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**PROPORTIONALITY, EQUALITY, AND DUALITY
IN BANKRUPTCY PROBLEMS WITH
NONTRANSFERABLE UTILITY**

By

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Proportionality, Equality, and Duality in Bankruptcy Problems with Nontransferable Utility

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Abstract

This paper analyzes bankruptcy problems with nontransferable utility as a generalization of bankruptcy problems with monetary estate and claims. Following the classical axiomatic theory of bankruptcy, we formulate some appropriate properties for NTU-bankruptcy rules and study their implications. We explore duality of bankruptcy rules and we derive several characterizations of the generalized proportional rule and the constrained relative equal awards rule.

Keywords: NTU-bankruptcy problem, axiomatic analysis, duality, proportional rule, constrained relative equal awards rule

JEL classification: C79, D63, D74

1 Introduction

In a bankruptcy problem with transferable utility (cf. O’Neill (1982)), claimants have individual claims on a deficient, monetary estate. Bankruptcy theory analyzes allocations of the estate among the claimants, taking into account their claims. Bankruptcy problems with transferable utility are well-studied both from an axiomatic as well as a game theoretic perspective (cf. Thomson (2003), Thomson (2013) and Thomson (2015)). Thomson (2013) states that, although the passage from TU to NTU is in general fraught with difficulties, an NTU generalization is worthwhile in the search for greater generality.

This paper studies bankruptcy problems with nontransferable utility in which claimants have incompatible claims and the estate corresponds to a set of feasible allocations of utility. Orshan, Valenciano, and Zarzuelo (2003) analyzed NTU-bankruptcy problems from a game theoretic perspective by showing that the intersection of the bilateral consistent prekernel and the core is nonempty for every smooth bankruptcy game. Estévez-Fernández, Borm, and Fiestras-Janeiro (2014) redefined NTU-bankruptcy games on the basis of convexity and compromise stability, allowing for a generalization of the characterization of TU-bankruptcy

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games. The current paper axiomatically approaches NTU-bankruptcy problems by formulating appropriate properties for bankruptcy rules and studying their implications, in line with the model of Estévez-Fernández et al. (2014).

The proportional rule for bankruptcy problems prescribes the Pareto efficient allocation which is proportional to the vector of claims. We study the proportional rule for NTU-bankruptcy problems and extend the axiomatic characterizations of Young (1988) and Chun (1988) using adequate generalizations of the properties composition down, composition up, self-duality, and path linearity.

The constrained equal awards rule for TU-bankruptcy problems divides the estate equally among the claimants such that they are not allocated more than their claims. In a bankruptcy problem with nontransferable utility, it makes sense to compare the claims in relation to the estate since claimants differ in their measure of utility. Therefore, we introduce the constrained *relative* equal awards rule for NTU-bankruptcy problems which takes into account the *relative* claims of the claimants, i.e., the claims in relation to their utopia values. We extend the axiomatic characterizations of Dagan (1996), Herrero and Villar (2002), Yeh (2004) and Yeh (2006) using generalizations of the properties symmetry, truncation invariance, conditional full compensation, and claim monotonicity. Interestingly, we show that the constrained relative equal awards rule also shares a characteristic feature with the serial cost sharing rule (cf. Moulin and Shenker (1992)) by extending its axiomatic characterization based on symmetry and independence of larger claims.

Two bankruptcy rules are called dual (cf. Aumann and Maschler (1985)) if one rule allocates awards in the same way as the other rule allocates losses. Two properties for bankruptcy rules are called dual (cf. Herrero and Villar (2001)) if for any two dual bankruptcy rules it holds that one rule satisfies one property if, and only if, the other rule satisfies the other property. We generalize the notions of dual bankruptcy rules and dual properties to the context of NTU-bankruptcy problems without explicitly formulating dual bankruptcy problems. In particular, we exploit duality to show that the proportional rule is self-dual and to adequately define the dual of the constrained relative equal awards rule, the constrained relative equal losses rule.

This paper is organized in the following way. Section 2 formally introduces bankruptcy problems with nontransferable utility and defines basic notions for NTU-bankruptcy rules. In Section 3, we explore duality and analyze dual properties for bankruptcy rules. Section 4 studies the proportional rule and Section 5 analyzes the constrained relative equal awards rule for bankruptcy problems with nontransferable utility. In Section 6, we formulate some concluding remarks and point out some suggestions for future research.

2 Bankruptcy Problems with Nontransferable Utility

Let N be a nonempty and finite set of *claimants*. For any $x, y \in \mathbb{R}_+^N$, we denote $x \leq y$ if $x_i \leq y_i$ for all $i \in N$, and $x < y$ if $x_i < y_i$ for all $i \in N$. A function $p: \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$ is called *increasing* if $p(x) \leq p(y)$ and $p(x) \neq p(y)$ for all $x, y \in \mathbb{R}_+^N$ with $x \leq y$ and $x \neq y$. The zero-vector $x \in \mathbb{R}_+^N$ with $x_i = 0$ for all $i \in N$ is denoted by 0^N . For any $E \subseteq \mathbb{R}_+^N$,

- the *comprehensive hull* is given by $\text{comp}(E) = \{x \in \mathbb{R}_+^N \mid \exists y \in E : y \geq x\}$;
- the *strong Pareto set* is given by $\text{SP}(E) = \{x \in E \mid \neg \exists y \in E, y \neq x : y \geq x\}$;
- the *weak Pareto set* is given by $\text{WP}(E) = \{x \in E \mid \neg \exists y \in E : y > x\}$.

For any $E \subseteq \mathbb{R}_+^N$ and any $t \in \mathbb{R}_+$, the set $tE \subseteq \mathbb{R}_+^N$ is given by $tE = \{tx \mid x \in E\}$.

A *bankruptcy problem with nontransferable utility* (cf. Estévez-Fernández et al. (2014)) is a triple (N, E, c) in which $E \subseteq \mathbb{R}_+^N$ is the *estate* satisfying the following conditions:

- E is nonempty, closed, and bounded;
- E is nonzero, i.e., $E \neq \{0^N\}$;
- E is comprehensive, i.e., $E = \text{comp}(E)$;
- E is non-leveled, i.e., $\text{SP}(E) = \text{WP}(E)$,

and $c \in \mathbb{R}_+^N \setminus E$ is the vector of *claims*. Note that we do not require convexity of E . Moreover, the conditions on E imply that $E \cap \mathbb{R}_{++}^N \neq \emptyset$. The estate corresponds to a set of feasible allocations of utility which are assumed to be normalized such that allocating nothing to a claimant corresponds to zero utility. The claim vector represents the individual utility claims on the estate. Let BR^N denote the class of all bankruptcy problems with nontransferable utility with claimant set N . For convenience, we denote an NTU-bankruptcy problem by $(E, c) \in \text{BR}^N$. Note that each TU-bankruptcy problem (cf. O’Neill (1982)) corresponds to an NTU-bankruptcy problem $(E, c) \in \text{BR}^N$ with $E = \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \leq M\}$ for some number $M \in \mathbb{R}_{++}$.

Throughout this paper, scaling the estate is an essential, fundamental operation which preserves its shape. Let $(E, c) \in \text{BR}^N$ and let $x \in \mathbb{R}_+^N \setminus \{0^N\}$. The scalar $\lambda^{E,x} \in \mathbb{R}_{++}$ is defined such that

$$x \in \text{WP}(\lambda^{E,x} E) \quad \text{and} \quad \frac{1}{\lambda^{E,x}} x \in \text{WP}(E).$$

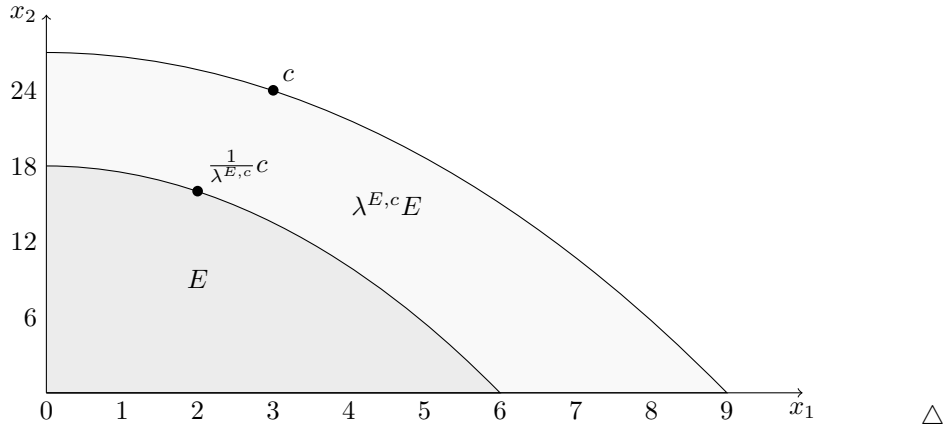
Note that the conditions on E imply that $\lambda^{E,x}$ exists and that $\lambda^{E,x}$ is increasing in x . We have $\lambda^{E,x} \leq 1$ if $x \in E$, and $\lambda^{E,x} > 1$ if $x \notin E$. For all $t \in \mathbb{R}_{++}$, we have

$$\lambda^{tE,x} = \frac{\lambda^{E,x}}{t} \quad \text{and} \quad \lambda^{E,tE} = t \lambda^{E,x}.$$

Note that $(tE, x) \in \text{BR}^N$ for all $t \in (0, \lambda^{E,x})$, and $(E, tx) \in \text{BR}^N$ for all $t \in (\frac{1}{\lambda^{E,x}}, \infty)$.

Example 1.

Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$. We have $\lambda^{E,c} = 1\frac{1}{2}$, $\lambda^{E,c} E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 3x_2 \leq 81\}$, and $\frac{1}{\lambda^{E,c}} c = (2, 16)$. This is depicted below.



A bankruptcy rule f is a function which assigns to any bankruptcy problem $(E, c) \in \text{BR}^N$ a payoff allocation $f(E, c) \in \text{WP}(E)$ for which $f(E, c) \leq c$. Moreover, for technical reasons, we assume that $f(E, c) = 0^N$ if $E = \{0^N\}$, and $f(E, c) = c$ if $c \in E$, for each pair (E, c) .

Let $(E, c) \in \text{BR}^N$, let $x \in \mathbb{R}_+^N \setminus \{0^N\}$, and let f be a bankruptcy rule. The *payoff path* of f from 0^N to x is the function $p_f^{E,x} : [0, \lambda^{E,x}] \rightarrow \mathbb{R}_+^N$ which is, for all $t \in [0, \lambda^{E,x}]$, defined by

$$p_f^{E,x}(t) = f(tE, x).$$

A bankruptcy rule f satisfies

- *path monotonicity* if $p_f^{E,c}$ is increasing on $[0, \lambda^{E,c}]$ for all $(E, c) \in \text{BR}^N$;
- *path continuity* if $p_f^{E,c}$ is continuous on $[0, \lambda^{E,c}]$ for all $(E, c) \in \text{BR}^N$.

