the solution point is used to generate a new reference point, the solutions of two problems are the same.

Axiom 1.10: Reference Point Convexity (R.CONV). Let $\Gamma = (p, X, r)$ in Σ, and for $\alpha \in (0, 1]$, let $\Gamma_\alpha = (p, X, \alpha r + (1-\alpha) f(\Gamma))$. Then $f(\Gamma) = f(\Gamma_\alpha)$.

The axiom is related to the literature of repeated games with "satisficing" players (Bendor, Mookherjee and Ray, 1995)[11], in which each player's aspiration level is endogenous and is consistent with his long run average payoff. Each player may adjust his aspiration level in period 1 based on his aspiration level in period 0 and the personal payoff experience in period 0.

1.6. Applications and Examples

1.6.1. Social Choice Problems

We now discuss the generation of feasible sets and interim reference points in economic contexts. A reference point can be generated separately from a social choice problem. We provide three scenarios in which an interim reference point naturally arises. (i) Contract obligations, i.e. creditors have debt claims. (ii) Subjective entitlements. When there is flexibility in a contract, each contracting party may interpret the same term differently in one's own favor and believe to be entitled. (iii) Repeated interaction and dynamic adjustment of aspirations. In a long-run relationship, players may adjust their aspirations based on past experiences.

Bankruptcy. Suppose two creditors divide the estate. There are two options, either creditor 1 or 2 obtains the estate. The feasible set is generated by the set of incentive compatible allocation mechanisms. Each creditor's interim reference utilities can be generated by his debt claim. For example, if creditor i has a claim equal to r_i, then the interim reference utility of creditor i with any type is given by r_i.

Contracts. Suppose one seller and one buyer trade one object. In period 1, the players know their private values of the object. Suppose the players can sign a contract at period 0. A flexible contract specifies a range of all potential transaction prices. The players then trade at one of these prices or there is no trade.[12] The feasible set is generated by the set of incentive compatible trading mechanisms.[13] Given the early contract prices, the seller is entitled with the highest price while the buyer is entitled with the lowest price. Then these subjective entitlements generate interim reference utilities. For example, the interim reference utility of each type of the seller is given by the highest price.[14]

Compromise. Suppose two players vote among three alternatives, L, M and R. Player 1's preference is given by $L \succ M \succ R$ and player 2's preference is given by $R \succ M \succ L$.

[11] An action of a player is deemed "satisfactory" if its current payoff exceeds some aspiration level held by the player.

[12] Here, the decision set is endogenously determined by a contract.

[13] This model, adapted to our social choice environment, is different from the model of Hart and Moore (2008). In their model, there is no incomplete information between the players.

[14] Hart and Moore (2008) introduces the notion of contracts as reference points. In their model, each player can aggrieve the other player if the final price diverges from his entitlement.

Each player receives utilities of 1 and 0 from his best and worst alternative, respectively, and privately observes his utility between 0 and 1 from alternative M. Players need to make decisions in period 0 and 1, and the utility shocks are independently distributed across players and across periods. In period 0, suppose that an equal randomization over L and R was chosen. In period 1, suppose L is eliminated and the past allocation is now unattainable. The players then need to select a decision between M and R given the past experience.

Consuming Public Goods. Suppose two players decide to consume some public good $l \in \{a, b\}$. The budget constraint in period 0 is given by $p^a q^a + p^b q^b \leq m$, where p^l, q^l are the unit price and the quantity of good l, and m is the budget. Suppose in period 0, the players experienced a consumption bundle (q_0^a, q_0^b). In period 1, the prices may change and the past bundle is not affordable. If player i with type j receives some private observed utility u_j^l from consuming one unit of good l, the interim reference utility of player i with type j is given by $u_j^a q_0^a + u_j^b q_0^b$.

The above examples require that the reference point has been formed at some earlier period before the decision making. On the other hand, the reference point can be generated endogenously by a social choice problem. If X is the set of incentive compatible utility allocations of some social choice problem, we define the utopia point of X, $r^*(X)$, by

$$r_{ij}^*(X) = \max_{x \in X} x_{ij} \tag{1.6.1}$$

for all $(i, j) \in \mathring{T}$. For $r^*(X) \notin X$, we have $r^*(X) \in \mathcal{R}(X)$.

1.6.2. Illustration of the Minimal Norm Solution

We provide two examples to illustrate the minimal norm solution.

Example 1.1: Let $N = \{1, 2\}$, $D = \{d_1, d_2\}$, $T_i = \{0, 1\}$, $i = 1, 2$. $(u_i)_{i \in N}$ is given by

$u_i(d, t_i)$	10	11	20	21
d_1	0	1	0	0
d_2	0	0	0	1

Suppose $\pi(t) = p_{1,t_1} \times p_{2,t_2}$ for all $t \in T$, and $p_{11} > p_{21}$.

The example fits a bankruptcy problem: Either creditor 1 or 2 obtains the asset and in any state, a creditor weakly prefers to obtain the asset. Now suppose each creditor has an ex ante claim equal to 1. The interim reference point generated by the ex ante claims is given by $r = (1, 1, 1, 1)$. Notice that it is weakly dominant for a player to report truthfully.[15] The set of IC mechanisms coincides with all constant mechanisms, $X = \{(-(1 - \delta_1), \delta_1, -\delta_1, 1 - \delta_1) : 0 \leq \delta_1 \leq 1\}$.

[15] While we assume that type 0 of a player reports the true type in case of indifference, we may instead assume that it receives a utility of $\epsilon > 0$ small from d_i and the incentive becomes strict.

First, the ex ante utilitarian rule is the solution to $\max_{x\in X}\sum_i\sum_j p_{ij}x_{ij}$, where ties are broken randomly and fairly. The ex ante utilitarian solution is given by $\delta_1 = 1$. The asset is allocated to creditor 1 for sure and creditor 2 obtains the asset with probability 0. Now $F(p, X, r) = (-(1-\delta_1^*), \delta_1^*, -\delta_1^*, 1-\delta_1^*)$, where

$$\delta_1^* = \frac{p_{11}}{p_{11}+p_{21}}.$$

Thus, $\delta_1^* > 1/2$. Creditor 2, whose value is drawn from a less favorable distribution, receives the asset with a lower but strictly positive probability.

Example 1.2: Let $N = \{1, 2\}$, $D = \{d_0, d_1\}$, $T_i = \{0, 1\}$, $i = 1, 2$. $(u_i)_{i\in N}$ is given by

$u_i(d, t_i)$	10	11	20	21
d_0	1/2	1/2	1/2	1/2
d_1	0	1	0	1

Assume $\pi(t) = p_{1,t_1}\times p_{2,t_2}$ for all $t\in T$ and for $i = 1, 2$, $(p_{i0}, p_{i1}) = (\underline{p}, \bar{p})$ with $\underline{p} < \bar{p}$. We assume the reference point is endogenously determined by the utopia point of the interim utility set.

Since the minimal norm program is symmetric with respect to players, the solution is symmetric.[16] We consider symmetric mechanisms. Denote $\mu(d_1|t) = \mu_1(t)$ for $t\in T$. We first use an intuitive technique to find a solution. For $t = (0, 0)$ and $t = (1, 1)$, choosing d_0 or d_1 for sure would be social welfare optimal. The question then would be the probabilities that d_0 is chosen if $t = (0, 1)$ and $t = (1, 0)$. Suppose $\mu_1(0, 1) = \mu_1(1, 0) = \alpha\in[0, 1]$. The symmetric interim utility vector is given by

$$(\underline{x}, \overline{x}) = (\underline{p}\frac{1}{2} + \bar{p}(1-\alpha)\frac{1}{2}, \underline{p}[\alpha + (1-\alpha)\frac{1}{2}] + \bar{p}). \tag{1.6.2}$$

It is easy to see that no matter what the other plays, it is a dominant strategy to report truthfully. One candidate solution mechanism μ^* is given by

$$(\mu_1^*(0,0), \mu_1^*(0,1), \mu_1^*(1,0), \mu_1^*(1,1)) = (0, \underline{p}, \underline{p}, 1). \tag{1.6.3}$$

To verify it is indeed the solution for the entire problem, let $Q = (\underline{Q}_i, \overline{Q}_i)_{i\in\{1,2\}}$ be the interim expected probabilities that d_1 is chosen. Denote $(\underline{Q}_i, \overline{Q}_i) = (\underline{Q}, \overline{Q})$, $i = 1, 2$. The set of IC expeceted probabilities is given by

$$\{(\underline{Q}, \overline{Q}) : 0 \le \underline{Q} \le \overline{Q} \le 1\}. \tag{1.6.4}$$

[16] Suppose X is symmetric. If there is a asymmetric solution $x'\in X$, then there is another solution $x''\in X$ by interchanging the players' labels. Since X is convex, $\frac{1}{2}x' + \frac{1}{2}x''\in X$ is also a solution.

The set of ex post feasible interim allocation probabilities (i.e. there exists a mechanism μ such that $(\underline{Q},\overline{Q})$ are the marginals) is given by[17]

$$\{(\underline{Q},\overline{Q}) : 0 \leq \underline{Q},\overline{Q} \leq 1, \quad \bar{p}\overline{Q} - \underline{p}\underline{Q} \leq \bar{p}^2\}. \tag{1.6.5}$$

Denote $\mathcal{Q}$ the intersection of these inequalities (Figure 1.1a). Denote $(x_{i0}, x_{i1}) = (\underline{x}, \overline{x})$, $i = 1, 2$, then $(\underline{x}, \overline{x}) = (\frac{1}{2}(1 - \underline{Q}), \frac{1}{2}(1 + \overline{Q}))$. The set of symmetric feasible allocations is given by

$$X^s = \{(\underline{x}, \overline{x}) \in \mathbb{R}^2 : \ \underline{x} + \overline{x} \geq 1, \ 0 \leq \underline{x} \leq 1/2 \leq \overline{x} \leq 1, \ 2\underline{p}\underline{x} + 2\bar{p}\overline{x} \leq 1 + \bar{p}^2\} \tag{1.6.6}$$

(Figure 1.1b). Denote IC, FE1, and FE2 for the inequalities in (1.6.6).

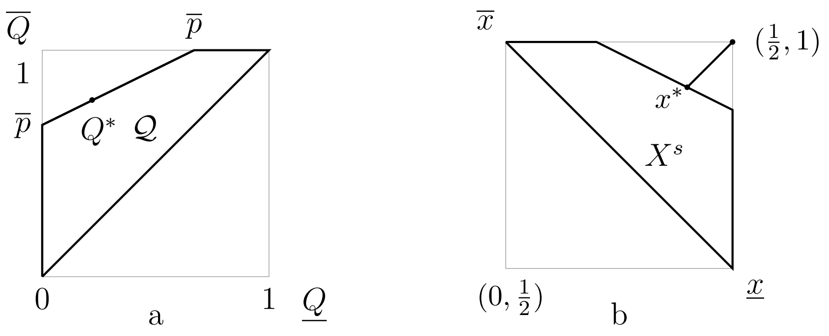


Figure 1.1

Let $r^s = (1/2, 1)$, then the minimal norm solution for (X^s, r^s) is given by $x^* = (\underline{x}^*, \overline{x}^*) = (\frac{1}{2} - \frac{1}{2}\underline{p}\bar{p}, 1 - \frac{1}{2}\underline{p}\bar{p})$. The ex post feasibility $\bar{p}\overline{Q} - \underline{p}\underline{Q} \leq \bar{p}^2$ (FE2) is binding. Finally, $Q^* = (\underline{Q}^*, \overline{Q}^*) = (\underline{p}\bar{p}, 1 - \underline{p}\bar{p})$. It is easy to see that the minimal norm solution is implemented by the stochastic mechanism given by (1.6.3).

1.7. Comparison to Literature

1.7.1. Complete Information

Yu (1973) considers a class of social choice problems with the endogenous reference points equal to the utopia points. To formalize such a problem, let N be the set of players, let Σ_0^N be the set of all nonempty convex compact subsets of $\mathbb{R}^N$ and let

$$\Sigma^N = \{X \subseteq \mathbb{R}^N : X = comp(C), \ \text{for some } \ C \in \Sigma_0^N\}, \tag{1.7.1}$$

where $comp(C) = C - \mathbb{R}_+^N$. A solution $f : \Sigma^N \to \mathbb{R}^N$ is such that for every $X \in \Sigma^N$, $f(X) \in X$. The utopia point $r^*(X)$ is defined by $r_i^*(X) = \max\{x_i : x \in X\}$ for all $i \in N$.

[17]The feasibility condition is given by Proposition 2.2 in Chapter 2.

The Yu solution is given by

$$Y(X) = \arg\min_{x \in X} \sum_{i \in N} (r_i^*(X) - x_i)^2. \tag{1.7.2}$$

Roth (1977) and Conley et al. (2014) show that there is no solution satisfying Pareto optimality, symmetry, together with IIA other than the utopia point (u-IIA), translation invariance and scale covariance. In particular, Roth (1977) shows that the Yu solution satisfies u-IIA, translation invariance but violates scale covariance.[18]

Conley et al. (2014) introduces the axiom of proportional losses for a class of transferable utility problems and the axiom of individual fairness.[19] They characterize the Yu solution with Pareto optimality, symmetry, u-IIA, translation invariance, the proportional losses, individual fairness and feasible set continuity.[20] The axioms of SCA, IF, and F.CONT are modified based on their axioms.

Voorneveld et al. (2011) provides a characterization of the Yu solution, by a consistency axiom, first used by Lensberg (1988). Denote $\mathcal{N}$ all nonempty finite subsets of natural numbers. By varying the number of players, the domain is now given by $\bar{\Sigma} = \cup_{N \subset \mathcal{N}} \Sigma^N$.

u-consistency: Let $X \in \Sigma^N$ and $I \subseteq N$, define $X_I^f \in \Sigma^I$ by

$$X_I^f := \{x \in \mathbb{R}^I : (x, f_{N \setminus I}(X)) \in X\}. \tag{1.7.4}$$

If $r_i^*(X_I^f) = r_i^*(X)$ for each $i \in I$, then $f_i(X_I^f) = f_i(X)$ for each $i \in I$.

The axiom considers a problem X with N a set of players and I a subset of N. Then, give players in $N \setminus I$ their utilities according to f in X and consider a reduced problem X_I^f for the remaining members in I. The solution f is *u*-consistent if the prescribed allocation to each member of I in the reduced problem X_I^f is the same as in the original game X. Our characterization does not use the *u*-consistency axiom and instead we use IF and F.CONT.

Rubinstein and Zhou (1999) considers a choice set $X \subset \mathbb{R}^N$ with an arbitrary refer-

[18] IIA other than the utopia point: If X and X' satisfy $r^*(X) = r^*(X')$ and $X \subseteq X'$, and if $f(X') \in X$, then $f(X) = f(X')$.

Translation Invariance: For any $z \in \mathbb{R}^N$, if $X' = X + z$, then $f(X') = f(X) + z$.

Scale Covariance: For any $h \in \mathbb{R}^N_{++}$, if $X' = h * X$, then $f(X') = h * f(X)$.

[19] The proportional losses: For any $(\lambda, w) \in \mathbb{R}^N_{++} \times \mathbb{R}$, let $X = \{x : \lambda \cdot x = w\} \cap [r^*(X) - \mathbb{R}^N_{++}]$. For any $h \in \mathbb{R}^N_{++}$, if $X' = h * X$, then

$$h_i[r_j^*(X) - f_j(X)][r_i^*(X') - f_i(X')] = h_j[r_i^*(X) - f_i(X)][r_j^*(X') - f_j(X')], \tag{1.7.3}$$

for all $i, j \in N$. Here $\lambda \cdot x$ denotes the standard inner product.

[20] Symmetry: For any permutation m of N and $x \in \mathbb{R}^N$, write $m(x)_i := x_{m(i)}$. If for all m, $m(X) = X$, then $f_i(X) = f_j(X)$ for all $i, j \in N$.

Feasible set continuity: f is continuous with respect to the Hausdorff metric on Σ^N.

ence point $r \in \mathbb{R}^N$. They obtain a characterization of the minimal Euclidean distance solution by a strong symmetry axiom and an IIA axiom. The strong symmetry axiom is strictly stronger than Nash's symmetry axiom, which only applies to the problems symmetric to the main diagonal.

Strong Symmetry. If X is symmetric with respect to some line $l(r, \lambda)$, then $f(X, r) \in l(r, \lambda)$.[21]

Strong symmetry requires that if a feasible set is symmetric with respect a line through the reference point, then the solution must lie on the line. A justification would be that the players' compromises over utility losses from the reference point force only the "centric" outcomes to be chosen.

1.7.2. Incomplete Information

There is a relatively small literature on two-person bargaining problems with incomplete information. Harsanyi and Selten (1972) (HS hereafter) first characterizes the generalized Nash product solution by a set of axioms. Myerson (1984) characterizes the incentive feasible neutral solution by the axioms of probability invariance, random dictatorship, and extension. Weidner (1992) characterizes the incentive feasible generalized Nash product solution by the axioms of HS (1972) and Myerson (1984).

1.7.2.1. The Harsanyi/Selten Solution

Harsanyi and Selten (1972) considers bargaining problems as a class of bases $(\pi, X, 0)$, where $\pi \in \Delta(T)$ is the prior, the bargaining set $X \subset \mathbb{R}^n$ is the convex hull of interim utility allocations from all strict equilibrium points of an extensive form game[22], and $0 \in \mathbb{R}^n$ is the disagreement point. Since none of their axioms involves changes in the disagreement point, we follow HS to abbreviate a problem $(\pi, X, 0)$ by (π, X). The HS solution L^* is defined by for each problem (π, X),

$$L^*(\pi, X) = \arg\max_{x \in X} \prod_{i \in N} \prod_{j \in T_i} x_{ij}^{p_{ij}}. \tag{1.7.5}$$

To compare their axioms with Axioms 1.1-1.8, we introduce their eight axioms (with a slightly different order).

Irrelevant Alternatives (IIA′). If $X \subseteq X'$ and $f(\pi, X') \in X$, then $f(\pi, X') = f(\pi, X)$.

[21] We provide a generalized strong symmetry of Rubinstein and Zhou (1999). We say X is p-symmetric with respect to $l(r, \lambda)$ if for any $z \in X$, there is $z' \in X$ such that $\arg\min_{x \in l(r,\lambda)} \|z - x\|_p = \frac{1}{2}(z + z')$.

p-Symmetry. If X is p-symmetric with respect to some line $l(r, \lambda)$, then $f(X, r) \in l(r, \lambda)$.
When there is no incomplete information or $p = (1, ..., 1)$, it reduces to strong symmetry.

[22] An equilibrium point s^* is strict if when a player t_i deviates from equilibrium strategy $s_i^*(t_i)$ to any other best reply of s_{-i}^*, then all players' interim utilities are not affected.

Pareto Optimality (PO). $f(\pi, X) \in PO(X)$.

The first two axioms are similar to Axioms 1.1-1.2, except that PO is replaced by WPO. Since the extensive form in HS is fixed, any change in the bargaining set results from varying utility functions. While it is unclear whether every extension can be generated in this way, they assume that an arbitrary extension always exists.

Player Symmetry (PS). If (π, X') is derived from (π, X) by interchanging two players, then $f(\pi, X')$ is derived from $f(\pi, X)$ by interchanging these two players.

Type Symmetry (TS). If (π, X') is derived from (π, X) by interchanging two types of a player, then $f(\pi, X')$ is derived from $f(\pi, X)$ by interchanging these types of the player.

PS and TS are the main difference from the axiom of symmetry for TU problems. It is partly due to the fact that HS define a hyperplane problem (π, X_0) by

$$X_0 = \{x \in \mathbb{R}^n_+ : 1 \cdot x \leq w\}, \tag{1.7.6}$$

while we define a hyperplane problem (π, X_1) by

$$X_1 = \{x \in \mathbb{R}^n_+ : \langle 1, x\rangle_p \leq w\} \tag{1.7.7}$$

for $w = |N|$. However, these two classes of hyperplane problems are closely related. Applying the HS solution to X_0 and X_1 gives $L^*(\pi, X_0) = p$ and $L^*(\pi, X_1) = (1, ..., 1)$. It follows that the HS solution satisfies an axiom of symmetry for TU problems adapted to bargaining problems.[23]

Splitting Types (ST$'$). If (π', X') is derived from (π, X) by splitting a type of a player with probability $\alpha \in (0, 1)$, then $f(\pi', X')$ is derived from $f(\pi, X)$ by splitting this type.

Profitability (PRO). $f(\pi, X) > 0$.

Linear Invariance (L.INV). For any $h \in \mathbb{R}^n_{++}$, then $f(\pi, h * X) = h * f(\pi, X)$.

Mixing Probabilities (MIX). If (π, X) and (π', X) with $\pi, \pi' \in \Delta(T)$ have the same solutions $x \in X$, and if $\pi'' = \alpha\pi + (1 - \alpha)\pi'$ for some $\alpha \in [0, 1]$, then $f(\pi'', X) = x$.

ST$'$ is defined by the operation on the priors rather than on the marginal probabilities, thus ST is much weaker than ST$'$. PRO is a requirement of strong individual rationality. IF is a counterpart to PRO in our problem, except that the disagreement point is replaced by the reference point. L.INV is another axiom different from ours.

[23] It is unclear unclear whether we can use this axiom to obtain an alternative characterization of the HS solution. Instead, we may use PS and TS to obtain an alternative characterization of the minimal norm solution.

HS require the solution being invariant to order preserving linear transformations in the interim utility space. The minimal norm solution violates this axiom.[24] Finally, while we do not use MIX for characterization, it is clear that the minimal norm solution satisfies this axiom.

1.7.2.2. Myerson's Neutral Bargaining Solution

Myerson (1984) defines a bargaining problem $\Gamma = (S, d^*)$ by a social choice problem $S = (\pi, D, (u_i)_{i \in N})$ and a disagreement option $d^* \in D$. Myerson proposes a set-valued concept that generalizes Nash's solution, called the neutral bargaining solutions. The neutral solutions are characterized by a dual system of equations and they have no explicit formula. Myerson shows that the neutral solutions satisfy a probability invariance axiom and an extension axiom.

The probability invariance axiom states that only the interim expected utility is decision-theoretically significant to the problem and that probabilities cannot be meaningfully defined separately from state-dependent utility functions.

Probability Invariance (P.INV). Let $S = (\pi, D, (u_i)_{i \in N})$ and $\tilde{S} = (\tilde{\pi}, D, (\tilde{u}_i)_{i \in N})$, $\pi, \tilde{\pi} \in \Delta(T)$. If

$$\tilde{\pi}_i(t_{-i}|t_i)\tilde{u}_i(d,t) = \pi_i(t_{-i}|t_i)u_i(d,t) \tag{1.7.8}$$

for all $d \in D$, $t \in T$, and $i \in N$, then $f(\tilde{S}, d^*) = f(S, d^*)$.

Since interim utilities always have probabilities multiplied by utilities, two social choice problems in the axiom have the same set of mechanisms and each mechanism generates the same incentive feasible allocation. Note that both the HS solution and the ex ante utilitarian solution[25] violate this axiom. By replacing the disagreement point by the reference point, the minimal norm solution also violates the axiom.

For the extension axiom, Myerson (1984) requires that any extension in the bargaining set must result from adding decisions.

DEFINITION 1.3: Let $S = (\pi, D, (u_i)_{i \in N})$ and $\tilde{S} = (\pi, \tilde{D}, (\tilde{u}_i)_{i \in N})$. $\tilde{S}$ is an extension of S, if $\tilde{D} \supseteq D$ and $\tilde{u}_i|_{D \times T} = u_i$ for all $i \in N$.

Extension. Let $\tilde{S} = (\pi, \tilde{D}, (\tilde{u}_i)_{i \in N})$ be an extension of $S = (\pi, D, (u_i)_{i \in N})$. If $f(\tilde{S}, d^*) \in \mathcal{U}(S)$, then $f(\tilde{S}, d^*) = f(S, d^*)$.[26]

As noted by Myerson (1984), there are social choice problems that give a larger set of feasible allocations than the original one, but that cannot be constructed from it by

[24] It is unclear whether any solution among the Myerson's neutral solutions satisfies the axiom.

[25] The ex ante utilitarian solution is the solution to $\max_{x \in \mathcal{U}(S)} \sum_i \sum_j p_{ij} x_{ij}$.

[26] Myerson's extension axiom also allows a sequential approximation, which makes it not entirely comparable to the one here.

adding new decisions. On the other hand, Lemma 1.1 shows that there are polytopes that cannot be generated by the incentive compatible allocations of any social choice problem. Such sets cannot be extensions of any social choice problem trivially. The following result, which is implied by either Myerson's comment (1984, p.468) or Lemma 1.1, indicates that compared to our IIA, the definition of Myerson's extension is very strong: The utility set generated by an extension of a social choice problem is with restrictions.

LEMMA 1.5: *There exist $\pi \in \Pi$, $S \in \mathcal{S}(\pi)$ and a polytope $X \subset \mathbb{R}^n$ such that $\mathcal{U}(S) \subsetneq X$ but X cannot be generated by any extension of S.*

Proof. See Appendix 1.A. □

1.8. Conclusion

In this chapter, we characterize the minimal norm solution by a set of axioms. We provide some examples of social choice problems with reference points to illustrate this solution. This solution can be further used to study bankruptcy problems, early contracting problems, or collective repeated consumption choices with incomplete information, where the reference point is either generated by contract obligations, or entitlements, or repeated interaction and choice outcomes. We also find that there are many avenues for future research:

1. The domain of the feasible sets. Myerson (1984) and Weidner (1992) define the domain of the feasible sets being generated by all social choice problems. If there exists some social choice problem $(\pi, D, (u_i)_{i\in N})$ that generates the feasible set X of a p-TU problem,[27] then we can obtain a characterization on this coarser domain.[28]

2. The probability invariance axiom. The TU axiom is inconsistent with P.INV because for two problems with different systems of marginals but the same feasible sets, TU requires the problems having different solutions while P.INV requires them having the same solutions. On the other hand, the minimal standard-norm solution does satisfy P.INV.

3. Weak/strong Pareto optimality. For generic utility functions, it can be seen that the class of choice problems containing nonempty WPO\PO has Lebesgue measure zero under complete information but has measure strictly positive under incomplete information. For a complete information problem, by slightly perturbing the players' utility functions, WPO\PO vanishes. However, with incomplete information, perturbing

[27] Some examples suggest that there may exist no social choice problem that generates a p-TU feasible set.

[28] In this case, IIA is replaced by a modified extension axiom, in which an extension can be obtained by varying $(D, (u_i)_{i\in N})$ arbitrarily. This axiom is stronger than IIA but weaker than Myerson's extension axiom.

the players' utility functions may not eliminate WPO\PO. For example, consider a one-person problem with $T = \{t, t'\}$ and $D = \{d_0, d_1\}$. The utility function at d_0 and d_1 are $(2, -1)$ and $(0, 1)$. Then, $\mathcal{U}(S) = \text{conv}\{(2, 1), (0, 1), (2, -1)\}$. A two-person example is Example 2 in Section 1.5. Thus, if $|N| \leq 2$, $|D| = 2$, and if all preference orderings over D are possible, WPO\PO exists generically. It is unclear whether this result generalizes.

4. The extension axiom. Myerson (1984) constructs extensions by adding decision options into the original decision set. In general, it is difficult to construct a required extension with the desirable geometric properties. Characterization of the minimal norm solution with the extension axiom is left for future work.

Appendix 1.A Proofs

Proof of Lemma 1.1. First note that for any $\pi \in \Pi$, if $X = \mathcal{U}(S)$ or $X = \mathcal{U}_c(S)$ for some $S \in \mathcal{S}(\pi)$, then X is a polytope. Since D is finite, $\mathcal{U}(S)$ is the intersection of finitely many linear inequalities, i.e. the incentive and feasibility conditions. The latter conditions imply that $\mathcal{U}(S)$ is bounded. $\mathcal{U}_c(S)$ is the convex hull of interim utility vectors at each $d \in D$. Hence in either case, $X \in \mathcal{X}_0$.

(i) Let $\pi \in \Pi$. For any polytope $X \subset \mathbb{R}^n$, the set of its extreme points, G, is finite. For every $g \in G$, define a decision d_g such that the interim utility vector at d_g is g, i.e. the utility function is defined by $\hat{u}_i(d_g, t) = g_i(t_i)$ for all $t \in T$ and $i \in N$. Then $X = \mathcal{U}_c(S)$ for $S = (\pi, D, (u_i)_{i\in N})$ and $X \subseteq \mathcal{X}_c(\pi)$. Hence, $\mathcal{X}_c = \mathcal{X}_0$.

(ii) Consider the simplest class of one-person social choice problems with $T = \{a, b\}$, $\pi = (\pi_a, 1 - \pi_a)$, $\pi_a \in (0, 1)$. The interim utility space is $\mathbb{R}^2$. Let X be the line segment between $(0, 1)$ and $(1, 0)$. We claim that there exists no social choice problem $S = (\pi, D, u)$ such that $\mathcal{U}(S) = X$, by varying D and u. Denote $u_d = (u(d, a), u(d, b))$ for $d \in D$.

First notice that $|D| \geq 2$. For every $d \in D$, it follows that $u_d \in X$, otherwise selecting d constantly yields an interim utility outside X. Moreover, because $\min_{d\in D} u(d, t) \leq x(t) \leq \max_{d\in D} u(d, t)$ for all $t \in T$, i.e. every interim utility is bounded by the bounds of the utility functions, it is necessary that the endpoints $(0, 1)$ and $(1, 0)$ correspond to the utility functions at some $d_0, d_1 \in D$.

Now define $S_0 = (\pi, D, u)$ with $D = \{d_0, d_1\}$ and u by $u_{d_0} = (0, 1)$ and $u_{d_1} = (1, 0)$. A simple calculation shows that $\mathcal{U}(S_0) = \text{conv}\{(0, 1), (1, 0), (1, 1)\}$. Hence, $\mathcal{U}(S_0) \neq X$.

Finally, every $\tilde{S}$ derived from S_0 by adding decisions to $\{d_0, d_1\}$ and defining utility functions at such new decisions, $\mathcal{U}(\tilde{S})$ must contain $\mathcal{U}(S_0)$ and $\mathcal{U}(\tilde{S}) \neq X$. Hence, there exists no social choice problem S such that $\mathcal{U}(S) = X$ and therefore $\mathcal{X} \subsetneq \mathcal{X}_0$.[29] $\square$

Proof of Lemma 1.2. Let $\lambda \in \mathbb{R}^n_{++}$. Consider the linear problem $h * \Gamma = (p, h * X, 0)$

[29] For this one-person case, a similar reasoning shows that for any polytope $X \subset \mathbb{R}^2$ with $(0, 1)$ and $(1, 0)$ being its extreme points and the line segment being part of its northeast boundary, there exists no social choice problem S such that $\mathcal{U}(S) = X$. For example, $X = \text{conv}\{(0, 1), (1, 0), (-1, 0)\}$ or $X = \text{conv}\{(0, 1), (1, 0), (0, 0), (-1, -1)\}$.

Moreover, it can be shown that it is impossible to construct any line segment except that the new utility functions satisfy $u(d_0, t) > u(d_1, t)$ for all $t \in T$, or $u(d_0, t) < u(d_1, t)$ for all $t \in T$, i.e. $(u(d_1, a), u(d_1, b)) = (1, 2)$.

obtained from some p-TU problem $\Gamma = (p, X, 0)$ with $\kappa \leq \min_{(i,j)} w/p_{ij} < w < 0$, where $h = \lambda^{-1}$. The relaxed Lagrangian (without any $x_{ij} \geq \kappa$) for the minimal norm problem with multiplier $\eta \leq 0$ is given by

$$\min_x \sum_i \sum_j p_{ij} x_{ij}^2 + \eta(w - \sum_i \sum_j \lambda_{ij} p_{ij} x_{ij}).$$

FOCs give necessary conditions

$$2p_{ij}\tilde{x}_{ij} - \eta\lambda_{ij}p_{ij} = 0, \text{ for all } (i,j) \in \mathring{T}.$$

If $\eta = 0$, then $\tilde{x}_{ij} = 0$ for all $(i,j) \in \Pi$, and $\tilde{x} \notin X$. Thus $\eta < 0$ and $\tilde{x}_{ij} = \eta\lambda_{ij}/2$, for all $(i,j) \in \mathring{T}$.

(i) TU and IF. Set $\lambda = (1, ..., 1)$, $\tilde{x}_{ij} = \eta/2$, for all $(i,j) \in \mathring{T}$. All constraints $\tilde{x}_{ij} \geq \kappa$ are not binding and $F(\Gamma) = \tilde{x}$. Thus, $\sum_j p_{ij}\tilde{x}_{ij} = \eta/2$ for all $i \in N$, and F is TU. $\eta < 0$ implies $F(\Gamma) < 0$. F is IF.

(ii) SCA. For each $\lambda \in \mathbb{R}^n_{++}$, we only need to show that for the relaxed solution $\tilde{x}$, all additional constraints $\lambda_{ij}\tilde{x}_{ij} \geq \kappa$ are not binding and thus $F(h * \Gamma) = \tilde{x}$.

Since $\tilde{x}_{ij} = \eta\lambda_{ij}/2$, for all $(i,j) \in \mathring{T}$, and $\langle \lambda, \tilde{x} \rangle_p = w$, we have $\eta = \frac{2w}{\|\lambda\|_p}$ and $\tilde{x}_{ij} = \frac{\lambda_{ij} w}{\|\lambda\|_p}$ for all $(i,j) \in \mathring{T}$. Hence

$$\lambda_{ij}\tilde{x}_{ij} = \frac{(\lambda_{ij})^2 w}{\|\lambda\|_p} > \frac{w}{p_{ij}} \geq \min_{(k,m)} \frac{w}{p_{km}} = \kappa, \tag{1.A.1}$$

for all $(i,j) \in \mathring{T}$. F is SCA.

