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door

Maria Lavrutich

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Maria Lavrutich
Tilburg
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Contents

1 Introduction[1]

This thesis contributes to the real options and industrial organization literature by investigating how competition, capacity choice and uncertainty affect firms' investment behavior. More specifically, it investigates how firms react to the threat of entry and when they choose to employ defensive tools rather than accommodate. Nowadays, these types of strategic interactions are inherent to many industries. The era of global growth and technological innovation creates perfect conditions for the emergence of new market players, and this motivates the incumbent firms to adjust their strategies. Naturally, when entering the market requires substantial irreversible investment, the timing of the decision to enter a new market is crucial in the uncertain economic environment. Therefore, this thesis applies real options theory to model strategic investment behavior of the firms under uncertainty.

The field of real options theory took off with the seminal book by Dixit and Pindyck (1994). The main idea of this book is that investment timing plays a crucial role in the decisions to undertake irreversible investment in an uncertain world. More precisely, the possibility to delay the investment and, therefore, to access additional information, creates an option value for the market participants. As a result, in the real options framework the optimal investment thresholds turn out to be above the so called Marshallian trigger points, which correspond to a zero net present value (NPV).

The early literature on real options usually focuses on investment decisions of a single firm. From the 1990s onwards, however, this field was extended by considering situations where more firms are active in one market. Firms face investment options, where, in case one firm invests, the value of the investment options of other firms are reduced because of the increased competition in this market. Adding competition to the real options framework provides an incentive for the firms to invest quickly in order to preempt investments of other firms, so that they are the first

[1] This chapter is based on Huberts *et al.* (2015).

in the market and gain (temporary) monopoly profits in this way. On the other hand, uncertainty and irreversibility generate the value of waiting effect, so that an interesting trade off arises. Smets (1991) is the first contribution in this area. He considers a framework of a duopoly market where the firms can enlarge an existing profit flow. Grenadier (2000) is an early survey of this literature. The survey by Huisman *et al.* (2004) focuses on identical firms in a duopoly context. That paper argues that, since firms are identical, it is natural to consider symmetric strategies. This results in obvious coordination problems in situations where it is only optimal for one firm to invest. Huisman *et al.* (2004) shows that application of mixed strategies, originally developed by Fudenberg and Tirole (1985b) for a deterministic framework, provides a meaningful way to deal with such coordination problems. The survey by Chevalier-Roignant and Trigeorgis (2011) provides an overview of the strategic real options literature where it explicitly considers first- versus second-mover advantage, the role of information, firm heterogeneity, capital increment size, and the number of competing firms. Azevedo and Paxson (2014) wrote a survey on game-theoretic aspects of real options models like degree of competition, asymmetries between firms, information structure, cooperation between firms, and market sharing.

Another important modification of the basic real options model arises when firms are allowed to choose not only the timing of the investment decision, but also the size of the investment. The real options literature concentrating on investment timing only has a standard result in that uncertainty generates a value of waiting with investment. When also size needs to be determined, the literature has a common result that in a more uncertain economic environment, firms invest later and in a larger capacity size (e.g., see Dangl (1999), Bar-Ilan and Strange (1999)). So, where from the traditional real options literature it could easily be concluded that uncertainty is bad for growth, this is not so clear anymore when also capacity size needs to be determined. Moreover, in a competitive framework capacity choice of a certain firm can influence the decision of the other firm to enter the market. For example, among the early models of capacity choice, Spence (1977) introduces a setting where the firm can deter entry by overinvestment. Wu (2007) studies incentives of the leader in a growing market to preempt the follower by investing in capacity. The main result of that paper is that under the assumption of uncertainty about the date at which the market starts to decline, the leader will choose a smaller capacity in order to take advantage of market decline, i.e. to stay longer in the market than the larger competitor. Huisman and Kort (2015) analyzes accommodation and deterrence strategies of the market leader in a duopoly setting. It introduces the overinvestment effect that arises due to possibility of the market leader to deter entry of its competitor, as a bigger level of the leader's capacity ensures that the follower invests later. Moreover, the length of the deterrence region becomes larger when market uncertainty is higher.

This happens because larger uncertainty generates more incentives for the follower to postpone the investment and, therefore, the leader can enjoy a longer monopoly period when implementing the entry deterrence strategy.

Additionally, capacity choice may serve as a tool to not deter the entry of a new firm, but also to induce the exit of an active firm. In the exit games the firm with a second mover advantage may have incentives to engage in predatory behavior. For example, Bayer (2007) presents a model, where the capacity choice of the entrant accelerated the exit decision of the incumbent. In general, capacity choice is not the only instrument of exit inducement. Empirical literature suggests that a common response to the threat of entry in certain industries is a price war, where the stronger firm is able to drive out its weaker opponent by driving the output prices down. Price wars have received a rather limited attention in the real options literature, which gives room for further research.

This thesis approaches this wide variety of economic problems and covers different aspects of strategic interactions between firms in a rigorous economic framework. It consists of three main chapters, where the continuous-time optimal stopping models under uncertainty with lumpy investment are solved using the techniques from real options theory.

In Chapter 2 a model with capacity choice is considered under the assumption that firms do not have access to all information about potential entrants. It generalizes Huisman and Kort (2015) in that it is permitted for a hidden third firm to enter the industry. The other two firms that are modeled as explicit players have no information about the exact investment timing of the hidden firm. As in Armada *et al.* (2011) they only have the knowledge that the hidden firm invests with a probability satisfying a Poisson jump process. Additionally, it is assumed that the firms hold a certain belief about the capacity of the hidden player. In this setting we analyze the effect of the hidden entrant on the capacity choice and investment timing of the two firms that are well informed about each other, operating in a limited market with only two places available. The main results of this chapter are associated with the fact that due to the fear of hidden entry the follower is more eager to invest and it becomes too costly to deter its entry. Thus, the deterrence strategy can only be implemented for a small market size when the investment is not particularly attractive for the follower. But when the market size is small, also for the leader it is not profitable to invest. Consequently, the entry accommodation strategy is implemented so that we have a simultaneous equilibrium even in the endogenous game, which is new in the literature.

Chapter 3 is focused on investment, capacity choice and exit decisions of the entrant in a duopoly setting. We show that larger firms incur larger fixed costs associated with the installed capacity, which may trigger their decision to exit the market earlier in the face of declining demand. In this setting, the second mover advantage of the entrant allows it to drive the incumbent firm out of the market if the latter has acquired too much capacity. The main result of this chapter is associated with the existence of a region of hysteresis, i.e. a gap between the investment regions of the entrant. If the market is large enough, the entrant chooses to coexist with its opponent in a duopoly. In a small market, however, the entrant has an incentive to force the incumbent out of the market and become a monopolist. For the case of an intermediate market size, it is optimal to wait until either one of the scenarios will occur. It is, however, not clear ex-ante which scenario will be realized due to the market uncertainty and, thus, the outcome of the game is dependent on the sample paths of the underlying stochastic process.

Chapter 4 considers exit inducement in an incumbent-entrant model in a different setting. While in Chapter 3 the capacity choice of the firms is the main instrument to stimulate the opponent's exit, Chapter 4 focuses on price wars. In particular, we allow for predatory behavior of firms in a market where their future profits are subject to firm-specific stochastic shocks. The predatory behavior is defined as driving the price down to the level of marginal production costs, i.e. a price war implies zero profits for the firms. In our model the profits accumulated over time serve as a proxy for firm's reputation. The firms are assumed to go bankrupt when their reputation is damaged, i.e. when their accumulated profits hit zero. This may raise an incentive for either of the firms to initiate a price war. Due to firm-specific uncertainty that affects firms' profits, neither of the firms can guarantee to be the last one standing. Therefore, sometimes the new firm may still be willing to take a chance and enter the market despite the threat of predation. Thus, in our model the firm specific uncertainty creates a rationale for a price war in a complete information setting.

2 Entry Deterrence and Hidden Competition[1]

This chapter studies strategic investment behavior of firms facing an uncertain demand in a duopoly setting. Firms choose both investment timing and the capacity level while facing additional uncertainty about market participants, which is introduced via the concept of hidden competition. We focus on the analysis of possible strategies of the firms in terms of their capacity choice and on the influence of hidden competition on these strategies.

We show that due to hidden competition, the follower is more eager to invest. As a result, an entry deterrence strategy of the leader becomes more costly, and it can only be implemented for smaller market size, leaving additional room for entry accommodation. The leader has incentives to prevent entry of the hidden competitor stimulating simultaneous investment if the hidden firm has a large capacity, and has more incentives to apply entry deterrence in the complementary case of a small capacity of the hidden player. In the first case overinvestment aimed to deter the follower's entry does not occur for a wide range of parameters values.

2.1 Introduction

Apple has recently been rumored to develop a project of creating its own branded electric car (probably self-driving). Several news reports[2] claim that Apple employees have been secretly working on the technology. Even though the Apple representatives decline to comment on this issue, there are some speculations about whether Apple will proceed with technology development and enter the market of electric cars in the future and if so when it can potentially happen. This raises the question of how the manufacturers of existing cars should respond to this news. The problem

[1]This chapter is based on Lavrutich *et al.* (2016).

[2]`http://www.huffingtonpost.com/entry/apple-electric-car-charging_us_5745c0e5e4b03ede44136d55`, `http://www.macrumors.com/roundup/apple-car/`.

is that, even though they have enough information about their current competitors, hardly anything is known about the new project of Apple. The main issue is that the technology Apple is working on has not been developed yet. As a result, it is hard to predict at what point in time it will exactly emerge, what are the investment costs, and what scale of the investment will be chosen by Apple. In this chapter we try to tackle this sort of problem where Apple is considered in the role of a hidden competitor.

Following Huisman and Kort (2015), we present a model, where in order to enter the market, firms invest in a production plant with a certain capacity, where the firms choose the investment scale. We extend the model of Huisman and Kort (2015) by relaxing the assumption that firms are fully informed about all market participants. In line with Armada *et al.* (2011) we incorporate an additional type of uncertainty in the model by introducing the concept of hidden competition. In particular we assume that, apart from the two competitors that are well informed about each other, a third, hidden firm, can enter the market at an unknown point in time. This can be related to Bobtcheff and Mariotti (2012) where it is assumed that this additional uncertainty is associated with the emergence of a new idea. If the technology is known, the investment timing can be predicted due to the rationality of the market players. However, the existing firms in the market can hardly infer when the new idea will come to life and how much time will it take to develop a new technology can be exactly developed, as, for example, in the case of the Apple electric car.

Consistent with Armada *et al.* (2011) we develop a model, where two positioned firms compete in the market with two places available, facing a possibility of a hidden entry. The entrance occasion of the hidden firm is modeled as an exogenous event driven by a Poisson jump process. Armada *et al.* (2011) demonstrates that hidden competition can exert significant influence on the firms' investment timing in the limited market. Namely, they show that, as the arrival rate of the hidden competitor rises, we can observe a decrease in the investment trigger for the follower on the one hand, and an increase in the investment trigger for the leader on the other hand. This means that if the probability that the hidden competitor enters the market is higher, the market leader will invest later, while the follower will invest sooner.

In this chapter the problem of investing in a market with hidden competition is approached from a different perspective. We examine how hidden competition affects the optimal strategies of the firms if they are allowed to choose the capacity level. As in Huisman and Kort (2015) we consider deterrence and accommodation strategies for the leader. We show that the deterrence region shrinks with the probability of hidden entry. This happens because the larger the probability that the hidden competitor can enter the market, the more eager the follower is to invest earlier, and therefore it is getting harder for the leader to deter entry. In fact, we show that hidden competition induces the positioned firms to enter the market tat the same time. In this case, unlike in Huisman and Kort (2015) and Armada *et al.* (2011) the firms may enter the market simultaneously after waiting.

The rest of this chapter is organized as follows. Section 2 is devoted to the analysis of the investment decisions of the positioned firms facing a threat of hidden entry on the market with two places available. We solve the game backwards, first determining the optimal investment trigger and the optimal capacity level for the follower. Then we continue by determining the optimal strategies of the firms when the roles of the leader and the follower are endogenously assigned. Section 3 summarizes the main results and concludes the chapter. The proofs of the propositions are presented in the Appendix.

2.2 Model setup

In the model two risk-neutral, *ex ante* identical firms make a market entry decision. When a firm becomes active on the market, it starts the production process after investing in a production plant with capacity of size $q > 0$. The investments costs are equal to δq, where $\delta > 0$.

The two firms that have full information about each other are called positioned firms. The positioned firm that invests first is called the leader and the second investor is called the follower. An important feature is that here the standard duopoly model is extended by incorporating the possibility of hidden entry. Like in Armada *et al.* (2011), we assume that at any moment in time, the positioned firms face the probability that a third firm can become active on the market. The information about this firm remains hidden for the positioned market players. Therefore, this firm is called the hidden competitor. In the analysis below we distinguish between two situations depending on whether the hidden or one of the positioned firm is the first investor. We first consider the scenario where the hidden player enters the market first. Later on, we examine the case when this role is taken by one of the positioned firm. It is assumed that the firms hold certain beliefs about the investment timing and size of the hidden competitor. In particular, the number of available places in the market, N, follows a Poisson jump process with arrival rate λ, so that the jump in N corresponds to the entry of the hidden firm. Moreover, as in the Apple example, we expect the hidden firm to have a new technology. In this case the positioned firms do not have all the information about the production process of the hidden player. Thus, we assume that firms have certain beliefs about the production process of the hidden firm. These beliefs are reflected by q_H, the capacity level that the hidden is presumed to install upon entering the market, which is treated as a parameter. Our analysis is aimed at identifying the effect of their beliefs about the presence and (or) size of the hidden player on the firm's strategies. Additionally, we follow Armada *et al.* (2011) by imposing that the market is big enough only for two firms[3]. This implies that the follower loses the option to invest if the hidden competitor enters the market earlier.

The market price of a unit of output is defined by the multiplicative inverse demand function:

$$P_t = X_t(1 - Q_t), \tag{2.1}$$

where Q_t is aggregate market output and X_t is a stochastic shock process that drives the uncertainty in the firm's profitability. It is assumed here that X_t evolves according to a geometric Brownian motion:

$$dX_t = \alpha X_t dt + \sigma X_t dZ_t, \tag{2.2}$$

where α is the constant drift, $\sigma > 0$ is the standard deviation, and dZ_t is the increment of a Wiener process. The discount rate, $r > 0$, is assumed to be larger than the drift, $r > \alpha$, otherwise waiting with investment would always be an optimal policy for the firms. Given this structure of the demand function, production optimization results in a fixed optimal quantity irrespective of the level of x. As a result, it is always optimal for the firms to produce up to capacity.

This specific choice of the demand structure is motivated by the desire to reflect the property that the market has limited size, which corresponds to the above assumption that maximally two firms can enter. A multiplicative demand function implies that, to avoid negative prices, the firms can increase their output only up to a certain level. Without loss of generality, in this model the maximum total market output is normalized to 1.

Denoting the leader's and the follower's capacity levels as q_L and q_F, respectively, the total output quantity given that the hidden firm has not entered the market yet can be written as

$$Q = q_L + q_F. \tag{2.3}$$

In the next sections we apply dynamic programming techniques to solve the optimal stopping problem for the positioned firms on the duopoly market described above.

[3]Here we can think of industries where the firms face significant barriers to entry, for example, due to strict government regulations, exclusive technology, limited resources, patents, or licenses.

2.3 The problem of the second investor

The problem is solved backwards starting with the decision of the second positioned investor when the hidden competitor has not invested yet. We determine its best response for a given strategy of the positioned leader. Then we analyze the strategy of the first investor, which has two choices in terms of investment timing: either to invest immediately and become a market leader or to wait with investment taking a risk of becoming the follower.

The optimal stopping problem of the firm looks as follows:

$$V_F^*(x, q_L) = \sup_{\tau, q_F} \mathbb{E}_x \left[\int_\tau^\infty \mathrm{e}^{-rt} x(1 - (q_F + q_L))q_F - \mathrm{e}^{-r\tau}\delta q_F \right], \tag{2.4}$$

where $V_F^*(x, q_L)$ denotes the value of the follower upon investment, and τ is a stopping time.[4]

Because of the Markovian nature of the underlying stochastic process, the solution will take the well-known form of the firs-passage time of an endogenously determined threshold. Denote by $x_F^*(q_L)$ the optimal investment threshold for the follower and by $q_F^*(q_L)$ the corresponding capacity level if the positioned leader is active on the market with a capacity level q_L. This implies that the follower will not enter the market until the stochastic component of the profit flow, x, reaches $x_F^*(q_L)$. On the contrary, for the values of x exceeding $x_F^*(q_L)$ investment becomes optimal and the follower enters the market immediately installing the capacity $q_F^*(x, q_L)$.

Thus, the range of x such that $x > x_F^*(q_L)$ is called the stopping region, while the one that satisfies $x < x_F^*(q_L)$ is called the continuation or waiting region. The optimal investment trigger is found using the fact that at the threshold value the firm's value of waiting is equal to the value of stopping, i.e. the firm is indifferent between entering the market and waiting for more information.

Under the assumption that a positioned firm is the leader, it is possible to derive the value function for the region where the follower waits with investment. As in Dixit and Pindyck (1994) we start with the Bellman equation

$$\mathbb{E}(dF) = rFdt. \tag{2.5}$$

Taking into account that with probability λdt the last available place on the market is occupied, Ito's lemma gives

$$\mathbb{E}(dF) = (1 - \lambda dt) \left(\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F(x)}{\partial x^2} dt + \alpha x \frac{\partial F(x)}{\partial x} dt \right) + \lambda dt(0 - F(x)) + o(dt). \tag{2.6}$$

Combining (2.5) and (2.6) we get the following partial differential equation:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F(x)}{\partial x^2} + \alpha x \frac{\partial F(x)}{\partial x} + \lambda[0 - F(x)] = rF(x), \tag{2.7}$$

[4] Throughout the thesis the following notation is used $\mathbb{E}_x[\,\cdot\,] = \mathbb{E}[\,\cdot\,|X_0 = x]$.

where the term $\lambda[0 - F(x)]$ represents the expected loss after the hidden entry in the interval of dt.

The solution of the partial differential equation above, and for the optimal stopping problem of the follower in general is described by the following proposition.

Proposition 2.1 *The follower's optimal capacity choice for a given level of the stochastic profitability shock, x, and the leader's capacity, q_L, is given by*

$$q_F^*(x, q_L) = \frac{1}{2}\left(1 - q_L - \frac{\delta(r-\alpha)}{x}\right)^+. \tag{2.8}$$

The value function of the follower takes the following form:

$$V_F^*(x, q_L) = \begin{cases} A(q_L)x^{\beta_1} & \text{if } x < x_F^*(q_L), \\ \\ \dfrac{[x(1-q_L) - \delta(r-\alpha)]^2}{4x(r-\alpha)} & \text{if } x \geq x_F^*(q_L), \end{cases} \tag{2.9}$$

with

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1, \tag{2.10}$$

$$A(q_L) = \left(\frac{(\beta_1 - 1)(1-q_L)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1} \frac{\delta(1-q_L)}{(\beta_1-1)(\beta_1+1)}. \tag{2.11}$$

The optimal investment trigger for the follower is given by

$$x_F^*(q_L) = \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)(1-q_L)}. \tag{2.12}$$

The above equations lead to the following optimal capacity level of the follower given the leader's capacity, q_L, at the optimal investment threshold

$$q_F^*(q_L) = \frac{1-q_L}{\beta_1+1}, \tag{2.13}$$

where $0 \leq q_L \leq 1$.

It is important to notice that for given capacity level of the leader, both the optimal capacity level and the investment trigger of the follower are decreasing with λ. The reason is that in the waiting region the follower faces the risk that the hidden competitor might enter the market before x reaches $x_F^*(q_L)$. If this is the case, the follower loses its option to invest. The bigger λ is, the more likely it is that such a situation can arise. Thus, the follower has an incentive to invest earlier for larger values of λ and, therefore, in a smaller capacity level. Moreover, the bigger the capacity level chosen by the leader the later the follower invests, while it will choose a smaller capacity level given the investment timing. This brings us to the problem of capacity choice by the market leader.

2.4 The problem of the first investor

In order to identify which strategy is optimal for the first investor, we solve the leader's investment decision problem. We start by determining the value of investing immediately, referring henceforth to it as the leader value. In case the leader decides to enter the market, it also has to decide upon the level of the capacity installment. As mentioned above, both the optimal investment trigger of the follower and its optimal capacity level depend on the capacity that the leader chooses. Essentially, there are two strategies available for the leader: install such a capacity that the follower enters the market either strictly later, or exactly at the same time as the leader. Following Huisman and Kort (2015), we call the former an entry deterrence strategy, and the latter an entry accommodation strategy. In what follows we solve the capacity optimization problem of the leader that undertakes an immediate investment and derive the leader value. Then we address the value of waiting with investment for the first entrant. As in Huisman (2001, Chapter 9), we construct the waiting curve, which represents the value of waiting with investment until the occurrence of the exogenous event, which in our case is the hidden entry. Lastly, we analyze the possible equilibria of this game.

2.4.1 The leader's deterrence strategy

The first strategy for the leader is to choose the capacity level in such a way that the follower will postpone its investment. First, we focus on the continuation region of the follower. Similarly to the previous section, the expected discounted revenue of the leader in the continuation region of the follower, denoted by $L(x)$, is determined by the following differential equation:

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 L(x)}{\partial x^2} + \alpha x \frac{\partial L(x)}{\partial x} - rL(x) + xq_L(1-q_L) + \lambda[\Phi_1(x) - L(x)] = 0, \quad (2.14)$$

where $\Phi_1(x) = \dfrac{xq_L(1-(q_L+q_H))}{r-\alpha}$ is the value function of the leader if the hidden firm occupies the follower's position. If the hidden competitor enters the market earlier than the follower, the leader value function will decrease in comparison to the standard case. This loss due to the hidden entry is captured by including the additional term in the differential equation, $\lambda[\Phi_1(x) - L(x)]$.

Next, using the fact that in the stopping region both positioned firms are present in the market we consider the following boundary conditions:

$$\lim_{x\to 0} L(x) = 0, \quad (2.15)$$

$$\lim_{x\to x_F} L(x) = \frac{x_F q_L(1-Q)}{r-\alpha}. \quad (2.16)$$

Combining these conditions and the expressions for q_F and $x_F^*(q_L)$, obtained in the previous section, we find the leader value in the deterrence region, i.e. its expected discounted revenues net of investment costs, i.e. $L(x) - \delta q_L$, in the continuation region of the follower:

$$\begin{aligned} V_L^{det}(x, q_L) =& x\frac{q_L(1-q_L)}{r-\alpha} - x\frac{q_L \lambda q_H}{(\lambda + r - \alpha)(r-\alpha)} - \delta q_L \\ & - \left(\frac{x(\beta_1 - 1)(1-q_L)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1} \left[\frac{\delta q_L}{(\beta_1 - 1)} - \frac{\delta(\beta_1+1)q_L\lambda q_H}{(\beta_1-1)(1-q_L)(\lambda+r-\alpha)}\right]. \end{aligned} \tag{2.17}$$

Recall that the follower will invest as soon as the stochastic process exceeds the value of the follower's trigger, $x_F^*(q_L)$. Thus, to implement the deterrence strategy the leader chooses q_L such that $x \leq x_F^*(q_L)$ given the current value of x .

Taking into account the expression for x_F^*, the deterrence strategy occurs when the leader chooses the capacity level such that

$$q_L > \hat{q}_L(x) = 1 - \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)x}. \tag{2.18}$$

Setting the derivative of the value function with respect to q_L to zero results into the following first order condition[5]:

$$\begin{aligned} \left(\frac{x(\beta_1 - 1)(1-q_L)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1} \frac{\delta}{(\beta_1-1)} \left[-\frac{(1-(\beta_1+1)q_L)}{(1-q_L)} + \frac{(\beta_1+1)\lambda q_H}{(\lambda+r-\alpha)}\frac{1-\beta_1 q_L}{(1-q_L)^2}\right] \\ + \frac{x(1-2q_L)}{r-\alpha} - \frac{x\lambda q_H}{(r-\alpha)(\lambda+r-\alpha)} - \delta = 0. \end{aligned} \tag{2.19}$$

The solution of equation (2.19) gives us the optimal capacity level for the leader under the deterrence strategy, $q_L^{det}(x)$. Therefore, the optimal value function of the leader in the deterrence region is $V_L^{det^*}(x) \equiv V_L^{det}(x, q_L^{det}(x))$. Further we will show that the leader can use the deterrence strategy if the value of the stochastic process x lies in the interval (x_1^{det}, x_2^{det}), where x_2^{det} is the biggest and x_1^{det} is the smallest possible value of the stochastic process that allows the leader to implement the deterrence strategy. The latter can be found by setting the capacity level to zero in the first order condition for the deterrence problem (equation (2.19)). In order to identify the biggest possible value of x for which deterrence is possible, x_2^{det}, recall that the leader uses this strategy only if the follower indeed enters later. This happens for those values of x that satisfy the following inequality: $x < x_F^*(q_L^{det})$. Therefore,

[5]Extensive numerical experiments show that the equation (2.19) has a single root, corresponding to a global maximum of the function $V_L^{det}(x, q_L)$.

x_2^{det} is defined by $x_F^*(q_L^{det}(x_2^{det})) = x_2^{det}$. These results are presented in the following proposition.

Proposition 2.2 *The lower bound of the deterrence region, x_1^{det}, is implicitly determined by the following equation:*

$$\left(\frac{x(\beta_1-1)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1}\frac{\delta}{(\beta_1-1)}\left[-1+\frac{(\beta_1+1)\lambda q_H}{(\lambda+r-\alpha)}\right] +\frac{x}{r-\alpha}-\frac{x\lambda q_H}{(r-\alpha)(\lambda+r-\alpha)}-\delta=0. \tag{2.20}$$

The upper bound of the deterrence region is given by

$$x_2^{det}=\frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}+\sqrt{\left(3+\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}\right)^2-8}\right]}. \tag{2.21}$$

It holds that $x_2^{det} > x_1^{det}$.

Note that x_1^{det} and x_2^{det} determine the feasible region for the deterrence strategy: for the values of the stochastic component of the profit flow, x, that fall into an interval (x_1^{det}, x_2^{det}), the leader will consider implementing the deterrence strategy.

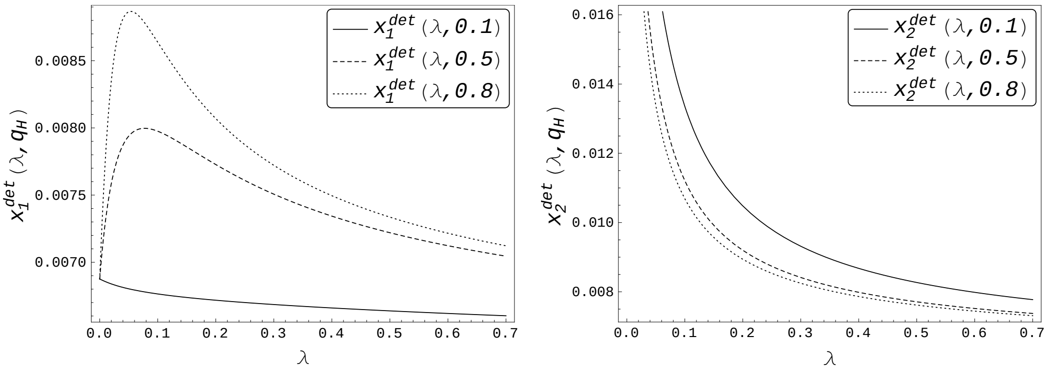
As can be seen, the upper and the lower bounds of the deterrence region depends on the parameter λ, the arrival rate of the hidden firm, and q_H, the capacity level of the hidden firm. Proposition 3 focuses on the latter characteristic.

Proposition 2.3 *An increase in the capacity level of the hidden firm, q_H, leads to an increase in the lower bound of the deterrence region, x_1^{det}, and to a decrease in its upper bound, x_2^{det}.*

Intuitively, the bigger the hidden firm is expected to be, the less is left for the leader after the division of market rents, and thus, the less appealing is the investment opportunity. Therefore on the one hand, a larger x is needed to convince the leader to enter such a market by installing a positive capacity. This explains an increasing pattern in x_1^{det}. On the other hand, since entry of the hidden firm with a large capacity is bad for the leader's profitability and there are only two entries possible, the leader has less incentive to deter the positioned follower's entry, causing x_2^{det} to decline. The next proposition focuses on the effect of λ.

Proposition 2.4 *An increase in the arrival rate of the hidden firm, λ, leads to a decrease in the upper bound of the deterrence region, x_2^{det}. If $q_H = 0$ the lower bound of the deterrence region, x_1^{det}, also decreases. For $q_H > 0$ the effect of an increase in λ on x_1^{det} is ambiguous.*

The effect of a change in the arrival rate, λ, on the upper and the lower bounds of the deterrence region is shown in Figure 2.1.



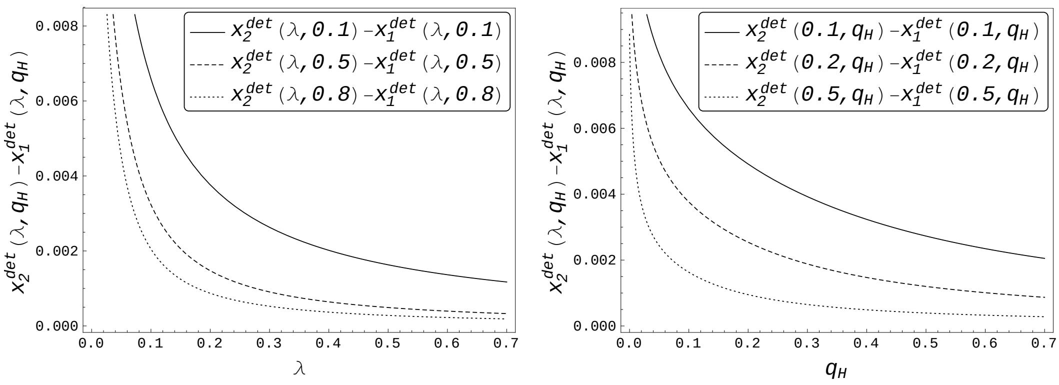
(a) $x_1^{det}(\lambda, q_H)$ *(b)* $x_2^{det}(\lambda, q_H)$

Figure 2.1: $x_1^{det}(\lambda, q_H)$ and $x_2^{det}(\lambda, q_H)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$, and $q_H = \{0.1, 0.5, 0.8\}$.

In Figure 2.1a, one can notice two differently directed effects of an increase in λ on x_1^{det}. On the one hand, for small q_H there is only a declining pattern to be observed. The reason is that for larger λ the leader is more willing to invest earlier in order to collect monopoly rents. On the other hand, for larger q_H numerical experiments reveal another effect of an increase in λ. In particular, when the value of q_H is sufficiently large, x_1^{det} is first increasing with λ. This indicates that the effect of declining profitability of the market dominates the advantage of investing earlier and collecting monopoly profits when the probability that the hidden firm enters the market is sufficiently small. However, after a certain point the latter effect becomes more dominant causing x_1^{det} to decrease with λ.

Considering the influence of a change in the arrival rate of the hidden firm, λ, on the upper bound of the deterrence region, x_2, we can conclude that a bigger risk of the hidden entry causes x_2^{det} to decline. This means that when the risk that the hidden firm will occupy the last available place on the market is higher, the follower is more eager to invest earlier. Hence, in this case, it is more difficult to ensure that the follower will enter the market strictly later than the leader and the deterrence region becomes smaller. This causes x_2^{det} to decrease with λ. Moreover, for larger values of λ this declining pattern is enhanced by the desire of the leader to invest in a smaller capacity due to a decreased profitability of the market. In addition, a smaller capacity does not prevent the follower to enter.

The dependence between the size of the deterrence region and the arrival rate of the hidden competitor and the expected size of the hidden firm is shown in Figure 2.2.



(a) For $\lambda = \{0.1, 0.2, 0.5\}$. *(b)* For $q_H = \{0.1, 0.5, 0.8\}$.

Figure 2.2: $x_2^{det}(\lambda, q_H) - x_1^{det}(\lambda, q_H)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, and $\delta = 0.2$.

In Figure 2.2a the decreasing effect of q_H is a direct implication of Propositions 2.3 and 2.4. The relation between the size of the deterrence region and the arrival rate λ cannot be described analytically due to the complexity of the expressions. Instead, numerous numerical experiments were carried out to investigate this dependence, allowing to conclude that a decrease in x_2^{det} dominates a decrease of x_1^{det} for a wide range of the parameter values causing the deterrence region to become smaller. The result is illustrated in Figure 2.2b. Huisman and Kort (2015) came to the conclusion that the deterrence interval expands with demand uncertainty, σ, which is also confirmed by our findings. However, in the presented setting yet another type of uncertainty is involved, namely the uncertainty about the market participants. The region where only the deterrence strategy is optimal tends to become smaller if this uncertainty is larger, or, in other words, if the risk that the hidden firm can enter the market is large. This region also becomes smaller for a larger capacity level of the hidden firm for the reason that the leader has less incentive to overinvest.

2.4.2 The leader's accommodation strategy

An entry deterrence strategy is not the only option for the leader to implement. In fact, the market can be big enough for both positioned firms to invest at the same time. The leader can choose such an investment scale, that the follower will enter the market immediately after the leader, which yields the following value

$$V_L^{acc}(x, q_L) = \frac{x q_L(1 - (q_L + q_F^*(x, q_L))}{r - \alpha} - \delta q_L, \tag{2.22}$$

where $q_F^*(x, q_L)$ is given by (2.8).

We call this strategy the accommodation strategy. The following proposition presents the optimal capacity of the leader, the corresponding value function, and the lower bound for the accommodation strategy.

Proposition 2.5 *Under the accommodation strategy the leader install the optimal capacity level $q_L^{acc}(x)$ given by*

$$q_L^{acc}(x) = \frac{1}{2}\left(1 - \frac{\delta(r-\alpha)}{x}\right), \tag{2.23}$$

and obtains the following value

$$V_L^{acc^*}(x) = \frac{[x - \delta(r-\alpha)]^2}{8x(r-\alpha)}. \tag{2.24}$$

The lower bound of the accommodation region, x_1^{acc}, is given by

$$x_1^{acc} = \frac{(\beta_1 + 3)}{(\beta_1 - 1)}\delta(r-\alpha). \tag{2.25}$$

Note that x_1^{acc} does not depend on the capacity level of the hidden firm, because under the assumption of a market with only two places available it is impossible for the third firm of any size to enter the market, given that the leader has entered and applies the accommodation strategy. However, the arrival rate λ still affects the lower bound of the accommodation region. Differentiating (2.25) with respect to λ we get[6]

$$\frac{\partial x_1^{acc}}{\partial \lambda} = -\frac{4\delta(r-\alpha)}{(\beta_1 - 1)^2} \cdot \frac{\partial \beta_1}{\partial \lambda} < 0, \tag{2.26}$$

The interpretation of the decline in x_1^{acc} with λ is straightforward. The bigger is the chance that the hidden firm can become active on the market, the earlier the positioned firms should undertake their investment, because the follower otherwise faces a high probability to lose its investment option.

2.4.3 The leader's boundary strategy

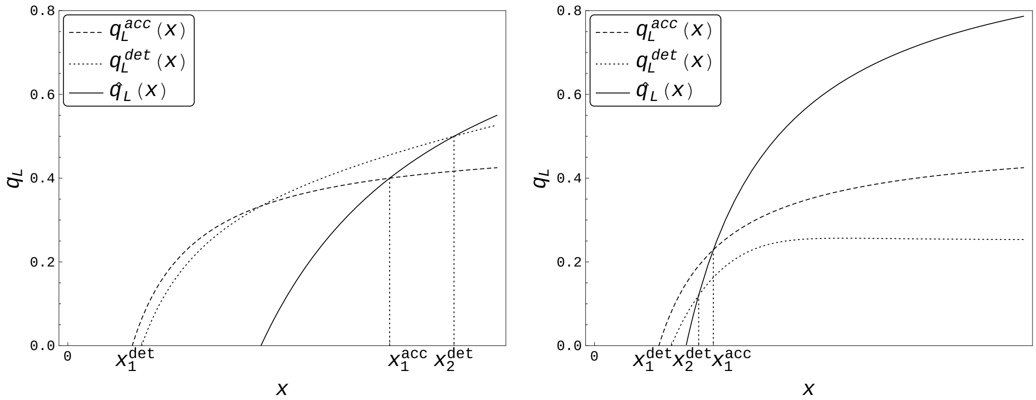
Recall that the boundary capacity, $\hat{q}_L(x)$, is the maximal capacity level of the leader that will stimulate the follower to enter the market immediately. For $q_L > \hat{q}_L(x)$ the follower will always postpone its investment, while for $q_L \leq \hat{q}_L(x)$ will enter the market for a given x. Earlier we referred to the strategy of the leader in the first case

[6]Here we use the observation that $\dfrac{\partial \beta_1}{\partial \lambda} = \dfrac{1}{\sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(\lambda + r)}} > 0.$

as deterrence strategy, while in the second – accommodation strategy. In line with Huisman and Kort (2015) this capacity level is given by

$$\hat{q}_L(x) = 1 - \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)x}. \tag{2.27}$$

The main difference, however, between the model presented by Huisman and Kort (2015) and the current modification is that the results of the latter are to a great extent influenced by two additional parameters associated with hidden competition. The key assumption of the presented model is that the positioned firms face a non zero probability of hidden entry, λdt. The expected investment size of the hidden player is represented by the parameter q_H. Figure 2.3 depicts the standard scenario with no hidden entries as well as the situation when the positioned firms face a positive probability that a hidden firm can enter the market by investing in a positive capacity. The capacity levels $q_L^{det}(x)$, $q_L^{acc}(x)$ and $\hat{q}_L(x)$ for both cases are presented as functions of the stochastic profitability shock, x. This specific example points out important differences of the presented setting with a standard duopoly model.



(a) For $\lambda = 0$, $q_H = 0$. *(b)* For $\lambda = 0.2$, $q_H = 0.2$.

Figure 2.3: The capacity levels $q_L^{det}(x)$, $\hat{q}_L(x)$ and $q_L^{acc}(x)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$, and different values of λ and q_H.

Earlier x_1^{det} was defined as the lower bound of the deterrence region. Therefore, in this figure x_1^{det} is determined by the intersection of $q_L^{det}(x)$ and the horizontal axis. To ensure that the follower invests later than the leader, the condition that the leader's capacity is bigger than $\hat{q}_L(x)$ has to be satisfied. In contrast, in order to implement the accommodation strategy the leader should choose a capacity level below $\hat{q}_L(x)$. Thus, the upper bound of the deterrence region, x_2^{det}, and the lower bound of accommodation region, x_1^{acc}, can be found as intersections of $\hat{q}_L(x)$ and $q_L^{det}(x)$ or $q_L^{acc}(x)$, respectively.

Figure 2.3a, where λ is equal to zero, resembles the result of Huisman and Kort (2015). Namely, the deterrence and accommodation regions intersect ($x_1^{acc} < x_2^{det}$). For the values of x below x_1^{acc} only deterrence can occur, in the region above x_2^{det} only accommodation is possible, whereas in the interval (x_1^{acc}, x_2^{det}) the leader chooses the strategy that brings the bigger value.

However, as the parameters associated with hidden competition sufficiently increase, the situation presented above changes. Figure 2.3b illustrates the case when $\lambda = 0.2$ and $q_H = 0.2$. As mentioned earlier, the parameters λ and q_H affect the boundaries of the feasible regions both for the deterrence and accommodation strategy. As can be seen, $\hat{q}_L$ shifts upwards, while $q_L^{det}(x)$ shifts downwards for every value of x, causing x_1^{acc} and x_2^{det} to change in such a way that now $x_1^{acc} > x_2^{det}$. The leader chooses deterrence if x lies in the interval between x_1^{det} and x_2^{det} and the accommodation strategy can only be implemented when x is bigger that x_1^{acc}. Yet, in the interval between x_1^{acc} and x_2^{det} neither a deterrence nor an accommodation optimal capacity level can be installed by the leader and in this region it is optimal for the leader to acquire a capacity equal to the boundary level $\hat{q}_L(x)$, i.e. the maximal capacity of the leader that induces simultaneous investment. This situation is illustrated in Figure 2.4.

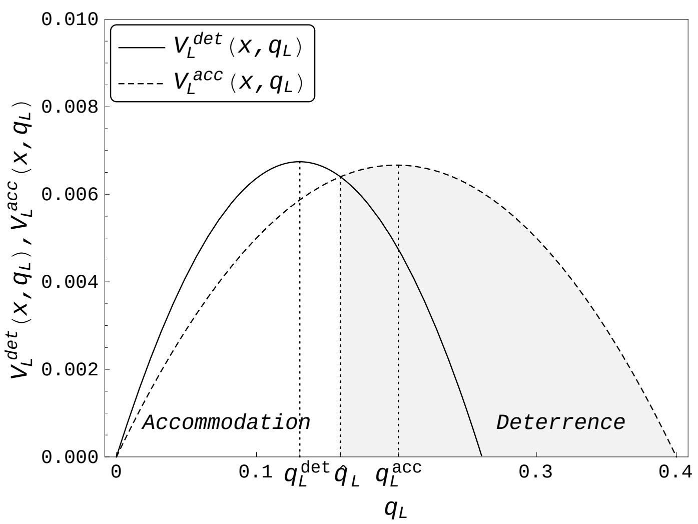


Figure 2.4: The value functions $V_L^{det}(x, q_L)$ and $V_L^{acc}(x, q_L)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$, $\lambda = 0.22$, $q_H = 0.2$, and where $x = 0.01$.

The intuition behind this result is as follows. In the presence of a high risk that the hidden firm will enter the market it is harder to deter the follower from occupying the last available place. Therefore, on the one hand we observe the shrinkage of the deterrence region, resulting in the fact that the optimal capacity level for which deterrence is optimal, $q_L^{det}(x)$, falls below $\hat{q}_L(x)$, which is in the accommodation region (see Figure 2.4). On the other hand, the optimum in terms of capacity choice cannot be reached for the accommodation strategy either, as the market is not yet big enough. This we see in Figure 2.4, where maximal accommodation profits are reached at $q_L^{acc}(x)$ greater than $\hat{q}_L(x)$, which is in the deterrence region. Therefore, the leader optimally invests at the boundary, i.e. choose the capacity level $\hat{q}_L(x)$ and enter the market simultaneously with the follower. The value of the leader in the latter case is denoted by $\widehat{V}_L(x)$ and is equal to $\hat{V}_L(x) \equiv V_L^{acc}(x, \hat{q}_L(x)) = \dfrac{\delta \hat{q}_L(x)}{\beta - 1}$.

Then the optimal leader value, $V_L^*(x)$, can be described as follows:

$$V_L^*(x) = \begin{cases} 0, & \text{if } \ 0 \leq x < x_1^{det}, \\ V_L^{det^*}(x), & \text{if } \ x_1^{det} \leq x < \min\{x_2^{det}, x_1^{acc}\}, \\ \widetilde{V}_L(x), & \text{if } \ \min\{x_2^{det}, x_1^{acc}\} \leq x < \max\{x_2^{det}, x_1^{acc}\}, \\ V_L^{acc^*}(x), & \text{if } \ x \geq \max\{x_2^{det}, x_1^{acc}\}, \end{cases} \tag{2.28}$$

where $\widetilde{V}_L(x) = \mathbb{1}_{\{x_2^{det} < x_1^{acc}\}} \hat{V}_L(x) + \mathbb{1}_{\{x_2^{det} > x_1^{acc}\}} \max\{V_L^{det^*}(x), V_L^{acc^*}(x)\}$.