We show that path monotonicity is a stronger property than path continuity.

Lemma 2.1.

Let f be a bankruptcy rule. If f satisfies path monotonicity, then, f satisfies path continuity.

Proof. Assume that f satisfies path monotonicity and suppose that f does not satisfy path continuity, i.e., there exists an $(E, c) \in \text{BR}^N$ for which $p_f^{E,c}$ is not continuous at a certain $\hat{t} \in [0, \lambda^{E,c}]$. Assume that $\hat{t} \in (0, \lambda^{E,c})$. Since f satisfies path monotonicity, $p_f^{E,c}$ is increasing on $[0, \lambda^{E,c}]$, which implies that

$$\lim_{t \uparrow \hat{t}} p_f^{E,c}(t) = \sup_{t \in [0, \hat{t})} p_f^{E,c}(t) \leq p_f^{E,c}(\hat{t}) \leq \inf_{t \in (\hat{t}, \lambda^{E,c}]} p_f^{E,c}(t) = \lim_{t \downarrow \hat{t}} p_f^{E,c}(t).$$

Since $p_f^{E,c}$ is not continuous at \hat{t} , either $\sup_{t \in [0, \hat{t})} p_f^{E,c}(t) \neq p_f^{E,c}(\hat{t})$ or $\inf_{t \in (\hat{t}, \lambda^{E,c}]} p_f^{E,c}(t) \neq p_f^{E,c}(\hat{t})$ (or both). Assume that $\sup_{t \in [0, \hat{t})} p_f^{E,c}(t) \neq p_f^{E,c}(\hat{t})$. Then, there exists a $p^* \in \mathbb{R}_+^N$ for which $\sup_{t \in [0, \hat{t})} p_f^{E,c}(t) \leq p^* \leq p_f^{E,c}(\hat{t})$ and $\sup_{t \in [0, \hat{t})} p_f^{E,c}(t) \neq p^* \neq p_f^{E,c}(\hat{t})$. Since $\lambda^{E,x}$ is increasing in x , this means that $t < \lambda^{E,p^*} < \hat{t}$ for all $t \in [0, \hat{t})$. This is not possible. Similarly, we can show that $\inf_{t \in (\hat{t}, \lambda^{E,c}]} p_f^{E,c}(t) \neq p_f^{E,c}(\hat{t})$ is not possible. Clearly, these arguments also apply to the cases $\hat{t} = 0$ and $\hat{t} = \lambda^{E,c}$. Hence, f satisfies path continuity. \square

3 Duality

In this section, we explore duality of bankruptcy rules and their properties. Two bankruptcy rules are called dual (cf. Aumann and Maschler (1985)) if one rule allocates awards in the same way as the other rule allocates losses. We generalize this idea to rules for bankruptcy problems with nontransferable utility.

Definition 3.1 (Dual Bankruptcy Rules).

The bankruptcy rules f and g are called *dual* if $f(E, c) = c - g(\lambda^{E, c - f(E, c)} E, c)$ and $g(E, c) = c - f(\lambda^{E, c - g(E, c)} E, c)$ for all $(E, c) \in \text{BR}^N$.

For any bankruptcy rule f and any $(E, c) \in \text{BR}^N$, we have $\lambda^{E, c - f(E, c)} \in (0, \lambda^{E, c})$ since $0 \leq f(E, c) \leq c$ and $0 \neq f(E, c) \neq c$, which means that $(\lambda^{E, c - f(E, c)} E, c) \in \text{BR}^N$.

For any two dual bankruptcy rules f and g , any $(E, c) \in \text{BR}^N$, and any $t \in (0, \lambda^{E, c})$, we have $\lambda^{tE, x} = \frac{\lambda^{E, x}}{t}$, which means that

$$f(tE, c) = c - g(\lambda^{tE, c - f(tE, c)} tE, c) = c - g(\lambda^{E, c - f(E, c)} E, c).$$

Clearly, duality is a symmetric relation between two bankruptcy rules. We show that each bankruptcy rule has at most one dual bankruptcy rule.

Lemma 3.1.

Let f , g , and h be three bankruptcy rules. If f and g are dual, and f and h are dual, then, $g = h$.

Proof. Assume that f and g are dual, and that f and h are dual. Let $(E, c) \in \text{BR}^N$. We can write

$$g(E, c) = c - f(\lambda^{E, c-g(E, c)} E, c) = h(\lambda^{E, c-f(\lambda^{E, c-g(E, c)} E, c)} E, c) = h(\lambda^{E, g(E, c)} E, c) = h(E, c),$$

where the first and third equality follow from duality of f and g , the second equality follows from duality of f and h , and the last equality follows from $g(E, c) \in \text{WP}(E)$, which implies that $\lambda^{E, g(E, c)} = 1$. Hence, $g = h$. \square

Definition 3.2 (Self-Dual Bankruptcy Rule).

A bankruptcy rule f satisfies *self-duality* if $f(E, c) = c - f(\lambda^{E, c-f(E, c)} E, c)$ for all $(E, c) \in \text{BR}^N$.

The remainder of this section studies relations between properties of two dual bankruptcy rules. Two properties for bankruptcy rules are called *dual* (cf. Herrero and Villar (2001)) if for any two dual bankruptcy rules, one property is satisfied by one rule if, and only if, the other property is satisfied by the other rule. A property for bankruptcy rules is a *self-dual* property if, for any two dual bankruptcy rules, the property is satisfied by one rule if, and only if, it is also satisfied by the other rule. First, we show that path monotonicity is a self-dual property.

Lemma 3.2.

Path monotonicity is self-dual.

Proof. Let f and g be two dual bankruptcy rules and assume that f satisfies path monotonicity. Let $(E, c) \in \text{BR}^N$. For all $t \in (0, \lambda^{E, c})$, we can write

$$p_f^{E, c}(t) = f(tE, c) = c - g(\lambda^{E, c-f(tE, c)} E, c) = c - p_g^{E, c}(\lambda^{E, c-f(tE, c)}) = c - p_g^{E, c}(\lambda^{E, c-p_f^{E, c}(t)}),$$

where the second equality follows from duality. Since $p_f^{E, c}$ is increasing on $(0, \lambda^{E, c})$ and $\lambda^{E, x}$ is increasing in x , we have $\lambda^{E, c-p_f^{E, c}(t)}$ is decreasing in t on $(0, \lambda^{E, c})$. Since $0 \leq p_g^{E, c}(t) \leq c$ and $0 \neq p_g^{E, c}(t) \neq c$, this means that $p_g^{E, c}$ is increasing on $[0, \lambda^{E, c}]$, so g satisfies path monotonicity. Hence, path monotonicity is self-dual. \square

Next, we study a self-dual symmetry property. The idea of equality, equity, or symmetry underlies many theories of economic justice (cf. Rawls (1971) and Young (1995)). The interpretation of symmetry depends on the underlying model. In a bankruptcy problem with nontransferable utility, claimants not only differ in their claims, but also differ in their measure of utility. It makes sense to compare their claims in relation to the estate. Preserving the most important characteristics of the estate, the maximal individual payoffs within the estate, or utopia values, appear to be a natural benchmark for a symmetry property. For $(E, c) \in \text{BR}^N$, the vector of *utopia values* $u^E \in \mathbb{R}_{++}^N$ is, for all $i \in N$, given by

$$u_i^E = \max\{x_i \mid x \in E\}.$$

Note that the conditions on E imply that this maximum exists and is positive. Moreover, $u^{tE} = tu^E$ for all $t \in \mathbb{R}_{++}$. The number $\frac{c_i}{u_i^E}$ is called the *relative claim* of $i \in N$.

Definition 3.3 (Relative Symmetry).

A bankruptcy rule f satisfies *relative symmetry* if $\frac{f_i(E,c)}{u_i^E} = \frac{f_j(E,c)}{u_j^E}$ for all $(E,c) \in \text{BR}^N$ and any $i, j \in N$ with $\frac{c_i}{u_i^E} = \frac{c_j}{u_j^E}$.

Note that for a TU-bankruptcy problem $(E,c) \in \text{BR}^N$ with $E = \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \leq M\}$ for some $M \in \mathbb{R}_{++}$, we have $u_i^E = M$ for all $i \in N$ and relative symmetry boils down to the classic property of equal treatment of claimants with equal claims.

Lemma 3.3.

Relative symmetry is self-dual.

Proof. Let f and g be two dual bankruptcy rules and assume that f satisfies relative symmetry. Let $(E,c) \in \text{BR}^N$, let $i, j \in N$ with $\frac{c_i}{u_i^E} = \frac{c_j}{u_j^E}$ and denote $d = \lambda^{E,c-g(E,c)}$. Then, we can write

$$\begin{aligned} \frac{g_i(E,c)}{u_i^E} &= \frac{c_i - f_i(dE,c)}{u_i^E} = \frac{c_i}{u_i^E} - d \frac{f_i(dE,c)}{du_i^E} = \frac{c_i}{u_i^E} - d \frac{f_i(dE,c)}{u_i^{dE}} \\ &= \frac{c_j}{u_j^E} - d \frac{f_j(dE,c)}{u_j^{dE}} = \frac{c_j}{u_j^E} - d \frac{f_j(dE,c)}{du_j^E} = \frac{c_j - f_j(dE,c)}{u_j^E} = \frac{g_j(E,c)}{u_j^E}, \end{aligned}$$

where the first equality follows from duality, the fourth equality follows from relative symmetry, and the last equality again from duality. This means that g satisfies relative symmetry. Hence, relative symmetry is self-dual. \square

Two other convenient properties for bankruptcy rules are composition down and composition up. Composition down implies that allocations can be used to derive solutions downwards on the payoff path to the claim vector and composition up implies that allocations can be used to derive solutions upwards on the payoff path to the claim vector.

Definition 3.4 (Composition Down).

A bankruptcy rule f satisfies *composition down* if for all $(E,c) \in \text{BR}^N$ and any $t \in (0,1)$, we have $f(tE,c) = f(tE, f(E,c))$.

Definition 3.5 (Composition Up).

A bankruptcy rule f satisfies *composition up* if for all $(E,c) \in \text{BR}^N$ and any $t \in (0,1)$, we have $f(E,c) = f(tE,c) + f(\lambda^{E,f(E,c)-f(tE,c)}E, c - f(tE,c))$.