(iii) ST. Let $\Gamma = (p, X, r)$ and $\Gamma' = (p', X', r')$. Suppose Γ' is obtained from Γ by splitting a type $s \in T_1$ with $\alpha \in (0,1)$. By definition, $F(\Gamma')$ is the solution to

$$\min_{x' \in X'} \; \alpha p_{1s}(r_{1s} - x'_{1a})^2 + (1-\alpha)p_{1s}(r_{1s} - x'_{1b})^2 + \sum_{j \in T_1 \setminus \{s\}} p_{1j}(r_{1j} - x'_{1j})^2 + \sum_{i \neq 1} \sum_{j \in T_i} p_{ij}(r_{ij} - x'_{ij})^2,$$

where for every $x' \in X'$, there exists $x \in X$ such that $x'_{ij} = x_{ij}$ for all $j \in T_i$, all $i \neq 1$, and $x'_{1j} = x_{1j}$ for all $j \in T'_1 \backslash \{a, b\}$, and $x'_{1a} = x'_{1b} = x_{1s}$. Hence, $F_{ij}(\Gamma') = F_{ij}(\Gamma)$ for all $j \in T_i$, all $i \neq 1$, and $F_{1j}(\Gamma') = F_{1j}(\Gamma)$ for all $j \in T'_1 \setminus \{a, b\}$, and $F_{1a}(\Gamma') = F_{1b}(\Gamma') = F_{1s}(\Gamma)$. □

We use the following lemma to obtain Lemma 1.3.

LEMMA 1.6: *Let $\Gamma = (p, X, 0)$ be a p-TU problem. If f satisfies IIA, SCA, WPO, and IF, then $f(\Gamma) = \alpha f(\Gamma^p)$ for some $\alpha > 0$.*

Proof of Lemma 1.6. Notice that by SCA, every p-TU problem Γ can be obtained from some p-TU problem $\Gamma_\kappa = (p, X_{w,\kappa}, 0)$ with $w = -|N|$ and some $\kappa \leq \min_{(i,j)} -|N|/p_{ij}$, by scaling with $h = (\beta, ..., \beta) \in \mathbb{R}^n_{++}$ for some $\beta > 0$.

We only need to show that for any $\kappa < \min_{(i,j)} -|N|/p_{ij}$, the problems Γ^p and Γ_κ have the same solution. First note that $X^p \subset X_{-|N|,\kappa}$. Notice that $WPO(X_{-|N|,\kappa}) = PO(X_{-|N|,\kappa})$. By IF and WPO, $f(\Gamma_\kappa) \in \{x : \langle 1, x\rangle_p = -|N|, x < 0\} \subset X^p$. By IIA, $f(\Gamma^p) = f(\Gamma_\kappa)$. □

Proof of Lemma 1.3. Let $\Gamma = (p, X, 0)$ be a p-TU problem and $e = f(\Gamma^p)$. By Lemma 1.6, $f(\Gamma) = \alpha e$ for some $\alpha > 0$. By IF, $e < 0$. Now consider $\lambda \in \mathbb{R}^n_{++}$ and let $h = \lambda^{-1}$. We have

$$h * X = \{x \in \mathbb{R}^n : \sum_i \sum_j \lambda_{ij} p_{ij} x_{ij} \leq w \text{ and } \lambda_{ij} x_{ij} \geq \kappa, \text{ for all } (i,j) \in \mathring{T}\}. \quad (1.A.2)$$

Since $f_{ij}(\Gamma) = \alpha e_{ij}$ for all $(i,j) \in \mathring{T}$, by SCA,

$$\frac{f_{ij}(h * \Gamma)}{f_{km}(h * \Gamma)} = \frac{h_{km} e_{ij}}{h_{ij} e_{km}} = \frac{\lambda_{ij} e_{ij}}{\lambda_{km} e_{km}}, \quad (1.A.3)$$

for all $(i,j), (k,m) \in \mathring{T}$. Hence, the solution $f(h * \Gamma)$ is on the line $l(0, \lambda * e)$. Because $WPO(h * X) = PO(h * X)$, by WPO, $f(h * \Gamma) \in l(0, \lambda * e) \cap \{x : \langle \lambda, x\rangle_p = w\}$, which is a unique point. On the other hand, a similar proof as Lemma 1.2 shows that the solution $\phi(p^e, h * X, 0)$ is on the line $l(0, \lambda * e)$. Since $\phi(p^e, h * X, 0)$ is WPO, $f(h * \Gamma) = \phi(p^e, h * X, 0)$. □

We use the following lemma to show Lemma 1.4.

LEMMA 1.7: *Suppose f satisfies IF, IIA, WPO, and SCA. Let $\Gamma = (p, X, 0)$ be in Σ. If $y = \phi(p^e, X, 0) < 0$, then $f(\Gamma) = y$.*

Proof of Lemma 1.7. Fix $\Gamma = (p, X, 0)$. Since f is IF, $e = f(\Gamma^p) < 0$. Let $y = \phi(p^e, X, 0)$. Then X and the ball $B_{p^e}(0, \|y\|_{p^e})$ has y as the unique point in common. Since X and $B_{p^e}(0, \|y\|_{p^e})$ are convex and compact, by a hyperplane separation theorem, there exists $H_{\lambda,w} = \{x : \langle \lambda, x\rangle_{p^e} = w\}$ that separates X and the ball $B_{p^e}(0, \|y\|_{p^e})$ and supports the ball at y, with the normal $\lambda = -y$. By assumption of Lemma 1.7, $y < 0$ and $\lambda \in \mathbb{R}^n_{++}$.

Construct a linear problem $h * \tilde{\Gamma} = (p, h * \tilde{X}, 0)$, where $h^{-1} = \lambda * (-e)^{-1}$ and $\tilde{\Gamma} = (p, \tilde{X}, 0)$ is a p-TU problem with $\tilde{w} = w$ and $\tilde{\kappa} < 0$ small enough such that $X \subset h * \tilde{X}$. Since f satisfies the axioms of Lemma 1.3, it implies $f(h * \tilde{\Gamma}) = \phi(p^e, h * \tilde{X}, 0) = y$. Since f is IIA and $y \in X$, $f(\Gamma) = f(h * \tilde{\Gamma}) = y$. □

Proof of Lemma 1.4. We consider three problems: a p^0-TU problem Γ^0 with complete information, a problem Γ obtained by splitting type 1 of player 1 from Γ^0, and a p-TU problem Γ^p which is an extension of Γ'.

Let $\Gamma^0 = (p^0, X^0, 0)$ be the normalized p^0-TU problem satisfying $N^0 = N$, $T_i^0 = \{1\}$ and $p_{i1}^0 = 1$ for all $i \in N$, and $w^0 = k^0 = -|N|$. Then by TU for Γ^0,

$$f_{i1}(\Gamma^0) = -1 \text{ for all } i \in N. \tag{1.A.4}$$

Now let $\Gamma = (p, X, r)$ be derived from Γ^0 by splitting type 1 of player 1 into a and b with $(\alpha, 1-\alpha)$. Pick any $x^0 \in X^0$ and define $x_{1a} = x_{1b} = x_{11}$, and $x_{i1} = x_{i1}^0$ for all $i \neq 1$. It is easy to see that

$$\begin{aligned} X = \{x \in \mathbb{R}^{|N|+1} :& \alpha x_{1a} + (1-\alpha)x_{1b} + \sum_{i \neq 1} x_{i1} \leq -|N|, \\ & x_{1a} = x_{1b}, \text{ and } x_{ij} \geq -|N|, \text{ for all } (i,j) \in \mathring{T}\}, \end{aligned} \tag{1.A.5}$$

and $r = (0, ..., 0)$. Now by ST for Γ^0 and Γ, and (1.A.4),

$$f_{1a}(\Gamma) = f_{1b}(\Gamma) = f_{11}(\Gamma^0) = -1, \quad f_{i1}(\Gamma) = f_{i1}(\Gamma^0) = -1, \text{ for all } i \neq 1. \tag{1.A.6}$$

Let $\Gamma^p = (p, X^p, 0)$ be the normalized p-TU problem. So $X \subset X^p$. Let $e = f(\Gamma^p) < 0$ and $y = \phi(p^e, X, 0)$. We claim that $y < 0$. The relaxed Lagrangian (dropping all $x_{ij} \geq -|N|$) with $\eta_1 \leq 0$ is given by

$$\min_x \frac{\alpha}{-e_{1a}} x_{1a}^2 + \frac{1-\alpha}{-e_{1b}} x_{1b}^2 + \sum_{i \neq 1} \frac{1}{-e_{i1}} x_{i1}^2 + \eta_1(-|N| - \alpha x_{1a} - (1-\alpha)x_{1b} - \sum_{i \neq 1} x_{i1}) + \eta_2(x_{1a} - x_{1b}).$$

FOCs give necessary conditions

$$\begin{aligned} &\frac{2\alpha}{e_{1a}} y_{1a} = -\eta_1 \alpha + \eta_2, \quad \frac{2(1-\alpha)}{e_{1b}} y_{1b} = -\eta_1(1-\alpha) - \eta_2, \\ &\frac{2}{e_{i1}} y_{i1} = -\eta_1, i = 2, ..., |N|. \end{aligned}$$

Sum the first two conditions and combine with the third,

$$\frac{\alpha}{e_{1a}} y_{1a} + \frac{(1-\alpha)}{e_{1b}} y_{1b} = \frac{1}{e_{i1}} y_{i1}, \tag{1.A.7}$$

for all $i \neq 1$. It is clear that $y_{1a} = y_{1b}$ and $\eta_2 \neq 0$. If $\eta_1 = 0$, then $y_{i1} = 0$ for all $i \neq 1$ and $y_{1a} = y_{1b} = 0$, contradiction. So, $\eta_1 < 0$ and $y_{i1} < 0$ for all $i \neq 1$. Then, $y_{1a} = y_{1b} < 0$. Hence, $y < 0$.

Now by Lemma 1.7, $y < 0$ implies $f(\Gamma) = y$ and therefore $y = (-1, ..., -1)$. By TU for Γ^p,

$$\alpha e_{1a} + (1-\alpha)e_{1b} = e_{i1} = -1, \tag{1.A.8}$$

for all $i \neq 1$. Combine the conditions above,

$$\frac{\alpha}{e_{1a}} + \frac{1-\alpha}{e_{1b}} = \frac{1}{\alpha e_{1a} + (1-\alpha)e_{1b}}. \tag{1.A.9}$$

Simplify the condition, we have for any $\alpha \in (0,1)$,

$$e_{1a} = e_{1b}. \tag{1.A.10}$$

Finally, we can apply this procedure repeatedly for $|T_1| > 2$ and for all $i \in N$. $\square$

LEMMA 1.8: *Let $\Gamma = (p, X, 0)$, $y = F(\Gamma) \leq 0$, and $y_{ij} = 0$ for some $(i,j) \in \mathring{T}$. There exists a sequence $\Gamma_k = (p, X_k, 0)$, $k = 1, 2, ...,$ such that $y_k = F(\Gamma_k) < 0$ for all $k = 1, 2, ...,$ and $X_k \to X$ in the Hausdorff metric, and $y_k \to y$.*

Proof. The proof is similar to Conley et al. (2014). Let $\lambda = -y$ and $\mathring{T}_0 = \{(i,j) \in \mathring{T} : \lambda_{ij} = 0\}$. Since $y \leq 0$, we can restrict attention to $\Gamma = (p, X, 0)$ where $X \subset \mathbb{R}^n_-$. To see this, let $\tilde{X} = X \cap \mathbb{R}^n_-$. Since F satisfies IIA, $y \in \tilde{X}$ implies $F(p, X, 0) = F(p, \tilde{X}, 0)$. Now define a sequence $\lambda_k = \lambda + \frac{1}{k}\sum_{(i,j)\in\mathring{T}_0} 1_{ij}$, $k = 1, 2, ...,$ where 1_{ij} is the vector with 1 on place (i,j) and 0 otherwise. Let

$$y_k = -\frac{\langle \lambda_k, \lambda \rangle_p}{\|\lambda_k\|_p}\lambda_k. \tag{1.A.11}$$

Notice that $y_k < y \leq 0$. Define $X_k = \text{conv}(X \cup \{y_k\})$ and $\Gamma_k = (p, X_k, 0)$. We claim $y_k = F(\Gamma_k)$. By the projection theorem, we only need to show $\langle -y_k, x\rangle_p \leq \langle -y_k, y_k\rangle_p$ for all $x \in X_k$.

We first claim that $\langle \lambda_k, x\rangle_p \leq \langle \lambda_k, y_k\rangle_p$ for all $x \in X$. To see this, note that $\langle \lambda_k, y_k\rangle_p = \langle \lambda_k, y\rangle_p$. Because $\langle \lambda, x\rangle_p \leq \langle \lambda, y\rangle_p$ for all $x \in X$, $\sum_{(i,j)\notin\mathring{T}_0} \lambda_{ij}p_{ij}x_{ij} \leq \sum_{(i,j)\notin\mathring{T}_0} \lambda_{ij}p_{ij}y_{ij}$. For $(i,j) \in \mathring{T}_0$, $\frac{1}{k}p_{ij}x_{ij} \leq 0 = \frac{1}{k}p_{ij}y_{ij}$. Hence,

$$\langle \lambda_k, x\rangle_p \leq \langle \lambda_k, y\rangle_p = \langle \lambda_k, y_k\rangle_p, \text{ for all } x \in X. \tag{1.A.12}$$

Then, since each $x \in X_k$ is a convex combination of some $x_1, ..., x_m \in X \cup \{y_k\}$, i.e. $x = \sum_m \alpha_m x_m$ for some nonnegative $(\alpha_1, ..., \alpha_m)$ with $\sum_m \alpha_m = 1$, we have $\langle \lambda_k, x\rangle_p = \sum_m \alpha_m \langle \lambda_k, x_m\rangle_p \leq \sum_m \alpha_m \langle \lambda_k, y_k\rangle_p = \langle \lambda_k, y_k\rangle_p$. Hence, for all $x \in X_k$,

$$\langle -y_k, x\rangle_p = \frac{\langle \lambda_k, \lambda\rangle_p}{\|\lambda_k\|_p}\langle \lambda_k, x\rangle_p \leq \frac{\langle \lambda_k, \lambda\rangle_p}{\|\lambda_k\|_p}\langle \lambda_k, y_k\rangle_p = \langle -y_k, y_k\rangle_p. \tag{1.A.13}$$

$\square$

Proof of Lemma 1.5. Consider the one-person problem in the proof of Lemma 1.1. Let $S = (\pi, D, u)$, where $D = \{d_0\}$ and $(u(d_0, a), u(d_0, b)) = (0, 1)$. Let X be the line segment between $(0,1)$ and $(1,0)$. From the proof of Lemma 1.1, it follows that there is no extension $\tilde{S} = (\pi, \tilde{D}, \tilde{u})$ of S such that (i) $\tilde{D} = \{d_0, d_1\}$ with $(\tilde{u}(d_1, a), \tilde{u}(d_1, b)) = (y_1, y_2) \in \mathbb{R}^2$ and (ii) $X = \mathcal{U}(\tilde{S})$. Similarly, it is impossible to extend $\text{conv}\{(0,1),(1,0),(1,1)\}$ to $\text{conv}\{(0,1),(1,0),(1,1),(2,-1)\}$ by adding any d_2 into $\tilde{D}$. $\square$

CHAPTER 2

Implementation of Vector-Valued Reduced Form Allocation Rules

2.1. Introduction

Myerson (1981) considers the problem faced by a seller $i = 0$ who has an object to sell, and who does not know the value that each risk neutral buyer $i \in \{1, ..., n\}$ might be willing to pay for the object. Let t_i be buyer i's value and assume $\tilde{t}_i$ is independently distributed with a continuous distribution F_i having support T_i, and let T be the set of vectors of buyers' values. The seller would like to find some auction procedure which gives him the highest expected revenue. The seller has to use a feasible allocation rule $q : T \to \Delta(\{0, ..., n\})$, which assigns to every value profile of buyers a lottery determining the winner of the object. The seller also has to respect the buyers' incentive constraints. Using the envelope condition, the expected payment for buyer i can be expressed by an interim allocation rule $Q_i : T_i \to [0, 1]$, which assigns to each type of this buyer an expected probability of winning. Let $\psi_i : T_i \to \mathbb{R}$ be the virtual valuation for buyer i defined as in Myerson (1981).[1] This allows one to express the seller's problem solely in terms of interim allocation rules, or

$$\max \sum_{i=1}^{n} E[\tilde{\psi}_i \tilde{Q}_i], \tag{2.1.1}$$

$$\begin{aligned} \text{s.t.(i)} \;\; & Q_i \;\text{ is incentive compatible for all }\; i = 1, ..., n, \\ \text{(ii)} \;\; & (Q_1, ..., Q_n) \;\text{ can be implemented by some feasible }\; q. \end{aligned}$$

In this chapter, we consider subproblems arise in the context of mechanism design problems similar to (or more general than) Myerson's optimal auction problem. Specifically, we study an implementation problem: Under what conditions can a system of interim allocation rules (or reduced forms) $(Q_1, ..., Q_n)$ be generated by some feasible allocation rule q? In Myerson's model, since the allocation rule can easily be optimized point-wise for each type profile, solving the implementation problem is not necessary for finding the solution to this optimal auction problem.

Now consider the problem of finding a revenue-maximizing auction for n risk averse buyers as in Maskin and Riley (1984). For each buyer i, a payment schedule (l_i, w_i) :

[1] $\psi_i(t_i) = t_i - (1 - F_i(t_i))/f_i(t_i)$ for all $t_i \in T_i$.

$T_i \to \mathbb{R}^2$ assigns to each type of this buyer a payment to the seller in case of losing and winning. The seller's problem can now be expressed as

$$\max \sum_{i=1}^{n} E[\tilde{l}_i(1-\tilde{Q}_i) + \tilde{w}_i\tilde{Q}_i], \tag{2.1.2}$$
$$\text{s.t.(i)} \ (Q_i, l_i, w_i) \text{ is incentive compatible for all } i = 1, ..., n,$$
$$\text{(ii)} \ (Q_1, ..., Q_n) \text{ can be implemented by some feasible } q.$$

Again, the implementation problem arises in the seller's problem. Different from the risk neutral case, the allocation rule cannot easily be optimized point-wise for each type profile. Hence, finding an optimal solution in terms of an interim allocation rule requires first solving for its implementability. This implementation problem has been solved by Border (1991).

Finally, consider a revenue-maximizing problem with two homogenous objects and n risk averse buyers. A feasible allocation rule assigns each type profile of the buyers a lottery over the winners of the two objects. For each buyer i, a vector-valued interim allocation rule $(Q_i^1, Q_i^2) : T_i \to [0,1]^2$ arises, which assigns to each type of this buyer expected probabilities of winning one object and two objects. For each buyer i, the expected payment schedule is given by $(l_i, w_i^1, w_i^2) : T_i \to \mathbb{R}^3$, which assigns to each type of this buyer a payment in case of losing, in case of winning one object, and in case of winning two objects. The seller's problem is expressed by

$$\max \sum_{i=1}^{n} E[\tilde{l}_i(1-\tilde{Q}_i^1-\tilde{Q}_i^2) + \tilde{w}_i^1\tilde{Q}_i^1 + \tilde{w}_i^2\tilde{Q}_i^2], \tag{2.1.3}$$
$$\text{s.t.(i)} \ (Q_i^1, Q_i^2, l_i, w_i^1, w_i^2) \text{ is incentive compatible for all } i = 1, ..., n,$$
$$\text{(ii)} \ (Q_1^1, Q_1^2, ..., Q_n^1, Q_n^2) \text{ can be implemented by some feasible } q.$$

For this optimal auction problem, the interim allocation rule for each player contains interim expected probabilities for different decision outcomes. Vector-valued reduced forms appear in the implementation problem.

The above examples show that mechanism design problems naturally give rise to implementation problems. In this chapter, we discuss implementation problems in general social choice environments.[2] We define a social choice problem by a finite set of players and for each player a finite individual type set, a common prior on the product type set, a finite set of social alternatives, and a set of players' utility functions, which depend on alternatives, monetary transfers, and type profiles. The examples include:

[2]Note that besides the implementability of allocation rules, we may require the implementability of payment rules to solve a mechanism design problem, which we do not consider in most part of the chapter.

(i) Auctions with externalities (Jehiel, Moldovanu and Stacchetti, 1999), in which a losing buyer's valuation is affected by the identity of the winner;

(ii) Package auctions and exchanges with complementary objects (Milgrom, 2007). In a package exchange problem, the players without cash can only reallocate their initial assets through random allocation mechanisms;

(iii) Voting schemes where monetary transfers are not possible and players have cardinal utility. The players must determine a fair compromise among different alternatives (Börgers and Postl, 2009).

In this chapter, we provide a necessary and sufficient condition for the implementability of vector-valued reduced form allocation rules. We first obtain a general characterization of the set of implementable reduced forms by an infinite number of linear inequalities. Then, we investigate the possibility of characterization by finitely many inequalities. We obtain a necessary condition by a class of inequalities with integral coefficients. This class is strictly larger than the class in Border (1991) and in Che, Kim and Mierendorff (2013). We then provide a condition such that the necessary condition becomes sufficient. For the two-player case, we formulate the implementation problem as a digraph multicommodity flow problem and establish an equivalence between these two problems. To deal with multi-unit auctions with group capacity constraints, Che, Kim and Mierendorff (2013) formulate their problem as a digraph single-commodity flow in a different way. The network flow result in this chapter is complementary to their results.

In many social choice problems, the system of reduced form has no full dimensionality. We further provide an implementability condition (and a necessary condition) on coordinate subspaces and use the necessary condition to study mechanism design problems without money. As a leading example, we study a package allocation problem as in Miralles (2012), but with non-additive valuations. There are two players and two objects. Each player initially owns one object and values two objects as complements. The players can keep their objects, or allocate both objects to one player, or exchange. Together with the incentive compatibility condition, we illustrate how the necessary condition for the implementability can be used to find the upper bound of the ex ante utilitarian social welfare (or ex ante trading surplus). In such a problem without money, both the incentive compatibility and the implementability conditions can be binding in an interim Pareto optimal solution. For this example, since a feasible allocation rule that implements the upper bound of the ex ante utilitarian welfare exists, the necessary condition is sufficient for the implementability.

The remainder of the chapter is organized as follows. Section 2.2 formulates the implementation problem. Section 2.3 provides a characterization for implementability (Theorem 2.1) and obtains a necessary condition by finitely many inequalities (Corollary 2.1). We provide a condition such that this necessary condition is also sufficient (Theorem 2.2) and also an implementability condition in coordinate subspaces (Theorem 2.3).

Sections 2.4 uses these implementability conditions to study mechanism design problems without money. Section 2.5 provides some discussion for mechanism design problems with general utility functions and the relationship to the implementation problems. Section 2.6 concludes.

2.2. The Problem

An implementation problem is given by $I = (N, D, (T_i)_{i\in N}, (\lambda_i)_{i\in N})$. There is a finite set of players N, $|N| \geq 2$, and a finite set of social alternatives D, $|D| \geq 2$. Let $\Delta(D)$ be the set of probability distributions over D. For each $i \in N$, T_i is a non-empty finite set of types, and denote the power set of T_i by 2^{T_i}. For each $i \in N$, λ_i is a probability measure on T_i. We assume $\lambda_i(t_i) > 0$ for all $t_i \in T_i$ and $i \in N$. The product type set is given by $T = \times_{i\in N} T_i$. Let λ be the product measure $\times_{i\in N}\lambda_i$ on T. For each $i \in N$, denote $T_{-i} = \times_{j\neq i} T_j$ and $\lambda_{-i}(t_{-i}) = \times_{j\neq i}\lambda_j(t_j)$. For any $A = \times_{i\in N} A_i \subseteq T$ and $j \in N$, we write $\lambda(A)$ as $\lambda(A_j \times A_{-j})$ occasionally. For $A_i \in 2^{T_i}$, denote $(A_i)^c = T_i \setminus A_i$. Denote by $\mathcal{I}$ the set of implementation problems.

Fix an $I \in \mathcal{I}$. Define the set of *feasible allocation rules* by

$$\mathcal{D}_0 = \left\{ q \in \mathbb{R}^{D\times T} \;\middle|\; q(d,t) \geq 0, \sum_{d\in D} q(d,t) = 1, \forall d \in D, t \in T \right\}.$$

That is, a feasible allocation rule assigns to each type profile a lottery over social alternatives. Hence for each type profile, it satisfies a probability simplex condition. Denote $k = |D| \times |T|$ and $\mathbb{R}^k$ the Euclidean space that contains $\mathcal{D}_0$. Define the set of *systems of reduced forms* by

$$\mathcal{D}_1 = \left\{ Q \in \times_{i\in N}\mathbb{R}^{D\times T_i} \;\middle|\; Q_i(d,t_i) \geq 0, \sum_{d\in D} Q_i(d,t_i) = 1, \forall d \in D,\, t_i \in T_i,\, i \in N \right\}.$$

That is, a system of reduced forms assigns to each type t_i of each player i a lottery over the set of social alternatives. Denote $l = \sum_{i\in N}(|D| \times |T_i|)$ and $\mathbb{R}^l$ the Euclidean space that contains $\mathcal{D}_1$. Now define a linear transformation $\Lambda : \mathbb{R}^k \to \mathbb{R}^l$ by, for any $x \in \mathbb{R}^k$,

$$(\Lambda x)_i(d,t_i) = \sum_{t_{-i}\in T_{-i}} x(d,t)\lambda_{-i}(t_{-i}), \tag{2.2.1}$$

for all $d \in D$, $t_i \in T_i$, and $i \in N$. For $q \in \mathcal{D}_0$, Λq is the system of interim expected probabilities generated by the allocation rule q and the belief system λ.

Definition 2.1: Let $I \in \mathcal{I}$. A system of reduced forms $Q \in \mathcal{D}_1$ is *implementable* if there exists a feasible allocation rule $q \in \mathcal{D}_0$ such that $Q = \Lambda q$.

Denote by $\mathcal{D}$ the set of implementable systems of reduced forms. The implementation problem (I) studies necessary and sufficient conditions for systems of reduced forms to be implementable.

2.3. Characterization

Theorem 2.1 obtains a general necessary and sufficient condition for the implementability. Corollary 2.1 provides a necessary condition with finitely many inequalities with integer coefficients. For problems with $|N| = |D| = 2$, Theorem 2.2 shows that the necessary condition in Corollary 2.1 is sufficient.

Before the statement of the theorems, we introduce some definitions. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ be the standard inner product. Now define a linear transformation $\Gamma : \mathbb{R}^l \to \mathbb{R}^l$ by, for each $x \in \mathbb{R}^l$,

$$(\Gamma x)_i(d, t_i) = \lambda_i(t_i) x_i(d, t_i), \tag{2.3.1}$$

for all $d \in D$, $t_i \in T_i$ and $i \in N$. For $Q \in \mathcal{D}_1$, ΓQ is the interim expected probability vector Q weighted by the system of marginal probabilities of λ.

2.3.1. Main Results

Theorem 2.1 is the main result of this chapter. For the case $|N| = |D| = 2$, the result of Theorem 2.1 was first proved by Strassen (1965). The result follows from the fact that $\Gamma\mathcal{D}$ is a convex and compact subset in an Euclidean space. Then a standard separation argument implies that $\Gamma\mathcal{D}$ is the intersection of all supporting half-spaces of the set itself.

Theorem 2.1: *Let $I \in \mathcal{I}$. $Q \in \mathcal{D}_1$ is implementable if and only if*

$$\langle f, \Gamma Q \rangle \leq \sup\{\langle f, \Gamma\Lambda q \rangle : q \in \mathcal{D}_0\} \text{ for all } f \in \mathbb{R}^l. \tag{2.3.2}$$

Proof. See Appendix 2.A. □

Theorem 2.1 has a social welfare interpretation in terms of utility functions. We discuss this intuition in detail in Section 2.4.

Notice that $\mathcal{D}_0$ is a polytope. By the Krein-Milman theorem, $\mathcal{D}$ is the closed convex hull of images of $\text{ext}(\mathcal{D}_0)$ under Λ, where $\text{ext}(\mathcal{D}_0)$ is the set of extreme points of $\mathcal{D}_0$. Hence, $\mathcal{D}$ has finitely many facets. For applications, it is important to characterize these facets of $\mathcal{D}$. Corollary 2.1 establishes a necessary condition that may lead to such a nested characterization. If it is also sufficient, then the normal vector of each facet of $\Gamma\mathcal{D}$ contains entries only from $\{-1, 0, +1\}$, and the other conditions with non-integral coefficients in $[-1, 1]$ are redundant.[3]

[3]It is well known that for $Ax \leq b$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, if A is totally unimodular and b is

Corollary 2.1: *Let $I \in \mathcal{I}$. If $Q \in \mathcal{D}_1$ is implementable, then*

$$\langle f, \Gamma Q \rangle \leq \sup\{\langle f, \Gamma\Lambda q\rangle : q \in \mathcal{D}_0\} \text{ for all } f \in \{-1, 0, +1\}^l. \tag{2.3.3}$$

Corollary 2.1 follows immediately from Theorem 2.1. The first interpretation of condition (2.3.3) is similar to Theorem 2.1. Alternatively, condition (2.3.3) shows that if for each player i with type t_i and alternative d, the ex ante probability $\lambda_i(t_i)Q_i(d, t_i)$ represents the quantity of a good (i, d, t_i), then given any price vector f, the right hand side is equal to the monetary value of the initial endowment and hence the total budget.

The network flow approach has been used to provide a sufficient condition for the implementability (Che, Kim and Mierendorff, 2013).[4] The following result, based on Hansel and Troallic (1978), applies a network flow approach to prove that if $|N| = |D| = 2$, the condition in Corollary 2.1 is sufficient. In Appendix 2.B (Lemma 2.2-2.4), we discuss the possibility to generalize this result and establish an equivalence between the implementation problem and a digraph multicommodity network flow problem.

Theorem 2.2: *Let $I \in \mathcal{I}$ with $|N| = |D| = 2$. Then* (2.3.3) *is necessary and sufficient for implementability.*

Proof. See Appendix 2.B. □

2.3.2. Results on Coordinate Subspaces

For many economic environments we discuss later, the system of reduced forms has no full dimensionality. In this subsection, we provide conditions for implementation problems on coordinate subspaces.

Fix an $I \in \mathcal{I}$ and some $\tilde{D}_i \subseteq D$ for each $i \in N$. We choose $\tilde{D} = (\tilde{D}_i)_{i\in N}$ as a system of essential alternatives, where $\tilde{D}_i$ is labelled as the set of essential alternatives and $\tilde{D}_i^c = D \setminus \tilde{D}_i$ is labelled as the set of inessential alternatives for each $i \in N$. Denote $\mathbb{R}^{\tilde{l}} = \times_{i\in N}\mathbb{R}^{\tilde{D}_i \times T_i}$, a coordinate subspace of $\mathbb{R}^l$. Denote $\mathbb{R}^{l-\tilde{l}}$ the complementary subspace of $\mathbb{R}^{\tilde{l}}$. Now define the projection of $\mathcal{D}$ onto the coordinate subspace $\mathbb{R}^{\tilde{l}}$ by

$$P_{\tilde{D}}(\mathcal{D}) = \{Q_{\tilde{D}} \in \mathbb{R}^{\tilde{l}} | \quad \exists Q_{D\setminus\tilde{D}} \in \mathbb{R}^{l-\tilde{l}} : (Q_{\tilde{D}}, Q_{D\setminus\tilde{D}}) \in \mathcal{D}\}. \tag{2.3.4}$$

Define an implementation problem on the coordinate subspace $\mathbb{R}^{\tilde{l}}$ as follows.

(I-$\tilde{D}$). Let $I \in \mathcal{I}$ and $\tilde{D} = (\tilde{D}_i)_{i\in N}$. For $Q_{\tilde{D}} \in \mathbb{R}^{\tilde{l}}$, find a feasible solution $q \in \mathcal{D}_0$ such that (i) there exists $Q \in \mathcal{D}$ such that q implements Q, and (ii) Q coincides with $Q_{\tilde{D}}$ on $\mathbb{R}^{\tilde{l}}$.

integral, then all extreme points of the feasible region are integral.

[4] Flows in networks are used to model traffic in a road system, circulation with demands, or fluids in pipes. The problem is to send as many trucks as possible, where roads have limits on the number of trucks per unit time (maximize the flows). On the other hand, the dual problem is to destroy some bridges to disconnect all routes while minimizing the cost of destroying the bridges (minimize the costs of cuts).

Theorem 2.1 implies the following implementability condition for coordinate subspaces.