The corresponding optimal capacity level, $q_L^*(x)$, is given by

$$q_L^*(x) = \begin{cases} 0, & \text{if } \ 0 \leq x < x_1^{det}, \\ q_L^{det}(x), & \text{if } \ x_1^{det} \leq x < \min\{x_2^{det}, x_1^{acc}\}, \\ \tilde{q}_L(x), & \text{if } \ \min\{x_2^{det}, x_1^{acc}\} \leq x < \max\{x_2^{det}, x_1^{acc}\}, \\ q_L^{acc}(x), & \text{if } \ x \geq \max\{x_2^{det}, x_1^{acc}\}, \end{cases} \tag{2.29}$$

where $\tilde{q}_L(x) = \mathbb{1}_{\{x_2^{det} > x_1^{acc}\}} \left(\mathbb{1}_{\{V_L^{det}(x) > V_L^{acc}(x)\}} q_L^{det}(x) + \mathbb{1}_{\{V_L^{det}(x) \leq V_L^{acc}(x)\}} q_L^{acc}(x) \right) + \mathbb{1}_{\{x_2^{det} < x_1^{acc}\}} \hat{q}_L(x)$.

Proposition 2.6 gives the condition under which $x_2^{det} < x_1^{acc}$ and, as a result, the boundary solution occurs.

Proposition 2.6 *When that $\lambda(8q_H - 1) > r - \alpha$, it holds that $x_2^{det} < x_1^{acc}$ and the leader invests in a capacity level being equal to $\hat{q}_L(x)$.*

The above condition is sufficient for the boundary region to exist. The obtained result entails that if the parameters reflecting the degree of the hidden competition, λ and q_H, become large enough, while the difference $r - \alpha$ is relatively low, the boundary region always exists. This can be interpreted in the following way. A high λ implies that there is a large risk of losing the last place on the market for the follower. When r is smaller, an investment results in a higher discounted cash flow stream, while a large α implies better market growth prospects. As a result, for smaller r or (and) larger α the follower is more reluctant to lose its investment option. Therefore, to secure the place in the market for larger λ, larger α and smaller r, the follower chooses simultaneous investment earlier, before the optimum for the accommodation strategy is reached. This guarantees the existence of the boundary strategy. A larger capacity of the hidden firm, q_H, is the reason that the leader wants to avoid entry of the hidden firm. It does so by pursuing a policy of investing simultaneously with the other positioned firm. This makes additional room for the entry accommodation, or, in other words, the boundary strategy.

2.5 Waiting curve

Consider the situation where the hidden competitor enters the market before the positioned firms and occupies the leader's position. In this case there is only one place left in the market, which is to be taken by one of the positioned firms. Naturally, the positioned firms will try to preempt each other in order to secure the last place in the market. Such a preemption game will lead to investment at zero-NPV threshold if x is low enough and yields zero value in expectation for both positioned firms. For higher values of x the firms will invest immediately and as a result one firm will occupy the last position in the market. As the firms are symmetric, in such case the positioned firms will obtain the last available place with equal probability. Then following Huisman (2001, Chapter 9) we derive the waiting curve as stated in the following proposition.

Proposition 2.7 *The waiting curve is given by*

$$W(x) = \begin{cases} \left(\frac{x}{x_M^*(q_H)}\right)^{\beta_1} \frac{\lambda(1-q_H)\,\delta\beta_2}{4\left(\beta_1^2-1\right)(\beta_2-\beta_1)(\lambda+r)} & \text{if } x < x_M^*(q_H), \\ \left(\frac{x}{x_M^*(q_H)}\right)^{\beta_2} \frac{\lambda(1-q_H)\,\delta\beta_1}{4(r+\lambda)\left(1-\beta_2^2\right)(\beta_1-\beta_2)} - \frac{\delta\lambda(1-q_H)}{4(r+\lambda)} & \\ \quad + \frac{\lambda x(1-q_H)^2}{8(r-\alpha)(r-\alpha+\lambda)} - \frac{\delta^2\lambda(r-\alpha)}{8x(\sigma^2-r-\alpha-\lambda)} & \text{if } x \geq x_M^*(q_H), \end{cases} \tag{2.30}$$

with the Marshallian investment trigger

$$x_M^*(q_H) = \frac{\delta(r-\alpha)}{(1-q_H)}. \tag{2.31}$$

The waiting curve represents the value for the firms if they both wait with investment until after the hidden entry occurs.

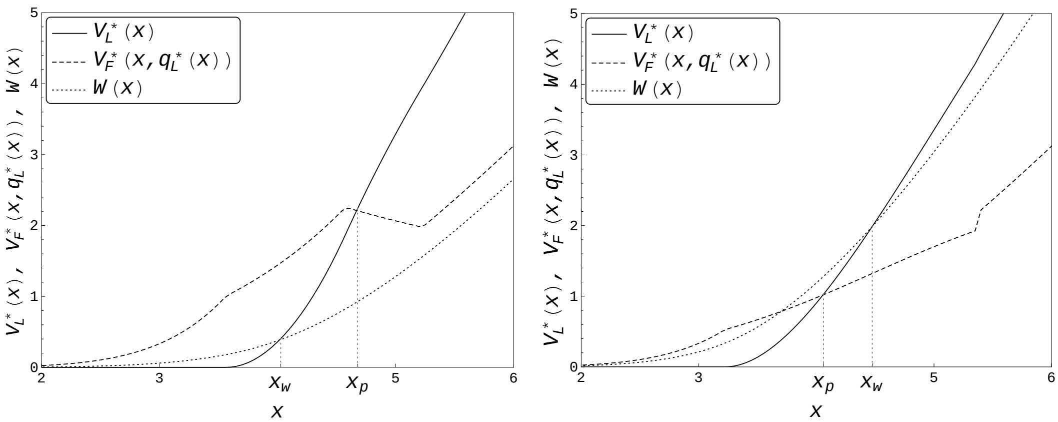
2.6 Equilibria

In this section we analyze the equilibria in the game where both positioned firms are allowed to invest first. In the traditional setting, where hidden competition is not considered (e.g. Huisman and Kort (2015)), a preemption equilibrium occurs. This equilibrium can be described as follows. When x increases, increasing market profitability creates incentives for the firms to preempt their rival and thereby to induce the second investor to enter later. The reward for the first entrant is a period of monopoly profits. As a result the firms engage in timing preemption. As long as the value of the first investor exceeds the value of the second investor, each positioned firm will have an incentive to invest a little earlier in order to become market leader. The preemption game stops as soon as the leader and the follower values are equalized (see e.g. Huisman and Kort (2015)). We denote the corresponding value of x by x_p and call it the preemption trigger. For x smaller than x_p it does not pay off to invest because the market is too small. It follows that one of the firms invests at once as soon as the stochastic process reaches x_p. The other firm will postpone its investment and enter the market as a follower once x reaches $x_F > x_p$.

The main difference of our model with the setting described above is the presence of hidden competition. In fact, the hidden firm can enter the market either after one of the positioned players has invested and become a follower, or it can enter as first and occupy the leader's position. The former possibility implies that the positioned firm that did not invest loses the option to enter. This is included into the value of the follower by construction. In order to determine the implications of the possibility that the hidden firm may enter the market first, we need to add the waiting curve to the analysis. Depending on the parameters of the hidden competitor, its location relative to the other value curves may vary. It is, however, possible to show that the waiting curve can never exceed the leader curve everywhere. This result is stated in the next proposition.

Proposition 2.8 *The waiting curve, $W(x)$, is lager than and the leader value, $V_L^*(x)$, for small values of x, and smaller for large values of x.*

In particular, the waiting curve can intersect the leader value before or after the preemption point. This is illustrated in Figure 2.5.



(a) $q_H = 0.2$ *(b)* $q_H = 0.05$

Figure 2.5: The value functions for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 100$, $\lambda = 0.25$, and different values of q_H.

Figure 2.5a shows that when the hidden firm is expected to acquire a large market share and enter with a large probability, waiting yields a relatively low value. The result is that the waiting curve intersects with the leader value before the preemption trigger. Denote this intersection point by x_w. When $x < x_w$ the firms naturally do not have any incentives to invest, as both waiting and being a follower yields higher value than investing immediately. For $x > x_w$ the equilibrium turns out to be exactly the same as in the subgame without the waiting curve. This is because investment with positive probability is not an equilibrium strategy if $x_w < x < x_p$, as the firms can always improve by investing with zero probability, while for $x > x_p$ both firms prefer to become the leader and invest at once. As a result, due to the preemption argument the firms' equilibrium strategy is to wait until the stochastic process hits x_p and invest afterwards.

Now consider the situation in Figure 2.5b, when the hidden firm is relatively small. In this case $x_w > x_p$. For the same reason as in the previous example no investment will occur before x_p. For $x > x_p$ the preemption argument still holds. Even though investing immediately yields a higher value than waiting only for $x > x_w$, the firms have an incentive to enter the market just before that and take the leader's position. Hence, the equilibrium strategy remains the same.

As a result, in both scenarios we have that for a low enough initial value of the stochastic process the first investor will always enter at the preemption point x_p. In principle, this point can be located in three different regions: where the leader applies either a deterrence or a boundary or an accommodation strategy. However, given that at the preemption point leader and follower values must match, the next proposition shows that the preemption point cannot be situated in the accommodation region.

Proposition 2.9 *If $x \geq x_1^{acc}$, the value of the leader always exceeds the value of the follower.*

Consequently, it is either the deterrence or the boundary capacity level that determines the preemption trigger, which is derived as being the intersection of the corresponding follower and leader value functions. These intersections are denoted by x_p^{det} and $\hat{x}_p$, respectively, and can be thus found by solving the forthcoming equations with respect to x

$$V_L^{det^*}(x) = V_F^*(x, q_L^{det}(x)), \tag{2.32}$$

$$\widehat{V}_L(x) = V_F^*(x, \hat{q}_L(x)). \tag{2.33}$$

Recall that if λ is small, the boundary strategy is irrelevant. Therefore, as in the benchmark model of Huisman and Kort (2015), where $\lambda = 0$, the preemption equilibrium always occurs in the deterrence region, implying that the first investor enters as soon as the stochastic process x hits the preemption trigger, x_p^{det}, while the second investor postpones its entry till x_F. However, unlike in Huisman and Kort (2015) for sufficiently large λ, it is also possible that the preemption trigger lies in the boundary region, where it is optimal for the firms to invest simultaneously at $\hat{x}_p$. The latter situation is illustrated in Figure 2.6.

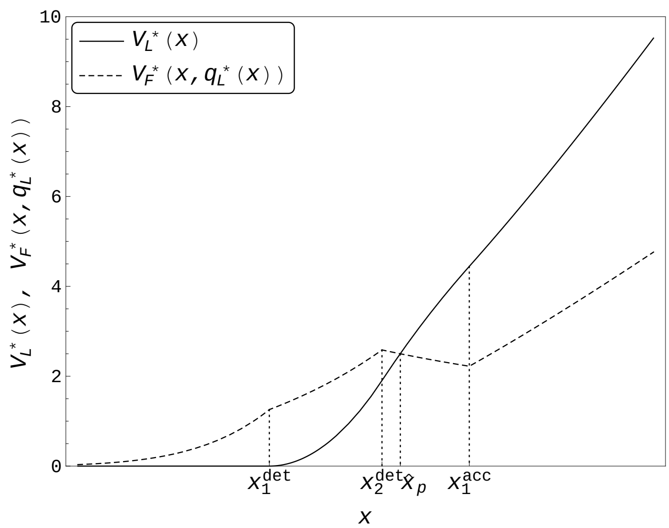


Figure 2.6: The value functions $V_L^*(x)$ and $V_F^*(x, q_L^*(x))$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 100$, $\lambda = 0.22$, $q_H = 0.25$.

As can be seen, in contrast to the standard result the follower value declines with x in the boundary region. This result is stated in the following proposition.

Proposition 2.10 *The value of the follower declines with x in the boundary region.*

The follower value can be affected by an increase in x in two ways: via investment timing and via capacity choice. In this problem the investment timing of the follower is always given. This is because the boundary capacity level of the leader is determined such that $x = x_F(q_L)$, implying that as the stochastic component of the demand function increases, the leader increases its capacity such that the new level of x exactly corresponds to the follower's investment threshold. Proposition 2.10 proves that the follower value is more influenced by the capacity effect, i.e. it declines as the capacity level of the leader increases than the increase in price for a given capacity due to the growth of x. As a result, the follower gets a lower value for larger x, due to an increase in the leader capacity level. This allows the leader and the follower values to intersect in the boundary region, implying that the preemption point is located in an interval where the firms invest simultaneously. Intuitively this result can be interpreted in the following way. If the degree of hidden competition is large, the value of the deterrence strategy decreases, as it becomes too costly to prevent entry of the second firm. As a result, it is optimal for the firms to wait till simultaneous investment is possible. However, even when the market is so big that the firms invest together at once, the concept of Stackelberg leadership implies that the leader has a first mover advantage and sets the capacity level first, causing a difference in payoffs of the first and second investor. This results into a slightly different preemption game, where each firm still has incentives to invest earlier in order to enjoy the first mover advantage and acquire a larger capacity level. However, in contrast to the standard preemption game, the entry of the positioned firms occurs at the same time.

Proposition 2.11 *For a given λ there exists a unique value of q_H, denoted by $\tilde{q}_H(\lambda)$, such that for $q_H \geq \tilde{q}_H(\lambda)$, preemption always occurs in the boundary region, while for $q_H < \tilde{q}_H(\lambda)$ we have preemption in the entry deterrence region:*

$$x_p(\lambda, q_H) = \begin{cases} x_p^{det}(\lambda, q_H) & \text{if } q_H < \tilde{q}_H(\lambda), \\ \\ \hat{x}_p(\lambda) & \text{if } q_H \geq \tilde{q}_H(\lambda). \end{cases} \tag{2.34}$$

$\tilde{q}_H(\lambda)$ is given by

$$\tilde{q}_H(\lambda) = \frac{(\lambda + r - \alpha)\beta_1}{\lambda(\beta_1 - 1)(\beta_1 + 2)}, \tag{2.35}$$

with β_1 defined by (2.10).

From Proposition 2.11 it follows that for $q_H \geq \tilde{q}_H(\lambda)$ both positioned firms invest simultaneously at the boundary capacity level, $\hat{q}_L(x)$, while similarly to the original model by Huisman and Kort (2015), for $q_H < \tilde{q}_H(\lambda)$ the first investor implements an entry deterrence strategy acquiring $q_L^{det}(x)$.

Intuitively, the larger is the hidden firm that is expected to enter the market, the more attractive is the boundary strategy for the positioned firms, as it guarantees that the hidden player loses the chance to invest at the moment that both firms enter. Hence, the leader is not exposed to the risk that it has to compete with a large hidden firm since the other positioned firm invests at the same time as the leader. This is confirmed by Proposition (2.12).

Proposition 2.12 *The capacity $\tilde{q}_H(\lambda)$ declines with λ.*

Thus, a larger capacity of the hidden player implies a larger range of λ and q_H for which simultaneous investment takes place. This is illustrated by the numerical example in Figure 2.7. As can be seen, for a larger capacity of the hidden firm the positioned firms are more willing to invest simultaneously, as by doing so they occupy all available places on the market and thus prevent the undesirable entry of a large hidden player.

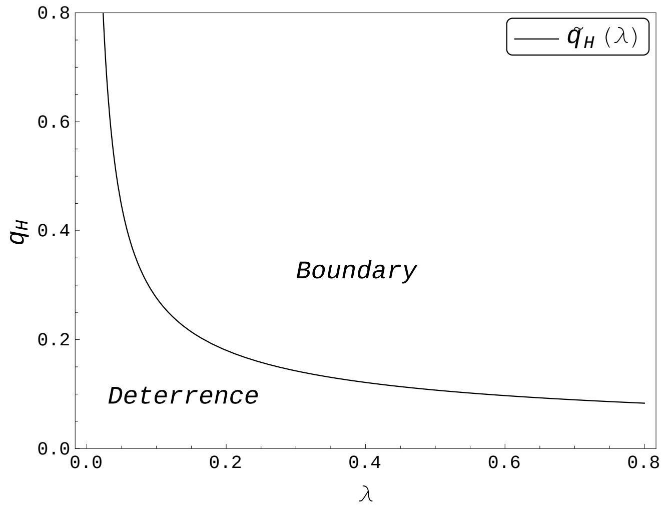


Figure 2.7: The possible strategies of the leader depending on λ and q_H for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$.

The next proposition states an optimal investment trigger and the corresponding capacity level of the positioned firms when $q_H \geq \tilde{q}_H(\lambda)$, i.e. when they enter the market simultaneously in the boundary region.

Proposition 2.13 *If $q_H \geq \tilde{q}_H(\lambda)$, the preemption trigger is equal to $\hat{x}_p$, which is given by*

$$\hat{x}_p = \frac{\delta(r-\alpha)(\beta_1+2)}{(\beta_1-1)}, \tag{2.36}$$

with the corresponding capacity level

$$\hat{q}_L(\hat{x}_p) = \frac{1}{\beta_1 + 2}. \tag{2.37}$$

This implies that the positioned firms not only invest at the same time, but also at the same capacity level. This is because the first mover advantage of the leader disappears due to the rent equalization property in the preemption game.

Differentiating (2.36) and (2.37) with respect to λ gives

$$\frac{\partial \hat{x}_p}{\partial \lambda} = -\frac{3\delta(r-\alpha)}{(\beta_1 - 1)^2} \cdot \frac{\partial \beta_1}{\partial \lambda} < 0, \tag{2.38}$$

$$\frac{\partial \hat{q}_L(\hat{x}_p)}{\partial \lambda} = -\frac{1}{(\beta_1 + 2)^2} \cdot \frac{\partial \beta_1}{\partial \lambda} < 0. \tag{2.39}$$

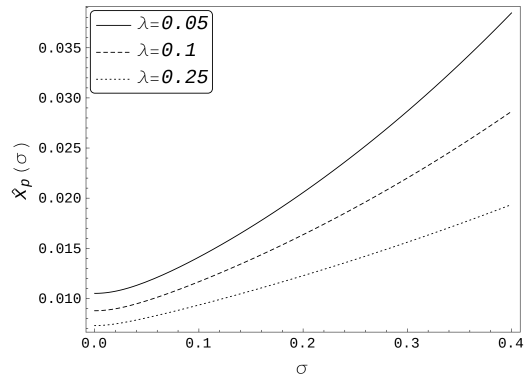
Thus, we observe the negative dependence between arrival rate λ and the preemption trigger $\hat{x}_p$, as well as the capacity level at this preemption point $\hat{q}_L(\hat{x}_p)$. To interpret this result we take a derivative of $\hat{q}_L(x)$ with respect to λ:

$$\frac{\partial \hat{q}_L(x)}{\partial \lambda} = \frac{2\delta(r-\alpha)}{(\beta_1 - 1)^2 x} \cdot \frac{\partial \beta_1}{\partial \lambda} > 0. \tag{2.40}$$

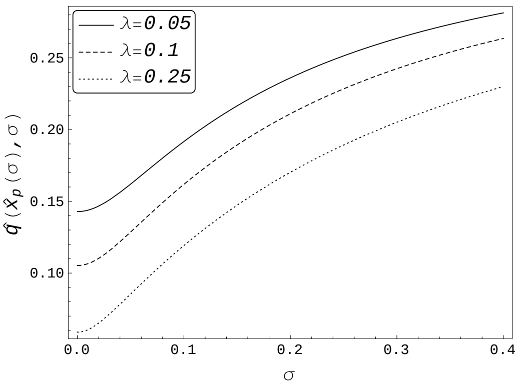
Recall that $\hat{q}_L(x)$ is the maximal capacity level of the leader such that the follower invests immediately ($x = x_F$). As can be seen from (2.40), the boundary capacity level for a given x is larger if λ increases. In other words the follower facing the threat of loosing the last available place of the market is willing to accommodate for a larger level of the leader's capacity for a given x. Consequently, the bigger is λ, the closer is the leader capacity level to the optimal level for the accommodation strategy leading to an increase in the leader value. The follower value, on the contrary, decreases with λ, as a result of an increase in the leader's capacity level. The increase of the leader value, together with the decrease in the follower value, results in the fact that the preemption point $\hat{x}_p$ decreases with λ. This lower value of $\hat{x}_p$ results in a lower output price at the moment of investment, which has a negative effect on the corresponding capacity level $\hat{q}_L(\hat{x}_p)$. According to (2.39), this negative effect dominates the positive effect an increasing λ has on $\hat{q}_L(x)$ (see (2.40)).

Note that the capacity of the hidden firm, q_H, does not exert an influence on the preemption point $\hat{x}_p$ in this case. This happens because applying the boundary strategy implies that both firms invest at once, occupying all available places on the market and therefore the third player, the hidden firm, loses the option to invest, i.e. to install capacity.

The analytical expressions for the preemption trigger, (2.36), and the capacity level, (2.37), corresponding to the boundary strategy can be used to analyze the effect of uncertainty. Figure 2.8 shows how a change in σ affects $\hat{x}_p$ and $\hat{q}_L(\hat{x}_p)$ for different values of λ.



(a) The preemption trigger $\hat{x}_p$



(b) The capacity level $\hat{q}_L(\hat{x}_p)$

Figure 2.8: The preemption trigger $\hat{x}_p$ and the corresponding capacity level $\hat{q}_L(\hat{x}_p)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$, $\lambda = \{0.05, 0.1, 0.25\}$.

As in Huisman and Kort (2015), both the preemption trigger and the corresponding quantity increase with uncertainty. This confirms the standard result in the real options literature. In the present model the increase in the investment threshold due to uncertainty is less for larger λ, as an increasing probability of the hidden entry induces earlier investment.

Consider now the case when $q_H < \tilde{q}_H(\lambda)$ and the first investor prevents an immediate entry of the second investor by installing the deterrence capacity, $q_L^{det}(x)$. In this case the leader invests at the moment x hits x_p^{det}, while the follower waits till x_F. These thresholds are described by the following proposition.

Proposition 2.14 *If $q_H < \tilde{q}_H(\lambda)$ the preemption trigger is equal to x_p^{det}, which is the solution with respect to x of*

$$V_L^{det^*}(x) \quad = \quad V_F^*(x, q_L^{det}(x)), \tag{2.41}$$

with the corresponding capacity level $q_L^{det}(x_p^{det})$, implicitly determined by (2.19).

Consider now the dependence between the preemption point x_p and the capacity of the hidden firm for a given λ. In Figure 2.9 the preemption point for the boundary strategy, $\hat{x}_p$, is not affected by the capacity of the hidden firm, because as mentioned earlier the simultaneous investment of the positioned firms implies that the hidden player loses the chance to install capacity. The effects of an increase in q_H on the deterrence preemption trigger, x_p^{det}, and the follower's investment threshold, x_F, result from changes in the leader and follower values.

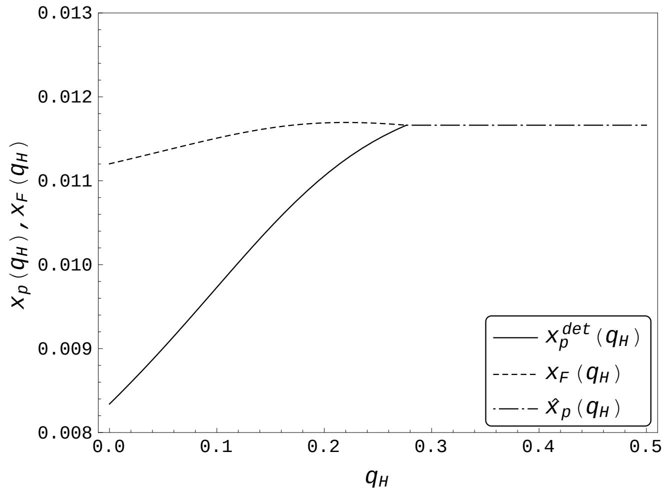


Figure 2.9: The preemption trigger $x_p(q_H)$ and the follower's trigger $x_F(q_H)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$, $\lambda = 0.1$.

Intuitively, the leader value is lower if the hidden firm is larger. This is because the market becomes less profitable given that when the hidden firm becomes active, it does so by installing a larger capacity. An increase in the capacity level of the hidden firm affects the follower value only through the leader's capacity choice. The bigger the potential entrant is, the more incentives the leader has to reduce its capacity and as a result to stimulate the follower to enter the market earlier in order to prevent the hidden entry. Hence, the follower value increases in x. Together with the decrease of the leader value, this shifts the preemption point to the right. The follower's entry threshold is only influenced indirectly by the capacity of the hidden firm through the leader's capacity choice at its optimal point in time.

Due to the described effects, the bigger the hidden firm that is expected to enter is, the later the leader invests, and later investment implies a larger capacity. On the other hand, the leader reduces the capacity for every level of x to stimulate the follower to enter the market. The combination of these two effects is the reason for the non-monotonicity in the optimal leader capacity at the preemption point as a function of q_H as demonstrated in Figure 2.10.

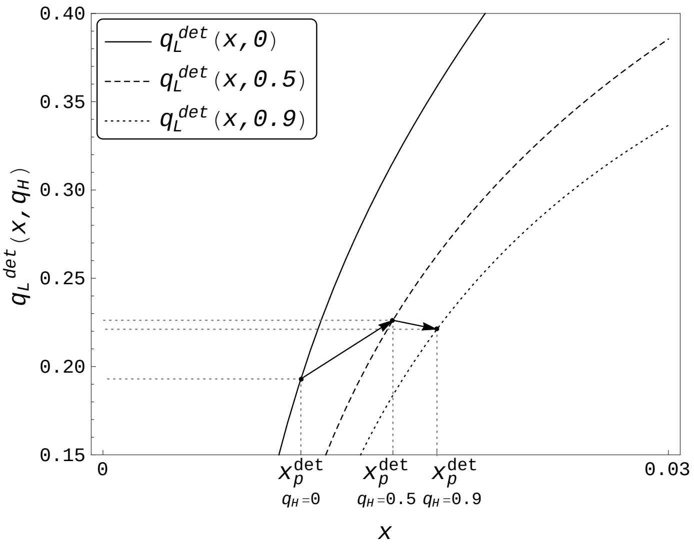


Figure 2.10: The capacity level $q_L^{det}(x, q_H)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$, $q_H = \{0, 0.5, 0.9\}$.

Due to this non-monotonic relationship between the capacity of the leader and the capacity of the hidden firm, the follower threshold first increases with q_H and then declines until it reaches the threshold for the boundary strategy as depicted in Figure 2.9. Now consider the influence of the arrival rate of the hidden firm on the firm's optimal investment thresholds under the deterrence strategy.

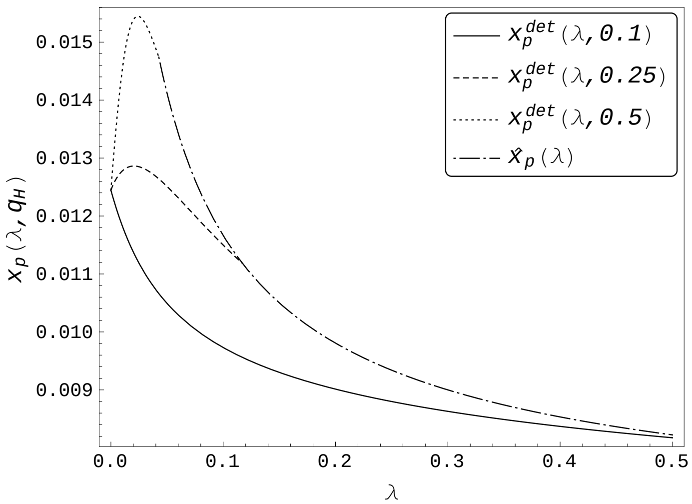


Figure 2.11: The preemption trigger $x_p(\lambda)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$ and $q_H = \{0.1, 0.25, 0.5\}$.

Figure 2.11 depicts the dependence between the preemption trigger, x_p, and the arrival rate of the hidden competitor, λ, for different levels of the capacity size of the hidden player. The dash-dotted line in the figure depicts the investment trigger related to the boundary region, whereas the other lines, each corresponding to a different capacity level of the hidden firm, represent the deterrence investment trigger, which is now the focus of our analysis. As it was shown before, for a larger capacity the region where it is optimal for the first investor to use the deterrence strategy shrinks. Moreover, the shape of x_p^{det} changes as the size of the hidden firm changes. Thus, to interpret this result it is reasonable to consider the scenarios of small and large q_H separately. In what follows we first consider the scenario with $q_H = 0$, followed by an analysis of a situation with large q_H.

2.6.1 Small hidden firm

If the capacity of the hidden firm is small its entry is beneficial for the leader. In the extreme case of $q_H = 0$ the advantage of hidden entry is particularly big as it implies that the follower loses the option to invest and the leader becomes a monopolist on the market forever. This scenario has an interesting interpretation. Namely, it can be interpreted as the game where the government with probability λdt restricts the number of places on the market to one by, for example, offering a patent monopoly to an innovating firm. Thus, an increase in the arrival rate of the hidden firm has a direct effect on the leader value, namely, the value increases for a given x due the attractiveness of the investment opportunity. On the other hand, a higher probability of hidden entry affects the follower's decision, resulting into an indirect effect on the leader value. In fact, facing the threat of losing the last available place on the market the follower is more eager to invest earlier. Therefore, it is more costly for the leader to perform the deterrence strategy, i.e. to induce the follower to invest later. Thus, for each level of x a larger capacity is needed to ensure that the follower indeed invests later[7]. This leads to a decrease in the leader value for a given level of x. The total effect on the leader value is determined by the predominance of one of these effects.

Intuitively, for small x, when the investment opportunity is unappealing, the follower is less eager to invest and the direct effect dominates. However once x becomes sufficiently large the investment becomes more attractive, making the credible deterrence of the follower's entry more difficult. Thus, the second effect becomes dominant leading to a decrease in the leader value. Moreover, for larger λ the second effect starts dominating earlier, as the follower becomes more aggressive facing a larger risk of the hidden entry.

As a result, of entry deterrence the follower value is always lower for larger λ, as it

[7]Although analytical expressions for $q_L^{det}(x)$ cannot be obtained, careful numerical simulations confirm that the presented relation holds for the considered range of λ, i.e. $\lambda < \lambda_p(q_H)$.

is forced to accelerate its investment, because the probability to lose its investment option is larger. Numerical experiments show that as a result of an increase in λ the decline in the follower value is always large enough to ensure that the intersection of the leader and follower value curves takes place for lower values of x. This results in the fact that the preemption point always declines with λ for relatively small capacity of the hidden firm Figure 2.12 illustrates this result.

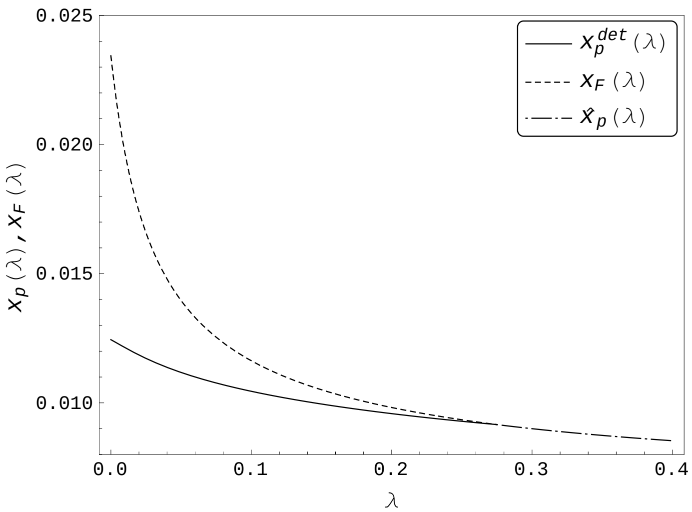


Figure 2.12: The preemption trigger $x_p(\lambda)$ and the follower's trigger $x_F(\lambda)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$, $q_H = 0.15$.

The implication for the follower's threshold is that it is influenced by a change in λ both by the desire to enter the market, and by the change in the optimal capacity level of the leader at the moment of investment. The optimal capacity of the leader is in turn affected by both preemption timing and the capacity choice that ensures credible deterrence. As discussed earlier an increase in λ leads to an upward shift in the leader capacity level for each value of x together with a decrease in the leader investment threshold as a result of the preemption effect. Therefore, if the decrease in investment timing is large enough, the leader capacity at the moment of investment will be lower for larger λ. However, for a relatively small decrease in the investment threshold together with a larger upward shift in the capacity curve an increase in λ can result in a larger optimal capacity of the leader at the moment of investment., making the follower more reluctant to invest. This could cause a non-monotonicity in the follower's investment threshold. However, in the considered model the risk associated with later investment is too high, as the opportunity to enter the market might be lost forever. As a result, an increase in λ causes the follower to become more aggressive and invest earlier for higher λ.

2.6.2 Large hidden firm

Now consider the situation when q_H is large. Here we restrict ourselves to scenarios where the firm's revenues are positive. Namely, we exclude the possibility of negative prices by restricting the considered range of the capacity of the hidden firm, namely $q_H \leq 1 - q_L$. For q_H being large the opposite situation occurs compared to the analysis of the situation with a small hidden firm. In the event of the hidden entry the leader is left with a relatively small market share, and thus wants the follower to invest earlier to prevent the hidden entry. Therefore, as λ increases, the leader reduces its capacity to tempt the follower to enter the market sooner, thus when the trigger value is lower.

The effect of an increase in λ on the value functions can again be decomposed in two parts. On the one hand, the larger the arrival rate the more likely it is that the hidden firm enters with large capacity, thus the lower is the leader value due to this direct effect. On the other hand, the larger is the arrival rate, the more eager is the follower to invest, which is good for the leader. Therefore, its value is increasing due to this indirect effect.

For small x investment is not attractive yet. Thus, an increase in λ exerts less influence on the follower's investment decision. Therefore, the direct effect associated with the entry of a large hidden player dominates, causing the leader value to decrease. For larger x the investment opportunity is more valuable and, as a result, the follower is willing to invest sooner to occupy the last available place on the market. This situation is favorable for the leader and, therefore, the second effect of an increasing leader value becomes more important. As in the previous case the larger is λ, the sooner the indirect effect becomes dominant. This happens because in the presence of larger hidden entry risk the desire of the follower to invest sooner outweighs the direct negative effect of possible hidden entry for lower x. On the other hand, the follower value in this case increases with λ, as the leader wants the follower to invest early and it does so by investing in smaller capacity. Hence, the leader rewards the follower even more for early investment.

Combing these results, we conclude that when the market is small, it is more likely that the leader value declines with λ while the follower value increases, resulting in an intersection point occurring for a larger value of x. When the market becomes more profitable, a larger λ leads to an increase in both follower and leader values, shifting their intersection point to the left.

As a result, the effect of an increase of λ on the preemption point for the deterrence strategy, which is determined by the intersection of the value functions, is non-monotonic. In particular as we observe in Figure 2.13 $x_p^{det}(\lambda)$ first increases with λ and then starts to decline.

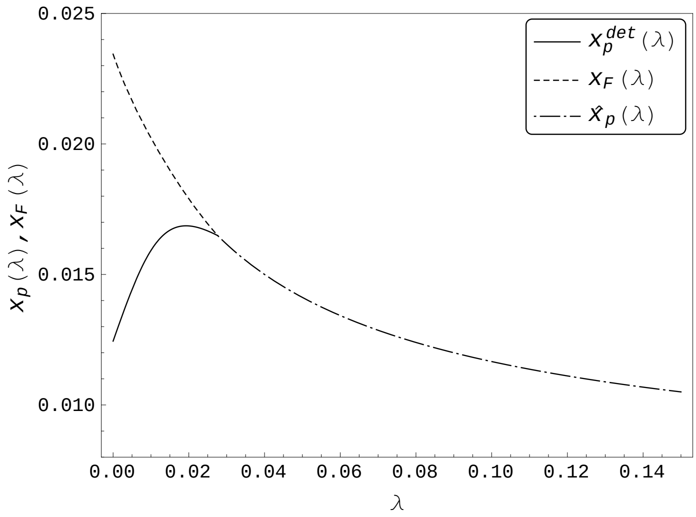


Figure 2.13: The preemption trigger $x_p(\lambda)$ and the follower's trigger $x_F(\lambda)$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\delta = 0.2$, $q_H = 0.7$.

For the follower the implication is the same as in the case of zero capacity of the hidden firm. Namely, increasing risk of the hidden entry induces the follower to invest earlier in order not to lose the option to enter the market and, as a result, the follower's trigger declines with λ.

2.7 Conclusion

This chapter examines firms' strategies when making an investment decision under uncertainty, which includes both timing and capacity level. After allowing the firms to choose the capacity level in the duopoly model with hidden competition, we found new effects of the possible entry occasion of the hidden firm. As a result, and in contrast with Huisman and Kort (2015), where the possibility of hidden entry was not present, the optimum for the accommodation capacity level is not available immediately after the deterrence region ends, and for a non-zero probability of the hidden entry a gap between the two strategies is generated. Intuitively, the deterrence region becomes smaller if the uncertainty about the market participants is high, because the follower facing the threat of losing its investment option is more eager to invest earlier and it is getting harder for the leader to deter this entry. On the other hand, the market is not yet big enough to acquire the optimal accommodation capacity level. Thus, in the gap region the leader chooses the strategy that maximizes its value, namely, invests at the boundary capacity level stimulating immediate investment of the follower. This strategy is equivalent to the entry accommodation in terms of timing, but, however, implies a smaller capacity level of the leader. Therefore, in the endogenous roles game, when a relatively large hidden player is expected to enter with a sufficiently large probability, the deterrence strategy becomes too costly to implement, and the first investor deterring the entry of the second firm always gets a lower value. As a result, both firms enter the market simultaneously.

Finally, it is important to point out the possibilities for further research related to this topic. First, it is worth mentioning that the obtained results are derived for the specific case of a market with limited places. On the one hand, the assumption of the restricted number of market participants can be relaxed by extending the model for a larger number of either positioned or hidden firms. On the other hand, it is also important to consider the more general setting where the follower does not lose the option to invest once the hidden competitor becomes active on this market. Intuitively, as long as the multiplicative demand function is used the firms are limited in capacity expansion, since too much capacity leads to negative prices. As long as we consider markets described by this specific demand structure, the positioned firms will always have incentives to install a capacity large enough to prevent entry of the hidden player. That is why it is interesting to consider how the assumption of hidden competition affects optimal investment behavior of firms on markets described by alternative demand functions with unlimited places available. Furthermore, a relevant extension would be to incorporate the possibility of multiple investments into the model similar to Boyer *et al.* (2012). This will allow to draw the conclusions about the industry development. Combining multiple investments with capacity optimization would extend Boyer *et al.* (2012).

Another interesting topic arises relaxing the assumption of the entry probability of the hidden firm being constant. In particular, further analysis could examine the influence of the entry decisions of the positioned firms on the arrival rate of the hidden firm. The more profitable is the market, the more attractive is it for potential entrants. Therefore, the mean arrival rate of the hidden rival may decline with every new entry, as the market becomes less profitable.

2.8 Appendix

Proof of Proposition 2.1 If the leader has already invested, the follower value, denoted by V_F, is given by

$$V_F(x, q_L, q_F) = \mathbb{E}_x\left[\int_0^\infty x q_F (1-Q)\mathrm{e}^{-rt}dt - \delta q_F\right] = \frac{x q_F(1-Q)}{r-\alpha} - \delta q_F. \quad (2.42)$$

Note that V_F in this case does not depend on λ, as in this case the positioned firm occupies the last available place in the market.

To determine the optimal quantity, the follower solves the following maximization problem, given the level of geometric Brownian motion, x :

$$\max_{q_F} \mathbb{E}_x\left[\int_0^\infty q_F x (1-Q)\mathrm{e}^{-rt}dt - \delta q_F\right]. \quad (2.43)$$

The first order condition for the follower in this case takes the following form:

$$\frac{\partial}{\partial q_F}\left[\frac{x}{r-\alpha}\left(1-(q_F+q_L)\right)q_F - \delta q_F\right] = 0. \quad (2.44)$$

Thus, the follower's optimal capacity level is equal to

$$q_F^*(x, q_L) = \frac{1}{2}\left(1 - q_L - \frac{\delta(r-\alpha)}{x}\right). \quad (2.45)$$

The total quantity, Q, takes now the following form

$$Q(q_L) = Q(q_F^*(q_L), q_L) = q_L + q_F^*(q_L) = \frac{1}{2}\left(1 + q_L - \frac{\delta(r-\alpha)}{x}\right). \quad (2.46)$$

Substituting the expression for q_F^* into the follower value function, we get

$$V_F(x, q_L, q_F^*(x, q_L)) = \frac{[x(1-q_L) - \delta(r-\alpha)]^2}{4x(r-\alpha)}. \quad (2.47)$$

The value function of the follower in the continuation region can be found by solving[8]

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F(x)}{\partial x^2} + \alpha x \frac{\partial F(x)}{\partial x} - (r+\lambda)F(x) = 0. \quad (2.48)$$

Denoting x_F is the trigger value for the follower, we consider the following boundary conditions:

$$\lim_{x \to 0} F(x) = 0. \quad (2.49)$$

[8]The problem is solved by applying dynamic programming methods presented in Dixit and Pindyck (1994).

$$\lim_{x \to x_F} F(x) = \frac{x_F q_F (1-Q)}{r-\alpha} - \delta q_F, \tag{2.50}$$

$$\lim_{x \to x_F} \frac{\partial F(x)}{\partial x} = \frac{q_F(1-Q)}{r-\alpha}. \tag{2.51}$$

Considering the condition (2.49) we can write the solution of the differential equation (2.48) as $F(x) = Ax^{\beta_1}$ with β_1 equal to[9]

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1. \tag{2.52}$$

From the value matching (2.50) and smooth pasting (2.51) conditions we get the expression for A

$$A(q_L, q_F) = \frac{1}{x_F^{\beta_1}(q_L, q_F)} \left(\frac{x_F(q_L, q_F) q_F (1 - (q_F + q_L))}{r-\alpha} - \delta q_F \right), \tag{2.53}$$

where the trigger value $x_F(q_L, q_F)$ is given by

$$x_F(q_L, q_F) = \frac{\beta_1 \delta (r-\alpha)}{(\beta_1 - 1)(1-Q)}. \tag{2.54}$$

At the moment of investment the optimal capacity of the follower $q_F^*(x, q_L)$ is given by

$$q_F^*(x, q_L) = \frac{1}{2}\left(1 - q_L - \frac{\delta(r-\alpha)}{x}\right). \tag{2.55}$$

Hence, the optimal investment trigger $x_F^*(q_L)$ and the follower's quantity $q_F^*(q_L)$ given the capacity of the leader q_L are defined by

$$x_F^*(q_L) = \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)(1-q_L)}, \tag{2.56}$$

$$q_F^*(q_L) = \frac{1-q_L}{\beta_1+1}. \tag{2.57}$$

Substituting the results in the expression for A gives

$$A(q_L) = \left(\frac{(\beta_1-1)(1-q_L)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1} \frac{\delta(1-q_L)}{(\beta_1-1)(\beta_1+1)}. \tag{2.58}$$

Therefore, the compact solution for the follower's problem is given by

$$V_F^*(x, q_L) = \begin{cases} A(q_L)x^{\beta_1} & \text{if } x < x_F^*(q_L), \\ \dfrac{[x(1-q_L) - \delta(r-\alpha)]^2}{4x(r-\alpha)} & \text{if } x \geq x_F^*(q_L). \end{cases} \tag{2.59}$$

[9]Here β_1 is the positive solution of the fundamental quadratic equation, i.e. $\frac{1}{2}\sigma^2 x {\beta_1}^2 + (\alpha - \frac{1}{2}\sigma^2)\beta_1 - (r+\lambda) = 0$.