We show that composition down and composition up are stronger properties than path monotonicity.

Lemma 3.4.

Let f be a bankruptcy rule.

(i) *If f satisfies composition down, then, f satisfies path monotonicity.*

(ii) *If f satisfies composition up, then, f satisfies path monotonicity.*

Proof. (i) Assume that f satisfies composition down. Let $(E,c) \in \text{BR}^N$ and let $t_1, t_2 \in [0, \lambda^{E,c}]$ such that $t_1 < t_2$. We show that $p_f^{E,c}(t_1) \leq p_f^{E,c}(t_2)$ and $p_f^{E,c}(t_1) \neq p_f^{E,c}(t_2)$. Assume that $t_1, t_2 \in (0, \lambda^{E,c})$. Then, $\frac{t_1}{t_2} \in (0,1)$ and we can write

$$\begin{aligned} p_f^{E,c}(t_1) &= f(t_1E, c) = f\left(\frac{t_1}{t_2}t_2E, c\right) = f\left(\frac{t_1}{t_2}t_2E, f(t_2E, c)\right) \\ &= f(t_1E, f(t_2E, c)) \leq f(t_2E, c) = p_f^{E,c}(t_2), \end{aligned}$$

where the third equality follows from composition down, and the inequality follows from the definition of a bankruptcy rule. Moreover, $p_f^{E,c}(t_1) \neq p_f^{E,c}(t_2)$ since $\text{WP}(t_1 E) \cap \text{WP}(t_2 E) = \emptyset$. Clearly, these arguments also apply to the cases $t_1 = 0$ and $t_2 = \lambda^{E,c}$.

(ii) Assume that f satisfies composition up. Let $(E, c) \in \text{BR}^N$ and let $t_1, t_2 \in [0, \lambda^{E,c}]$ such that $t_1 < t_2$. We show that $p_f^{E,c}(t_1) \leq p_f^{E,c}(t_2)$ and $p_f^{E,c}(t_1) \neq p_f^{E,c}(t_2)$. Assume that $t_1, t_2 \in (0, \lambda^{E,c})$. Then, $\frac{t_1}{t_2} \in (0, 1)$ and we can write

$$\begin{aligned} p_f^{E,c}(t_2) &= f(t_2 E, c) = f\left(\frac{t_1}{t_2} t_2 E, c\right) + f\left(\lambda^{t_2 E, f(t_2 E, c) - f\left(\frac{t_1}{t_2} t_2 E, c\right)} t_2 E, c - f\left(\frac{t_1}{t_2} t_2 E, c\right)\right) \\ &= f(t_1 E, c) + f(\lambda^{E, f(t_2 E, c) - f(t_1 E, c)} E, c - f(t_1 E, c)) \\ &\geq f(t_1 E, c) = p_f^{E,c}(t_1), \end{aligned}$$

where the second equality follows from composition up, and the inequality follows from the definition of a bankruptcy rule. Moreover, $p_f^{E,c}(t_2) \neq p_f^{E,c}(t_1)$ since $\text{WP}(t_2 E) \cap \text{WP}(t_1 E) = \emptyset$. Clearly, these arguments also apply to the cases $t_1 = 0$ and $t_2 = \lambda^{E,c}$. \square

Finally, we show that composition down and composition up are dual properties.

Lemma 3.5.

Composition down and composition up are dual.

Proof. Let f and g be two dual bankruptcy rules.

Assume that f satisfies composition down. Then, we know from Lemma 3.4 that f satisfies path monotonicity. Hence, by Lemma 3.2, g satisfies path monotonicity. Let $(E, c) \in \text{BR}^N$, let $t \in (0, 1)$, and denote $d = \lambda^{E, c - g(E, c)}$ and $d' = \lambda^{E, c - g(tE, c)}$. Then, we have $d < d'$ since $g(tE, c) \leq g(E, c)$ and $g(tE, c) \neq g(E, c)$. We can write

$$\begin{aligned} g(E, c) - g(tE, c) &= (c - f(dE, c)) - (c - f(d'E, c)) \\ &= f(d'E, c) - f(dE, c) \\ &= f(d'E, c) - f(dE, f(d'E, c)) \\ &= f(d'E, c) - \left(f(d'E, c) - g(\lambda^{E, f(d'E, c) - f(dE, f(d'E, c))} E, f(d'E, c))\right) \\ &= g(\lambda^{E, g(E, c) - g(tE, c)} E, c - g(tE, c)), \end{aligned}$$

where the first equality follows from duality, the third equality follows from composition down taking into account that $\frac{d}{d'} \in (0, 1)$, and the fourth equality follows from duality. This means that g satisfies composition up.

Reversely, assume that g satisfies composition up. Then, we know from Lemma 3.4 that g satisfies path monotonicity. Hence, by Lemma 3.2, f satisfies path monotonicity. Let $(E, c) \in \text{BR}^N$, let $t \in (0, 1)$, and denote $d = \lambda^{E, c - f(E, c)}$ and $d' = \lambda^{E, c - f(tE, c)}$. Then, we have $d < d'$ since $f(tE, c) \leq f(E, c)$ and $f(tE, c) \neq f(E, c)$. We can write

$$\begin{aligned} f(tE, c) &= c - g(d'E, c) \\ &= c - \left(g(dE, c) + g(\lambda^{E, g(d'E, c) - g(dE, c)} E, c - g(dE, c))\right) \\ &= f(E, c) - g(\lambda^{E, f(E, c) - f(tE, c)} E, f(E, c)) \\ &= f(E, c) - \left(f(E, c) - f(\lambda^{E, f(E, c) - g(\lambda^{E, f(E, c) - f(tE, c)} E, f(E, c))} E, f(E, c))\right) \\ &= f(\lambda^{E, f(tE, c)} E, f(E, c)) \\ &= f(tE, f(E, c)), \end{aligned}$$

where the first equality follows from duality, the second equality follows from composition up taking into account that $\frac{d}{d'} \in (0, 1)$, the third and fourth equality follow from duality and the last equality follows from $f(tE, c) \in \text{WP}(tE)$, which implies that $\lambda^{E, f(tE, c)} = t$. This means that f satisfies composition down. \square

4 The Proportional Rule

This section introduces the proportional rule for bankruptcy problems with nontransferable utility and provides three axiomatic characterizations.

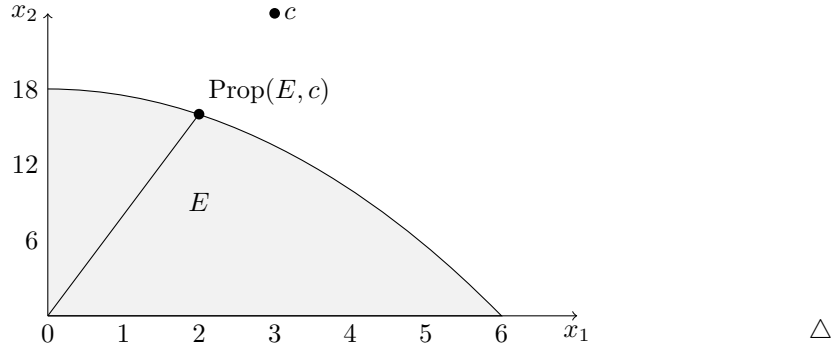
Definition 4.1 (The Proportional Rule).

For all $(E, c) \in \text{BR}^N$, the *proportional rule* Prop is defined by

$$\text{Prop}(E, c) = \frac{1}{\lambda^{E, c}} c.$$

Example 2.

Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$ as in Example 1. We have $\lambda^{E, c} = 1\frac{1}{2}$, which means that $\text{Prop}(E, c) = \frac{1}{\lambda^{E, c}} c = \frac{2}{3} c = (2, 16)$.



The characterization of the proportional rule for TU-bankruptcy problems in terms of composition down and self-duality, or composition up and self-duality (cf. Young (1988)), can be extended to bankruptcy problems with nontransferable utility.

Theorem 4.1.

- (i) *The proportional rule is the unique bankruptcy rule satisfying composition down and self-duality.*
- (ii) *The proportional rule is the unique bankruptcy rule satisfying composition up and self-duality.*

Proof. Since (ii) follows directly from (i) and Lemma 3.5, it suffices to prove only (i).

First, we show that the proportional rule satisfies composition down and self-duality. Let $(E, c) \in \text{BR}^N$ and let $t \in (0, 1)$. Then, we can write

$$\begin{aligned} \text{Prop}(tE, \text{Prop}(E, c)) &= \frac{1}{\lambda^{tE, \text{Prop}(E, c)}} \text{Prop}(E, c) = \frac{1}{\lambda^{tE, \frac{1}{\lambda^{E, c}} c}} \left(\frac{1}{\lambda^{E, c}} c \right) \\ &= \lambda^{E, c} \frac{1}{\lambda^{tE, c}} \left(\frac{1}{\lambda^{E, c}} c \right) = \frac{1}{\lambda^{tE, c}} c = \text{Prop}(tE, c). \end{aligned}$$

Hence, the proportional rule satisfies composition down.

Moreover, we can write

$$\begin{aligned} \text{Prop}(\lambda^{E,c-\text{Prop}(E,c)}E, c) &= \frac{1}{\lambda^{\lambda^{E,c-\text{Prop}(E,c)}E, c}}c = \frac{\lambda^{E,c-\text{Prop}(E,c)}}{\lambda^{E,c}}c = \frac{\lambda^{E,(1-\frac{1}{\lambda^{E,c}})c}}{\lambda^{E,c}}c \\ &= \left(1 - \frac{1}{\lambda^{E,c}}\right) \frac{\lambda^{E,c}}{\lambda^{E,c}}c = c - \frac{1}{\lambda^{E,c}}c = c - \text{Prop}(E, c). \end{aligned}$$

Hence, the proportional rule satisfies self-duality.

Second, we show that the proportional rule is the only bankruptcy rule satisfying composition down and self-duality. Let $(E, c) \in \text{BR}^N$ and let $x \in \mathbb{R}_+^N \setminus \{0^N\}$. Let f be a bankruptcy rule satisfying composition down and self-duality. Then, we know from Lemma 3.4 that f satisfies path monotonicity. Hence, by Lemma 2.1, f satisfies path continuity. This means that for any $s \in [0, \sum_{i \in N} x_i]$, there exists a unique $t \in [0, \lambda^{E,x}]$ for which $\sum_{i \in N} f_i(tE, x) = s$.