THEOREM 2.3: *Let $I \in \mathcal{I}$ and $\tilde{D} = (\tilde{D}_i)_{i \in N}$. Define $\tilde{C} = \{f \in \mathbb{R}^l : f_i(d, t_i) = 0$ for all $d \in \tilde{D}_i^c, t_i \in T_i, i \in N\}$. Then,*
(i) $Q_{\tilde{D}} \in P_{\tilde{D}}(\mathcal{D})$ if and only if (2.3.2) *holds only for all $f \in \tilde{C}$.*
(ii) If $Q_{\tilde{D}} \in P_{\tilde{D}}(\mathcal{D})$, then (2.3.2) *holds for all $f \in \tilde{C} \cap \{-1, 0, +1\}^l$.*

Proof. See Appendix 2.B. □

For certain classes of problems on coordinate subspaces, the linear inequalities given by $\{-1, 0, +1\}^l$ in Theorem 2.3 (ii) turn out to be necessary and sufficient. In these cases, we can further investigate whether some subclass of these linear inequalities is redundant. In Proposition 2.1, Border (1991) shows that for standard single object auctions without externalities, all inequalities other than the class $\{0, +1\}^l$ are redundant. However, as soon as we depart from the environment with single object auctions without externalities, some inequalities other than $\{0, +1\}^l$ are not redundant in general. In Proposition 2.2, we illustrate this point by a two-person two-alternative voting problem. For illustration, we consider only symmetric allocation rules and symmetric reduced forms for symmetric players.

First consider the single object allocation problem in Myerson (1981), with the seller 0 and the bidders $\{1, ..., n\}$. Let $\tilde{D}_0 = \{\emptyset\}$ and $\tilde{D}_i = \{i\}$ for all $i \in N \setminus \{0\}$. Then a system of essential reduced forms is given by $(Q_i(i, \cdot))_{i \in N \setminus \{0\}}$.

PROPOSITION 2.1: *(Border, 1991) Let $N = D = \{0, 1, ..., n\}$, $\tilde{D}_0 = \{\emptyset\}$, and $\tilde{D}_i = \{i\}$ for all $i \in N \setminus \{0\}$. Then, $(Q_i(i, \cdot))_{i \in N \setminus \{0\}}$ is implementable if and only if for all $A_i \in 2^{T_i}$, $i \in N \setminus \{0\}$,*

$$\sum_{i \in N \setminus \{0\}} \sum_{t_i \in A_i} Q_i(i, t_i) \lambda_i(t_i) \leq \lambda(\bigcup_{i \in N \setminus \{0\}} A_i \times T_{-i}). \tag{2.3.5}$$

Proof. See Appendix 2.C. □

The result has the following interpretation. The sum of the ex ante expected probability of winning of all buyers with types drawn from $(A_i)_{i \in N \setminus \{0\}}$, is bounded above by the ex ante probability that there exists at least one buyer i who draws a type from A_i. For this allocation problem, all inequalities containing the entry -1 are redundant. In Appendix 2.C, we show how we can eliminate these redundant inequalities.

EXAMPLE 2.1: (Border, 1991) Let $n = 2$. For $i = 1, 2$, $T_i = \{0, 1\}$ and $\lambda_i(0) = \lambda_0, \lambda_i(1) = \lambda_1 = 1 - \lambda_0$. In this environment, a symmetric allocation rule requires $q(1, (x, y)) = q(2, (y, x))$ for all $x, y \in \{0, 1\}$. By symmetry, $Q_1(1, \cdot) = Q_2(2, \cdot) = Q_a$.

From Proposition 2.1, Q_a is implementable if and only if

$$0 \leq Q_{a,0}, Q_{a,1} \leq 1, \tag{2.3.6}$$

$$Q_{a,0}\lambda_0 + Q_{a,0}\lambda_0 \leq 1 - \lambda_1^2, \tag{2.3.7}$$

$$Q_{a,1}\lambda_1 + Q_{a,1}\lambda_1 \leq 1 - \lambda_0^2, \tag{2.3.8}$$

$$Q_{a,0}\lambda_0 + Q_{a,1}\lambda_1 + Q_{a,0}\lambda_0 + Q_{a,1}\lambda_1 \leq 1, \tag{2.3.9}$$

and other inequalities are redundant. For $\lambda_0 = 1/2$, Figure 2.1 illustrates the set of implementable Q_a.

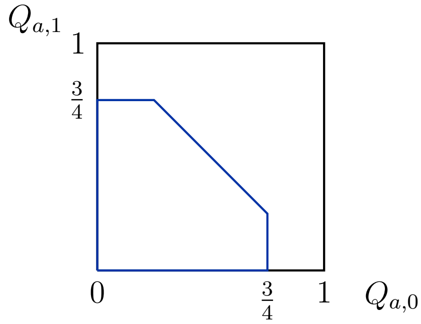


Figure 2.1

Consider a two-person two-alternative voting problem. Let $N = \{1, 2\}$, $D = \{0, b\}$, and $\tilde{D}_i = \{b\}$, $i = 1, 2$. A system of essential reduced forms is given by $(Q_1(b, \cdot), Q_2(b, \cdot))$.

Proposition 2.2: *Let $N = \{1, 2\}$, $D = \{0, b\}$, and $\tilde{D}_i = \{b\}$, $i = 1, 2$. Then $(Q_1(b, \cdot), Q_2(b, \cdot))$ is implementable if and only if for all $\overline{B}_i, \underline{B}_i \in 2^{T_i}$, $\overline{B}_i \cap \underline{B}_i = \emptyset$, $i = 1, 2$,*

$$\sum_{i \in N} \left(\sum_{t_i \in \overline{B}_i} Q_i(b, t_i)\lambda_i(t_i) - \sum_{t_i \in \underline{B}_i} Q_i(b, t_i)\lambda_i(t_i) \right) \leq \sum_{i \in N} \lambda(\overline{B}_i \times (\underline{B}_{-i})^c). \tag{2.3.10}$$

Proof. See Appendix 2.C. □

In contrast to Border's condition, the implementability condition for this voting problem requires the entry -1 to appear in the linear inequalities. The intuition for this result is as follows: In Myerson (1981), selecting alternative i with a higher interim expected probability tightens the (probabilistic) budget for alternative j. $Q_i(i, \cdot)$ and $Q_j(j, \cdot)$ are competing. In the voting problem, however, selecting alternative b with a higher interim expected probability for player i relaxes the budget that b can be selected for player j. Hence, $Q_i(b, \cdot)$ and $Q_j(b, \cdot)$ are of common interest.

Example 2.2: Let T and λ be as assumed in Example 2.1. In this case, a symmetric allocation rule requires $q(b, (x, y)) = q(b, (y, x))$ for all $x, y \in \{0, 1\}$. With symmetry,

$Q_1(b,\cdot) = Q_2(b,\cdot) = Q_b$. By Proposition 2.2, Q_b is implementable if and only if

$$0 \le Q_{b,0}, Q_{b,1} \le 1, \tag{2.3.11}$$
$$Q_{b,0}\lambda_0 - Q_{b,1}\lambda_1 + Q_{b,0}\lambda_0 - Q_{b,1}\lambda_1 \le 2\lambda_0^2, \tag{2.3.12}$$
$$Q_{b,1}\lambda_1 - Q_{b,0}\lambda_0 + Q_{b,1}\lambda_1 - Q_{b,0}\lambda_0 \le 2\lambda_1^2, \tag{2.3.13}$$

and other inequalities are redundant. For $\lambda_0 = \frac{1}{2}$, the set of implementable Q_b is illustrated in Figure 2.2.

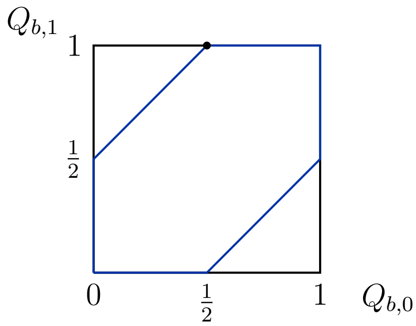


Figure 2.2

For each λ_0, the corner point $(1,1)$ is feasible but the corner points $(1,0)$ and $(0,1)$ are ruled out. For $\lambda_0 = 1/2$, the unique symmetric allocation rule that implements $(\frac{1}{2}, 1)$ is given by

$$q(b,(0,0)) = 0, q(b,(0,1)) = q(b,(1,0)) = q(b,(1,1)) = 1. \tag{2.3.14}$$

2.4. Social Choice Problems without Monetary Transfers

In this section, we discuss social choice problems without transfers. We first provide two examples of such problems: a voting problem and a package allocation problem. We show that the implementation conditions in Section 2.3 can be used to find the solutions for certain social welfare objectives in such problems (mechanism design problems). We delay the discussion of social choice problems with money in Section 2.5.

A social choice problem without monetary transfers is given by $S = (I, (v_i)_{i\in N})$, where $I = (N, D, (T_i)_{i\in N}, (\lambda_i)_{i\in N})$, and player $i \in N$ has a valuation function $v_i : D \times T_i \to \mathbb{R}$ given by $v_i(d, t_i)$ for $d \in D$ and $t_i \in T_i$. Notice that we restrict to the class of private values. We further discuss this assumption in Section 2.5.

Let $S = (I, (v_i)_{i\in N})$. An allocation mechanism is given by a feasible allocation rule $q : T \to \Delta(D)$. Given that the other players always report truthfully, the interim expected utility of player i with type $t_i \in T_i$ from reporting $\hat{t}_i \in T_i$ is given by

$$U_i(q, \hat{t}_i|t_i) = \sum_{t_{-i}\in T_{-i}} \sum_{d\in D} q(d, (\hat{t}_i, t_{-i}))v_i(d, t_i)\lambda_{-i}(t_{-i}). \tag{2.4.1}$$

We say q is incentive compatible if truth telling by all players is a Bayesian Nash equilibrium. Denote the interim utility by $U_i(q|t_i) = U_i(q, t_i|t_i)$. If there is some status quo $d_* \in D$, it further requires q to satisfy the individual rationality condition, or $U_i(q|t_i) \geq U_i(d_*|t_i)$ for all $t_i \in T_i$ and $i \in N$. Let $M(S)$ be the set of incentive compatible (and individual rational if any) mechanisms for S. A mechanism design (MD) problem (with a linear social welfare function) is selecting some $q \in M(S)$ to maximize

$$\sum_{i\in N} \sum_{t_i\in T_i} \tilde{f}_i(t_i)\lambda_i(t_i)U_i(q|t_i), \tag{2.4.2}$$

for some interim welfare weight system $\tilde{f} \in \times_{i\in N}\mathbb{R}^{T_i}_{++}$. In particular, $\tilde{f} = (1, ..., 1)$ corresponds to the ex ante utilitarian social welfare function.

2.4.1. Bounds on Social Welfare

To see the welfare implication of Theorem 2.1, let $S = (I, (v_i)_{i\in N})$ and $f_i(d, t_i) = \tilde{f}_i(t_i)v_i(d, t_i)$ for all $d \in D$, $t_i \in T_i$, and $i \in N$, where $\tilde{f} \in \times_{i\in N}\mathbb{R}^{T_i}_{++}$ is a system of interim welfare weights. Then, the left hand side of (2.3.2) is given by (2.4.2), which corresponds to the linear social welfare function with interim welfare weights $\tilde{f}$.

Theorem 2.1 states that the dual variable of reduced form allocation rules is utility functions. For $S = (I, (v_i)_{i\in N})$, the left hand side of (2.3.2), which corresponds to the $\tilde{f}$-weighted linear social welfare from some allocation rule, is bounded above by the maximal social welfare from all feasible allocation rules. Hence, Theorem 2.1 obtains the bounds on the social welfare for all linear social welfare functions by fixing $(v_i)_{i\in N}$ and varying $\tilde{f}$, and the bounds on the social welfare for all MD problems without money by varying $(v_i)_{i\in N}$ and $\tilde{f}$. For $\tilde{f} = (1, ..., 1)$, it provides an upper bound of the ex ante utilitarian social welfare for all MD problems without money.

2.4.2. Values and Implementation on Coordinate Subspaces

Let $S = (I, (v_i)_{i\in N})$. The system of valuation functions $(v_i)_{i\in N}$ induces a system of essential decisions $\tilde{D}$ and an implementation problem (I-$\tilde{D}$) in the following two cases:

1. Suppose for all $i \in N$ and $d \in D$, $v_i(d, t_i) \neq 0$ for some $t_i \in T_i$. Then, $Q_i(d, \cdot)$ has influence on player i's utility. By the probability simplex condition, for some $d_0 \in D$ and all $i \in N$, $Q_i(d_0, \cdot) = 1 - \sum_{d\neq d_0} Q_i(d, \cdot)$. We can choose d_0 as the *normalized* alternative and the others as essential alternatives.

2. Suppose for some $i \in N$ and $d \in D$, $v_i(d, t_i) = 0$ for all $t_i \in T_i$. For such an alternative d, $Q_i(d, \cdot)$ does not influence player i's utility and $Q_i(d, \cdot)$ is free. We say d is *irrelevant* for player i. Then, we can choose such alternatives as inessential alternatives for player i.

To illustrate these two cases, we disucss voting problems (Proposition 2.3) and package allocation problems (Proposition 2.4) in the following subsections. We consider only symmetric allocation rules and symmetric reduced forms for symmetric players.

2.4.3. Example: Voting

Consider a two-person $n+1$-alternative voting problem with $N = \{1, 2\}$ and $D = \{0, 1, ..., n\}$, $n \geq 1$. Suppose for all $i \in N$ and $d \in D$, $v_i(d, t_i) \neq 0$ for some $t_i \in T_i$. Let $\tilde{D}_i = D \setminus \{0\}$, $i = 1, 2$. A system of essential reduced forms is given by $(Q_1(d, \cdot), Q_2(d, \cdot))_{d \in D \setminus \{0\}}$. The following condition for voting problems with three or more alternatives follows from Lemma 2.4 in Appendix 2.B.

PROPOSITION 2.3: *Let $N = \{1, 2\}$, $D = \{0, 1, ..., n\}$, and $\tilde{D}_i = D \setminus \{0\}$, $i = 1, 2$. If $(Q_1(d, \cdot), Q_2(d, \cdot))_{d \in D \setminus \{0\}}$ is implementable, then for all $\overline{B}_i \in 2^{T_i}$, $i = 1, 2$,*

$$\sum_{d \in D \setminus \{0\}} \sum_{i \in N} \left(\sum_{t_i \in \overline{B}_i} Q_i(d, t_i) \lambda_i(t_i) - \sum_{t_i \in (\overline{B}_i)^c} Q_i(d, t_i) \lambda_i(t_i) \right) \leq 2\lambda(\overline{B}_1 \times \overline{B}_2). \tag{2.4.3}$$

Proof. See Appendix 2.C. □

In particular, a three-alternative problem corresponds to the compromise problem of Börgers and Postl (2009). Suppose player 1's preference is given by $1 \succ 0 \succ 2$ and player 2's preference is given by $2 \succ 0 \succ 1$. Each player receives utility 1 from his best and 0 from his worst alternative. Each player receives t_i between 0 and 1 from the alternative 0. The interim utility for player i with type t_i is given by $Q_i(0, t_i)t_i + Q_i(i, t_i)$.

A system of essential reduced forms in the problem of Borgers and Postl is given by $(Q_1(0, \cdot), Q_1(1, \cdot), Q_2(0, \cdot), Q_2(2, \cdot))$. Then, it is easy to see that for each player i, replacing $Q_i(j, \cdot)$ by $1 - Q_i(i, \cdot) - Q_i(0, \cdot)$ in (2.4.3) of Proposition 2.3 provides a necessary condition for the implementability to the problem of Börgers and Postl (2009).

2.4.4. Example: Package Allocation

Miralles (2012) studies the allocation of two homogeneous objects among a set of players without money and finds the mechanism that maximizes the ex ante utilitarian welfare. Since players have additive valuations, Miralles applies Proposition 2.1 separately to each object. We now consider a related problem with complementary valuations, which forces the implementability condition to be defined on the set of all social alternatives.

Let $N = \{1,2\}$ and $D = \{0,1,2,b\}$. Each player has no money and initially owns one object, or $d_* = 0$. For $i = 1,2$, the choice i means that player i obtains both objects and the choice b indicates the exchange of objects. For $i = 1,2$, player i has the value of 0 if no object is obtained, i.e. for all $d \neq i, b$ and $t_i \in T_i$, $v_i(d,t_i) = 0$, and player i has a complementary valuation for the package, i.e. for all $t_i \in T_i$,

$$v_i(b,t_i) - v_i(j,t_i) < v_i(i,t_i) - v_i(0,t_i).$$

For $(I\text{-}\tilde{D})$ derived from $(I,(v_i)_{i\in N})$, we have $\tilde{D}_i = \{i,b\}$, $i = 1,2$. A system of essential reduced forms is given by $(Q_1(1,\cdot),Q_2(2,\cdot),Q_1(b,\cdot),Q_2(b,\cdot))$.

Proposition 2.4: *Let $N = \{1,2\}$, $D = \{0,1,2,b\}$, and $\tilde{D}_i = \{i,b\}$, $i = 1,2$. If $(Q_1(1,\cdot),Q_2(2,\cdot),\ Q_1(b,\cdot),Q_2(b,\cdot))$ is implementable, then for all $A_i,\overline{B}_i,\underline{B}_i \in 2^{T_i}$, $\overline{B}_i \cap \underline{B}_i = \emptyset$, $i = 1,2$,*

$$\begin{aligned}&\sum_{i\in N}\left(\sum_{t_i\in A_i} Q_i(i,t_i)\lambda_i(t_i) + \sum_{t_i\in\overline{B}_i} Q_i(b,t_i)\lambda_i(t_i) - \sum_{t_i\in\underline{B}_i} Q_i(b,t_i)\lambda_i(t_i)\right)\\ &\quad\le \lambda(\bigcup_{i\in N} A_i\times T_{-i}) + \sum_{i\in N}\lambda(\overline{B}_i\times(\underline{B}_{-i})^c) - \lambda\left((\bigcup_{i\in N} A_i\times T_{-i})\bigcap(\bigcup_{i\in N}\overline{B}_i\times(\underline{B}_{-i})^c)\right).\end{aligned} \tag{2.4.4}$$

Proof. See Appendix 2.C. □

For two-person problems, Proposition 2.4 provides a nested condition based on Proposition 2.1 and 2.2. By setting $\overline{B}_i,\underline{B}_i = \emptyset$ for $i = 1,2$, we have the condition in Proposition 2.1. By setting $A_i = \emptyset$ for $i = 1,2$, we have the condition in Proposition 2.2.

Example 2.3: Let T and λ be specified as in Example 2.1. In this case, a symmetric allocation rule requires $q(1,(x,y)) = q(2,(y,x))$ and $q(b,(x,y)) = q(b,(y,x))$ for all $x,y \in \{0,1\}$. By symmetry, $Q_1(1,\cdot) = Q_2(2,\cdot) = Q_a$ and $Q_1(b,\cdot) = Q_2(b,\cdot) = Q_b$. From Proposition 2.4, the bound of the set of implementable reduced forms is given by (2.3.6), (2.3.7), (2.3.8), (2.3.9), (2.3.11), (2.3.12), (2.3.13), and

$$Q_{b,0}\lambda_0 + 2Q_{a,0}\lambda_0 \le 1 - \lambda_1^2, \tag{2.4.5}$$

$$Q_{b,1}\lambda_1 + 2Q_{a,1}\lambda_1 \le 1 - \lambda_0^2, \tag{2.4.6}$$

$$Q_{b,0}\lambda_0 + Q_{a,0}\lambda_0 + Q_{a,1}\lambda_1 \le 1 - \lambda_0\lambda_1, \tag{2.4.7}$$

$$Q_{b,1}\lambda_1 + Q_{a,0}\lambda_0 + Q_{a,1}\lambda_1 \le 1 - \lambda_0\lambda_1, \tag{2.4.8}$$

$$Q_{b,0}\lambda_0 + Q_{b,1}\lambda_1 + 2Q_{a,1}\lambda_1 + 2Q_{a,1}\lambda_1 \le 1. \tag{2.4.9}$$

Compared to the auction and voting problems, the package exchange problem requires many more inequalities. These linear inequalities form a 4-polytope. For $\lambda_0 = 1/2$, the system of inequalities is given by

$$
\begin{aligned}
&0 \leq Q_{a,0}, Q_{a,1} \leq 3/4, \\
&0 \leq Q_{b,0}, Q_{b,1} \leq 1, \\
&Q_{a,0} + Q_{a,1} \leq 1, \\
&Q_{b,0} - Q_{b,1} \leq 1/2, \\
&Q_{b,1} - Q_{b,0} \leq 1/2, \\
&Q_{b,0} + 2Q_{a,0} \leq 3/2, \\
&Q_{b,1} + 2Q_{a,1} \leq 3/2, \\
&Q_{b,0} + Q_{a,0} + Q_{a,1} \leq 3/2, \\
&Q_{b,1} + Q_{a,0} + Q_{a,1} \leq 3/2, \\
&Q_{b,0} + Q_{b,1} + 2Q_{a,0} + 2Q_{a,1} \leq 2.
\end{aligned}
$$

By Fourier-Motzkin elimination,[5] we reduce one of the four variables and compute the projections of this 4-polytope onto the three-dimensional subspaces: (1) $(Q_{b,1}, Q_{a,0}, Q_{a,1})$, (2) $(Q_{a,0}, Q_{a,1}, Q_{b,0})$, (3) $(Q_{a,1}, Q_{b,0}, Q_{b,1})$, and (4) $(Q_{b,0}, Q_{b,1}, Q_{a,0})$. These projections correspond to four 3-polytopes. For $\lambda_0 = 1/2$, we depict them in Figure 2.3 below.

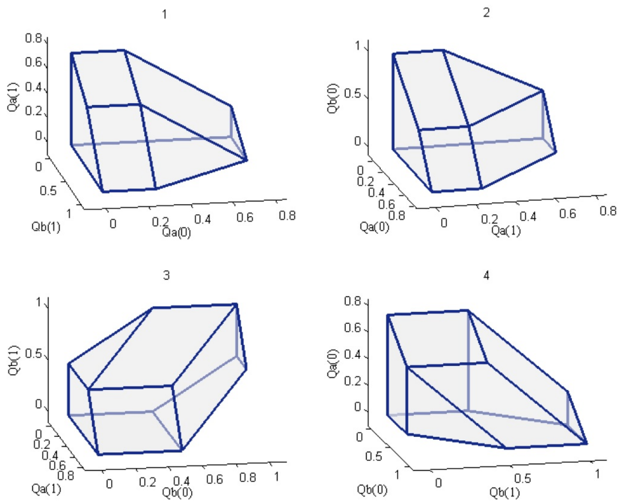


Figure 2.3

2.4.4.1. Ex Ante Efficient Solutions to Package Allocation Problems

In the package exchange problem, we further assume that the symmetric valuation profile is given by for each player $i = 1, 2$,

[5]The elimination of a set of variables $\mathcal{V}$ from a system of linear inequalities refers to the creation of another system of linear inequalities, but without the variables in $\mathcal{V}$, such that both systems have the same solutions over the remaining variables.

	0	i	j	b
$t_i = 0$	0	1	0	x
$t_i = 1$	0	2	0	y

where $(x, y) \in \mathbb{R}^2$ are parameters. We assume that for any type of a player, the valuation of the objects are complementary. We consider four cases where the players may be better off or worse off from exchanging their own objects, depending on (x, y): $(1, 1), (-1, 1), (1, -1), (-1, 1)$.

With the necessary condition (2.4.4), we are ready to characterize the bound that contains the set of incentive feasible interim utility allocations. Now, the (symmetric) interim utility vector, incentive compatibility and individual rationality conditions are given by

$$U_0 = Q_{a,0} + xQ_{b,0} \geq Q_{a,1} + xQ_{b,1}, \tag{2.4.10}$$

$$U_1 = 2Q_{a,1} + yQ_{b,1} \geq 2Q_{a,0} + yQ_{b,0}, \tag{2.4.11}$$

$$U_0 \geq 0, U_1 \geq 0. \tag{2.4.12}$$

Together with (2.4.4), we have a linear system of six variables,

$$(Q_{b,0}, Q_{b,1}, Q_{a,0}, Q_{a,1}, U_0, U_1) \in \mathbb{R}^6.$$

By Fourier-Motzkin elimination, we reduce the first four variables and obtain the bound of the interim utility set $\mathcal{U}(x, y)$ given by

$$\begin{aligned}
\mathcal{U}(1,1) &= \{U \in \mathbb{R}^2_+ : U_0 - U_1 \leq 0,\ \ U_0 + U_1 \leq 2, -\frac{5}{2}U_0 + U_1 \leq 0,\ \ U_1 \leq \frac{5}{4}\ \},\\
\mathcal{U}(-1,1) &= \{U \in \mathbb{R}^2_+ : 2U_0 - U_1 \leq 0,\ \ 2U_0 + U_1 \leq 2,\ \ U_1 \leq \frac{3}{2}\ \},\\
\mathcal{U}(1,-1) &= \{U \in \mathbb{R}^2_+ : -2U_0 + U_1 \leq 0,\ \ 2U_0 + U_1 \leq \frac{5}{2},\ \ U_0 \leq 1\ \},\\
\mathcal{U}(-1,-1) &= \{U \in \mathbb{R}^2_+ : -\frac{4}{3}U_0 + U_1 \leq \frac{1}{2},\ \ 2U_0 - U_1 \leq 0, -\frac{2}{3}U_0 + U_1 \leq \frac{2}{3}\ \}.
\end{aligned}$$

For each constellation of (x, y), the upper bound of the ex ante (utilitarian) efficient allocations are defined as the solutions to $\max_{U \in \mathcal{U}(x,y)} U_0 + U_1$. Figure 2.4 illustrates the bounds of interim utility sets and the corresponding upper bounds of the ex ante efficient solutions in the interim utility space.

In all cases, $(U_0, U_1) = (\frac{1}{2}, 1)$ is incentive feasible, i.e. reallocating the bundle by tossing a coin. The incentive conditions (2.4.10), (2.4.11), and the feasibility condition

$$Q_{b,0} + Q_{b,1} + 2Q_{a,0} + 2Q_{a,1} \leq 2, \tag{2.4.13}$$

are binding. In the cases $(-1,1),(-1,-1)$, this utility vector is on the strong Pareto set, but in the cases $(1,1)$, $(1,-1)$, it is Pareto dominated. This is because in the latter cases, reporting type 1 is punished more severely by exchanging the objects and this sustains a better outcome.

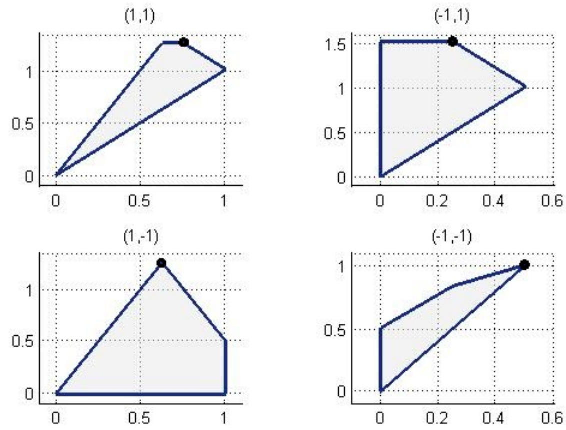


Figure 2.4

In the case $(1,1)$, the ex ante efficient solutions are given by the line segment between $(\frac{3}{4},\frac{5}{4})$ and $(1,1)$. In the cases $(-1,1)$, $(1,-1)$, $(-1,-1)$, the solutions are uniquely given by $(\frac{1}{4},\frac{3}{2})$, $(\frac{5}{8},\frac{5}{4})$, and $(\frac{1}{2},1)$, respectively. We provide the systems of reduced forms that implement these utility vectors.

Case $(1,1)$. The utility vector $(\frac{3}{4},\frac{5}{4})$ is implemented by

$$(Q_{a,0},Q_{a,1},Q_{b,0},Q_{b,1})=(0,\frac{1}{2},\frac{3}{4},\frac{1}{4}).$$

The feasibility conditions (2.4.13) and $Q_{b,0}-Q_{b,1}\leq 1/2$ are binding.

Case $(-1,1)$. The utility vector$(\frac{1}{4},\frac{3}{2})$ is implemented by

$$(Q_{a,0},Q_{a,1},Q_{b,0},Q_{b,1})=(\frac{1}{8},\frac{1}{2},0,\frac{1}{2}).$$

The feasibility conditions (2.4.13) and $Q_{b,1}+2Q_{a,1}\leq 3/2$ are binding.

Case $(1,-1)$. The utility vector $(\frac{5}{8},\frac{5}{4})$ is implemented by

$$(Q_{a,0},Q_{a,1},Q_{b,0},Q_{b,1})=(\frac{1}{4},\frac{5}{8},\frac{1}{2},0).$$

Now the feasibility condition (2.4.13) and the incentive condition (2.4.10) are binding.

Case $(-1,-1)$. The utility vector $(\frac{1}{2},1)$ is implemented by

$$(Q_{a,0},Q_{a,1},Q_{b,0},Q_{b,1})=(\frac{1}{2},\frac{1}{2},0,0).$$

It turns out that all these reduced forms are implementable. Hence, for these cases, the necessary condition (2.4.4) is necessary and sufficient for the implementability. We provide feasible allocation rules that implement these reduced forms in Table 2.1. For all cases and all type profiles, (q_0, q_1, q_2, q_b) are listed. For all cases except $(-1, -1)$, the solution allocation rule is unique.

$(1,1)$	0	1
0	$(0,0,0,1)$	$(0,0,\frac{1}{2},\frac{1}{2})$
1	$(0,\frac{1}{2},0,\frac{1}{2})$	$(0,\frac{1}{2},\frac{1}{2},0)$

$(-1,1)$	0	1
0	$(0,\frac{1}{2},\frac{1}{2},0)$	$(0,0,1,0)$
1	$(0,1,0,0)$	$(0,0,0,1)$

$(1,-1)$	0	1
0	$(0,0,0,1)$	$(0,\frac{1}{4},\frac{3}{4},0)$
1	$(0,\frac{3}{4},\frac{1}{4},0)$	$(0,\frac{1}{2},\frac{1}{2},0)$

$(-1,-1)$	0	1
0	$(0,\frac{1}{2},\frac{1}{2},0)$	$(0,\frac{1}{2},\frac{1}{2},0)$
1	$(0,\frac{1}{2},\frac{1}{2},0)$	$(0,\frac{1}{2},\frac{1}{2},0)$

Table 2.1

Observe that in the cases $(1,1)$ $(1,-1)$, and $(-1,-1)$, the solution allocation rules are stochastic for asymmetric reports. The result shows that for mechanism design problems without monetary transfers, some extreme points of the interim utility set correspond to non-deterministic allocation rules.

2.5. Discussion

In this section, we provide some remarks about the results in Sections 2.3 and 2.4. First, we consider social choice problems with money and with general utility functions in which the implementation problems we studied arise. Second, we compare the implementability of reduced form allocation rules and reduced form values in the literature. Finally, we investigate reduction of redundant inequalities in a package allocation example.

2.5.1. Non-Quasilinear Utility

A social choice problem with transfers is given by $S = (I, (v_i)_{i\in N})$, where player $i \in N$ has a valuation function $v_i : D \times \mathbb{R} \times T_i \to \mathbb{R}$ given by $v_i(d, m, t_i)$ for $d \in D$, monetary payment $m \in \mathbb{R}$, and $t_i \in T_i$. Fix such an S. An allocation mechanism $\mu = (q, m)$ is given by a feasible allocation rule $q : T \to \Delta(D)$ together with a set of payment rules $m = (m_i)_{i\in N}$, where $m_i : D \times T \to \mathbb{R}$ for all $i \in N$. Notice that the payment rules depend on both the reporting profiles and the decisions. Given that the other players always report truthfully, the interim expected utility of player i with type $t_i \in T_i$ from reporting $\hat{t}_i \in T_i$ is given by

$$U_i(\mu, \hat{t}_i|t_i) = \sum_{t_{-i}\in T_{-i}} \sum_{d\in D} q(d, (\hat{t}_i, t_{-i}))v_i(d, m_i(d, (\hat{t}_i, t_{-i})), t_i)\lambda_{-i}(t_{-i}). \tag{2.5.1}$$

For a social choice problem with transfers, a mechanism design problem is defined similarly as that for a social choice problem without transfers.

There are some classes of social choice problems with money in which the implementation problems (with or without full dimensionality) arise as subproblems. This depends on assumptions on the valuation functions and the restriction of payment rules. We discuss two classes: (1) Quasilinear utility; and (2) Nonlinear utility, deterministic payments, and no ex post balanced budget.

1. Quasilinear utility. For each player $i \in N$, $v_i(d, m, t_i) = \tilde{v}_i(d, t_i) - m$ for some function $\tilde{v}_i : D \times T_i \to \mathbb{R}$. For $\mu = (q, m)$, we can set $m_i(d, t) = m_i(t)$ for all $d \in D$, $t \in T$ and $i \in N$. For each player i with type t_i, the interim utility from μ is given by

$$U_i(\mu|t_i) = \sum_{d \in D} Q_i(d, t_i)\tilde{v}_i(d, t_i) - M_i(t_i), \tag{2.5.2}$$

where

$$Q = \Lambda q \ \text{ and } \ M_i(t_i) = \sum_{t_{-i} \in T_{-i}} m_i(t)\lambda_{-i}(t_{-i}). \tag{2.5.3}$$

This environment is the same as Jehiel, Moldovanu and Stacchetti (1999), when one buyer imposes an externality on the other buyers in a single-object auction. Notice that the separability of the expected probabilities and the payments continues to hold when players have budget constraints. Vohra (2011) uses the reduced form approach to study the single object auctions with budget-constrained buyers. The condition here can be used to study the multi-unit auctions with budget-constrained buyers.

