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Abstract

In this paper minimum coloring games are considered. We characterize the type of conflict graphs inducing simple or three-valued simple minimum coloring games. We provide an upper bound on the number of maximum cliques of conflict graphs inducing such games. Moreover, a characterization of the core is provided in terms of the underlying conflict graph. In particular, in case of a perfect conflict graph the core of an induced three-valued simple minimum coloring game equals the vital core.

Keywords: minimum coloring games, simple games, three-valued simple games
JEL classification number: C71, C44

1   Introduction

Consider a set of agents who all need access to some type of facility, but some agents might be in conflict. All facilities are similar, but if two agents are in conflict, they cannot have access to the same facility. The total costs are linearly increasing with the number of facilities used, so the aim is to find the minimum number of facilities that can serve all agents. This problem can be modelled by an undirected graph, called the conflict graph, in which the vertices represent the agents. Two vertices are adjacent if and only if the corresponding agents are in conflict. Next, we color all vertices in such a way that any two adjacent nodes receive different colors. Finding the minimum number of facilities such that every agent has non-conflicting access to some facility, is equivalent to finding a coloring of this conflict graph that uses the smallest number of colors. This combinatorial optimization problem is known as the minimum coloring problem. A survey on minimum coloring problems can, for example, be found in Randerath and Schiermeyer (2004) and Pardalos, Mavridou, and Xue (1999). An application of the minimum coloring problem is, for example, scheduling courses at secondary schools, where some courses are compulsory and other courses are electives. Courses can be scheduled in any order, but pairs of courses are in conflict in the sense that they can not be assigned to the same time slot if there is a student who has chosen both courses.

Assuming that in the initial situation no agents share facilities, optimal cooperation in sharing facilities between non-conflicting agents will lead to minimal joint costs. To

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analyze how to divide the minimal joint costs among the agents, Deng, Ibaraki, and Nagamochi (1999) introduced minimum coloring games. A minimum coloring cost game can be seen as an example of a combinatorial optimization or operations research game. In a combinatorial optimization game, the value of each coalition is obtained by solving a combinatorial optimization problem on the corresponding substructure. A survey on operations research games can be found in Borm, Hamers, and Hendrickx (2001).

In Deng et al. (1999) the existence of core elements is investigated for the more general class of combinatorial optimization cost games where the value of a coalition is defined by an integer program. They showed that such games have a non-empty core if and only if the associated linear program has an integer optimal solution. Moreover, in case of bipartite conflict graphs, they characterized the core of the induced minimum coloring games as the convex hull of the characteristic vectors of the edges in the conflict graph. Deng, Ibaraki, Nagamochi, and Zang (2000) studied total balancedness of minimum coloring games and other combinatorial optimization games. They showed that a minimum coloring game is totally balanced if and only if the underlying conflict graph is perfect. In Okamoto (2003) concave minimum coloring games are characterized in terms of forbidden subgraphs. Moreover, for this case an explicit formula of the Shapley value is provided. In Bietenhader and Okamoto (2006) core largeness, extendability, and exactness of minimum coloring games are considered. Okamoto (2008) characterized the core of minimum coloring games on perfect conflict graphs as the convex hull of the characteristic vectors of the maximum cliques in the conflict graph, which is a generalization of the result by Deng et al. (1999). Additionally, Okamoto (2008) also investigated the nucleolus, the compromise value and the Shapley value of a minimum coloring game. The most recent work on minimum coloring games is by Hamers, Miquel, and Norde (2014). They provided a necessary and sufficient condition for a conflict graph such that the induced minimum coloring game has a population monotonic allocation scheme.

The minimum coloring games studied in the works above are cost games. However, if we assume that in the initial situation no agents share facilities, i.e., every vertex has its own color, then cooperation in sharing facilities between non-conflicting agents will lead to cost savings. In this paper we define minimum coloring games as cost savings games instead of cost games and we focuss on conflict graphs inducing coalitional cost savings that are only 0 and 1, or 0, 1 and 2. Games for which the only possible values for a coalition are 0 and 1 are called simple. Simple games are introduced by Von Neumann and Morgenstern (1944) and further studied by Shapley and Shubik (1954). Games for which the only possible values for a coalition are 0, 1 and 2 are called three-valued simple. The class of three-valued simple games is a natural extension of the class of simple games. Three-valued simple games are introduced and studied by Musegaas, Borm, and Quant (2015).