Let $t \in (0, \lambda^{E,x})$. For all $t' \in (0, t)$, we can write

$$p_f^{E, p_f^{E,x}(t)}(t') = f(t'E, p_f^{E,x}(t)) = f(t'E, f(tE, x)) = f(t'E, x) = p_f^{E,x}(t'),$$

where the third equality follows from composition down. Given a vector $y \in \mathbb{R}_+^N \setminus \{0^N\}$ on the payoff path of f from 0^N to x , this means that any vector on the payoff path of f from 0^N to y is also on the payoff path of f from 0^N to x . Moreover, we can write

$$x - p_f^{E,x}(t) = x - f(tE, x) = f(\lambda^{E,x-f(tE,x)}E, x) = f(\lambda^{E,x-p_f^{E,x}(t)}E, x) = p_f^{E,x}(\lambda^{E,x-p_f^{E,x}(t)}),$$

where the second equality follows from self-duality. Given a vector $y \in \mathbb{R}_+^N \setminus \{0^N\}$ on the payoff path of f from 0^N to x , this means that $x - y$ is also on the payoff path of f from 0^N to x .

Let $\hat{t} \in (0, \lambda^{E,x})$ such that $\sum_{i \in N} f_i(\hat{t}E, x) = \frac{1}{2} \sum_{i \in N} x_i$. Then, $f(\hat{t}E, x)$ and $x - f(\hat{t}E, x)$ are both on the payoff path of f from 0^N to x . We can write

$$\sum_{i \in N} (x_i - f_i(\hat{t}E, x)) = \sum_{i \in N} x_i - \sum_{i \in N} f_i(\hat{t}E, x) = \sum_{i \in N} x_i - \frac{1}{2} \sum_{i \in N} x_i = \frac{1}{2} \sum_{i \in N} x_i.$$

This implies that $x - f(\hat{t}E, x) = f(\hat{t}E, x)$, so $f(\hat{t}E, x) = \frac{1}{2}x$. This means that $\frac{1}{2}x$ is on the payoff path of f from 0^N to x .

In particular, $\frac{1}{2}c$ is on the payoff path of f from 0^N to c . Moreover, $\frac{1}{4}c$ is on the payoff path of f from 0^N to $\frac{1}{2}c$, which implies that $\frac{1}{4}c$ and $\frac{3}{4}c$ are on the payoff path of f from 0^N to c . Continuing this reasoning, we have that $\frac{m}{2^n}c$ is on the payoff path of f from 0^N to c for any $n \in \mathbb{N}$ and any $m \in \mathbb{N}$, $m \leq 2^n$. Since f satisfies path continuity, this means that tc is on the payoff path of f from 0^N to c for any $t \in [0, 1]$. In other words, we can write $f(tE, c) = \frac{1}{\lambda^{tE,c}}c = \text{Prop}(tE, c)$. Hence, $f = \text{Prop}$. \square

Chun (1988) used a linearity axiom to characterize the proportional rule. We extend this characterization by showing that the proportional rule is the only bankruptcy rule with a linear payoff path for any bankruptcy problem with nontransferable utility.

Definition 4.2 (Path Linearity).

A bankruptcy rule f satisfies *path linearity* if $f(\theta E + (1-\theta)tE, c) = \theta f(E, c) + (1-\theta)f(tE, c)$ for all $(E, c) \in \text{BR}^N$, any $t \in [0, \lambda^{E,c}]$, and any $\theta \in (0, 1)$.

Theorem 4.2.

The proportional rule is the unique bankruptcy rule satisfying path linearity.

Proof. First, we show that the proportional rule satisfies path linearity. Let $(E, c) \in \text{BR}^N$, let $t \in [0, \lambda^{E,c}]$, and let $\theta \in (0, 1)$. If $t = 0$, we can write

$$\begin{aligned} \text{Prop}(\theta E + (1 - \theta)tE, c) &= \text{Prop}(\theta E, c) = \frac{1}{\lambda^{\theta E, c}}c = \theta \frac{1}{\lambda^{E, c}}c \\ &= \theta \text{Prop}(E, c) = \theta \text{Prop}(E, c) + (1 - \theta) \text{Prop}(tE, c). \end{aligned}$$

If $t > 0$, we can write

$$\begin{aligned} \text{Prop}(\theta E + (1 - \theta)tE, c) &= \text{Prop}((\theta + (1 - \theta)t)E, c) = \frac{1}{\lambda^{(\theta + (1 - \theta)t)E, c}}c \\ &= \frac{\theta + (1 - \theta)t}{\lambda^{E, c}}c = \theta \frac{1}{\lambda^{E, c}}c + (1 - \theta) \frac{t}{\lambda^{E, c}}c \\ &= \theta \frac{1}{\lambda^{E, c}}c + (1 - \theta) \frac{1}{\lambda^{tE, c}}c = \theta \text{Prop}(E, c) + (1 - \theta) \text{Prop}(tE, c). \end{aligned}$$

Hence, the proportional rule satisfies path linearity.

Second, we show that the proportional rule is the only bankruptcy rule satisfying path linearity. Let f be a bankruptcy rule satisfying path linearity. Let $(E, c) \in \text{BR}^N$. Then, we can write

$$\begin{aligned} f(E, c) &= f\left(\frac{1}{\lambda^{E, c}}\lambda^{E, c}E + \left(1 - \frac{1}{\lambda^{E, c}}\right)0\lambda^{E, c}E, c\right) \\ &= \frac{1}{\lambda^{E, c}}f(\lambda^{E, c}E, c) + \left(1 - \frac{1}{\lambda^{E, c}}\right)f(0\lambda^{E, c}E, c) \\ &= \frac{1}{\lambda^{E, c}}f(\lambda^{E, c}E, c) + \left(1 - \frac{1}{\lambda^{E, c}}\right)f(\{0^N\}, c) \\ &= \frac{1}{\lambda^{E, c}}c + \left(1 - \frac{1}{\lambda^{E, c}}\right)0^N \\ &= \frac{1}{\lambda^{E, c}}c \\ &= \text{Prop}(E, c). \end{aligned}$$

Hence, $f = \text{Prop}$. □

5 The Constrained Relative Equal Awards Rule

This section introduces the constrained relative equal awards rule for bankruptcy problems with nontransferable utility and provides four axiomatic characterizations. The constrained relative equal awards rule generalizes the constrained equal awards rule for TU-bankruptcy problems which divides the estate equally among the claimants such that they are not allocated more than their claims. Following our interpretation of equality and symmetry in bankruptcy problems with nontransferable utility as discussed in Section 3, it makes sense to define a bankruptcy rule which divides the estate *relatively* equal among the claimants such that they are not allocated more than their claims.

Definition 5.1 (The Constrained Relative Equal Awards Rule).

For all $(E, c) \in \text{BR}^N$ and any $i \in N$, the *constrained relative equal awards rule* CREA is defined by

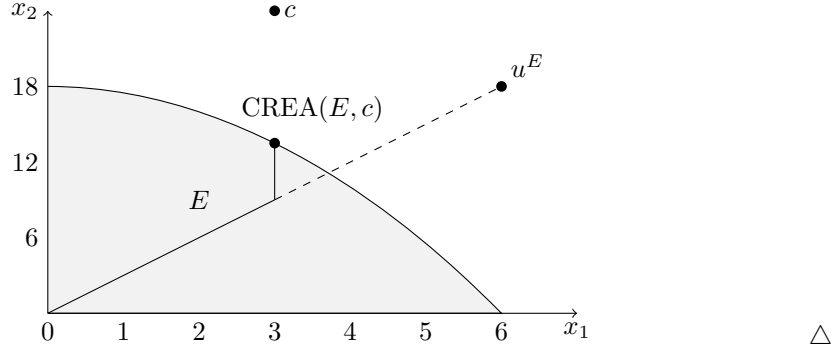
$$\text{CREA}_i(E, c) = \min\{c_i, \alpha^{E,c} u_i^E\},$$

where $\alpha^{E,c} \in (0, 1)$ is such that $\text{CREA}(E, c) \in \text{WP}(E)$.

Note that the conditions on E imply that $\alpha^{E,c}$ is well-defined. Moreover, for a TU-bankruptcy problem $(E, c) \in \text{BR}^N$ with $E = \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \leq M\}$ for some $M \in \mathbb{R}_{++}$, we have $u_i^E = M$ for all $i \in N$ and the constrained relative equal awards rule coincides with the classic constrained equal awards rule.

Example 3.

Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$ as in Example 1. We have $u^E = (6, 18)$ and $\alpha^{E,c} = \frac{3}{4}$. Moreover, $\text{CREA}_1(E, c) = \min\{c_1, \alpha^{E,c} u_1^E\} = \min\{3, 4\frac{1}{2}\} = 3$ and $\text{CREA}_2(E, c) = \min\{c_2, \alpha^{E,c} u_2^E\} = \min\{24, 13\frac{1}{2}\} = 13\frac{1}{2}$.



Throughout this section, we refer to the Appendix for the derivations of the specific properties stated for the constrained relative equal awards rule. Inspired by Dagan (1996), we axiomatically characterize the constrained relative equal awards rule using the properties relative symmetry, composition up, and truncation invariance. The truncation invariance property considers only the truncated claims of the claimants as relevant. For any $(E, c) \in \text{BR}^N$, the vector of *truncated claims* $\hat{c}^E \in \mathbb{R}_+^N$ is, for all $i \in N$, given by

$$\hat{c}_i^E = \min\{c_i, u_i^E\}.$$

Definition 5.2 (Truncation Invariance).

A bankruptcy rule f satisfies *truncation invariance* if $f(E, \hat{c}^E) = f(E, c)$ for all $(E, c) \in \text{BR}^N$.

Theorem 5.1.

The constrained relative equal awards rule is the unique bankruptcy rule satisfying relative symmetry, composition up, and truncation invariance.

Proof. From Lemma A.2, Lemma A.4, and Lemma A.5, we know that the constrained relative equal awards rule satisfies relative symmetry, composition up, and truncation invariance. Let f be a bankruptcy rule satisfying relative symmetry, composition up, and truncation invariance. Then, we know from Lemma 3.4 that f satisfies path monotonicity. Hence, by Lemma 2.1, f satisfies path continuity.