2. Non-Quasilinear utility, deterministic payments, and no ex post balanced budget. We assume each player's payment rule is *deterministic*, i.e. for all $d \in D$, $t, t' \in T$, and $i \in N$,

$$m_i(d, t) = m_i(d, t'), \ \text{if} \ t_i = (t')_i. \tag{2.5.4}$$

In other words, each player's payment is independent of other players' reports. For player $i \in N$ with type $t_i \in T_i$, the interim expected payment at $d \in D$ is given by $M_i(d, t_i) = m_i(d, t)$ for all $t \in T$. Hence, for each player i with type t_i, the interim utility from the mechanism μ is given by

$$U_i(\mu|t_i) = \sum_{d \in D} Q_i(d, t_i) v_i(d, M_i(d, t_i), t_i), \ \text{ where } \ Q = \Lambda q. \tag{2.5.5}$$

In this case, the separability of the expected probabilities and the composition of valuation functions and payments holds.

To see that the requirement on no ex post budget balance cannot be dropped, consider the bilateral trade problem of Myerson and Satterthwaite (1983). Let $N = D = \{1, 2\}$ and let T_1, T_2 be finite subsets of $[0, 1]$. The seller (player 1) is risk neutral and has linear utility, i.e. for all m and t_1, $v_1(1, m, t_1) = t_1 - m$ and $v_1(2, m, t_1) = -m$. Now, the buyer (player 2) is constant absolute risk averse, i.e. for all $m \in \mathbb{R}$ and $t_2 \in T_2$,

$$v_2(2, m, t_2) = 1 - e^{-(t_2 - m)} \text{ and } v_2(1, m, t_2) = 1 - e^{m}. \tag{2.5.6}$$

Assume the payment rule for the buyer is deterministic and is given by $(M_2(1, \cdot), M_2(2, \cdot))$. The payment rule for the seller is given by $M_1(\cdot)$. An ex post budget balanced payment rule requires that for all $t_1 \in T_1$,

$$M_1(t_1) = -\sum_{t_2 \in T_2} \sum_{d \in D} q(d, t) M_2(d, t_2) \lambda_2(t_2). \tag{2.5.7}$$

Hence, if ex post balanced budget is required, the joint feasibility condition on allocation rules and payment rules becomes a relevant issue.

The deterministic payment is less restrictive in case of independent beliefs than correlated beliefs. For example, Maskin and Riley (1984) and Matthews (1983) show that for a subclass of constant relative risk averse utility functions, the optimal single-unit auctions have the deterministic payments. Crémer and McLean (1988) shows that for risk neutral buyers, the optimal single object auctions with correlated beliefs require correlated payments.

2.5.2. Comparison to Reduced Form Values

Goeree and Kushnir (2013) is the first paper to discuss the implementability of reduced forms for social choice problems with Quasilinear utility and interdependent values. They define the reduced form value for each player by taking the sum of interim expected values (over all alternatives) generated by the products of valuation functions and allocation probabilities. A system of reduced form values contains one reduced form value function for each player. In the following discussion, we restrict the comparison to private values.

1. Quasilinear utility. For all $i \in N$, $v_i(d, m, t_i) = \tilde{v}_i(d, t_i) - m$ for some function $\tilde{v}_i$. Define a linear map $\Phi_v : \mathbb{R}^k \to \times_{i \in N} \mathbb{R}^{T_i}$ by, for any $x \in \mathbb{R}^k$,

$$(\Phi_v x)_i(t_i) = \sum_{t_{-i} \in T_{-i}} \sum_{d \in D} x(d, t) \tilde{v}_i(d, t_i) \lambda_{-i}(t_{-i}), \tag{2.5.8}$$

for all $t_i \in T_i$ and $i \in N$. A system of reduced form values is given by $V = (V_i)_{i \in N} \in \times_{i \in N} \mathbb{R}^{T_i}$. Then, V is implementable if there exists $q \in \mathcal{D}_0$ such that $V = \Phi_v q$.

Since a system of reduced form values is a lower-dimensional object than a system of reduced form allocation rules, Theorem 2.1 implies their Proposition 2.1, by setting $f_i(d, t_i) = \tilde{f}_i(t_i) \tilde{v}_i(d, t_i)$ for all $d \in D$, $t_i \in T_i$, $i \in N$, for some $\tilde{f} \in \times_{i \in N} \mathbb{R}^{T_i}$, and then varying $\tilde{f}$. In contrast, Theorem 2.1 is obtained from their reduced form value implementability conditions by further varying valuation functions and hence social choice problems.

The advantages of our approach are: (i) Theorem 2.1 does not need any information on valuation functions and has fewer data as input. Hence, the same condition applies to all problems; (ii) The set of implementable reduced form allocation rules is easier to compute, because it does not depend on valuation functions. A disadvantage is that we have more reduced form variables and thus more linear inequalities. At present, which model achieves a lower computational complexity remains to be investigated.[6]

2. Non-Quasilinear utility and deterministic payment rules. Let $m = (m_i)_{i\in N}$ be a system of deterministic payment rules. Define a linear map $\Phi_{v,m} : \mathbb{R}^k \to \times_{i\in N}\mathbb{R}^{T_i}$ by, for any $x \in \mathbb{R}^k$,

$$(\Phi_{v,m}x)_i(t_i) = \sum_{t_{-i}\in T_{-i}} \sum_{d\in D} x(d,t)v_i(d, M_i(d,t_i), t_i)\lambda_{-i}(t_{-i}), \tag{2.5.9}$$

for all $t_i \in T_i$ and $i \in N$.

A system of reduced form utilities is given by $U = (U_i)_{i\in N} \in \times_{i\in N}\mathbb{R}^{T_i}$. U is implementable if there exists $q \in \mathcal{D}_0$ such that $U = \Phi_{v,m}q$. Because $\Phi_{v,m}$ is parameterized by m, the feasibility condition on U also contains m. In contrast to the reduced form utility approach, the feasibility condition on Q is parameterized by neither v nor m.

If no balanced budget or only ex ante balanced budget is required, then the reduced form utility approach requires working with feasibility conditions on allocation rules and payments simultaneously. On the other hand, the reduced form allocation rule approach has the advantage of the separation of the feasibility condition on allocation rules and payments.

2.5.3. Reduction of Inequalities

Border (1991) investigates a further reduction of (2.3.5) in Proposition 2.1, by the class of the upper contour sets of reduced forms instead of all characteristic functions (all measurable subsets). For single object auctions without externalities, Border (1991)'s Proposition 2.3.2 proves that such smaller class is sufficient. For two-alternative voting problems, we can show that such smaller class is also sufficient, by referring to a proof of Theorem 4 of Gutmann et al. (1991).

In contrast to single object auctions without externalities or voting problems, we observe that in the package exchange example, Gutmann et al. (1991)'s approach cannot be applied here to obtain a further reduction.

To see this, we consider a subclass of inequalities of (2.4.4). Let T_1 and T_2 be finite subsets in $\mathbb{R}$. Fix $(A_1, A_2, \underline{B}_1, \overline{B}_2, \underline{B}_2, Q)$ such that $\underline{B}_1 = \overline{B}_2 = \emptyset$ and vary $\overline{B}_1$. Denote $B_1 = \overline{B}_1$, $B_2 = (\underline{B}_2)^c$. We now compute the boundary subset B_1 such that (2.4.4) holds

[6] In algorithmic mechanism design and computer science literature, some recent papers discuss the computational complexity of Border's theorem, e.g. Gopalan, Nisan and Roughgarden (2015).

with equality. Then, consider

$$\begin{aligned}
\max_{B_1 \in 2^{T_1}} \mathcal{J}(B_1) =& \sum_{i\in N}\sum_{t_i \in A_i} Q_i(i,t_i)\lambda_i(t_i) + \sum_{t_1\in B_1} Q_1(b,t_1)\lambda_1(t_1) - \sum_{t_2 \in (B_2)^c} Q_2(b,t_2)\lambda_2(t_2) \\
&- \lambda(\bigcup_{i\in N}(A_i \times T_{-i}) \bigcup B_1 \times B_2) \\
=& \sum_{t_1\in B_1} Q_1(b,t_1)\lambda_1(t_1) - \lambda(B_1 \setminus A_1 \times B_2 \setminus A_2) - \lambda(\bigcup_{i\in N} A_i \times T_{-i}) \qquad (2.5.10) \\
&+ \sum_{i\in N}\sum_{t_i\in A_i} Q_i(i,t_i)\lambda_i(t_i) - \sum_{t_2\in(B_2)^c} Q_2(b,t_2)\lambda_2(t_2) \\
=& \sum_{t_1\in B_1\setminus A_1} [Q_1(b,t_1) - \lambda_2(B_2\setminus A_2)]\lambda_1(t_1) + \sum_{t_1 \in B_1\cap A_1} Q_1(b,t_1)\lambda_1(t_1) + \mathit{constant}.
\end{aligned} \tag{2.5.11}$$

The solution is given by

$$B_1^*(A_1, A_2, B_2, Q) := A^* \cup A_1, \tag{2.5.12}$$

where

$$A^* = \{t_1 \in T_1 : Q_1(b,t_1) \geq \lambda_2(B_2 \setminus A_2)\}. \tag{2.5.13}$$

Since $Q_1(b,t_1)$ and A_1 can be arbitrary, if A^* and A_1 are "disconnected", i.e. there exists $t_1 \in T_1$ such that

$$\max A^* < t_1 < \min A_1, \tag{2.5.14}$$

then B_1^* may not be of the form $\{t_1 \in T_1 : Q_1(b,t_1) \geq \beta_1\}$ for some $\beta_1 \in \mathbb{R}$. For example, let $Q_1(b,\cdot)$ be strictly increasing in t_1. Then,

$$A^* \subset [(Q_1(b,\cdot))^{-1}(\lambda_2(B_2\setminus A_2)), \max T_1]. \tag{2.5.15}$$

For $A_1 = \{\min T_1\}$ and $A^* \cup A_1 \neq T_1$, B_1^* is "disconnected". On the other hand, $Q_1(b,\cdot)$ is strictly monotone and all its upper contour sets are "connected".

2.6. Conclusion

In this chapter, we obtain a characterization of the implementability conditions for social choice problems with vector-valued reduced forms. We provide a necessary and sufficient condition for the implementability and a necessary condition by a class of finitely many linear inequalities. We also provide a characterization for the implementability on coordinate subspaces. We then use these conditions to study mechanism design problems without money given certain welfare objectives. In a two-person two-object allocation example, we show how the implementability condition can be used to find the bound of incentive feasible interim utility allocations and the upper bound of the ex ante utilitarian social welfare.

The results in this chapter provide an important intermediate step to find the solutions to mechanism design problems for a wide class of social choice environments, when the ex post allocation rules cannot be easily optimized and the implementation problems arise as subproblems. These include package allocations with complementary valuations and auctioning multiple objects with risk averse buyers.

Finally, the implementability condition requires a large number of inequalities and the reduction of redundant inequalities is still possible for specific problems. As the cardinalities of the type sets increase, the number of inequalities increases very quickly and the computational burden becomes a highlighted issue. We leave these interesting problems for future research.

Appendix 2.A Proof of Theorem 2.1

LEMMA 2.1: *(i)* $\mathcal{D}_0$ *is convex and compact.*
(ii) $\Lambda : \mathbb{R}^k \to \mathbb{R}^l$ *and* $\Gamma : \mathbb{R}^l \to \mathbb{R}^l$ *are continuous.*
(iii) $\mathcal{D}$ *and* $\Gamma\mathcal{D}$ *are convex and compact.*

Proof of Lemma 2.1. (i) $\mathcal{D}_0$ is obviously convex. $q \in \mathcal{D}_0$ if and only if $q(\cdot, t) \in \Delta(D)$ for all $t \in T$. Since T is finite, $\mathcal{D}_0 = \prod_{t \in T} \Delta(D)$ is a product of finitely many compact subsets $\Delta(D) \subset \mathbb{R}^{|D|}$. By Tychonoff's theorem, $\mathcal{D}_0$ is compact. (ii) Λ and Γ are linear maps on finite-dimensional spaces and thus continuous. (iii) By the continuous mapping theorem, the continuous image of a compact set is compact. Hence $\mathcal{D}$ is compact. Since $\mathcal{D}_0$ is convex and Λ is linear, $\mathcal{D}$ is convex. A similar analysis implies that $\Gamma\mathcal{D}$ is also compact and convex. $\square$

Proof of Theorem 2.1. (Only If) Suppose $Q \in \mathcal{D}$, then there exists $q \in \mathcal{D}_0$ that implements Q. Since $q \in \mathcal{D}_0$, we have $\langle f, \Gamma Q \rangle$ is bounded above by $\sup_{q \in \mathcal{D}_0} \langle f, \Gamma \Lambda q \rangle$, for all $f \in \mathbb{R}^l$.

(If) We show that if $Q^* \notin \mathcal{D}$, then Q^* must violate condition (2.3.2) for some $f \in \mathbb{R}^l$. Now suppose $Q^* \notin \mathcal{D}$. $\lambda_i(t_i) > 0$ for all $t_i \in T_i$ and $i \in N$ implies $\Gamma Q^* \notin \Gamma\mathcal{D}$. Since $\Gamma\mathcal{D}$ is nonempty, closed and convex, by a hyperplane separation theorem, there exists $f \in \mathbb{R}^l$ such that $\langle f, \Gamma Q^* \rangle > \sup_{Q \in \mathcal{D}} \langle f, \Gamma Q \rangle$, which contradicts to (2.3.2). $\square$

Appendix 2.B Sufficient Condition: A Digraph Multicommodity Flow Problem

An implementation problem $I = (N, D, (T_i)_{i \in N}, (\lambda_i)_{i \in N})$ with $|N| = 2$ can be formulated by constructing a digraph network flow problem. The elements of network flow problems are given in Appendix 2.D.

(i) If $|D| = 2$, (I) corresponds to a 1-commodity flow problem. An application of the max flow-min cut theorem implies (2.3.3) in Corollary 2.1 is sufficient for the implementability. Moreover, the max flow-min cut theorem provides a much tighter condition than (2.3.3).

(ii) If $|D| \geq 3$, (I) corresponds to a $(|D|-1)$-commodity flow problem. Due to the lack of max flow-min cut theorems for such problems, it is unclear whether (2.3.3) is sufficient for the implementability.

We now formulate (I) pinto a multicommodity flow problem. Let $D = \{0, 1, ..., n\}$. First, single out $d = 0$ and define by change of variables for all $d = 1, ..., n$,[7]

$$\begin{aligned} F_d(t_1, t_2) =& q(d, (t_1, t_2))\lambda(t_1, t_2), \quad && \forall (t_1, t_2) \in T, \\ u_d(t_1) =& \lambda_1(t_1)Q_1(d, t_1), && \forall t_1 \in T_1, \\ v_d(t_2) =& \lambda_2(t_2)Q_2(d, t_2), && \forall t_2 \in T_2. \end{aligned}$$

Lemma 2.2 below shows that (I) is equivalent to the following problem $(I1)$.

$(I1)$. Given $(u_d, v_d)_{d=1}^n \in \mathbb{R}_+^{|T_1|\times|T_2|\times n}$ and $\sum_{t_1\in T_1} u_d(t_1) = \sum_{t_2\in T_2} v_d(t_2)$, for all $d = 1, ..., n$, find a feasible solution $F \in \mathbb{R}^{|T|\times n}$ such that for all $d = 1, ..., n$,

$$\begin{aligned} 0 \leq \sum_{d=1}^n F_d(t_1, t_2) &\leq \lambda(t_1, t_2), \qquad && \forall (t_1, t_2) \in T, \\ \sum_{t_2\in T_2} F_d(t_1, t_2) &= u_d(t_1), && \forall t_1 \in T_1, \\ \sum_{t_1\in T_1} F_d(t_1, t_2) &= v_d(t_2), && \forall t_2 \in T_2. \end{aligned}$$

Lemma 2.2: *Let $N = \{1, 2\}, D = \{0, 1, ..., n\}$, with $n \geq 1$, then (I) is equivalent to $(I1)$.*

Proof of Lemma 2.2. For $Q \in \mathcal{D}_1$ to be implementable, we have for all $d = 0, ..., n$,

$$\sum_{t_1\in T_1} \lambda_1(t_1)Q_1(d, t_1) = \sum_{t_2\in T_2} \lambda_2(t_2)Q_2(d, t_2). \tag{2.B.1}$$

Then we can restrict attention to all $Q \in \mathcal{D}_1$ that satisfy this condition. Also notice that

$$\begin{aligned} q(0, t) &= 1 - \sum_{d=1}^n q(d, t), \forall t \in T, \\ Q_i(0, t_i) &= 1 - \sum_{d=1}^n Q_i(d, t_i), \forall t_i \in T_i, i = 1, 2. \end{aligned}$$

Hence, (I) is equivalent to the following problem $(I2)$.

$(I2)$. Given $(Q_1(d, \cdot), Q_2(d, \cdot))_{d=1}^n \in \mathbb{R}_+^{|T_1|\times|T_2|\times n}$ and (2.B.1), find a feasible solution

[7] We abuse notation to denote a valuation function by v_i and a utility function by u_i for player i.

$(q(d,\cdot))_{d=1}^n \in \mathbb{R}^{|T|\times n}$ such that for all $d = 1, ..., n$,

$$\begin{aligned} 0 \leq \sum_{d=1}^{n} q(d,t) \leq 1, \forall t \in T, \\ \sum_{t_2 \in T_2} q(d,t)\lambda_2(t_2) = Q_1(d,t_1), \forall t_1 \in T_1, \\ \sum_{t_1 \in T_1} q(d,t)\lambda_1(t_1) = Q_2(d,t_2), \forall t_2 \in T_2. \end{aligned}$$

By change of variables, for $d = 1, ..., n$, let

$$q(d,\cdot) \to F_d, \;\; Q_1(d,\cdot) \to u_d, \;\; Q_2(d,\cdot) \to v_d.$$

It is easy to see that $(I2)$ is equivalent to $(I1)$. $\square$

Now let $\{1, ..., n\}$ be the commodities and construct a supply digraph $G_n = (V, A)$ with vertexes

$$V = \cup_{d=1}^n \{r_d\} \cup T_1 \cup T_2 \cup_{d=1}^n \{s_d\}, \tag{2.B.2}$$

and arcs A from each r_d to each t_1, from each t_1 to each t_2, and from each t_2 to each s_d.

The corresponding demand digraph $H_n = (V', R)$ is given by

$$V' = \{r_1, s_1, ..., r_n, s_n\}, \tag{2.B.3}$$

and $R = \{(r_d, s_d)_{d=1}^n\}$, where R contains all source-sink pairs of the commodities. Figure 2.B.1 illustrates (G_2, H_2) for $T_1 = \{t_1, t_1'\}$ and $T_2 = \{t_2, t_2'\}$.

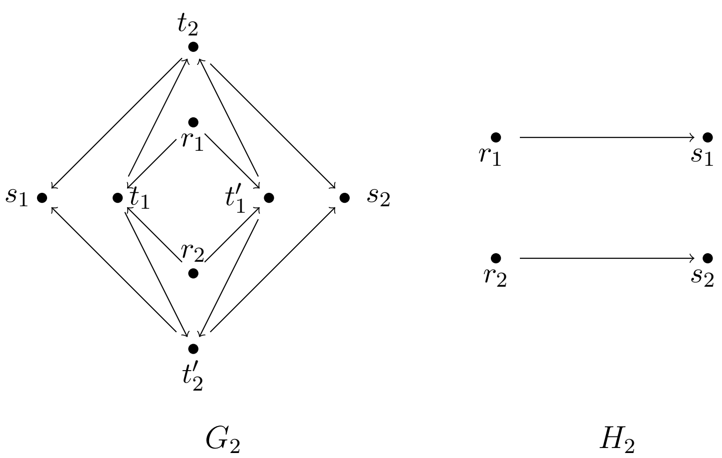


Figure 2.B.1

Now the capacity function $c : A \to \mathbb{R}_+$ is defined by for all $d = 1, ..., n$,

$$\begin{aligned} c(t_1, t_2) &= \lambda(t_1, t_2), \quad && \forall (t_1, t_2) \in T, \\ c(r_d, t_1) &= u_d(t_1), && \forall t_1 \in T_1, \\ c(t_2, s_d) &= v_d(t_2), && \forall t_2 \in T_2. \end{aligned}$$

The demand function $\phi : R \to \mathbb{R}_+$ is defined by for all $d = 1, ..., n$,

$$\phi(r_d, s_d) = \sum_{t_1 \in T_1} u_d(t_1) = \sum_{t_2 \in T_2} v_d(t_2). \tag{2.B.4}$$

A flow[8] $f = (f_d)_{d=1}^n$ with $f_d : A \to \mathbb{R}_+$, $d = 1, ..., n$, satisfies

$$0 \leq \sum_{d=1}^{n} f_d(a) \leq c(a), \forall a \in A, \tag{2.B.5}$$

$$\sum_{a \in \delta^{in}(k)} f_d(a) = \sum_{a \in \delta^{out}(k)} f_d(a), \forall k \in V, k \neq r_d, s_d, d = 1, ..., n, \tag{2.B.6}$$

where $\delta^{in}(k)$ is the set of arcs entering k and $\delta^{out}(k)$ is the set of arcs leaving k.

The value of a flow f is given by

$$val(f_d) = f_d(\delta^{out}(r_d)) = f_d(\delta^{in}(s_d)), d = 1, ..., n. \tag{2.B.7}$$

A *feasible* flow is a flow f with value

$$val(f_d) = \phi(r_d, s_d), d = 1, ..., n. \tag{2.B.8}$$

The n-commodity flow problem (G_n, H_n, c, ϕ) is to find a feasible flow. For $U \subseteq V$, denote by $cap(\delta_A^{out}(U))$ the total capacity of arcs of A leaving U and by $\phi(\delta_R^{out}(U))$ the total demands of arcs of R leaving U.

LEMMA 2.3: $(I1)$ *has a feasible solution if and only if for the problem* (G_n, H_n, c, ϕ) *there exists a feasible flow.*

Proof of Lemma 2.3. (Only If) Suppose $(I1)$ has a solution F^*. Define $f = (f_d)_{d=1}^n$ by

$$\begin{aligned} f_d(t_1, t_2) &= F_d^*(t_1, t_2), \ \ \forall (t_1, t_2) \in T, \\ f_d(r_d, t_1) &= u_d(t_1), \forall t_1 \in T_1, \\ f_d(t_2, s_d) &= v_d(t_2), \forall t_2 \in T_2, \end{aligned}$$

[8] We abuse notation to denote a flow by f. In other parts in the chapter, f is used as an element of $\mathbb{R}^l$.

$$f_d(r_j, t_1) = 0 \text{ if } j \neq d, \forall t_1 \in T_1,$$
$$f_d(t_2, s_j) = 0 \text{ if } j \neq d, \forall t_2 \in T_2.$$

Then it is easy to see that f is a flow and it is a solution to (2.B.8).

(If) Suppose there exists a flow f^* satisfying (2.B.8). For any flow f, $val(f_d) \leq \sum_{t_1 \in T_1} u_d(t_1) = \sum_{t_2 \in T_2} v_d(t_2)$, for all $d = 1, ..., n$. Then, f^* must satisfy

$$f_d^*(r_d, t_1) = u_d(t_1) = \sum_{t_2 \in T_2} f_d^*(t_1, t_2), \ \ \forall t_1 \in T_1,$$
$$f_d^*(t_2, s_d) = v_d(t_2) = \sum_{t_1 \in T_1} f_d^*(t_1, t_2), \ \ \forall t_2 \in T_2,$$
$$0 \leq \sum_{d=1}^{n} f_d^*(t_1, t_2) \leq c(t_1, t_2), \ \ \forall (t_1, t_2) \in T.$$

Hence, f^* restricted to arcs in T is a solution to $(I1)$. □

The following result shows that the cut condition of the multicommodity flow problem (G_n, H_n, c, ϕ) is a subclass of linear inequalities in condition (2.3.3). It translates the proof of sufficiency of condition (2.3.3) in a two-person implementation problem into the proof of a generalized max flow-min cut theorem in a specific digraph. On the other hand, there is no existing maximum flow-min cut theorem for digraph multicommodity flow problems except for some very specific digraphs, see Schrijver (2013).

Lemma 2.4: *Let $N = \{1, 2\}$ and $D = \{0, 1, ..., n\}$, $n \geq 1$. Consider the n-commodity digraph flow problem (G_n, H_n, c, ϕ).*

(i) The cut condition

$$\phi(\delta_R^{out}(U)) \leq cap(\delta_A^{out}(U)), \ \ \textit{for all} \quad U \subseteq V, \tag{2.B.9}$$

is a subclass of (2.3.3)*, and it is equivalent to*

$$\sum_{d \in D'} [\sum_{t_1 \in B_1} u_d(t_1) - \sum_{t_1 \in T_1 \backslash B_1} u_d(t_1) + \sum_{t_2 \in B_2} v_d(t_2) - \sum_{t_2 \in T_2 \backslash B_2} v_d(t_2)] \leq 2 \sum_{t_1 \in B_1, t_2 \in B_2} \lambda(t_1, t_2), \tag{2.B.10}$$

for all $D' \subseteq \{1, ..., n\}$, $B_1 \subseteq T_1, B_2 \subseteq T_2$.

(ii) If the cut condition (2.B.9) *is necessary and sufficient for a feasible flow, then* (2.B.9) *is necessary and sufficient for the implementability.*

Proof of Lemma 2.4. We first prove part (i). To see this, note that for any $U \subseteq V$,

$$\phi(\delta_R^{out}(U)) = \sum_{d \in D'} \phi(r_d, s_d) = \sum_{d \in D'} \sum_{t_1 \in T_1} u_d(t_1), \tag{2.B.11}$$

where $D' = \{d \in \{1, ..., n\} : r_d \in U, s_d \notin U\}$, and

$$cap(\delta_A^{out}(U)) = \sum_{(r_d,t_1)\in S_1} c(r_d, t_1) + \sum_{(t_1,t_2)\in S_0} c(t_1, t_2) + \sum_{(t_2,s_d)\in S_2} c(t_2, s_d)$$
$$= \sum_{(r_d,t_1)\in S_1} u_d(t_1) + \sum_{(t_1,t_2)\in S_0} \lambda(t_1, t_2) + \sum_{(t_2,s_d)\in S_2} v_d(t_2), \qquad (2.B.12)$$

where $S_1 = \{(r_d, t_1) : r_d \in U, t_1 \notin U\}$, $S_2 = \{(t_2, s_d) : t_2 \in U, s_d \notin U\}$, $S_0 = \{(t_1, t_2) : t_1 \in U, t_2 \notin U\}$. Now the cut condition for U becomes

$$\sum_{d\in D'} \sum_{t_1\in T_1} u_d(t_1) \leq \sum_{(r_d,t_1)\in S_1} u_d(t_1) + \sum_{(t_2,s_d)\in S_2} v_d(t_2) + \sum_{(t_1,t_2)\in S_0} \lambda(t_1, t_2). \qquad (2.B.13)$$

Let $B_1 = T_1 \cap U$, $B_2 = T_2 \cap U$, $D'' = \{d \in \{1, ..., n\} : r_d \in U, s_d \in U\}$, and $D''' = \{d \in \{1, ..., n\} : r_d \notin U, s_d \notin U\}$. The cut condition for U is rewritten as

$$\sum_{d\in D'} \sum_{t_1\in T_1} u_d(t_1) \leq \sum_{d\in D'\cup D''} \sum_{t_1\in T_1\setminus B_1} u_d(t_1) + \sum_{d\in D'\cup D'''} \sum_{t_2\in B_2} v_d(t_2) + \sum_{t_1\in B_1, t_2\in T_2\setminus B_2} \lambda(t_1, t_2). \qquad (2.B.14)$$

We claim that the cut condition for every U satisfying $D'' \neq \emptyset$ or $D''' \neq \emptyset$ is implied by the cut condition for $\tilde{U}$ derived from U satisfying $\tilde{D}' = D' \cup D'' \cup D'''$ and $\tilde{D}'' = \tilde{D}''' = \emptyset$, and $\tilde{S}_1 = S_1$, $\tilde{S}_2 = S_2$. Let $\tilde{U} = \{r_d\}_{d\in\tilde{D}'} \cup \tilde{S}_1 \cup \tilde{S}_2$, $\tilde{B}_1 = B_1$, and $\tilde{B}_2 = B_2$. The cut condition for $\tilde{U}$ is given by

$$\sum_{d\in\tilde{D}'} \sum_{t_1\in T_1} u_d(t_1) \leq \sum_{d\in\tilde{D}'} \sum_{t_1\in T_1\setminus\tilde{B}_1} u_d(t_1) + \sum_{d\in\tilde{D}'} \sum_{t_2\in\tilde{B}_2} v_d(t_2) + \sum_{t_1\in\tilde{B}_1, t_2\in T_2\setminus\tilde{B}_2} \lambda(t_1, t_2), \qquad (2.B.15)$$

or

$$\sum_{d\in D'} \sum_{t_1\in T_1} u_d(t_1) + \sum_{d\in D''\cup D'''} \sum_{t_1\in T_1} u_d(t_1) \leq \sum_{d\in D'\cup D''} \sum_{t_1\in T_1\setminus B_1} u_d(t_1) + \sum_{d\in D'''} \sum_{t_1\in T_1\setminus B_1} u_d(t_1)$$
$$+ \sum_{d\in D'\cup D'''} \sum_{t_2\in B_2} v_d(t_2) + \sum_{d\in D''} \sum_{t_2\in B_2} v_d(t_2) + \sum_{t_1\in B_1, t_2\in T_2\setminus B_2} \lambda(t_1, t_2). \qquad (2.B.16)$$

Notice that

$$\sum_{d\in D''} \sum_{t_1\in T_1} u_d(t_1) > -\sum_{d\in D'''} \sum_{t_1\in B_1} u_d(t_1) + \sum_{d\in D''} \sum_{t_2\in B_2} v_d(t_2) \qquad (2.B.17)$$

implies that the condition for $\tilde{U}$ is tighter than the condition for U. Now the cut condition (2.B.9) is given by

$$\sum_{d\in D'} [\sum_{t_1\in T_1} u_d(t_1) - \sum_{t_1\in T_1\setminus B_1} u_d(t_1) - \sum_{t_2\in B_2} v_d(t_2)] \leq \sum_{t_1\in B_1, t_2\in T_2\setminus B_2} \lambda(t_1, t_2), \qquad (2.B.18)$$

for all $D' \subseteq \{1, ..., n\}$, $B_1 \subseteq T_1, B_2 \subseteq T_2$. Equivalently, (2.B.10) holds for all $D' \subseteq$

$\{1, ..., n\}$, $B_1 \subseteq T_1, B_2 \subseteq T_2$. Hence, the cut condition corresponds to the condition in (2.3.3) with all f satisfying $f_i(d, t_i) = f_i(d', t_i)$ for all $d, d' \in \{1, ..., n\}$, $t_i \in T_i$, $i = 1, 2$. Therefore, (2.B.9) is a subclass of (2.3.3).

(ii) In general, (2.B.9) is only a necessary condition for the existence of a feasible flow in general digraph multicommodity flow problems, see Schrijver (2013). If (2.B.9) is also sufficient for the feasibility, then by Lemma 2.3 and Lemma 2.4 (i), (2.B.9) is also sufficient for the implementability. □

Proof of Theorem 2.2. Let $N = \{1, 2\}$ and $D = \{0, 1\}$. Construct the digraph $G_1 = (V, A)$ with $r, s \in V$ (since $n = 1$, we drop the subscript $d = 1$ for r_1, s_1 and all other variables). For any flow f,

$$val(f) \leq cap(\delta_A^{out}(U)), \forall U \subset V, r \in U, s \notin U. \tag{2.B.19}$$

In particular,

$$val(f) \leq cap(\delta_A^{out}(\{r\})) = \sum_{t_1 \in T_1} u(t_1). \tag{2.B.20}$$

By Ford-Fulkerson Theorem (Lemma 2.5 in Appendix 2.D) and Lemma 2.3,

$$\sum_{t_1 \in T_1} u(t_1) = \sum_{t_2 \in T_2} v(t_2) = \max_f val(f) \leq cap(\delta_A^{out}(U)), \tag{2.B.21}$$

for all $U \subset V$, $r \in U, s \notin U$.

For $U = \{r\} \cup S_1 \cup S_2$ where $S_1 \subseteq T_1, S_2 \subseteq T_2$, the cut condition is given by

$$\begin{aligned}\sum_{t_1 \in T_1} u(t_1) &\leq \sum_{t_1 \notin S_1} c(r, t_1) + \sum_{t_1 \in S_1, t_2 \notin S_2} c(t_1, t_2) + \sum_{t_2 \in S_2} c(t_2, s) \\ &= \sum_{t_1 \notin S_1} u(t_1) + \sum_{t_1 \in S_1, t_2 \notin S_2} \lambda(t_1, t_2) + \sum_{t_2 \in S_2} v(t_2).\end{aligned} \tag{2.B.22}$$

The implementability condition is equivalent to

$$\sum_{i \in N} \left(\sum_{t_i \in S_i} Q_i(1, t_i)\lambda_i(t_i) - \sum_{t_i \in (S_i)^c} Q_i(1, t_i)\lambda_i(t_i) \right) \leq \sum_{i \in N} \lambda(S_i \times S_{-i}), \tag{2.B.23}$$

for all $S_i \in T_i$, $i = 1, 2$.