In this paper we investigate two features of simple and three-valued simple minimum coloring games. First, we characterize the type of conflict graphs inducing such games. For this characterization, a distinction is made between perfect and imperfect conflict graphs, and the concept of maximum clique is used. We show that simple minimum coloring games are always induced by perfect graphs, while three-valued simple minimum coloring games can be induced by both perfect and imperfect graphs. In particular, there is only one type of imperfect conflict graphs inducing three-valued simple minimum coloring games. We also provide an upper bound on the number of maximum cliques for conflict graphs inducing simple or three-valued simple games. Second, we characterize the core in terms of the underlying conflict graph for these games. This characterization is

also based on the concept of maximum clique. Since simple minimum coloring games are always induced by perfect graphs, the characterization of the core is readily derived. We show that for three-valued simple minimum coloring games induced by imperfect conflict graphs, the core is empty. On the other hand, for three-valued simple minimum coloring games induced by perfect conflict graphs, we show that the core equals the vital core, as introduced by Musegaas et al. (2015).

The organization of this paper is as follows. Section 2 recalls basic definitions from graph theory and formally introduces minimum coloring games. In Section 3 simple minimum coloring games are investigated. Finally, Section 4 analyzes three-valued simple minimum coloring games.

2 Minimum coloring games

In this section we recall basic definitions from graph theory and formally define minimum coloring games. We also provide a survey of game-theoretic characteristics of minimum coloring games and in particular recall the characterization of the core of minimum coloring games associated to perfect conflict graphs.

Let $G = (N, E)$ be an *undirected graph* G, represented by a pair (N, E), where $N = \{1, \ldots, n\}$ is a set of vertices and $E \subset \{\{i, j\} \mid i, j \in N, i \neq j\}$ is a set of edges. The graph G is called *complete* if $E = \{\{i, j\} \mid i, j \in N, i \neq j\}$, that is every two vertices are adjacent. K_n denotes the complete graph on n vertices. For $S \subset N$, the *subgraph* of G induced by S is the graph $G[S] = (S, E_S)$ where $E_S = \{\{i, j\} \in E \mid i, j \in S\}$. The *complement* of G is the graph $\overline{G} = (N, \overline{E})$ where $\overline{E} = \{\{i, j\} \mid i, j \in N, i \neq j, \{i, j\} \notin E\}$. In this paper we only consider graphs that are connected on N, i.e., every pair of vertices is linked via a sequence of consecutive edges in E. However, note that it still might happen that some induced subgraph $G[S]$ is not connected on S via E_S. For conflict graphs, $\{i, j\} \in E$ is interpreted as i and j being in conflict.

A *coloring* of G is a mapping $\gamma : N \to \mathbb{N}$ such that $\gamma(i) \neq \gamma(j)$ for every $\{i, j\} \in E$. The natural numbers assigned to the vertices correspond to the colors assigned to the vertices. A *minimum coloring* of G is a coloring γ that uses the smallest number of colors, i.e., a coloring for which $|\{\gamma(i) \mid i \in N\}|$ is minimal. The number of colors in a minimum coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. The problem of finding a minimum coloring of a conflict graph (N, E) is called a *minimum coloring problem*.

The chromatic number of a conflict graph is strongly related to the concept of a clique, where a *clique* in G is a set $S \subset N$ such that $G[S] = K_{|S|}$. A *maximum clique* of G is a clique S of the largest possible size, i.e., a clique for which $|S|$ is maximal. The number of vertices in a maximum clique is called the *clique number* of G and is denoted by $\omega(G)$. We denote the set of all maximum cliques in G by $\Omega(G)$, i.e.,

$$\Omega(G) = \left\{ S \subset N \mid G[S] = K_{\omega(G)} \right\}.$$

Note that all vertices in a maximum clique are mutually adjacent and therefore they each must receive a different color in a minimum coloring, so

$$\chi(G) \geq \omega(G), \tag{1}$$

for any conflict graph G.

With N a non-empty finite set of players, a *transferable utility (TU) game* is a function $v : 2^N \to \mathbb{R}$ which assigns a number to each coalition $S \in 2^N$, where 2^N denotes the collection of all subsets of N. The value $v(S)$ denotes the highest joint monetary payoff or cost savings the coalition S can jointly generate by means of optimal cooperation without any help of the players in $N \backslash S$. By convention, $v(\emptyset) = 0$. Let TU^N denote the class of all TU-games with player set N.