Let $(E, c) \in \text{BR}^N$ and suppose that $f(tE, c) \neq \text{CREA}(tE, c)$ for some $t \in [0, \lambda^{E,c}]$. Let $\hat{t} = \inf\{t \in [0, \lambda^{E,c}] \mid f(tE, c) \neq \text{CREA}(tE, c)\}$. Since f and CREA satisfy path continuity, we have $\hat{t} \in [0, \lambda^{E,c})$ and $f(\hat{t}E, c) = \text{CREA}(\hat{t}E, c)$. Assume that $\hat{t} \in (0, \lambda^{E,c})$. Denote $N = \{1, \dots, n\}$ such that $\frac{c_1}{u_1^E} \leq \dots \leq \frac{c_n}{u_n^E}$ and let $k \in N$ such that $f_i(\hat{t}E, c) = c_i$ for all $i < k$, and $f_i(\hat{t}E, c) = \hat{t}\alpha^{\hat{t}E,c}u_k^E < c_i$ for all $i \geq k$. Let $m = \min\{\|x\| \mid x \in \text{WP}(E)\}$ and take $\varepsilon \in (0, m(\frac{c_k}{u_k^E} - \frac{f_k(\hat{t}E, c)}{u_k^E}))$. Note that the conditions on E imply that m exists. Since f satisfies path continuity, there exists a $\delta > 0$ such that $\|f(tE, c) - f(\hat{t}E, c)\| < \varepsilon$ for all $t \in (\hat{t}, \min\{\hat{t} + \delta, \lambda^{E,c}\})$. Let $t \in (\hat{t}, \min\{\hat{t} + \delta, \lambda^{E,c}\})$. Since f satisfies path monotonicity, we have $\lambda^{E, f(tE, c) - f(\hat{t}E, c)} \in (0, \lambda^{E,c})$. Denote $d = \lambda^{E, f(tE, c) - f(\hat{t}E, c)}$. We can write

$$m \left(\frac{c_k}{u_k^E} - \frac{f_k(\hat{t}E, c)}{u_k^E} \right) > \varepsilon > \|f(tE, c) - f(\hat{t}E, c)\| = \|f(dE, c - f(\hat{t}E, c))\| \geq dm,$$

where the equality follows from composition up taking into account that $\frac{\hat{t}}{t} \in (0, 1)$. This means that $d < (\frac{c_k}{u_k^E} - \frac{f_k(\hat{t}E, c)}{u_k^E})$. Let $\tilde{u}^{dE} \in \mathbb{R}_+^N$ be given by

$$\tilde{u}_i^{dE} = \begin{cases} 0 & \text{if } i < k; \\ u_i^{dE} & \text{if } i \geq k. \end{cases}$$

For all $i < k$, we can write

$$\tilde{u}_i^{dE} = 0 = c_i - c_i = c_i - f_i(\hat{t}E, c) = c_i - \text{CREA}_i(\hat{t}E, c).$$

For all $i \geq k$, we can write

$$\begin{aligned} \tilde{u}_i^{dE} &= u_i^{dE} = du_i^E < \left(\frac{c_k}{u_k^E} - \frac{f_k(\hat{t}E, c)}{u_k^E} \right) u_i^E \leq \left(\frac{c_i}{u_i^E} - \frac{\hat{t}\alpha^{\hat{t}E,c}u_k^E}{u_k^E} \right) u_i^E \\ &= c_i - \hat{t}\alpha^{\hat{t}E,c}u_i^E &= c_i - f_i(\hat{t}E, c) &= c_i - \text{CREA}_i(\hat{t}E, c). \end{aligned}$$

Then, we can write

$$\begin{aligned} f(dE, c - f(\hat{t}E, c)) &= f(dE, \tilde{u}^{dE}) &= \frac{1}{\lambda^{dE, \tilde{u}^{dE}}} \tilde{u}^{dE} \\ &= \text{CREA}(dE, \tilde{u}^{dE}) = \text{CREA}(dE, c - \text{CREA}(\hat{t}E, c)), \end{aligned}$$

where the first equality follows from truncation invariance, the second and third equality follow from relative symmetry, and the last equality follows again from truncation invariance. We can write

$$\begin{aligned} f(tE, c) &= f(\hat{t}E, c) + f(dE, c - f(\hat{t}E, c)) \\ &= \text{CREA}(\hat{t}E, c) + \text{CREA}(dE, c - \text{CREA}(\hat{t}E, c)) \\ &= \text{CREA}(tE, c), \end{aligned}$$

where the first and the last equality follow from composition up. This contradicts the definition of \hat{t} . Clearly, the same type of arguments also apply to the case $\hat{t} = 0$, even without using composition up. Hence, $f(tE, c) = \text{CREA}(tE, c)$ for all $t \in [0, \lambda^{E,c}]$. \square

The second axiomatic characterization is in the spirit of Yeh (2006), who showed that the constrained equal awards rule for TU-bankruptcy problems is the only bankruptcy rule that satisfies claim monotonicity and a property which requires that the claimants with small enough claims are fully compensated. We generalize this idea to a conditional full compensation property for NTU-bankruptcy rules based on the relative claims, and characterize the constrained relative equal awards rule in terms of claim monotonicity and conditional full compensation.

Definition 5.3 (Claim Monotonicity).

A bankruptcy rule f satisfies *claim monotonicity* if $f_i(E, c') \geq f_i(E, c)$ for all $(E, c) \in \text{BR}^N$, any $i \in N$, and any $c' \in \mathbb{R}_+^N$ with $c'_i \geq c_i$ and $c'_j = c_j$ for all $j \in N \setminus \{i\}$.

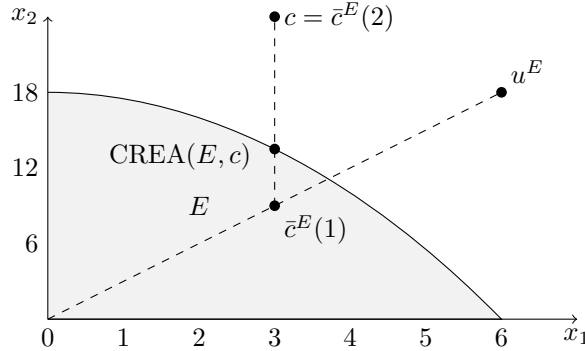
Definition 5.4 (Conditional Full Compensation).

A bankruptcy rule f satisfies *conditional full compensation* if $f_i(E, c) = c_i$ for all $(E, c) \in \text{BR}^N$ and any $i \in N$ for which $\bar{c}^E(i) \in E$, where

$$\bar{c}^E(i) = \left(\min \left\{ \frac{c_i}{u_i^E}, \frac{c_j}{u_j^E} \right\} u_j^E \right)_{j \in N}.$$

Example 4.

Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$ as in Example 1. We have $u^E = (6, 18)$, which implies that $\frac{c_1}{u_1^E} = \frac{1}{2}$ and $\frac{c_2}{u_2^E} = \frac{1}{3}$, which means that $\bar{c}^E(1) = \frac{c_1}{u_1^E} u^E = (3, 9)$ and $\bar{c}^E(2) = c = (3, 24)$. Hence, $\bar{c}^E(1) \in E$ and $\bar{c}^E(2) \notin E$.



Theorem 5.2.

The constrained relative equal awards rule is the unique bankruptcy rule satisfying claim monotonicity and conditional full compensation.

Proof. From Lemma A.1 and Lemma A.6, we know that the constrained relative equal awards rule satisfies claim monotonicity and conditional full compensation. Let f be a bankruptcy rule satisfying claim monotonicity and conditional full compensation. Let $(E, c) \in \text{BR}^N$.

First, we show that $f_i(E, c) = \text{CREA}_i(E, c)$ for all $i \in N$ with $\text{CREA}_i(E, c) = c_i$. Let $i \in N$ with $\text{CREA}_i(E, c) = c_i$. For all $j \in N$, we can write

$$\bar{c}_j^E(i) = \min \left\{ \frac{c_i}{u_i^E}, \frac{c_j}{u_j^E} \right\} u_j^E \leq \min \left\{ \alpha^{E,c}, \frac{c_j}{u_j^E} \right\} u_j^E = \min \{c_j, \alpha^{E,c} u_j^E\} = \text{CREA}_j(E, c).$$

This implies that $\bar{c}^E(i) \in E$ since E is comprehensive. Since f satisfies conditional full compensation, then, $f_i(E, c) = c_i$.

Now, suppose that $f(E, c) \neq \text{CREA}(E, c)$. Since E is non-leveled, and both $f(E, c) \in \text{WP}(E)$ and $\text{CREA}(E, c) \in \text{WP}(E)$, this implies that there exists a $j \in N$ for which $f_j(E, c) < \text{CREA}_j(E, c) = \alpha^{E,c} u_j^E$. Let $k \in \text{argmin}_{j \in N} \{ \frac{f_j(E, c)}{u_j^E} \mid \text{CREA}_j(E, c) = \alpha^{E,c} u_j^E \}$. We have $f_k(E, c) < \alpha^{E,c} u_k^E < c_k$ since $f_k(E, c) = \text{CREA}_k(E, c)$ if $\text{CREA}_k(E, c) = c_k$. Let $b \in \mathbb{R}_+^N$ be given by $b_k = \alpha^{E,c} u_k^E$ and $b_j = c_j$ for all $j \in N \setminus \{k\}$. We have $f(E, c) \leq b \leq c$. For all $j \in N$, we can write

$$\bar{b}_j^E(k) = \min \left\{ \frac{b_k}{u_k^E}, \frac{b_j}{u_j^E} \right\} u_j^E = \min \left\{ \alpha^{E,c}, \frac{c_j}{u_j^E} \right\} u_j^E = \min \{ c_j, \alpha^{E,c} u_j^E \} = \text{CREA}_j(E, c).$$

This implies that $\bar{b}^E(k) \in E$. Then, $f_k(E, b) = b_k$ since f satisfies conditional full compensation. This means that $f_k(E, c) < f_k(E, b)$, which contradicts that f satisfies claim monotonicity. Hence, $f(E, c) = \text{CREA}(E, c)$. \square

Next, we generalize the characterization of Herrero and Villar (2002) and Yeh (2004) by showing that the constrained relative equal awards rule is the only bankruptcy rule satisfying composition down and conditional full compensation.

Theorem 5.3.

The constrained relative equal awards rule is the unique bankruptcy rule satisfying composition down and conditional full compensation.

Proof. From Lemma A.3 and Lemma A.6, we know that the constrained relative equal awards rule satisfies composition down and conditional full compensation. Let f be a bankruptcy rule satisfying composition down and conditional full compensation. Then, we know from Lemma 3.4 that f satisfies path monotonicity. Hence, by Lemma 2.1, f satisfies path continuity. Let $(E, c) \in \text{BR}^N$.