On the other hand, the condition in Corollary 2.1 is given by (See Proposition 2.2),

$$\sum_{i \in N} \left(\sum_{t_i \in \overline{S}_i} Q_i(1, t_i)\lambda_i(t_i) - \sum_{t_i \in \underline{S}_i} Q_i(1, t_i)\lambda_i(t_i) \right) \leq \sum_{i \in N} \lambda(\overline{S}_i \times (\underline{S}_{-i})^c), \tag{2.B.24}$$

for all $\overline{S}_i, \underline{S}_i \in 2^{T_i}$, $\overline{S}_i \cap \underline{S}_i = \emptyset$, $i = 1, 2$. Hence, (2.B.23) is a subclass of (2.B.24) by setting

$\overline{S}_i = S_i$ and $\underline{S}_i = (S_i)^c$, $i = 1, 2$. Since (2.B.23) is sufficient for the implementability, (2.B.24) is also sufficient for the implementability. □

Proof of Theorem 2.3. We define a linear map $\tilde{\Lambda} : \mathbb{R}^k \to \mathbb{R}^{\tilde{l}}$ by, for any $x \in \mathbb{R}^k$,

$$(\tilde{\Lambda}x)_i(d, t_i) = \sum_{t_{-i} \in T_{-i}} x(d, t)\lambda_{-i}(t_{-i}), \tag{2.B.25}$$

for all $d \in \tilde{D}_i$, $t_i \in T_i$, and $i \in N$. Then

$$P_{\tilde{D}}(\mathcal{D}) = \tilde{\Lambda}\mathcal{D}_0. \tag{2.B.26}$$

Define a linear map $\tilde{\Gamma} : \mathbb{R}^{\tilde{l}} \to \mathbb{R}^{\tilde{l}}$ by, for each $x \in \mathbb{R}^{\tilde{l}}$,

$$(\tilde{\Gamma}x)_i(d, t_i) = \lambda_i(t_i)x_i(d, t_i), \tag{2.B.27}$$

for all $d \in \tilde{D}_i$, $t_i \in T_i$, and $i \in N$. Then, a similar step as Theorem 2.1 shows that $P_{\tilde{D}}(\mathcal{D})$ and $\tilde{\Gamma}P_{\tilde{D}}(\mathcal{D})$ are convex and compact. A hyperplane separation theorem shows that $Q_{\tilde{D}} \in P_{\tilde{D}}(\mathcal{D})$ if and only if

$$\langle f_{\tilde{D}}, \tilde{\Gamma}Q_{\tilde{D}}\rangle \le \sup\{\langle f_{\tilde{D}}, \tilde{\Gamma}\tilde{\Lambda}q\rangle : q \in \mathcal{D}_0\}, \quad \text{for all} \quad f_{\tilde{D}} \in \mathbb{R}^{\tilde{l}}. \tag{2.B.28}$$

This corresponds to condition (2.3.2) with all $f \in \mathbb{R}^l$ satisfying $f_i(d, t_i) = 0$ for all $d \in \tilde{D}_i^c$, $t_i \in T_i$, and $i \in N$. □

Appendix 2.C Proof of Proposition 2.1-2.4

Proof of Proposition 2.1. See Che, Kim and Mierendorff (2013) for a recent proof of the condition (2.3.5) based on a single-commodity network flow problem. Here we show that if the necessary condition given by all $f \in \{-1, 0, +1\}^l$ is sufficient for the implementability, the necessary condition given by all $f \in \{0, +1\}^l$ is also sufficient.

Since $(Q_i(i, \cdot))_{i \in N \setminus \{0\}}$ are essential, by Theorem 2.3, we set $f_0(d, \cdot) = 0$ for all $d \in D$, and $f_i(d, \cdot) = 0$ for all $d \neq i$, $i \in N \setminus \{0\}$ in (2.3.3). Denote this subclass of f by $\mathcal{C}$.

For any $\tilde{f} \in \mathcal{C}$ that satisfies $\tilde{f}$ has some coordinate being -1, i.e. $\tilde{f}_j(j, \cdot)(t_*) = -1$ for some $j \in N \setminus \{0\}$ and $t_j = t_*$, define $\hat{f} \in \mathcal{C}$ by replacing $\hat{f}_j(j, t_*) = 0$ while $\hat{f} = \tilde{f}$ for all other coordinates. We show that in (2.3.3), the condition given by $\hat{f}$ is tighter than the condition given by $\tilde{f}$. For every $t \in T^*$ and $f \in \{-1, 0, +1\}^l$,

$$\max_{d \in D}\{\sum_{i \in N} f_i(d, t_i)\} = \max\{0, \max_{i \in N \setminus \{0\}} f_i(i, t_i)\}. \tag{2.C.1}$$

Let $T^* = \{t \in T : t_j = t_*\}$. For any $t \in T^*$, $d = j$ is not an optimal solution to (2.C.1) given $\tilde{f}$. Then for each $t \in T^*$,

$$\max_{i\in N}\{\tilde{f}_i(i,t_i)\} = \max_{i\in N}\{\hat{f}_i(i,t_i)\}. \tag{2.C.2}$$

Then,

$$L(\tilde{f}) = -Q_j(j,t_*)\lambda_j(t_*) - \sum_{t\in T^*}\max_{i\in N}\{\tilde{f}_i(i,t_i)\}\lambda(t) + K, \tag{2.C.3}$$

and

$$L(\hat{f}) = -\sum_{t\in T^*}\max_{i\in N}\{\hat{f}_i(i,t_i)\}\lambda(t) + K, \tag{2.C.4}$$

where

$$K = \sum_{i\in N}\sum_{t_i\in T_i}\tilde{f}_i(i,t_i)Q_i(i,t_i)\lambda_i(t_i) + Q_j(j,t_*)\lambda_j(t_*) - \sum_{t\in T\setminus T^*}\max_{i\in N}\{\tilde{f}_i(i,t_i)\}\lambda(t). \tag{2.C.5}$$

It is easy to see that $L(\tilde{f}) \leq L(\hat{f})$ since $Q_j(j,t_*) \geq 0$. Repeat this procedure and replace all -1 by 0. This completes the proof. □

For general implementation problems, the extreme allocation rules that generate the extreme points of the set of implementable reduced forms generalize the "hierarchical allocation rules" of Border (1991) for single object auctions without externalities.

Definition 2.2: $q^* \in \mathcal{D}_0$ is a *generalized hierarchical allocation rule*, if there exists $f \in \mathbb{R}^l$ such that $\sup\{\langle f, \Gamma\Lambda q\rangle : q \in \mathcal{D}_0\}$ is attained at q^*.

Denote by $h(f) = \sup\{\langle f, \Gamma\Lambda q^*\rangle : q \in \mathcal{D}_0\}$ the value of the support function at f. The proofs of Propositions 2.2 and 2.4 provide a characterization of the generalized hierarchical allocation rules for social choice problems in Section 2.4.

In the remainder of the proofs, we represent a vector in $\{-1,0,+1\}^l$ by a system of sign functions. For any $\mathcal{A}_i = (\overline{A}_i, \underline{A}_i) \in 2^{T_i}\times 2^{T_i}$ such that $\overline{A}_i \cap \underline{A}_i = \emptyset$, define a sign function by $\chi_{\mathcal{A}_i} = \chi_{\overline{A}_i} - \chi_{\underline{A}_i}$. Then it is easy to see that if $f \in \{-1,0,+1\}^l$, then for $i \in N$, $t_i \in T_i$, $f_i(d,t_i) = \chi_{\mathcal{A}_i}(t_i)$ for some $\mathcal{A}_i$.

Proof of Proposition 2.2. The formal proof has been given by Theorem 2.2. We now compute the hierarchical allocation rules. Let $\overline{B}_i, \underline{B}_i \in 2^{T_i}$, $\overline{B}_i \cap \underline{B}_i = \emptyset$, $i = 1,2$. Consider $f = (f_1(b,\cdot), f_2(b,\cdot)) = (\chi_{\overline{B}_1} - \chi_{\underline{B}_1}, \chi_{\overline{B}_2} - \chi_{\underline{B}_2})$. For each $t \in T$, consider the point-wise maximization problem given by

$$\max_{q\in\mathcal{D}_0}\sum_{i\in N}(\chi_{\overline{B}_i}(t_i) - \chi_{\underline{B}_i}(t_i))q(b,t). \tag{2.C.6}$$

In Table 2A.1, the first two columns denote type profiles, i.e. $(1,1)$ corresponds to $t \in \overline{B}_1 \times \overline{B}_2$. The third column provides the point-wise solution and the last column is the maximum value of this problem. The set of all hierarchical allocation rules with the normal vector f, denote $\mathcal{D}_0^*(f)$, is found by combining the solutions for all type profiles in Table 2A.1.

In this case, for any selection $\tilde{q}^* \in \mathcal{D}_0^*(f)$, the value of the support function at f is given by

$$\begin{aligned}\sum_{t\in T}\sum_{i\in N}(\chi_{\overline{B}_i} - \chi_{\underline{B}_i})\tilde{q}^*(b,t)\lambda(t) &= \lambda(\overline{B}_1 \times \overline{B}_2) + \lambda(\bigcup_{i\in N} \overline{B}_i \times (\underline{B}_{-i})^c) \\ &= \lambda(\overline{B}_1 \times (\underline{B}_2)^c) + \lambda((\underline{B}_1)^c \times \overline{B}_2).\end{aligned}$$

$f_1(b,t_1)$	$f_2(b,t_2)$	$q^*(b,t)$	Value
1	1	1	2
1	0	1	1
1	-1	$[0,1]$	0
0	1	1	1
0	0	$[0,1]$	0
0	-1	0	0
-1	1	$[0,1]$	0
-1	0	0	0
-1	-1	0	0

Table 2A.1

□

Proof of Proposition 2.3. See Lemma 2.4 in Appendix 2.B, by setting $|D| = 3$. □

Proof of Proposition 2.4. Since $(Q_1(1,\cdot), Q_2(2,\cdot), Q_1(b,\cdot), Q_2(b,\cdot))$ are essential, by Theorem 2.3, we set $f_1(0,\cdot)$, $f_2(0,\cdot)$, $f_1(2,\cdot)$, and $f_2(1,\cdot)$ to be 0 in (2.3.3).

$$\max_{q\in\mathcal{D}_0}\sum_{i\in N} f_i(i,t_i)q(i,t) + \sum_{i\in N} f_i(b,t_i)q(b,t) = \max\{0, f_1(1,t_1), f_2(2,t_2), f_1(1,t_1) + f_2(2,t_2)\}.$$

First notice that a similar argument as Proposition 2.1 implies that we need only to consider $f = (f_1(1,\cdot), f_2(2,\cdot), f_1(b,\cdot), f_2(b,\cdot))$ where $f_i(i,t_i) \geq 0$, for all $t_i \in T_i$, $i = 1,2$. Let $A_i, \overline{B}_i, \underline{B}_i \in 2^{T_i}$, $\overline{B}_i \cap \underline{B}_i = \emptyset$, $i = 1,2$. Consider $f = (\chi_{A_1}, \chi_{A_2}, \chi_{\overline{B}_1} - \chi_{\underline{B}_1}, \chi_{\overline{B}_2} - \chi_{\underline{B}_2})$. For each $t \in T$, consider the point-wise maximization problem

$$\max_{q\in\mathcal{D}_0}\sum_{i\in N}\chi_{A_i}(t)q(i,t) + \sum_{i\in N}(\chi_{\overline{B}_i}(t_i) - \chi_{\underline{B}_i}(t_i))q(b,t). \tag{2.C.7}$$

The solutions are given in Table 2A.2. The first two columns denote type profiles, i.e. $(1,1,1,1)$ corresponds to $(A_1 \cap \overline{B}_1) \times (A_2 \cap \overline{B}_2)$. The third column provides the point-wise solution and the last column is the point-wise maximum value of this problem. All hierarchical allocation rules, or $\mathcal{D}_0^*(f)$, are found by combining the solutions for all type profiles.

$f_1(1,t_1)$	$f_1(b,t_1)$	$f_2(2,t_2)$	$f_2(b,t_2)$	$(q^*(1,t),q^*(2,t),q^*(b,t))$	Value
1	1	1	1	$q_b=1$	2
1	1	1	0	$q_1+q_2+q_b=1$	1
1	1	1	-1	$q_1+q_2=1$	1
1	1	0	1	$q_b=1$	2
1	1	0	0	$q_1+q_b=1$	1
1	1	0	-1	$q_1=1$	1
1	0	1	1	$q_1+q_2+q_b=1$	1
1	0	1	0	$q_1+q_2=1$	1
1	0	1	-1	$q_1+q_2=1$	1
1	0	0	1	$q_1+q_b=1$	1
1	0	0	0	$q_1=1$	1
1	0	0	-1	$q_1=1$	1
1	-1	1	1	$q_1+q_2=1$	1
1	-1	1	0	$q_1+q_2=1$	1
1	-1	1	-1	$q_1+q_2=1$	1
1	-1	0	1	$q_1=1$	1
1	-1	0	0	$q_1=1$	1
1	-1	0	-1	$q_1=1$	1
0	1	1	1	$q_b=1$	2
0	1	1	0	$q_2+q_b=1$	1
0	1	1	-1	$q_2=1$	1
0	1	0	1	$q_b=1$	2
0	1	0	0	$q_b=1$	1
0	1	0	-1	$0\le q_1+q_2+q_b\le 1$	0
0	0	1	1	$q_2+q_b=1$	1
0	0	1	0	$q_2=1$	1
0	0	1	-1	$q_2=1$	1
0	0	0	1	$q_b=1$	1
0	0	0	0	$0\le q_1+q_2+q_b\le 1$	0
0	0	0	-1	$0\le q_1+q_2\le 1,\ q_b=0$	0
0	-1	1	1	$q_2=1$	1
0	-1	1	0	$q_2=1$	1
0	-1	1	-1	$q_2=1$	1
0	-1	0	1	$0\le q_1+q_2+q_b\le 1$	0
0	-1	0	0	$0\le q_1+q_2\le 1,\ q_b=0$	0
0	-1	0	-1	$0\le q_1+q_2\le 1,\ q_b=0$	0

Table 2A.2

Case 1. $t\in T'=\overline{B}_1\times\overline{B}_2$. For such a profile, the probability weighted value is $2\lambda(t)$.

Case 2. $t\in T''$, the intersection of $(A_1)^c\times(A_2)^c$ and $T\setminus((\overline{B}_1\times(\underline{B}_2)^c)\cup(\underline{B}_1)^c\times\overline{B}_2)$. For such a profile, the probability weighted value is 0.

Case 3. $t\in T\setminus(T'\cup T'')$. For such a profile, the probability weighted value is $\lambda(t)$.

Now, compared to $\lambda(T) = 1$, where each type profile counts once, the support function counts $t \in T'$ twice and $t \in T''$ zero times. The value of the support function at f is given by

$$h(f) = 1 + \lambda(T') - \lambda(T''). \tag{2.C.8}$$

Denote

$$T_A = (A_1)^c \times (A_2)^c, \qquad T_B = (\overline{B}_1 \times (\underline{B}_2)^c) \cup ((\underline{B}_1)^c \times \overline{B}_2).$$

Then, $T_A^c = \cup_{i \in N}(A_i \times T_{-i})$, $T'' = T_A \cap T_B^c$, and

$$h(f) = 1 + \lambda(T') - \lambda(T_A \cap T_B^c) = \lambda(T') + \lambda((T_A \cap T_B^c)^c) = \lambda(T') + \lambda(T_A^c \cup T_B).$$

Furthermore,

$$\begin{aligned}
\lambda(T') + \lambda(T_A^c \cup T_B) &= \lambda(T_A^c) + (\lambda(T') + \lambda(T_B)) - \lambda(T_A^c \cap T_B) \\
&= \lambda(T_A^c) + \lambda(\overline{B}_1 \times (\underline{B}_2)^c) + \lambda((\underline{B}_1)^c \times \overline{B}_2) - \lambda(T_A^c \cap T_B) \\
&\leq \lambda(T_A^c) + \lambda(\overline{B}_1 \times (\underline{B}_2)^c) + \lambda((\underline{B}_1)^c \times \overline{B}_2).
\end{aligned}$$

Hence, compared to the conditions in Proposition 2.1 and 2.2, the condition in Proposition 2.4 is tighter. $\square$

Appendix 2.D Single and Multicommodity Flow Problems

We first introduce a single commodity maximum flow problem. Let $G = (V, A)$ be a directed graph and let $r, s \in V$ be the source and the sink. For any $k \in V$, denote $\delta^{in}(k)$ as the set of the arcs entering k and $\delta^{out}(k)$ the set of arcs leaving k. Let $c : A \to \mathbb{R}_+$ be a capacity function. A function $f : A \to \mathbb{R}_+$ is an $r - s$ flow if

$$0 \leq f(a) \leq c(a), \forall a \in A, \tag{2.D.1}$$

$$\sum_{a \in \delta^{in}(k)} f(a) = \sum_{a \in \delta^{out}(k)} f(a), \forall k \in V \setminus \{r, s\}. \tag{2.D.2}$$

The value of an $r - s$ flow f is given by

$$val(f) = \sum_{a \in \delta^{out}(r)} f(a) - \sum_{a \in \delta^{in}(r)} f(a). \tag{2.D.3}$$

So, the value is the net amount of flow leaving r ($\delta^{in}(r) = \emptyset$). It is also equal to the net amount of flow entering s.

A set C of arcs is a $r - s$ cut if $C = \delta^{out}(U)$ for some subset $U \subset V$ with $r \in U$ and

$s \notin U$. For such a U, let

$$cap(\delta_A^{out}(U)) = \sum_{a \in \delta^{out}(U)} c(a). \tag{2.D.4}$$

The 1-commodity maximum flow problem now is to find an $r-s$ flow of maximum value.

LEMMA 2.5: *(Ford-Fulkerson, 1956) For any one-commodity flow problem with digraph $G = (V, A)$, $r, s \in V$ and $c : A \to \mathbb{R}_+$, the maximum flow is equal to the minimum cut, or*

$$\max_{f:\ r-s\ flow} val(f) = \min_{U:\ r-s\ cut} cap(\delta_A^{out}(U)). \tag{2.D.5}$$

Now we introduce a multicommodity flow problem. Let $G = (V, A)$ be a supply digraph with multiple commodities $d = 1, ..., n$, sources and sinks $(r_d, s_d)_{d=1}^n$, and the capacity function $c : A \to \mathbb{R}_+$. The corresponding demand digraph $H = (V', R)$ is given by $V' = \{r_1, s_1, ..., r_n, s_n\}$ and $R = \{(r_d, s_d)_{d=1}^n\}$, which contains all source-sink pairs. A demand function is given by $\phi : R \to \mathbb{R}_+$. A flow $f = (f_d)_{d=1}^n$ contains $f_d : A \to \mathbb{R}_+$, $d = 1, ..., n$, satisfying

$$0 \leq \sum_{d=1}^{n} f_d(a) \leq c(a), \forall a \in A, \tag{2.D.6}$$

$$\sum_{a \in \delta^{in}(k)} f_d(a) = \sum_{a \in \delta^{out}(k)} f_d(a), \forall k \in V, k \neq r_d, s_d, d = 1, ..., n. \tag{2.D.7}$$

The value of the flow f is given by

$$val(f_d) = f_d(\delta^{out}(r_d)) = f_d(\delta^{in}(s_d)), d = 1, ..., n. \tag{2.D.8}$$

For $U \subseteq V$, denote $cap(\delta_A^{out}(U))$ the total capacity of arcs of A leaving U, and denote $\phi(\delta_R^{out}(U))$ the total demands of arcs of R leaving U. A flow f subject to c with value ϕ is called feasible. A problem (G, H, c, ϕ) is to find a feasible flow.

Chapter 3

Nonexistence of Monotone Solutions in Two-Person Bargaining Problems with Incomplete Information

3.1. Introduction

For bargaining problems between a buyer and a seller with incomplete information, Myerson and Satterthwaite (1983) shows that if the seller and the buyer have independent beliefs with a common support, there is no ex post (Pareto) efficient, ex post budget balanced, individually rational, and incentive compatible trading mechanism. In the ex ante efficient mechanism that maximizes the trading surplus, each player's interim utility is responsive to his valuation of the object.[1] However, in this environment, trade is not always ex post efficient.[2] This raises a question: If the disagreement payoffs are private information for both players and trade is always ex post efficient, does there exist a trading mechanism that is ex post efficient and responsive to the players' disagreement payoffs?

To illustrate this problem, let us consider a simple example. There are two risk neutral players that jointly produce a private good from complementary inputs, one from each player. Assume that it is common knowledge that producing the object and allocating it to one player yields utility 3 for this player and 0 for the other player. The costs of inputs are privately observed by each player and independently drawn from 0 and 1 with equal probabilities.

For this bargaining problem with incomplete information, several incentive compatible and individually rational solutions have been proposed. The ex ante utilitarian solution (Myerson and Satterthwaite, 1983), which maximizes the ex ante trading surplus, requires producing the object with probability one and yields interim utility 3/2 for both types of the players. The ex ante utilitarian solution is ex post efficient but not monotonic with respect to the costs at the interim stage. The generalized Nash so-

[1] For illustration, suppose the seller and the buyer have independent private values and $\tilde{v}_s, \tilde{v}_b \sim U[0,1]$. In the ex ante efficient mechanism, trade occurs if $v_b > v_s + 1/4$ and no trade occurs otherwise. The interim utilities are given by $U_s(v_s) = U_s(1) - \int_{v_s}^{1} Q_s(x)dx$ and $U_b(v_b) = U_b(0) + \int_0^{v_b} Q_b(x)dx$, where $Q_s(x) = \max\{0, \frac{3}{4} - x\}$ and $Q_b(x) = \min\{1, \frac{1}{4} + x\}$ for $x \in [0,1]$. Hence, U_s is increasing in v_s and U_b is increasing in v_b.

[2] There exists some $v_b < v_s$ such that trade is inefficient.

lution (Harsanyi and Selten, 1975, Myerson, 1979), which maximizes the Nash product weighted by the marginal probabilities of the types, requires producing the object with a probability less than one and yields interim utility 3/4 for the low type and 3/2 for the high type of each player.[3] The generalized Nash solution is monotonic but not ex post efficient. Do these drawbacks disqualify these solutions? We show that the answer is No, because there is no solution that is both ex post efficient and monotone.

Requiring each player's utility outcome being responsive to his disagreement payoff is a desirable property. In case of an exchange economy, it requires that a player is rewarded for having a larger initial endowment. The monotonicity is consistent with but weaker than an equal sharing of the trading surplus between the players. The players may accept the monotonicity as an egalitarian criterion. In the axiomatic theory of bargaining, the Nash solution, the Kalai-Smorodinsky solution, and the egalitarian solution are weakly responsive to the disagreement payoffs (Thomson, 1987). For bargaining procedures with complete information, Crawford (1979) studies a class of multistage procedures where the right to propose a division is auctioned off to players. A prominent aspect of this procedure is that any change in the status quo is reflected in the final outcome.[4] Hence, if there is complete information, the most well-known bargaining solutions are monotonic in the disagreement payoffs and "disagreement point monotonicity" has even been proposed as an axiom.

In this chapter, we consider bargaining problems with possibly correlated information on the disagreement payoffs of the players. When the status quo is always ex post inefficient, requiring ex post efficiency of the solution implies that the disagreement is never implemented. The uncertainty of the disagreement payoffs has no material consequence but it may still have strategic meaning. To characterize such monotonicity for bargaining with incomplete information, we introduce the following property on utility outcomes: Each player's equilibrium interim utility is non-constant and is nondecreasing in one's own disagreement payoff throughout the support. We find that if the players' beliefs are independent, then each player's interim utility is independent of the type of this player. Thus, searching for any possibility result of non-constant interim utility must go beyond independent beliefs.[5] In some contexts, the players' disagreement payoffs are positively correlated, i.e. either (i) both players face similar outside market conditions

[3] A random allocation mechanism that implements the generalized Nash solution is as follows: If both players report 0, then allocate the object to each player with probability 1/2; If only one player reports 1, with probability 1/2, allocate the object to this player, and with probability 1/2, there is no joint production; If both players report 1, then with probability 1, there is no joint production.

[4] In contrast, Rubinstein (1982)'s bargaining procedure requires that a slight increase in a sufficiently low status quo does not affect the outcome.

[5] In Myerson and Satterthwaite's environment, the correlation between players' beliefs might overcome the impossibility result. By extending the full extraction of surplus result in Crémer and McLean (1988) to a continuum of states, McAfee and Reny (1992) shows that for a broad class of correlated beliefs, there exists some ex post efficient and individually rational mechanism, provided that ex post budget balance is replaced by ex ante budget balance.

or (ii) they have been in some agreement in which they share some common interest. In other contexts, the players' disagreement payoffs can be negatively correlated; for example, when the players play a zero-sum game in case of conflict.

With positively correlated beliefs, it is not obvious that a better outside option will improve one's interim expected utility in an ex post efficient mechanism. After all, a player who draws a higher disagreement payoff has a more pessimistic belief on the other's disagreement payoff and his interim expected utility may hence turn out to be lower. With negatively correlated beliefs, a player with a better disagreement payoff is more optimistic and the interim expected utility is more likely to be higher. However, we find this is not the case even for negatively correlated beliefs, provided that trade is always ex post efficient. The main results show that for finite type sets, incentive compatible, ex post efficient and monotone solutions do not exist for both positively and negatively correlated beliefs, if (i) The support of beliefs contains two types for each player, and (ii) The beliefs have uniform marginal probabilities.

Myerson (1979, 1985) implicitly mentions the tradeoff between ex post efficiency and interim monotonicity. Myerson (1979) considers a two-person public project example with one-sided private information, where building the project is always ex post efficient. He shows that the incentive compatible generalized Nash solution yields a monotone but inefficient utility allocation. Myerson (1985) considers a version of Akerlof's lemon problem, in which only the seller has private information related to the quality of the object. The buyer always values the object higher than the seller and trade is always ex post efficient. Myerson shows that for one of the neutral bargaining solutions he propose, the seller's interim utility is also monotone but the outcome is ex post inefficient. This chapter develops Myerson's observations into more general impossibility results for two-sided uncertainty and correlated beliefs. The implications are twofold. First, it reflects a conflict of ex post efficiency and interim utility monotonicity: Either the players have to forgo some gains from agreement or they have to accept a constant division even if their outside options turn out to be different. Second, it provides a theoretical model for why the breakdown of an obviously beneficial agreement is observed so often in the lab (Roth, 1995).

Our model relates to Börgers and Postl (2009), which considers a modified Myerson and Satterthwaite environment where (i) players' ordinal preferences are common knowledge while cardinal preferences are private information, and (ii) there is no status quo. While our assumption on players' ordinal preferences is similar to Börgers and Postl (2009), we consider a different assumption on the cardinal preferences. It is worth to notice while the two models differ in the assumptions on the supports for private values, the models have the same set of decisions options and the same feasibility problem. In both models, to characterize the set of incentive compatible mechanisms, the same feasibility problem arises. While the ex ante utilitarian solution in Börgers and Postl (2009) is still open, the ex ante utilitarian solution in the model of this chapter is trivial.

The remainder of this chapter is as follows. Section 3.2 introduces the model. Sections 3.3 and 3.4 discuss efficiency and monotonicity separately. Section 3.5 characterizes the non-constant ex post efficient solutions. Section 3.6 discusses the ex post properties and the model with more than two players. Section 3.7 concludes.

3.2. Model

Two players have the opportunity to jointly produce one unit of a private good from complementary inputs, one from each player. The set of social alternatives is $D = \{d_0, d_1, d_2\}$. The choice d_0 is the disagreement point, in which case players $i = 1, 2$ receive payoffs $t = (t_1, t_2) \in T \subset \mathbb{R}^2$. For $i = 1, 2$, the choice d_i means that producing the object and allocating it to player i, which yields utility $V_i > 0$ for player i and 0 for the other player. We assume that (V_1, V_2) are commonly known by the players and that each player i privately observes his own disagreement payoff t_i. The players bargain over which decision to select and randomization is allowed. We now introduce the model's assumptions.

Assumption 3.1: $(\tilde{t}_1, \tilde{t}_2)$ has a joint probability density $f : T \to \mathbb{R}_{++}$, where $T = S \times S$ and either (i) $S = [0, 1]$ and f is a continuous density function, or (ii) $S = \{s_1, ..., s_n\} \subset [0, 1]$ for some $n \geq 2$.

In Section 3.6, we discuss other support assumptions.

Assumption 3.2: $f(s, s') = f(s', s)$, for all $s, s' \in S$.

For $i = 1, 2$, denote $f_i(t_i)$ the marginal probability of player i, or $\int_S f(t)dt_j$, for $j \neq i$,[6] and $f_j(t_j|t_i)$ the conditional probability of player i, or $f(t)/f_i(t_i)$. Independent beliefs correspond to $f(t) = f_1(t_1)f_2(t_2)$ for all $t \in T$. Denote $\mathcal{F}$ the set of beliefs satisfying the assumptions 3.1 and 3.2.

Assumption 3.3: For $i = 1, 2$, player i has a von Neumann-Morgenstern utility function $u_i : \Delta(D) \times S \to \mathbb{R}$ given by $u_i(q, t_i) = q_0 t_i + q_i V_i$, where $q = (q_0, q_1, q_2) \in \Delta(D)$, and $V_1 = V_2 = V$.

We assume the players have private values on their disagreement payoffs. $V_1 = V_2$ is a normalization. Since a player has a vNM utility function, we can normalize his utility from the best alternative to V and the worst alternative to 0. Then, the symmetric supports in Assumption 3.1 require some interpersonal comparison of utilities.

We say $q \in \Delta(D)$ is ex post (Pareto) efficient at $t \in T$ if there exists no $q' \in \Delta(D)$ such that $u_i(q, t_i) \leq u_i(q', t_i)$, with at least one inequality strict. $q \in \Delta(D)$ is ex post

[6] Here we use the integral symbol for both continuous and finite supports.

utilitarian efficient at $t \in T$ if q is a solution to $\max u_1(q, t_1) + u_2(q, t_2)$. We now introduce the final assumption in this chapter.

ASSUMPTION 3.4: $V > 2$.

Note that Assumption 3.4 implies that for every $t \in T$, every $q' \in \Delta(D)$ with $q'_0 > 0$ is ex post Pareto dominated: there exists $q \in \Delta(D)$ with $q_0 = 0$ such that $u_i(q', t_i) < u_i(q, t_i)$, $i = 1, 2$. Hence, ex post Pareto efficiency and ex post utilitarian efficiency coincide in our model, which requires that for each state, d_0 is selected with probability 0. The linearity of the utility functions then implies that an increase in q_1 requires the same decrease in q_2 and the transfer rate is 1:1 for the players. This feasibility constraint mimics a balanced budget constraint in the quasilinear case, except that transfers are now bounded. In Börgers and Postl (2009), it is assumed that $V = 1$. In that case, the status quo is not always ex post inefficient, which is similar to the support condition of Myerson and Satterthwaite (1983).

A random allocation mechanism is given by $q : T \to \Delta(D)$. If player $i = 1, 2$ reports $\hat{t}_i$ instead of his true type t_i, while the other player is honest, then i's interim expected utility is given by[7]

$$U_i(q, \hat{t}_i | t_i) = \int_S u_i(q(\hat{t}_i, t_j), t_i) f_j(t_j | t_i) dt_j. \tag{3.2.1}$$

We say q is incentive compatible (IC) if truthful reporting by both players constitutes a Bayesian equilibrium. Denote the interim utility under truthful reporting by $U_i(q|t_i) = U_i(q, t_i|t_i)$. We say q is individually rational (IR) if $U_i(q|t_i) \geq t_i$, for all $t_i \in S$, $i = 1, 2$. q is ex post efficient (EFF) if $q_1(t) + q_2(t) = 1$, for all $t \in T$. Notice that if q is ex post efficient, then q is fully determined by q_1.

In general, a mechanism may be asymmetric among players. Since our environment is symmetric, i.e. it has symmetric beliefs and symmetric utility functions of players, Lemma 3.1 shows that we can restrict our attention to symmetric mechanisms (SYM), where $q_1(s, s') = q_2(s', s)$, $q_0(s, s') = q_0(s', s)$, for all $s, s' \in S$.

LEMMA 3.1: *Let $f \in \mathcal{F}$. Consider any asymmetric mechanism q. If q satisfies IC, IR and $U_i(q|t_i)$ is (weakly) increasing in t_i on S for $i = 1, 2$, then there exists a symmetric mechanism $\tilde{q}$ that satisfies IC, IR and $U_i(\tilde{q}|t_i)$ is (weakly) increasing in t_i on S for $i = 1, 2$.*

Proof. Define $q^* = (q_0, q_2, q_1)$. Then, q^* is also IC and IR. Since q, q^* are IC, IR, and the integral operator is linear, $\tilde{q} = 1/2q + 1/2q^*$ is IC, IR and SYM. Since for $i = 1, 2$, $U_i(q|t_i)$ and $U_i(q^*|t_i)$ are (weakly) increasing in t_i on S, $U_i(\tilde{q}|t_i) = 1/2U_i(q|t_i) + 1/2U_i(q^*|t_i)$ is (weakly) increasing in t_i on S. □

[7] For a pair of types (t_i, t_j), the first coordinate in q corresponds to player i. For a pair of types without specifying players i and j, the first coordinate in q corresponds to player 1.

In the remainder of this chapter, we say a mechanism is feasible if it satisfies incentive compatibility, individual rationality, and symmetry. The (symmetric) constant solution is defined by $q(t) = (0, 1/2, 1/2)$ for all $t \in T$. We say a feasible mechanism q is interim utility equivalent to the constant solution if $U_i(q|t_i) = V/2$, for all $t_i \in S$, $i = 1, 2$.

3.3. Existence of Efficient Solutions

In Myerson and Satterthwaite (1983), the status quo (no trade) is ex post efficient for some states of the world. In their case, there does not exist any incentive compatible, individually rational, budget balanced, and ex post efficient mechanism. In our model, the Myerson and Satterthwaite impossibility does not arise, because the constant solution is ex post efficient.