If we assume that initially every vertex has its own color, then the minimum coloring of G results in optimal cost savings for N as a whole. To tackle the allocation problem of these cost savings one can analyze an associated TU-game v^G to a minimum coloring problem with conflict graph $G = (N, E)$, where the set of players is the set of vertices. For a coalition $S \subset N$, $v(S)$ reflects the maximal cost savings this coalition can generate, i.e., the number of colors that are saved with respect to the initial situation where $|S|$ colors were used. Hence, the value of coalition S is obtained by solving the minimum coloring problem with conflict graph $G[S]$. Correspondingly, the *minimum coloring game* $v^G \in \text{TU}^N$ induced by the conflict graph $G = (N, E)$ is defined by

$$v^G(S) = |S| - \chi(G[S]),$$

for all $S \subset N$.

In the following, we give an example of a minimum coloring game and discusses the concepts of clique number and chromatic number.

Example 2.1. Consider the conflict graph $G = (N, E)$ with $N = \{1, \ldots, 5\}$ as depicted in Figure 1. Note that $\Omega(G) = \{\{1, 3, 5\}, \{1, 4, 5\}, \{2, 4, 5\}\}$, so using (1) we have $\chi(G) \geq \omega(G) = 3$. Consider the following coloring of G with three colors given by the function $\gamma : N \to \{1, 2, 3\}$ with

$$\gamma(1) = \gamma(2) = 1,$$
$$\gamma(3) = \gamma(4) = 2,$$
$$\gamma(5) = 3.$$

We may conclude that this coloring γ is a minimum coloring of G and $\chi(G) = 3$. As a consequence, the value of the grand coalition N is given by

$$v^G(N) = 5 - \chi(G) = 2.$$

For coalition $\{1, 2\}$, the induced subgraph $G[\{1, 2\}]$ contains no edges and thus only one color is needed to color the vertices, i.e., $\chi(G[\{1, 2\}]) = 1$. Hence, the value of coalition $\{1, 2\}$ is given by

$$v^G(\{1, 2\}) = 2 - \chi(G[\{1, 2\}]) = 1.$$

The game v^G is given by

$$v^G(S) = \begin{cases} 2 & \text{if } \{1, 2, 3, 4\} \subset S, \\ 1 & \text{if } \{1, 2\} \subset S \text{ or } \{2, 3\} \subset S \text{ or } \{3, 4\} \subset S, \text{ and } \{1, 2, 3, 4\} \not\subset S, \\ 0 & \text{otherwise}, \end{cases}$$

for all $S \subset N$. $\triangle$

A game $v \in \text{TU}^N$ is called *monotonic* if

$$v(S) \leq v(T),$$

for all $S, T \in 2^N$ with $S \subset T$. Minimum coloring games are integer valued nonnegative monotonic games as is seen in the following proposition.
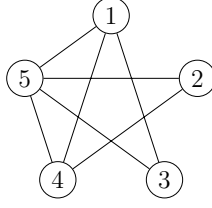
Figure 1: The conflict graph of Example 2.1

Proposition 2.1. *Let $G = (N, E)$ be a graph. Then, v^G is integer valued, nonnegative and monotonic.*

Proof. Integer valuedness and nonnegativity of v^G are straightforward consequences of the definition. As for monotonicity, let $S, T \in 2^N$ with $S \subset T$. Note that if a minimum coloring for $G[S]$ uses $\chi(G[S])$ colors, then at most $|T \backslash S|$ additional colors are necessary for a minimum coloring of $G[T]$, i.e., $\chi(G[T]) \leq \chi(G[S]) + |T \backslash S|$. As a consequence,

$$v^G(S) = |S| - \chi(G[S]) \leq |S| - \chi(G[T]) + |T \backslash S| = |T| - \chi(G[T]) = v^G(T). \qquad \square$$

The *core $C(v)$* of a game $v \in \mathrm{TU}^N$ is defined as the set of all allocations $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$ (*efficiency*) and $\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N$ (*stability*). Hence, the core consists of all possible allocations of $v(N)$ for which no coalition has an incentive to leave the grand coalition. Consequently, if the core is empty, then it is not possible to find a stable allocation of $v(N)$.