Proof of Proposition 2.2 Define $\psi(x, q_H)$ as the left hand side of the the equation (2.19) in the case when $q_L = 0$:

$$\psi(x,q_H) = \left(\frac{x(\beta_1-1)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1} \frac{\delta}{(\beta_1-1)}\left[-1+\frac{(\beta_1+1)\lambda q_H}{(\lambda+r-\alpha)}\right] + \frac{x}{r-\alpha} - \frac{x\lambda q_H}{(r-\alpha)(\lambda+r-\alpha)} - \delta. \tag{2.60}$$

Therefore, x_1^{det} is implicitly determined by the equation $\psi(x, q_H) = 0$.
Note that

$$\psi(0, q_H) = -\delta < 0, \tag{2.61}$$

$$\psi(x_F^*(0), q_H) = \frac{\delta}{\beta_1 - 1} > 0, \tag{2.62}$$

$$\frac{\partial\psi(x,q_H)}{\partial x} = \frac{(\lambda+r-\alpha)-\lambda q_H}{(\lambda+r-\alpha)(r-\alpha)} - \frac{(\lambda+r-\alpha)-(\beta_1+1)\lambda q_H}{(\lambda+r-\alpha)(r-\alpha)}\frac{\beta_1}{\beta_1+1}\left(\frac{x(\beta_1-1)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1-1}. \tag{2.63}$$

Differentiating (2.63) with respect to x we get:

$$\frac{\partial^2\psi(x,q_H)}{\partial x^2} = x^{\beta_1-2}\frac{\beta_1\delta\left(\frac{(\beta_1-1)}{(\beta_1+1)\delta(r-\alpha)}\right)^{\beta_1}((\beta_1+1)\lambda q_H-(\lambda+r-\alpha))}{(\lambda+r-\alpha)}. \tag{2.64}$$

For $x \geq 0$ the function in (2.64) is either monotonically increasing or monotonically decreasing depending on the combination of the parameter values.

Consider $x \in (0, x_F^*(0))$. Evaluating the first derivative $\frac{\partial\psi(x,q_H)}{\partial x}$ at the upper bounds of this interval we obtain

$$\left.\frac{\partial\psi(x,q_H)}{\partial x}\right|_{x=0} = \frac{\lambda(1-q_H)+r-\alpha}{(\lambda+r-\alpha)(r-\alpha)} \geq 0, \tag{2.65}$$

$$\left.\frac{\partial\psi(x,q_H)}{\partial x}\right|_{x=x_F^*(0)} = \frac{\left(\beta_1^2-1\right)\lambda q_H+\lambda+r-\alpha}{(\beta_1+1)(\lambda+r-\alpha)(r-\alpha)} \geq 0. \tag{2.66}$$

Given the monotonicity of $\frac{\partial^2\psi(x,q_H)}{\partial x^2}$ we can conclude that for $x \in (0, x_F^*(0))$, $\frac{\partial\psi(x,q_H)}{\partial x} > 0$. From this fact in combination with the results of (2.61) and (2.62) we deduce that x_1^{det} exists and $x_1^{det} \in (0, x_F^*(0))$.

The expression for x_2^{det} is determined using the expression for $\hat{q}_L(x)$ given by equation (2.18), and the first order condition (2.19):

$$x_2^{det} = \frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}+\sqrt{\left(3+\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}\right)^2-8}\right]}. \tag{2.67}$$

Let $B = \frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)} > 0$, then we can rewrite x_2^{det} as

$$x_2^{det} = \frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]}, \tag{2.68}$$

where $1-B+\sqrt{1+6B+B^2} < 4$, as $\sqrt{1+6B+B^2} < \sqrt{9+6B+B^2} = B+3$. Then the following holds

$$x_2^{det} > \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)} = x_F^*(0). \tag{2.69}$$

Since $x_1^{det} \in (0, x_F^*(0))$, (2.69) allows us to conclude that $x_2^{det} > x_1^{det}$.

Proof of Proposition 2.3 Here we fist examine the effect of an increase in q_H on x_1^{det} and then on x_2^{det}. Consider equation (2.60), which implicitly determines x_1^{det}. Applying the implicit function theorem to (2.60) we get

$$\frac{\mathrm{d}x_1^{det}}{\mathrm{d}q_H} = -\frac{\left.\frac{\partial\psi(x,q_H)}{\partial q_H}\right|_{x=x_1^{det}}}{\left.\frac{\partial\psi(x,q_H)}{\partial x}\right|_{x=x_1^{det}}}. \tag{2.70}$$

As $\frac{\partial\psi(x)}{\partial x} > 0$ from Proposition 2.2, in order to show that $\frac{\mathrm{d}x_1^{det}}{\mathrm{d}q_H} > 0$ it is sufficient to demonstrate that $\frac{\partial\psi(x,q_H)}{\partial q_H} < 0$.

$$\begin{aligned}\frac{\partial\psi(x,q_H)}{\partial q_H} &= \left(\frac{x(\beta_1-1)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1}\frac{\delta(\beta_1+1)\lambda}{(\beta_1-1)(\lambda+r-\alpha)} - \frac{x\lambda}{(r-\alpha)(\lambda+r-\alpha)}\\ &= \frac{x\lambda}{(r-\alpha)(\lambda+r-\alpha)}\left(\left(\frac{x(\beta_1-1)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1-1}-1\right).\end{aligned} \tag{2.71}$$

For $x < \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)} = x_F^*(0)$, it holds that $\left(\frac{x(\beta_1-1)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1-1} < 1$. Therefore, $\left.\frac{\partial\psi(x,q_H)}{\partial q_H}\right|_{x=x_1^{det}} < 0$ and $\frac{\mathrm{d}x_1^{det}}{\mathrm{d}q_H} > 0$.

Recall that the upper bound of the deterrence region is given by

$$x_2^{det} = \frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}+\sqrt{\left(3+\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}\right)^2-8}\right]}. \quad (2.72)$$

Differentiating with respect to q_H gives

$$\frac{\partial x_2^{det}(q_H)}{\partial q_H} = \frac{-\frac{4\delta(r-\alpha)(\beta_1+1)^2\lambda}{\lambda+r-\alpha}\left(-1+\frac{3+\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}}{\sqrt{\left(3+\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}\right)^2-8}}\right)}{\left(1-\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}+\sqrt{\left(3+\frac{(\beta_1+1)(\beta_1-1)\lambda q_H}{(\lambda+r-\alpha)}\right)^2-8}\right)^2}. \quad (2.73)$$

We can rewrite (2.73) using the notation introduced in the proof of Proposition 2.2 as follows

$$\frac{\partial x_2^{det}(q_H)}{\partial q_H} = -\frac{4\delta(r-\alpha)(\beta_1+1)^2\lambda\left(-1+\frac{3+B}{\sqrt{(3+B)^2-8}}\right)}{(\lambda+r-\alpha)\left(1-B+\sqrt{(3+B)^2-8}\right)^2}. \quad (2.74)$$

Since $\left(-1+\frac{3+B}{\sqrt{(3+B)^2-8}}\right)>0$, we can conclude that $\frac{\partial x_2^{det}(q_H)}{\partial q_H}<0$.

Proof of Proposition 2.4 In the notation introduced in the proof of Proposition 2.2 x_2^{det} is given by

$$x_2^{det} = \frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]}. \quad (2.75)$$

For simplicity denote let $D(B) = 1-B+\sqrt{1+6B+B^2}$, then the derivative of x_2^{det} with respect to λ takes the following form

$$\begin{aligned}\frac{\partial x_2^{det}(\lambda)}{\partial\lambda} &= \frac{-\frac{\partial\beta_1(\lambda)}{\partial\lambda}\frac{8\delta(r-\alpha)}{(\beta_1-1)^2}D(B)-\left(\frac{\partial D(B)}{\partial\beta_1}\frac{\partial\beta_1}{\partial\lambda}+\frac{\partial D(B)}{\partial\lambda}\right)\frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}}{D(B)^2}\\ &= -\frac{4\delta(r-\alpha)}{(\beta_1-1)D(B)^2}\left[\frac{\partial\beta_1(\lambda)}{\partial\lambda}\frac{2D(B)}{(\beta_1-1)}+\left(\frac{\partial D(B)}{\partial\beta_1}\frac{\partial\beta_1}{\partial\lambda}+\frac{\partial D(B)}{\partial\lambda}\right)(\beta_1+1)\right],\end{aligned} \quad (2.76)$$

where

$$\frac{\partial D(B)}{\partial\lambda} = \frac{(\beta_1^2-1)(r-\alpha)q_H}{(\lambda+r-\alpha)^2}\left(-1+\frac{3+B}{\sqrt{(3+B)^2-8}}\right)>0, \quad (2.77)$$

$$\frac{\partial D(B)}{\partial \beta_1} = \frac{2\beta_1 \lambda q_H}{(\lambda + r - \alpha)} \left(-1 + \frac{3+B}{\sqrt{(3+B)^2 - 8}} \right) > 0, \tag{2.78}$$

and

$$\frac{\partial \beta_1}{\partial \lambda} = \frac{1}{\sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(\lambda + r)}} > 0. \tag{2.79}$$

Note that $D(B) = 1 - B + \sqrt{1 + 6B + B^2}$ is positive given the range of B, because $D(0) = 2 > 0$ and the function $D(B)$ is monotonically increasing with B:

$$\frac{\partial D(B)}{\partial B} = \frac{B+3}{\sqrt{1+6B+B^2}} - 1 = \sqrt{\frac{9+6B+B^2}{1+6B+B^2}} - 1 > 0. \tag{2.80}$$

Thus, all the terms inside the brackets in (2.76) are positive and we conclude that $\frac{\partial x_2^{det}(\lambda)}{\partial \lambda} < 0$.

Now we want to show that x_1^{det} is decreasing with λ if $q_H = 0$. Applying the implicit function theorem to (2.60) we get

$$\frac{\mathrm{d}x_1^{det}}{\mathrm{d}\lambda} = -\frac{\left.\frac{\partial \psi(x, \beta_1(\lambda))}{\partial \lambda}\right|_{x = x_1^{det}}}{\left.\frac{\partial \psi(x, \lambda)}{\partial x}\right|_{x = x_1^{det}}}. \tag{2.81}$$

Using the fact that $\frac{\partial \psi(x,\lambda)}{\partial x} > 0$ it is sufficient to prove that $\left.\frac{\partial \psi(x, \beta_1(\lambda))}{\partial \lambda}\right|_{x = x_1^{det}} > 0$. The latter can be written as

$$\frac{\partial \psi(x, \beta_1(\lambda))}{\partial \lambda} = \frac{\partial \psi(x, \beta_1(\lambda))}{\partial \beta_1} \frac{\partial \beta_1}{\partial \lambda}, \tag{2.82}$$

where $\frac{\partial \beta_1}{\partial \lambda} > 0$, so that we only need to consider $\frac{\partial \psi(x, \beta_1(\lambda))}{\partial \beta_1}$, which is given by

$$\frac{\partial \psi(x, \beta_1(\lambda))}{\partial \lambda} = -\frac{\delta \left(\frac{(\beta_1 - 1)x}{(\beta_1+1)\delta(r-\alpha)}\right)^{\beta_1} \left((\beta_1 + 1)\log\left(\frac{(\beta_1-1)x}{(\beta_1+1)\delta(r-\alpha)}\right) + 1\right)}{\beta_1^2 - 1}. \tag{2.83}$$

Consider

$$g(\beta_1, x) = (\beta_1 + 1)\log\left(\frac{(\beta_1 - 1)x}{\delta(r-\alpha)(\beta_1+1)}\right) + 1. \tag{2.84}$$

Since $g(\beta_1, x)$ monotonically increases with x and $g(\beta_1, \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}) = 0$, for $x < \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}$ it holds that $g(\beta_1, x) < 0$ and $\frac{\partial \psi(x, \beta_1(\lambda))}{\partial \lambda} > 0$. Therefore, in order to demonstrate the negative relation between x_1^{det} and λ for $q_H = 0$ it is sufficient to show that $x_1^{det} < \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}$.

If $q_H = 0$, x_1^{det} is implicitly determined by $\psi_1(x_1^{det}, \beta_1) = 0$, where

$$\psi_1(x, \beta_1) = -\left(\frac{x(\beta_1 - 1)}{\delta(r-\alpha)(\beta_1 + 1)}\right)^{\beta_1} \frac{\delta}{(\beta_1 - 1)} + \frac{x}{r-\alpha} - \delta. \tag{2.85}$$

Plugging in $x = \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}$ we get

$$\psi_1\left(\frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}, \beta_1\right) = \frac{\delta}{(\beta_1-1)}\left(-\mathrm{e}^{-\frac{\beta_1}{\beta_1+1}} + (\beta_1+1)\mathrm{e}^{-\frac{1}{\beta_1+1}} - \beta_1 + 1\right). \tag{2.86}$$

Note that as β_1 goes to infinity it holds that

$$\lim_{\beta_1\to\infty} \psi_1\left(\frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}, \beta_1\right) = 0, \tag{2.87}$$

while the derivative with respect to β_1 is

$$\frac{\partial\psi_1\left(\frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}, \beta_1\right)}{\partial\beta_1} = -\frac{(\beta_1+3)\left(\left[\mathrm{e}^{\frac{\beta_1}{\beta_1+1}} - \mathrm{e}^{\frac{1}{\beta_1+1}}\right]\beta_1 + \mathrm{e}^{\frac{\beta_1}{\beta_1+1}}\right)}{e\left(\beta_1^2-1\right)^2} < 0. \tag{2.88}$$

Thus, $\dfrac{\partial\psi_1(\frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}, \beta_1)}{\partial\beta_1} > 0$. Moreover, which $\psi_1(x, \beta_1)$ is increasing with x, because $\dfrac{\partial\psi_1(x,\beta_1)}{\partial x} = \dfrac{1}{(r-\alpha)}\left(1 - \dfrac{\beta_1}{\beta_1+1}\left(\dfrac{x(\beta_1-1)}{\delta(r-\alpha)(\beta_1+1)}\right)^{\beta_1-1}\right) > 0$ for $x < \dfrac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}$. Hence, $x_1^{det} < \frac{\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\mathrm{e}^{-\frac{1}{\beta_1+1}}$. As a result, we can conclude that $\left.\dfrac{\partial\psi(x,\beta_1(\lambda))}{\partial\lambda}\right|_{x=x_1^{det}} > 0$ and $\dfrac{\mathrm{d}x_1^{det}}{\mathrm{d}\lambda} < 0$.

Proof of Proposition 2.5 In the stopping region the value function for the leader looks as follows

$$V_L^{acc}(x, q_L) = \frac{xq_L(1 - (q_L + q_F^*(x, q_L))}{r-\alpha} - \delta q_L. \tag{2.89}$$

Substituting the follower's optimal capacity level, $q_F^*(x, q_L) = \frac{1}{2}\left(1 - q_L - \frac{\delta(r-\alpha)}{x}\right)$, and maximizing with respect to q_L gives the following first order condition:

$$\frac{\partial V_L^{acc}(x, q_L)}{\partial q_L} = \frac{x}{2(r-\alpha)}(1 - 2q_L) - \frac{\delta}{2} = 0. \tag{2.90}$$

Thus, the capacity level of the leader can be written as

$$q_L^{acc}(x) = \frac{1}{2}\left(1 - \frac{\delta(r-\alpha)}{x}\right). \tag{2.91}$$

The next step is to substitute the resulting expression into (2.89) to obtain the value of the accommodation strategy for the leader

$$V_L^{acc^*}(x) = \frac{\left[x - \delta(r-\alpha)\right]^2}{8x(r-\alpha)}. \tag{2.92}$$

The accommodation strategy is implemented by the leading firm when the optimal capacity level of the leader, $q_L^{acc}(x)$, is such that the other positioned firm follows immediately after, that is when

$$x_F^*(q_L^{acc}(x)) \leq x. \tag{2.93}$$

Let x_1^{acc} denote the lower bound of the accommodation region, i.e. the level of the stochastic profitability shock such that

$$x_1^{acc} = x_F^*(q_L^{acc}(x_1^{acc})). \tag{2.94}$$

Given equation (2.91), the optimal trigger is the solution of

$$x = \frac{2\delta(r-\alpha)(\beta_1+1)x}{(\beta_1-1)[x+\delta(r-\alpha)]}. \tag{2.95}$$

Solving (2.94) we get

$$x_1^{acc} = \frac{(\beta_1+3)}{(\beta_1-1)}\delta(r-\alpha). \tag{2.96}$$

Proof of Proposition 2.6 Using the notation introduced in the proof of Proposition 2.2 we can write x_2^{det} as

$$x_2^{det} = \frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]}. \tag{2.97}$$

We can write the difference between the upper bound of the deterrence region and the lower bound of the accommodation region as

$$\begin{aligned} x_2^{det} - x_1^{acc} &= \frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]} - \frac{\delta(r-\alpha)(\beta_1+3)}{(\beta_1-1)} \\ &= \frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)}\left[\frac{4\beta_1+4}{\left[1-B+\sqrt{1+6B+B^2}\right]} - (\beta_1+3)\right] \end{aligned}$$

$$= \frac{4\delta(r-\alpha)(\beta_1+1)\left[(\beta_1+3)\left(3+B-\sqrt{1+6B+B^2}\right)-8\right]}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]}. \quad (2.98)$$

From the derivations in Proposition 2.4 it follows that $\frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]} > 0$. Thus, $x_2^{det} - x_1^{acc} < 0$ when

$$(\beta_1+3)\left(3+B-\sqrt{1+6B+B^2}\right)-8 < 0. \quad (2.99)$$

Using the property that $\beta_1 > 1$ we can rewrite the inequality as follows

$$3+B-\frac{8}{\beta_1+3} < \sqrt{1+6B+B^2}. \quad (2.100)$$

The expressions on both sides of the last inequality are positive as for $\beta_1 > 1$ it holds that $\frac{8}{\beta_1+3} < 2$. Therefore, for x_2^{det} to be smaller than x_1^{acc} it is sufficient to prove that

$$\left(3+B-\frac{8}{\beta_1+3}\right)^2 < 1+6B+B^2, \quad (2.101)$$

$$9+6B+B^2-(3+B)\frac{16}{\beta_1+3}+\frac{64}{(\beta_1+3)^2} < 1+6B+B^2, \quad (2.102)$$

$$8-\frac{48}{\beta_1+3}+\frac{64}{(\beta_1+3)^2} < B\frac{16}{\beta_1+3}, \quad (2.103)$$

$$B > \frac{\beta_1^2-1}{2(\beta_1+3)}. \quad (2.104)$$

Substituting the expression for B we get

$$\frac{(\beta_1^2-1)\lambda q_H}{(\lambda+r-\alpha)} > \frac{\beta_1^2-1}{2(\beta_1+3)}, \quad (2.105)$$

$$\frac{\lambda q_H}{(\lambda+r-\alpha)} > \frac{1}{2(\beta_1+3)}. \quad (2.106)$$

Given the restrictions on β_1, one can see that $\frac{1}{2(\beta_1+3)} < \frac{1}{8}$. Therefore, as long as $\frac{\lambda q_H}{(\lambda+r-\alpha)} > \frac{1}{8}$ (or equivalently $\lambda(8q_H-1) > r-\alpha$), inequality (2.106) always holds, implying that $x_2^{det} < x_1^{acc}$.

Proof of Proposition 2.7 Consider first the game, when the hidden competitor takes the leader's position. In this case the firms invest immediately and with probability 0.5 one of them becomes the follower. Denote the capacity level by q_S and by S the expected value of the positioned firm, which is given by

$$S(x, q_S) = \frac{1}{2}\left(\frac{xq_S(1-(q_H+q_S)}{r-\alpha} - \delta q_S\right). \tag{2.107}$$

The capacity level that maximizes the this value function is

$$q_S^*(x, q_H) = \frac{1}{2}\left(1 - q_H - \frac{\delta(r-\alpha)}{x}\right). \tag{2.108}$$

Plugging back the optimal capacity level into the value function yields

$$S^*(x, q_H) = \frac{[x(1-q_H) - \delta(r-\alpha)]^2}{8x(r-\alpha)}. \tag{2.109}$$

As argued in Section 2.5 the investment yields zero value in expectation if x is below the Marshallian (or zero-NPV) investment trigger, which we denote by $x_M^*(q_H)$. It can be found by setting $S^*(x)$ to zero, which gives

$$x_M^*(q_H) = \frac{\delta(r-\alpha)}{(1-q_H)}. \tag{2.110}$$

As a result, the value of waiting for the hidden entry satisfies the following system

$$\begin{cases} \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W(x)}{\partial x^2} + \alpha x \frac{\partial W(x)}{\partial x} - rW(x) + \lambda[0 - W(x)] = 0 & \text{if } x < x_M^*(q_H), \\ \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W(x)}{\partial x^2} + \alpha x \frac{\partial W(x)}{\partial x} - rW(x) + \lambda[S^*(x) - W(x)] = 0 & \text{if } x \geq x_M^*(q_H). \end{cases} \tag{2.111}$$

The solution of this system is the following function

$$W(x) = \begin{cases} Bx^{\beta_1} & \text{if } x < x_M^*(q_H), \\ Cx^{\beta_2} - \frac{\delta\lambda(1-q_H)}{4(r+\lambda)} + \frac{\lambda x(1-q_H)^2}{8(r-\alpha)(r-\alpha+\lambda)} - \frac{\delta^2\lambda(r-\alpha)}{8x(\sigma^2-r-\alpha-\lambda)} & \text{if } x \geq x_M^*(q_H), \end{cases} \tag{2.112}$$

where the constants B and C can be found by applying smooth pasting and value matching conditions at $x = x^*(q_H)$.

The resulting waiting curve is defined as follows

$$W(x) = \begin{cases} \left(\frac{x}{x_M^*(q_H)}\right)^{\beta_1} \frac{\lambda(1-q_H)\delta\beta_2}{4\left(\beta_1^2-1\right)(\beta_2-\beta_1)(\lambda+r)} & \text{if } x < x_M^*(q_H), \\ \left(\frac{x}{x_M^*(q_H)}\right)^{\beta_2} \frac{\lambda(1-q_H)\delta\beta_1}{4(r+\lambda)\left(1-\beta_2^2\right)(\beta_1-\beta_2)} - \frac{\delta\lambda(1-q_H)}{4(r+\lambda)} & \text{if } x \geq x_M^*(q_H). \\ \quad + \frac{\lambda x(1-q_H)^2}{8(r-\alpha)(r-\alpha+\lambda)} - \frac{\delta^2\lambda(r-\alpha)}{8x(\sigma^2-r-\alpha-\lambda)} & \end{cases} \tag{2.113}$$

Proof of Proposition 2.8 On the one hand, note that for small values of x the waiting curve is always above the leader value. This is because for $x \leq x_1^{det}$ it holds that $V_L^*(x) = 0$, while for $x < x_M^*(q_H)$, $W(x) = \left(\frac{x}{x_M^*(q_H)}\right)^{\beta_1} \frac{\lambda(1-q_H)\delta\beta_2}{4\left(\beta_1^2-1\right)(\beta_2-\beta_1)(\lambda+r)} > 0$. Thus, for $0 < x < \min\{x_1^{det}, x_M^*(q_H)\}$, $W(x) > V_L^*(x)$.

On the other hand, consider the leader value and the waiting curve as x goes to infinity. As follows from the previous analysis, for x being large enough the leader implements the accommodation strategy and $V_L^*(x) = \dfrac{[x-\delta(r-\alpha)]^2}{8x(r-\alpha)}$, while $W(x) = \left(\frac{x}{x_M^*(q_H)}\right)^{\beta_2} \frac{\lambda(1-q_H)\delta\beta_1}{4(r+\lambda)\left(1-\beta_2^2\right)(\beta_1-\beta_2)} - \frac{\delta\lambda(1-q_H)}{4(r+\lambda)} + \frac{\lambda x(1-q_H)^2}{8(r-\alpha)(r-\alpha+\lambda)} - \frac{\delta^2\lambda(r-\alpha)}{8x(\sigma^2-r-\alpha-\lambda)}$.
Hence, for $x > \max\{x_1^{acc}, x_M^*(q_H)\}$ the difference $V_L^*(x) - W(x)$ is given by

$$V_L^*(x) - W(x) = \frac{1}{8}\left(\frac{x\left(\lambda q_H\left(2-q_H\right)+r-\alpha\right)}{(r-\alpha)(r-\alpha+\lambda)} - \frac{2\delta(\lambda q_H + r)}{\lambda + r} + \frac{\delta^2\left(r-\alpha+\frac{\lambda}{\sigma^2-\alpha-\lambda-r}\right)}{x}\right) - \left(\frac{(1-q_H)x}{\delta(r-\alpha)}\right)^{\beta_2} \frac{\beta_1\delta\lambda(1-q_H)}{4\left(1-\beta_1^2\right)(\beta_1-\beta_2)(\lambda+r)}. \tag{2.114}$$

Since $\beta_2 < 0$ and $\frac{(\lambda q_H(2-q_H)+r-\alpha)}{(r-\alpha)(r-\alpha+\lambda)} > 0$, we can conclude that

$$\lim_{x\to\infty}\{V_L^*(x) - W(x)\} = \infty. \tag{2.115}$$

Therefore, it holds that $V_L^*(x) \geq W(x)$ for $x \to \infty$. The result that $W(x) > V_L^*(x)$ for small x, while $V_L^*(x) \geq W(x)$ for large x, allows to establish that there always exists at least one intersection between $V_L^*(x)$ and $W(x)$.

Proof of Proposition 2.9 Recall that the value of the first investor implementing accommodation and the corresponding capacity level are given by

$$V_L^{acc^*}(x) = \frac{(x-\delta(r-\alpha))^2}{8x(r-\alpha)}, \tag{2.116}$$

and

$$q_L^{acc}(x) = \frac{1}{2}\left(1 - \frac{\delta(r-\alpha)}{x}\right). \tag{2.117}$$

The value of the stochastic process that triggers investment of the follower can be found by substituting $q_L^{acc}(x)$ into the expression for $x_F(q_L)$:

$$x = \frac{2(\beta_1+1)\delta x(r-\alpha)}{(\beta_1-1)(\delta(r-\alpha)+x)}. \tag{2.118}$$

Solving for x we get

$$x_F^{acc} = \frac{(\beta_1+3)}{(\beta_1-1)}\delta(r-\alpha). \tag{2.119}$$

Note that $x_F^{acc} = x_1^{acc}$. This means that it is never optimal for the follower to wait in the accommodation region. Thus, we consider only the value of stopping, which is equal to

$$V_F^*(x, q_L^{acc}(x)) = \frac{(x - \delta(r - \alpha))^2}{16x(r - \alpha)}. \tag{2.120}$$

Hence, for $x \geq x_1^{acc}$

$$V_L^{acc^*}(x) = \frac{(x - \delta(r - \alpha))^2}{8x(r - \alpha)} > \frac{(x - \delta(r - \alpha))^2}{16x(r - \alpha)} = V_F^*(x, q_L^{acc}(x)). \tag{2.121}$$

Proof of Proposition 2.10 The follower value in the stopping region can be written as

$$V_F(x, q_L, q_F^*(x, q_L)) = q_F^*(x, q_L)\left(\frac{x(1 - (q_L(x) + q_F^*(x, q_L)))}{r - \alpha} - \delta\right). \tag{2.122}$$

Plugging in the boundary capacity level $\hat{q}_L(x)$ we obtain:

$$V_F^*(x, \hat{q}_L(x)) = \frac{\delta^2(r - \alpha)}{x(\beta_1 - 1)^2}, \tag{2.123}$$

so that the value of the follower is clearly decreasing with x.

Proof of Proposition 2.11 The threshold $\hat{x}_p$ is determined by the intersection of the leader's and follower's curves:

$$\widehat{V}_L(x, \hat{q}_L(x)) = V_F^*(x, \hat{q}_L(x)), \tag{2.124}$$

where

$$\hat{q}_L(x) = 1 - \frac{\delta(r - \alpha)(\beta_1 + 1)}{(\beta_1 - 1)x}. \tag{2.125}$$

For the capacity level $\hat{q}_L(x)$ the follower gets exactly the same value in both stopping and continuation regions which is equal to

$$V_F^*(x, \hat{q}_L(x)) = \frac{\delta^2(r - \alpha)}{x(\beta_1 - 1)^2}, \tag{2.126}$$

while the leader's value becomes

$$\widehat{V}_L(x) = \frac{\delta}{\beta_1 - 1}\left(1 - \frac{\delta(r - \alpha)(\beta_1 + 1)}{(\beta_1 - 1)x}\right). \tag{2.127}$$

Solving $\widehat{V}_L(x) = V_F^*(x, \hat{q}_L(x))$ for x we get

$$\hat{x}_p = \frac{\delta(r - \alpha)(\beta_1 + 2)}{(\beta_1 - 1)}. \tag{2.128}$$

First, note that the threshold $\hat{x}_p$ is only relevant if it lies in the feasible region of the boundary strategy. In particular, it should hold that $x_2^{det} < \hat{x}_p < x_1^{acc}$. It is easy to see that $\hat{x}_p < x_1^{acc}$, as $\hat{x}_p = \frac{\delta(r-\alpha)(\beta_1+2)}{(\beta_1-1)} < \frac{\delta(r-\alpha)(\beta_1+3)}{(\beta_1-1)} = x_1^{acc}$. Yet the relation between x_2^{det} and $\hat{x}_p$ depends on the hidden competition parameters λ and q_H.

Using the notation introduced in the proof of Proposition 2.2 we can write

$$\begin{aligned} x_2^{det} - \hat{x}_p &= \frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]} - \frac{\delta(r-\alpha)(\beta_1+2)}{(\beta_1-1)} \\ &= \frac{\delta(r-\alpha)}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]}\left[(\beta_1+2)\left(3+B-\sqrt{1+6B+B^2}\right)-4\right]. \end{aligned} \tag{2.129}$$

Due to the fact that $\frac{4\delta(r-\alpha)(\beta_1+1)}{(\beta_1-1)\left[1-B+\sqrt{1+6B+B^2}\right]}$ is positive, $x_2^{det} - \hat{x}_p < 0$ when

$$(\beta_1+2)\left(3+B-\sqrt{1+6B+B^2}\right)-4<0. \tag{2.130}$$

Using the property that $\beta_1 > 1$ we can rewrite the inequality as

$$3+B-\frac{4}{\beta_1+2} < \sqrt{1+6B+B^2}. \tag{2.131}$$

The expressions on both sides of this inequality are positive as for $\beta_1 > 1$ it holds that $\frac{4}{\beta_1+2} < \frac{4}{3}$. Therefore, for x_2^{det} to be smaller than $\hat{x}_p$ it is enough to prove that

$$\left(3+B-\frac{4}{\beta_1+2}\right)^2 < 1+6B+B^2, \tag{2.132}$$

$$9+6B+B^2-(3+B)\frac{8}{\beta_1+2}+\frac{16}{(\beta_1+2)^2} < 1+6B+B^2, \tag{2.133}$$

$$8-\frac{24}{\beta_1+2}+\frac{16}{(\beta_1+2)^2} < B\frac{8}{\beta_1+2}, \tag{2.134}$$

$$B > \frac{\beta_1(\beta_1+1)}{(\beta_1+2)}. \tag{2.135}$$

Substituting the expression for B we get

$$\frac{(\beta_1^2-1)\lambda q_H}{(\lambda+r-\alpha)} > \frac{\beta_1(\beta_1+1)}{(\beta_1+2)}, \tag{2.136}$$

$$q_H > \frac{(\lambda + r - \alpha)\beta_1}{\lambda(\beta_1 - 1)(\beta_1 + 2)}. \tag{2.137}$$

This implies that for $x_2^{det} < \hat{x}_p$. Thus, there exists unique q_H, which we denote by $\tilde{q}_H$, such that for $q_H > \tilde{q}_H$, it holds that $x_2^{det} < \hat{x}_p$ and the preemption trigger lies in the boundary region.

Proof of Proposition 2.12 The capacity level $\tilde{q}_H$ is determined by

$$\tilde{q}_H(\lambda) = \frac{(\lambda + r - \alpha)\beta_1}{\lambda(\beta_1 - 1)(\beta_1 + 2)}. \tag{2.138}$$

Taking the derivative with respect to λ and taking into account that $\frac{\partial \beta_1}{\partial \lambda} > 0$ we get

$$\frac{\partial \tilde{q}_H(\lambda)}{\partial \lambda} = -\frac{1}{\lambda\left(\beta_1 - 1\right)\left(\beta_1 + 2\right)} \left(\frac{(\lambda + r - \alpha)\left(\beta_1^2 + 2\right)}{\left(\beta_1 - 1\right)\left(\beta_1 + 2\right)} \frac{\partial \beta_1}{\partial \lambda} + \frac{\beta_1 (r - \alpha)}{\lambda} \right) < 0. \tag{2.139}$$

Therefore, we conclude that $\tilde{q}_H(\lambda)$ decreases with λ.

Proof of Proposition 2.13 From the proof of Proposition 2.11 $\hat{x}_p$ is given by

$$\hat{x}_p = \frac{\delta(r - \alpha)(\beta_1 + 2)}{(\beta_1 - 1)}. \tag{2.140}$$

Substituting the above expression into (2.125) we get

$$\hat{q}_L(\hat{x}_p) = \frac{1}{(\beta_1 + 1)}. \tag{2.141}$$

Proof of Proposition 2.14 The leader implements the deterrence strategy and its capacity choice is such that the follower delays its investment. Therefore, the preemption trigger is defined as the first intersection of the leader value and the follower value of waiting. At this point one of the firms enters the market as a leader, whereas its rival waits till the follower's optimal investment moment. The leader value and its capacity under the deterrence strategy are derived in Section 2.4.1 while the follower value of waiting as well as its optimal timing and optimal capacity choice are given in Proposition 1. Due to the complexity of these functions the explicit solution for the preemption trigger cannot be obtained.

3 Capacity Choice in a Duopoly with Endogenous Exit[1]

Applying the real options framework, this chapter investigates the investment decision of an entrant given that an incumbent is already active. Both firms have an option to exit this market if the demand level falls too low. The combination of three decision components, capacity choice, entry and exit timing, results into multiple trigger strategies for the entrant. In particular, in the presence of a large incumbent, it can either choose to coexist with its rival in a duopoly or (eventually) monopolize the market by installing a sufficiently large capacity. The former scenario is realized when the market is large, while the latter occurs when the market is small. When the market is of intermediate size, a hysteresis region emerges where the entrant does not take any actions and prefers to postpone investment.

3.1 Introduction

Traditional real options models address the question of investment timing in uncertain markets applying dynamic programming techniques. Most of these models associate stopping with the decision to enter the new market by undertaking an irreversible investment. The common assumption in such models is that firms can temporarily suspend their operations in the case of negative profit flow and later resume it at no cost if the market profitability increases. This means that after investment firms stay in the market, irrespectively of the realized demand patterns. In reality, resumption of the firms' operations is rarely costless and sometimes even impossible.[2] As a result, a negative demand shock may trigger their decision to exit the market forever. Irreversibility of exit decisions in uncertain markets allows to treat them as real options.

Exit options have received limited attention in the literature. Ghemawat and Nale-

[1]This chapter is based on Lavrutich (2015).

[2]For example due to the loss of the team of professionals.

buff (1985) and Fudenberg and Tirole (1986b) analyze the exit game in a duopoly with asymmetric firms in a deterministic setting. The early literature on stochastic monopoly models includes Mossin (1968), that focuses on combined entry and exit strategies, later generalized by Dixit (1989), and Alvarez (1998, 1999) that study the optimal exit strategy of a firm operating with a fixed capacity. Another contribution, which focuses on both entry and exit decisions is Kwon (2010). He finds that in a monopoly with declining demand the investment threshold deceases in volatility due to the presence of the exit option. Adkins and Paxson (2016) considers a three-factor stochastic real options model to investigate the decision to invest and/or abandon a single project. They conclude that the option to abandon the project in the operational stage causes the investment threshold to decrease, as it makes the investment opportunity more attractive. Hagspiel *et al.* (2016) analyzes the effect of flexibility on exit and entry decisions of a monopolist. If a firm is assumed to be able to adjust its production levels, it invests in a smaller capacity level than an inflexible firm. Moreover, the difference between the capacity choices of a flexible and an inflexible firm is larger in a highly uncertain environment.

A continuous time duopoly setting with the option to exit was investigated among others by Lambrecht (2001), who presents a model of strategic interactions of firms that have both entry and exit options. He explicitly derives entry and exit thresholds, and investigates how the exit order is influenced by different economic factors. He shows that, consistent with earlier findings, the firm that has a lower monopoly exit threshold leaves the market last. Additionally, he modifies the model by assuming that financially distressed firms can decrease their debt through debt exchange offers. As a result, a reversed bankruptcy order of the firms may appear. Murto (2004) examines exit decisions under uncertainty in a duopoly game with asymmetric firms in a declining market. He shows that when market uncertainty is sufficiently low, there is a unique equilibrium where the larger firm exits the market earlier. However, in a highly uncertain environment there exists an empty span between the exit regions of the firms, i.e. the two exit regions have an empty intersection. Within this span neither of them leaves the market and a reversed exit order may appear. As a result, the equilibrium is no longer unique and it is not clear which firm is first to exit the market. Ruiz-Aliseda (2006) studies an entry/exit game in a duopoly market that first expands until some random moment in time and then starts declining. He finds that the monopolist does not exit as long as the market grows. After the market matures and starts declining no firm enters the market anymore and the incumbent ultimately leaves. In case both players are active when the market reaches maturity the firm with higher sunk costs exits first. Bayer (2007) presents a model where firms consider an option to increase their capacity in order to stimulate sooner exit of the opponent. In this model the firm with the larger capacity exits last. This is due to the assumption that the production costs are fixed and do not depend on the capacity level. In such a setting predatory behavior occurs in a more competitive and a less uncertain market.

Similar to Lambrecht (2001) we investigate the combination of firms' exit and entry decisions in a duopoly. The main difference, however, is that in our model capacity choice is considered. Namely, in order to become active in the market, firms can freely choose the scale of their investment. This in turn affects not only their investment decision, but also the exit order. We adopt the approach for capacity optimization used in the monopoly model of Dangl (1999), and later extended by Huisman and Kort (2015) to a duopoly scenario. In the setting where firms are able to choose both timing and size of their investment, Huisman and Kort (2015) shows that the firm with a larger capacity invests at a higher investment threshold. Moreover, they demonstrate that the market leader overinvests in capacity in order to ensure that its rival enters the market later and installs a smaller capacity.

Here we extend Huisman and Kort (2015) by incorporating the exit option into the model. This triggers a second mover advantage for the firm that enters the market last, the entrant, as it can influence the exit game. This chapter focuses on the analysis of the investment strategies of the entrant given that the first investor is already active in the market with a certain capacity.

As in Lambrecht (2001) and Murto (2004), we demonstrate that the firm with the larger capacity level exits the market first. As a result, in the presence of a sufficiently large incumbent the entrant has an incentive to drive the incumbent out of the market by installing a relatively large capacity with size still below the incumbent's capacity level. This may result in a non-monotonicity in the entrant's expected entry time with respect to the size of the incumbent. In particular, the entrant's expected entry time first increases as a result of the decrease in the output price similar to Huisman and Kort (2015), yet then starts declining as the entrant anticipates sooner exit of the incumbent.

In addition, we show that the introduction of an exit option leads to a multiple trigger strategy of the entrant. This result is associated with the existence of a so called region of hysteresis. This region corresponds to a gap between the investment regions of the entrant. In particular, if the market is large enough, or, in other words, exit by one of the firms is unlikely to occur soon, the entrant chooses to coexist with its opponent in a duopoly. In a small market, however, given that it is already optimal for the monopolist to enter, the entrant has an incentive to monopolize the market by driving its rival out. For the case of an intermediate market size, it is optimal to wait until either of the scenarios is profitable. As a result, the entrant does not undertake any actions and prefers to postpone investment.

The hysteresis region can be related to the inaction region of Decamps *et al.*

(2006). The latter extends Dixit (1993a) and studies a single firm's decision to invest in alternative projects with an uncertain cash flow. For each project there is a certain region of the output price that triggers the firm's investment. The main similarity with our finding is associated with the fact that the optimal investment intervals of two projects do not intersect, creating the inaction region. In the inaction region the firm does not invest yet and it is unknown in which of the two projects it will eventually invest.

The chapter is organized as follows. Section 2 is devoted to the analysis of the investment decisions of the monopolist that has an option to exit the market. Section 3 discusses the exit order of the firms in a duopoly context and specifies the solution for the entrant's entry-exit problem. Section 4 summarizes the main results and concludes the chapter. The proofs of the propositions are presented in the Appendix.

3.2 Monopoly

Consider the investment problem of a monopolist, that faces a possibility to undertake an irreversible investment in a plant with a certain capacity. Once the investment is made the firm becomes active on the market and launches the production process. The market for the final output is characterized by uncertain demand, specified by a multiplicative inverse demand function:

$$P_t = X_t(1 - \eta Q_t), \tag{3.1}$$

with $\eta > 0$, Q_t total market output, and X_t a stochastic shock, which follows a geometric Brownian motion:

$$dX_t = \alpha X_t dt + \sigma X_t dZ_t, \tag{3.2}$$

where α and σ are the drift and volatility parameters, respectively, and Z_t is a Wiener process. The firm is assumed to be risk neutral with a discount rate r. Moreover, it should hold that $r > \alpha$, otherwise the discounted value of the future revenue stream is infinite and the firm always prefers to delay investment.

We assume that the firm that becomes active on the market always produces up to capacity and henceforth we will refer to Q as the capacity level.[3]. The investment costs the firm bears are proportional to the capacity and are given by δQ, where $\delta > 0$ is the unit investment cost. Apart from the investment costs that are incurred only at the moment of investment, the fixed production costs proportional to capacity, cQ, are paid by the firm in each period with $c > 0$. In practice the firm can produce below capacity. However, as argued in Goyal and Netessine (2007), firms often have

[3]The capacity level must be such that $0 \leq Q \leq \frac{1}{\eta}$, otherwise the output price is negative.

incentive to produce up to capacity facing large fixed costs due to production ramp-ups or engagements with certain suppliers. Additionally, fixed costs may also trigger firm's decision to exit when demand is declining. Here we can also think of the costs for labor, regular maintenance of machinery or rent for production spaces, laboratories, etc. The implications of relaxing the capacity clearance assumption are examined in detail in Hagspiel *et al.* (2016).

Once invested, the firm faces the possibility to exit the market at no cost when the demand level is too low. The presence of exit costs does not significantly influence the problem and for simplicity they are normalized to zero. The exit decision is assumed to be irreversible, i.e. production cannot be resumed once being shut down. Hence, the problem of the potential market entrant consists of the optimal choice of investment timing, capacity level, and exit timing.

In the presented setting the firm holds an option to exit when it is active, while if the firm has not entered yet, it holds an investment option. For the idle firm there exists an optimal investment trigger, which we denote by $X_M^I(Q)$, such that once it is reached by the stochastic process, x, the firm is indifferent between investing capacity Q and waiting. Thus, for $x \geq X_M^I(Q)$ the monopolist enters the market, forgoing its investment option, V_0^M, for the operating project value, V_1^M, and pays sunk investment costs δQ. After the firm has entered the market, it possesses the option to abandon the project, i.e. exit the market. The optimal level of x to exercise such an option is denoted by $X_M^E(Q)$.

First, consider the situation where the capacity level of the firm is given. The value of the firm and the optimal thresholds in this case are summarized by the following proposition.

Proposition 3.1 *The value of the idle and active monopolist for a given level of the stochastic process, x, and capacity, Q, and are given by* (3.3) *and* (3.4)*, respectively:*

$$V_0^M(x,Q) = \frac{\beta_2}{\beta_2-\beta_1}\left(\frac{x}{X_M^I(Q)}\right)^{\beta_1}\left(\frac{X_M^I(Q)(1-\eta Q)Q}{r-\alpha}\left(1-\frac{1}{\beta_2}\right)-\left(\frac{c}{r}+\delta\right)Q\right), \tag{3.3}$$

$$V_1^M(x,Q) = \left(\frac{x}{X_M^E(Q)}\right)^{\beta_2}\frac{cQ}{r(1-\beta_2)}+\frac{x(1-\eta Q)Q}{r-\alpha}-\frac{cQ}{r}, \tag{3.4}$$

where the optimal exit threshold $X_M^E(Q)$ for a given capacity choice Q is given by

$$X_M^E(Q) = \frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-\eta Q)}, \tag{3.5}$$

and the optimal investment threshold $X_M^I(Q)$ is the solution of

$$\frac{(\beta_1-\beta_2)c}{(1-\beta_2)\beta_1 r}\left(\frac{x}{X_M^E(Q)}\right)^{\beta_2}+\left(1-\frac{1}{\beta_1}\right)\frac{x(1-\eta Q)}{r-\alpha}-\frac{c}{r}-\delta=0, \tag{3.6}$$

and β_1, β_2 are given by

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1, \tag{3.7}$$

$$\beta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0. \tag{3.8}$$

Intuitively, the monopolist chooses its capacity level Q such that the investment yields the highest possible value until it is optimal to exit, i.e. as long as $x > X_M^E(Q)$. Using (3.5) we can express the latter condition in terms of Q, namely as $Q < \widetilde{Q}_M(x)$, defined by

$$\widetilde{Q}_M(x) = \frac{1}{\eta}\left(1 - \frac{\beta_2(r-\alpha)c}{r(\beta_2 - 1)x}\right). \tag{3.9}$$

This means that if the capacity of the firm is too large, $Q \geq \widetilde{Q}_M(x)$, the firm is not able to bear the current production costs for a given market profitability and it does not expect to compensate for this in the future. Hence, will exit the market immediately. Hence, the optimal capacity level of the monopolist is found maximizing the value of an operating project $V_1(x, Q)$ with respect to Q such that $Q < \widetilde{Q}_M(x)$. This results in the following proposition.