First, we show that $f_i(E, c) = \text{CREA}_i(E, c)$ for all $i \in N$ with $\text{CREA}_i(E, c) = c_i$. Let $i \in N$ with $\text{CREA}_i(E, c) = c_i$. For all $j \in N$, we can write

$$\bar{c}_j^E(i) = \min \left\{ \frac{c_i}{u_i^E}, \frac{c_j}{u_j^E} \right\} u_j^E \leq \min \left\{ \alpha^{E,c}, \frac{c_j}{u_j^E} \right\} u_j^E = \min \{ c_j, \alpha^{E,c} u_j^E \} = \text{CREA}_j(E, c).$$

This implies that $\bar{c}^E(i) \in E$ since E is comprehensive. Since f satisfies conditional full compensation, then, $f_i(E, c) = c_i$.

Now, suppose that $f(E, c) \neq \text{CREA}(E, c)$. Since E is non-leveled, and both $f(E, c) \in \text{WP}(E)$ and $\text{CREA}(E, c) \in \text{WP}(E)$, this implies that there exists a $j \in N$ for which $f_j(E, c) < \text{CREA}_j(E, c) = \alpha^{E,c} u_j^E$. Let $k \in \text{argmin}_{j \in N} \{ \frac{f_j(E, c)}{u_j^E} \mid \text{CREA}_j(E, c) = \alpha^{E,c} u_j^E \}$. We have $f_k(E, c) < \alpha^{E,c} u_k^E < c_k$ since $f_k(E, c) = \text{CREA}_k(E, c)$ if $\text{CREA}_k(E, c) = c_k$.

Assume that $f_k(E, c) > 0$. Let $x \in \mathbb{R}_+^N \setminus \{0^N\}$ be given by

$$x_j = \begin{cases} c_j & \text{if } \text{CREA}_j(E, c) = c_j; \\ \frac{f_k(E, c)}{u_k^E} u_j^E & \text{if } \text{CREA}_j(E, c) < c_j. \end{cases}$$

For all $j \in N$ with $\text{CREA}_j(E, c) < c_j$, we have $x_j = \frac{f_k(E, c)}{u_k^E} u_j^E < \alpha^{E,c} u_j^E = \text{CREA}_j(E, c)$.

This implies that $\lambda^{E,x} < \lambda^{E, \text{CREA}(E, c)} = 1$.

For all $j \in N$, we can write

$$\begin{aligned} \overline{f(E, c)}_j^{\lambda^{E,x}E}(k) &= \min \left\{ \frac{f_k(E, c)}{u_k^{\lambda^{E,x}E}}, \frac{f_j(E, c)}{u_j^{\lambda^{E,x}E}} \right\} u_j^{\lambda^{E,x}E} = \min \left\{ \frac{f_k(E, c)}{u_k^{\lambda^{E,x}E}} u_j^{\lambda^{E,x}E}, f_j(E, c) \right\} \\ &= \min \left\{ \frac{f_k(E, c)}{u_k^E} u_j^E, f_j(E, c) \right\} \leq \min \left\{ \frac{f_k(E, c)}{u_k^E} u_j^E, c_j \right\} \leq x_j. \end{aligned}$$

This implies that $\overline{f(E, c)}_j^{\lambda^{E,x}E}(k) \in \lambda^{E,x}E$ since $\lambda^{E,x}E$ is comprehensive. Then, we have $f_k(\lambda^{E,x}E, c) = f_k(\lambda^{E,x}E, f(E, c)) = f_k(E, c)$ since f satisfies composition down and conditional full compensation. Since f satisfies path monotonicity, this means that

$$f_k(tE, c) = f_k(E, c) \text{ for all } t \in [\lambda^{E,x}, 1].$$

Let $\hat{t} \in (1, \lambda^{E,c})$ be such that

$$f_k(E, c) < f_k(\hat{t}E, c) < \alpha^{E,c} u_k^E.$$

Such a \hat{t} exists since f satisfies path continuity. Let $y \in \mathbb{R}_+^N \setminus \{0^N\}$ be given by

$$y_j = \begin{cases} c_j & \text{if } \text{CREA}_j(E, c) = c_j; \\ \frac{f_k(\hat{t}E, c)}{u_k^E} u_j^E & \text{if } \text{CREA}_j(E, c) < c_j. \end{cases}$$

For all $j \in N$ with $\text{CREA}_j(E, c) < c_j$, we have $y_j = \frac{f_k(\hat{t}E, c)}{u_k^E} u_j^E < \alpha^{E,c} u_j^E = \text{CREA}_j(E, c)$.

This implies that $\lambda^{E,y} < \lambda^{E, \text{CREA}(E, c)} = 1$. Moreover, we have $x \leq y$ and $x \neq y$, which means that $\lambda^{E,y} \in (\lambda^{E,x}, 1)$. This implies that $f_k(\lambda^{E,y}E, c) = f_k(E, c)$.

For all $j \in N$, we can write

$$\begin{aligned} \overline{f(\hat{t}E, c)}_j^{\lambda^{E,y}E}(k) &= \min \left\{ \frac{f_k(\hat{t}E, c)}{u_k^{\lambda^{E,y}E}}, \frac{f_j(\hat{t}E, c)}{u_j^{\lambda^{E,y}E}} \right\} u_j^{\lambda^{E,y}E} = \min \left\{ \frac{f_k(\hat{t}E, c)}{u_k^{\lambda^{E,y}E}} u_j^{\lambda^{E,y}E}, f_j(\hat{t}E, c) \right\} \\ &= \min \left\{ \frac{f_k(\hat{t}E, c)}{u_k^E} u_j^E, f_j(\hat{t}E, c) \right\} \leq \min \left\{ \frac{f_k(\hat{t}E, c)}{u_k^E} u_j^E, c_j \right\} \leq y_j. \end{aligned}$$

This implies that $\overline{f(\hat{t}E, c)}_j^{\lambda^{E,y}E}(k) \in \lambda^{E,y}E$ since $\lambda^{E,y}E$ is comprehensive. Then, we have $f_k(\lambda^{E,y}E, c) = f_k(\lambda^{E,y}E, f(\hat{t}E, c)) = f_k(\hat{t}E, c)$ since f satisfies composition down and conditional full compensation. This means that $f_k(E, c) = f_k(\hat{t}E, c)$, which contradicts the definition of \hat{t} . Clearly, the same type of arguments also apply to the case $f_k(E, c) = 0$, even without defining x . Hence, $f(E, c) = \text{CREA}(E, c)$. \square

Interestingly, the constrained relative equal awards rule also shares a characteristic feature with the serial cost sharing mechanism (cf. Moulin and Shenker (1992)). We show this by formulating a fourth characterization of the constrained relative equal awards rule based on relative symmetry and independence of larger relative claims.

Definition 5.5 (Independence of Larger Relative Claims).

A bankruptcy rule f satisfies *independence of larger relative claims* if $f_i(E, c') = f_i(E, c)$ for all $(E, c) \in \text{BR}^N$, any $i \in N$, and any $c' \in \mathbb{R}_+^N$ for which $c'_j = c_j$ for all $j \in N$ with $\frac{c_j}{u_j^E} \leq \frac{c_i}{u_i^E}$ and $c'_j \geq c_j$ for all $j \in N$ with $\frac{c_j}{u_j^E} > \frac{c_i}{u_i^E}$.

Theorem 5.4.

The constrained relative equal awards rule is the unique bankruptcy rule satisfying relative symmetry and independence of larger relative claims.

Proof. From Lemma A.2 and Lemma A.7, we know that the constrained relative equal awards rule satisfies relative symmetry and independence of larger relative claims. Let f be a bankruptcy rule satisfying relative symmetry and independence of larger relative claims. We show that $f = \text{CREA}$. Let $(E, c) \in \text{BR}^N$. Denote $N = \{1, \dots, n\}$ such that $\frac{c_1}{u_1^E} \leq \dots \leq \frac{c_n}{u_n^E}$ and let $k \in N$ such that $\text{CREA}_i(E, c) = c_i$ for all $i < k$, and $\text{CREA}_i(E, c) = \alpha^{E,c} u_k^E < c_i$ for all $i \geq k$. For all $i < k$, we can write

$$f_i(E, c) = f_i(E, \text{CREA}(E, c)) = \text{CREA}_i(E, c),$$

where the first equality follows from independence of larger relative claims. For all $i \geq k$, let $c^i \in \mathbb{R}_+^N$ be given by

$$c_j^i = \begin{cases} c_j & \text{if } \frac{c_j}{u_j^E} \leq \frac{c_i}{u_i^E}; \\ \frac{c_i}{u_i^E} u_j^E & \text{if } \frac{c_j}{u_j^E} > \frac{c_i}{u_i^E}. \end{cases}$$

We can write

$$f_k(E, c) = f_k(E, c^k) = \alpha^{E,c^k} u_k^E = \text{CREA}_k(E, c^k) = \text{CREA}_k(E, c),$$

where the first equality follows from independence of larger relative claims, the second and third equality follow from relative symmetry, and the last equality follows again from independence of larger relative claims. Clearly, the same argument can now be applied to claimant $k + 1$. Continuing this reasoning, we have $f_i(E, c) = \text{CREA}_i(E, c)$ for all $i \geq k$. Hence, $f(E, c) = \text{CREA}(E, c)$. \square

Finally, we use duality in relation to the constrained relative equal awards rule. We introduce the constrained relative equal losses rule for NTU-bankruptcy problems as a generalization of the constrained equal losses rule for TU-bankruptcy problems and show that the constrained relative equal awards rule and the constrained relative equal losses rule are dual. This means that the constrained relative equal losses rule also satisfies relative symmetry, composition down and composition up.

Definition 5.6 (The Constrained Relative Equal Losses Rule).

For all $(E, c) \in \text{BR}^N$ and any $i \in N$, the *constrained relative equal losses rule* CREL is defined by

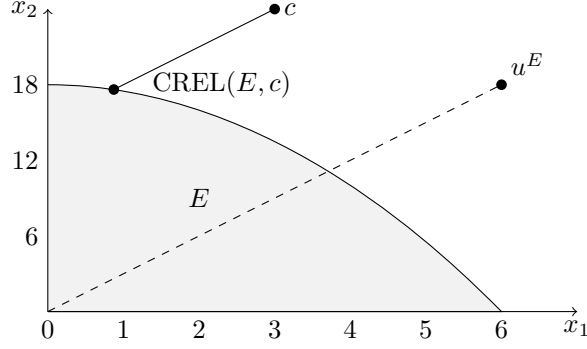
$$\text{CREL}_i(E, c) = \max\{0, c_i - \beta^{E,c} u_i^E\},$$

where $\beta^{E,c} \in \mathbb{R}_{++}$ is such that $\text{CREL}(E, c) \in \text{WP}(E)$.