Okamoto (2008) characterized the core of minimum coloring games induced by perfect conflict graphs, where a graph $G = (N, E)$ is called *perfect* if

$$\omega(G[S]) = \chi(G[S]),$$

for all $S \subset N$. If a graph is not perfect, then it is called *imperfect*.

Theorem 2.2 (cf. Okamoto (2008)). *Let $G = (N, E)$ be a perfect graph. Then,* [1]

$$C(v^G) = Conv\left(\left\{ e^{N \backslash S} \mid S \in \Omega(G) \right\}\right).$$

Chudnovsky, Robertson, Seymour, and Thomas (2006) characterized perfect graphs. This characterization uses the notation of cycle graphs, where a *cycle graph C_n* is a graph $G = (N, E)$ for which there exists a bijection $f : \{1, \dots, n\} \to N$ such that

$$E = \{\{f(i), f(i+1)\} \mid i \in \{1, \dots, n-1\}\} \cup \{f(1), f(n)\}.$$

An *odd cycle graph* is a cycle graph C_n where n is odd.

Theorem 2.3 (cf. Chudnovsky et al. (2006)). *A graph is perfect if and only if it does not contain an odd cycle graph of length at least five, or a complement of such graph, as an induced subgraph.*

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[1]For $S \in 2^N \backslash \{\emptyset\}$, the *characteristic vector $e^S \in \mathbb{R}^N$* is defined as

$$e_i^S = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in N$.

Example 2.2. Reconsider the conflict graph $G = (N, E)$ in Figure 1. Since G does not contain an odd cycle graph of length at least five, or a complement of such graph as an induced subgraph, we know from Theorem 2.3 that G is a perfect conflict graph. Since the set $\Omega(G)$ of maximum cliques is given by

$$\Omega(G) = \{\{1, 3, 5\}, \{1, 4, 5\}, \{2, 4, 5\}\},$$

it follows from Theorem 2.2 that

$$C(v^G) = \text{Conv}(\{(0, 1, 0, 1, 0), (0, 1, 1, 0, 0), (1, 0, 1, 0, 0)\}). \qquad \triangle$$

3 Simple minimum coloring games

In this section we consider simple minimum coloring games. First, we characterize the type of conflict graphs inducing minimum coloring games that are simple. After that, we consider in more detail these conflict graphs and analyze the core of the induced minimum coloring games.

Recall that a game $v \in \text{TU}^N$ is called *simple* if

(i) $v(S) \in \{0, 1\}$ for all $S \subset N$,

(ii) $v(N) = 1$,

(iii) v is monotonic.

Let SI^N denote the class of all simple games with player set N. The following theorem gives a necessary and sufficient condition, in terms of the chromatic number, for a conflict graph to induce a simple minimum coloring game.

Theorem 3.1. *Let $G = (N, E)$ be a graph. Then, $v^G \in \text{SI}^N$ if and only if $\chi(G) = n - 1$.*

Proof. ("$\Rightarrow$") Let $v^G \in \text{SI}^N$. Then, $v^G(N) = 1$ and consequently $\chi(G) = n - v^G(N) = n - 1$.
("$\Leftarrow$") Let $\chi(G) = n - 1$. Then, $v^G(N) = n - \chi(G) = 1$. According to Proposition 2.1 v^G is integer valued, nonnegative and monotonic, so in particular $v^G(S) \in \{0, 1\}$ for all $S \subset N$, which implies $v^G \in \text{SI}^N$. $\qquad \square$

Proposition 3.3 provides an upper bound on the number of maximum cliques for conflict graphs inducing simple games. This proposition also states that conflict graphs inducing simple games are perfect. In the proof of this proposition we use the following lemma, which gives the clique number and the chromatic number for odd cycle graphs of length at least five and their complements. The proof of this lemma is straightforward and therefore omitted.

Lemma 3.2. *Let $k \in \mathbb{N}$ with $k \geq 2$. Then, $\omega(C_{2k+1}) = 2$, $\chi(C_{2k+1}) = 3$, $\omega(\overline{C_{2k+1}}) = k$ and $\chi(\overline{C_{2k+1}}) = k + 1$.*

Proposition 3.3. *Let $G = (N, E)$ be a graph. If $v^G \in \text{SI}^N$, then* [2]

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[2] Note that these are only sufficient conditions and not necessary conditions. Consider for example the conflict graph in Figure 4 in Section 4.1. This conflict graph is perfect and has two maximum cliques. However, this conflict graph does not induce a simple game because the value of the grand coalition in the induced minimum coloring game is 2.