Proposition 3.2 *The optimal capacity level of the monopolist, $Q_M^*(x)$, for a given level of x is implicitly determined by*

$$\left(\frac{x}{X_M^E(Q)}\right)^{\beta_2} \frac{c\,(1-\eta Q(1+\beta_2))}{r(1-\beta_2)(1-\eta Q)} + \frac{x(1-2\eta Q)}{r-\alpha} - \frac{c}{r} - \delta = 0. \tag{3.10}$$

The optimal investment trigger, $X_M^{I^*}$ satisfies

$$-\left(\frac{r\beta_1(\beta_2-1)X_M^{I^*}}{(\beta_1+1)\beta_2(r-\alpha)c}\right)^{\beta_2} \frac{(\beta_1-\beta_2)c}{\beta_1(\beta_2-1)r} + \frac{(\beta_1-1)X_M^{I^*}}{(\beta_1+1)(r-\alpha)} - \left(\frac{c}{r} + \delta\right) = 0, \tag{3.11}$$

and the corresponding capacity level is equal to

$$Q_M^* \equiv Q_M^*(X_M^{I^*}) = \frac{1}{\eta(\beta_1+1)}. \tag{3.12}$$

The optimal exit trigger $X_M^{E^*}$ is given by

$$X_M^{E^*} = \frac{\beta_2(\beta_1+1)(r-\alpha)c}{\beta_1(\beta_2-1)r}. \tag{3.13}$$

It follows from the above proposition that the production costs are crucial for the decision of the firm to exit the market. Intuitively, the larger the production costs are, the larger losses the firm faces when the demand level becomes low. This causes the exit threshold of the monopolist to increase with c. Note, however, that the optimal capacity level does not depend on the production costs incurred by the firm, while the optimal investment threshold increases with c. This is due to the particular choice of the demand function and the assumption that the production costs the firm bears in each period are fixed. Therefore, given that there are no strategic effects involved in the firm's decision, the firm will respond to an increase in the production costs in the same way as to an increase in the investment costs, namely, by postponing its investment decision, while keeping the capacity choice unaffected.

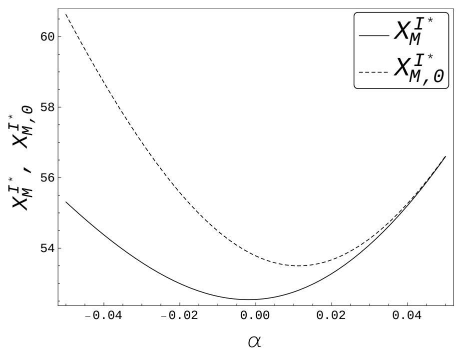
Proposition 3.3 *The optimal exit threshold $X_M^{E^*}$, as well as the optimal capacity level Q_M^* decreases with both σ and α, and increases with r.*

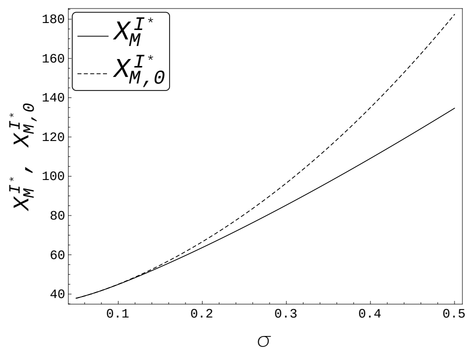
As follows from the above proposition, in a more uncertain environment the firm decreases its exit threshold. This finding is consistent with the traditional result in the real options literature. A larger drift, α, means that the market is showing stronger market growth prospects, which triggers a later exit decision. A larger r implies that the future payoffs are discounted more heavily. As a result, the discounted cash flow steam from the project is lower and the firm prefers to leave the market for higher x.

Proposition 3.4 *The optimal investment threshold, $X_M^{I^*}$, is lower than this threshold in the model without an exit option, $X_{M,0}^{I^*}$. Moreover, $X_{M,0}^{I^*}$ increases with σ and exhibits a U-shaped non-monotonic behavior with respect to α and r.*

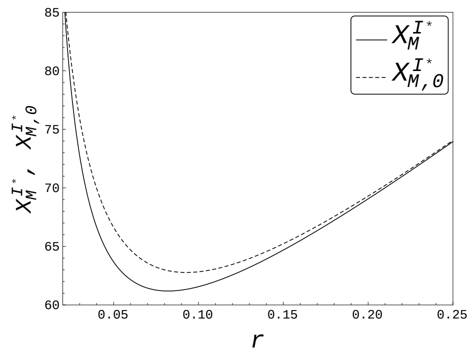
As stated in Proposition 3.4, the firm that has an option to exit invests earlier than the firm that stays in the market forever upon investment. This is because the existence of the exit option increases the attractiveness of the investment opportunity, as the firm can avoid staying longer in a market with negative cash flows.

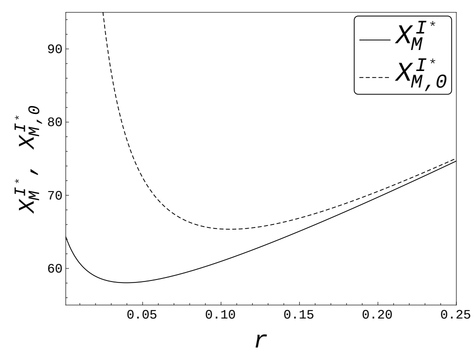
Due to the complexity of the expression for $X_M^{I^*}$, the effects of the different parameter values cannot be easily obtained. However, numerical experiments show that when the parameters of the model are changed, $X_M^{I^*}$ exhibits a similar behavior to $X_{M,0}^{I^*}$, the threshold in the model without the exit decision. Because of this we consider the analytical results for the latter and provide a numerical comparison between $X_{M,0}^{I^*}$ and $X_M^{I^*}$. Figure 3.1 illustrates the sensitivity of the entry threshold with respect to the different parameter values for both models with and without exit. In the standard real options model where the capacity level is fixed, the entry threshold decreases with α and increases with both r and σ. As it can be seen in Figure 3.1, in our model the effect of uncertainty, σ, remains the same, meaning that the firms will postpone the investment in a more uncertain environment. On the other hand, we find a non-monotonic U-shaped relation between the entry thresholds and both r and α. In the case of the discount rate, r, this is due to the presence of the production costs that are incurred at each moment in time. As r increases the firm discounts not only its future profits more heavily, but also its costs. The former effect results in an increase in the investment threshold, because the firm values future payoffs less, which makes the project less attractive. The later effect leads to a decrease in the threshold, as the project is becoming relatively less costly. This results in a U-shaped type of function as illustrated in Figures 3.1c and 3.1d. A similar shape of the entry threshold is observed when sensitivity with respect α is considered. The reason is now, however, that we add capacity optimization into the model. As proved in the Appendix (see proof of Proposition 3.3) in the model with fixed capacity there is only a decreasing effect to be observed. This can be explained by the fact that the investment opportunity becomes more attractive, given the better market growth prospects. If the firm is allowed to choose the capacity level it will choose a larger quantity for larger α to anticipate on future growth, which in turn gives an incentive to delay its investment decision.





(a) $\sigma = 0.15, r = 0.1$. *(b)* $r = 0.05, \alpha = 0.02$.





(c) $\sigma = 0.2, \alpha = 0.02$. *(d)* $\sigma = 0.2, \alpha = -0.02$.

Figure 3.1: The optimal investment thresholds $X_M^{I^*}$ and $X_{M,0}^{I^*}$ for the set of parameter values: $\eta = 1$, $c = 20$, $\delta = 100$, and different values of r, α, and σ.

As it can also be seen from Figure 3.1 the difference between the exit threshold in the model with the exit option, $X_M^{I^*}$, and in the model without the exit option, $X_{M,0}^{I^*}$, is the largest for large values of σ, small values of α, intermediate values of r in the case of positive α and small values of r in the case of negative α. Intuitively, in the more uncertain environment, i.e. when σ is large, the firms value the option to exit the market more. Similarly, the smaller the drift in the geometric Brownian motion, α, and, as a result, the worse are the growth prospects, the bigger role plays the exit decision. The effect of the discount rate, however, differs for positive and negative α, as illustrated in Figures 3.1c and 3.1d. Note that in both figures the investment thresholds $X_M^{I^*}$ and $X_{M,0}^{I^*}$ approach each other as r increases. This is because if the discount rate is large the firm just cares about the immediate cash flows, while the exit decision is relatively far in the future. Thus, the presence of the option to exit does not influence the firm's investment decision a lot. For positive α the investment thresholds are very close to each other also for small r, given that the condition $r > \alpha$ is satisfied. In this scenario, if r is approaching α from above, there exists a dominating revenue term, i.e. the revenues from the project become v large and exit is rather unlikely. Though, if α is negative, while r is close to zero, the difference between $X_M^{I^*}$ and $X_{M,0}^{I^*}$ becomes large again. In this case exit is more important given negative growth prospects and low discount rate.

3.3 Duopoly

In the duopoly the incumbent is the firm that made the first move in the investment game. Its capacity level is denoted by Q_L. When demand is high enough, the second firm, the entrant, also becomes active on the market. It decides upon its capacity level, Q_F, which maximizes its value function given the level acquired by the incumbent. As mentioned earlier, in what follows we concentrate on the entrant's investment and exit decision for a given capacity choice of the incumbent, which is already operating in the market. The problem is solved backwards starting with the exit game for a given capacity choice. After determining the optimal exit timing and specifying the value of the active firm, we consider the capacity optimization problem. Lastly, the entry problem is discussed.

3.3.1 Exit decision

Once both firms have undertaken their investment the exit game starts. It continues until demand becomes low enough to trigger the exit of one of the firms. Several papers, e.g. Lambrecht (2001), demonstrate that the firm with a lower monopoly threshold exists last in the equilibrium. Similar to Ghemawat and Nalebuff (1985) and Murto (2004), in the current setting this is the firm with the lower capacity level. Intuitively, such a firm incurs lower production costs, that in the face of declining demand induces smaller losses. As an example consider the airline industry. The carriers with more excess capacity that incur higher labor and operating costs suffer more from the downturn in demand. For instance, in 2001 several large airlines in the US, including US Airways and United, declared bankruptcy due to the drastically declining passenger demand after the 9/11 attacks. As another example, Delta and Northwestern, the third and fourth largest carriers in the US, filed for bankruptcy protection in 2005, as a result of intense competition with the low-cost airlines that were driving the prices down. The lower labor and operating costs enable the low-cost airlines to surpass the crisis period better (Lawton (2003), Flouris and Walker (2005)). In this example another reason for bankruptcy, apart from excess capacity, could be inability to raise additional funds when demand is declining. This problem is the main focus of Chapter 4, which discusses exit due to damages to a firm's reputation.

We specify the exit order of the firms according to the following proposition.[4]

Proposition 3.5 *The firm with the larger capacity level exits first at the optimal duopoly exit threshold, X_D^E, determined by*

$$X_D^E(Q_L, Q_F) = \frac{\beta_2 c(r-\alpha)}{r(\beta_2 - 1)(1 - \eta(Q_L + Q_F))}, \tag{3.14}$$

while the firm with the smaller capacity level exits once x hits the optimal monopoly exit threshold, X_M^E, given by

$$X_M^E(Q_i) = \frac{\beta_2 c(r-\alpha)}{r(\beta_2 - 1)(1 - \eta Q_i)}, \tag{3.15}$$

with $i = L$ if the smaller firm has entered the market first and $i = F$ if it has entered last.

If the firms install the same capacity level, $Q_L = Q_F$, either of the firms exits first.

The crucial feature of the model with both exit and investment option is that the entrant has a second mover advantage. Namely, in the case of a sufficiently large capacity level of the incumbent, the entrant can install a capacity large enough to force the incumbent out of the market for specific values of the market size. Thus, the optimization problem of the entrant should incorporate that the exit order depends on its capacity choice. In particular, the entrant can reply to a certain capacity level of

[4]Due to the fact that in our model the follower undertakes an entry decision, the exit game starts for values of x that are large enough to rule out the possibility of the gap equilibrium shown by Murto (2004).

the incumbent, Q_L, with either larger, $Q_F > Q_L$, smaller $Q_F < Q_L$, or equal capacity level, $Q_L = Q_F$. Each of these cases leads to different exit order scenarios. This, in turn, affects the value functions, because they must incorporate the possibility that in case of a market decline one firm is going to exit the market, while the other becomes a monopolist.

Hence, the entrant's exit strategy is specified as follows. When it enters the market as the larger firm, i.e. $Q_F > Q_L$, it exits at the duopoly exit threshold $X_D^E(Q_L, Q_F)$. Intuitively, the larger is the capacity that the entrant acquires, the earlier will it exit the market, which is confirmed by (3.14). This brings us to the capacity level, $\widetilde{Q}_F(x, Q_L)$, that leads to the immediate displacement of the entrant from the market.

$$\widetilde{Q}_F(x, Q_L) = \frac{1}{\eta}\left(1 - \frac{\beta_2 c(r-\alpha)}{r(\beta_2 - 1)x}\right) - Q_L. \tag{3.16}$$

When the entrant is the smaller firm, i.e. $Q_F < Q_L$, the incumbent exits first at $X_D^E(Q_L, Q_F)$ and the entrant enjoys monopoly profits until x hits the monopoly exit threshold, $X_M^E(Q_L, Q_F)$. In this case $\widetilde{Q}_F(x, Q_F)$ has a different interpretation, namely, it is the capacity level such that once installed by the entrant the incumbent is forced out of the market. Clearly, if the entrant is a smaller firm, it will never install this capacity level. This is reflected in its optimal capacity choice, which we consider later.

If $Q_F = Q_L$ the game is symmetric, and, as shown by Murto (2004), the exit order is not identified, ruling out the mixed strategies.[5]

The described strategies of the entrant are illustrated in Table 1, where we denote by V_1^B the value of the larger entrant (B stands for "big"), by V_1^S the value of the smaller entrant (S stands for "small"), and by V_1^M the value of the monopolist, defined in the previous section.

Conditions	$Q_F < \widetilde{Q}_F(x, Q_L)$ $x < X_D^E(Q_L, Q_F)$	$Q_F \geq \widetilde{Q}_F(x, Q_L)$ $x \geq X_D^E(Q_L, Q_F)$
$Q_F > Q_L$ entrant exits first	$V_1^B - \delta Q_F$	$-\delta Q_F$
$Q_F < Q_L$ entrant exits last	$V_1^S - \delta Q_F$	$V_1^M - \delta Q_F$
$Q_F = Q_L$ Exit order unclear	$\lambda V_1^B + (1-\lambda)V_1^S - \delta Q_F$	$(1-\lambda)V_1^M - \delta Q_F$

Table 3.1: Value of the entrant for the different capacity levels of the incumbent, where λ is the probability that the entrant exits first in the symmetric game.

[5]Steg and Thijssen (2015), for example, focus on the equilibrium in mixed strategies.

The value functions of the entrant, introduced in Table 3.1, are characterized in Proposition 3.6.

Proposition 3.6 *The value of the active entrant is given by*

$$V_1(x,Q_L,Q_F) = \mathbb{1}_{\{Q_F \geq \widetilde{Q}_F\}}\Big((1-\lambda)V_1^M(x,Q_F)\mathbb{1}_{\{Q_F=Q_L\}} + V_1^M(x,Q_F)\mathbb{1}_{\{Q_F<Q_L\}}\Big) + \mathbb{1}_{\{Q_F<\widetilde{Q}_F\}}\Big(V_1^B(x,Q_L,Q_F)\mathbb{1}_{\{Q_F>Q_L\}} + V_1^S(x,Q_L,Q_F)\mathbb{1}_{\{Q_F<Q_L\}} + (\lambda V_1^B(x,Q_L,Q_F) + (1-\lambda)V_1^S(x,Q_L,Q_F))\mathbb{1}_{\{Q_F=Q_L\}}\Big) - \delta Q_F, \quad (3.17)$$

with λ introduced earlier, and where V_1^B is the value of the large entrant defined by

$$V_1^B(x,Q_L,Q_F) = \left(\frac{x}{X_D^E(Q_L,Q_F)}\right)^{\beta_2}\frac{cQ_F}{r(1-\beta_2)} + \frac{XQ_F(1-\eta(Q_L+Q_F))}{r-\alpha} - \frac{cQ_F}{r}, \quad (3.18)$$

V_1^S is the value of the smaller entrant determined by

$$V_1^S(x,Q_L,Q_F) = \left(\frac{x}{X_M^E(Q_F)}\right)^{\beta_2}\frac{cQ_F}{r(1-\beta_2)} + \left(\frac{x}{X_D^E(Q_L,Q_F)}\right)^{\beta_2}\frac{X_D^E(Q_L,Q_F)\eta Q_L Q_F}{r-\alpha} + \frac{XQ_F(1-\eta(Q_L+Q_F))}{r-\alpha} - \frac{cQ_F}{r}, \quad (3.19)$$

and V_1^M is a value of the active monopolist given by (3.4)*.*

Note that both $V_1^B(x,Q_L,Q_F)$ and $V_1^S(x,Q_L,Q_F)$ contain the duopoly revenue net of production costs, reflected by the last two terms in (3.18) and (3.19), respectively. In (3.18) the first term represents the exit option. In (3.19) the exit option of the entrant is represented by the second term and the first term corrects for the fact that the duopoly revenues are replaced by the monopoly revenues once the incumbent exits.

In the symmetric game the value of the entrant is a weighted average of the values under different exit orders, $V_1^B(x,Q_L,Q_F)$ and $V_1^S(x,Q_L,Q_F)$. Hence, this value is always smaller than the maximum of $V_1^B(x,Q_L,Q_F)$ and $V_1^S(x,Q_L,Q_F)$. As the value functions are continuous, it will never be optimal to choose the exact same capacity as the incumbent. Therefore, the symmetric game will never occur. Now, since $\mathbb{1}_{\{Q_F=Q_L\}} = 0$, the value of the active entrant can be rewritten in the following way:

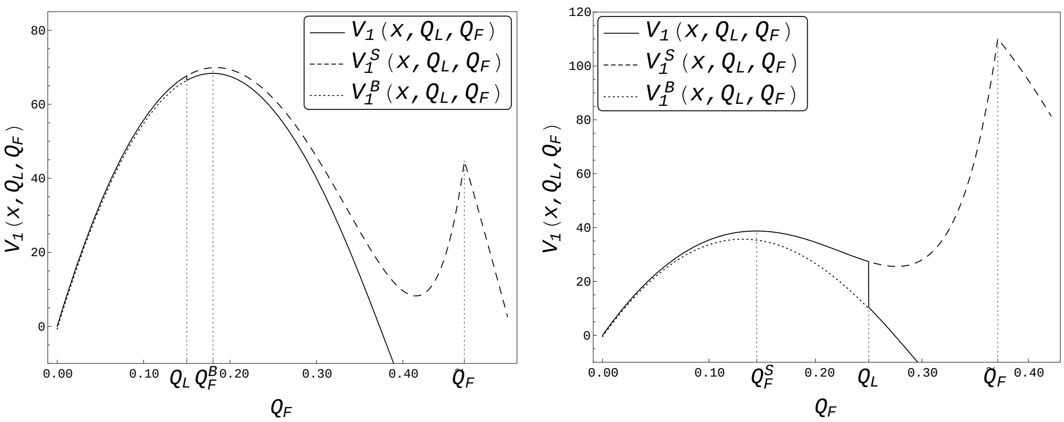
$$\begin{aligned} V_1(x,Q_L,Q_F) &= \mathbb{1}_{\{Q_F<\widetilde{Q}_F\}}\Big(V_1^B(x,Q_L,Q_F)\mathbb{1}_{\{Q_F>Q_L\}} + V_1^S(x,Q_L,Q_F)\mathbb{1}_{\{Q_F<Q_L\}}\Big) \\ &+ \mathbb{1}_{\{Q_F\geq\widetilde{Q}_F\}}\Big(V_1^M(x,Q_F)\mathbb{1}_{\{Q_F<Q_L\}}\Big). \end{aligned} \quad (3.20)$$

3.3.2 Capacity optimization

In order to find the optimal response of the entrant to a given capacity level acquired by the incumbent, Q_L, we maximize $V_1(x, Q_L, Q_F) - \delta Q_F$ with respect to Q_F:

$$Q_F^*(x, Q_L) = \underset{Q_F}{\operatorname{argmax}}\{V_1(x, Q_L, Q_F) - \delta Q_F\}. \tag{3.21}$$

The exit order is endogenously determined by the firms' relative capacity size, because the firm with the smaller capacity always exits last. However, the strategy of being the last firm to exit is not always preferable, as it requires a relatively low capacity level. We illustrate this situation below.



(a) For $Q_L = 0.15$. *(b)* For $Q_L = 0.25$.

Figure 3.2: The value function of the entrant as a function of Q_F for a given capacity level of the incumbent for the parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\eta = 1$, $\delta = 100$, $c = 50$, $x = 66$.

Figure 3.2 shows the value function of the entrant for different values of the incumbent's capacity level. The dotted curves in both figures correspond to the value that the entrant gets if it exits first, while the dashed curves show the value when it exits last. Note that the latter value is not a unimodal function of capacity, Q_F. Instead, it has a spike for large values of Q_F. This is because, given that the entrant exits last, it becomes a monopolist as soon as Q_F reaches $\widetilde{Q}_F(x, Q_F)$. Thus, anticipation of a sooner monopoly position causes the entrant's value to increase as Q_F approaches $\widetilde{Q}_F(x, Q_F)$. This value, however, can only be reached when the incumbent's capacity is larger. Due to the exit order constraint the entrant can end up on the upper curve only if it becomes the smallest firm in the market, $Q_F < Q_L$. The complementary case of $Q_F > Q_L$ yields the value that corresponds to the lower curve. Thus, the solid parts of the curves represent the actual value of the entrant. In Figure 3.2a the capacity level of the incumbent is small, meaning that the entrant is relatively close to the monopoly situation. In this case the value of becoming the last firm to exit is smaller than the value of installing a larger capacity and exiting last. On the contrary, when the capacity of the firm that already operates in the market is large, the exit order becomes more important. Figure 3.2b shows that for larger Q_L the entrant prefers to install a smaller capacity and exit last. Thus, we can conclude that the firm faces a trade-off between staying longer in the market and installing a larger capacity. In fact, if Q_L is small a larger value is obtained by installing a larger capacity and as a consequence forgoing the potential monopoly position. Proposition 3.7 shows the entrant's capacity choice for each level of the incumbent's capacity.

Proposition 3.7 *For relatively large values of x the optimal capacity level of the entrant depending on the capacity level of the incumbent is given by*

$$Q_F^*(x, Q_L) = Q_{F,D}^*(x, Q_L)\mathbb{1}_{\{Q_L \leq \bar{Q}_4(x)\}} + Q_{F,M}^*(x, Q_L)\mathbb{1}_{\{Q_L > \bar{Q}_4(x)\}}, \tag{3.22}$$

where $Q_{F,D}^*$ denotes the optimal capacity of the entrant when it enters as a duopolist, while $Q_{F,M}^*$ is the optimal capacity level when it becomes a monopolist upon entry. These capacity levels are given by

$$Q_{F,D}^*(x, Q_L) = \begin{cases} Q_F^B(x, Q_L) & \text{if } Q_L < \bar{Q}_1(x), \\ Q_L - \varepsilon & \text{if } Q_L \in (\bar{Q}_1(x), \bar{Q}_2(x)] \cup (\bar{Q}_3(x), \bar{Q}_4(x)], \\ Q_F^S(x, Q_L) & \text{if } \bar{Q}_L \in (\bar{Q}_2(x), \bar{Q}_3(x)], \end{cases} \tag{3.23}$$

where $\varepsilon > 0$ is a small value and the capacity levels of the entrant $Q_F^B(x, Q_L)$ and $Q_F^S(x, Q_L)$ implicitly determined by the first order conditions (3.24) *and* (3.25)*, respectively:*

$$\frac{\partial(V_1^B(x, Q_L, Q_F) - \delta Q_F)}{\partial Q_F} = 0, \tag{3.24}$$

$$\frac{\partial(V_1^S(x, Q_L, Q_F) - \delta Q_F)}{\partial Q_F} = 0, \tag{3.25}$$

and

$$Q_{F,M}^*(x, Q_L) = \begin{cases} \tilde{Q}_F(x, Q_L) & \text{if } \bar{Q}_L \in (\bar{Q}_4(x), \bar{Q}_5(x)], \\ Q_M(x) & \text{if } \bar{Q}_L > \bar{Q}_5(x). \end{cases} \tag{3.26}$$

The expressions for $\bar{Q}_1(x)$, $\bar{Q}_2(x)$, $\bar{Q}_3(x)$, $\bar{Q}_4(x)$, $\bar{Q}_5(x)$ are given in the Appendix.

A numerical example illustrating the optimal capacity choice of the entrant for a given x is presented in Figure 3.3.

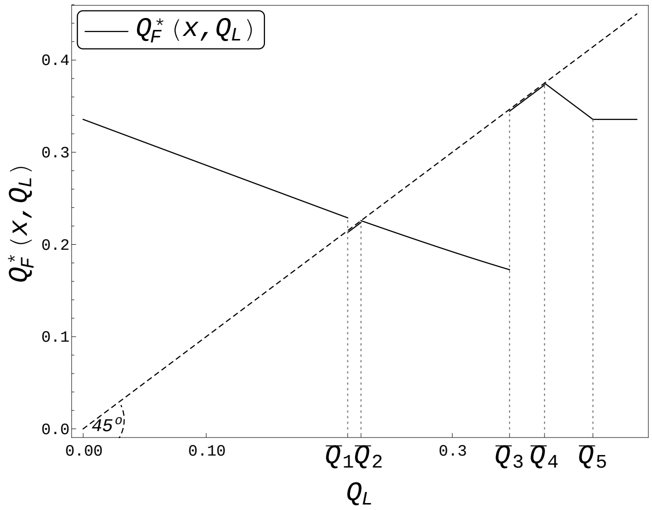


Figure 3.3: The optimal reaction of the entrant given the incumbent's capacity level for the parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\eta = 1$, $\delta = 100$, $c = 50$, $x = 100$.

Apart from illustrating the optimal capacity choice of the entrant, Figure 3.3 also helps to infer which exit order corresponds to a certain capacity clevel of the incumbent. Namely, the entrant will choose to leave the market first when its optimal capacity level is above the 45° line, otherwise it acquires a market share smaller than the incumbent and exits last.

As can be seen, the entrant prefers to exit first only when the capacity level of the incumbent is relatively small, i.e. $Q_L < \bar{Q}_1(x)$. Then the entrant can obtain a large revenue by installing a larger capacity. The large revenue outweighs the advantage of leaving the market last and, consequently, the entrant behaves as a large duopolist.

In addition, if the entrant observes that the incumbent has installed a capacity of a considerable size, it is not possible to obtain such a large revenue that it would be still profitable to leave the market first. As a result, the entrant chooses to be a small duopolist in order to stay longer on the market for $Q_L \in (\bar{Q}_2(x), \bar{Q}_3(x)]$.

Note that for both the large and small duopolist the optimal capacity level mostly decreases with Q_L, because a larger capacity of the first investor reduces the output price for a given capacity of the entrant. However, Figure 3.3 shows that for some intervals the entrant's capacity increases with the incumbent's capacity.

The increasing parts of the curves correspond to the scenarios, where the entrant chooses to mimic the incumbent's behavior and acquires capacity just below Q_L. When $Q_L \in (\bar{Q}_1(x), \bar{Q}_2(x)]$, the share of the incumbent is large enough to stimulate the entrant to leave last. However, leaving last would require that the capacity of the entrant satisfies the constraint, $Q_F < Q_L$. As a result, the optimum of the small duopolist cannot be reached. In this case the entrant maximizes its revenue in a constrained duopoly using a mimicking strategy, with the optimal choice $Q_F = Q_L - \varepsilon$.

When Q_L hits $\bar{Q}_3(x)$ we observe a relatively large discontinuous upward jump in the entrant's optimal capacity. This result corresponds to the findings in Kwon and Zhang (2015), namely, that if the capacity of one firm is large enough, it becomes optimal for its rival to increase the capacity to force such a firm out of the market. Thus, in the regions where $Q_L \in (\bar{Q}_3(x), \bar{Q}_4(x)]$ and $Q_L \in (\bar{Q}_4(x), \bar{Q}_5(x)]$ the capacity of the incumbent is so large, and, consequently, the output price is so low, that the entrant acts to force a soon or immediate exit of the incumbent, respectively. Anticipating the incumbent's (almost) immediate exit, the entrant behaves as a constrained monopolist, i.e. it installs a capacity level such that on the one hand the incumbent exits first, and, on the other hand, that it gets the largest possible monopoly value for itself. As a result, it chooses $Q_L - \varepsilon$ and $\widetilde{Q}_F(x)$ for the two regions, respectively. Note that mimicking strategy arises here for a different reason than in the case of smaller levels of the incumbent's capacity. Namely, for $Q_L \in (\bar{Q}_3(x), \bar{Q}_4(x)]$ the duopoly exit threshold is relatively close, yet due to the capacity constraint for the small firm the immediate exit of its rival cannot be triggered. Thus, the entrant installs the largest capacity available to ensure that this threshold is hit as soon as possible.

In the last region, i.e. where $Q_L > \bar{Q}_5(x)$, the entrant becomes an unconstrained monopolist as acquiring the monopoly capacity level is enough to ensure the incumbent leaves the market.

In the above formulation we concentrate on the capacity strategy of the entrant for a given x. Naturally, the problem can be reversed and the boundaries of the strategic regions can be defined in terms of x for a given level of Q_L using the inverse function of $\bar{Q}_i(x)$ with $i = 1, .., 5$. The capacity choice of the entrant in two dimensions, thus as a function of both x and Q_L, is illustrated in Figure 3.4.

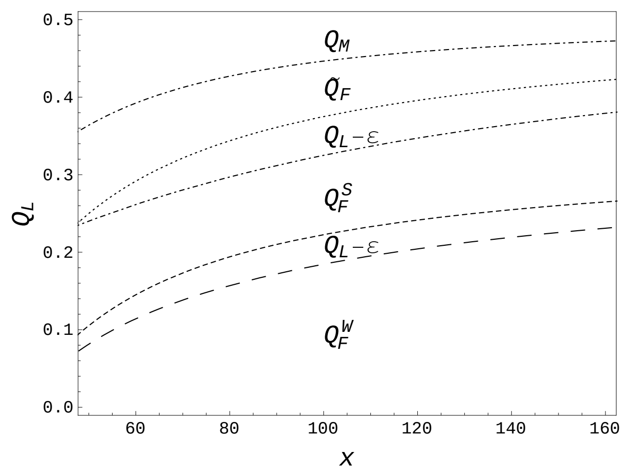


Figure 3.4: The optimal capacity strategies of the entrant for the set of parameter values: $r = 0.05$, $\alpha = 0.01$, $\sigma = 0.17$, $\eta = 1$, $\delta = 100$, and $c = 50$.

Consider, for example, $Q_L = 0.4$. Then if x is small enough the entrant chooses a capacity level to maximize monopoly profits, because the rival leaves the market anyhow. For x a little larger the entrant overinvests to force the exit of the incumbent. As x increases even further the market moves away from the duopoly exit trigger and the entrant prefers to capture larger immediate profits. Thus, it gradually increases its capacity as exit becomes further away and thus less crucial to take into account in its investment decision. Consequently, apart from a direct effect on revenue, the market profitability also indirectly influences the firm's exit strategies through capacity choice. In particular, exit decisions are hastened in less profitable markets and delayed in more profitable markets.

3.3.3 Entry decision

Given the optimal exit timing and the optimal capacity choice, the entrant determines its optimal investment timing, or, in other words, its investment threshold, which we denote by X_F^I. It does so by solving the following optimal stopping problem

$$V^*(x, Q_L) = \sup_{\tau_1} \mathbb{E}_x \left[\mathrm{e}^{-r\tau_1} \Big(V_1(x, Q_L, Q_F^*(x, Q_L)) - \delta Q_F^*(x, Q_L) \Big) \right], \tag{3.27}$$

where τ_1 is a stopping time, $Q_F^*(x, Q_L)$ is given by (3.22). Here $V^*(x, Q_L)$ equals to the value of the active firm in the stopping region and to the value of the option to invest in the continuation region. The value of this option corresponds to the value of an idle firm, V_0, which together with the investment threshold, X_F^I, is defined by the following proposition.

Proposition 3.8 *The value of the idle entrant is given by*

$$V_0(x, Q_L) = A_F(Q_L) x^{\beta_1}, \tag{3.28}$$

where $A_F(Q_L) = \widehat{A}_F(X_F^I(Q_L), Q_L, Q_F^*(X_F^I(Q_L), Q_L))$ *with*

$$\begin{aligned} \widehat{A}_F(x, Q_L, Q_F) = & \frac{\beta_2}{\beta_2 - \beta_1} \left(\frac{1}{x} \right)^{\beta_1} \Bigg(\frac{x(1 - \eta(Q_L + Q_F))Q_F}{r - \alpha} \left(1 - \frac{1}{\beta_2} \right) - \left(\frac{c}{r} + \delta \right) Q_F \\ & - \frac{x}{\beta_2} \left(\frac{\partial V_1(x, Q_L, Q_F)}{\partial Q_F} - \delta \right) \frac{\partial Q_F^*(x, Q_L)}{\partial x} \Bigg). \end{aligned} \tag{3.29}$$

The optimal investment trigger $X_F^I(Q_L)$ *is a solution with respect to* x *of*

$$h(x, Q_L, Q_F^*(x, Q_L)) = 0, \tag{3.30}$$

where

$$h(x, Q_L, Q_F) = \left(1 - \frac{\beta_2}{\beta_1}\right) \frac{B_F(Q_L, Q_F)}{Q_F} x^{\beta_2} + \left(1 - \frac{1}{\beta_1}\right) \frac{x(1 - \eta(Q_L + Q_F))}{r - \alpha} - \frac{c}{r} - \delta$$

$$- \frac{x}{\beta_1 Q_F} \left(\frac{\partial V_1(x, Q_L, Q_F)}{\partial Q_F} - \delta\right) \frac{\partial Q_F}{\partial x} = 0, \qquad (3.31)$$

and $B_F(Q_L, Q_F)$ *is*

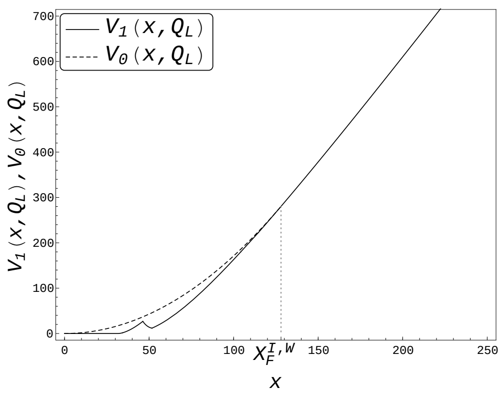
$$\begin{aligned} B_F(Q_L, Q_F) &= B_F^B(Q_L, Q_F)\mathbb{1}_{\{Q_F > Q_L\}} + B_F^S(Q_L, Q_F)\mathbb{1}_{\{Q_F < Q_L\}} \\ &+ (\lambda B_F^B(Q_L, Q_F) + (1 - \lambda) B_F^S(Q_L, Q_F))\mathbb{1}_{\{Q_F = Q_L\}}, \end{aligned} \qquad (3.32)$$

with λ *being the probability that the entrant exits first in a symmetric game, and*

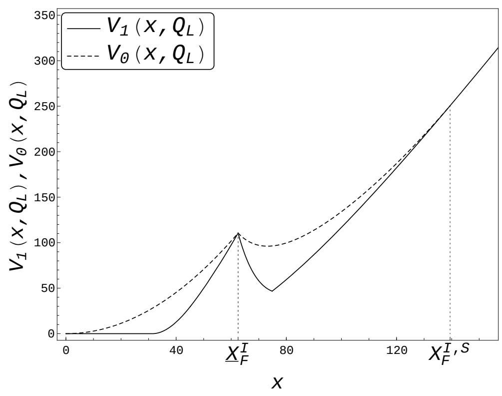
$$B_F^B(Q_L, Q_F) = \left(\frac{1}{X_D^E(Q_L, Q_F)}\right)^{\beta_2} \frac{cQ_F}{r(1 - \beta_2)}, \qquad (3.33)$$

$$B_F^S(Q_L, Q_F) = \left(\frac{1}{X_M^E(Q_F)}\right)^{\beta_2} \frac{cQ_F}{r(1 - \beta_2)} + \left(\frac{1}{X_D^E(Q_L, Q_F)}\right)^{\beta_2} \frac{X_D^E(Q_L, Q_F)\eta Q_L Q_F}{r - \alpha}. \qquad (3.34)$$

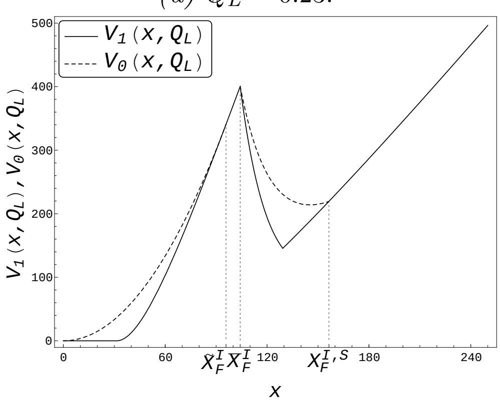
Extensive numerical experiments show that for reasonable parameter values the level of x at the moment of investment is large enough to satisfy Proposition 3.7. The intuition behind this is that for smaller x the option value exceeds the value of investing immediately and the firm waits till the market is large enough. Therefore, the optimal capacity choice at the moment of entry is consistent with the results of Proposition 3.7. Clearly, the piecewise structure of the entrant's optimal capacity strongly affects its optimal investment timing. For example, it can end up either being a duopolist or a monopolist upon entry. Naturally, these cases correspond to two different value curves with the latter scenario resulting in higher profits. As a result, multiple investment thresholds corresponding to different value functions may be available to the entrant as long as they are consistent with the boundaries for the capacity choice, i.e. capacity at the moment of entry should fall into the specific regions defined in Proposition 3.7. In order to find the optimal strategy of the entrant for each level of x, we first need to establish whether it is optimal to invest immediately given the available capacity strategy for the particular value of x. If this is not the case we need to determine until what moment it is optimal to wait with investment, or in other words, the threshold corresponding to the capacity strategy, which will eventually trigger the investment. We do this by comparing the values of the options to invest for different capacity strategies. Figure 3.5 presents entrant values as functions of market profitability for different values of the incumbent's capacity choice.



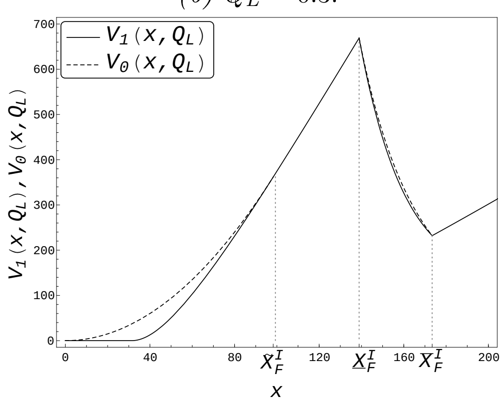
(a) $Q_L = 0.23$.



(b) $Q_L = 0.3$.



(c) $Q_L = 0.38$.



(d) $Q_L = 0.41$.

Figure 3.5: The value functions of the entrant for the set of parameter values: $r = 0.05$, $\alpha = 0.2$, $\sigma = 0.1$, $\eta = 1$, $\delta = 100$, $c = 20$, and different values of Q_L.

The solid line in Figure 3.5 represent the value of an active entrant. If x is relatively small the entrant is able to obtain the monopoly value, which corresponds to the increasing part of the entrant curve before the spike. It is possible because current output price is low, so it is easier to force the incumbent out of the market. In a large market this strategy is too costly for the entrant, as a larger capacity is needed to stimulate the incumbent's immediate exit. Thus, the entrant operates in a duopoly. The declining part of the duopoly value after the spike is associated with the mimicking strategy. As stated earlier, under this strategy the entrant's capacity is given by $Q_F = Q_L - \varepsilon$. Thus, for a given level of Q_L the capacity of the entrant is also fixed in such a way that it always exits last. An increase in market profitability has two effects on the entrant's value in this case. On the one hand, a larger x means that exit of its rival is farther in the future, affecting the firm's value negatively. On the other hand, a larger x increases its revenues, leading to an increase in value. The mimicking strategy enters the optimal capacity of the entrant twice – for small $Q_L \in (\bar{Q}_1, \bar{Q}_2]$ and for large $Q_L \in (\bar{Q}_3, \bar{Q}_4]$. In the first case an increasing effect prevails as the exit is far away in the future so that the firms care more about higher revenues. In the latter case the exit of the large firm is relatively close, and therefore the negative effect of an increase in market size dominates. Due to the latter we observe a decline in the entrant value in the region connecting the strategies of monopolist and duopolist.

The dashed lines in Figure 3.5 correspond to the values of the idle entrant, or, in other words, to the value of waiting. It follows that it is optimal for the firm to wait with investment if the dashed line lies above the solid line. Intuitively, each capacity strategy results in a different investment timing. For example, in Figure 3.5a it is optimal for the entrant to invest as a large firm. The investment threshold of a large duopolist is denoted by $X_F^{I,W}$. In this case the capacity of the incumbent is so small, that waiting for a larger market and capturing a larger market share yields a higher value. Once the capacity of the incumbent increases, taking into account its exit becomes a more valuable strategy. There exists a maximal market size such that for a given Q_L the entrant can enter as a monopolist, which is denoted by $\underline{X}_F^I$. Since each strategy is admissible only in a particular region in terms of Q_L and x, it may happen that the optimal investment moment given a particular strategy lies outside the admissible boundaries.

First consider the case of low initial x, i.e. $X \in [0, \underline{X}_F^I]$. If the monopoly threshold lies beyond $\underline{X}_F^I$, given that it aims at the monopoly position, the entrant chooses to enter the market at the next best alternative, namely, at $\underline{X}_F^I$. This happens if the capacity of the incumbent is relatively small, as in this case the entrant has more incentive to wait for a larger market, as shown in Figure 3.5b. If the capacity of the incumbent increases further, it hastens the entrant's investment decision, making the optimal thresholds first leading to a constrained and then leading to an unconstrained monopoly available in the corresponding strategic regions. Thus, for low initial x the entrant waits either until X_M^I in the unconstrained monopoly region, or until $\widetilde{X}_F^I$ in the case of constrained monopoly. The latter situation is illustrated in Figures 3.5c and 3.5d.

If the initial value of x is so large that forcing the incumbent out of the market immediately upon entry is not possible anymore, i.e. $x > \underline{X}_F^I$, the entrant has an option to wait either until the market is large enough to enter as a duopolist or until the market is low enough to enter as a monopolist. Thus, in Figures 3.5b and 3.5c waiting also pays off if $X \in (\underline{X}_F^I, X_F^{I,S})$, where $X_F^{I,S}$ is the optimal investment threshold of a small entrant. If the capacity of the incumbent is larger, the optimal investment threshold of the small entrant in a duopoly declines and it may occur below the starting point of the corresponding strategy, which we denote by $\overline{X}_F^I$, see Figure 3.5d. Clearly, in this case for $X \in (\underline{X}_F^I, \overline{X}_F^I)$ the entrant can either enter with a mimicking capacity, or wait until either $\underline{X}_F^I$ or $\overline{X}_F^I$ is hit. It turns out that in this case waiting for either of the thresholds to be hit is a preferable strategy as it leads to a larger value.