LEMMA 3.2: *Let $f \in \mathcal{F}$. There exists a feasible and ex post efficient mechanism.*

Notice that there may exist other feasible and ex post efficient mechanisms. To see the multiplicity, suppose $f(t) = 1$ for all $t \in [0,1]^2$. Consider a class of feasible and ex post efficient mechanisms given by

$$q_1(t) = \frac{1}{2} + \frac{1}{2\|h\|_\infty} h(t_1 - t_2) \text{ for all } t \in [0,1]^2, \tag{3.3.1}$$

where

(i) $h : [-1,1] \to \mathbb{R}$ is continuous, and
(ii) $h(x) = h(x+1)$ for all $x \in [-1,0]$, and
(iii) $h(x) = -h(-x)$ for all $x \in [-1,0]$, and
(iv) $\|h\|_\infty = \max_{x \in [-1,1]} h(x)$.

Then, such a mechanism is a solution to

$$\int_0^1 q_1(t) f_2(t_2) dt_2 = \frac{1}{2}, \text{ for all } t_1 \in [0,1], \tag{3.3.2}$$

$$\int_0^1 q_2(t) f_1(t_1) dt_1 = \frac{1}{2}, \text{ for all } t_2 \in [0,1]. \tag{3.3.3}$$

To see the result intuitively, notice that for each reporting profile, the mechanism prescribes a fair lottery plus a probability premium determined by the difference of the players' reports. The condition (ii) is used for incentive compatibility while the conditions (iii) and (iv) are required by ex post feasibility. Player 1 receives the same utility by reporting 1 or 0, and the expected value of the premium (multiplied by $2\|h\|_\infty$) is given by

$$\int_0^1 h(1-x)dx = \int_0^1 h(-x)dx = \int_{-1/2}^{1/2} h(x)dx = 0. \tag{3.3.4}$$

Similarly, for any other report in $[0, 1]$, the expected value of the premium is also zero. Hence, player 1 is indifferent among all reports.[8]

3.4. Existence of Monotone Solutions

For this bargaining problem, we introduce an interim monotonicity property: At the interim stage, each player's interim expected utility is (i) non-constant and (ii) nondecreasing everywhere in his disagreement payoff.

DEFINITION 3.1: Let $f \in \mathcal{F}$ and let q be a feasible mechanism. We say q is interim monotone (I-M) if, (i) for $i = 1, 2$ and all $\overline{u} \in \mathbb{R}$, the event $\{t_i \in S : U_i(q|t_i) = \overline{u}\}$ has a probability measure less than 1, and (ii) for $i = 1, 2$, $U_i(q|t_i') \leq U_i(q|t_i)$ for all $t_i' \leq t_i$, $t_i, t_i' \in S$.

We discuss the extension to the ex post utility profile and ex post monotonicity in Section 3.6. Notice that for $\alpha \in [0, 1/2)$, the lottery mechanism $q(t) = (1 - 2\alpha, \alpha, \alpha)$ for all $t \in T$, which is independent of reports, yields an interim monotone allocation. The following result is immediate.

LEMMA 3.3: *Let $f \in \mathcal{F}$. There exists a feasible and interim monotone mechanism.*

3.5. The Impossibility Results

In Sections 3.3 and 3.4, we have shown that there exist (i) some feasible and ex post efficient solution which is not interim monotone, and (ii) some feasible and interim monotone solution which is not ex post efficient. In this section, we show that the incompatibility of efficiency and interim monotonicity holds for all independent beliefs and for a broad class of correlated beliefs.

3.5.1. Independence

The example in Section 3.3 illustrates that the set of feasible and ex post efficient mechanisms can be quite large for independent beliefs. However, the next result shows that all these mechanisms yield the same interim utility.

[8] There exists another class of asymmetric solutions. For $a \in [-1, 1]$, define

$$q_1^a(t) = \frac{1}{2}(1 + a - 2a(t_1 + t_2) + 4at_1t_2).$$

Then, q^a is a solution to this problem. To see the intuition, let $a = 1$. If player 2 reports $t_2 > 1/2$, then to maximize q_1, player 1 will report 1. If player 2 reports $t_2 < 1/2$, then player 1 will report 0. Hence, a higher report may increase or decrease the probability of winning for player 1. For player 2 with types uniformly distributed on $[0, 1]$, the mechanism ensures that player 1 is indifferent between these reports.

Proposition 3.1: *Let $f \in \mathcal{F}$ and let the beliefs be independent. If a feasible mechanism is ex post efficient, then it is interim utility equivalent to the constant solution.*

Proof. Let q be a feasible and ex post efficient solution. $q_0(t) = 0$ for all $t \in T$ implies that for $i = 1, 2$, $u_i(q(\hat{t}_i, t_j), t_i) = u_i(q(\hat{t}_i, t_j), t'_i)$, for all $t_j, \hat{t}_i, t_i, t'_i \in S$. Independence implies $f_j(t_j|t_i) = f_j(t_j)$, for all $t_j, t_i \in S$. Thus, for all $\hat{t}_i, t_i, t'_i \in S$, $U_i(q, \hat{t}_i|t_i) = U_i(q, \hat{t}_i|t'_i)$. IC implies $U_i(q|\cdot)$ is a constant and by SYM, it is $V/2$. □

Notice that ex post efficiency and independence imply that a player's disagreement payoff has neither payoff consequences nor strategic consequences. In other words, a player's interim utility from any report is independent of his true type.

Proposition 3.1 is robust to an increase in the number of players. Let N be a finite set of players, $|N| \geq 3$, and assume that the assumptions from Section 3.2, appropriately modified, continue to hold. A similar analysis shows that if $V > |N|$, the impossibility result remains true.

3.5.2. Correlation

Proposition 3.1 implies that any possibility result for interim non-constant solutions requires correlated beliefs. In this subsection, we obtain an impossibility result for two classes of correlated beliefs on finite supports. Firstly, this result holds if only two values are possible. Secondly, it holds if the beliefs have the uniform marginals.

Proposition 3.2: *Let $S = \{0, 1\}$. If a feasible mechanism is ex post efficient, then it is interim utility equivalent to the constant solution.*

Proof. Let q be a feasible and ex post efficient solution. Consider IC for player 1 of type $s \in S$ (multiplied by $f_1(s)/V$),

$$\sum_{t_2 \in S} f(s, t_2) q_1(s, t_2) \geq \sum_{t_2 \in S} f(s, t_2) q_1(s', t_2), \quad \text{for} \quad s' \neq s. \tag{3.5.1}$$

In other words, for $s' \neq s$,

$$f(s, s)(q_1(s, s) - q_1(s', s)) \geq f(s, s')(q_1(s', s') - q_1(s, s')). \tag{3.5.2}$$

By SYM and EFF, $q_1(s, s') = 1 - q_2(s, s') = 1 - q_1(s', s)$ and $q_1(s, s) = q_1(s', s') = 1/2$. For $q_1(s, s') < 1/2$, IC for type s is violated. For $q_1(s, s') > 1/2$, IC for type s' is violated. Hence, $q_1(s, s') = q_1(s', s) = 1/2$. Each player's interim utility must be a constant. □

Intuitively, in case of reporting profile $(1, 1)$ or $(0, 0)$, the winning probability for each player is $1/2$. In case of reporting profile $(0, 1)$ or $(1, 0)$, one of the players must be rewarded with a winning probability more than $1/2$. Suppose the player who reports 1 is rewarded. Then, it is a dominant strategy for any player to report 1. Hence, there exists no mechanism that rewards one player while punishes the other in case the players make different reports.

This result is independent of positively and negatively correlated beliefs. It counters the intuition that having negatively correlated beliefs makes it easier to have a monotone solution.

PROPOSITION 3.3: *Let $S = \{s_1, ..., s_n\}$, $n > 2$, and*

$$\sum_{t_2 \in S} f(t) = 1/n, \text{ for all } t_1 \in S, \tag{3.5.3}$$

$$\sum_{t_1 \in S} f(t) = 1/n, \text{ for all } t_2 \in S. \tag{3.5.4}$$

If a feasible mechanism is ex post efficient, then it is interim utility equivalent to the constant solution.

Proof. Let q be a feasible and ex post efficient solution. IC for player 1 of type $s \in S$ (multiplied by $f_1(s)/V$) is the same as (3.5.1). SYM implies $q_1(s, s') = q_2(s', s)$ for all $s, s' \in S$. EFF implies $q_2(s, s') = 1 - q_1(s, s')$, and $q_1(s, s) = 1/2$ for all $s, s' \in S$, Sum over $n \times (n - 1)$ inequalities for all $s' \in S$ and all $s \in S$, we have

$$\begin{aligned} &2\sum_{s\in S} f(s,s)q_1(s,s) + \frac{1}{2}\sum_{s\in S}\sum_{t_2\neq s} f(s,t_2)[q_1(s,t_2) + q_1(t_2,s)] \\ &\geq \sum_{s\in S} f(s,s) + \sum_{s\in S}\sum_{t_2\neq s} f(s,t_2)q_1(s,s) + \sum_{k=1}^{n}[\sum_{t_2\neq s_{k+1}} f(s_k,t_2) - \sum_{t_2\neq s_k} f(s_{k+1},t_2)]q_1(s_{k+1},s_k) \\ &= \sum_{s\in S} f(s,s) + \frac{1}{2}\sum_{s\in S}\sum_{t_2\neq s} f(s,t_2) + \sum_{k=1}^{n}[f_1(s_k) - f_1(s_{k+1})]q_1(s_{k+1},s_k), \end{aligned} \tag{3.5.5}$$

where $s_{n+1} := s_1$ and the last equality uses the fact that

$$\begin{aligned} \sum_{t_2\neq s_{k+1}} f(s_k,t_2) - \sum_{t_2\neq s_k} f(s_{k+1},t_2) &= \sum_{t_2\neq s_{k+1}} f(s_k,t_2) + f(s_k,s_{k+1}) - \sum_{t_2\neq s_k} f(s_{k+1},t_2) - f(s_k,s_{k+1}) \\ &= f_1(s_k) - f_1(s_{k+1}). \end{aligned} \tag{3.5.6}$$

Equivalently,

$$0 \geq \sum_{k=1}^{n}[f_1(s_k) - f_1(s_{k+1})]q_1(s_{k+1},s_k). \tag{3.5.7}$$

Because $f_1(s_k) = f_1(s_{k+1})$ for all $k = 1, ..., n$,

$$\sum_{k=1}^{n}[f_1(s_k) - f_1(s_{k+1})]q_1(s_{k+1},s_k) = 0. \tag{3.5.8}$$

Hence, all $2n \times (n-1)$ inequalities are binding. Each player's interim utility must be a constant. By SYM, this constant is $V/2$. □

The class of correlated beliefs with the uniform marginals is not too "small" compared to the class of independent beliefs, in terms of the number of free parameters in beliefs. For beliefs without restrictions, the probability simplex condition implies the total degrees of freedom are $n^2 - 1$. For the class of independent beliefs,

$$f_1(s_1), ..., f_1(s_{n-1}), f_2(s_1), ..., f_2(s_{n-1}),$$

are $2(n-1)$ parameters. The degrees of freedom are given by

$$n^2 - 1 - 2(n-1).$$

For the class of correlated beliefs with the uniform marginals,

$$f(s_1, s_1), ..., f(s_1, s_{n-1}), ..., f(s_{n-1}, s_1), ..., f(s_{n-1}, s_{n-1}),$$

are $(n-1)^2$ parameters. The degrees of freedom are given by

$$n^2 - 1 - (n-1)^2.$$

For $n = 3$, the two classes of beliefs have the same degrees of freedom. For $n > 3$, the class of independent beliefs has larger degrees of freedom.

The class of symmetric beliefs with uniform marginals corresponds to $n \times n$ doubly stochastic matrices satisfying symmetry, by multiplying n for each $f(t)$ in (3.5.3). The Birkhoff theorem shows that the set of $n \times n$ doubly stochastic matrices is the convex hull of the set of $n \times n$ permutation matrices.[9]

An increase in the number of types introduces more variation in the beliefs and may expand the set of feasible outcomes. However, Proposition 3.3 shows that if beliefs have uniform marginals, pooling all incentive constraints of a player altogether makes all such constraints binding. The impossibility result holds again for both positively and negatively correlated beliefs.

Remarks. In Propositions 3.2 and 3.3, ex post efficiency is indispensable for our results. We consider two ways to relax it to ϵ-ex post inefficiency, for $\epsilon > 0$ small. We may define a feasible mechanism q as ϵ-ex post inefficient, if $\{t \in T : q_0(t) > 0\}$ has Lebesgue measure ϵ (on $\mathbb{R}^2$). Alternatively, we may require q to satisfy $q_0(t) = \epsilon$ for all $t \in T$. For the second definition, even in case of independent beliefs, interim monotonicity is easily reconciled with other constraints. A solution mechanism is simple: $q = (\epsilon, 1/2 - \epsilon/2, 1/2 - \epsilon/2)$. In this case, each player i gets $\epsilon t_i + (1-\epsilon)V/2$ for all type profiles.

[9] Katz (1970) shows that the set of $n \times n$ symmetric doubly stochastic matrices is the convex hull of the set of all matrices of form $\frac{1}{2}(P + P^\top)$, where P is an $n \times n$ permutation matrix and $P^\top$ is its transpose.

3.6. Extensions

3.6.1. Ex Post Properties

We now consider stronger solution concepts by requiring a mechanism being robust to beliefs. We say a mechanism q is dominant strategy incentive compatible (DSIC), if truthful reporting by both players constitutes a dominant strategy equilibrium, or $u_i(q(t_i, t_j), t_i) \geq u_i(q(t'_i, t_j), t_i)$ for all $t_i, t'_i, t_j \in S$, $i = 1, 2$.

In many contexts, the players may compare their utility outcomes after they know the state. To describe such ex post utility comparison, we introduce an ex post monotonicity property. At the ex post stage, the expected utility is (i) non-constant and (ii) nondecreasing everywhere in disagreement payoff.

Definition 3.2: Let $f \in \mathcal{F}$ and let q be a feasible mechanism. We say q is ex post monotone (EP-M) if, (i) for $i = 1, 2$ and all $\overline{u} \in \mathbb{R}$, the event $\{t \in T : u_i(q(t), t_i) = \overline{u}\}$ has a probability measure (on $\mathbb{R}^2$) less than 1, and (ii) if for $i = 1, 2$, $u_i(q(t_i, t'_j), t'_i) \leq u_i(q(t_i, t_j), t_i)$ for all $t'_i \leq t_i$, $t_j \leq t'_j$, and $t_i, t'_i, t_j, t'_j \in S$.

By definition, ex post non-constant is necessary for interim non-constant and ex post (interim) monotonicity, while ex post monotonicity is neither necessary nor sufficient for interim monotonicity without further specifying beliefs.[10]

Ex post monotonicity also implies that for all states in which one player has a higher disagreement payoff, he receives weakly more than the other player. An example is the ex post Egalitarian solution: for $i = 1, 2$ and all $t \in T$, $u_i(q(t), t_i) = t_i + (V - t_i - t_j)/2$.

In Proposition 3.4 below, we find that if strong conditions such as dominant strategy incentive compatibility or ex post monotonicity are required, we have impossibility results on the ex post non-constant solutions.[11]

Proposition 3.4: *Let $f \in \mathcal{F}$. If a feasible mechanism is (i) ex post efficient and dominant strategy incentive compatible, or (ii) ex post efficient and ex post monotone, then it is the constant solution.*

Proof. (i) Take $t'_i > t_i$. DSIC implies $u_i(q(t_i, t_j), t_i) \geq u_i(q(t'_i, t_j), t_i)$, for all $t_j, \in S$. By EFF, $u_i(q(t_i, t_j), t_i) \geq u_i(q(t'_i, t_j), t_i) = u_i(q(t'_i, t_j), t'_i)$, for all $t_j \in S$, and $u_i(q(t'_i, t_j), t'_i) \geq u_i(q(t_i, t_j), t'_i) = u_i(q(t_i, t_j), t_i)$, for all $t_j \in S$. Hence, $u_i(q(t_i, t_j), t_i) = u_i(q(t'_i, t_j), t'_i)$, for all $t_j \in S$. By SYM and EFF, $q_1(t) = q_2(t) = 1/2$ for all $t \in T$.

[10] It is worth noting that de Clippel (2012) introduces the following interim egalitarian criterion, which is much stronger that I-M because the latter requires the interim surplus to be equalized across players and types and thus the interim utility is a positive affine transformation of disagreement payoffs, i.e. a feasible mechanism q is interim egalitarian, if for all t_1, t_2, $U_1(q|t_1) - t_1 = U_2(q|t_2) - t_2$.

[11] Recent papers show that an interim utility of any Bayesian mechanism can by obtained with a dominant strategy mechanism with the same ex ante social surplus, in a linear IPV environment (Manelli and Vincent, 2010; Gershkov, Goeree, Kushnir, Moldovanu and Shi, 2012). Because we do not allow transfers, the equivalence does not apply in our environment.

(ii) Take $t_i' > t_i$. By EP-M and EFF, for all $t_j \in S$, $u_i(q(t_i,t_j),t_i) \le u_i(q(t_i',t_j),t_i') = u_i(q(t_i',t_j),t_i)$. Notice that $u_i(q(t_i,t_j),t_i) = Vq_i(t_i,t_j)$ is bounded everywhere and integrable, $U_i(q|t_i) \le \int_S u_i(q(t_i',t_j),t_i)f_j(t_j|t_i)dt_j = U_i(q,t_i'|t_i)$. By IC, it implies $U_i(q|t_i) \ge U_i(q,t_i'|t_i)$. Hence, $u_i(q(t_i,t_j),t_i) = u_i(q(t_i',t_j),t_i')$ for almost all $t_j \in S$. By SYM, $q_1(t) = q_2(t) = 1/2$, a.e. $t \in T$. □

To interpret part (ii) of Proposition 3.4, notice that for $i = 1, 2$, ex post monotonicity requires that the probability of player i obtaining the object is nondecreasing in player i's report, given any report of the other player. Hence, player i has incentive to overreport. The result immediately follows a lack of incentive compatibility for the ex post Egalitarian solution, which is ex post efficient and ex post monotone.

3.6.2. Three Players

We now discuss how the results change if there are more than two players. For the case of correlated beliefs, we provide a three-player example where Proposition 3.2 does not generalize.

EXAMPLE 3.1: Suppose $V > 3$ and $f : \{0,1\}^3 \to \mathbb{R}_{++}$ satisfies the following conditions:[12]

(i) $f_{010} = f_{001} = f_{100}$, $f_{011} = f_{110} = f_{101}$, $f_i(0) = f_i(1)$, $i = 1, 2, 3$,

(ii) $f_{000} > f_{011}$, $f_{111} > f_{100}$,

(iii) $\frac{1}{3}f_{000} + f_{001} < \frac{1}{3}f_{111} + f_{110}$,

(iv) $f_{110} > f_{100}$.

Then there exists a feasible and ex post efficient mechanism that delivers an interim monotone utility.

Proof. Let $q : \{0,1\}^3 \to \Delta(\{0,1,2,3\})$ be the mechanism given by,

$$q(0,0,0) = q(1,1,1) = (0,\frac{1}{3},\frac{1}{3},\frac{1}{3}), \quad q(1,1,0) = q(0,0,1) = (0,\frac{1}{2},\frac{1}{2},0), \tag{3.6.1}$$
$$q(1,0,0) = q(0,1,1) = (0,0,\frac{1}{2},\frac{1}{2}), \quad q(1,0,1) = q(0,1,0) = (0,\frac{1}{2},0,\frac{1}{2}).$$

It is easily shown that given conditions (i)-(iv) on the beliefs, q is feasible, EFF and I-M.[13] □

[12] For simplicity, denote $f_{xyz} = f(x,y,z)$ for $x, y, z \in \{0,1\}$.

[13] The incentive condition for player 1 with type 0 and 1 (multiplied by $f_1(0)$ and $f_1(1)$),

$$f_{000}\tfrac{1}{3} + f_{001}\tfrac{1}{2} + f_{010}\tfrac{1}{2} \ge f_{001}\tfrac{1}{2} + f_{010}\tfrac{1}{2} + f_{011}\tfrac{1}{3}, \text{ and}$$

$$f_{101}\tfrac{1}{2} + f_{110}\tfrac{1}{2} + f_{111}\tfrac{1}{3} \ge f_{100}\tfrac{1}{3} + f_{101}\tfrac{1}{2} + f_{110}\tfrac{1}{2}.$$

The interim monotonicity requires

$$\tfrac{1}{3}f_{000} + f_{001} < \tfrac{1}{3}f_{111} + f_{110}.$$

Intuitively, a player is punished if he is the only person who reports 1 or 0. He will be rewarded if he belongs to the majority. The beliefs assumption allows the mediator to better identify which player is more likely to be a deviator.

Condition (ii) ensures incentive compatibility. For a player with type 0, if the other two players draw $(0,1)$ or $(1,0)$, then a deviation to report 1 will not change his allocation. If the others draw $(0,0)$, the deviation loses an allocation probability of $1/3$. If the others draw $(1,1)$, the deviation wins an allocation probability of $1/3$. The profitability of the deviation depends on the likelihoods of $(0,0,0)$ and $(0,1,1)$. A similar analysis applies to a player with type 1. Condition (iii) ensures interim monotonicity. The condition implies a higher ex ante utility of type 1 compared to type 0. If condition (iv) holds, q is individually rational.

A one-parameter family of beliefs is as follows. For some $\epsilon \in (0, \frac{1}{12}]$, let

$$f_{000} = \frac{1}{2} - 4\epsilon, \;\; f_{111} = \frac{1}{2} - 5\epsilon, \;\; f_{100} = \epsilon, \;\; f_{110} = 2\epsilon, \tag{3.6.2}$$

For each player i, the interim utility vector is given by

$$(U_i(0), U_i(1)) = (\frac{1}{3}V(1-2\epsilon), \frac{1}{3}V(1+2\epsilon)). \tag{3.6.3}$$

As $\epsilon \to 0$, the beliefs become almost perfectly correlated and the mechanism q remains a solution. In case $\epsilon = 0$, condition (iii) is violated. Hence, the existence of a monotone solution does not satisfy a continuity in beliefs.

The result shows that varying the number of players expands the support of beliefs, the set of allocation rules, and hence the set of interim utility outcomes.

3.6.3. A Triangular Support

Instead of assuming the unit square support for the beliefs, we may consider a triangular support $T = \{t \in \mathbb{R}^2 : t_1 > 0, t_2 > 0, t_1 + t_2 < 1\}$, and the joint density f is continuous and strictly positive on the support. Then it remains true that at all states, the status quo is ex post Pareto dominated for $V > 2$. With this support assumption, the previous results on correlated beliefs may no longer hold. First, a distribution with uniform marginals cannot have an everywhere positive density on the triangle support. Second, a player with a type greater than 1/2 knows that the other player's type is lower than 1/2.

The individual rationality requires

$$V(\tfrac{1}{3}f_{000} + f_{001}) \geq 0, \text{ and } V(\tfrac{1}{3}f_{111} + f_{110}) \geq f_i(1) = f_{111} + f_{101} + f_{110} + f_{100}.$$

3.7. Conclusion

In this chapter, we study a bargaining problem with incomplete information when the status quo is ex post inefficient for all states of the world. We discuss the existence of ex post efficient and monotone bargaining solutions, for which each player's interim utility is responsive to his disagreement payoff. The assumption on the ex post inefficient status quo makes Myerson and Satterthwaite's impossibility not to arise, for example, the constant mechanism is a solution. However, if non-constant solutions are considered, tensions between efficiency and monotonicity arise, since a mechanism must now satisfy: (i) incentive compatibility, (ii) feasibility and efficiency, i.e. any punishment for a player must be fully translated into rewards for the other, and (iii) some variation in utility allocation.

For independent beliefs, we obtain an impossibility result. For correlated beliefs on finite supports, we obtain an impossibility result for two classes of beliefs. In particular, we show that if the support contains only two points for each player, the impossibility result holds for all beliefs. We also find that if stronger solution concepts such as dominant strategy incentive compatibility or ex post weak monotonicity are required, a stronger nonexistence result holds.

We also study how the results change when there are more than two players. With independent beliefs, the impossibility result remains true irrespective of the number of players. With correlated beliefs, we provide an example of the positive result for three players. The implications are twofold: (i) An increase in the number of players enlarges the set of mechanisms such that an efficient and monotone solution arises.[14] In case of two players, it is difficult for the arbitrator to identify a deviator if their reports are not aligned, even if beliefs are correlated. In case of three players, the arbitrator can reward the majority. (ii) It suggests an empirical prediction: In the negotiations where trade is always ex post efficient and the players' outside options are correlated, the fully efficient outcome is more likely to arise for large groups than for small groups.[15]

[14] Here we consider only the grand coalition and the single player coalitions.

[15] Isaac, Walker and Arlington (1994) studies some experiments on voluntary contribution mechanisms and provides some evidence that a group's ability to provide the efficient level of a pure public good is positively related to group size.

Chapter 4

Efficient Mechanisms for Bilateral Trading

4.1. Introduction

Myerson and Satterthwaite (1983) (MS thereafter) introduces a bilateral trade problem in which a seller has an indivisible object to sell and a buyer wants to buy it. The seller and the buyer have private information on their values of the object, which are independently drawn from a common support. The authors prove the nonexistence of an incentive compatible, individually rational, balanced budget, and ex post efficient trading mechanism in this environment. They also study the optimal trading mechanism that maximizes the ex ante trading surplus. In their context, each player observes his private value perfectly. However, for many other contexts, there is some intermediary that controls the information that is accessible to the players, and the players only have coarse information when making a decision.

Suppose the intermediary can release less information to the players compared to the MS information structure. What is the optimal information structure and the trading procedure consistent with this information structure that maximizes the ex ante trading surplus? If we look at broader classes of trading environments, for which classes is the MS information structure (not) optimal? In this chapter, we provide a partial answer to the first question, by showing that the MS information structure is not optimal. We show that there exist some coarser partitions (for both the seller and the buyer) such that we can find a trading mechanism that attains a higher ex ante trading surplus than the optimal mechanism of the MS.

There are at least two countervailing effects when we vary the information partitions. A pair of finer partitions increases the possibility of efficient information aggregation and efficient trade, and thus the ex ante trading surplus. However, the finer partitions worsen the individual rationality and the incentive compatibility constraints, and thus restrict the set of feasible trading outcomes. If the second effect dominates the first, then coarse partitions increase the ex ante trading surplus.

That a coarse information structure may increase the ex ante social welfare has been noticed by Hirshleifer (1971). In the context of an exchange economy for risk, he assumes that the players are risk averse and that the endowments of wealth from different states differ across the players. If the players trade in complete markets for contingent claims ex ante, then they will share some of the risk. If the players perfectly learn the state before they trade, then there will be no trade at all. The no trade allocation is ex ante Pareto dominated by the allocation of risk with no information. In his model, the decisions are made either ex ante or ex post. This differs from mechanism design environments where the decisions are made at the interim stage.

Bergemann and Pesendorfer (2007) considers the problem of designing information structures and auctions. The seller aims to choose the accuracy by which bidders learn their values and the auction to maximize his revenue. They show that an optimal information structure that maximizes the seller's revenue exists, which is represented by monotone partitions being asymmetric across players. Bergemann, Brook and Morris (2015) characterizes the information structure which minimizes the seller's revenue for any value distribution, in a first-price auction. In these auction models, coarser partitions of the buyers always lower the ex ante social welfare.

The chapter also relates to the recent literature on Bayesian persuasion. Kamenica and Gentzkow (2011) considers a problem of a sender who commits to a signal structure in order to persuade the receiver to take the sender's preferred action. The authors characterize the optimal signal for any given set of preferences and prior beliefs. Roesler and Szentes (2016) considers a problem where the buyer designs his own information structure in order to induce the seller to charge his preferred price. The buyer can choose a costless, unbiased signal about his true value. The seller makes a take-it-or-leave-it offer to the buyer, knowing the joint distribution of the buyer's value and signal but not their realizations. Compared to theirs, our model is a classical mechanism design problem with two-sided incomplete information.

The remainder of this chapter is organized as follows. Section 4.2 introduces the model. Section 4.3 presents the main results. Section 4.4 concludes.

4.2. Model

In Myerson and Satterthwaite (1983), the seller (player 1) owns an indivisible object and the buyer (player 2) wants to buy it. Each player $i = 1, 2$ has a value $\tilde{v}_i$ distributed according to a c.d.f. $F_i(\cdot)$ with continuous and positive density $f_i(\cdot)$ over the support $[0, 1]$. The players' values are assumed to be independent. The players are risk neutral and have quasi-linear utility given by $u_1(q, m, v_1) = -qv_1 + m$, and $u_2(q, m, v_2) = qv_2 - m$, where $q \in [0, 1]$ is the probability of trade, $m \in \mathbb{R}$ is the monetary payment from the buyer to the seller, and v_i is the value of player i.

In the MS information structure, each player knows his own value at the time of bargaining, but considers the other's value as random. For player i, a partition (of $[0, 1]$) is defined by a family of sets T_i such that (i) $\emptyset \notin T_i$, and (ii) $\bigcup_{A \in T_i} A = [0, 1]$, and (iii) if $A \in T_i$ and $B \in T_i$, then $A \cap B = \emptyset$. For $v_i \in [0, 1]$, denote $t_i(v_i)$ be the element of

T_i that contains v_i and denote $\tilde{t}_i$ the corresponding signal induced by $\tilde{v}_i$. A partition T_i is monotone if for each $A \in T_i$, A is convex. A partition T_i is said to be finite if T_i is a finite set.

For each player i with type $t_i \in T_i$, his conditional belief on v_i is given by $F_i(v_i|t_i) = \lambda(\tilde{v}_i \leq v_i, \tilde{t}_i = t_i)/\lambda(\tilde{t}_i = t_i)$, where λ is the Lebesgue measure. Denote $E(\tilde{v}_i|t_i) = \int_0^1 v_i dF_i(v_i|t_i)$.

For the players, a pair of partitions is given by $T = (T_1, T_2)$. The MS information structure $\overline{T} = (\overline{T}_1, \overline{T}_2)$ is then given by $t_i(v_i) = v_i$, for all $v_i \in [0,1], i = 1, 2$. We say T is coarser than $\overline{T}$ if T_1 and T_2 are partitions and $T_i \neq \overline{T}_i$ for some i.

For a pair of partitions T, a direct mechanism is given by $\mu = (q, m) : T \to [0,1] \times \mathbb{R}$. The allocation rule q assigns to each type profile a probability of trade and the payment rule m assigns to each type profile a monetary payment from the buyer to the seller. Given the other player reports truthfully, the interim expected utility of player i with type $t_i \in T_i$ from reporting $t'_i \in T_i$ is given by

$$U_i(\mu, t'_i|t_i) = \int_0^1 \int_0^1 u_i(\mu(t'_i, t_j(v_j)), v_i) dF_i(v_i|t_i) dF_j(v_j). \tag{4.2.1}$$

We say that μ is incentive compatible if truth telling by both players is a Bayesian equilibrium. For an incentive compatible mechanism μ, denote the interim utility by $U_i(\mu|t_i) = U_i(\mu, t_i|t_i)$. μ is individually rational if $U_i(\mu|t_i) \geq 0$, for all $t_i \in T_i$, $i = 1, 2$. For any T, a mechanism is T-feasible if it is incentive compatible and individual rational. Let $M(T)$ be the set of T-feasible mechanisms.

A mechanism $\mu = (q, m)$ is ex post efficient, if for all $v \in [0,1]^2$,

$$q(t_1(v_1), t_2(v_2)) = \begin{cases} 0 & \text{if } v_1 \geq v_2, \\ 1 & \text{otherwise.} \end{cases}$$

A mechanism μ is (constrained) ex ante efficient if

$$\mu \in \arg\max_{\mu' \in M(T)} \sum_{i=1}^{2} E[U_i(\mu'|\tilde{t}_i)]. \tag{4.2.2}$$

That is, μ yields the highest ex ante trading surplus. We say an allocation rule q is ex ante efficient if $\mu = (q, m)$ is ex ante efficient. The efficiency loss of μ is given by the ratio of the ex ante trading surplus from μ to that from the ex post efficient allocation.

Myerson and Satterthwaite (1983) shows that the presence of private information and voluntary participation implies the impossibility of ex post efficiency. Intuitively, the seller types with high reservation values and the buyer types with low reservation values have little incentive to participate in a trading mechanism at the interim stage. To induce them to participate, the third party must pay them some incentive costs. In an ex post efficient mechanism, the incentive costs are so large such that voluntary participation is not attainable without any subsidy from the third party.

LEMMA 4.1: *(Myerson-Satterthwaite) (i) There exists no $\overline{T}$-feasible and ex post efficient mechanism. (ii) Suppose $v_1 + \frac{F_1(v_1)}{f_1(v_1)}$ and $v_2 - \frac{1-F_2(v_2)}{f_2(v_2)}$ are strictly increasing. Let*

$$c_1(v_1, \alpha) = v_1 + \alpha \frac{F_1(v_1)}{f_1(v_1)}, c_2(v_2, \alpha) = v_2 - \alpha \frac{1 - F_2(v_2)}{f_2(v_2)}. \tag{4.2.3}$$

Then there exists some $\alpha \in (0, 1)$ such that

$$q^\alpha(v_1, v_2) = \begin{cases} 0 & \text{if } c_1(v_1, \alpha) \geq c_2(v_2, \alpha), \\ 1 & \text{otherwise.} \end{cases} \tag{4.2.4}$$

is ex ante efficient.