(i) G *is perfect,*

(ii) $|\Omega(G)| \leq 2$.

Proof. Let $v^G \in \mathrm{SI}^N$. Then, according to Theorem 3.1, $\chi(G) = n - 1$.

Part (i): Suppose that G is not perfect. Then, according to Theorem 2.3, there exists an $S \subset N$ such that $G[S] = C_{2k+1}$ or $G[S] = \bar{C}_{2k+1}$ with $k \geq 2$. Then, using Lemma 3.2, we have

$$v^G(S) = |S| - \chi(G[S]) = 2k + 1 - 3 = 2k - 2 \geq 2 > v^G(N)$$

in case $G[S] = C_{2k+1}$, or

$$v^G(S) = |S| - \chi(G[S]) = 2k + 1 - (k+1) = k \geq 2 > v^G(N),$$

in case $G[S] = \bar{C}_{2k+1}$, which both contradict monotonicity of v^G. Hence, G is perfect.

Part (ii): Since G is perfect (see part (i)) and v^G is simple, we have $\omega(G) = \chi(G) = n-1$. Suppose $|\Omega(G)| > 2$ and let k, l and m be three distinct vertices such that $N\backslash\{k\}$, $N\backslash\{l\}$ and $N\backslash\{m\}$ are maximum cliques of G. Since $G[N\backslash\{k\}] = K_{n-1}$ and $G[N\backslash\{l\}] = K_{n-1}$, we have

$$\{\{i,j\}|i,j \in N, i \neq j\}\backslash\{k,l\} \subset E.$$

Moreover, since $G[N\backslash\{m\}] = K_{n-1}$ and $\{k,l\} \subset N\backslash\{m\}$, we have $\{k,l\} \in E$. This implies $G = K_n$ which contradicts $\chi(G) = n - 1$. Hence, $|\Omega(G)| \leq 2$. $\square$

Due to the fact that conflict graphs are assumed to be connected on N, we may conclude from Theorem 3.1 that a conflict graph inducing a simple game has at least three vertices. Moreover, from the previous proposition in combination with Theorem 3.1 we may conclude that a conflict graph $G = (N, E)$ inducing a simple game has at least one and at most two maximum cliques of size $n - 1$. So, there are two classes of conflict graphs on n vertices inducing a simple game. The first class consists of the conflict graphs with one maximum clique of size $n - 1$. Note that this class consists of $n - 3$ different conflict graphs (up to isomorphism)[3], because the vertex that is not in the maximum clique is adjacent to at least one vertex (because G is assumed to be connected on N) and at most $n - 3$ vertices (because otherwise there are two maximum cliques). For an illustration with six vertices, see Figure 2(a), (b) and (c). The second class consists of the conflict graphs with two maximum cliques of size $n - 1$. Note that this class consists of a unique conflict graph (up to isomorphism), namely the conflict graph with exactly one pair of vertices not being adjacent. For an illustration with six vertices, see Figure 2(d). Hence, for given $n \geq 3$, there are $n - 2$ different conflict graphs (up to isomorphism) on n vertices inducing a simple game.

Using Theorem 2.2 and Proposition 3.3, one derives the following description of the core for simple minimum coloring games.

Corollary 3.4. *Let $G = (N, E)$ be a graph and let $v^G \in SI^N$.*

(i) If $\Omega(G) = \{N\backslash\{i\}\}$, then

$$C(v^G) = e^{\{i\}}.$$

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[3]Two graphs $G = (N, E)$ and $G' = (N', E')$ are called isomorphic if there exists a bijection $f : N \to N'$ such that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$.
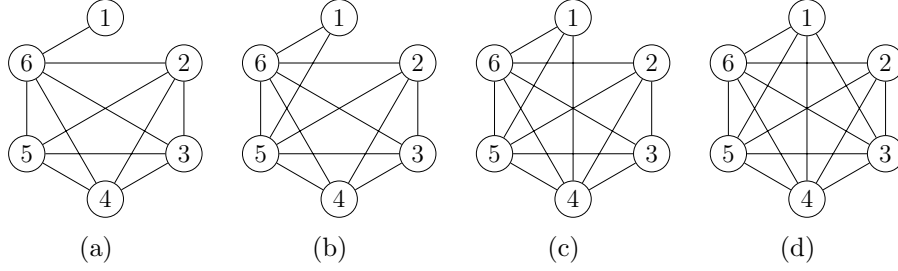
Figure 2: All conflict graphs (up to isomorphism) on six vertices inducing a simple game