We conclude that the introduction of the exit option together with being able to choose the capacity level has resulted in the following entrant's investment behavior. If the capacity of the incumbent is relatively large the entrant has three investment thresholds. Two of them trigger investment providing a monopoly position for the entrant. The first threshold occurs if the initial market size is so low that the entrant waits until it is profitable enough to enter as a monopolist. The second one is present for intermediate market size, where the entrant has an option to wait until the incumbent's exit threshold is close enough, so that the incumbent is expected to exit soon. The last investment threshold corresponds to the standard case in the real options models, i.e. when the market is large enough for the two firms to operate together in a duopoly. Hence, in the presence of a large incumbent in a small market the entrant waits until the monopoly threshold is hit and then invests immediately as long as the monopoly strategy is available. In the case of an intermediate market size it waits until either monopoly or duopoly threshold is hit, i.e. until the exit is close enough to force the incumbent out or until the market is large enough to coexist with the incumbent. Following the recent literature, we will refer to the region between these thresholds as inaction region or hysteresis region (Decamps *et al.* (2006)). Lastly, in a large market the entrant invests immediately.

If the incumbent sets its capacity at an intermediate level, the only difference with the previous case is that the thresholds that lead to the monopoly situation merge into one. Thus, the entrant waits for the same moment to invest both in the cases of small or intermediate market. At the threshold the entrant overinvests to trigger immediate exit of the incumbent.

If the capacity of the incumbent is small, stimulating its exit becomes so costly that the entrant prefers to wait until a duopoly is profitable and we are back in the situation of one investment threshold.

The investment thresholds of the entrant described above are presented in the following proposition.

Proposition 3.9 *There exist a threshold which leads to immediate investment in a duopoly $X_{F,D}^I(Q_L)$. The entrant's monopolization strategy becomes available only if the capacity of the incumbent is sufficiently large. In this case there exist two additional investment thresholds: $X_{F,\underline{M}}^I(Q_L)$ that leads to the monopoly once being* *hit by x from below, and $X^I_{F,\overline{M}}(Q_L)$ that triggers the monopoly situation from above.*

The investment thresholds of the entrant described above as well as the entrant's strategies for different capacity levels of the incumbent are illustrated in Figure 3.6. The dark gray areas in this figure represent the combinations of x and Q_L such that the entrant only waits for an increase in x and then enters with the capacity level, indicated right below the figure. The light gray area corresponds to the hysteresis region, where the entrant waits either for a decline in market profitability or an increase and enters at either of the two investment thresholds. In the white parts of the graph the entrant invests immediately with the capacity level indicated right above the figure.

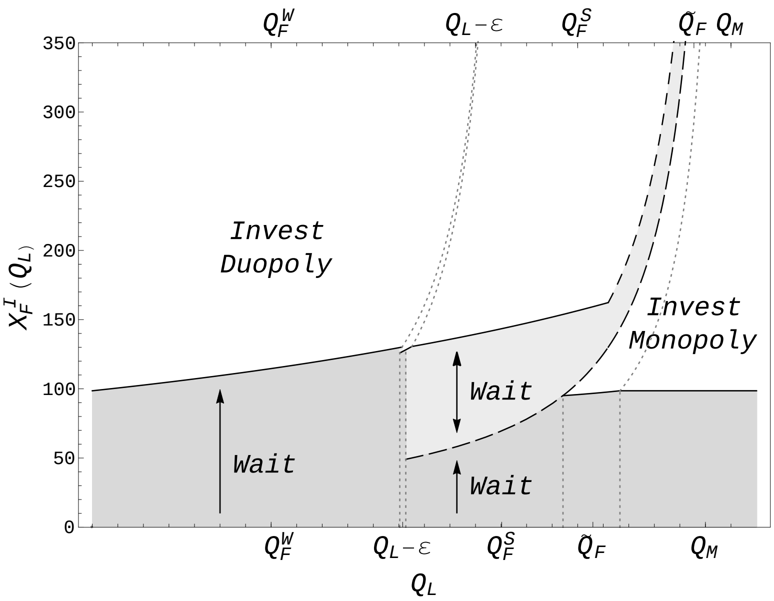


Figure 3.6: The optimal investment strategy of the entrant for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.17$, $\eta = 1$, $\delta = 100$, and $c = 20$.

Note that in the region where the entrant sets the capacity level equal to $\widetilde{Q}_F$, the optimal investment threshold lies below the monopoly level. Recall that $\widetilde{Q}_F$ is the capacity level under a constrained monopoly. In this case the entrant chooses its capacity in such a way that the duopoly exit threshold X^E_D is immediately hit and the incumbent is forced out of the market. Evidently, as the capacity installed by incumbent increases the entrant is able to drive it out for larger values of x. As we observe in Figure 3.6, the entrant gradually increases its entry timing until the monopoly trigger can be reached.

Another interesting result arises when we allow α, the constant drift in the Brownian motion, to be negative. In this case we observe non-monotonicity in the entrant's duopoly threshold. If the capacity level of the incumbent is low and, thus exit is still far away, the duopoly threshold increases with the incumbent's capacity as in Figure 3.6. However, once the incumbent's capacity becomes sufficiently large, the entrant chooses to be a smaller firm in order to exit last. At the same time this means that the exit of the larger firm, the incumbent, becomes closer. Anticipating sooner exit of its rival, the entrant has an incentive to enter the market sooner and its investment threshold decreases. This situation is illustrated in Figure 3.7.

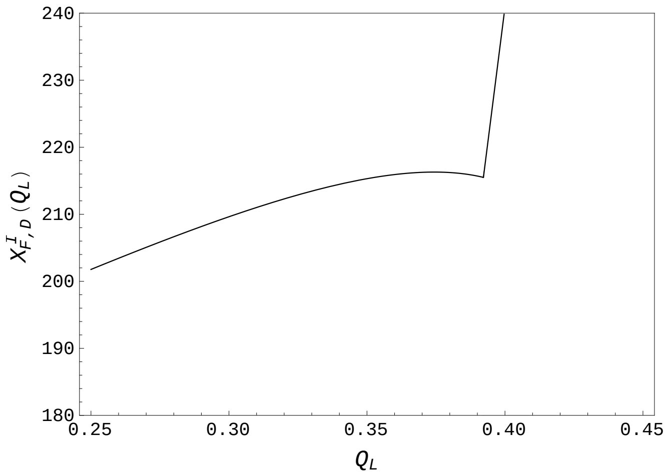


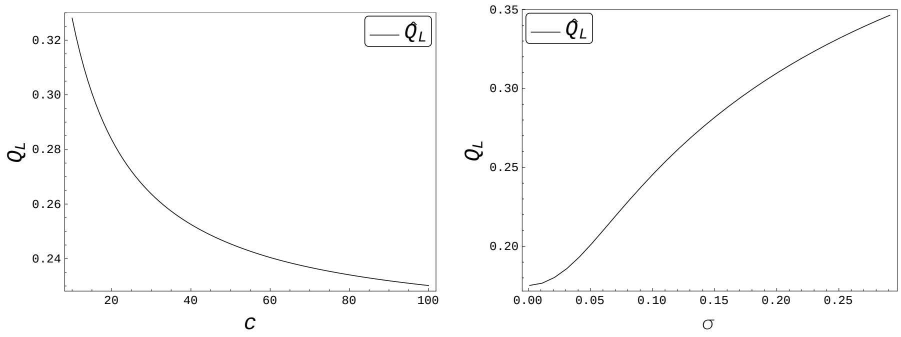
Figure 3.7: The optimal investment strategy of the the entrant for the set of parameter values: $r = 0.05$, $\alpha = -0.03$, $\sigma = 0.17$, $\eta = 1$, $\delta = 100$, and $c = 50$.

In general, the entrant considers a monopolization strategy only if the capacity of the incumbent exceeds a certain threshold. As stated in Proposition 3.9, the entrant never has an incentive to become a monopolist if the incumbent's capacity is sufficiently large. This is because, given that the market share of the opponent is small, in order to induce a monopoly scenario the entrant either needs to wait until the market is low enough or to install a large enough capacity. In both cases the entrant exits rather soon itself. Therefore, for small Q_L the entrant prefers to extract greater duopoly rents above becoming a monopolist for a short period. Denote by $\widehat{Q}_L$ the minimal value of the incumbent's capacity for which in a small market benefits of monopolization outweigh the disadvantages of the sooner exit, or in other words, $\widehat{Q}_L$ is the smallest incumbent's capacity level for which the hysteresis region occurs. Then $\widehat{Q}_L$ is the solution of the following system

$$\begin{cases} V_M\left(\frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-2\eta Q_L)}, Q_L\right) - \delta Q_L = V_0\left(\frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-2\eta Q_L)}, Q_L, Q^*_{F,D}\left(x, Q_L\right)\right), \\ h(x, Q_L, Q^*_{F,D}(x, Q_L)) = 0, \end{cases} \tag{3.35}$$

where $h(x, Q_L, Q_F)$ is defined by (3.31).

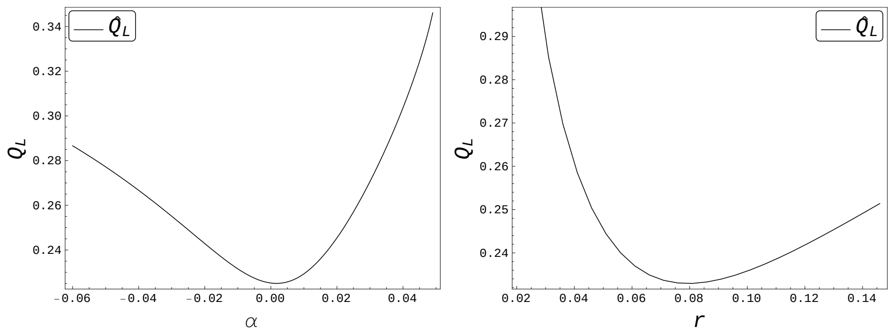
Figures 3.8 and 3.9 illustrate how the capacity level $\widehat{Q}_L$ changes with respect to different parameter values.



(a) $\sigma = 0.1$. *(b)* $c = 50$.

Figure 3.8: The capacity level $\widehat{Q}_L$ for the set of parameter values: $r = 0.05$, $\alpha = 0.02$, $\eta = 1$, $\delta = 100$, and different values of c and σ.

As can be seen, an increase in production costs, c, results in a smaller level of $\widehat{Q}_L$. This is because larger production costs increase the exit triggers of the firms. Consequently, a smaller capacity of the incumbent is needed to ensure that monopolization is profitable. An increase in market uncertainty, σ has an opposite effect. The standard result in the real options literature (see e.g. Dixit and Pindyck (1994)) is that the firms delay their decisions for higher uncertainty. In particular, the decision to exit the market is delayed, that is why a larger σ implies that a larger capacity of the incumbent is needed to trigger the entrant's monopolization strategy. The effects of α and r are non-monotonic as it can be seen from Figure 3.9.



(a) $r = 0.05$. *(b)* $\alpha = 0.02$.

Figure 3.9: The capacity level $\widehat{Q}_L$ for the set of parameter values: $\sigma = 0.1$, $\eta = 1$, $c = 50$, $\delta = 100$, and different values of r and α.

Consider now the effect of a change in the drift, α, illustrated in Figure 3.9a. When α is positive the firms expect the market to grow in the future and, as a result, to move away from the exit threshold. Therefore, as α increases the monopolization strategy brings less benefits for the entrant and $\widehat{Q}_L$ increases. When α is negative, this means on the one hand that both firms expect to exit the market soon, while the less negative α becomes, the longer the monopoly period of the entrant is anticipated. Thus, we can see a decline in $\widehat{Q}_L$ for negative α, because the monopolization strategy becomes more attractive. A similar type of non-monotonicity is to be observed considering the effect of the discount rate if α is positive (see Figure 3.9b). On the one hand, for large r the firms discount their future payoffs more heavily, or put differently, care more about the present rather than the exit decisions in the future. Thus, the entrant needs a larger capacity installed by the incumbent to consider the monopoly scenario. On the other hand, for relatively small r another effect comes into the picture. Namely, the exit trigger of the incumbent increases with r, and it becomes easier to drive it out of the market, i.e. less capacity is needed. Note, that for negative α the latter effect disappears as that would mean that the entrant expects to exit sooner itself and to stay for a shorter period in the monopoly. As a result, for negative α the capacity level $\widehat{Q}_L$ as a function of the discount rate, r, exhibits only increasing behavior.

3.4 Conclusion

This chapter analyzes entry and exit decisions of the firm in an existing market under uncertainty. In the presence of an incumbent the entrant launches its market operations by undertaking an investment in a certain capacity. In our model the entrant decides not only about its optimal investment threshold but also about the exit threshold and its optimal capacity level. Thus, the duopoly model with capacity optimization (Huisman and Kort (2015)) is modified to incorporate an option to exit. The entrant, while observing the existing quantity in the market installed by the incumbent, can influence the exit order by choosing capacity. This is because we show that the firm with the larger capacity exits first. Thus, in order to stay longer in the market the entrant has to choose a capacity below the incumbent's level. As a result, different strategies are available for the entrant in terms of its capacity choice. In particular, it can choose to mimic the behavior of the incumbent and install a capacity level that is just below the incumbent's capacity. When the incumbent's capacity is large enough the entrant is able to boost its capacity such that the incumbent exits immediately. The latter is crucial for the main result of this chapter. Namely, that in contrast with the basic model, the entrant has multiple investment thresholds. Now it not only has an option to enter as a duopolist but also to monopolize the market by forcing the exit of the incumbent. The first situation appears when the market is big enough for the firms to coexist. The second scenario occurs when the market is sufficiently small, so that it is relatively easy to drive the competitor out by installing a large enough capacity. However, for an intermediate market size a gap between the two strategies occurs, generating a hysteresis region. Intuitively, within this region the market is too small to coexist in a duopoly, but yet too big to make monopolization of the market profitable for the entrant. Furthermore, for negative market growth prospects the entrant's investment trigger exhibits a non-monotonicity with respect to the capacity of the incumbent. At first the investment trigger increases with the incumbent's capacity. However, once the incumbent's capacity becomes sufficiently large, it starts declining, as the entrant anticipates sooner exit of the incumbent and is eager to invest sooner.

We have derived important results for the sensitivity of the optimal investment and exit thresholds with respect to the different parameter values for the monopoly case. The monopolist exits the market later if: the economic environment is more uncertain, i.e. σ is large; the market shows better growth prospects, i.e. α is large; the firm discounts the future payoff less, i.e. r is small. The monopolist enters the market later when σ is large, while the relation between the entry timing and both α and r is non-monotonic. For small α the investment threshold declines, as the market becomes more attractive, while for large α it increases, because the firm has incentives to install a larger capacity level in order to account for the future growth. A larger r, on the one hand, implies a smaller discounted revenue stream, but on the other hand also a smaller discounted production cost stream. The latter effect dominates for the small r so that the firm will enter earlier, while the former dominates for larger r inducing the firm to postpone the investment. Furthermore, we have shown that if the exit option is present the monopolist enters the market earlier.

Lastly, it is important to indicate the possibilities for further research. This chapter is focused on the decisions of the entrant entering an existing market. However, it is also interesting to examine the case of a new market where both firms have an option to invest. In this way we include the decision of the incumbent in the analysis. In addition, the obtained results are derived for the specific case when firms produce up to capacity. This assumption can be relaxed by allowing the firms to leave some capacity idle when the demand level decreases. Moreover, different demand functions could be considered.

3.5 Appendix

Proof of Proposition 3.1 The entrant solves the following optimal stopping problem

$$V^*(x) = \sup_{\tau_1 < \tau_2, Q} \mathbb{E}_x \left[\int_{\tau_1}^{\tau_2} \mathrm{e}^{-rt} \left(x(1-\eta Q)Q - cQ \right) - \mathrm{e}^{-r\tau_1} \delta Q \right]. \quad (3.36)$$

where τ_1 and τ_2 are stopping times, corresponding to the entry and exit decisions, respectively.

As in Dixit and Pindyck (1994), in order to solve the optimal stopping problem in (3.36), we split it up into exit and entry problems. The exit problem is to find the optimal time to abandon leave the market, τ_2^*, such that

$$\mathbb{E}_x \left[\int_0^{\tau_2^*} \mathrm{e}^{-rt} \left(x(1-\eta Q)Q - cQ \right) \right] = \sup_{\tau_2} \mathbb{E}_x \left[\int_0^{\tau_2} \mathrm{e}^{-rt} \left(x(1-\eta Q)Q - cQ \right) \right]. \quad (3.37)$$

The solution of the problem is represented by the value function of the active firm is denoted by $V_1^M(x, Q)$ and the optimal exit threshold denoted by X_M^E.

The entry problem is to find the optimal time to undertake an investment, τ_1^*, and the optimal capacity level, Q_M^*, such that

$$\mathbb{E}_x \left[\mathrm{e}^{-r\tau_1^*} (V_1(x, Q_M^*) - \delta Q_M^*) \right] = \sup_{\tau_1, Q} \mathbb{E}_x \left[\mathrm{e}^{-r\tau_1} (V_1(x, Q) - \delta Q) \right]. \quad (3.38)$$

Denote the value of the idle firm by $V_0^M(x)$. Following the solution procedure in Dixit and Pindyck (1994), the value functions of the idle and the active firms take the following forms

$$V_0^M(x) = A_1 x^{\beta_1} + A_2 x^{\beta_2}, \quad (3.39)$$

$$V_1^M(x, Q) = B_1 x^{\beta_1} + B_2 x^{\beta_2} + \frac{x(1-\eta Q)Q}{r-\alpha} - \frac{cQ}{r}, \quad (3.40)$$

with A_1, A_2, B_1 and B_2 being constants, β_1, β_2 – the roots of the fundamental quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0$. To rule out the possibility of speculative bubbles it should hold that $B_1 = 0$, while $A_2 = 0$ due to the boundary condition at $x = 0$, namely, $V_0^M(0) = 0$. Thus, we can rewrite (3.39) and (3.40) as

$$V_0^M(x) = A_1 x^{\beta_1}, \quad (3.41)$$

$$V_1^M(x, Q) = B_2 x^{\beta_2} + \frac{x(1-\eta Q)Q}{r-\alpha} - \frac{cQ}{r}. \quad (3.42)$$

Consider the optimal exit threshold, X_M^E, and the optimal investment threshold, X_M^I. The following boundary conditions must hold[6]

$$\begin{cases} V_1^M(X_M^I, Q)) - \delta Q = V_0^M(X_M^I), \\ \left.\dfrac{\partial V_1^M(x,Q)}{\partial x}\right|_{x=X_M^I} + \dfrac{\partial V_1(X_M^I,Q)}{\partial Q}\left.\dfrac{\partial Q}{\partial x}\right|_{x=X_M^I} = \left.\dfrac{\partial V_0^M(x)}{\partial x}\right|_{x=X_M^I}, \\ V_1^M(X_D^E, Q) = 0, \\ \left.\dfrac{\partial V_1^M(x,Q)}{\partial x}\right|_{x=X_D^E} = 0. \end{cases} \tag{3.43}$$

Plugging in the values of the idle and active monopolist from (3.41) and (3.42) we get[7]

$$\begin{cases} -A_1 X_M^{I^{\beta_1}} + B_2 X_M^{I^{\beta_2}} + \dfrac{X_M^I(1-\eta Q)Q}{r-\alpha} - \dfrac{cQ}{r} - \delta Q = 0, \\ -\beta_1 A_1 X_M^{I^{\beta_1 - 1}} + \beta_2 B_2 X_M^{I^{\beta_2 - 1}} + \dfrac{(1-\eta Q)Q}{r-\alpha} = 0, \\ B_2 X_M^{E^{\beta_2}} + \dfrac{X_M^E(1-\eta Q)Q}{r-\alpha} - \dfrac{cQ}{r} = 0, \\ \beta_2 B_2 X_M^{E^{\beta_2 - 1}} + \dfrac{(1-\eta Q)Q}{r-\alpha} = 0. \end{cases} \tag{3.44}$$

Solving for A_1, B_2, X_M^I and X_M^E leads to[8]

$$\begin{cases} \dfrac{(\beta_1-\beta_2)c}{(1-\beta_2)\beta_1 r}\left(\dfrac{X_M^I}{X_M^E(Q)}\right)^{\beta_2} + \left(1 - \dfrac{1}{\beta_1}\right)\dfrac{X_M^I(1-\eta Q)}{r-\alpha} - \dfrac{c}{r} - \delta = 0, \\ A_1(Q) = \dfrac{\beta_2}{\beta_2 - \beta_1}\left(\dfrac{1}{X_M^I}\right)^{\beta 1}\left(\dfrac{X_M^I(1-\eta Q)Q}{r-\alpha}\left(1-\dfrac{1}{\beta_2}\right) - \left(\dfrac{c}{r}+\delta\right)Q\right), \\ X_M^E(Q) = \dfrac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-\eta Q)}, \\ B_2(Q) = \left(\dfrac{1}{X_M^E(Q)}\right)^{\beta_2}\dfrac{cQ}{r(1-\beta_2)}, \end{cases} \tag{3.45}$$

and the corresponding values of the idle and active monopolist:

$$V_0^M(x,Q) = A_1(Q)x^{\beta_1}, \tag{3.46}$$

[6]Here we refer to $Q(x)$ as Q. Thus, in the first two equations of the systems (3.43), (3.44) and (3.45) it is evaluated at $x = X_M^I$, while in the last two – at $x = X_M^E$.

[7]Given that $\frac{\partial V_1(x,Q)}{\partial Q} = 0$ for the optimal Q.

[8]The fact that $A_1(Q) > 0$ and the uniqueness of the optimal investment trigger X_M^I are verified numerically.

$$V_1^M(x,Q) = \left(\frac{x}{X_M^E(Q)}\right)^{\beta_2} \frac{cQ}{r(1-\beta_2)} + \frac{x(1-\eta Q)Q}{r-\alpha} - \frac{cQ}{r}. \quad (3.47)$$

Proof of Proposition 3.2 The monopolist maximizes the value of being active on the market:

$$\begin{aligned} \underset{Q}{\text{maximize}} \quad & V_1^M(x,Q) - \delta Q \\ \text{s.t.} \quad & Q < \widetilde{Q}_M(x). \end{aligned}$$

We now want to show that this function has a single maximum in the feasible region of the investment problem. We start by considering the first and the second order derivatives defined below:

$$\frac{\partial(V_1^M(x,Q)-\delta Q)}{\partial Q} = \left(\frac{x}{X_M^E(Q)}\right)^{\beta_2} \frac{c\,(1-\eta Q(1+\beta_2))}{r(1-\beta_2)(1-\eta Q)} + \frac{x(1-2\eta Q)}{r-\alpha} - \frac{c}{r} - \delta, \quad (3.48)$$

$$\frac{\partial^2(V_1(x,Q)-\delta Q)}{\partial Q^2} = \left(\frac{r(\beta_2-1)}{\beta_2 c}\right)^{\beta_2-1} \left(\frac{x}{r-\alpha}\right)^{\beta_2} \frac{\eta(2-(\beta_2+1)\eta Q)}{(1-\eta Q)^{2-\beta_2}} - \frac{2\eta x}{r-\alpha}. \quad (3.49)$$

In order for the firm to enter the market it should hold that $x > X_M^E(Q)$; otherwise it will immediately exit. In order for the monopolist to enter the market with positive capacity the following should hold $x > X_M^E(0) = \frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)}$. Given this and the fact that $\beta_2 < 0$ it holds that $\left.\frac{\partial^2(V_1(x,Q)-\delta Q)}{\partial Q^2}\right|_{Q=0} = \frac{2\eta x}{r-\alpha}\left(\left(\frac{r(\beta_2-1)x}{\beta_2 c(r-\alpha)}\right)^{\beta_2-1} - 1\right) < 0$. Moreover, $\lim_{Q\to\frac{1}{\eta}} \frac{\partial^2(V_1(x,Q)-\delta Q)}{\partial Q^2} = \infty$.[9] Note that the second order derivative increases with Q and has a single root, as $\frac{\partial^3(V_1(x,Q)-\delta Q)}{\partial Q^3} = \left(\frac{(\beta_2-1)rx}{\beta_2 c(r-\alpha)}\right)^{\beta_2} \frac{c\eta^2\beta_2(-3+(\beta_2+1)\eta Q)}{r(1-\eta Q)^{3-\beta_2}} > 0$. This means that the first order derivative is first declining with Q and then is increasing. Translating the condition $x > X_M^E(Q)$ in terms of the capacity level it should hold that $0 \le Q < \widetilde{Q}_M(x)$. Note that by construction $\left.\frac{\partial(V_1(x,Q)-\delta Q)}{\partial Q}\right|_{Q=\widetilde{Q}_M(x)} = -\delta < 0$. This implies that there exist two possibilities depending on the sign of $\left.\frac{\partial(V_1(x,Q)-\delta Q)}{\partial Q}\right|_{Q=0}$: either the function $V_1^M(x,Q) - \delta Q$, which takes the value of zero for zero capacity, is first increasing and then decreasing or is strictly decreasing for the considered range of Q. In the latter scenario the optimal capacity choice is 0, meaning that the firm will forgo its investment option. In the former scenario there exists a single maximum defined by the first order condition.[10]

[9] Capacity is defined such that $0 \le Q \le \frac{1}{\eta}$ so that the prices cannot be negative.

[10] The uniqueness of the solution of (3.50) is verified numerically.

The resulting optimal capacity level $Q_M^*(x)$ is determined by[11]

$$\left(\frac{x}{X_M^E(Q)}\right)^{\beta_2} \frac{c\,(1-\eta Q(1+\beta_2))}{r(1-\beta_2)(1-\eta Q)} + \frac{x(1-2\eta Q)}{r-\alpha} - \frac{c}{r} - \delta = 0. \tag{3.50}$$

The optimal capacity level at the investment threshold can be found by solving the following system:

$$\begin{cases} \dfrac{(\beta_1-\beta_2)c}{(1-\beta_2)\beta_1 r}\left(\dfrac{x}{X_M^E(Q)}\right)^{\beta_2} + \left(1-\dfrac{1}{\beta_1}\right)\dfrac{x(1-\eta Q)}{r-\alpha} - \dfrac{c}{r} - \delta = 0, \\ \left(\dfrac{x}{X_M^E(Q)}\right)^{\beta_2} \dfrac{c\,(1-\eta Q(1+\beta_2))}{r(1-\beta_2)(1-\eta Q)} + \dfrac{x(1-2\eta Q)}{r-\alpha} - \dfrac{c}{r} - \delta = 0. \end{cases} \tag{3.51}$$

Combining the equations in the above system we get

$$\frac{1-\eta Q(\beta_1+1)}{\beta_1-\beta_2}\left(\frac{(1-\beta_2)x}{r-\alpha} + \frac{\beta_2}{1-\eta Q}\left(\frac{c}{r}+\delta\right)\right) = 0. \tag{3.52}$$

From (3.52) it follows that either $Q_M^* = \dfrac{1}{\eta(\beta_1+1)}$ or the expression in the parentheses is equal to zero, i.e. $X_M^I(Q) = \dfrac{\beta_2(r-\alpha)}{(\beta_2-1)(1-\eta Q)}\left(\dfrac{c}{r}+\delta\right)$. Plugging the latter back into (3.51) and solving for Q gives $Q^* = \dfrac{1}{\eta(\beta_2+1)}$. If $\beta_2 < -1$ this gives a negative capacity, whereas the complementary case when $\beta_2 > -1$, leads to the negative prices, $P(x) = \dfrac{x\beta_2}{\beta_2+1} < 0$. Therefore, we conclude that

$$Q_M^* = \frac{1}{\eta(\beta_1+1)}. \tag{3.53}$$

The corresponding X_M^{I*} is implicitly defined by[12]

$$-\left(\frac{\beta_1(\beta_2-1)rx}{(\beta_1+1)\beta_2(r-\alpha)c}\right)^{\beta_2}\frac{(\beta_1-\beta_2)c}{\beta_1(\beta_2-1)r} + \frac{(\beta_1-1)x}{(\beta_1+1)(r-\alpha)} - \left(\frac{c}{r}+\delta\right) = 0. \tag{3.54}$$

The optimal exit threshold X_M^{E*} is given by

$$X_M^{E*} = \frac{\beta_2(\beta_1+1)(r-\alpha)c}{\beta_1(\beta_2-1)r}. \tag{3.55}$$

[11] Given that the second order condition for a maximum, $\frac{\partial^2(V_1(x,Q)-\delta Q)}{\partial Q^2}\Big|_{Q=Q_M^*(x)} < 0$, is satisfied. The first order condition (3.50) also ensures that the smooth pasting condition is correctly specified as $\frac{\partial V_1(X_M^I,Q)}{\partial Q} = 0$ for the optimal capacity level Q.

[12] We choose the root such that $x > \frac{(\beta_1+1)\beta_2(r-\alpha)}{\beta_1(\beta_2-1)}\left(\frac{c}{r}+\delta\right)$, otherwise $A_1(Q)$ is negative.

Proof of Proposition 3.3 In what follows we obtain the explicit expressions for the derivatives of the optimal exit threshold with respect to different parameter values. We first consider the exit threshold for the fixed capacity level, X_M^E, then we look at the optimal exit threshold, $X_M^{E^*}$, i.e. the exit threshold for the optimal capacity level.

1. **Sensitivity with respect to the drift, $\boldsymbol{\alpha}$.**

The derivative of X_M^E, the exit threshold for the fixed capacity level, with respect to α is given by the following expression

$$\frac{\partial X_M^E}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-\eta Q)}\right) = -\frac{c}{r(1-\eta Q)}\left(\frac{\frac{\partial \beta_2}{\partial \alpha}(r-\alpha)+\beta_2(\beta_2-1)}{(\beta_2-1)^2}\right). \tag{3.56}$$

Note first that $\frac{\partial \beta_2}{\partial \alpha} = \frac{\beta_2}{\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}$ and $\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2} = -\beta_2\sigma^2-\left(\alpha-\frac{\sigma^2}{2}\right)$. Plugging this back yields

$$\begin{aligned}
\frac{\partial X_M^E}{\partial \alpha} &= \frac{c\beta_2}{r(1-\eta Q)}\left(\frac{-(r-\alpha)-(\beta_2-1)\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= \frac{c\beta_2}{r(1-\eta Q)}\left(\frac{-(r-\alpha)-(\beta_2-1)\left(-\beta_2\sigma^2-\left(\alpha-\frac{\sigma^2}{2}\right)\right)}{(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= \frac{c\beta_2}{r(1-\eta Q)}\left(\frac{\frac{1}{2}\sigma^2\beta_2^2+\left(\alpha-\frac{\sigma^2}{2}\right)\beta_2-r+\frac{\sigma^2}{2}\left(\beta_2^2-2\beta_2+1\right)}{(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= \frac{c\beta_2\sigma^2}{2r(1-\eta Q)\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} < 0.
\end{aligned} \tag{3.57}$$

The derivative of the optimal capacity level with respect to α is given by

$$\frac{\partial Q^*}{\partial \alpha} = \frac{\beta_1}{(\beta_1+1)^2\eta\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} > 0. \tag{3.58}$$

Consider now the optimal exit threshold for the optimal capacity level $Q^* = \frac{1}{\eta(\beta_1+1)}$. Applying the multiplication rule we can write the following

$$\frac{\partial X_M^{E^*}}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\frac{\beta_2(r-\alpha)c}{r(\beta_2-1)(1-\eta Q^*)}\right)$$

$$= \left(\frac{1}{1-\eta Q^*}\right)\frac{\partial}{\partial\alpha}\left(\frac{\beta_2(r-\alpha)c}{r(\beta_2-1)}\right) + \frac{c\beta_2(r-\alpha)}{r(\beta_2-1)}\frac{\partial}{\partial\alpha}\left(\frac{1}{1-\eta Q^*}\right)$$
$$= \frac{c(\beta_1+1)\beta_2\sigma^2}{2r\beta_1\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} + \frac{1}{\beta_1\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\frac{c\beta_2(r-\alpha)}{r(\beta_2-1)}$$
$$= \frac{c\beta_2}{r\beta_1\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\left(\frac{(\beta_1+1)\sigma^2}{2}+\frac{(r-\alpha)}{(\beta_2-1)}\right) \tag{3.59}$$

Here the expression in the brackets can be simplifies as follows

$$\frac{(\beta_1+1)\sigma^2}{2}+\frac{r-\alpha}{\beta_2-1} = \frac{r+\frac{\sigma^2}{2}+\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}-(r-\alpha)}{\frac{\alpha+\frac{\sigma^2}{2}+\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}} = \sigma^2. \tag{3.60}$$

As a result, it holds that

$$\frac{\partial X_M^{E^*}}{\partial\alpha} = \frac{c\beta_2\sigma^2}{r\beta_1\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} < 0. \tag{3.61}$$

2. **Sensitivity with respect to the uncertainty, $\boldsymbol{\sigma}$.**

The derivative of the exit trigger for a given capacity level with respect to σ is negative:

$$\frac{\partial X_M^E}{\partial\sigma} = \frac{\partial X_M^E}{\partial\beta_2}\frac{\partial\beta_2}{\partial\sigma} < 0, \tag{3.62}$$

because $\frac{\partial\beta_2}{\partial\sigma} = \frac{2(r-\alpha\beta_2)}{\sigma\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} > 0$, and $\frac{\partial X_M^E}{\partial\beta_2} = -\frac{c(r-\alpha)}{r(\beta_2-1)^2(1-\eta Q)} < 0$.

Note that

$$\frac{\partial\beta_1}{\partial\sigma} = -\frac{2\left(r-\alpha\beta_1\right)}{\sigma\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} < 0, \tag{3.63}$$

as

$$r-\alpha\beta_1 = r - \frac{-\alpha\left(\alpha-\frac{\sigma^2}{2}\right)+\alpha\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2}$$
$$= \frac{\sqrt{\left(\alpha\left(\alpha-\frac{\sigma^2}{2}\right)+r\sigma^2\right)^2}-\sqrt{\left(\alpha\left(\alpha-\frac{\sigma^2}{2}\right)\right)^2+2\alpha^2r\sigma^2}}{\sigma^2}$$
$$= \frac{\sqrt{\alpha\left(\alpha-\frac{\sigma^2}{2}\right)^2+2\alpha^2r\sigma^2+\sigma^4r(r-\alpha)}-\sqrt{\left(\alpha\left(\alpha-\frac{\sigma^2}{2}\right)\right)^2+2\alpha^2r\sigma^2}}{\sigma^2} > 0. \tag{3.64}$$

Hence,

$$\frac{\partial Q^*}{\partial \sigma} = \frac{\partial Q^*}{\partial \beta_1}\frac{\partial \beta_1}{\partial \sigma} = \frac{2\left(r-\alpha\beta_1\right)}{\eta\sigma(\beta_1+1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} > 0. \tag{3.65}$$

Taking into account the above observations we can derive the expression for the optimal exit threshold. Applying the multiplication rule we get

$$\begin{aligned}
\frac{\partial X_M^{E*}}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left(\frac{\beta_2(r-\alpha)c}{r(\beta_2-1)(1-\eta Q^*)}\right)\\
&= \left(\frac{1}{1-\eta Q^*}\right)\frac{\partial}{\partial \sigma}\left(\frac{\beta_2(r-\alpha)c}{r(\beta_2-1)(1-\eta Q^*)}\right) + \frac{c\beta_2(r-\alpha)}{r(\beta_2-1)}\frac{\partial}{\partial \sigma}\left(\frac{1}{1-\eta Q^*}\right)\\
&= -\frac{2c(r-\alpha)(\beta_1+1)(r-\alpha\beta_2)}{r(\beta_2-1)^2\beta_1\sigma\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} + \frac{2\left(r-\alpha\beta_1\right)}{\sigma\beta_1^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\frac{c\beta_2(r-\alpha)}{r(\beta_2-1)}\\
&= \frac{2c(r-\alpha)}{r(\beta_2-1)\beta_1\sigma\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\left(-\frac{(\beta_1+1)(r-\alpha\beta_2)}{(\beta_2-1)} + \frac{\left(r-\alpha\beta_1\right)\beta_2}{\beta_1}\right)
\end{aligned} \tag{3.66}$$

The last expression can be simplified using the fact that $\beta_1\beta_2 = -\frac{2r}{\sigma^2}$ and $\beta_1+\beta_2 = 1-\frac{2\alpha}{\sigma^2}$ as the roots of the fundamental quadratic equation:

$$\begin{aligned}
-&\frac{(\beta_1+1)(r-\alpha\beta_2)}{(\beta_2-1)} + \frac{\left(r-\alpha\beta_1\right)\beta_2}{\beta_1}\\
&= \frac{2r}{\beta_1}\left(\frac{\beta_2 r-\alpha\beta_1\beta_2}{2r} - \frac{(\beta_1+1)(\beta_1 r-\alpha\beta_1\beta_2)}{2(\beta_2-1)r}\right)\\
&= \frac{2r}{\beta_1}\left(\frac{1}{2}\left(\frac{2\alpha}{\sigma^2}+\beta_2\right) - \frac{(\beta_1+1)\left(\frac{2\alpha}{\sigma^2}+\beta_1\right)}{2(\beta_2-1)}\right)\\
&= \frac{2r}{\beta_1}\frac{\left(-\frac{2\alpha\beta_1}{\sigma^2}+\frac{2\alpha\beta_2}{\sigma^2}-\frac{2\alpha}{\sigma^2}-\frac{2\alpha}{\sigma^2}-\beta_1^2-\beta_1+\beta_2^2-\beta_2\right)}{2(\beta_2-1)}\\
&= \frac{2r}{\beta_1}\frac{\left((\beta_2-\beta_1)\left(\frac{2\alpha}{\sigma^2}+\beta_1+\beta_2\right)-\frac{4\alpha}{\sigma^2}-(\beta_1+\beta_2)\right)}{\beta_1 2(\beta_2-1)}\\
&= \frac{2r}{\beta_1}\frac{\left(\frac{2\alpha}{\sigma^2}-\frac{2\alpha}{\sigma^2}+\beta_2+\beta_2-1-1\right)}{2(\beta_2-1)}\\
&= \frac{2r}{\beta_1}.
\end{aligned} \tag{3.67}$$

Therefore, the optimal exit trigger declines with uncertainty.

$$\frac{\partial X_M^{E*}}{\partial \sigma} = \frac{4c(r-\alpha)}{r(\beta_2-1)\beta_1^2\sigma\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} < 0. \tag{3.68}$$

3. Sensitivity with respect to the discount rate, r.

To determine the sign of the derivative with respect to the discount rate we use the following observations. Fisst, that $\beta_1\beta_2 = -\frac{2r}{\sigma^2}$, second, $\beta_1 + \beta_2 = 1 - \frac{2\alpha}{\sigma^2}$, third, $\sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2} = -\beta_2\sigma^2 - \left(\alpha - \frac{\sigma^2}{2}\right)$. Lastly we also use the following expression $\frac{\partial\beta_2}{\partial r} = -\frac{1}{\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}$. As a result, for the exit threshold for the fixed capacity level we obtain

$$\begin{aligned}
\frac{\partial X_M^E}{\partial r} &= \frac{c}{r(1-\eta Q)}\left(\frac{\beta_2(\beta_2-1)\frac{\alpha}{r} - \frac{\partial\beta_2}{\partial r}(r-\alpha)}{(\beta_2-1)^2}\right)\\
&= \frac{c}{r(1-\eta Q)}\left(\frac{\frac{\alpha}{r}(\beta_2-1)\beta_2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}+(r-\alpha)}{(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= \frac{c}{r(1-\eta Q)}\left(\frac{\frac{\alpha}{r}\left((\beta_2-1)\beta_2\left(-\left(\alpha-\frac{\sigma^2}{2}\right)-\beta_2\sigma^2\right)+\frac{r^2}{\alpha}-r\right)}{(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= \frac{c}{r(1-\eta Q)}\left(\frac{\frac{\alpha}{r}\left(\beta_2^2\left(-\alpha-\beta_2\sigma^2\right)+\beta_2^2\sigma^2+\frac{r^2}{\alpha}+\frac{1}{2}\sigma^2\beta_2^2+\left(\alpha-\frac{\sigma^2}{2}\right)\beta_2-r\right)}{(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= \frac{c}{r(1-\eta Q)}\left(-\frac{\beta_2\sigma^2\left(\frac{2\alpha^2\beta_2}{r\sigma^2}+\frac{2\alpha\beta_2^2}{r}-\frac{2\alpha\beta_2}{r}-\frac{2r}{\beta_2\sigma^2}\right)}{2(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= \frac{c}{r(1-\eta Q)}\left(-\frac{\beta_2\sigma^2\left(-\frac{4\alpha^2}{\sigma^4}-\frac{4\alpha\beta_2}{\sigma^2}+\frac{4\alpha}{\sigma^2}+\beta_1^2\right)}{2\beta_1(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= \frac{c}{r(1-\eta Q)}\left(-\frac{\beta_2\sigma^2\left(\left(-\frac{2\alpha}{\sigma^2}-\beta_2+1\right)^2-\left(-\frac{2\alpha}{\sigma^2}-\beta_2+1\right)^2+(\beta_2-1)^2\right)}{2\beta_1(\beta_2-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)\\
&= -\frac{\beta_2 c\sigma^2}{2\beta_1 r(1-\eta Q)\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} > 0. \qquad (3.69)
\end{aligned}$$

Now, for the optimal capacity level it holds that

$$\frac{\partial Q^*}{\partial r} = -\frac{1}{(\beta_1+1)^2\eta\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} < 0. \qquad (3.70)$$

Taking into account (3.60) and (3.70) and applying the multiplication rule, it can be shown that the optimal exit threshold increases with r:

$$\begin{aligned}
\frac{\partial X_M^{E*}}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{\beta_2(r-\alpha)c}{r(\beta_2-1)(1-\eta Q^*)}\right) \\
&= \left(\frac{1}{1-\eta Q^*}\right)\frac{\partial}{\partial r}\left(\frac{\beta_2(r-\alpha)c}{r(\beta_2-1)(1-\eta Q^*)}\right) + \frac{c\beta_2(r-\alpha)}{r(\beta_2-1)}\frac{\partial}{\partial r}\left(\frac{1}{1-\eta Q^*}\right) \\
&= -\frac{\beta_2(\beta_1+1)c\sigma^2}{2\beta_1^2 r\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} - \frac{1}{\beta_1^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\frac{c\beta_2(r-\alpha)}{r(\beta_2-1)} \\
&= -\frac{\beta_2 c\sigma^2}{\beta_1^2 r\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} > 0.
\end{aligned} \tag{3.71}$$

Proof of Proposition 3.4 Denote the investment threshold in the model without exit option by $X_{M,0}^I$. As it follows from the analysis in Huisman and Kort (2015), the optimal investment threshold for the fixed capacity $X_{M,0}^I$ is given by

$$X_{M,0}^I = \frac{\beta_1\left(\frac{c}{r}+\delta\right)(r-\alpha)}{(\beta_1-1)(1-\eta Q)}, \tag{3.72}$$

while the substitution of the optimal capacity level $Q^* = \frac{1}{\eta(\beta+1)}$ gives

$$X_{M,0}^{I*} = \frac{(\beta_1+1)\left(\frac{c}{r}+\delta\right)(r-\alpha)}{(\beta_1-1)}. \tag{3.73}$$

Note that in the model with exit option the optimal investment threshold X_M^{I*} is implicitly defined by:

$$-\left(\frac{\beta_1(\beta_2-1)rx}{(\beta_1+1)\beta_2(r-\alpha)c}\right)^{\beta_2}\frac{(\beta_1-\beta_2)c}{\beta_1(\beta_2-1)r} + \frac{(\beta_1-1)x}{(\beta_1+1)(r-\alpha)} - \left(\frac{c}{r}+\delta\right) = 0, \tag{3.74}$$

whereas $X_{M,0}^{I*}$ solves the equation which contains the latter two terms of (3.74), namely:

$$\frac{(\beta_1-1)x}{(\beta_1+1)(r-\alpha)} - \left(\frac{c}{r}+\delta\right) = 0. \tag{3.75}$$

Given that the first term of (3.74) is positive it holds that

$$X_M^{I*} < X_{M,0}^{I*}. \tag{3.76}$$

Now we will show how a change in α influences the investment threshold. As pointed out in the main text, we observe the non-monotonic behavior of the derivative of the entry threshold with respect to α. In fact, the reason behind that is that we incorporate capacity optimization into the model. In order to demonstrate this we first consider the sign of this derivative when the capacity level is fixed. Note that $\frac{\partial \beta_1}{\partial \alpha} = -\frac{\beta_1}{\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}$ and $\sqrt{\left(\alpha - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2} = \beta_1\sigma^2 + \left(\alpha - \frac{\sigma^2}{2}\right)$.

$$
\begin{aligned}
\frac{\partial X^I_{M,0}}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left(\frac{\beta_1\left(\frac{c}{r}+\delta\right)(r-\alpha)}{(\beta_1-1)(1-\eta Q)}\right) = -\frac{\frac{c}{r}+\delta}{1-\eta Q}\left(\frac{\beta_1}{\beta_1-1}+\frac{(r-\alpha)\frac{\partial\beta_1}{\partial\alpha}}{(\beta_1-1)^2}\right) \\
&= -\frac{\frac{c}{r}+\delta}{1-\eta Q}\left(\frac{\beta_1}{\beta_1-1}+\frac{(r-\alpha)\frac{-\beta_1}{\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}}{(\beta_1-1)^2}\right) \\
&= -\frac{\left(\frac{c}{r}+\delta\right)\beta_1}{(1-\eta Q)}\left(\frac{(\beta_1-1)\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}-(r-\alpha)}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right) \\
&= -\frac{\left(\frac{c}{r}+\delta\right)\beta_1}{(1-\eta Q)}\left(\frac{(\beta_1-1)\left(\beta_1\sigma^2+\left(\alpha-\frac{\sigma^2}{2}\right)\right)-(r-\alpha)}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right) \\
&= -\frac{\left(\frac{c}{r}+\delta\right)\beta_1}{(1-\eta Q)}\left(\frac{\frac{1}{2}\sigma^2\beta_1^2+\left(\alpha-\frac{\sigma^2}{2}\right)\beta_1-r+\frac{1}{2}(\beta_1-1)^2\sigma^2}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right) \\
&= -\frac{\left(\frac{c}{r}+\delta\right)\beta_1\sigma^2}{2(1-\eta Q)\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} < 0. \qquad (3.77)
\end{aligned}
$$

However, once we add capacity optimization the derivative is no longer monotonic in α

$$
\begin{aligned}
\frac{\partial X^I_{M,0}}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left(\frac{(\beta_1+1)\left(\frac{c}{r}+\delta\right)(r-\alpha)}{(\beta_1-1)}\right) = \left(\frac{c}{r}+\delta\right)\left(-\frac{\beta_1+1}{\beta_1-1}-\frac{2(r-\alpha)\frac{\partial\beta_1}{\partial\alpha}}{(\beta_1-1)^2}\right) \\
&= \left(\frac{c}{r}+\delta\right)\left(\frac{-(\beta_1^2-1)\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}+2\beta_1(r-\alpha)}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right) \\
&= \left(\frac{c}{r}+\delta\right)\left(\frac{-(\beta_1^2-1)\left(\beta_1\sigma^2+\left(\alpha-\frac{\sigma^2}{2}\right)\right)+2\beta_1(r-\alpha)}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right) \\
&= \left(\frac{c}{r}+\delta\right)\left(\frac{2\beta_1\left(\frac{1}{2}\sigma^2\beta_1^2+\beta_1\left(\alpha-\frac{\sigma^2}{2}\right)-r\right)-(\beta_1-1)^2\left(\alpha-\frac{\sigma^2}{2}\right)}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right)
\end{aligned}
$$

$$= \quad -\frac{\left(\frac{c}{r}+\delta\right)\left(\alpha-\frac{\sigma^2}{2}\right)}{\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}. \tag{3.78}$$

The expression above is negative when $\alpha < \frac{\sigma^2}{2}$, and when $\alpha > \frac{\sigma^2}{2}$ it is positive.