Note that the conditions on E imply that $\beta^{E,c}$ is well-defined. Moreover, for a TU-bankruptcy problem $(E, c) \in \text{BR}^N$ with $E = \{x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i \leq M\}$ for some $M \in \mathbb{R}_{++}$, we have $u_i^E = M$ for all $i \in N$ and the constrained relative equal losses rule coincides with the classic constrained equal losses rule.

Example 5.

Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$ as in Example 1. We have $u^E = (6, 18)$ and $\beta^{E,c} = 1 - \frac{1}{6}\sqrt{15}$. Moreover, $\text{CREL}_1(E, c) = \max\{0, c_1 - \beta^{E,c} u_1^E\} = \max\{0, \sqrt{15} - 3\} = \sqrt{15} - 3$ and $\text{CREL}_2(E, c) = \max\{0, c_2 - \beta^{E,c} u_2^E\} = \max\{0, 3\sqrt{15} + 6\} = 3\sqrt{15} + 6$.



△

Proposition 5.5.

The constrained relative equal awards rule and the constrained relative equal losses rule are dual.

Proof. Let $(E, c) \in \text{BR}^N$. First, we show that $\text{CREA}(E, c) = c - \text{CREL}(\lambda^{E, c - \text{CREA}(E, c)} E, c)$. Denote $d = \lambda^{E, c - \text{CREA}(E, c)}$. Assume that $d\beta^{dE, c} \leq \alpha^{E, c}$. For all $i \in N$, we can write

$$\begin{aligned} \text{CREL}_i(dE, c) &= \max\{0, c_i - \beta^{dE, c} u_i^{dE}\} = c_i - \min\{c_i, d\beta^{dE, c} u_i^E\} \\ &\geq c_i - \min\{c_i, \alpha^{E, c} u_i^E\} = c_i - \text{CREA}_i(E, c). \end{aligned}$$

This means that we have $\text{CREL}(dE, c) \geq c - \text{CREA}(E, c)$. Since E is non-leveled, and both $\text{CREL}(dE, c) \in \text{WP}(dE)$ and $c - \text{CREA}(E, c) \in \text{WP}(dE)$, this implies that we have $\text{CREL}(dE, c) = c - \text{CREA}(E, c)$. Clearly, the same type of arguments also apply to the case $d\beta^{dE, c} > \alpha^{E, c}$.

Second, we show that $\text{CREL}(E, c) = c - \text{CREA}(\lambda^{E, c - \text{CREL}(E, c)} E, c)$. Denote $d' = \lambda^{E, c - \text{CREL}(E, c)}$. Assume that $d'\alpha^{d'E, c} \leq \beta^{E, c}$. For all $i \in N$, we can write

$$\begin{aligned} \text{CREA}_i(d'E, c) &= \min\{c_i, \alpha^{d'E, c} u_i^{d'E}\} = c_i - \max\{0, c_i - d'\alpha^{d'E, c} u_i^E\} \\ &\leq c_i - \max\{0, c_i - \beta^{E, c} u_i^E\} = c_i - \text{CREL}_i(E, c). \end{aligned}$$

This means that we have $\text{CREA}(d'E, c) \leq c - \text{CREL}(E, c)$. Since E is non-leveled, and both $\text{CREA}(d'E, c) \in \text{WP}(d'E)$ and $c - \text{CREL}(E, c) \in \text{WP}(d'E)$, this implies that we have $\text{CREA}(d'E, c) = c - \text{CREL}(E, c)$. Clearly, the same type of arguments also apply to the case $d'\alpha^{d'E, c} > \beta^{E, c}$. Hence, CREA and CREL are dual. □

6 Concluding Remarks

This section formulates some concluding remarks on our model and results for bankruptcy problems with nontransferable utility and points out some suggestions for future research.

First, note that our interpretation of equality is quite specific. In a bankruptcy problem with nontransferable utility, claimants differ in their measure of utility and their claims are therefore incomparable. For that reason, it makes sense to compare their claims in relation to the estate. In particular, we formulate the relative symmetry property in terms of the claims relative to the utopia values as given by the estate. Besides, this approach ensures that all considered notions are invariant under individual rescaling of utility.

Second, this paper focuses on proportionality, equality, and duality in the analysis of the proportional rule and the constrained relative equal awards rule for bankruptcy problems with nontransferable utility. Moreover, we have shown how to define a constrained relative equal losses rule such that it is the dual of the constrained relative equal awards rule. Future research could study generalizations of other well-known bankruptcy rules for TU-bankruptcy problems, such as the random arrival rule (cf. O'Neill (1982)), the Talmud rule (cf. Aumann and Maschler (1985)) and the adjusted proportional rule (cf. Curiel, Maschler, and Tijs (1987)).

Finally, this paper interprets bankruptcy problems with nontransferable utility as a generalization of bankruptcy problems with transferable utility. Alternatively, bankruptcy problems with nontransferable utility can be interpreted as a new approach to bargaining problems with claims (cf. Chun and Thomson (1992)) with the zero vector as disagreement point. Chun and Thomson (1992) extended the classical bargaining problem of Nash (1950) with a vector of claims and studied these problems using axiomatic bargaining theory. The proportional rule for NTU-bankruptcy problems coincides with their proportional solution for bargaining problems with claims. Moreover, the constrained relative equal awards rule for NTU-bankruptcy problems can be considered as an extension of the solution of Kalai and Smorodinsky (1975) to bargaining problems with claims. Future research could study the relation between NTU-bankruptcy rules and solutions for bargaining problems with claims in detail. In particular, one could use axiomatic bargaining theory to characterize rules for bankruptcy problems with nontransferable utility, similar to Dagan and Volij (1993).

Appendix: Properties of the CREA rule

Lemma A.1.

The constrained relative equal awards rule satisfies claim monotonicity.

Proof. Let $(E, c) \in \text{BR}^N$, let $i \in N$, and let $c' \in \mathbb{R}_+^N$ such that $c'_i \geq c_i$ and $c'_j = c_j$ for all $j \in N \setminus \{i\}$. We show that $\text{CREA}_i(E, c') \geq \text{CREA}_i(E, c)$.

If $\text{CREA}_i(E, c) = \alpha^{E, c} u_i^E$, then, clearly $\text{CREA}(E, c') = \text{CREA}(E, c)$. Assume that $\text{CREA}_i(E, c) = c_i$, i.e., $\frac{c_i}{u_i^E} \leq \alpha^{E, c}$.

If $\text{CREA}_i(E, c') = c'_i$, then,

$$\text{CREA}_i(E, c') = c'_i \geq c_i = \text{CREA}_i(E, c).$$

Assume that $\text{CREA}_i(E, c') = \alpha^{E, c'} u_i^E$ and take $t = \frac{c_i}{u_i^E}$. For all $j \in N$,

$$\min \{c'_j, t u_j^E\} = \min \left\{ c_j, \frac{c_i}{u_i^E} u_j^E \right\} \leq \min \{c_j, \alpha^{E, c} u_j^E\} = \text{CREA}_j(E, c).$$

Since E is comprehensive, this means that $(\min\{c'_j, t u_j^E\})_{j \in N} \in E$, which implies that $t \leq \alpha^{E, c'}$. Consequently,

$$\text{CREA}_i(E, c') = \alpha^{E, c'} u_i^E \geq t u_i^E = \frac{c_i}{u_i^E} u_i^E = c_i = \text{CREA}_i(E, c).$$

□

Lemma A.2.

The constrained relative equal awards rule satisfies relative symmetry.

Proof. Let $(E, c) \in \text{BR}^N$ and let $i, j \in N$ with $\frac{c_i}{u_i^E} = \frac{c_j}{u_j^E}$. We show that $\frac{\text{CREA}_i(E, c)}{u_i^E} = \frac{\text{CREA}_j(E, c)}{u_j^E}$.

We have

$$\begin{aligned} \frac{\text{CREA}_i(E, c)}{u_i^E} &= \frac{\min\{c_i, \alpha^{E, c} u_i^E\}}{u_i^E} = \min\left\{\frac{c_i}{u_i^E}, \frac{\alpha^{E, c} u_i^E}{u_i^E}\right\} = \min\left\{\frac{c_i}{u_i^E}, \alpha^{E, c}\right\} \\ &= \min\left\{\frac{c_j}{u_j^E}, \alpha^{E, c}\right\} = \min\left\{\frac{c_j}{u_j^E}, \frac{\alpha^{E, c} u_j^E}{u_j^E}\right\} = \frac{\min\{c_j, \alpha^{E, c} u_j^E\}}{u_j^E} \\ &= \frac{\text{CREA}_j(E, c)}{u_j^E}. \end{aligned}$$

□

Lemma A.3.

The constrained relative equal awards rule satisfies composition down.

Proof. Let $(E, c) \in \text{BR}^N$ and let $t \in (0, 1)$. We show that $\text{CREA}(tE, \text{CREA}(E, c)) = \text{CREA}(tE, c)$.

We have $(\min\{c_i, \alpha^{tE, c} u_i^{tE}\})_{i \in N} \in tE$, which implies that $(\min\{\frac{c_i}{t}, \alpha^{tE, c} u_i^E\})_{i \in N} \in E$. Since E is comprehensive, $(\min\{c_i, \alpha^{tE, c} u_i^E\})_{i \in N} \leq (\min\{\frac{c_i}{t}, \alpha^{tE, c} u_i^E\})_{i \in N}$ implies that $(\min\{c_i, \alpha^{tE, c} u_i^E\})_{i \in N} \in E$. This means that $\alpha^{tE, c} \leq \alpha^{E, c}$.

Assume that $\alpha^{tE, \text{CREA}(E, c)} \leq \alpha^{tE, c}$. Then, for all $i \in N$,

$$\begin{aligned} \text{CREA}_i(tE, \text{CREA}(E, c)) &= \min\{\text{CREA}_i(E, c), \alpha^{tE, \text{CREA}(E, c)} u_i^{tE}\} \\ &= \min\{\min\{c_i, \alpha^{E, c} u_i^E\}, t \alpha^{tE, \text{CREA}(E, c)} u_i^E\} \\ &= \min\{c_i, \alpha^{E, c} u_i^E, t \alpha^{tE, \text{CREA}(E, c)} u_i^E\} \\ &\leq \min\{c_i, \alpha^{E, c} u_i^E, t \alpha^{tE, c} u_i^E\} \\ &= \min\{c_i, t \alpha^{tE, c} u_i^E\} \\ &= \min\{c_i, \alpha^{tE, c} u_i^{tE}\} \\ &= \text{CREA}_i(tE, c). \end{aligned}$$

This means that we have $\text{CREA}(tE, \text{CREA}(E, c)) \leq \text{CREA}(tE, c)$. Since E is non-leveled, and both $\text{CREA}(tE, \text{CREA}(E, c)) \in \text{WP}(tE)$ and $\text{CREA}(tE, c) \in \text{WP}(tE)$, this implies that $\text{CREA}(tE, \text{CREA}(E, c)) = \text{CREA}(tE, c)$.