The parameter α reflects the Lagrangian multiplier of the individual rationality constraint of the players. A larger α implies a higher shadow price and hence a larger incentive distortion. For $\tilde{v}_1, \tilde{v}_2 \sim U[0, 1]$, $\alpha = \frac{1}{3}$ and the ex ante efficient allocation rule is given by

$$q^\alpha(v_1, v_2) = \begin{cases} 0 & \text{if } v_1 \geq v_2 - \frac{1}{4}, \\ 1 & \text{if } v_1 < v_2 - \frac{1}{4}, \end{cases} \tag{4.2.5}$$

The MS solution yields the ex ante trading surplus 0.14. The efficiency loss is 15.6% and the probability of no trade is 71.9%.

4.3. Results

The following result shows that even if we consider coarser partitions than those in MS, ex post efficiency is not possible. The reason is as follows: A pair of coarse partitions requires that at least one player will receive a coarse signal with positive probability. Then, for a T-feasible mechanism, there exists an event such that either (i) the mechanism requires no trade while trade is efficient, or (ii) the mechanism requires trade while no trade is efficient. In other words, any pair of coarser partitions aggregates information inefficiently with positive probability.

PROPOSITION 4.1: *If T is a pair of monotone partitions, there does not exist a T-feasible mechanism that is ex post efficient.*

Proof. Consider at least player 1 has a coarser partition T_1 with element $E_1 = [a_1, b_1]$ for some $a_1 < b_1$. Let l be the unique line segment $(E_1 \times [0, 1]) \cap \{v \in [0, 1]^2 : v_1 = v_2\}$ and let $[a_2, b_2]$ be its projection on T_2. Then $a_2 = a_1$ and $b_2 = b_1$. Let $A = [a_1, b_1] \times [a_2, b_2]$. For every $E_2 \in T_2$, the convexity of E_2 implies that E_2 is either a singleton or an interval being closed, or open, or half open. Hence, E_2 is measurable. We have two cases.

Case 1. There exists an element $E_2 \in T_2$ such that $p(([0,1] \times E_2) \cap A) > 0$, where p is the Lebesgue measure in $\mathbb{R}^2$. Denote $A_2 = ([0,1] \times E_2) \cap A$. Then, $\{v \in [0,1]^2 : v_1 = v_2\}$ separates A_2 into $A_{21} = \{v \in A_2 : v_2 > v_1\}$ and $A_{22} = \{v \in A_2 : v_2 < v_1\}$, with $p(A_{21}) > 0$, and $p(A_{22}) > 0$. On the other hand, an allocation rule $q(t_1, \cdot)$ must be measurable with respect to T_2, for all $t_1 \in T_1$. So, the decision q is constant on $E_1 \times E_2$, which implies q is constant on A_2, or $q(A_{21}) = q(A_{22})$. This contradicts to ex post efficient $q^*(A_{21}) = 1 \neq 0 = q^*(A_{22})$.

Case 2. For each element $E_2 \in T_2$, $p(([0,1] \times E_2) \cap A) = 0$. For every $E_2 \cap [a_2, b_2] \neq \emptyset$, $E_2 \cap [a_2, b_2]$ is a singleton. Then player 2 must perfectly observe v_2 for some $[c_2, d_2] \subset (a_2, b_2)$, $c_2 < d_2$. Ex post efficiency requires $q([a_1, b_1], v_2) \in \{0, 1\}$ for all $v_2 \in [c_2, d_2]$. Then, there exists $B_2 \subseteq [c_2, d_2]$ such that $q([a_1, b_1], v_2) = \chi_{B_2}(v_2)$ for all $v_2 \in [c_2, d_2]$. Since $q([a_1, b_1], \cdot)$ is Lebesgue measurable on $[c_2, d_2]$, B_2 is a measurable subset. Let $A_2 = [a_1, b_1] \times B_2$.

If $p(A_2) > 0$. then $\{v \in [0,1]^2 : v_1 = v_2\}$ separates A_2 into $A_{21} = \{v \in A_2 : v_2 > v_1\}$ and $A_{22} = \{v \in A_2 : v_2 < v_1\}$, with $p(A_{21}) > 0$, $p(A_{22}) > 0$. A similar analysis as Case 1 implies inefficiency occurs with positive probability.

If $p(A_2) = 0$, then $C_2 = [a_1, b_1] \times ([c_2, d_2] \setminus B_2)$ satisfies $p(C_2) > 0$. A similar analysis as before implies inefficiency occurs with positive probability.

Finally, for an element with half-open or open intervals of T_1, the analysis is similar to the closed interval case above, since they differ only in zero measure events. □

Proposition 4.2 below is the main result of this chapter.

PROPOSITION 4.2: *Assume $\tilde{v}_1, \tilde{v}_2 \sim U[0,1]$. There exists a pair of partitions T and a T-feasible mechanism (q, m) that yields a higher ex ante trading surplus than the MS ex ante efficient mechanism.*

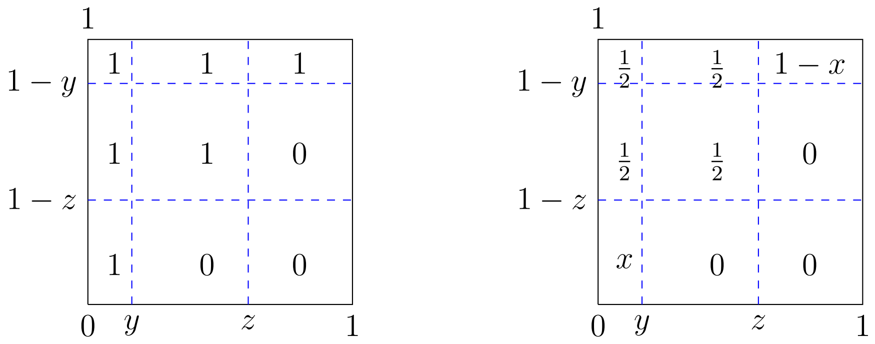
Proof. We provide an example of such an information structure and the corresponding mechanism. Let $y = 0.166$ and $z = 0.606$. Consider a pair of finite monotone partitions $T = (T_1, T_2)$ given by

$$T_1 = \{[0, y], (y, z], (z, 1]\} \text{ and } T_2 = \{[0, 1-z], (1-z, 1-y], (1-y, 1]\}.$$

We denote these elements of partitions (or types) by $\underline{t}_i, \hat{t}_i, \bar{t}_i$ for $i = 1, 2$. For this T, we define the mechanism (q, m) as follows.

The allocation rule q is given by Figure 4.1a). The horizontal axis is the seller's value and the vertical axis is the buyer's value. By construction, the lowest type of the seller (and the highest type of the buyer) always trade. The highest type of the seller (and the lowest type of the buyer) trade with a strictly positive probability.

The payment rule m is given by Figure 4.1b). Let $x = 0.197$. The payment rule requires: (i) In case of trade and the report profile is not $(\underline{t}_1, \underline{t}_2)$ or $(\bar{t}_1, \bar{t}_2)$, the buyer pays the seller 1/2. (ii) In case of trade and the report profile is $(\underline{t}_1, \underline{t}_2)$, then the buyer pays the seller x; if the report profile is $(\bar{t}_1, \bar{t}_2)$, then the buyer pays the seller $1-x$. (iii) In case of no trade, there is no payment.



a) Allocation rule *q* *b*) Payment rule *m*

Figure 4.1

It is easy to check that such a mechanism is T-feasible. The ex ante trading surplus from this scheme is 0.159.[1] The efficiency loss is 4.6% and the probability of no trade is 51%. □

The driving forces for this result are as follows. First, note that such a mechanism (q, m) is not $\overline{T}$-feasible, because in that case, the individual rationality condition for the seller with $v_1 = z$ is violated, or

$$-y + y(1-x) < 0. \tag{4.3.1}$$

For the partitions T, this condition is replaced by

$$-yE(\tilde{v}_1|(z,1]) + y(1-x) \geq 0. \tag{4.3.2}$$

Hence, a coarser partition relaxes the individual rationality constraint for the seller with higher values. Instead of perfectly observing his value and opting out with probability one, the seller with a value in $[z, 1]$ is only informed his conditional expected value, hence he has more incentive to participate. A similar analysis applies to the buyer.

[1]The ex ex ante trading surplus is given by

$$\begin{aligned}&\int_0^1 \int_0^1 (v_2 - v_1)\mathbb{1}\{v_2 > v_1\} f_1(v_1) f_2(v_2) dv_1 dv_2 \\ =&\frac{1}{2} - \frac{1}{2}y^2 - \frac{1}{2}(z^2 - y^2)z - \frac{1}{2}(1-z)^2(z-y) - \frac{1}{2}y(1-z^2) - \frac{1}{2}(1-y)^2(1-z).\end{aligned}$$

Second, for the partitions in MS, the incentive compatibility condition for the seller with $v_1 = z$ is given by

$$-z \cdot z + z\frac{1}{2} \geq -y \cdot z + y(1-x). \tag{4.3.3}$$

For the partitions T, this condition is replaced by

$$-zE(\tilde{v}_1|(y,z]) + z\frac{1}{2} \geq -yE(\tilde{v}_1|(y,z]) + y(1-x). \tag{4.3.4}$$

A coarser partition relaxes the incentive constraint for the seller with a value equal to z and thus also for the seller with a value lower than z. A similar analysis applies to the buyer.

Hence, T generates new utility allocations that are not attainable under $\overline{T}$. On the other hand, T eliminates some utility allocations that were attainable under $\overline{T}$. The players must bear some losses in the ex ante trading surplus from aggregating information imprecisely. These two effects finally determine the gains in the ex ante trading surplus from a coarser information structure.

4.4. Conclusion

In this chapter, we show that the original information partition in MS is not the optimal information partition that maximizes the ex ante trading surplus. With uniform priors, we construct a pair of coarser partitions and a feasible mechanism that outperforms the ex ante efficient solution of the MS information partition.

The main message is that if the mechanism designer can control the information structure of the players[2], the original information partition from MS will not arise. The result has implications in practice. Firstly, there are many situations such that a player has only access to coarse information about his own preferences. For a one-to-one matching market, the value from matching a partner will be fully revealed after a match. For experience goods, a buyer usually has a rough idea about the consumption experience at the time of purchase. Secondly, some intermediaries indeed have some control over players' information. Two-sided platforms, such as B2C platforms and dating websites, provide users access to a substantial number of opportunities in which they are interested. Before matching with their trading partners, a buyer has to decide how much and what to search about his value for the good, and a seller has to decide how much to learn about his outside option. For an isolated pair of partners, the profit-maximizing platform may concern whether the existing trading partners benefit from coarser information on

[2]Hurkens and Vulkan (2006) shows that games with exogenous information structures that have a unique Nash equilibrium are robust to endogenization of the information structures. Hence, such games can be analyzed using an exogenous information structure even when they in fact describe economic situations where information gathering seems natural.

their matching values and outside options.[3]

It remains to be investigated whether coarsening the partition of only one player can improve efficiency. For one-side uncertainty on the buyer's value, the answer is no. An ex post efficient mechanism exists in the case that the buyer perfectly knows his value[4] but not in the case with coarse partitions.

Finally, we notice that in contrast to the MS environment, in the auction environment of Myerson (1981), coarser information always reduces ex ante efficiency. It is because misallocation of the object must occur with positive probability under coarser partitions. A natural question then arises: Start from the players perfectly knowing their private values and consider a pair of coarser partitions, is the result in this chapter specific for the bargaining problem of the MS? Does it remain true for other classes of mechanism design problems than MS? Characterization of all classes of such problems is an interesting question and left for future work.

[3]Hagiu and Jullien (2011) studies how intermediaries can use information on consumers characteristics in order to affect matching between firms and consumers. They study the sources of an intermediary's incentives not to optimize the search process by which consumers find the stores (sellers) that the intermediary provides access to.

[4]For illustration, assume $v_1 = a \in (0, 1)$ and $\tilde{v}_2 \sim U[0, 1]$, ex post efficiency requires the seller posts a price a and the buyer accepts or rejects such a price.

Chapter 5

Bounds on Revenue of Auctions with Two Complements and Resale

5.1. Introduction

In this chapter, we consider a seller's revenue maximization problem with two complementary objects, for example, spectrum licenses. In practice, large companies might value multiple licenses to serve large geographical locations more than the sum of the values of the separate licenses because the marginal cost of serving a larger area can be lower. The auctions widely used by the governments, e.g. simultaneous ascending auctions (SAA),[1] may suffer from an exposure problem: Bidders face the risk of paying too much for part of a package of licenses when the rest of the package is won by other bidders. The exposure problem[2] generates inefficient outcomes.[3] For illustration, consider a seller who wants to allocate two licenses A and B efficiently. Bidder 1 values the packages (A,B,AB) as complements, $(1, 2, 6)$, while bidder 2 values them as substitutes, $(4, 4, 4)$. In an SAA, bidder 2 demands at least one unit until the prices reach $(4, 4)$. But then bidder 1 would better not win AB at such prices.

Inefficient initial allocation of spectrum resources introduces resale opportunities. A private seller may organize his own auction to sell its spectrum holdings. Unlike the auctioning of government-owned assets, revenue maximization, rather than efficiency, is a private seller's objective. In this case, a revenue-maximizing auction (Myerson, 1981) may lead to splitting licenses among buyers inefficiently and neither buyer obtains the synergy. A natural question then is: Given a resale market among the buyers, can the initial seller obtain the revenue that he could get if such resale would be banned?

[1] Vickrey auctions are often considered as a means of efficiently allocating spectrums. However, due to its complexity of pricing rule and other several drawbacks (Milgrom, 2004), it is rarely observed in practice.

[2] Szentes and Rosenthal (2003) mentions various measures used to soften exposure problems: in spectrum auctions the simultaneous designs typically involve ascending prices, which allow bidders time to assess gradually the likelihood of successfully acquiring various combinations of spectrum blocks; and provisions for bid withdrawals are often included.

[3] Consider the Netherlands DCS-1800 auction in 1998. Eighteen lots were offered for sale. Two of the lots were designed to be large enough that a new entrant could use them to establish a new wireless telephone business. Alternatively, a new entrant who acquired perhaps four or six small licenses could combine them to support entry. The smaller licenses would therefore likely be complements for the new entrants, but substitutes for the incumbents. Finally, the final prices per unit of bandwidth for the two large lots were more than twice as high as for any of the sixteen smaller lots. The entrants, willing to pay high prices for large spectrums, were reluctant to bid for small spectrums.

While resale can restore full efficiency when there are no market frictions, Myerson and Satterthwaite (1983) shows that, if resale takes place under incomplete information and players' beliefs have overlapping supports, then no resale mechanism can produce the efficient outcome. Cramton, Gibbons and Klemperer (1987), on the other hand, shows that if players have relatively symmetric ownerships, then efficient resale is possible. In this chapter, we investigate a resale market with possibly symmetric ownerships and study its effects on the seller's revenue.

To motivate our model, consider the following example. Assume that the seller has two identical objects which he himself does not value. Buyer $i = 1, 2$ attaches a value equal to i for a single unit, and this is common knowledge. While buyer 1 has additive values for the units, buyer 2 values the units as complements with his value of the bundle being $\tilde{t}_2$, which is uniformly distributed on $[4, 6]$. If resale is impossible, Myerson's optimal auction asks buyer 2 to report a price and the bundle is allocated to this buyer if the reported price is higher than 4.5. Otherwise, the seller splits the units between the buyers at prices equal to the single unit values, so 1 and 2. Hence, the maximal expected revenue is 4.125.

Now assume that buyer 1 can sell his unit to buyer 2 after the seller's auction. To implement the expected revenue 4.125, the adjustment to Myerson's auction is simple: The seller splits the units with a price of 2 to buyer 2 and a price of 2.125 to buyer 1. Since there is no new information, buyer 1 will resell his unit to the other at a price equal to 2.5. Buyer 2 accepts if $t_2 - 2.5 \geq 2$. The cutoff value in allocation is exactly 4.5. Notice that buyer 1 receives a total utility of zero since the trading surplus and the utilities of seller and buyer 2 are the same as before.[4] The example illustrates that the Myerson revenue is implementable if one buyer has additive values with complete information and the other has complementary values, given a monopoly resale market. But what would happen if buyer 1 also has complementary values for units?

To answer this question, we consider a two-buyer two-object environment in which the buyers have one-dimensional private information on the bundle. The Vickrey auction allocates the bundle to one of the two buyers ex post efficiently and serves as a lower bound for the seller's revenue. The Myerson auction splits licenses among the buyers with positive probability. We assume fully transparent auction outcomes, i.e. both the bids and the allocation of the objects are announced publicly. At the resale stage, a mediator offers a resale mechanism that maximizes the surplus of the initial buyers given the auction outcome. We find that the Myerson revenue is unattainable and the maximal revenue is bounded above by the revenue from a modified Myerson auction (MMA). This auction requires selling the bundle with personalized reserve prices and the seller withholding one object in case the reserve prices are not met.

[4] Alternatively, with probability 1/4, buyer 1 does not sell the object and loses 9/8, and with probability 3/4, he sells and gains 3/8.

Several papers discuss single object auctions followed by resale under complete information. For asymmetric independent private values (IPV) bidders, Gupta and Lebrun (1999) shows that revenue ranking between the first and second price auctions depends on how the surplus is divided in the resale market. Haile (2003) considers a model with bidders having only noisy information regarding their true values, which are revealed to them only after the auction. There is again no general revenue ranking. A second strand of literature investigates single object auctions followed by resale under uncertainty. Ausubel and Cramton (1999) introduces the concept of "perfect resale", by assuming that all gains from trade are exhausted in resale. They characterize the optimal auction with perfect resale and IPV bidders, and find that it is optimal to assign goods to those with the highest values. Their result continues to hold in the multi-unit case. Hafalir and Krishna (2008) investigates the first price auction with resale for asymmetric IPV bidders, where the winner can resell the object to the loser through a monopoly offer. They find that the outcome of this game yields a higher seller's revenue than the efficient outcome in the second price auction.

Zheng (2002) is the first paper to consider optimal sale of one object with resale and without any restrictions on the number of periods, i.e. the current owner in each round can choose the current auction and cannot control future resale. The paper shows that under some conditions the seller can still achieve the Myerson revenue. Zhang and Wang (2015) considers one regular buyer with private values and one publicly known value buyer, with resale structured by a stochastic ultimatum game where the probabilities of being the proposer determines the buyers' bargaining powers. They find that the seller's revenue is increasing in the publicly known value buyer's bargaining power in the resale market. Myerson revenue is attainable only if this buyer has full bargaining power.

There are only a few papers that discuss auctioning complementary objects with resale.[5] In a framework introduced by Leufkens, Peeters, and Vermeulen (2006), Xu, Levin, and Ye (2015) studies sequential second price auctions followed by resale for two complements and two bidders. Very different from our setting, they assume that no bidder knows his valuation for the second object during the first auction, while it is common knowledge that winning the first auction increases this valuation by a factor. They find that if the loser of the first item makes an offer to the winner, a monotone equilibrium exists and the expected revenue to the seller can either increase or decrease

[5]For multi-unit auctions without resale, Ausubel and Cramton (1998) observes that bidders may have an incentive to reduce demand, i.e., to bid for fewer units than they actually want, in order to pay a lower price for the objects they do win. Bukhchandani and Huang (1989) analyzes a multi-unit discriminatory or uniform price auction with common values and resale. They examine the information linkage between auction and resale through announced bids. Hafalir and Kurnaz (2015) considers multi-unit discriminatory auctions with resale where symmetric IPV bidders have single-unit demands. When the winner (which turns out to be unique in a symmetric equilibrium) uses the optimal mechanism in the resale stage, there may not exist a symmetric and monotone equilibrium if there are more than two units.

with resale.

This chapter contributes to the literature in several aspects. First, we investigate the optimal auction with two complements and one-dimensional private values, where inefficient splits open the possibility of resale. Compared to previous works, we introduce an extreme resale procedure: a centralized market in which the third party organizes resale mechanisms for the buyers. Second, this chapter establishes the impossibility of obtaining the Myerson revenue. In contrast to single object models, we find that the two-object models make the rule of selecting the re-seller after splits a crucial modeling assumption. Since there is no natural generalization of the winner's optimal mechanism to two-object case, we conjecture that unattainability of the Myerson revenue remains to hold for other reasonable resale markets, which might indicate an intrinsic difference between one and two object environments.

The chapter is organized as follows. Section 5.2 introduces the model. Section 5.3 introduces the Vickrey and Myerson auctions. Section 5.4 investigates resale games. It provides the solution to the mediator's problem and the solution to the seller's problem. Theorem 5.1 is the main result of this chapter. Section 5.5 discusses some other resale market. Section 5.6 concludes.

5.2. Model

The seller has two identical units for sale, to which he attaches value 0. There are two (initial) buyers. Each buyer $i = 1, 2$ has a value of a_i for a single unit and a value $t_i \in [\underline{t}_i, \bar{t}_i]$ for the bundle. We assume that the values a_1 and a_2 are commonly known with $0 < a_1 < a_2$, and that t_i is private information of buyer i. We assume $\tilde{t}_i$, $i = 1, 2$, are independently distributed according to F_i with absolutely continuous density $f_i > 0$ on $[\underline{t}_i, \bar{t}_i]$. Denote $T_i = [\underline{t}_i, \bar{t}_i]$ and $T = T_1 \times T_2$. We assume F_i is regular: $J_i(t_i) = t_i - \frac{1-F_i(t_i)}{f_i(t_i)}$ is strictly increasing in t_i. We also assume the following support condition throughout.

Assumption 5.1: $\underline{t}_1 > a_1 + a_2$ and $\underline{t}_2 > 2a_2$.

This condition states that bundling is always more efficient than splitting the objects among the buyers. If the condition is not satisfied, then a buyer who obtains the bundle may find it profitable to resell one unit to the other buyer. Assumption 5.1 is stronger than that each buyer values two units as complements.[6] Denote $a_{12} = a_1 + a_2$.

Let $D_0 = \{0, 01, 02, 1, 2, s\}$ be the set of all partitions of the objects, i.e. either the seller keeps both objects ($d = 0$), or the seller splits with buyer i ($d = 0i$), or one buyer gets the bundle ($d = i$), or the buyers split the objects ($d = s$).

[6] If buyer 1 only has small complementarity, or $\underline{t}_1 \in [2a_1, a_1 + a_2)$, then our main result (Theorem 5.1) may not hold. We further discuss this point after Theorem 5.1.

Assumption 5.2: The seller can commit to not resell any unit after the initial sale.

The assumption is strong since if the seller himself initially withholds some unit(s), he has an incentive to resell.[7] So we assume that the resale takes place only among the buyers. If Assumption 5.2 holds, then $d = 0$ and 01 are redundant, i.e. 0 and 01 are chosen with zero probability by a revenue-maximizing seller. This is because 0 and 01 are always dominated by 02 for the seller's revenue, and since the seller can implement 02 whenever he wants. Hence, throughout this chapter, we restrict our attention to $D_1 = \{02, 1, 2, s\}$.

For the seller, a direct mechanism (auction) is given by $(q, m) : T \to \Delta(D_1) \times \mathbb{R}^2$. The allocation rule q contains the allocation of objects among the players for each reporting profile, and the payment rule $m = (m_1, m_2)$ specifies the payments from the buyers to the seller for each reporting profile. Denote by $\Delta(T_i)$ the set of all probability distributions on T_i. A behavioral reporting strategy for buyer i is given by $\sigma_i : T_i \to \Delta(T_i)$ satisfying $\int_{T_i} \sigma_i(\hat{t}_i|t_i)d\hat{t}_i = 1$, for all t_i. Here $\sigma_i(t_i)$ can be degenerate with one or more mass points and the integral abuses notation.[8]

Definition 5.1: An auction (q, m) is BNE-feasible, if it has at least one BNE.

Denote M_0 the set of BNE-feasible auctions and denote $A_0 = \{q : (q, m) \in M_0\}$ the set of BNE-feasible allocation rules, which consists of final allocation rules that the seller can implement if resale between the players would be forbidden.

5.3. The Vickrey and Myerson Auctions

From Assumption 5.1, the ex post efficient allocation rule q^e requires that for all $t \in T$, if $t_i > t_j$, then buyer i obtains the bundle and ties are broken randomly fairly. Any splitting or withholding is inefficient. In this setting, the Vickrey auction is simple: the buyers simultaneously report their values for the bundle and if $t_i > t_j$, then buyer i wins the bundle and pays t_j to the seller. The truthful equilibrium of the Vickrey auction provides a lower bound on the seller's revenue.[9]

[7] In two period models without discounting, we may assume that if resale is between the seller and the initial buyers and only the seller can sell, then further resale among the initial buyers are impossible. If the seller cannot commit to not resell, then the seller can simply wait until the second period and then sell both units. If the seller is further assumed to have to sell at least one unit in the first period, then after selling one unit, given Assumption 5.1, selling the additional unit to the winner maximizes the seller's revenue. In the final allocation, the seller's withholding one unit is implemented with probability 0 but the buyers' splits can be implemented with a positive probability. Whether the seller's revenue is higher or lower than that in Theorem 5.1 is ambiguous.

[8] Instead of restricting buyers' action sets to type sets, the seller may allow buyer i to choose actions from some abstract set A_i where $T \subseteq A_i$. However, given our assumption on the disclosure rule, using other action sets does not expand the seller's implementable outcomes. This is because if players randomize over actions in equilibrium, the seller can incorporate such randomization and offer an outcome equivalent direct mechanism. It is unclear whether the claim generalizes to other disclosure rules.

[9] When resale is possible, truthful reporting may no longer be a dominant strategy equilibrium action.

The Myerson auction (MA) is defined as the incentive compatible and individually rational auction that maximizes the seller's revenue, when resale (by the seller and among the buyers) is forbidden. The allocation rule from the MA is the Myerson allocation rule. Denote by R_0 the revenue from the MA.

In the remainder of the chapter, we assume the following,

ASSUMPTION 5.3: $J_i(\underline{t}_i) = \underline{t}_i - \frac{1}{f_i(\underline{t}_i)} < a_{12}$, $i = 1, 2$.

This assumption requires that $J_i(\underline{t}_i)$ is not too large such that the virtual valuation of the bundle for a buyer with the lowest type falls below the virtual valuation of splits (e.g. a_{12}).[10] For J_i strictly increasing and $J_i(\bar{t}_i) > a_{12}$, there is a unique $x_i^* \in (\underline{t}_i, \bar{t}_i)$ such that $J_i(x_i^*) = a_{12}$.

PROPOSITION 5.1: *The Myerson auction* (q^*, m^*) *is given by*[11]

$$q^*(t) = \begin{cases} \delta_i & \text{if } J_i(t_i) > \max\{J_j(t_j), a_{12}\}, \\ \delta_s & \text{if } a_{12} > \max_i J_i(t_i), \end{cases}$$

and for $i = 1, 2$,

$$m_i^*(t) = t_i q_i^*(t) + a_i q_s^*(t) - \int_{\underline{t}_i}^{t_i} q_i^*(t_j, x_i) dx_i. \tag{5.3.1}$$

Proof. See Appendix 5.A. □

The Myerson auction requires that the seller never keeps an object. One source of inefficiency is that the buyers split the two objects for some states. With positive probability, the buyers' values lie in the region where $a_{12} > \max_i J(t_i)$ but $\max_i t_i > a_{12}$. The objects then fail to be allocated to the buyer with the higher complementarity. Another source of inefficiency is that some buyer with a higher virtual value (rather than true value) obtains the bundle. If beliefs are symmetric, i.e. $F_1 = F_2$, the second inefficiency vanishes.

5.3.1. The Modified Myerson Auction

For the seller, a natural response to avoid resale after splits is to avoid splitting altogether. We define the modified Myerson auction (MMA) as the auction that maximizes the seller's revenue given that splitting is chosen with probability zero. This auction will play an important role for our analysis of resale. Denote

$$A_1 = \{q | q(t) \in \Delta(D_1), q(t) \neq d_s \text{ for almost all } t \in T\}, \tag{5.3.2}$$

[10] If Assumption 5.3 does not hold, i.e. if for at least one of the buyers the virtual value is always higher than a_{12}, the problem is trivial because the optimal auction always sells two units as a bundle.

[11] δ_d denotes that decision d is chosen with probability 1. δ_i denotes $i \neq j$ being chosen with probability 1.

and denote by R_1 the maximal revenue from $M_1 = \{(q,m) \in M_0 | q \in A_1\}$.

PROPOSITION 5.2: *The modified Myerson auction* (q^{**}, m^{**}) *is given by*
(i) If $\max_i J_i(\underline{t}_i) < a_2$, *then,*

$$q^{**}(t) = \begin{cases} \delta_i & \text{if } J_i(t_i) > \max\{J_j(t_j), a_2\}, \\ \delta_{02} & \text{if } a_2 > \max_i J_i(t_i), \end{cases}$$

and

$$m_i^{**}(t) = t_i q_i^{**}(t) + a_2 q_{02}^{**}(t) \mathbb{1}_{\{i=2\}} - \int_{\underline{t}_i}^{t_i} q_i^{**}(x_i, t_j) dx_i. \tag{5.3.3}$$

(ii) If $\max_i J_i(\underline{t}_i) > a_2$, *then,*

$$q^{**}(t) = \delta_i \quad \text{if } J_i(t_i) > J_j(t_j),$$

and

$$m_i^{**}(t) = t_i q_i^{**}(t) - \int_{\underline{t}_i}^{t_i} q_i^{**}(x_i, t_j) dx_i. \tag{5.3.4}$$

The revenue from the MMA is strictly lower than that from the MA, i.e. $R_1 < R_0$.

Proof. See Appendix 5.A. □

In Proposition 5.2 (i), the MMA requires that the objects to be allocated the buyer with the highest virtual valuation, given that the alternative of splits is eliminated. It is implemented by a generalized second price auction for the bundle with reserve bid a_2, personalized reserve prices $r_i = J_i^{-1}(a_2)$, $i = 1, 2$, and the following payment rule:

$$m_i(b) = \begin{cases} J_i^{-1}(\max\{a_2, b_j\}) & \text{if } b_i > \max\{a_2, b_j\}, \\ a_2 & \text{if } \max\{b_1, b_2\} < a_2 \text{ and } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The buyers bid their virtual valuations in equilibrium. In case of winning the bundle, buyer i's payment does not depend on his own bid and will be at least $r_i > \underline{t}_i$. In case no buyer bids above a_2, the seller sells one object to buyer 2 for a price of a_2. Thus, the seller may withhold one unit with positive probability. In Proposition 5.2(ii), the lowest virtual valuations of the bundle are sufficiently high and the modified Myerson auction always sells both units. Note that the MMA is ex post efficient if beliefs are symmetric.

Compared to the MA, all types of the buyers in the MMA are weakly better off. This is because $q_i^{**}(t) \geq q_i^*(t)$ for all $t \in T$ and the lowest types always receive zero. Hence for the buyers, the MMA interim Pareto dominates the MA. The seller is strictly worse off. The effect on the social surplus is ambiguous.

EXAMPLE 5.1: Suppose $a_1 = 1$, $a_2 = 2$, $\tilde{t}_1 \sim U[4, 8]$, and $\tilde{t}_2 \sim U[5, 9]$. Then $J_1(t_1) = 2t_1 - 8$ and $J_2(t_2) = 2t_2 - 9$. A1 holds since $J_1(4) < J_2(5) < 3$. Moreover, $\max\{J_1(4), J_2(5)\} < 2$, hence we are in case (i) of Proposition 5.2. The reserve prices are given by $r_1 = 5$ and $r_2 = 5.5$.

5.4. Resale

Now we are moving to the case with resale possibility. In case the seller splits with buyer 2 ($d = 02$), given Assumption 5.2, no resale occurs. Hence, we restrict attention to $D = \{1, 2, s\}$. To specify the degree of transparency after the auction, let S_i be the set of possible messages that the seller can send to buyer i. Throughout this chapter, we assume high transparency of the auction outcome. That is, each buyer observes the "bids" (or reports) as well as the decision. Since they know the payment rule, it is not needed to tell them the payments. Formally, $S_i = T \times D$ and $\eta_i(\hat{t}, d) = (\hat{t}, d)$ for all $\hat{t}$ and d. The information disclosure rule is given by $\eta = (\eta_1, \eta_2)$. A system of posterior beliefs $\mu = (\mu_1, \mu_2)$ is given by $\mu_i : T \times D \to \Delta(T_j)$, where μ_i is the belief of buyer i's on buyer j's type given the reports and the decision.

The resale market is centralized and organized by a strategic mediator, whose objective is to maximize the total expected surplus of the buyers. We assume that this mediator also observes the auction outcome. Hence, the mediator chooses the resale procedure given the initial decision d and the belief system μ. While the mediator relies on type reports in the seller's auction to update beliefs, in the resale procedure, he "ask" for type reports again.

A resale mechanism is given by $(\phi, p) : T \to \Delta(D) \times \mathbb{R}^2$, where the reallocation rule ϕ assigns to each reporting profile a vector of reallocating probabilities of the objects, and the payment rule $p = (p_1, p_2)$ assigns to each reporting profile the payments from the initial buyers to the mediator. Given such a resale mechanism, we assume that the buyers participate in the resale voluntarily, and the buyers simultaneously make reports to the mechanism, which determines the final allocation. The mediator runs no budget deficit ex post and the buyers must balance the budget themselves.

5.4.1. Resale Games

Let the seller's auction $(q, m) \in M_0$ be given and consider the resale game $G(q, m)$. The timeline of this game is as follows. (i) The buyers make reports to the seller's auction. (ii) The outcome $(\hat{t}, d)$ is publicly announced. (iii) The mediator chooses a resale mechanism. (iv) The buyers make reports to the mediator. (v) The reports determine the final outcome.