Similarly, we observe a non-monotonic behavior of the entry threshold when a change in r is considered. However, in this case the non-monotonicity comes from the fact that we add production costs that are discounted each period. The derivative of the entry threshold with respect to r is

$$\begin{aligned}
\frac{\partial X^I_{M,0}}{\partial r} =& \frac{\partial}{\partial r}\left(\frac{(\beta_1+1)\left(\frac{c}{r}+\delta\right)(r-\alpha)}{(\beta_1-1)}\right) \\
=& \frac{\beta_1+1}{\beta_1-1}\left(-\frac{c(r-\alpha)}{r^2}+\frac{c}{r}+\delta\right)-\left(\frac{c}{r}+\delta\right)\frac{2(r-\alpha)}{(\beta_1-1)^2}\frac{\partial\beta_1}{\partial r} \\
=& \left(\frac{\alpha c}{r^2}+\delta\right)\left(\frac{\beta_1+1}{\beta_1-1}-\frac{2r(r-\alpha)\left(\frac{c+\delta r}{\alpha c+\delta r^2}\right)}{\left(\beta_1^2-1\right)\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right) \\
=& \left(\frac{\alpha c}{r^2}+\delta\right)\left(\left(\beta_1^2-1\right)-\frac{2(r-\alpha)-2(r-\alpha)+2r(r-\alpha)\left(\frac{c+\delta r}{\alpha c+\delta r^2}\right)}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right) \\
=& \left(\frac{\alpha c}{r^2}+\delta\right)\left(\frac{\left(\beta_1^2-1\right)\left(\beta_1\sigma^2+\alpha-\frac{\sigma^2}{2}\right)-2(r-\alpha)}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}+\frac{2(r-\alpha)\left(1-r\left(\frac{c+\delta r}{\alpha c+\delta r^2}\right)\right)}{\left(\frac{\beta_1+1}{\beta_1-1}\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right.}\right) \\
=& \left(\frac{\alpha c}{r^2}+\delta\right)\left(1+\frac{\sigma^2}{\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}-\frac{2(r-\alpha)^2\frac{c}{\alpha c+\delta r^2}}{(\beta_1-1)^2\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right).
\end{aligned} \tag{3.79}$$

Evidently, when $c = 0$ the last term in the brackets disappears and the derivative is always positive for the considered set of parameter values:

$$\frac{\partial X^I_{M,0}}{\partial r} = \delta\left(1+\frac{\sigma^2}{\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}\right) > 0. \tag{3.80}$$

Lastly, consider the sensitivity of the entry threshold with respect to uncertainty. Taking into account that $\frac{\partial\beta_1}{\partial\sigma}$ is given by (3.63), we have

$$\frac{\partial X^I_{M,0}}{\partial\sigma} = \frac{\partial X^I_{M,0}}{\partial\beta_1}\frac{\partial\beta_1}{\partial\sigma} = \frac{4(r-\alpha)\left(\frac{c}{r}+\delta\right)(r-\alpha\beta_1)}{(\beta_1-1)^2\sigma\sqrt{\left(\alpha-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}} > 0. \tag{3.81}$$

Proof of Proposition 3.5 Similarly to the monopoly case, the value of the firm i, which constitutes a duopoly with firm j and possesses an exit option is given by

$$V_1^D(x, Q_i, Q_j) = B_i x^{\beta_2} + \frac{x(1-\eta(Q_i+Q_j))Q_i}{r-\alpha} - \frac{cQ_i}{r}. \tag{3.82}$$

Applying the boundary conditions

$$\begin{cases} V_1^D(X_D^E, Q_i, Q_j) = 0, \\ \left.\dfrac{\partial V_1^D(x, Q_i, Q_j)}{\partial x}\right|_{x=X_D^E} = 0, \end{cases} \tag{3.83}$$

we obtain the following duopoly exit threshold

$$X_D^E(Q_i, Q_j) = \frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-\eta(Q_i+Q_j))}. \tag{3.84}$$

As can be seen the capacity levels of the firms i and j enter the expression for the exit threshold only as a sum. Thus, we can conclude that both firms have the same duopoly exit threshold. Yet the monopoly thresholds are different if $Q_i \neq Q_j$. This can be seen from the expression for the monopoly threshold (3.85), which is derived using (3.45):

$$X_M^E(Q_i) = \frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-\eta Q_i)}. \tag{3.85}$$

Moreover, if $Q_i > Q_j$, then $X_M^E(Q_i) > X_M^E(Q_j)$ and visa versa. It is now easy to show that if $Q_i > Q_j$ then firm i will always exit first at the duopoly threshold. This scenario is indeed an equilibrium, as if j exits at the duopoly threshold, $X_D^E(Q_i, Q_j)$, it is optimal for firm i to leave once x hits $X_M^E(Q_i)$. The opposite scenario, however, is not an equilibrium. This is because if j exits at its monopoly threshold, $X_M^E(Q_j)$, there is still the region where firm i prefers to stay on the market and get monopoly profits, namely, when $X \in (X_M^E(Q_i), X_M^E(Q_j)]$. Hence, the firm with a larger capacity exits the market first. Applying this result for the incumbent-entrant setting we obtain (3.14) and (3.15). Note that when the firms are of the same size, $Q_L = Q_F$, both strategies are equilibrium strategies, and as a result, in a symmetric game it is unclear which firm exits first when we consider pure strategies.

Proof of Proposition 3.6 As it follows from Proposition 5 the value of the active entrant is

$$V_1(x, Q_F, Q_L) = B_F x^{\beta_2} + \frac{x(1-\eta(Q_F+Q_L))Q_F}{r-\alpha} - \frac{cQ_F}{r}, \tag{3.86}$$

where B_F differs depending on the exit order. If the entrant is a larger firm, it will leave the market once x hits $X_D^E(Q_L, Q_F)$, which is the optimal exit threshold for a large firm in a duopoly. Thus, it must hold that

$$V_1(X_D^E(Q_L, Q_F), Q_F) = 0, \tag{3.87}$$

which after plugging in (3.86) and solving for B_F yields

$$B_F^B(Q_L, Q_F) = \left(\frac{1}{X_D^E(Q_L, Q_F)}\right)^{\beta_2} \frac{cQ_F}{r(1-\beta_2)}. \tag{3.88}$$

In the complementary case, when the entrant is the smaller firm, it becomes a monopolist as x hits $X_D^E(Q_L, Q_F)$, and then exits at $X_M^E(Q_F)$ in case x declines further. Hence, the following condition must be satisfied

$$V_1(X_D^E(Q_L, Q_F), Q_F) = V_1^M(X_D^E(Q_L, Q_F), Q_F), \tag{3.89}$$

where $V_1^M(x, Q)$ is given by (3.47). This gives the following expression for B_F

$$B_F^S(Q_L, Q_F) = \left(\frac{1}{X_M^E(Q_F)}\right)^{\beta_2} \frac{cQ_F}{r(1-\beta_2)} + \left(\frac{1}{X_D^E(Q_L, Q_F)}\right)^{\beta_2} \frac{X_D^E(Q_L, Q_F)\eta Q_L Q_F}{r-\alpha}. \tag{3.90}$$

Let λ be the probability that the entrant exits first in a symmetric game, then

$$\begin{aligned} B_F(Q_L, Q_F) &= B_F^B(Q_L, Q_F)\mathbb{1}_{\{Q_F > Q_L\}} + B_F^S(Q_L, Q_F)\mathbb{1}_{\{Q_F < Q_L\}} \\ &+ (\lambda B_F^B(Q_L, Q_F) + (1-\lambda)B_F^S(Q_L, Q_F))\mathbb{1}_{\{Q_F = Q_L\}}. \end{aligned} \tag{3.91}$$

To obtain the value of the large entrant we plug in $B_F^B(Q_L, Q_F)$ from (3.88) into (3.86), for the value of the small follower – and $B_F^S(Q_L, Q_F)$ from (3.90), which gives

$$V_1^B(x, Q_L, Q_F) = \left(\frac{x}{X_D^E(Q_L, Q_F)}\right)^{\beta_2} \frac{cQ_F}{r(1-\beta_2)} + \frac{x(1-\eta(Q_L+Q_F))Q_F}{r-\alpha} - \frac{cQ_F}{r}, \tag{3.92}$$

$$\begin{aligned} V_1^S(x, Q_L, Q_F) = \left(\frac{x}{X_M^E(Q_F)}\right)^{\beta_2} \frac{cQ_F}{r(1-\beta_2)} + &\left(\frac{x}{X_D^E(Q_L, Q_F)}\right)^{\beta_2} \frac{X_D^E(Q_L, Q_F)\eta Q_L Q_F}{r-\alpha} \\ + &\frac{XQ_F(1-\eta(Q_L+Q_F))}{r-\alpha} - \frac{cQ_F}{r}. \end{aligned} \tag{3.93}$$

The value of the symmetric game, when the incumbent and the entrant are of the same size, depends on the probability that either of the firms will leave the market first. For the entrant in this case it is a weighted average of the two scenarios,

V_1^B and V_1^S, with respect to the probability of leaving the market first given that the incumbent has already invested. This yields the value for the active entrant determined by

$$V_1(x, Q_L, Q_F) = \begin{cases} V_1^B(x, Q_L, Q_F) & \text{if } Q_F > Q_L, \\ V_1^S(x, Q_L, Q_F) & \text{if } Q_F < Q_L, \\ \lambda V_1^B(x, Q_L, Q_F) + (1-\lambda) V_1^S(x, Q_L, Q_F) & \text{if } Q_F = Q_L, \end{cases} \quad (3.94)$$

which is equivalent to

$$V_1(x, Q_L, Q_F) = V_1^B \mathbb{1}_{\{Q_F > Q_L\}} + (\lambda V_1^B + (1-\lambda) V_1^S) \mathbb{1}_{\{Q_F = Q_L\}} + V_1^S \mathbb{1}_{\{Q_F < Q_L\}}. \quad (3.95)$$

Proof of Proposition 3.7 Firstly, it is possible to show that if $x \to \infty$, then $V_1^S(x, Q_L, Q_F) = V_1^B(x, Q_L, Q_F)$. For $x < \infty$ it holds that

$$V_1^S(x, Q_L, Q_F) - V_1^B(x, Q_L, Q_F) = \frac{cQ_F}{r(1-\beta_2)} \left(\frac{x}{X_D^E(Q_L, Q_F)} \right)^{\beta_2} \left[\left(\frac{1-\eta Q_F}{1-\eta(Q_L+Q_F)} \right)^{\beta_2} - 1 - \frac{\eta Q_L \beta_2}{1-\eta(Q_L+Q_F)} \right]. \quad (3.96)$$

For $\beta_2 < 0$ and $Q_F > 0$, the first two multipliers in the difference are positive, thus, it has the same sign as the expression in the square brackets, which we denote by $g(Q_L)$. Note first that $g(0) = 0$. Moreover, taking a derivative with respect to Q_L gives

$$\frac{\partial g(Q_L)}{\partial Q_L} = \frac{-\beta_2 \eta (1-\eta Q_F)}{(1-\eta(Q_F+Q_L))^2} \left(1 - \left(\frac{1-\eta Q_F}{1-\eta(Q_F+Q_L)} \right)^{\beta_2 - 1} \right) > 0. \quad (3.97)$$

The derivative above is positive, because $\beta_2 - 1 < -1$ and $\frac{1-\eta Q_F}{1-\eta(Q_L+Q_F)} > 1$. Given (3.97), the value of the symmetric game, being a weighted average of $V_1^B(x, Q_L, Q_L)$ and $V_1^S(x, Q_L, Q_L)$, is always smaller or equal than the value of the small firm. Thus, for some positive probability to exit first in a symmetric game, λ, it is always possible to find an ε small enough to ensure that installing a capacity $Q_L - \varepsilon$ and, hence, becoming a small firm, brings a larger value.

Case 1: Big follower. Analogous to the monopoly case the follower value $V_1^B(x, Q_L, Q_F)$ for $Q_F < \widetilde{Q}_F(x, Q_L)$[13] can be proved to have a single maximum[14], which is defined by the following first order condition

$$\frac{\partial (V_1^B(x, Q_L, Q_F) - \delta Q_F)}{\partial Q_F} = 0, \quad (3.98)$$

[13] The case $Q_F \geq \widetilde{Q}_F(x, Q_L)$ is not relevant for $V_1^B(x, Q_L, Q_F)$, because then the large firm exits the market and its value is equal to 0.

[14] The proof is completely analogous to the proof in Proposition 3.2 for the monopoly case.

or rewritten

$$\left(\frac{x}{X_D^E(Q_L,Q_F)}\right)^{\beta_2}\frac{c}{r(1-\beta_2)}\left(1-\frac{\beta_2\eta Q_F}{1-\eta(Q_F+Q_L)}\right)+\frac{x(1-\eta Q_L-2\eta Q_F)}{r-\alpha}-\frac{c}{r}-\delta=0. \tag{3.99}$$

We denote the capacity level corresponding to (3.99) by Q_F^B.

Case 2: Small follower. The value of the small follower, $V_1^S(x,Q_L,Q_F)-\delta Q_F$, is no longer a unimodal function of its capacity choice. Its shape and, as a result, the location of the maximum may change depending on the parameter values. In order to describe the behavior of this function we need to determine the signs of its first and second order derivatives with respect to Q_F defined below:

$$\begin{aligned}\frac{\partial(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{\partial Q_F}&=\left(\frac{x}{X_M^E(Q_F)}\right)^{\beta_2}\frac{c(1-\eta Q_F(\beta_2+1))}{r(1-\beta_2)(1-\eta Q_F)}+\frac{X\,(1-\eta Q_L-2\eta Q_F)}{r-\alpha}\\&-\left(\frac{x}{X_D^E(Q_L,Q_F)}\right)^{\beta_2}\frac{c\beta_2\eta Q_L(1-\eta(\beta_2 Q_F+Q_L))}{r(1-\beta_2)(1-\eta(Q_F+Q_L))^2}-\frac{c}{r}-\delta,\end{aligned} \tag{3.100}$$

$$\begin{aligned}\frac{\partial^2(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{(\partial Q_F)^2}&=\left(\frac{x}{X_M^E(Q_F)}\right)^{\beta_2}\frac{\beta_2 c\eta(2-\eta Q_F(\beta_2+1))}{r(\beta_2-1)(1-\eta Q_F)^2}-\frac{2\eta x}{r-\alpha}\\&-\left(\frac{x}{X_D^E(Q_L,Q_F)}\right)^{\beta_2}\frac{\beta_2 c\eta^2 Q_L(2-2\eta Q_L-\beta_2 Q_F)}{r(1-\eta(Q_F+Q_L))^3}.\end{aligned} \tag{3.101}$$

The sign of the expressions above cannot be uniquely determined. However, it is possible to describe the behavior of these two functions for the different parameter values. First, we can show that (3.101) is a strictly increasing function of Q_F. Consider the derivative $\frac{\partial^3(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{(\partial Q_F)^3}$ given by expression (3.102). It is always positive for $x>0$ and $Q_L>0$, because $\beta_2<0$ and the price is non-negative, $1-\eta(Q_L+Q_F)\geq 0$:

$$\frac{c\beta_2\eta^2}{r}\left(\frac{x}{X_M^E(Q_F)}\right)^{\beta_2}\left(\frac{(\beta_2+1)\eta Q_L-3}{(1-\eta Q_L)^3}+\frac{(2-\beta_2)\eta Q_L(\beta_2\eta Q_L-3(1-\eta Q_L))}{(1-\eta(Q_L+Q_F))^{4-\beta_2}(1-\eta Q_L)^{\beta_2}}\right)\geq 0. \tag{3.102}$$

Moreover, rewriting (3.101) as

$$\begin{aligned}&\left(\frac{(\beta_2-1)rx}{\beta_2 c(r-\alpha)}\right)^{\beta_2}\frac{\beta_2 c\eta}{(\beta_2-1)r}\left(\frac{2-(\beta_2+1)\eta Q_F}{(1-\eta Q_F)^{2-\beta_2}}-\frac{(\beta_2-1)\eta Q_L(2-\beta_2\eta Q_F-2\eta Q_L)}{1-\eta Q_F-\eta Q_L)^{3-\beta_2}}\right)\\&-\frac{2\eta x}{r-\alpha},\end{aligned} \tag{3.103}$$

it can be seen that $\lim\limits_{Q_F\to-\infty}\frac{\partial^2(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{(\partial Q_F)^2}=-\infty$ and $\lim\limits_{Q_F\to\frac{1}{\eta}}\frac{\partial^2(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{(\partial Q_F)^2}=\infty$. This means the second order derivative (3.101) always has a single root in the interval $(-\infty,\frac{1}{\eta}]$. Thus, we can conclude that the first order derivative (3.100) is a convex function of Q_F with a single minimum reached in $Q_F\in(-\infty,\frac{1}{\eta}]$. Such a function in general may have either two roots (when its minimum is smaller than zero), one root (when its minimum value is exactly zero), or none (when the minimum value is positive). On the other hand, the negative roots will not affect the shape of the value function as it is only defined for $Q_F\geq 0$. Thus, the sign of the value function to a large extent depends on the location of the minimum of its first order derivative. We will now demonstrate how it changes as we increase Q_L and/or x.

First, we can show that the first order derivative (3.100) evaluated at its minimum is an increasing function of Q_L for $Q_F\geq 0$ and decreasing for $Q_F<0$. Consider first the derivative of $\frac{\partial(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{\partial Q_F}$ with respect to Q_L, which equals to

$$\frac{\eta x}{r-\alpha}\left(1-\frac{(\beta_2-1)\eta\left(Q_F(1-\eta Q_F-\beta_2\eta Q_L)+Q_L(1-\eta Q_L)\right)}{(1-\eta Q_F-\eta Q_L)^2}\right)\left(\frac{x}{X_D^E(Q_L,Q_F)}\right)^{\beta_2-1}-1. \tag{3.104}$$

At the minimum point the second order condition should be satisfied, i.e. (3.101) should be larger than zero. Dividing (3.101) by -2 and adding it to (3.104) yields

$$\frac{2(1-\eta(Q_F+Q_L))^2+(\beta_2-1)\eta Q_F(2\eta Q_F+\beta_2\eta Q_L-2)}{(1-\eta(Q_F+Q_L))^{3-\beta_2}} \\ +\left(\frac{(\beta_2-1)rx}{\beta_2 c(r-\alpha)}\right)^{\beta_2-1}\frac{\eta X((\beta_2+1)\eta Q_F-2)}{2(r-\alpha)(1-\eta Q_F)^{2-\beta_2}}. \tag{3.105}$$

The sign of (3.105) is the same as the sign of (3.104) taking into account that the latter is evaluated at $\underset{Q_F}{\operatorname{argmin}}\frac{\partial(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{\partial Q_F}$. We can demonstrate now that for $Q_F\geq 0$ the derivative of (3.105) with respect to Q_L is positive:

$$\frac{(\beta_2-2)(\beta_2-1)\eta\left(3\eta Q_F(1-\eta Q_F)-\beta_2\eta^2 Q_F Q_L\right)}{(1-\eta(Q_F+Q_L))^{4-\beta_2}}+\frac{2(1-\beta_2)\eta}{(1-\eta(Q_F+Q_L))^{2-\beta_2}}>0, \tag{3.106}$$

and that (3.105) evaluated at minimum $Q_L=0$ is non-negative:

$$(\beta_2-1)\eta(-Q_F)(1-\eta Q_F)^{\beta_2-2}\geq 0. \tag{3.107}$$

This means that for $Q_F\geq 0$ the expression given by (3.105) is always positive and so is (3.104) evaluated at $\underset{Q_F}{\operatorname{argmin}}\frac{\partial(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{\partial Q_F}$. Note, however, that for $Q_F<0$ the sign of both (3.106) and (3.107) changes and the derivative of (3.105) with respect to Q_L becomes negative. Thus, we can conclude that as Q_L increases, the minimum of (3.100) as a function of Q_L increases for $Q_F \geq 0$ and decreases for $Q_F < 0$.

In order to determine how a change in Q_L affects the location of the minimum of (3.100) we set the second order condition (3.103) to zero and apply the implicit function theorem. This gives

$$\frac{dQ_F}{dQ_L} = \frac{-\beta_2\eta Q_F(1-\eta Q_F) + \left((\beta_2-1)^2+1\right)\eta^2 Q_F Q_L + 2(1-\eta Q_L)(1-\eta Q_F-(\beta_2-1)\eta Q_L)}{(1-\eta(Q_F+Q_L))^{4-\beta_2}\dfrac{(\beta_2+1)\eta Q_F - 3}{(1-\eta Q_F)^{3-\beta_2}} + (\beta_2-2)\eta Q_L(3(1-\eta Q_L)-\beta_2\eta Q_F)} < 0. \tag{3.108}$$

Hence, an increase in Q_L results into a decrease in the location of the minimum of the first order derivative in (3.100) together with an increase in its y-coordinate. This allows us to conclude that the largest possible value of the minimum of the first order derivative (3.100) is reached for $Q_F = 0$. This allows to determine the number of roots that the first order derivative has for different values of Q_L and x. Namely, if the second order condition is satisfied for $Q_F = 0$, i.e. $\frac{\partial^2}{\partial Q_F^2} V_1^S(x, Q_L, Q_F)\Big|_{Q_F=0} = 0$, and at the same time

1. $\frac{\partial}{\partial Q_F} V_1^S(x, Q_L, Q_F)\Big|_{Q_F=0} \leq 0$, then the first order derivative always has two roots (or one when it touches the x-axis),
2. $\frac{\partial}{\partial Q_F} V_1^S(x, Q_L, Q_F)\Big|_{Q_F=0} > 0$, then the first order derivative has two roots for small Q_L (or one when it touches the x-axis) and none for large Q_L.

Now it is possible to find a specific value of the market size, x, to distinguish these two scenarios. First, consider the derivative of $\frac{\partial V_1^S(x,Q_L,Q_F)}{\partial Q_F}\Big|_{Q_F=0}$ with respect to Q_L:

$$\frac{\partial}{\partial Q_L}\left(\frac{\partial V_1^S(x,Q_L,Q_F)}{\partial Q_F}\Big|_{Q_F=0}\right) = \frac{\eta x}{r-\alpha}\left(-1 + \left(\frac{(\beta_2-1)rx}{\beta_2 c(r-\alpha)}\right)^{\beta_2-1}\frac{(1-\beta_2\eta Q_L)}{(1-\eta Q_L)^{2-\beta_2}}\right). \tag{3.109}$$

This function is clearly increasing in Q_L. Moreover, if $Q_L = 0$ its value is negative, while for $Q_L \to \frac{1}{\eta}$ it becomes infinitely large. Thus, $\frac{\partial V_1^S(x,Q_L,Q_F)}{\partial Q_F}\Big|_{Q_F=0}$ has a single minimum, that can be found by setting (3.109) to zero. This gives us the following expression $X = \left(\frac{1-\beta_2\eta Q_L}{1-\eta Q_L}\right)^{\frac{1}{1-\beta_2}} \frac{\beta_2 c(r-\alpha)}{(\beta_2-1)r(1-\eta Q_L)}$, which we can plug into $\frac{\partial}{\partial Q_F} V_1^S(x, Q_L, Q_F)\Big|_{Q_F=0}$ and get its value at the minimum Q_L. Setting the obtained result to zero will give us the value of Q_L such that this function exactly touches the x-axis and, thus, has a single root:

$$\frac{\beta_2 c}{(\beta_2-1)r}\left(\frac{\eta Q_L}{1-\beta_2\eta Q_L}-\frac{(1-\eta Q_L)^{-\beta_2+1}}{\beta_2(1-\beta_2\eta Q_L)}+1\right)\left(\frac{1-\beta_2\eta Q_L}{1-\eta Q_L}\right)^{\frac{1}{1-\beta_2}}-\frac{c}{r}-\delta=0. \quad (3.110)$$

Note that the above expression is zero for $Q_L=0$ and goes to infinity as $Q_L\to\frac{1}{\eta}$. Together with the fact that its derivative with respect to Q_L is positive, and that $\frac{\beta_2 c\eta(2-\beta_2\eta Q_L)\left(1-(1-\eta Q_L)^{1-\beta_2}\right)\left(\beta_2-\frac{\beta_2-1}{1-\eta Q_L}\right)^{\frac{1}{1-\beta_2}}}{(\beta_2-1)r(1-\eta Q_L)(1-\beta_2\eta Q_L)^2}>0$, it allows to conclude that the solution of (3.110) is unique. As a result, there exists a unique $\mathring{X}=\left(\frac{1-\beta_2\eta Q_L}{1-\eta Q_L}\right)^{\frac{1}{1-\beta_2}}\frac{\beta_2 c(r-\alpha)}{(\beta_2-1)r(1-\eta Q_L)}$, such that for $x>\mathring{X}$ the condition 2 is satisfied for all Q_L, i.e. $\frac{\partial}{\partial Q_F}V_1^S(x,Q_L,Q_F)\Big|_{Q_F=0}$ is always positive.

Moreover, evaluating (3.101) at $Q_F=0$ we get

$$\left.\frac{\partial^2 V_1^S(x,Q_L,Q_F)}{(\partial Q_F)^2}\right|_{Q_F=0}=$$
$$\frac{2\eta x}{r-\alpha}\left(\frac{(\beta_2-1)rx}{\beta_2 c(r-\alpha)}\right)^{\beta_2-1}\left(1-\left(\frac{(\beta_2-1)rx}{\beta_2 c(r-\alpha)}\right)^{1-\beta_2}-\frac{(\beta_2-1)\eta Q_L}{(1-\eta Q_L)^{2-\beta_2}}\right), \quad (3.111)$$

which has the same sign as the last expression in brackets. Note that its last part, $\frac{(\beta_2-1)\eta Q_L}{(1-\eta Q_L)^{2-\beta_2}}$, is monotonically decreasing with Q_L, because its derivative is negative, i.e. $(\beta_2-1)\eta\left(\frac{1-(\beta_2-1)\eta Q_L}{(1-\eta Q_L)^{3-\beta_2}}\right)<0$. The other part, $1-\left(\frac{(\beta_2-1)rx}{\beta_2 c(r-\alpha)}\right)^{1-\beta_2}$, is a constant with respect to Q_L. Therefore, there exists unique Q_L, denoted by $\breve{Q}_L$, such that if $Q_L<\breve{Q}_L$ then $\left.\frac{\partial^2 V_1^S(x,Q_L,Q_F)}{(\partial Q_F)^2}\right|_{Q_F=0}<0$, and if $Q_L\geq\breve{Q}_L$ then $\left.\frac{\partial^2 V_1^S(x,Q_L,Q_F)}{(\partial Q_F)^2}\right|_{Q_F=0}\geq 0$.

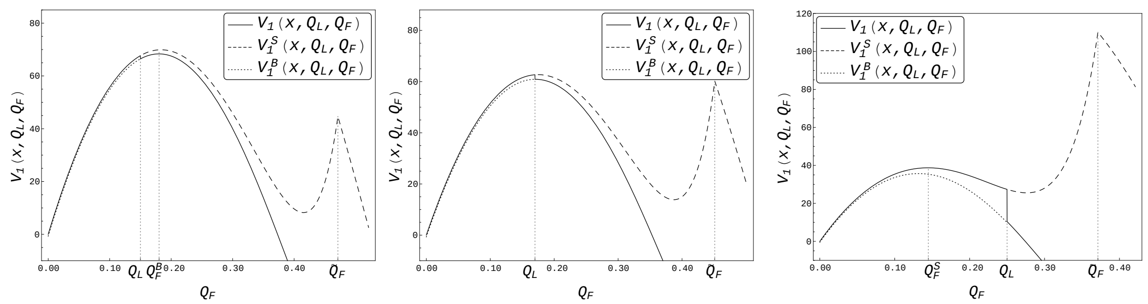
This means that for $Q_L\geq\breve{Q}_L$ the function $V_1^S(x,Q_L,Q_F)-\delta Q_F$ is convex. Its first order derivative with respect to Q_F is strictly increasing and positive for the considered parameter values. Hence, keeping in mind that in order to be a small follower the firms capacity should be smaller than the capacity of the leader, $Q_F<Q_L$, for $Q_L\geq\breve{Q}_L$ the firm will always invest just before Q_L.[15]

If $Q_L<\breve{Q}_L$, the sign of the second order derivative changes from negative to positive, so that $V_1^S(x,Q_L,Q_F)-\delta Q_F$ is convex for small values of Q_F and concave for large values of Q_F. As showed earlier the first order derivative may have either two or (one) roots (for small Q_L) or none (for large Q_L). In the latter case $V_1^S(x,Q_L,Q_F)-\delta Q_F$ is again a strictly increasing function of Q_F. Thus, there exists a critical value of Q_L such that for values above it the first order derivative is always positive. This Q_L can be determined by simultaneously setting to zero the first and second derivatives of $V_1^S(x,Q_L,Q_F)-\delta Q_F$, given by (3.100) and (3.101), respectively. Finally, for Q_L smaller than the critical value $V_1^S(x,Q_L,Q_F)-\delta Q_F$ is decreasing for the intermediate values of Q_F and increasing if Q_F is either small or large.

[15]In this case investing just before Q_L, i.e. at $Q_L-\varepsilon$ is ε-optimal for $\varepsilon>0$.

To summarize, the value function of the small follower starts from the origin and has a polynomial shape. Depending on the combination of the parameter values it can either have two turning points (e.g. for small Q_L) or none (e.g. for large Q_L). In the latter case the value function is strictly increasing and given the constraint $Q_F < Q_L$, its maximum is located at $Q_F = Q_L - \varepsilon$. If Q_L is relatively small the follower curve increases until the first turning point, then coming to its minimum and from there on it starts increasing again reaching the boundary at $\widetilde{Q}_F(x, Q_F)$. For this case the location of the maximum can be determined differently depending on the level of Q_L, which is crucial for the constraint in the optimization problem. In particular, the follower's optimum can be reached either at the first turning point, i.e. at the boundary $\widetilde{Q}_F(x, Q_F)$ or according to the following first order condition $\frac{\partial(V_1^S(x,Q_L,Q_F)-\delta Q_F)}{\partial Q_F} = 0$.

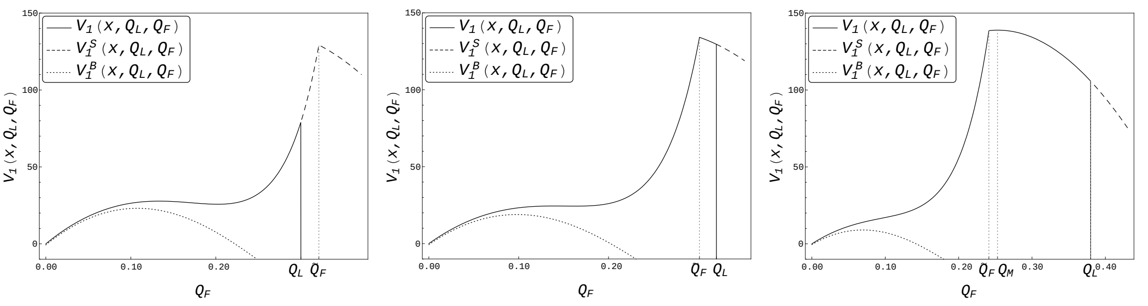
The capacity level that is defined by the above first order condition is denoted in this case by Q_F^S. It is implicitly determined by solving (3.100) and choosing the solution such that (3.101) is negative. The illustration of the value functions and the possibles locations of their maximums for different values of the incumbent's capacity is presented in Figure 3.10.



(a) Maximum is at Q_F^B for $Q_L = 0.15$.

(b) Maximum is at $Q_L - \varepsilon$ for $Q_L = 0.17$

(c) Maximum is at Q_F^S for $Q_L = 0.25$



(d) Maximum is at $Q_L - \varepsilon$ for $Q_L = 0.3$

(e) Maximum is at $\widetilde{Q}_F$ for $Q_L = 0.32$

(f) Maximum is at Q_M for $Q_L = 0.38$

Figure 3.10: Illustration of possible locations of the maximum for the parameter values: $r = 0.05$, $\alpha = 0.02$, $\sigma = 0.1$, $\eta = 1$, $\delta = 100$, $c = 50$, $x = 66$.

In all the cases except (a) the entrant replies with a smaller capacity than the leader and exits last. Next we define the specific capacity levels of the incumbent to distinguish between the scenarios shown in Figure 3.10.[16] The maximal capacity level of the incumbent such that it is optimal for the entrant to become a large firm and exit last $\bar{Q}_1(x)$ is the solution of

$$V_1^B(x, Q_L, Q_F^B(x, Q_L)) = V_1^S(x, Q_L, Q_L) \tag{3.112}$$

with respect to Q_L subject to $Q_L < Q_F^B(x, Q_L)$.

The minimum capacity level of the incumbent that ensures that the entrant chooses to be a small firm $\bar{Q}_2(x)$ is

$$Q_F^S(x, Q_L) = Q_L, \tag{3.113}$$

while the maximum level $\bar{Q}_3(x)$ is the solution of

$$V_1^S(x, Q_L, Q_F^S(x, Q_L)) = V_1^S(x, Q_L, Q_L) \tag{3.114}$$

with respect to Q_L subject to $Q_L > Q_F^S(x, Q_L)$.

The minimal capacity level of the incumbent that leads to a constrained monopoly scenario, i.e. when the entrant drives the incumbent out of the market by installing $\widetilde{Q}_F(x)$, is given by

$$\bar{Q}_4(x) = \frac{1}{2\eta}\left(1 - \frac{\beta_2 c(r-\alpha)}{r(\beta_2 - 1)x}\right). \tag{3.115}$$

The unconstrained monopoly appears when the incumbent's capacity level is larger than $\bar{Q}_5(x)$, meaning that the entrant is able to drive the incumbent out by installing the optimal monopoly capacity $\bar{Q}_5(x)$, which is implicitly determined by

$$Q_M(x) = \widetilde{Q}_F(x, Q_L). \tag{3.116}$$

Proof of Proposition 3.8 Assuming that the incumbent has already entered the market, we obtain the values of active and idle entrant similarly to the monopoly case:

$$V_0(x) = A_F x^{\beta_1}, \tag{3.117}$$

$$V_1(x, Q_L, Q_F) = B_F(Q_L, Q_F)x^{\beta_2} + \frac{x(1 - \eta(Q_F + Q_L))Q_F}{r - \alpha} - \frac{cQ_F}{r}, \tag{3.118}$$

where $B_F(Q_L, Q_F)$ is given by (3.91) and A_F is a to be determined from the system (3.119).

[16]The order of these capacity levels is verified numerically.

For the investment problem the following boundary conditions must hold[17]

$$\begin{cases} V_1(X_F^I, Q_L, Q_F) - \delta Q_F = V_0(X_F^I, Q_L, Q_F), \\ \left(\frac{\partial V_1(x, Q_L, Q_F)}{\partial x} + \frac{\partial \left(V_1(x, Q_L, Q_F) - \delta Q_F\right)}{\partial Q_F} \frac{\partial Q_F}{\partial x} \right) \Big|_{x=X_F^I} = \frac{\partial V_0(x, Q_L, Q_F)}{\partial x} \Big|_{x=X_F^I}, \end{cases} \tag{3.119}$$

which can be written as

$$\begin{cases} -A X_F^{I\beta_1} + B_F(Q_L, Q_F) X_F^{I\beta_2} + \frac{X_F^I(1-\eta(Q_F+Q_L))Q_F}{r-\alpha} - \frac{cQ_F}{r} - \delta Q_F = 0, \\ -\beta_1 A_F X_F^{I\beta_1 - 1} + \beta_2 B_F(Q_L, Q_F) X_F^{I\beta_2 - 1} + \frac{(1-\eta(Q_F+Q_L))Q_F}{r-\alpha} \\ \qquad + \left(\frac{\partial V_1(X_F^I, Q_L, Q_F)}{\partial Q_F} - \delta \right) \frac{\partial Q_F}{\partial x} \Big|_{x=X_F^I} = 0. \end{cases} \tag{3.120}$$

with $\frac{\partial V_1(X_F^I, Q_L, Q_F)}{\partial Q_F} = \frac{\partial B_F(Q_L, Q_F)}{\partial Q_F} + \frac{X_F^I(1-\eta(2Q_F+Q_L))}{r-\alpha} - \frac{c}{r}$.

Therefore, the investment threshold of the entrant $X_F^I(Q_L)$ is the solution with respect to x of

$$\left(1 - \frac{\beta_2}{\beta_1}\right) \frac{B_F(Q_L, Q_F^*(x, Q_L))}{Q_F^*(x, Q_L)} x^{\beta_2} + \left(1 - \frac{1}{\beta_1}\right) \frac{x(1-\eta(Q_L + Q_F^*(x, Q_L)))}{r-\alpha} - \frac{c}{r} - \delta$$
$$- \frac{x}{\beta_1 Q_F^*(x, Q_L)} \left(\frac{\partial V_1(x, Q_L, Q_F)}{\partial Q_F} \Big|_{Q_F = Q_F^*(x, Q_L)} - \delta \right) \frac{\partial Q_F^*(x, Q_L)}{\partial x} \Big|_{x=X_F^I} = 0, \tag{3.121}$$

and $A_F(Q_L) = \widehat{A}_F(X_F^I(Q_L), Q_L, Q_F^*(X_F^I(Q_L), Q_L))$ with

$$\widehat{A}_F(x, Q_L, Q_F) = \frac{\beta_2}{\beta_2 - \beta_1} \left(\frac{1}{x}\right)^{\beta_1} \left(\frac{x(1-\eta(Q_L+Q_F))Q_F}{r-\alpha} \left(1 - \frac{1}{\beta_2}\right) - \left(\frac{c}{r} + \delta\right) Q_F \right.$$
$$\left. - \frac{x}{\beta_2} \left(\frac{\partial V_1(x, Q_L, Q_F)}{\partial Q_F} - \delta \right) \frac{\partial Q_F^*(x, Q_L)}{\partial x} \right). \tag{3.122}$$

Proof of Proposition 3.9 If $Q_L > \bar{\bar{Q}}_1$, the threshold which leads to the immediate investment in a duopoly is given by

$$X_{F,D}^I(Q_L) = \overline{X}_F^I(Q_L), \tag{3.123}$$

where $\overline{X}_F^I(Q_L)$ is the inverse function of $\bar{Q}_3(x)$:

$$\overline{X}_F^I(Q_L) = (\bar{Q}_3(x))^{-1}, \tag{3.124}$$

[17]The systems (3.119) and (3.120) hold for $Q_F^*(x, Q_L)$ (given by (3.22)) evaluated at $x = X_F^I$. To simplify the notation we refer to $Q_F^*(x, Q_L)$ as Q_F.

and $\bar{\bar{Q}}_1$ is found by solving

$$h\left(\overline{X}_F^I(Q_L), Q_L, Q^*_{F,D}\left(\overline{X}_F^I(Q_L), Q_L\right)\right) = 0, \tag{3.125}$$

where $h(x, Q_L, Q_F)$ is defined by (3.31).