Clearly, the same type of arguments also apply to the case $\alpha^{tE, \text{CREA}(E, c)} > \alpha^{tE, c}$. □

Lemma A.4.

The constrained relative equal awards rule satisfies composition up.

Proof. Let $(E, c) \in \text{BR}^N$ and let $t \in (0, 1)$. We show that $\text{CREA}(E, c) = \text{CREA}(tE, c) + \text{CREA}(\lambda^{E, \text{CREA}(E, c)} - \text{CREA}(tE, c) E, c - \text{CREA}(tE, c))$.

We have $(\min\{c_i, \alpha^{tE, c} u_i^{tE}\})_{i \in N} \in tE$, which implies that $(\min\{\frac{c_i}{t}, \alpha^{tE, c} u_i^E\})_{i \in N} \in E$. Since E is comprehensive, $(\min\{c_i, \alpha^{tE, c} u_i^E\})_{i \in N} \leq (\min\{\frac{c_i}{t}, \alpha^{tE, c} u_i^E\})_{i \in N}$ implies that $(\min\{c_i, \alpha^{tE, c} u_i^E\})_{i \in N} \in E$. This means that $\alpha^{tE, c} \leq \alpha^{E, c}$.

For all $i \in N$,

$$\text{CREA}_i(tE, c) = \min\{c_i, \alpha^{tE, c} u_i^{tE}\} = \min\{c_i, t\alpha^{tE, c} u_i^E\} \leq \min\{c_i, \alpha^{E, c} u_i^E\} = \text{CREA}_i(E, c).$$

Since $\text{CREA}(tE, c) \neq \text{CREA}(E, c)$, this implies that $\lambda^{E, \text{CREA}(E, c) - \text{CREA}(tE, c)} \in (0, \lambda^{E, c})$. Denote $d = \lambda^{E, \text{CREA}(E, c) - \text{CREA}(tE, c)}$ and define $L, H \subseteq N$ by

$$L = \{i \in N \mid \text{CREA}_i(tE, c) = c_i\} \quad \text{and} \quad H = \{i \in N \mid \text{CREA}_i(tE, c) = t\alpha^{tE, c} u_i^E\}.$$

For all $i \in L$, we can write

$$\text{CREA}_i(dE, c - \text{CREA}(tE, c)) = 0 = c_i - c_i = \text{CREA}_i(E, c) - \text{CREA}_i(tE, c).$$

Assume that $d\alpha^{dE, c - \text{CREA}(tE, c)} \leq \alpha^{E, c} - t\alpha^{tE, c}$. For all $i \in H$, we can write

$$\begin{aligned} \text{CREA}_i(dE, c - \text{CREA}(tE, c)) &= \min\{c_i - \text{CREA}_i(tE, c), \alpha^{dE, c - \text{CREA}(tE, c)} u_i^{dE}\} \\ &= \min\{c_i - t\alpha^{tE, c} u_i^E, d\alpha^{dE, c - \text{CREA}(tE, c)} u_i^E\} \\ &\leq \min\{c_i - t\alpha^{tE, c} u_i^E, \alpha^{E, c} u_i^E - t\alpha^{tE, c} u_i^E\} \\ &= \min\{c_i, \alpha^{E, c} u_i^E\} - t\alpha^{tE, c} u_i^E \\ &= \text{CREA}_i(E, c) - \text{CREA}_i(tE, c). \end{aligned}$$

This means that $\text{CREA}(dE, c - \text{CREA}(tE, c)) \leq \text{CREA}(E, c) - \text{CREA}(tE, c)$. Since E is non-leveled, and both $\text{CREA}(dE, c - \text{CREA}(tE, c)) \in \text{WP}(dE)$ and $\text{CREA}(E, c) - \text{CREA}(tE, c) \in \text{WP}(dE)$, this implies that $\text{CREA}(dE, c - \text{CREA}(tE, c)) = \text{CREA}(E, c) - \text{CREA}(tE, c)$.

Clearly, the same type of arguments also apply to the case $d\alpha^{dE, c - \text{CREA}(tE, c)} > \alpha^{E, c} - t\alpha^{tE, c}$. \square

Lemma A.5.

The constrained relative equal awards rule satisfies truncation invariance.

Proof. Let $(E, c) \in \text{BR}^N$. We show that $\text{CREA}(E, \hat{c}^E) = \text{CREA}(E, c)$.

First, for all $t \in (0, 1)$ and any $i \in N$, we have

$$\min\{\hat{c}_i^E, tu_i^E\} = \min\{\min\{c_i, u_i^E\}, tu_i^E\} = \min\{c_i, u_i^E, tu_i^E\} = \min\{c_i, tu_i^E\}.$$

This implies that $\alpha^{E, \hat{c}^E} = \alpha^{E, c}$. For all $i \in N$, this means that

$$\begin{aligned} \text{CREA}_i(E, \hat{c}^E) &= \min\{\hat{c}_i^E, \alpha^{E, \hat{c}^E} u_i^E\} = \min\{\min\{c_i, u_i^E\}, \alpha^{E, c} u_i^E\} = \min\{c_i, u_i^E, \alpha^{E, c} u_i^E\} \\ &= \min\{c_i, \alpha^{E, c} u_i^E\} = \text{CREA}_i(E, c). \end{aligned}$$

\square

Lemma A.6.

The constrained relative equal awards rule satisfies conditional full compensation.

Proof. Let $(E, c) \in \text{BR}^N$ and let $i \in N$ such that $\bar{c}^E(i) \in E$. We show that $\text{CREA}_i(E, c) = c_i$ if $\text{CREA}_i(E, c) = \alpha^{E, c} u_i^E$.

Assume that $\text{CREA}_i(E, c) = \alpha^{E, c} u_i^E$. Then, we have $\alpha^{E, c} \leq \frac{c_i}{u_i^E}$. For all $j \in N$, we can write

$$\text{CREA}_j(E, c) = \min\{c_j, \alpha^{E, c} u_j^E\} = \min\left\{\alpha^{E, c}, \frac{c_j}{u_j^E}\right\} u_j^E \leq \min\left\{\frac{c_i}{u_i^E}, \frac{c_j}{u_j^E}\right\} u_j^E = \bar{c}_j^E(i).$$

Since E is non-leveled and $\text{CREA}(E, c) \in \text{WP}(E)$, this implies that we have $\text{CREA}(E, c) = \bar{c}^E(i)$. This means that $\text{CREA}_i(E, c) = c_i$. \square

Lemma A.7.

The constrained relative equal awards rule satisfies independence of larger relative claims.

Proof. Let $(E, c) \in \text{BR}^N$, let $i \in N$, and let $c' \in \mathbb{R}_+^N$ such that $c'_j = c_j$ for all $j \in N$ with $\frac{c_j}{u_j^E} \leq \frac{c_i}{u_i^E}$, and $c'_j \geq c_j$ for all $j \in N$ with $\frac{c_j}{u_j^E} > \frac{c_i}{u_i^E}$. We show that $\text{CREA}_i(E, c') = \text{CREA}_i(E, c)$.

For all $j \in N$,

$$\min\{c_j, \alpha^{E, c'} u_j^E\} \leq \min\{c'_j, \alpha^{E, c'} u_j^E\} = \text{CREA}_j(E, c').$$

Since E is comprehensive, this means that $(\min\{c_j, \alpha^{E, c'} u_j^E\})_{j \in N} \in E$, which implies that $\alpha^{E, c'} \leq \alpha^{E, c}$.

First, consider the case $\frac{c_i}{u_i^E} < \alpha^{E, c'} \leq \alpha^{E, c}$. Then,

$$\text{CREA}_i(E, c') = \min\{c'_i, \alpha^{E, c'} u_i^E\} = c'_i = c_i = \min\{c_i, \alpha^{E, c} u_i^E\} = \text{CREA}_i(E, c).$$

Second, consider the case $\alpha^{E, c'} \leq \frac{c_i}{u_i^E} \leq \alpha^{E, c}$. For all $j \in N$,

$$\begin{aligned} \text{CREA}_j(E, c') &= \min\{c'_j, \alpha^{E, c'} u_j^E\} = \min\left\{\frac{c'_j}{u_j^E}, \alpha^{E, c'}\right\} u_j^E \\ &\leq \min\left\{\frac{c'_j}{u_j^E}, \frac{c_i}{u_i^E}\right\} u_j^E = \min\left\{\frac{c_j}{u_j^E}, \frac{c_i}{u_i^E}\right\} u_j^E \\ &\leq \min\left\{\frac{c_j}{u_j^E}, \alpha^{E, c}\right\} u_j^E = \min\{c_j, \alpha^{E, c} u_j^E\} \\ &= \text{CREA}_j(E, c). \end{aligned}$$

This means that we have $\text{CREA}(E, c') \leq \text{CREA}(E, c)$. Since E is non-leveled, and both $\text{CREA}(E, c') \in \text{WP}(E)$ and $\text{CREA}(E, c) \in \text{WP}(E)$, this implies that we have $\text{CREA}(E, c') = \text{CREA}(E, c)$. In particular, we have $\text{CREA}_i(E, c') = \text{CREA}_i(E, c)$.

Finally, consider the case $\alpha^{E, c'} \leq \alpha^{E, c} < \frac{c_i}{u_i^E}$. For all $j \in N$,

$$\begin{aligned} \text{CREA}_j(E, c') &= \min\{c'_j, \alpha^{E, c'} u_j^E\} = \min\left\{\frac{c'_j}{u_j^E}, \alpha^{E, c'}\right\} u_j^E \\ &\leq \min\left\{\frac{c'_j}{u_j^E}, \alpha^{E, c}\right\} u_j^E = \min\left\{\frac{c_j}{u_j^E}, \alpha^{E, c}\right\} u_j^E \\ &= \min\{c_j, \alpha^{E, c} u_j^E\} = \text{CREA}_j(E, c). \end{aligned}$$

This means that we have $\text{CREA}(E, c') \leq \text{CREA}(E, c)$. Since E is non-leveled, and both $\text{CREA}(E, c') \in \text{WP}(E)$ and $\text{CREA}(E, c) \in \text{WP}(E)$, this implies that we have $\text{CREA}(E, c') = \text{CREA}(E, c)$. In particular, we have $\text{CREA}_i(E, c') = \text{CREA}_i(E, c)$. \square

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