Formally, a strategy α for the mediator specifies a resale mechanism $\alpha(h)$ for all $h = (\hat{t}, d)$. For each buyer $i = 1, 2$, a strategy (σ_i, β_i) specifies the first period reports $\sigma_i(h_i) \in \Delta(T_i)$ for all $h_i = t_i$ and the second period reports $\beta_i(h_i) \in \Delta(T_i)$ for all $h_i = (t_i, \hat{t}, d, (\phi, p))$.

Definition 5.2: Let $(q, m) \in M_0$. A PBE of $G(q, m)$ contains a strategy profile $e^* = (\sigma^*, \alpha^*, \beta^*)$ and a belief system μ^* satisfying:

(i) Given μ^*, every i and $h_i = (t_i, \hat{t}, d, (\phi, p))$,

$$\beta_i^*(h_i) \in \operatorname*{argmax}_{s_i \in \Delta(T_i)} U_i(s_i|h_i, \mu^*). \tag{5.4.1}$$

(ii) Given μ^*, β^* and every $h = (\hat{t}, d)$,

$$\alpha^*(h) \in \operatorname*{argmax}_{(\phi,p)\in\Phi(h,\mu^*)} \sum_i E[U_i(\tilde{t}_i|h, (\phi, p), \mu^*, \beta^*], \tag{5.4.2}$$

where $\Phi(h, \mu^*)$ is the set of incentive compatible, individual rational and ex post balanced budget resale mechanisms in the usual sense, and $U_i(t_i|h, (\phi, p), \mu^*, \beta^*)$ is the continuation payoff of buyer i with type t_i from (ϕ, p) assuming both buyers follow β^*.

(iii) Given μ^*, α^*, β^*, every i and t_i,

$$\sigma_i^*(t_i) \in \operatorname*{argmax}_{s_i \in \Delta(T_i)} U_i(s_i|t_i, \mu^*, \alpha^*, \beta^*). \tag{5.4.3}$$

(iv) Given e^* and every $(\hat{t}, d)$, $\tilde{F} := \mu^*(\hat{t}, d)$ satisfies the following requirements (1)-(3):

(1) $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ are statistically independent, where $\tilde{f}_i$ is the density (if any) of $\tilde{F}_i$ supported by E_i and $E = E_1 \times E_2$. This follows from the fact that σ^* is independent.

(2) If $\hat{t}$ is in the support of σ^*, then

$$\tilde{f}_i(t_i|\hat{t}_i, e^*) = \frac{\sigma_i^*(\hat{t}_i|t_i) f_i(t_i)}{\int_{T_i} \sigma_i^*(\hat{t}_i|x) f_i(x) dx}. \tag{5.4.4}$$

(3) If $\hat{t}$ is not in the support of σ^*, then the players must form beliefs after such off-path histories. Any restriction on the beliefs does not affect our results, because our analysis is based on the equilibrium paths and the results are robust to the specifications of off-path beliefs.

Definition 5.3: Let $(q, m) \in M_0$. $G(q, m)$ is feasible, if it has at least one PBE.

We say an auction (q, m) is feasible if $G(q, m)$ is feasible and denote M the set of all feasible auctions. For $(q, m) \in M$, a *final* allocation rule generated by a PBE (e^*, μ^*) of $G(q, m)$ is given by $o : T \to \Delta(D_1)$. Note that $M \subseteq M_0$. For $(q, m) \in M$, $G(q, m)$ has a BNE and hence it satisfies incentive compatibility and individual rationality associated with the BNE-feasible auctions. On the other hand, some BNE-feasible auction with resale may have no PBE. A direct consequence is that the maximal revenue with resale is not higher than the maximal revenue without resale.

Finally, we introduce the notion of a resale-proof equilibrium for a given resale game.

Definition 5.4: Let $G(q, m)$ be a feasible resale game. A PBE (e^*, μ^*) is resale-proof, if resale does not arise on any equilibrium path.

5.4.2. Optimal Resale Mechanisms

Let $G(q, m)$ be a feasible resale game and let (e^*, μ^*) be a PBE of this game. We provide a characterization of the solutions to the mediator's problem for each equilibrium path of this game. First note that due to the equilibrium play, the equilibrium beliefs may be no longer regular: They can be absolutely continuous with respect to the Lebesgue measure or have atoms or a mixture of absolutely continuous parts and atoms. Also, the supports may not be convex and virtual valuations may not be monotone.[12] For these cases, while complete characterization of the mediator's solutions can be difficult, we can obtain necessary conditions for the solutions.

A public history $(\hat{t}, d)$ is an equilibrium path, if e^* induces $(\hat{t}, d)$ with positive probability. We first establish that if $(\hat{t}, s)$ is an equilibrium path, then irrespective of the equilibrium beliefs $\tilde{F}$, an ex post efficient, individually rational and balanced budget resale mechanism exists. Since this mechanism yields the highest social surplus, $\alpha^*(\hat{t}, s)$ must be utility-equivalent to this efficient mechanism. In this case, splits are chosen with zero probability.

To construct such a mechanism given $\tilde{F}$ and $d = s$, we introduce a class of modified Vickrey-Clarke-Groves (MVCG) mechanism, which requires the player with a higher report obtains the bundle and for each reporting profile, a player pays the difference between the expected VCG payments of the two players (up to a constant). In contrast to VCG mechanisms, the class of MVCG mechanisms depends on the beliefs $\tilde{F}$.

Let (q^e, p^v) be the VCG mechanism defined by the efficient allocation rule q^e and the payments from the buyers to the mediator,

$$p_i^v(t) = t_j \mathbb{1}_{\{t_i > t_j\}} + \frac{1}{2} t_j \mathbb{1}_{\{t_i = t_j\}}, \text{ for all } t \in E, i = 1, 2. \tag{5.4.6}$$

The interim expected payment from the VCG mechanism (q^e, p^v) is given by[13]

$$P_i^v(t_i) = \int_{E_j} p_i^v(t) d\tilde{F}_j(t_j), \text{ for all } t_i \in E_i, i = 1, 2. \tag{5.4.8}$$

[12] In non-absolutely continuous cases, $\tilde{F}_i$ can be decomposed into an absolutely continuous part $\tilde{G}_i(x)$ with support $E_i' \subset E_i$ and a singular part with mass points $\{x_{ik}\} \subset E_i$, where

$$d\tilde{F}_i(x) = \left(1 - \sum_k \alpha_{ik}\right) \tilde{g}_i(x) dx + \sum_k \alpha_{ik} \delta_{x_{ik}}(dx). \tag{5.4.5}$$

[13] For beliefs not absolutely continuous, we have

$$P_i^v(t_i) = \left(1 - \sum_k \alpha_{jk}\right) \int_{E_j'} p_i^v(t_i, x) \tilde{g}_j(x) dx + \sum_k \alpha_{jk} p_i^v(t_i, x_{jk}). \tag{5.4.7}$$

The class of MVCG mechanisms $\{(q^e, p^*(\cdot|\pi))\}$ is parameterized by the constants $\pi = (\pi_1, \pi_2)$, $\pi_1 + \pi_2 = 0$, with the payments given by

$$p_i^*(t|\pi) = P_i^v(t_i) - P_j^v(t_j) - \pi_j, \text{ for all } t \in E, i = 1, 2. \tag{5.4.9}$$

It is easy to see that an MVCG is ex post efficient, ex post budget balanced, and incentive compatible. Following Krishna and Perry (2000) and Kos and Manea (2009), we show that an individually rational MVCG mechanism exists, which implies the following result.

LEMMA 5.1: *Let $G(q, m)$ be a feasible resale game and let (e^*, μ^*) be a PBE. If $(\hat{t}, s)$ is an equilibrium path, then $\alpha^*(\hat{t}, s)$ implements splits with zero probability.*

Proof. See Appendix 5.A. □

We now establish that for $i = 1, 2$, if $(\hat{t}, i)$ is an equilibrium path, then irrespective of the equilibrium beliefs, the re-seller's virtual cost (the re-seller's inflated value of the bundle due to incentive distortion) is always higher than the virtual valuation of splits. Hence, any resale mechanism with the highest social surplus again requires splits to be chosen with zero probability. $\alpha^*(\hat{t}, i)$ must be utility-equivalent to such a mechanism.

LEMMA 5.2: *Let $G(q, m)$ be a feasible resale game and let (e^*, μ^*) be a PBE. For $i =$ 1, 2, if $(\hat{t}, i)$ is an equilibrium path, then $\alpha^*(\hat{t}, i)$ implements splits with zero probability.*

Proof. See Appendix 5.A. □

5.4.3. The Solution to The Seller's Problem

We are able to establish the following impossibility result on the Myerson revenue. With optimal resale mechanisms, the seller's revenue is bounded above by the revenue from the MMA in Proposition 5.2. If prior beliefs are symmetric, this bound is tight. If beliefs are asymmetric, it is unclear whether the revenue can be strictly lower.

THEOREM 5.1: *Let R^* be the maximal revenue from all feasible auctions.*
(i) If $F_1 \neq F_2$, then $R^* \leq R_1 < R_0$.
(ii) If $F_1 = F_2$, then $R^* = R_1 < R_0$.

Proof. (i) By Lemma 5.1 and 5.2, for any feasible resale game $G(q, m)$ and its PBE (e^*, μ^*), if $(\hat{t}, s)$ or $(\hat{t}, i)$ is an equilibrium path, then splits cannot arise with positive probability in a final allocation on this path. Hence, the final allocation rule o must satisfy $o(t) \neq d_s$ for almost all $t \in T$. Now, the set of feasible final allocation rules, or

$$A = \{o | o \text{ is generated by a feasible } G(q, m) \text{ with some PBE } (e^*, \mu^*)\} \tag{5.4.10}$$

is a subset of A_1 in (5.3.2). Moreover, besides the incentive compatibility and the individual rationality associated with the static mechanisms, a PBE of a feasible resale game requires sequential rationality constraints at the resale stage.

Recall that R_1 is the maximal revenue from $\{(q,m) \in M_0 | q \in A_1\}$. Then the revenue ranking $R^* \leq R_1$ holds, because $A \subseteq A_1$ and the seller's revenues are the same for the two allocation rules $o \in A$ and $q \in A_1$ satisfying $o = q$ while leaving $\underline{t}_i$, $i = 1, 2$, the same expected utilities. The proof is complete if $R_1 < R_0$, which has been proven in Proposition 5.2.

(ii) If $F_1 = F_2$, then consider the MMA as the seller's auction. Truthful reports are part of a resale-proof equilibrium: For every equilibrium path $(\hat{t}, i)$, the mediator puts probability one on $\hat{t}_i > \hat{t}_j$. For every off-equilibrium path $(\hat{t}, d)$, the mediator may again put probability one on $\hat{t}_i > \hat{t}_j$. Hence, for all public histories, no resale mechanism will be offered. Hence, $R^* = R_1$. □

Proposition 5.3: *Let the MMA be the seller's auction.*
(i) If $F_1 \neq F_2$, then truthful reports are not part of a resale-proof equilibrium.
(ii) If $F_1 = F_2$, then truthful reports are part of a resale-proof equilibrium.

Proof. (i) Suppose truthful reports are part of a resale-proof equilibrium. $F_1 \neq F_2$ implies that there exists at least one i such that on the equilibrium path $(\hat{t}, i)$, $\hat{t}_j > \hat{t}_i$ occurs with positive probability. The mediator puts mass point on $\hat{t}$. An optimal resale mechanism sets a constant trading price between $\hat{t}_j$ and $\hat{t}_i$, independent of reports. Both players will accept this price and the resale occurs, which contradicts to resale-proofness. (ii) has been shown by Theorem 5.1(ii). □

The result in Proposition 5.3(i) does not rule out the possibility that the MMA has some untruthful reporting equilibrium that implements R_1.

If beliefs are symmetric, the MMA is a solution to the seller's problem while there exist other auctions that yield the same revenue. For example, we may define a second price auction for the bundle with some reserve price. If both bids are below the reserve, then sell one object to buyer 2 at price a_2. The advantage of this auction is that for any beliefs, its truthful equilibrium is resale-proof.[14]

Theorem 5.1 may not hold if $\underline{t}_1 > a_{12}$ is replaced by $2a_1 < \underline{t}_1 < a_{12}$. In this case, on the equilibrium path after buyer i obtains the bundle, splits may have the highest virtual valuation for some type profiles. The mediator may implement splits with positive probability. On the other hand, the optimal resale mechanism may remove some inefficiency. In this case, whether the Myerson revenue is attainable is unclear.

[14] However, such truthful equilibrium must yield a revenue lower than R_1. Similar to the MMA, it does not rule out the possibility that this auction has some other equilibrium with equilibrium resale that implements R_1.

5.5. Other Resale Market

Theorem 5.1 depends on the assumptions that (i) the buyers have symmetric bargaining powers[15] and (ii) they move simultaneously at the resale stage. In this case, Theorem 5.1 shows that the seller's revenue is strictly lower than the Myerson revenue. In this section, we show that this result is not specific for sequentially optimal centralized resale markets. Other resale markets with simultaneous moves may lead to multiple equilibrium outcomes, with some of them having low revenue for the seller.

In order to focus on the inefficiency caused by splits and characterize the equilibria explicitly, we assume that the buyers have symmetric prior beliefs $F_1 = F_2 = F$ on $[\underline{t}, \overline{t}]$ in the remainder of this section. Denote R^e the revenue from the truthful BNE of the Vickrey auction. Without resale possibility, the seller can guarantee R^e by running either one of the following auctions:

(FPA) If buyer i bids above the reserve price $r = \underline{t}$ and is the high bidder, he wins the bundle and pays his own bid. If no buyer meets the reserve, the seller withholds both objects.[16]

(MFPA) If buyer i bids above the reserve price $r = \underline{t}$ and is the high bidder, he wins the bundle and pays his own bid. If no buyer meets the reserve, the buyers split the objects with the prices equal to (a_1, a_2).

Now suppose the resale market is organized by the McAfee bidding procedure (M) introduced in McAfee (1992). At $d = i$, the re-seller i sets an ultimatum price for the bundle. At $d = s$, the two buyers simultaneously submit their bids for buying the other's unit, i.e. $[0, \infty)$. The high bidder wins the other's unit and pays his own bid to the other bidder. Ties are broken randomly.

We observe that compared to FPA, allowing the possibility of splits in MFPA introduces multiple equilibria and a possibly lower revenue than R^e. We mention the following result without proof.

(Multiple Efficient Outcomes) Suppose

$$a_2 < \underline{t} - \frac{1}{2}\left(\overline{t} - \int_{\underline{t}}^{\overline{t}} [F(x)]^2 dx\right). \tag{5.5.1}$$

(i) In FPA-M, there exists a resale-proof and efficient equilibrium outcome in which both buyers bid for the bundle as in a single object first price auction. The seller's revenue is R_2.

(ii) In MFPA-M, the resale-proof outcome of FPA-M remains an equilibrium out-

[15] That is, the buyers' welfare weights are equal in the mediator's problem. It remains to be investigated whether the seller obtains a higher or lower revenue if the buyers' weights are asymmetric.

[16] For simplicity, we assume that the bid is 0 when it is below the reserve.

come. Moreover, there exists another efficient equilibrium outcome in which both buyers always bid below the reserve and split. The resale is efficient. The seller's revenue is a_{12}.

Hence, the efficient allocation can be implemented both without equilibrium resale and with equilibrium resale. The efficient allocation does *not have to* be implemented with zero probability of resale. For single unit auctions, we are not aware of any similar results.

The multiplicity of efficient equilibria suggests a coordination problem between the buyers.[17] For the buyers, the low revenue outcome interim Pareto dominates the resale-proof outcome. That is, all types of the buyers are better off in the low revenue outcome at the beginning of the seller's auction. Hence, the low revenue outcome is more likely to be played by the buyers. For the seller that maximizes the revenue given efficiency, eliminating splits from the set of decision options avoids the multiplicity and guarantees the efficient revenue.[18]

It is worth to note that Assumption 5.1 implies that splits are always inefficient and there is no reason for the seller to allow splits in the first place. However, consider an alternative environment where splitting is efficient for some valuation profiles, for example,

$$\underline{t} < a_{12} < \underline{t} + \epsilon \text{ for some } \epsilon > 0 \text{ small.}$$

In this environment, an efficient allocation requires some splits for the lower values. We find that the previous analysis remains true. For such a case, completely eliminating splits has a second-order loss in the efficiency but a first-order gain in the revenue.[19]

5.6. Conclusion

In this chapter, we investigate the optimal selling procedure for two complementary objects, when inter-buyer resale cannot be prohibited. We assume that the buyers have independent private values on the bundle and that one buyer acquiring the bundle is efficient. Assuming that the auction bids and the allocation of objects are publicly observable, we discuss the seller's attainable revenue, assuming that a mediator chooses

[17]McAfee and McMillan (1992) note that in practice a bidding ring's own "knockout auction" often happens after rather than before the legitimate auction. Garratt, Troger and Zheng (2009) construct a family of non-value-bidding equilibria for an English auction that allows inter-bidder resale, for independent private value environments.

[18]Notice that the multiplicity of equilibria does not only arise for this resale game but it may arise in any of feasible resale games. If the seller's move is taken into account, then the equilibria must be selected in a way that maximizes the seller's revenue to ensure the equilibrium existence.

[19]Consider the seller runs FPA with $r = a_{12}$, or (ii) MFPA with $r = a_{12}$. For some parameters, the game MFPA-M has multiple equilibrium outcomes. It is also unclear whether the seller can eliminate the multiple outcomes without eliminating splits.

a resale procedure that maximizes the buyers' surplus.

We characterize the modified Myerson auction (MMA) that completely eliminates splits, which is a natural response to prevent resale. It requires selling the bundle with two personalized reserve prices, and selling one object to the buyer with the higher single unit value in case no buyer bids above his reserve price. We show that, in case of resale, the Myerson revenue is unattainable, and the maximal revenue is bounded above by the revenue from the modified Myerson auction. Hence, this chapter provides an example of resale markets where symmetric bargaining powers of buyers and simultaneous bidding in the resale market are conducive to a low revenue at the auction stage.

This chapter is related to Ausubel and Cramton (1999), which characterizes the optimal mechanisms when the resale market is perfect in the sense that any inefficiency will be corrected in the resale market. Given such a perfect resale market, they find that the seller should induce an efficient allocation directly in the initial market. In contrast, our impossibility result on the Myerson revenue does not rely on a resale market to be ex post efficient. Instead, we find that if the inefficiency of a social alternative is commonly known by the resale participants and their bargaining powers are symmetric, then it will be corrected. In case of optimal resale mechanisms, a perfect resale market followed by splits exists and is implemented by a modified VCG mechanism.

Many interesting points have not been discussed in this chapter. We assume that the level of post-auction transparency is given rather than being chosen by the seller. Hence, our model provides a lower bound for the seller's performance. With moderate degrees of transparency, each buyer observes the outcome of the initial allocation, his own bid, and payment. In this case, post-auction beliefs are no longer common knowledge and the mediator may even require players to report their beliefs. With a minimal transparency, we have private winners and payments, where each buyer observes his own bid, his own winner identity and payment. In this case, there is no public history for the players to coordinate and organize the resale followed by splits. Calzolari and Pavan (2006) studies revenue maximizing auctions with resale and allows the seller to disclose information to the re-seller. They find that it is impossible to maximize revenue with a deterministic selling procedure (also see Bergemann and Pavan (2015) for a survey). We leave these cases for future research.

Appendix 5.A Proofs

Proof of Proposition 5.1. Let $(q, m) \in M_0$ and $U_i(\hat{t}_i; t_i) = (Q_{0i}(\hat{t}_i) + Q_s(\hat{t}_i))a_i + Q_i(\hat{t}_i)t_i - \bar{M}_i(\hat{t}_i)$, where Q_d is the expected probability that d is chosen, and $\bar{M}_i$ is the expected payment for buyer $i = 1, 2$. As Lemma 5.2 in Myerson (1981), (q, m) is incentive compatible if and only if for each i, Q_i is nondecreasing and $U_i(t_i) = U_i(\underline{t}_i) + \int_{\underline{t}_i}^{t_i} Q_i(x)dx$. The interim expected payment of buyer i is given by

$$\bar{M}_i(t_i) = (Q_{0i}(t_i) + Q_s(t_i))a_i + Q_i(t_i)t_i - \int_{\underline{t}_i}^{t_i} Q_i(x)dx - U_i(\underline{t}_i). \tag{5.A.1}$$

The expected revenue for the seller, $\sum_i E[\bar{M}_i(\tilde{t}_i)]$ is given by

$$E[v(d, \tilde{t})q_d(\tilde{t})] - \sum_i U_i(\underline{t}_i), \tag{5.A.2}$$

where the sum of buyers' virtual valuations $v(d, t)$ for $d \in D_0$ is given by

$$v(0, t) = 0, v(0i, t) = a_i, v(s, t) = a_{12}, v(i, t) = J_i(t_i), \text{for all } t \in T. \tag{5.A.3}$$

Setting $U_i(\underline{t}_i) = 0$, $i = 1, 2$, leaves no rent for the buyers with the lowest types. The maximization requires the objects to be allocated in a way that pointwise maximizes the sum of buyers' virtual valuations. Because J_i, $i = 1, 2$ is regular, it follows that $q_i^*(t)$ is nondecreasing in t_i for all $t_j \in T_j$. Now Q_i^* is nondecreasing implies q^* is indeed incentive compatible. □

Proof of Proposition 5.2. The seller's problem is similar to Proposition 5.1 except $d = s$ is eliminated. Then $d = 02$, which was dominated by $d = s$, now may have the highest sum of virtual valuations, i.e. a_2 among all decisions for some $t \in T$. In either case (i) or (ii), q_i^{**} is nondecreasing in t_i for all $t_j \in T_j$. Hence q^{**} is incentive compatible. □

Proof of Lemma 5.1. (q^e, p^v) is efficient and dominant strategy incentive compatible. The ex post surplus is $\sum_i p_i^v(t) = \max_i t_i - a_{12} \geq 0$, for all $t \in E$. Thus, this mechanism runs an ex ante surplus,

$$E[\sum_i p_i^v(\tilde{t})] - a_{12} \geq 0. \tag{5.A.4}$$

Then, if $\tilde{F}$ is absolutely continuous or discrete, by Theorem 2 of Krishna and Perry (2000) and Theorem 5.1 of Kos and Manea (2009), there exists an efficient and individually rational mechanism that balances the budget. Now suppose $\tilde{F}_i$ is general. Consider the modified VCG mechanisms $(q^e, p^*(\cdot|\pi))$ parameterized by $\pi = (\pi_1, \pi_2)$ satisfying $\pi_1 + \pi_2 = 0$.

The interim pavements for p^v and $p^*(\cdot|\pi)$ differ up to constants and thus $(q^e, p^*(\cdot|\pi))$ is incentive compatible as well. Also, $(q^e, p^*(\cdot|\pi))$ is balanced budget. Finally, because (q^e, p^v) runs an ex ante surplus, there exists (π_1^*, π_2^*) such that

$$E[P_j^*(\tilde{t}_j|\pi^*)] + \pi_j^* \geq a_i, \text{ for } \quad j = 1, 2. \tag{5.A.5}$$

Hence $(q^e, p^*(\cdot|\pi^*))$ is also individually rational. □

Proof of Lemma 5.2. Denote $\underline{c}_i = \inf\{t_i : \tilde{F}_i(t_i) > 0\}$ and $\bar{c}_i = \sup\{t_i : \tilde{F}_i(t_i) < 1\}$ and similarly for j. If the gains from resale are common knowledge, i.e. $\underline{c}_j > \bar{c}_i$, then the problem is simple, i.e. the players trade at a price equal to $(\underline{c}_j - \bar{c}_i)/2$. We focus on the case with no common known gains from resale.

i. Suppose $\tilde{F}_i$ is absolutely continuous on $[\underline{c}_i, \bar{c}_i]$ and $\tilde{f}_i > 0$, and this also holds for j. The mediator's problem is similar to Myerson and Satterthwaite (1983). For any incentive compatible, individually rational, and budget balanced $(\phi, p) \in \Phi(\hat{t}, i, \mu^*)$, by the envelope theorem $U_i(t_i) = U_i(\bar{c}_i) + \int_{t_i}^{\bar{c}_i} \Psi_i(x)dx$ and $U_j(t_j) = U_j(\underline{c}_j) + \int_{\underline{c}_j}^{t_j} \Psi_j(x)dx$, where Ψ_i, Ψ_j are the expected probabilities of resale. Then (ϕ, p) is individually rational if and only if[20]

$$U_i(\bar{c}_i) + U_j(\underline{c}_j) \geq a_{12}. \tag{5.A.6}$$

The mediator's problem is given by

$$\max_{(\phi,p)\in\Phi(\hat{t},i,\mu^*)} \int_E t_1\phi_1(t) + t_2\phi_2(t) + a_{12}\phi_s(t)d\tilde{F}(t). \tag{5.A.7}$$

By the ex post balanced budget, the sum of ex ante expected payments, $\sum_i E[P_i(\tilde{t}_i)]$, is zero, and we have

$$U_i(\bar{c}_i) + U_j(\underline{c}_j) = \int_E \sum_{d\in\{i,j,s\}} \tilde{v}(d,t)\phi_d(t)d\tilde{F}(t), \tag{5.A.8}$$

where

$$\tilde{v}(i,t) = t_i + \frac{\tilde{F}_i(t_i)}{\tilde{f}_i(t_i)}, \tilde{v}(j,t) = t_j - \frac{1 - \tilde{F}_j(t_j)}{\tilde{f}_j(t_j)}, \tilde{v}(s,t) = a_{12}, \tag{5.A.9}$$

[20] Notice that $U_i(t_i) - t_i$ is decreasing in t_i and $U_j(t_j) - a_j$ is increasing in t_j. Hence, the individual rationality condition for $t_i = \bar{c}_i$ and $t_j = \underline{c}_j$ implies the condition holds strictly for other types.

for all $t \in E$. Combine with $p_s(t) = 1 - \sum_i p_i(t)$ for all $t \in E$, we rewrite (5.A.6) as

$$\int_E [\tilde{v}(i,t) - a_{12}]\phi_i(t) + [\tilde{v}(j,t) - a_{12}]\phi_j(t) d\tilde{F}(t) \geq 0. \tag{5.A.10}$$

Let $\lambda \geq 0$ be the Lagrange multiplier for (5.A.10), then we have

$$(1+\lambda)\int_E [t_i + \frac{\lambda}{1+\lambda}\frac{\tilde{F}_i(t_i)}{\tilde{f}_i(t_i)} - a_{12}]\phi_i(t) + [t_j - \frac{\lambda}{1+\lambda}\frac{1-\tilde{F}_j(t_j)}{\tilde{f}_j(t_j)} - a_{12}]\phi_j(t) d\tilde{F}(t) + a_{12}. \tag{5.A.11}$$

For $\underline{t}_i > a_{12}$, at least buyer i (the re-seller) has a virtual cost always higher than the virtual valuation of splits, or

$$\tilde{J}_i(t_i) = t_i + \frac{\lambda}{1+\lambda}\frac{\tilde{F}_i(t_i)}{\tilde{f}_i(t_i)} - a_{12} > 0, \tag{5.A.12}$$

for all $t_i \in E_i$.

Following Myerson (1981), let $\tilde{h}_i(x) = \tilde{J}(\tilde{F}_i^{-1}(x))$ and $\tilde{H}_i(x) = \int_0^x \tilde{h}_i(z)dz$. The convex hull of $\tilde{H}_i$ is given by

$$\tilde{G}_i(x) = \min\{w\tilde{H}_i(x_1) + (1-w)\tilde{H}_i(x_2) : w, x_1, x_2 \in [0,1], wx_1 + (1-w)x_2 = x\}, \tag{5.A.13}$$

with derivative $\tilde{g}_i(x) = \tilde{G}_i'(x)$. The ironed virtual cost is defined by $\bar{J}_i(t_i) = \tilde{g}(\tilde{F}_i(t_i))$. We have two cases: (i) $\tilde{J}_i(t_i)$ is nondecreasing on $[\underline{c}_i, \underline{c}_i + \epsilon)$. For x around 0, $\tilde{G}_i(x) = \tilde{H}_i(x)$ and $\tilde{g}_i(x) = \tilde{h}_i(x)$ and $\bar{J}_i(\underline{c}_i) = \tilde{J}_i(\underline{c}_i) > 0$. (ii) $\tilde{J}_i(t_i)$ is decreasing on $[\underline{c}_i, \underline{c}_i + \epsilon)$. For x around 0, $\tilde{G}_i(x) < \tilde{H}_i(x)$. An ironed region is given by $[\underline{c}_i, c_i^*]$ for some $c_i^* > \underline{c}_i$. For all x in this region, $\bar{J}_i(x) = \bar{J}_i(\underline{c}_i) < \tilde{J}_i(\underline{c}_i)$. But

$$\int_{\underline{c}_i}^{c_i^*} \bar{J}_i(\underline{c}_i) d\tilde{F}_i(x) = \int_{\underline{c}_i}^{c_i^*} \tilde{J}_i(x) d\tilde{F}_i(x) > 0. \tag{5.A.14}$$

In both cases, for all $x \in E_i$, $\bar{J}_i(x) \geq \bar{J}_i(\underline{c}_i) > 0$.

ii. Suppose $\tilde{F}_i : \mathbb{R} \to [0,1]$ is not absolutely continuous, supported by $E_i \subseteq [\underline{c}_i, \overline{c}_i]$. Player i's generalized virtual cost can be defined by Monteiro and Svaiter (2010). We replace the Lebesgue measure by the bounded signed measure ν such that $\nu([\underline{c}_i, \overline{c}_i]^c) = 0$ and

$$\nu((-\infty, x]) = \int_{-\infty}^x (t_i - a_{12}) d\tilde{F}_i(t_i) + \int_{-\infty}^x \frac{\lambda}{1+\lambda}\tilde{F}_i(t_i)\chi_{E_i}(t_i) dt_i. \tag{5.A.15}$$

Let $H_\nu(x) = \nu((-\infty, x])$. Denote $\Gamma = \{(y,z) \in \mathbb{R}^2 : y + z\tilde{F}_i(x) \leq H_\nu(x), \text{ for all } x \in \mathbb{R}\}$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ the generalized convex hull of H_ν by

$$\varphi(x) = \sup\{y + z\tilde{F}_i(x) : (y,z) \in \Gamma\}, \tag{5.A.16}$$

and $l_i(x) : [\underline{c}_i, \overline{c}_i] \to \mathbb{R}$ the generalized virtual valuation of H_ν by

$$l_i(x) = \inf\{z : y + z\tilde{F}_i(x) = \varphi(x), \exists(y, z) \in \Gamma\}. \tag{5.A.17}$$

Theorem 4 in Monteiro and Svaiter (2010) shows that l_i is nondecreasing and for every nondecreasing and measurable function $g_i : [\underline{c}_i, \overline{c}_i] \to \mathbb{R}$,

$$\int_{[\underline{c}_i, \overline{c}_i]} g_i(t_i) d\nu(t_i) = \int_{[\underline{c}_i, \overline{c}_i]} g_i(t_i) l_i(t_i) d\tilde{F}_i(t_i). \tag{5.A.18}$$

Now we need to show that $l_i(x) \geq 0$ for almost all $x \in [\underline{c}_i, \overline{c}_i]$. First, $t_i - a_{12} \geq 0$ implies ν is a nonnegative measure and $H_\nu(\cdot)$ is a nondecreasing function. $(H_\nu(\underline{c}_i), 0) \in \Gamma$ implies $\varphi(x) \geq H_\nu(\underline{c}_i)$ for all $x \geq \underline{c}_i$ and $\varphi(\underline{c}_i) = H_\nu(\underline{c}_i)$.

Case 1. If $\underline{c}_i$ is an atom, then $\tilde{F}_i(\underline{c}_i) > 0$. By Proposition 4 in Monteiro and Svaiter (2010),

$$l(\underline{c}_i) = \frac{\varphi(\underline{c}_i) - \varphi(\underline{c}_i-)}{\tilde{F}_i(\underline{c}_i) - \tilde{F}_i(\underline{c}_i-)} = \frac{\varphi(\underline{c}_i)}{\tilde{F}_i(\underline{c}_i)} > 0, \tag{5.A.19}$$

where $\varphi(\underline{c}_i-) = \min\{0, H_\nu(\underline{c}_i)\} = 0$.

Case 2. If $\underline{c}_i$ is not an atom, $\tilde{F}_i(x) > 0$, for all $x \in (\underline{c}_i, \underline{c}_i + \epsilon_1]$, $\epsilon_1 > 0$ small. Fix an x in this interval, let $(y, z) \in \mathbb{R}^2$ be such that (i) $y + z\tilde{F}_i(x) = \varphi(x)$ and (ii) $z < 0$. Note that $H_\nu(\underline{c}_i) \leq \varphi(x) \leq H_\nu(x)$ and $H_\nu(\cdot)$ is nondecreasing on $(\underline{c}_i, x]$. For $z < 0$, $y + z\tilde{F}(\cdot)$ is a nonincreasing function on $(\underline{c}_i, x]$. Then, there exists $\epsilon_2 \in (0, \epsilon_1)$ such that $y + z\tilde{F}_i(x) > H_\nu(x)$, for $x \in [\underline{c}_i, \underline{c}_i + \epsilon_2]$. Hence, $(y, z) \notin \Gamma$.

To summarize, l_i is nondecreasing on $[\underline{c}_i, \overline{c}_i]$, and $l_i(x) \geq 0$ for almost all $x \in [\underline{c}_i, \underline{c}_i + \epsilon]$, $\epsilon > 0$ small. The optimal resale mechanism selects splits with zero probability. □

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