If $Q_L \leq \bar{\bar{Q}}_1$, $X^I_{F,D}(Q_L)$ is a solution with respect to x of

$$h(x, Q_L, Q^*_{F,D}(x, Q_L)) = 0. \tag{3.126}$$

The entrant monopolization strategy becomes available only if the capacity of the incumbent satisfies $Q_L \geq \widehat{Q}_L$, where capacity level $\widehat{Q}_L$ can be found by solving the following system

$$\begin{cases} V_M\left(\frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-2\eta Q_L)}, Q_L\right) - \delta Q_L = V_0\left(\frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-2\eta Q_L)}, Q_L, Q^*_{F,D}\left(x, Q_L\right)\right), \\ h(x, Q_L, Q^*_{F,D}(x, Q_L)) = 0. \end{cases} \tag{3.127}$$

In this case the entrant's investment threshold that leads to the monopoly once being hit by x from below is

$$X^I_{F,\underline{M}}(Q_L) = \frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-2\eta Q_L)} \quad \text{if } Q_L \leq \bar{\bar{Q}}_2, \tag{3.128}$$

and $\bar{\bar{Q}}_2$ is found by solving

$$h\left(\frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-2\eta Q_L)}, Q_L, Q^*_M\left(\frac{\beta_2 c(r-\alpha)}{r(\beta_2-1)(1-2\eta Q_L)}\right)\right) = 0. \tag{3.129}$$

If $Q_L > \bar{\bar{Q}}_2$ the thresholds $X^I_{F,\underline{M}}(Q_L)$ is the solution with respect to x of

$$h(x, Q_L, Q^*_M(x)) = 0. \tag{3.130}$$

The situation when monopoly is triggered from above corresponds to the threshold

$$X^I_{F,\overline{M}}(Q_L) = \overline{X}_F^I(Q_L). \tag{3.131}$$

4 Predatory Pricing under Uncertainty[1]

In this chapter we develop a stochastic dynamic model of predatory pricing. When profits evolve stochastically, a negative demand shock can lead to bankruptcy for firms, that cannot immediately raise external capital. An assumption that firms' accumulated profits determine its reputation creates incentives for market incumbents to use predatory pricing strategies in order to keep new players out of the industry. Applying game theoretic and dynamic programming techniques, we show that firms may initiate a price war that could drive the opponent out of the market. Because of uncertainty the new player may wish to take a chance and enter based on the probability of success. Therefore, the realized market structure may vary for different sample paths of the stochastic process.

4.1 Introduction

Contentions that firms use aggressive pricing to drive the opponents out of the market are not uncommon, and yet the early literature on the topic failed to explain the rationality behind such behavior. The early contributions explained this phenomenon in two ways. The first one is known as the *deep pocket argument* introduced by McGee (1958) and later studied by Telser (1966) and Benoit (1983). In these models a more resourceful incumbent is able to drive a financially constrained entrant out of the market by the means of aggressive pricing. Their main conclusion is, however, that under perfect information no price war will be observed in equilibrium due to the temporary nature of price cuts. The second explanation is associated with the so-called, *chain store paradox*. In the formulation of Selten (1978) an incumbent's incentives to predate come from reputational considerations. More specifically, by initiating a price war when facing the first out of potentially many entrants an

[1]This chapter is based on Lavrutich and Thijssen (2016).

incumbent establishes reputation as an aggressive firm in order to prevent further entries. Selten (1978) presents the unintuitive result that starting a price war is not a viable strategy from the perspective of standard game theoretic approach. This finding is known in the literature as the chain store paradox.

More recent studies bridge the gap between theory and practice by incorporating market imperfections into the analysis. For example, Milgrom and Roberts (1982), Kreps and Wilson (1982), and Benoit (1984) find that incomplete (or imperfect) information about firms' payoffs resolves the chain store paradox. According to Fudenberg and Tirole (1985a) and Poitevin (1989) asymmetric information in financial markets can trigger predatory pricing behavior due to the fact that financiers are typically not aware of an entrant's true profitability. Alternatively, the entrant may be uncertain about its own costs and thus, is unable to perfectly predict its future profits. In Fudenberg and Tirole (1986a), the entrant's inference about its future profitability is based on the current profits and, as a result, the incumbent may "jam" this signal using a predatory pricing strategy. Saloner (1987) considers a three-stage game where two incumbents compete under asymmetric information about production costs, given the possibility of a future merger. In this model an informed firm is willing to signal low-cost type using output expansions in order to facilitate better takeover terms. Bolton and Scharfstein (1990) shows that mitigating agency problems between the firm and its financiers creates incentives for predatory behavior.

In this chapter we go back to a complete information setting. We investigate if firms have an incentive to use predatory pricing strategies in a dynamic setting where firms' profits are subject to stochastic shocks. In our model an entrant becomes active in the existing market by undertaking an irreversible investment given the uncertainty about its future profit stream. Upon entry, either of the firms may decide to initiate a price war. Consequently, the entrant could be driven out of the market or even abstain from entering the market in the first place. Our analysis primarily focuses on the question as to whether the entrant would exercise its option to invest given the predation threat.

In this regard, this chapter is also related to the real options theory and contributes to this strand of the literature in two ways. Firstly, we enhance the existing body of research by providing a more general model of predatory pricing under uncertainty. Even though a large bulk of real options studies focuses on entry deterrence strategies (e.g. Smets (1991), Spence (1977), Boyer *et al.* (2004) and Huisman and Kort (2015)) or exit games (e.g. Lambrecht (2001), Murto (2004)), predatory pricing has gained a rather limited attention. Among the few real options contributions that explicitly use the notions of either aggressive pricing or predatory behavior are Gryglewicz (2009), that considers a stochastic limit pricing model under asymmetric information, and Bayer (2007), where predation is defined as forcing the entrant out of the market by installing a large capacity. We present a complete information model, where the firm-specific uncertainty can provide the rationale for aggressive pricing.

This is related to our second contribution, which is more of a methodological nature. In this model we depart from the assumption of perfectly correlated shocks for market participants. In the standard real options models firms are subject to the same uncertainty regarding the future profits. As a result, their deterministic actions allow to predict the outcome of the game. In our model deterministic actions change the probabilistic environment. More precisely, firms built up their reputation, which is determined by their accumulated profits. A firm goes bankrupt when its reputation becomes too low, or, in other words, its accumulated profits are depleted. Given that firms are subject to potential future losses, the instantaneous profit inflow is subject to firm-specific uncertainty. In this setting an incumbent firm cannot guarantee that initiating a price war will eventually lead to exit of its opponent. On the contrary, aggressive pricing alters the probability of staying alive. Hence, the final outcome of the game is determined by the paths of the underlying stochastic process. Depending on the amount of accumulated profits, this may create incentives for an established firm to take a chance and initiate a price war, while for a new firm the uncertainty may create an incentive to enter despite the possibility of a price war.

4.2 Model

In our model we consider two risk neutral profit maximizing firms. One firm is already operating in the market as a monopolist. The other firm faces a possibility to enter this market by undertaking an irreversible investment that creates a revenue stream. The firms are assumed to be symmetric from the perspective of the production process, i.e. they incur the same marginal costs. The net revenue accumulated over time, when the firm is active, determines each firm's reputation. Additionally, market participants are subject to firm-specific uncertainty. If the accumulated profits that serve as a proxy for a firm's reputation are driven down to zero due to a series of negative shocks, this firm suffers from a severe reputational damage. In this case the firm goes bankrupt and leaves the market. Therefore, the accumulated profit of firm i, denoted by X_{it}, is modeled as an arithmetic Brownian motion. The instantaneous profit inflow for firm i, thus, satisfies the following differential equation

$$dX_{it} = \pi_i dt + \sigma_i dB_{it}, \tag{4.1}$$

where π_i is firm's i current profit, σ_i is a volatility parameter, and B_i is a Wiener process. Similarly to DeMarzo and Sannikov (2006) and Morellec *et al.* (2014) this formulation the drift of the Brownian is selected by the firms, while the volatility is not. As mentioned earlier, we relax the assumption of perfectly correlated profitability shocks for different firms. In reality a firm's future profitability may depend on various firm-specific factors, as e.g. firm-level productivity shocks, dependence on certain suppliers, particular contractual agreements, as well as on different characteristics of the firm's location, such as local labor markets condition or regulatory restrictions. Moreover, the new firm may potentially face different uncertainty than the incumbent firm that has already been operating in the market. Therefore, rather than assuming the same uncertainty for the incumbent and the entrant, we model the shocks to be firm-specific, i.e. processes B_{1t} and B_{2t} are uncorrelated[2].

In the present model the timeline of the game runs as follows. First, the entrant undertakes an irreversible investment decision for a certain level of the incumbent's accumulated profit. Second, when the new firm chooses to enter, the firms decide whether they are going to coexist in a duopoly or initiate a price war. In the latter case the predator drives the price down to the level of marginal costs. Due to the symmetry between firms, the price war implies zero instantaneous profits for both of them each period. In this setting the incumbent chooses only whether to fight or accommodate, while the entrant's decision involves an additional component, namely investment timing. Lastly, if the accumulated profit of either of the firms hits zero, this firm goes bankrupt and leaves the market, while its opponent becomes a monopolist.

To derive a subgame perfect equilibrium, we solve the model backwards starting with the stage where both firms are operating in the market. We derive the probability distributions of bankruptcy times and investigate under which conditions the firms are willing to engage in a price war. In the later sections we solve the optimal stopping problem of an entrant.

4.3 Duopoly subgame

In this section we examine the situation when both firms are already active in the market. When the entrant has already undertaken investment, the only decision left for the firms is whether to initiate a price war or not. Let $i = 1$ denote a firm's position as incumbent, and $i = 2$ as entrant. The choice of strategy is denoted by k, i.e. whether to accommodate (A) or to fight (F).

If both firms are choosing to coexist in a duopoly, their period profits equal to π_A. If either of the firms chooses to fight, their profits are driven down to zero, $\pi_F = 0$. In that way by behaving aggressively, the predator is not only harming the opponent

[2]The traditional models usually only account for industry specific uncertainty. However, as industry specific shocks are the same for both firms, they do not change the odds of winning the predation game. Hence, their inclusion does not significantly influence the main results.

but also itself. A change in period profits also alters firms' bankruptcy times that are defined as follows. Denote by x_i the accumulated profit of firm i at the beginning of the subgame. We assume that the new player has a certain level of initial reputation. Here we can think of existing brands entering a new market. In this example the profit in the old market would determine its reputation on the new market. Then the bankruptcy time τ_i, i.e. the time when the firm i's accumulated profit hits zero, is given by

$$\tau_i = \inf\{t > 0 : X_{it} = -x_i\}. \tag{4.2}$$

If the incumbent goes bankrupt first, i.e. if $\tau_1 < \tau_2$, it leaves the market with no profits. If the entrant goes bankrupt first, i.e. $\tau_1 > \tau_2$, the entrant leaves the market and the incumbent becomes a monopolist. This is reflected in the value function of the firm i for a given strategy k, V_i^k, which equals to

$$V_i^k(x_i, x_j) = x_i + \frac{\pi_k}{r} + \mathbb{E}^k_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}\mathbb{1}_{\tau_i<\tau_j}\right]\left(-\frac{\pi_k}{r}\right) + \mathbb{E}^k_{x_i,x_j}\left[\mathrm{e}^{-r\tau_j}\mathbb{1}_{\tau_i>\tau_j}\right]\left(\frac{\pi_M}{r} - \frac{\pi_k}{r}\right), \tag{4.3}$$

where r is the discount rate, π_M is monopoly profit, and $k \in \{A, F\}$. Denote the stochastic discount factor of the incumbent going bankrupt first, i.e. $\mathbb{E}^k_{x_1,x_2}\left[\mathrm{e}^{-r\tau_1}\mathbb{1}_{\tau_1<\tau_2}\right]$, by Q_1^k, and the stochastic discount factor of the entrant going bankrupt first, i.e. $\mathbb{E}^k_{x_1,x_2}\left[\mathrm{e}^{-r\tau_2}\mathbb{1}_{\tau_1>\tau_2}\right]$, by Q_2^k. Note that the accumulated profit of the entrant at the beginning of the duopoly subgame are always fixed to the level of the initial reserves. This is because the new firm is not producing before entering the market. The reserves of the incumbent firm, instead, differ depending on the entry time of the new firm. Thus, the stochastic discount factors and, therefore, the values of the firms are functions of x_1 only, while x_2 enters as a parameter. Using the proposed notation we can rewrite the value function in (4.3) as

$$V_i^k(x_i) = \frac{\pi_k}{r} + Q_i^k(x_i)\left(-\frac{\pi_k}{r}\right) + Q_j^k(x_i)\left(\frac{\pi_M}{r} - \frac{\pi_k}{r}\right). \tag{4.4}$$

where the stochastic discount factors are defined in the following proposition.

Proposition 4.1 *The stochastic discount factor of the incumbent going bankrupt first, given the strategy k, is given by*

$$Q_1^k(x_1, x_2) = \int_0^\infty \mathrm{e}^{-rt}\frac{x_1}{\sigma_1\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(x_1+\pi_k t)^2}{2\sigma_1^2 t}}\left(\Phi\left(\frac{x_2+\pi_k t}{\sigma_2\sqrt{t}}\right) - \mathrm{e}^{-\frac{2\pi_k x_2}{\sigma_2^2}}\Phi\left(\frac{-x_2+\pi_k t}{\sigma_2\sqrt{t}}\right)\right)dt, \tag{4.5}$$

and the stochastic discount factor of the entrant going bankrupt first, given the strategy k, is given by

$$Q_2^k(x_1,x_2) = \int_0^\infty \mathrm{e}^{-rt}\frac{x_2}{\sigma_2\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(x_2+\pi_k t)^2}{2\sigma_2^2 t}}\left(\Phi\left(\frac{x_1+\pi_k t}{\sigma_1\sqrt{t}}\right) - \mathrm{e}^{-\frac{2\pi_k x_1}{\sigma_1^2}}\Phi\left(\frac{-x_1+\pi_k t}{\sigma_1\sqrt{t}}\right)\right)dt, \tag{4.6}$$

where $k \in \{A, F\}$, and Φ is the CDF of the standard normal distribution.

As pointed out in Rogers and Shepp (2006), "finding the expectation (or the law) of a functional of a Brownian path is usually either quite straightforward .. or quite impossible". From this perspective, a model without firm-specific volatility is a good example of the former. However, departing from the assumption that firms are not subject to exactly the same shock process, immediately makes the derivation of analytical probabilities quite impossible. For this reason some results of this chapter are presented in the form of numerical examples.

The goal now is to determine for which values of x_1 do firms undertake a decision to initiate a price war. The firm would prefer to behave aggressively if the value of fighting exceeds the value of accommodating. Therefore, in order to determine fighting and accommodation regions, we consider the differences in values of accommodation and fighting for both firms: for the incumbent, i.e. $D_1^{AF}(x_1) = V_1^A(x_1) - V_1^F(x_1)$, and for the entrant, defined by $D_2^{AF}(x_1) = V_2^A(x_1) - V_2^F(x_1)$:

$$D_1^{AF}(x_1) = \frac{\pi_A}{r}\left(1 - Q_1^A(x_1)\right) + Q_2^A(x_1)\frac{\pi_M - \pi_A}{r} - Q_2^F(x_1)\frac{\pi_M}{r}, \tag{4.7}$$

$$D_2^{AF}(x_1) = \frac{\pi_A}{r}\left(1 - Q_2^A(x_1)\right) + Q_1^A(x_1)\frac{\pi_M - \pi_A}{r} - Q_1^F(x_1)\frac{\pi_M}{r}. \tag{4.8}$$

Lemma 1 *If $\frac{\pi_M}{\pi_M - \pi_A} < \frac{3\pi_A + \sqrt{\pi_A^2 + 2r\sigma_2^2}}{\sqrt{2r\sigma_2^2}}$, there exists a unique value of x_2 such that $\lim_{x_1\to\infty} D_1^{AF}(x_1) = 0$, and for smaller values of x_2 $\lim_{x_1\to\infty} D_1^{AF}(x_1) < 0$, while for larger of x_2 values $\lim_{x_1\to\infty} D_1^{AF}(x_1) \geq 0$. If $\frac{\pi_M}{\pi_M - \pi_A} > \frac{3\pi_A + \sqrt{\pi_A^2 + 2r\sigma_2^2}}{\sqrt{2r\sigma_2^2}}$, then $\lim_{x_1\to\infty} D_1^{AF}(x_1) > 0$. Moreover, it always holds that $\lim_{x_1\to\infty} D_2^{AF}(x_1) > 0$.*

Figure 4.1 illustrates the functions $D_1^{AF}(x_1)$ and $D_2^{AF}(x_1)$ defined by (4.7) and (4.8). The solid curve represents the difference between accommodation and fighting for the incumbent, while the dashed line that for the entrant[3]. As long as either of the firms prefers the fighting strategy, i.e. when either of the curves is negative, the firms engage in a price war. The fighting regions are colored gray.

[3]Numerical experiments indicate that both functions have at most one zero, or in other words, there is at most one value of x_1 for each firm, such that they are indifferent between fighting and accommodation.

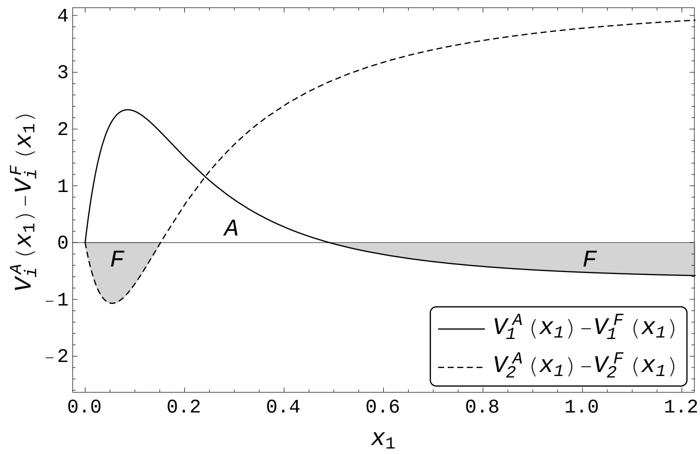
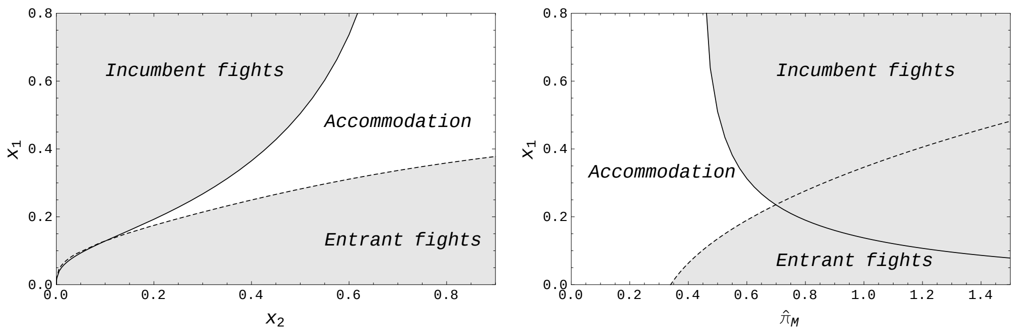


Figure 4.1: The difference between accommodation and fighting for the set of parameter values: $x_2 = 0.3$, $\pi_A = 0.3$, $\pi_M = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, and $r = 0.05$.

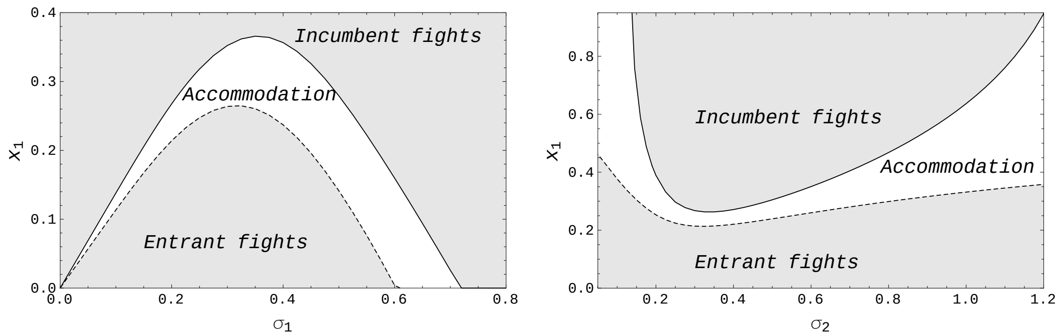
Intuitively, for large values of x_1 the incumbent firm is more willing to fight a new firm, as the probability to win the game is relatively large. The opposite holds for the entrant: the weaker the incumbent is, the more likely the entrant is to survive a price war, hence, it prefers to fight for small x_1. For the intermediate values of x_1 neither of the firms is willing to initiate a price war, therefore, the accommodation scenario occurs. Naturally, the regions where firms implement either of the strategies change depending on the parameter set. The next figures illustrate how a change in different parameter values influences the firms' choice of strategy.



(a) For different values of x_2, with $\pi_M = 0.65$. *(b)* For different values of π_M, with $x_2 = 0.3$.

Figure 4.2: Strategic regions for the set of parameter values: $\pi_M = 0.65$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$ and $r = 0.05$.

As can be seen in Figure 4.2a, if the entrant is stronger, i.e. x_2 is large, it is willing to engage in a price war with a stronger incumbent. The incumbent, on the contrary, is willing to postpone the price war until its accumulated profit is larger when facing a stronger entrant. Figure 4.2b shows that the larger the monopoly profit, or in other words, the larger is the difference between duopoly and monopoly profits is, the more appealing the fighting strategy is for both firms. In fact there exists a region, where the “prize" at the end of a price war is so large, that the firms would always inevitably end up fighting.



(a) For different values of σ_1, with $\sigma_2 = 0.3$. *(b)* For different values of σ_2, with $\sigma_1 = 0.2$.

Figure 4.3: Strategic regions for the set of parameter values: $x_2 = 0.3$, $\pi_A = 0.3$, $\pi_M = 0.65$, and $r = 0.05$.

In Figures 4.3a and 4.3b the effect of the firm-specific uncertainty on the fighting and accommodation regions is twofold. On the one hand, larger σ_1 implies that the incumbent is going bankrupt sooner and with larger probability. Thus, at first we observe a similar effect in Figure 4.2a, namely, the incumbent starts fighting for larger x_1, while the entrant stops fighting for larger x_1. However, when σ_1 becomes sufficiently large, we observe the opposite effect. This is because the uncertainty negatively affects the probability that both firms survive forever. In particular, for very large values of σ_1, the event of bankruptcy while accommodating is almost as likely as while fighting. In this case, the entrant prefers to accommodate, as there is no reason to give up its period profits, given that the incumbent is very likely to go bankrupt first anyway. The incumbent, in turn, has an incentive to drive the profits of the opponent to zero as a last resort. Thus, the incumbent is wiling to initiate a fight earlier, while the entrant gives this strategy up earlier. A similar reasoning can be applied to explain the non-monotonicity in Figure 4.3b.

To summarize, the firms may end up in either of the three scenarios depending of the parameter values: both firms engage in a price war for any x_1, both firms are willing to accommodate for any x_1, or they alternate between strategies depending on the current value of x_1. Importantly, the outcome here largely affects the optimal stopping problem of the entrant, which we introduce in the subsequent section.

4.4 The entrant's optimal stopping problem

In the subgame where the entrant is not active yet, it faces an optimal stopping problem, where it decides whether to wait or to invest immediately. In other words, the entrant chooses a stopping time that maximizes its value

$$F^*(x_1) = \sup_\tau \mathbb{E}_{x_1}\left[\mathrm{e}^{-r\tau}\left(V_2(x_1) - I\right)\right], \tag{4.9}$$

where I denotes the investment costs.

If the incumbent is a monopolist on the market, the entrant observes the following process for the incumbent's accumulated profit:

$$X_{1t} = \pi_M t + \sigma_1 B_{1t}. \tag{4.10}$$

The state space of this stochastic process can be split up in a continuation and a stopping region due to the Markovian property. As a result, of the irreversibility of the investment decision, the value of the entrant in the stopping region is the expected value of the active firm, $V_2(x_1)$. Denote the value function in the continuation region by, $W(x_1)$, then

$$F^*(x_1) = \begin{cases} W(x_1) \text{ for } x_1 \in C, \\ V_2(x_1) \text{ for } x_1 \notin C, \end{cases} \tag{4.11}$$

where the continuation and stopping regions are defined as

$$C = \{x_1 \in \mathbb{R} | F^*(x_1) > V_2(x_1)\}, \tag{4.12}$$

$$\mathbb{R} \setminus C = \{x_1 \in \mathbb{R} | F^*(x_1) = V_2(x_1)\}. \tag{4.13}$$

The optimal stopping time can be written as

$$\tau^* = \inf\{t > 0; X_{1t} \notin C\}. \tag{4.14}$$

Typically, if the value function $V_2(x_1)$ is decreasing, the continuation region can be written as $C = (X_1^*, \infty)$, and the solution can be expressed in terms of the optimal investment threshold, X_1^*. This threshold can be found by applying the value matching and smooth pasting conditions. However, in our model at the moment the firm decides to stop, the drift of the Brownian motion changes. As a result, the value function in the stopping region is based on the new underlying process. This results in a more complex relation between stopping and waiting values. For this reason, in order to solve the optimal stopping problem we need to apply a more general approach.

In particular, solving the optimal stopping problem in (4.9) is equivalent to the problem of finding the function F^*, which is the smallest superharmonic function dominating V_2 (Peškir and Shiryaev (2006)), i.e. the following conditions must be satisfied:

$$\begin{cases} \mathcal{L}F^* - rF^* \leq 0, \ (F^* \text{ minimal}) \ , \\ F^* \geq V_2, \ (F^* > V_2 \text{ on } C \ \& \ F^* = V_2 \text{ on } \mathbb{R} \setminus C), \end{cases} \tag{4.15}$$

where $\mathcal{L}$ is the infinitesimal generator of x_1.[4] In the literature the superharmonicity condition in (4.15) is often satisfied automatically and is rarely checked. In our model, however, it plays an important role due to the convexity of V_2.

It follows from (4.15) that for the candidate optimal threshold the following conditions have to be satisfied:

$$\begin{cases} \mathcal{L}F^* - rF^* = 0 \text{ on } C, \\ \left.\dfrac{\partial V_2}{\partial x_1}\right|_{\partial C} = \left.\dfrac{\partial F^*}{\partial x_1}\right|_{\partial C}, \\ F^* \geq V_2, \ (F^* > V_2 \text{ on } C \ \& \ F^* = V_2 \text{ on } \mathbb{R} \setminus C), \end{cases} \tag{4.16}$$

where the last two conditions represent the smooth pasting and value matching conditions at the boundary of the continuation region, ∂C. The first condition correspondingly reduces to

$$\mathcal{L}W(x_1) - rW(x_1) = 0, \tag{4.17}$$

or alternatively,

$$\frac{1}{2}\sigma_1^2 \frac{\partial^2 W(x_1)}{\partial X_1^2} + \pi_M \frac{\partial W(x_1)}{\partial x_1} - rW(x_1) = 0. \tag{4.18}$$

Solving the above equation for $W(x_1)$ and determining the bounds of the continuation region yield a solution for the optimal stopping problem in (4.9). In the next section we present the solution for the fighting region. Later we use numerical experiments to illustrate the solution for the region where only accommodation occurs. The primary focus of this chapter is the former scenario, when for any reputation of the incumbent firm the price war will be initiated, either by the incumbent itself or by the new entrant. Several reasons account for that interest. Firstly, it gives new important insights on how to solve the optimal stopping problem when the superharmonicity property of the gain function is not always satisfied. Secondly, the case based on fighting alone features interesting and unique results that are not observed

[4] $\mathcal{L}f(x) = \lim\limits_{t \to 0} \dfrac{\mathbf{E}_x[f(X_t)] - f(x)}{t}$.

in the standard models. Thirdly, the simplified expressions of the survival probabilities allow to derive more analytical results. Lastly, this scenario serves as a ground for the more general case when depending on the value of x_1 both accommodation and fighting may occur. The latter we leave to future research.

4.4.1 Fighting region

Consider the situation when one of the firms initiates a price war immediately upon the investment of the entrant. In this case the value of stopping, i.e. V_2^F for the entrant simplifies to

$$V_2^F(x_1) = Q_1^F(x_1)\frac{\pi_M}{r}. \tag{4.19}$$

Note that if x_1 is infinitely large, investment yields zero value in expectation and the option to invest in this case is zero. This results in the following boundary condition: $\lim_{x_1\to\infty} W(x_1) = 0$, which implies that the option value of the entrant takes the following form

$$W_F(x_1) = A\mathrm{e}^{-\beta_M x_1}, \tag{4.20}$$

where $\beta_M = \frac{\pi_M+\sqrt{\pi_M^2+2\sigma_1^2 r}}{\sigma_1^2} > 0$. Applying the value matching and smooth pasting conditions we find the implicit equation that determines candidate solutions for the optimal investment trigger, as stated in the following proposition.

Proposition 4.2 *Let the function g be given by*

$$g(x_1) = -\frac{\pi_M}{r}\left(\frac{1}{\beta_M}\frac{\partial Q_1^F(x_1)}{\partial x_1} + Q_1^F(x_1)\right) + I. \tag{4.21}$$

The candidates for the optimal investment threshold satisfy the following equation

$$g(x_1) = 0. \tag{4.22}$$

In order to determine the stopping region for this problem, we need to take into account an additional requirement in (4.15). Namely, the superharmonicity condition must be satisfied in the stopping region. This implies the following

$$\mathcal{L}V_2^F(x_1) - rV_2^F(x_1) \leq 0 \text{ for } x_1 \notin C. \tag{4.23}$$

where

$$\mathcal{L}V_2^F(x_1) - rV_2^F(x_1) = \frac{1}{2}\sigma_1^2\frac{\partial^2 V_2^F(x_1)}{\partial X_1^2} + \pi_M\frac{\partial V_2^F(x_1)}{\partial x_1} - rV_2^F(x_1). \tag{4.24}$$

If for the candidate investment threshold, there exists a region, $(0, X_1^*)$, where the superharmonicity condition is not satisfied, the NPV process in this region is a sub-martingale. This means that, in expectation, the value of investing later will grow indefinitely and it will never be optimal for the new firm to enter. In order to identify such cases, it is useful to express $\mathcal{L}V_2^F(x_1) - rV_2^F(x_1)$ in terms of the implicit function that determines candidate thresholds in (4.21). This result is presented in the following proposition.

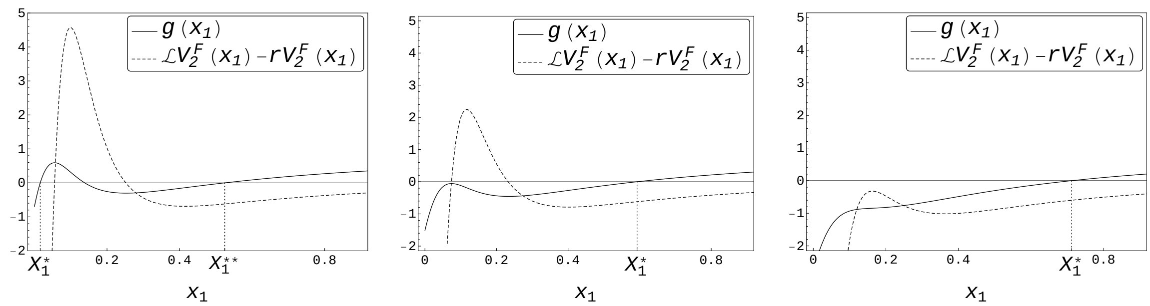
Proposition 4.3 *The expression in* (4.24) *can be simplified as follows:*

$$\mathcal{L}V_2^F(x_1) - rV_2^F(x_1) = -\frac{\beta_M \sigma_1^2}{2} \frac{\partial g(x_1)}{\partial x_1} + rg(x_1), \tag{4.25}$$

where $g(x_1)$ *is given by* (4.21)*. In particular, at the optimal investment threshold it holds that*

$$\mathcal{L}V_2^F(X_1^*) - rV_2^F(X_1^*) = -\frac{\beta_M \sigma_1^2}{2} \frac{\partial g(x_1)}{\partial x_1}\bigg|_{x_1 = X_1^*} < 0. \tag{4.26}$$

It follows from the above proposition that whenever $g(x_1)$ is a strictly increasing function, the superharmonicily condition will always be satisfied. This case corresponds to the unique optimal investment threshold. However, if there exists a region where $g(x_1)$ is increasing, the superharmonicity condition can no longer be guaranteed. Numerical experiments show that equation (4.22) has at most three solutions. Moreover, it is possible to show that the candidate solution with negative derivative can always be eliminated (see proof of Proposition 4.2). Therefore, there may exist either one, X_1^*, or two candidate solution for the optimal investment threshold, $\tilde{X}_1^* < \tilde{X}_1^{**}$, depending on the parameter values. In the latter case there always exists a region in $(\tilde{X}_1^*, \tilde{X}_1^{**})$, where $V_2^F(x_1)$ is not superharmonic.



(a) For $x_2 = 0.09$. *(b)* For $x_2 = 0.1$. *(c)* For $x_2 = 0.12$.

Figure 4.4: The functions $g(x_1)$ (solid) and $\mathcal{L}V(x_1) - rV(x_1)$ (dashed) for $\pi_M = 0.5$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r = 0.05$, $I = 0.8$, and different values of x_2.

Figure 4.4a illustrates the situation in which two candidate solutions for the optimal stopping problem exist and only for the smallest one the function $F^*(x_1)$ is superharmonic. In Figure 4.4b the superharmonicity property is violated. Figure 4.4c shows the case, where the function $g(x_1)$ is strictly increasing with a unique solution and, thus, $F^*(x_1)$ is superharmonic. Note that the only difference in the parameter values between these three cases is in x_2. Intuition suggests that the weaker the entrant is, the smaller the value of x_1 that triggers its investment is. In Figure 4.4c, i.e. when the entrant is relatively strong, we observe the standard case with a unique optimal threshold, X_1^*, and the continuation region of the form (X_1^*, ∞). However, when the new firm has a relatively poor reputation, the continuation and stopping regions are not trivial anymore, i.e. there exists an interval where the superharmonicity condition is not satisfied. As mentioned earlier, this implies that the value process is a submartingale and the firm has an incentive to postpone its investment. The incentive disappears when x_1 is either sufficiently low or sufficiently high, i.e. the entrant is willing to invest, when x_1 lies outside of the non-superharmonic interval. This implies a need of an additional continuation region. The option value in this region is bounded both from above and below. Given that $\mathcal{L}F^*(x_1) - rF^*(x_1) = 0$ on C, the additional option value must be represented by $\hat{W}(x_1) = B_1 \mathrm{e}^{-\beta_M x_1} + B_2 \mathrm{e}^{\gamma_M x_1}$, where $\beta_M = \frac{\pi_M + \sqrt{\pi_M^2 + 2\sigma_1^2 r}}{\sigma_1^2} > 0$ and $\gamma_M = \frac{-\pi_M + \sqrt{\pi_M^2 + 2\sigma_1^2 r}}{\sigma_1^2} > 0$. The constants B_1 and B_2 are to be determined from the value matching and smooth pasting conditions. These boundary conditions also give two additional thresholds, $\hat{X}_1^*$ and $\hat{X}_1^{**}$, such that $(\hat{X}_1^*, \hat{X}_1^{**}) \subset C$. The above findings are summarized in the next proposition.

Proposition 4.4 *The additional option value is given by*

$$\hat{W}(x_1) = B_1(\hat{X}_1^*) \mathrm{e}^{-\beta_M x_1} + B_2(\hat{X}_1^*) \mathrm{e}^{\gamma_M x_1}, \tag{4.27}$$

with $B_1(\hat{X}_1^*) = \frac{\mathrm{e}^{\beta_M \hat{X}_1^*}}{\beta_M + \gamma_M} \left(\gamma_M (V_2^F(\hat{X}_1^*) - I) - \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1} \Big|_{x_1 = \hat{X}_1^*} \right)$ *and* $B_2(\hat{X}_1^*) = \frac{-\mathrm{e}^{-\gamma_M \hat{X}_1^*} g(\hat{X}_1^*)}{\beta_M + \gamma_M}$.

The optimal thresholds of the inaction region, $\hat{X}_1^*$ *and* $\hat{X}_1^{**}$*, are implicitly determined by the following system*

$$\begin{cases} \mathrm{e}^{-\gamma_M \hat{X}_1^*} g(\hat{X}_1^*) = \mathrm{e}^{-\gamma_M \hat{X}_1^{**}} g(\hat{X}_1^{**}), \\ \mathrm{e}^{\beta_M \hat{X}_1^*} \left(\gamma_M (V_2^F(\hat{X}_1^*) - I) - \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1} \Big|_{x_1 = \hat{X}_1^*} \right) = \\ \qquad \mathrm{e}^{\beta_M (\hat{X}_1^{**})} \left(\gamma_M (V_2^F(\hat{X}_1^{**}) - I) - \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1} \Big|_{x_1 = \hat{X}_1^{**}} \right). \end{cases} \tag{4.28}$$

The relation between the thresholds of the inaction region, $\hat{X}_1^*$ *and* $\hat{X}_1^{**}$*, and the candidate investment thresholds in Proposition 4.2,* $\tilde{X}_1^*$ *and* $\tilde{X}_1^{**}$*, can be described as follows.*

1. *If* $g(\hat{X}_1^*) < 0$ *(and as a result* $B_2(\hat{X}_1^*) < 0$*), then* $\hat{X}_1^* < \tilde{X}_1^* < \hat{X}_1^{**} < \tilde{X}_1^{**}$.

2. *If* $g(\hat{X}_1^*) > 0$ *(and as a result* $B_2(\hat{X}_1^*) > 0$*), then* $\tilde{X}_1^* < \hat{X}_1^* < \tilde{X}_1^{**} < \hat{X}_1^{**}$.

3. *If* $g(\hat{X}_1^*) = 0$ *(and as a result* $B_2(\hat{X}_1^*) = 0$ *and* $B_1(\hat{X}_1^*) = A(\tilde{X}_1^*)$*), then* $\tilde{X}_1^* = \hat{X}_1^*$, *and* $\tilde{X}_1^{**} = \hat{X}_1^{**}$.

Numerical experiments show that if there exist two candidate solutions for the optimal investment threshold, $\tilde{X}_1^*$ and $\tilde{X}_1^{**}$, then the following is observed. If $\hat{X}_1^* < \tilde{X}_1^*$, then $\hat{W}(x_1) < W(x_1)$ for $x_1 > \tilde{X}_1^*$, which implies that $\hat{W}(\tilde{X}_1^{**}) < W(\tilde{X}_1^{**}) = V_2^F(\tilde{X}^{**})$. Thus, the option value $A(\tilde{X}_1^*)\mathrm{e}^{-\beta_M x_1}$ does not dominate the gain function, and, as a result, is not a solution to the optimal stopping problem. In the complementary case, it holds that $\hat{X}_1^{**} > \tilde{X}_1^{**}$ and neither of $A(\tilde{X}_1^{**})\mathrm{e}^{-\beta_M x_1}$ and $\hat{W}(x_1)$ is dominating. Therefore, the smallest superharmonic function that dominates $V_2^F(x_1)$ in this case is $A(\tilde{X}_1^*)\mathrm{e}^{-\beta_M x_1}$. Taking into account the above observations, we conclude that the continuation region of the optimal stopping problem is given by:

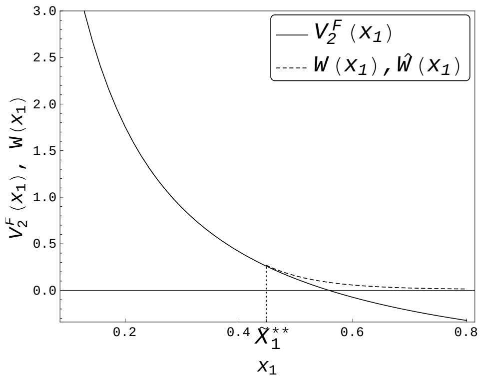
1. $(\tilde{X}_1^{**}, +\infty)$, if there exists a unique optimal threshold, X_1^*, and the gain function is superharmonic for $x_1 < X_1^*$,

2. $(\hat{X}_1^*, \hat{X}_1^{**}) \cup (\tilde{X}_1^{**}, +\infty)$, if the superharmonicity property is not satisfied and $\hat{X}_1^{**} \leq \tilde{X}_1^{**}$,

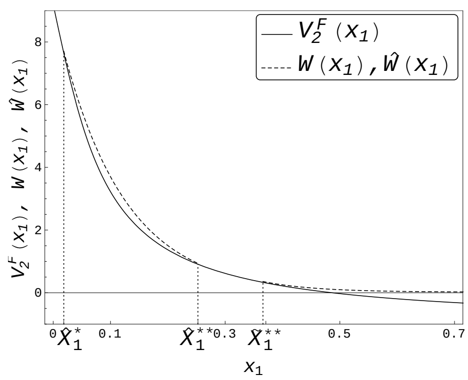
3. $(\tilde{X}_1^*, +\infty)$, if the superharmonicity property is not satisfied and $\hat{X}_1^{**} > \tilde{X}_1^{**}$.

Intuitively, the need of an additional continuation region can be explained by the non standard relation between the values of stopping and waiting. Before entry has occurred, the incumbent is a monopolist and, without facing any competition, is earning positive profits each period. Hence, there always exists a possibility that accumulated profit of the incumbent will never fall below a certain level. The longer the entrant waits in this case, the larger this level is. Naturally, for the entrant it is more attractive to enter when the incumbent is weak. Therefore, the new firm is facing the following trade-off. It can either wait until the larger expected payoff from a price war with a weaker incumbent is possible, even though the probability of reaching this state could be rather small; or to obtain a lower payoff by entering for a larger threshold, which can be reached with a higher probability. These investment thresholds correspond to the two candidate solutions of (4.22).

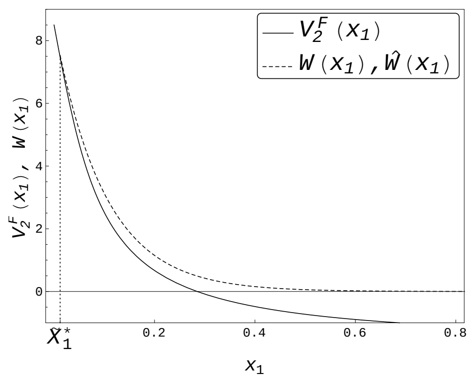
It follows from Proposition 4.3 that if there exist two candidate solutions for the threshold, there always exists an inaction region, where it is still worth waiting for the higher payoff. If x_1 hits the lower bound of this region it is optimal to enter, while at the upper bound the firm gives up waiting for the lower threshold, because the probability of reaching it is too small and, thus, invests. In the complementary cases, this region does not exist for two reasons. Either the investment opportunity is so valuable that it is never optimal to wait too long and the firm enters for the relatively high values of x_1, or the investment is only worth undertaking when x_1 is extremely small. The three described situations are illustrated in the Figure 4.5.



(a) For $x_2 = 0.12$ and $I = 0.8$.



(b) For $x_2 = 0.1$ and $I = 0.8$.



(c) For $x_2 = 0.1$ and $I = 1.5$.

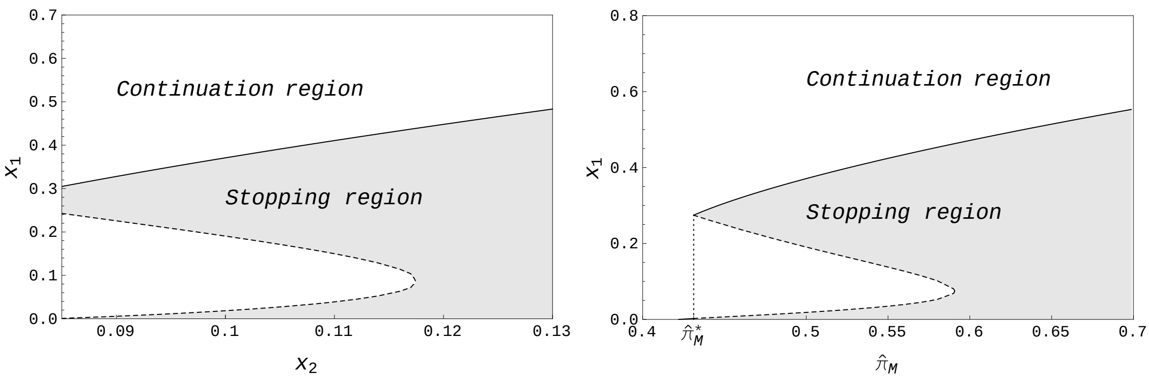
Figure 4.5: Continuation and stopping regions for the set of parameter values: $\pi_M = 0.5$, $\sigma_1 = 0.3$, $\sigma_2 = 0.4$, and $r = 0.05$.

In Figure 4.5a the investment costs are relatively small, while the initial reputation of the entrant is relatively high, therefore, it is worth entering only for a relatively large x_1. In Figure 4.5c the firm's reputation is relatively poor and the investment is costly, thus, it is only willing to enter if the probability of winning is relatively large. Figure 4.5b represents the intermediate scenario, where the inaction region occurs. For intermediate values of x_1 the firm has incentives to wait for the lower threshold, while when x_1 becomes sufficiently large it gives up this opportunity and enters the market. If x_1 becomes even larger the expected payoff is so small that the firm prefers to wait.

The next figures show how different parameter sets affect the stopping (gray) and continuation (white) regions, as well as investment thresholds. The solid curves represent the standard investment thresholds, $\tilde{X}_1^*$ and $\tilde{X}_1^{**}$, while the dashed curves stand for the thresholds of the inaction region, $\hat{X}_1^*$ and $\hat{X}_1^{**}$. The parameter sets in this case are chosen in such a way that it is optimal to fight for both firms.[5] In Figures 4.6, 4.7 and 4.8 the parameter values for which the jump in the optimal threshold occurs are marked with an asterisk. Notably, even though there exists a jump in the optimal threshold, the optimal value function F^* changes continuously. The reason is the fact that at the jump from the lower threshold, $\tilde{X}_1^*$, to the upper threshold, $\tilde{X}_1^{**}$, (or visa versa), the inaction thresholds are exactly equal to the standard thresholds: $\tilde{X}_1^* = \hat{X}_1^*$, and $\tilde{X}_1^{**} = \hat{X}_1^{**}$. Additionally, the option value in the inaction region, the value of the option to wait for $\tilde{X}_1^*$, and the value of the option to wait for $\tilde{X}_1^{**}$ are represented by the same function. Hence, at the jump the firm is indifferent between waiting for the lower threshold, for the upper threshold, or for either of the two.

Consider first the effects of x_2, the initial reputation of an entrant, and π_M, the monopoly profit, identified in Figure 4.6. This figure is to be read as follows. Suppose that $x_2 = 0.11$, then there exist four possible scenarios depending on the value of x_1:

1. if $x_1 > 0.41$, then the entrant waits until the solid line is hit from above;
2. if $0.15 \leq x_1 \leq 0.41$, then the entrant invests immediately;
3. if $0.04 < x_1 < 0.15$, the entrant waits until the dashed line is hit;
4. if $x_1 \leq 0.039$, then the entrant invests immediately.



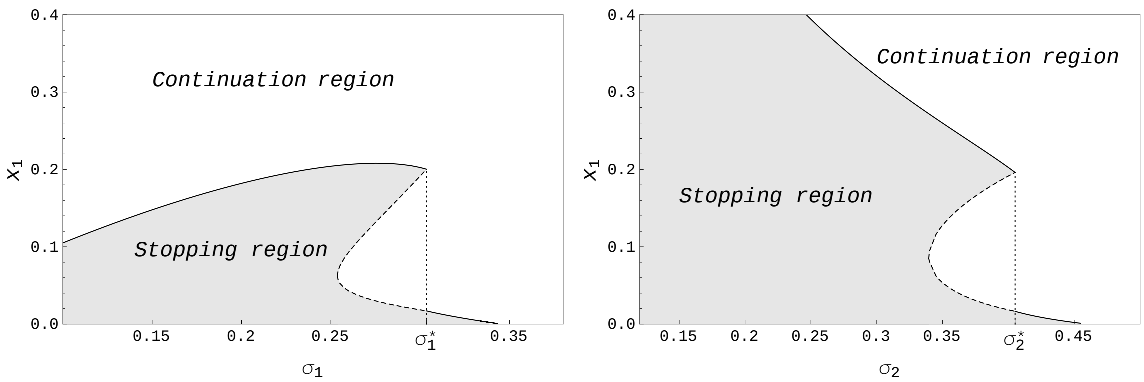
(a) For different values of x_2, with $\pi_M = 0.5$. *(b)* For different values of π_M, with $x_2 = 0.1$.

Figure 4.6: Continuation and stopping regions for the set of parameter values: $\pi_A = 0.3$, $\sigma_1 = 0.3$, $\sigma_2 = 0.4$, $r = 0.05$, and $I = 0.8$.

[5] Note that duopoly profits do not affect the inaction region, while the investment costs do not affect the trade-off between accommodating and fighting. Therefore, it is possible to find multiple scenarios where the inaction region exists by adjusting these parameters.

A larger x_2 implies that the entrant is stronger and willing to stand against a stronger incumbent. Thus, the investment threshold increases with x_2. A similar effect is observed when π_M is increased. Namely, the investment opportunity in this case becomes more valuable and the new firm enters the market sooner. In both cases the increased attractiveness of the investment opportunity causes the inaction region to become smaller.

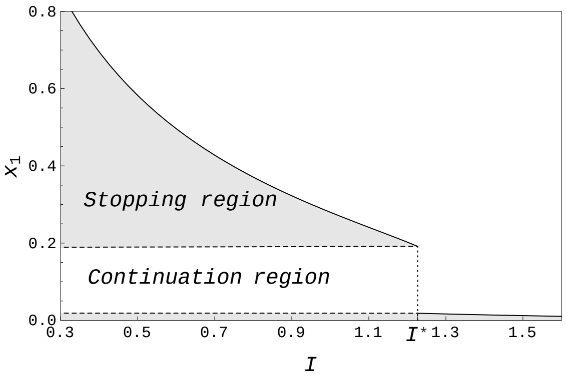
Figure 4.7 illustrates the effect of firm-specific uncertainty on the continuation and stopping regions of the entrant. When the volatility parameters are considered, we observe a non-monotonic behavior of the optimal investment threshold in σ_1. This is because an increase in the volatility of the incumbent firm has two opposite effects on the threshold. On the one hand, in a more uncertain environment the option to invest becomes more valuable, thus, the entrant is less willing to give up this flexibility and prefers to invest later. On the other hand, a larger volatility of the opponent implies larger probability to win the price war, which makes the investment more appealing for large σ_1. This creates incentives to enter earlier, i.e. for larger x_1. For the threshold $\tilde{X}_1^{**}$ the former effect is dominating for larger values of σ_1. For the smaller trigger, $\tilde{X}_1^{*}$, only the latter effect is observed.



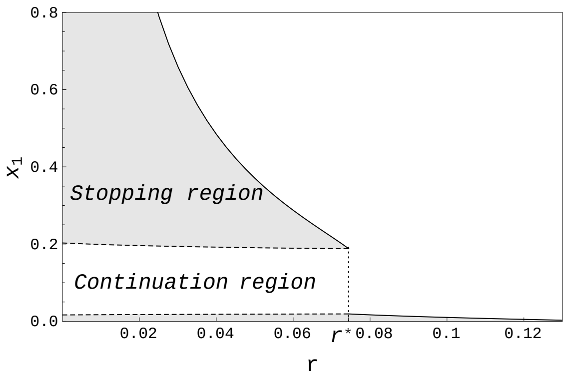
(a) For different values of σ_1, with $\sigma_2 = 0.4$. *(b)* For different values of σ_2, with $\sigma_1 = 0.3$.

Figure 4.7: Continuation and stopping regions for the set of parameter values: $x_2 = 0.1$, $\pi_A = 0.3$, $\pi_M = 0.5$, $r = 0.05$, and $I = 1.2$.

As can be seen in Figure 4.7b, for σ_2 this trade-off does not exist, because a larger uncertainty of the entrant decreases its probability of winning the price war. Therefore, in this case the effect works in the opposite direction and both triggers decline with σ_2. The inaction region in Figure 4.8b becomes larger for larger σ_1 and σ_2, explained by the value of waiting effect earlier on. For σ_1 this effect is more pronounced because when the incumbents profits are more volatile it is more likely to reach lower thresholds for the entrant.



(a) For different values of I.



(b) For different values of r.

Figure 4.8: Continuation and stopping regions for the set of parameter values: $x_2 = 0.1$, $\pi_A = 0.3$, $\pi_M = 0.5$, $\sigma_1 = 0.3$, $\sigma_2 = 0.4$, $r = 0.05$, and $I = 0.8$.

In Figure 4.8 the investment costs and the discount rate show similar influence on the optimal thresholds. In particular, the entrant prefers to wait longer for larger investment costs, as the investment opportunity becomes less attractive. Similarly, for larger values of r, the entrant discounts its future profits more heavily, thus, prefers to invest later. Additionally, we observe a slight expansion of the inaction region, as a result of an increase in I and a decrease in r. As mentioned earlier, a more costly investment reduces the entrant's incentives to enter the market, making it worthwhile to wait longer for more favorable investment conditions. An increase in discount rate implies that the entrant values future payoff less, therefore the inaction region shrinks with r.

4.4.2 Accommodation region

This section presents the scenarios when the accommodation strategy is chosen. Here we focus on the implications of the differences between the accommodation and fighting scenarios for the optimal stopping problem of the entrant. We start with looking back at the value function of the entrant under accommodation strategy[6].

$$V_2^A(x_1) = \frac{\pi_A}{r}\left(1 - Q_2^A(x_1)\right) + Q_1^A(x_1)\frac{\pi_M - \pi_A}{r}, \tag{4.29}$$

with the stochastic discount factors defined in Proposition 4.1.

Its major difference with the aggressive pricing strategy is associated with the fact that the firms obtain positive duopoly profits until either of them goes bankrupt. This means that the accumulated profits are now represented by drifted Brownian

[6]Recall that x_2 enters the stochastic discount factors, Q_1^A and Q_2^A, and, thus, the value function, V_2^A, as a parameter.

motions, with $\pi_A > 0$. Due to the positive drift of the accumulated profit process, the first bankruptcy event occurs later than in the predation game. Moreover, unlike during the price war, there exists a positive probability that none of the firms ever goes bankrupt and they coexist in a duopoly forever.

Following the approach of the previous section, we determine the candidates for the optimal investment threshold and identify whether the superharmonicity condition is satisfied. Technically, these cases look quite similar, but taking a deeper look at the matter we find a few significant differences. One of the distinctive features of the accommodation scenario is that for certain parameter sets the entrant is willing to invest even when the incumbent has extremely strong reputation. The above findings are summarized in the next proposition.

Proposition 4.5 *Let the function g_A be given by*

$$g_A(x_1) = -\frac{\pi_A}{r}\left(1 - \frac{1}{\beta_M}\frac{\partial Q_2^A(x_1)}{\partial x_1} - Q_2^A(x_1)\right) - \frac{\pi_M - \pi_A}{r}\left(\frac{1}{\beta_M}\frac{\partial Q_1^A(x_1)}{\partial x_1} + Q_1^A(x_1)\right) + I. \tag{4.30}$$

Then the candidates for the optimal thresholds are determined by

$$g_A(x_1) = 0. \tag{4.31}$$

In addition, it holds that

$$\lim_{x_1 \to \infty} g_A(x_1) = -\frac{\pi_A}{r}\left(1 - \mathrm{e}^{-\beta_2^A x_2}\right) + I. \tag{4.32}$$

When $\lim_{x_1 \to \infty} g_A(x_1) < 0$, the new firm prefers to invest even for infinitely large values of x_1. Intuitively, in this case the incumbent is certainly going to leave the market last. Then, if the entrant's expected discounted earnings until bankruptcy exceed its investment costs, it undertakes an investment despite the infinitely large accumulated profits of the opponent. In other words, if the duopoly situation is quite favorable, it does not matter how strong the incumbent firm is, the new firm always has an incentive to enter the market.

Recall, however, that firms are particularly inclined to use the accommodation strategy when the benefits of being in a monopoly position are not too large in comparison to duopoly, i.e. for small values of π_M. In this case violation of the superharmonicity condition is more likely to occur. This means that when x_1 falls below a certain level, it becomes optimal to wait until it hits an even lower level, which brings a larger payoff. Hence, the entrant may choose to invest for small values of x_1, wait for intermediate values, and stop for large values. In the extreme case, the first stopping region may consist of only one point, $x_1 = 0$. This situation is illustrated in Figure 4.9 .

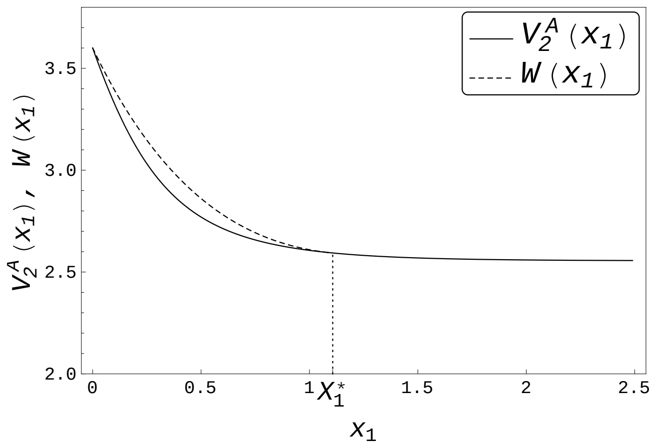


Figure 4.9: The continuation and the stopping values of the entrant for the set of parameter values: $x_2 = 0.2$, $\pi_A = 0.2$, $\pi_M = 0.22$, $\sigma_1 = 0.6$, $\sigma_2 = 0.3$, $r = 0.05$, and $I = 0.8$.

Figure 4.9 shows the gain function and the option value of the entrant for small monopoly payoff and relatively large volatility parameter of the incumbent. Here, the stream of duopoly profits until bankruptcy make the investment so appealing that the new firm enters even for extremely large values of x_1. On the other hand, for a small level of the incumbent's accumulated profit we observe a steep increase in the value function, which makes it optimal to wait until zero is hit. The reason is that in this region the incumbent is so weak, and thus, is very likely to leave the market soon, so that the entrant is motivated to wait until this happens. Hence, one of the implications of the model, which at first seems rather counterintuitive, is that in the accommodation region the firm has incentives to wait for small x_1 and to invest for large x_1.

4.5 Conclusion

This chapter studies the stochastic predatory pricing game, where the entrant undertakes an irreversible investment in a market with an existing incumbent. The future profits in this market are subject to firm-specific stochastic shocks. The market participants are subject to to reputational risks and may predatory strategy to drive the opponent out of the market. Due to the firm specific nature of uncertainty in the model, the winner of the price war cannot be determined *ex ante*, and the outcome of the game depends on the realizations of the underlying stochastic process. This translates the firms' decisions into probabilistic terms, namely, the choice of a certain strategy is now based on survival probabilities. This study offers several contributions to the literature.

First, based on the hitting time distributions of two uncorrelated Brownian motions, we derive stochastic discount factors and identify when the firms have incentives for predatory behavior. We find that even though the success of aggressive pricing strategy is not guaranteed, both firms are willing to engage in a price war in certain scenarios. From this perspective, our model contributes to the bulk of theoretic literature on predatory pricing, explaining the rationale behind price wars by the presence of uncertainty in the future profits. This model goes in line with the reputational argument, particularly, we demonstrate that firms with a strong reputation, i.e. with large accumulated profits, are more inclined to price aggressively facing a weaker opponent. In addition, the firms have more incentives to predate for a larger monopoly payoff, i.e. a larger "prize" for the winner of the predation game. The effect of the volatility parameters on predatory behavior exhibits non-monotonic behavior. This is because when one player's volatility parameter is relatively small, its increase reduces the probability to be the last one standing in the price war, as a result, the firm postpones the decision to engage in predatory pricing. If the firm is more uncertain about its future profits a further increase in the volatility, drives down the probability to stay in the market forever when accommodating, so that the chances survival under fighting and accommodation strategies are becoming closer. Facing an imminent bankruptcy the firm accelerates its decision to price aggressively to reduce the chances of survival of its opponent. The opposite effect is observed when the uncertainty of the rival firm is considered.

Second, we solve the optimal stopping problem of the entrant for the fighting and accommodation scenarios. This problem, however, differs from the standard real options models due to the fact that the new firm's investment decision is based upon the value of its opponent's accumulated profit. In addition, the entrant's investment changes the drift of the stochastic process, represented by the firm's current profits. It is either driven down to zero, if one of the firms decides to fight, or to the level of duopoly profits, if both firms prefer to accommodate. Intuition suggests that in this setting the entrant postpones its investment if the incumbent's accumulated profit is large and enters, when it falls below the optimal threshold. Even though this situation indeed occurs for certain parameter sets, in general it is not always the case. In such cases, we find that the continuation region may not be a connected set. In fact, the continuation region then consists of two disjoint parts, separated from the stopping set by three thresholds: two of them trigger investment from above, while the third one – from below. In the accommodation scenario it is even possible that the state space of the stochastic process consists of only stopping and inaction regions, while the standard continuation region does not exist. The inaction region occurs due to the complex relation between probability of survival for a certain value of the opponent's accumulated profit and the probability of ever reaching this value.

If the incumbent is alone in the market the process for it's accumulated profit has a positive drift and there exists a positive probability that it never falls below a certain level. The entrant then faces a choice between two opportunities, which under certain parameter sets are equally valuable. It can either invest for a relatively large value of the incumbent's accumulated profit level, which implies a modest expected payoff, but a high probability of obtaining it; or it can wait until the accumulated profit of the incumbent decreases, substantially increasing its expected gain, but facing a smaller probability that this will ever happen. The inaction region represents the values of the incumbent's accumulated profit such that the entrant prefers to wait either until the accumulated profit is low enough so that the entry is optimal, or until it is so large, that the probability of such a decrease is sufficiently low.

We demonstrate that the inaction region becomes larger when the attractiveness of the investment opportunity decreases, i.e. when the entrant cares more about the expected payoff. This happens for relatively large values of investment costs and discount rate, or relatively small values of the initial reputation of the entrant or the monopoly payoff. The inaction region also becomes larger in a more uncertain environment, when the entrant values its flexibility more. The incumbent's uncertainty, however, has a non-monotonic effect on the entrant's investment threshold. This is because, on the one hand, due to the above argument, a larger uncertainty creates incentives to postpone the investment. On the other hand, the more uncertain is the incumbent about its future payoff, the larger is the survival probability of the entrant, thus, it is willing to invest sooner. One should note, however, that if the investment becomes sufficiently unattractive, the inaction region disappears and the investment is only possible for the lower trigger, i.e. only when the survival probability is sufficiently large. At this point we observe a jump in the optimal investment threshold as the parameters change. Nevertheless, the jump does not occur in the optimal value function, which changes continuously.

Lastly, it is important to discuss the possibilities for further research. A valuable addition to our study would be to consider the optimal stopping problem when both fighting and accommodation occurs. The solution in this case is largely complicated by the presence of the jump in the value function of the entrant. This jump occurs at the point when the incumbent decides to switch from fighting to accommodation. In addition, the decision of the entrant to change the strategy does not create a discontinuity in the value function itself, but in its derivative. Nevertheless, this analysis may yield more interesting results. Another potential extension of our framework is generalizing the model to include the correlation between the profit processes of the firm.

4.6 Appendix

Proof of Proposition 4.1 Let X_{1t} and X_{2t} be two uncorrelated Brownian motions with constant drifts with the statring values of zero:

$$X_{it} = \mu_i t + \sigma_t B_{it}, \tag{4.33}$$

and

$$\tau_i = \inf\{t > 0 : X_{it} = -x_i\}, \tag{4.34}$$

with $i = \{1, 2\}$.

Given the first passage time density of Brownian motion (see Karatzas and Shreve (1991)), the survival probability is

$$\Pr(\tau_i > t) = \int_t^\infty \frac{x_i}{\sigma_i\sqrt{2\pi s^3}} \mathrm{e}^{-\frac{(x_i+\mu_i s)^2}{2\sigma_i^2 s}} ds. \tag{4.35}$$

Alternatively, consider the minimum process

$$\underline{M}_t^i = \min_{0<s<t} X_{is}. \tag{4.36}$$

Then the CDF and the survival probability can be obtained as follows

$$\Pr(\tau_i < t) = \Pr(\underline{M}_t^i < -x_i) = 1 - \Phi\left(\frac{x_i + \mu_i t}{\sigma_i\sqrt{t}}\right) + \mathrm{e}^{-\frac{2\mu_i x_i)_i}{\sigma_i^2}} \Phi\left(\frac{-x_i + \mu_i t}{\sigma_i\sqrt{t}}\right), \tag{4.37}$$

$$\Pr(t < \tau_i < \infty) + \Pr(\tau_i = \infty) = \Phi\left(\frac{x_i + \mu_i t}{\sigma_i\sqrt{t}}\right) + \mathrm{e}^{-\frac{2\mu_i x_i}{\sigma_i^2}} \Phi\left(\frac{-x_i + \mu_i t}{\sigma_i\sqrt{t}}\right), \tag{4.38}$$

where Φ is the CDF of the standard normal distribution.

Using the above CDF we can derive the following

$$\begin{aligned}\mathbb{E}_{x_1,x_2}\left[\mathrm{e}^{-r\tau_1}\mathbb{1}_{\tau_1<\tau_2}\right] &= \int_0^\infty \mathrm{e}^{-rt}\frac{x_1}{\sigma_1\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(x_1+\mu_1 t)^2}{2\sigma_1^2 t}} dt \int_t^\infty \frac{x_2}{\sigma_2\sqrt{2\pi s^3}}\mathrm{e}^{-\frac{(x_2+\mu_2 t)^2}{2\sigma_2^2 s}} ds \\ &= \int_0^\infty \mathrm{e}^{-rt}\frac{x_1}{\sigma_1\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(x_1+\mu_1 t)^2}{2\sigma_1^2 t}} \left(\Phi\left(\frac{x_2 + \mu_2 t}{\sigma_2\sqrt{t}}\right) - \mathrm{e}^{-\frac{2\mu_2 x_2}{\sigma_2^2}} \Phi\left(\frac{-x_2 + \mu_2 t}{\sigma_2\sqrt{t}}\right)\right) dt.\end{aligned} \tag{4.39}$$

Substituting the means of the Brownian motions with the duopoly profits, π_k, and interchanging τ_1 and τ_2 yields the result of the proposition.

Proof of Lemma 1 First, note that

$$\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}\mathbb{1}_{\tau_i<\tau_j}\right] = \mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}|\tau_i < \tau_j\right] P(\tau_i < \tau_j). \tag{4.40}$$

It follows from the proof of Proposition 4.1, that the probability that firm 1 goes bankrupt before firm 2 is given by

$$\Pr(\tau_1<\tau_2)=\int_0^\infty \frac{x_1}{\sigma_1\sqrt{2\pi\tau_1^3}}e^{-\frac{(x_1+\mu_1\tau_1)^2}{2\sigma_1^2\tau_1}}\left(\Phi\left(\frac{x_2+\mu_2\tau_1}{\sigma_2\sqrt{\tau_1}}\right)-\mathrm{e}^{-\frac{2\mu_2x_2}{\sigma_2^2}}\Phi\left(\frac{-x_2+\mu_2\tau_1}{\sigma_2\sqrt{\tau_1}}\right)\right)d\tau_1,$$

and the probability that firm 2 goes bankrupt before firm 1 is given by

$$\Pr(\tau_1>\tau_2)=\left(1-e^{-\frac{2\mu_1x_1}{\sigma_1^2}}\right)\left(1-e^{-\frac{2\mu_2x_2}{\sigma_2^2}}\right)-\Pr(\tau_1<\tau_2). \tag{4.41}$$

Additionally, it holds that

$$\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}\mathbb{1}_{\tau_i<\tau_j}\right]+\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}\mathbb{1}_{\tau_i>\tau_j}\right]=\mathbb{E}_{x_i}\left[\mathrm{e}^{-r\tau_i}\right], \tag{4.42}$$

where $\mathbb{E}_{x_i}\left[\mathrm{e}^{-r\tau_i}\right]=\mathrm{e}^{-\beta_i x_i}$, with $\beta_i=\frac{\mu_i+\sqrt{\mu_i^2+2\sigma_i^2 r}}{\sigma_1^2}$. Taking into account (4.40) we can rewrite (4.42) as

$$\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}|\tau_i<\tau_j\right]P(\tau_i<\tau_j)+\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}|\tau_i>\tau_j\right]P(\tau_i>\tau_j)=\mathrm{e}^{-\beta_i x_i}. \tag{4.43}$$

If $x_i=0$, then $P(\tau_i<\tau_j)=1$ and $P(\tau_i>\tau_j)=0$. Therefore, $\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}\mathbb{1}_{\tau_i<\tau_j}\right]=\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}|\tau_i<\tau_j\right]=1$. If $x_j=0$, then the opposite holds and $\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}\mathbb{1}_{\tau_i<\tau_j}\right]=0$.

If $x_i\to\infty$, then $\lim_{x_i\to\infty}P(\tau_i<\tau_j)=0$ and $\lim_{x_i\to\infty}P(\tau_i>\tau_j)=1$. Hence, $\lim_{x_i\to\infty}\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}\mathbb{1}_{\tau_i<\tau_j}\right]=0$. If $x_j\to\infty$, then the opposite holds: $\lim_{x_j\to\infty}P(\tau_i<\tau_j)=1$ and $\lim_{x_j\to\infty}P(\tau_i>\tau_j)=0$. Hence, $\lim_{x_j\to\infty}\mathbb{E}_{x_i,x_j}\left[\mathrm{e}^{-r\tau_i}\mathbb{1}_{\tau_i<\tau_j}\right]=\mathrm{e}^{-\beta_i x_i}$.

Now recall that the difference between accommodation and fighting values for the incumbent is given by

$$D_1^{AF}(x_1)=\frac{\pi_A}{r}\left(1-Q_1^A(x_1)\right)+Q_2^A(x_1)\frac{\pi_M-\pi_A}{r}-Q_2^F(x_1)\frac{\pi_M}{r}, \tag{4.44}$$

where $Q_1^k(x_1)=\mathbb{E}^k_{x_1,x_2}\left[\mathrm{e}^{-r\tau_1}\mathbb{1}_{\tau_1<\tau_2}\right]$ and $Q_2^k(x_1)=\mathbb{E}^k_{x_1,x_2}\left[\mathrm{e}^{-r\tau_2}\mathbb{1}_{\tau_2<\tau_1}\right]$. It follows from the above observations that

$$D_1^{AF}(0)=0, \tag{4.45}$$

and

$$\begin{aligned}\lim_{x_1\to\infty}D_1^{AF}(x_1) &= \frac{\pi_A}{r}+\mathrm{e}^{-\beta_2^A x_2}\frac{\pi_M-\pi_A}{r}-\mathrm{e}^{-\beta_2^F x_2}\frac{\pi_M}{r}\\ &= \frac{\pi_A}{r}+\mathrm{e}^{-\beta_2^A x_2}\frac{\pi_M-\pi_A}{r}-\mathrm{e}^{-\beta_2^F x_2}\frac{\pi_M}{r}+\frac{\pi_M}{r}-\frac{\pi_M}{r}\\ &= \left(1-\mathrm{e}^{-\beta_2^F x_2}\right)\frac{\pi_M}{r}-\left(1-\mathrm{e}^{-\beta_2^A x_2}\right)\frac{\pi_M-\pi_A}{r}\end{aligned}$$

$$= \left(1 - e^{-\beta_2^F x_2}\right) \frac{\pi_M - \pi_A}{r} \left(\frac{\pi_M}{\pi_M - \pi_A} - \frac{1 - e^{-\beta_2^A x_2}}{1 - e^{-\beta_2^F x_2}} \right). \quad (4.46)$$

Note that $\frac{\pi_M}{\pi_M - \pi_A} > 1$ and $\frac{1-e^{-\beta_2^A x_2}}{1-e^{-\beta_2^F x_2}} > 1$. Moreover,

$$\lim_{x_2 \to 0} \frac{1 - e^{-\beta_2^A x_2}}{1 - e^{-\beta_2^F x_2}} = \frac{\beta_2^A}{\beta_2^F} = \frac{\pi_A + \sqrt{\pi_A^2 + 2r\sigma_2^2}}{\sqrt{2r\sigma_2^2}}, \quad (4.47)$$

and

$$\frac{\partial}{\partial x_2} \left(\frac{1 - e^{-\beta_2^A x_2}}{1 - e^{-\beta_2^F x_2}} \right) = \frac{e^{x_2\left(\beta_2^F - \beta_2^A\right)} \left(\beta_2^F \left(1 - e^{\beta_2^A x_2}\right) - \beta_2^A \left(1 - e^{\beta_2^F x_2}\right)\right)}{\left(e^{\beta_2^F x_2} - 1\right)^2} < 0. \quad (4.48)$$

Hence, if $\frac{\pi_M}{\pi_M - \pi_A} > \frac{\pi_A + \sqrt{\pi_A^2 + 2r\sigma_2^2}}{\sqrt{2r\sigma_2^2}}$, the difference is always positive, i.e. the incumbent prefers to accommodate even for infinite values of x_1. In the complementary case, when $\frac{\pi_M}{\pi_M - \pi_A} \leq \frac{\pi_A + \sqrt{\pi_A^2 + 2r\sigma_2^2}}{\sqrt{2r\sigma_2^2}}$ there exists a unique x_2 such that if x_1 goes to infinity, for the smaller values of the entrant's accumulated profits the incumbent will choose to fight, while for the larger values – to accommodate.

The difference between the value functions for the entrant is defined as follows

$$D_2^{AF}(x_1) = \frac{\pi_A}{r} \left(1 - Q_2^A(x_1)\right) + Q_1^A(x_1) \frac{\pi_M - \pi_A}{r} - Q_1^F(x_1) \frac{\pi_M}{r}. \quad (4.49)$$

Similarly to the previous vase, we conclude that

$$D_2^{AF}(0) = 0 \quad (4.50)$$

and

$$\lim_{x_1 \to \infty} D_2^{AF}(x_1) = \frac{\pi_A}{r} \left(1 - e^{-\beta_2^A x_2}\right) \geq 0. \quad (4.51)$$

Proof of Proposition 4.2 The value matching and smooth pasting conditions give

$$\begin{cases} Ae^{-\beta_M X_1^*} = V_2^F(X_1^*) - I, \\ -A\beta_M e^{-\beta_M X_1^*} = \left.\frac{\partial V_2^F(x_1)}{\partial x_1}\right|_{x_1 = X_1^*}, \end{cases} \quad (4.52)$$

where $V_2^F(x_1) = Q_1^F(x_1)\frac{\pi_M}{r}$, and $\frac{\partial V_2^F(x_1)}{\partial x_1} = \frac{\partial Q_1^F(x_1)}{\partial x_1}\frac{\pi_M}{r}$.

Solving the above system for A and X_1^* we get

$$A(X_1^*) = -e^{\beta_M X_1^*} \frac{\pi_M}{r\beta_M} \left.\frac{\partial Q_1^F(x_1)}{\partial x_1}\right|_{x_1 = X_1^*} \quad (4.53)$$

and

$$-\frac{\pi_M}{r}\left(\frac{1}{\beta_M}\frac{\partial Q_1^F(x_1)}{\partial x_1}\bigg|_{x_1=X_1^*} + Q_1^F(X_1^*)\right) + I = 0. \tag{4.54}$$

Additionally, if there exists a solution, X_n^*, such that $\frac{\partial g(x_1)}{\partial x_1}\Big|_{x_1=X_n^*} < 0$, it is possible to show that X_n^* is never the optimal investment trigger. Namely, if for $x_1 < X_n^*$ the function $V(x_1)$ is steeper than $W(x_1)$, i.e. $\left|\frac{\partial V_2^F(x_1)}{\partial x_1}\right| > \left|\frac{\partial W(x_1)}{\partial x_1}\right|$, then the option value lies below the investment value, and, thus, X_n^* cannot be the optimal investment threshold. The difference of the absolute values of the derivatives can be written as follows

$$\begin{aligned}\left|\frac{\partial V_2^F(x_1)}{\partial x_1}\right| - \left|\frac{\partial W(x_1)}{\partial x_1}\right| &= -\frac{\partial Q_1^F(x_1)}{\partial x_1}\frac{\pi_M}{r} + \mathrm{e}^{-\beta_M(x_1-X_1^*)}\frac{\pi_M}{r}\frac{\partial Q_1^F(x_1)}{\partial x_1}\bigg|_{x_1=X_n^*} \\ &= \mathrm{e}^{-\beta_M x_1}\frac{\pi_M}{r}(f(X_n^*) - f(x_1)),\end{aligned} \tag{4.55}$$

where $f(x_1) = \mathrm{e}^{\beta_M}\frac{\partial Q_1^F(x_1)}{\partial x_1}$. Consider now its derivative

$$\frac{\partial f(x_1)}{\partial x_1} = \beta_M \mathrm{e}^{\beta_M}\left(\frac{1}{\beta_M}\frac{\partial^2 Q_1^F(x_1)}{\partial X_1^2} + \frac{\partial Q_1^F(x_1)}{\partial x_1}\right). \tag{4.56}$$

From (4.54) $\frac{\partial g(x_1)}{\partial x_1}$ always has the opposite sign to $\frac{\partial f(x_1)}{\partial x_1}$. Thus, in the neighborhood of X_n^* it holds that $\frac{\partial f(x_1)}{\partial x_1} > 0$ and, as a result, $\left|\frac{\partial V(x_1)}{\partial x_1}\right| - \left|\frac{\partial W(x_1)}{\partial x_1}\right| > 0$ for $x_1 < X_n^*$. Hence, X_n^* is not an optimal investment trigger.

Proof of Proposition 4.3 It follows from (4.52) that for any function V_2 (i.e. both for fighting and accommodating) it holds that $g(x_1) = -\frac{1}{\beta_M}\frac{\partial V_2(x_1)}{\partial x_1} - V_2(x_1) + I$ and $\frac{\partial g(x_1)}{\partial x_1} = -\frac{1}{\beta_M}\frac{\partial^2 V_2(x_1)}{\partial x_1^2} - \frac{\partial V_2(x_1)}{\partial x_1}$.

Then we can derive the following relation

$$\begin{aligned}\mathcal{L}V_2(x_1) - rV_2(x_1) &= \frac{1}{2}\sigma_1^2\frac{\partial^2 V_2(x_1)}{\partial X_1^2} + \pi_M\frac{\partial V_2(x_1)}{\partial x_1} - r(V_2(x_1) - I) \\ &= \frac{1}{2}\sigma_1^2\frac{\partial^2 V_2(x_1)}{\partial X_1^2} + \frac{1}{\beta_M}\frac{\partial V_2(x_1)}{\partial x_1}\left(\frac{1}{2}\sigma_1^2\beta_M^2 - r\right) - r(V_2(x_1) - I) \\ &= \frac{1}{2}\sigma_1^2\beta_M\left(\frac{1}{\beta_M}\frac{\partial^2 V_2(x_1)}{\partial X_1^2} + \frac{\partial V_2(x_1)}{\partial x_1}\right) \\ &\quad -r\left(\frac{1}{\beta_M}\frac{\partial V_2(x_1)}{\partial x_1} + r(V_2(x_1) - I)\right) \\ &= -\frac{\beta_M\sigma_1^2}{2}\frac{\partial g(x_1)}{\partial x_1} + rg(x_1).\end{aligned} \tag{4.57}$$

Moreover, at X_1^* it holds that

$$\mathcal{L}V_2(X_1^*) - rV_2(X_1^*) = -\frac{\beta_M \sigma_1^2}{2} \frac{\partial g(x_1)}{\partial x_1}\bigg|_{x_1 = X_1^*}. \tag{4.58}$$

Hence, we can conclude that $\mathcal{L}V_2(X_1^*) - rV_2(X_1^*) \geq 0$, when $\frac{\partial g(x_1)}{\partial x_1}\Big|_{x_1=X_1^*} \leq 0$ and $\mathcal{L}V_2(X_1^*) - rV_2(X_1^*) < 0$, when $\frac{\partial g(x_1)}{\partial x_1}\Big|_{x_1=X_1^*} > 0$. It means that if (4.54) has a solution with a negative derivative, there always exist a region where superharmonicity is not satisfied.

Proof of Proposition 4.4 In order to find the option value we need to consider the boundary conditions of value matching and smooth pasting at the two candidate thresholds, $\hat{X}_1^*$ and $\hat{X}_1^{**}$:

$$\begin{cases} B_1 \mathrm{e}^{-\beta_M \hat{X}_1^*} + B_2 \mathrm{e}^{\gamma_M \hat{X}_1^*} = \; V_2^F(\hat{X}_1^*) - I, \\ -B_1\beta_M \mathrm{e}^{-\beta_M \hat{X}_1^*} + B_2\gamma_M \mathrm{e}^{\gamma_M \hat{X}_1^*} = \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1}\Big|_{x_1=\hat{X}_1^*} \\ B_1 \mathrm{e}^{-\beta_M \hat{X}_1^{**}} + B_2 \mathrm{e}^{\gamma_M \hat{X}_1^{**}} = \; V_2^F(\hat{X}_1^{**}) - I, \\ -B_1\beta_M \mathrm{e}^{-\beta_M \hat{X}_1^{**}} + B_2\gamma_M \mathrm{e}^{\gamma_M \hat{X}_1^{**}} = \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1}\Big|_{x_1=\hat{X}_1^{**}}, \end{cases} \tag{4.59}$$

where $V_2^F(x_1) = Q_1^F(x_1)\frac{\pi_M}{r}$, and $\frac{\partial V_2^F(x_1)}{\partial \hat{x}_1} = \frac{\partial Q_1^F(x_1)}{\partial x_1}\frac{\pi_M}{r}$. Further simplification yields

$$\begin{cases} B_1 = \mathrm{e}^{\beta_M \hat{X}_1^*} \frac{1}{\beta_M+\gamma_M}\left(\gamma_M(V_2^F(\hat{X}_1^*) - I) - \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1}\Big|_{x_1=\hat{X}_1^*}\right), \\ B_2 = \mathrm{e}^{-\gamma_M \hat{X}_1^*} \frac{1}{\beta_M+\gamma_M}\left(\beta_M(V_2^F(\hat{X}_1^*) - I) + \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1}\Big|_{x_1=\hat{X}_1^*}\right), \\ \mathrm{e}^{-\gamma_M \hat{X}_1^*}\left(\beta_M(V_2^F(\hat{X}_1^*) - I) + \frac{\partial V_2^F(x_1)}{\partial x_1}\Big|_{x_1=\hat{X}_1^*}\right) = \\ \qquad \mathrm{e}^{-\gamma_M \hat{X}_1^{**}}\left(\beta_M(V_2^F(\hat{X}_1^{**}) - I) + \frac{\partial V_2^F(x_1)}{\partial x_1}\Big|_{x_1=\hat{X}_1^{**}}\right), \\ \mathrm{e}^{\beta_M \hat{X}_1^*}\left(\gamma_M(V_2^F(\hat{X}_1^*) - I) - \frac{\partial V_2^F(x_1)}{\partial x_1}\Big|_{x_1=\hat{X}_1^*}\right) = \\ \qquad \mathrm{e}^{\beta_M \hat{X}_1^{**}}\left(\gamma_M(V_2^F(\hat{X}_1^{**}) - I) - \frac{\partial V_2^F(x_1)}{\partial x_1}\Big|_{x_1=\hat{X}_1^{**}}\right), \end{cases} \tag{4.60}$$

Note now that the system can be simplified further using the function $g(x_1)$, defined by (4.21).

$$\begin{cases} B_1 = \mathrm{e}^{\beta_M \hat{X}_1^*} \frac{1}{\beta_M+\gamma_M} \left(\gamma_M (V_2^F(\hat{X}_1^*) - I) - \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1}\Big|_{x_1=\hat{X}_1^*} \right), \\ B_2 = -\mathrm{e}^{-\gamma_M \hat{X}_1^*} \frac{1}{\beta_M+\gamma_M} g(\hat{X}_1^*), \\ \mathrm{e}^{-\gamma_M \hat{X}_1^*} g(\hat{X}_1^*) = \mathrm{e}^{-\gamma_M \hat{X}_1^{**}} g(\hat{X}_1^{**}), \\ \mathrm{e}^{\beta_M \hat{X}_1^*} \left(\gamma_M (V_2^F(\hat{X}_1^*) - I) - \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1}\Big|_{x_1=\hat{X}_1^*} \right) = \\ \qquad \mathrm{e}^{\beta_M (\hat{X}_1^{**})} \left(\gamma_M (V_2^F(\hat{X}_1^{**}) - I) - \frac{\partial V_2^F(x_1)}{\partial \hat{x}_1}\Big|_{x_1=\hat{X}_1^{**}} \right). \end{cases} \tag{4.61}$$

Consider the function $\mathrm{e}^{-\gamma_M x_1} g(x_1)$. This function has the same zeros as $g(x_1)$. Moreover, as $\mathrm{e}^{-\gamma_M x_1} < 1$, it is always smaller than $g(x_1)$ if $g(x_1) > 0$, and larger than $g(x_1)$ if $g(x_1) < 0$. This observation allows us to conclude that if there exists a solution, $(\hat{X}_1^*, \hat{X}_1^{**})$[7], then the following holds:

1. If $g(\hat{X}_1^*) < 0$ (and $g(\hat{X}_1^{**}) < 0$), then it holds that $\hat{X}_1^* < \tilde{X}_1^* < \hat{X}_1^{**} < \tilde{X}_1^{**}$, where $\tilde{X}_1^*$ and $\tilde{X}_1^{**}$ are the solutions of $g(x_1) = 0$,[8]

2. If $g(\hat{X}_1^*) > 0$ (and $g(\hat{X}_1^{**}) > 0$), then it holds that $\tilde{X}_1^* < \hat{X}_1^* < \tilde{X}_1^{**} < \hat{X}_1^{**}$.

3. If $g(\hat{X}_1^*) = 0$ (and $g(\hat{X}_1^{**}) = 0$), then it holds that $\tilde{X}_1^* = \hat{X}_1^*$ and $\tilde{X}_1^{**} = \hat{X}_1^{**}$.

Proof of Proposition 4.5 The value matching and smooth pasting can be written as

$$\begin{cases} A_A \mathrm{e}^{-\beta_M X_{1A}^*} = \frac{\pi_A}{r} \left(1 - Q_2^A(X_{1A}^*) \right) + Q_1^A(X_{1A}^*) \frac{\pi_M - \pi_A}{r} - I, \\ -A_A \beta_M \mathrm{e}^{-\beta_M X_{1A}^*} = -\frac{\pi_A}{r} \frac{\partial Q_2^A(x_1)}{\partial x_1}\Big|_{x_1=X_{1A}^*} + \frac{\pi_M-\pi_A}{r} \frac{\partial Q_1^A(x_1)}{\partial x_1}\Big|_{x_1=X_{1A}^*}, \end{cases} \tag{4.62}$$

where X_{1A}^* is the optimal investment trigger.

Solving (4.62) for A_A yields $A_A = \hat{A}_A(X_{1A}^*)$, with

$$\hat{A}_A(x_1) = \frac{\mathrm{e}^{\beta_M x_1}}{\beta_M} \left(\frac{\pi_A}{r} \frac{\partial Q_2^A(x_1)}{\partial x_1} - \frac{\pi_M - \pi_A}{r} \frac{\partial Q_1^A(x_1)}{\partial x_1} \right), \tag{4.63}$$

and the candidates for the optimal investment trigger are determined by the following implicit equation:

$$-\frac{\pi_A}{r} \left(1 - \frac{1}{\beta_M} \frac{\partial Q_2^A(x_1)}{\partial x_1} - Q_2^A(x_1) \right) - \frac{\pi_M - \pi_A}{r} \left(\frac{1}{\beta_M} \frac{\partial Q_1^A(x_1)}{\partial x_1} + Q_1^A(x_1) \right) = -I. \tag{4.64}$$

[7] Using similar argument as in the proof of Proposition 4.2 we can neglect the solutions with a negative derivative of $\mathrm{e}^{-\gamma_M x_1} g(x_1)$.

[8] If $g(x_1) = 0$ has a unique solution, X_1^*, then $\hat{X}_1^* < \hat{X}_1^{**} < X_1^*$.

Denoting this function by $g_A(x_1)$ and taking the limit to infinity we arrive at the result in the proposition:

$$\lim_{x_1 \to \infty} g_A(x_1) = -\frac{\pi_A}{r}\left(1 - \mathrm{e}^{-\beta_2^A x_2}\right) + I. \tag{4.65}$$

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