(ii) If $\Omega(G) = \{N\backslash\{i\}, N\backslash\{j\}\}$ *with* $i \neq j$, *then*

$$C(v^G) = Conv\left(\{e^{\{i\}}, e^{\{j\}}\}\right).$$

Example 3.1. Consider the conflict graph $G = (N, E)$ in Figure 2(a), (b) or (c). Since $\Omega(G) = \{\{2, 3, 4, 5, 6\}\}$ and $v^G \in \mathrm{SI}^N$, it follows from Corollary 3.4 that

$$C(v^G) = \{(1, 0, 0, 0, 0, 0)\}.$$

Next, consider the conflict graph $G = (N, E)$ in Figure 2(d). Since

$$\Omega(G) = \{\{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}\},$$

and $v^G \in \mathrm{SI}^N$, it follows from Corollary 3.4 that

$$C(v^G) = \mathrm{Conv}\left(\{(1, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0)\}\right). \qquad \triangle$$

4 Three-valued simple minimum coloring games

In this section we characterize the type of conflict graphs inducing three-valued simple minimum coloring games. For this, a distinction is made between perfect and imperfect conflict graphs. After that, we characterize the core in terms of the underlying conflict graph for these games.

Musegaas et al. (2015) defined a game $v \in \mathrm{TU}^N$ to be *three-valued simple* if

(i) $v(S) \in \{0, 1, 2\}$ for all $S \subset N$,

(ii) $v(N) = 2$,

(iii) v is monotonic.

Let TSI^N denote the class of all three-valued simple games with player set N. The following theorem gives a necessary and sufficient condition, in terms of the chromatic number, for a conflict graph to induce a three-valued simple game.

Theorem 4.1. *Let* $G = (N, E)$ *be a graph. Then,* $v^G \in TSI^N$ *if and only if* $\chi(G) = n{-}2$.[4]

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[4]Note that Theorem 3.1 and Theorem 4.1 can be generalized for a more general class of integer valued, nonnegative and monotonic games.

Proof. ("⇒") Let $v^G \in \text{TSI}^N$. Then, $v^G(N) = 2$ and consequently $\chi(G) = n - v^G(N) = n - 2$.

("⇐") Let $\chi(G) = n - 2$. Then, $v^G(N) = n - \chi(G) = 2$. According to Proposition 2.1 v^G is integer valued, nonnegative and monotonic, so in particular $v^G(S) \in \{0, 1, 2\}$ for all $S \subset N$, which implies $v^G \in \text{TSI}^N$. □

From now on, we distinguish between two classes of conflict graphs inducing three-valued simple minimum coloring games: perfect conflict graphs (Section 4.1) and imperfect conflict graphs (Section 4.2). For both classes, we consider in more detail the structure of these conflict graphs and the cores of the induced minimum coloring games.

4.1 Three-valued simple minimum coloring games induced by perfect conflict graphs

In this section we consider three-valued simple minimum coloring games induced by perfect conflict graphs. We show that for these games the core equals the vital core, introduced by Musegaas et al. (2015).

We start with providing an upper bound on the number of maximum cliques for perfect conflict graphs inducing three-valued simple games.

Proposition 4.2. *Let $G = (N, E)$ be a perfect graph. If $v^G \in \text{TSI}^N$, then $|\Omega(G)| \leq 4$.*[5]

Proof. Let $v^G \in \text{TSI}^N$. Then, using Theorem 4.1, $\chi(G) = n - 2$. Hence, due to the fact that G is assumed to be connected on N, we have $n \geq 4$. Moreover, since G is perfect, we have $\omega(G) = \chi(G) = n - 2$, so at least two pairs of vertices are not adjacent in G. Without loss of generality we can assume that either $\{1, 2\} \notin E$ and $\{3, 4\} \notin E$, or $\{1, 2\} \notin E$ and $\{2, 3\} \notin E$. Therefore, we distinguish between the following two cases:

(a) Assume $\{1, 2\} \notin E$ and $\{3, 4\} \notin E$. Then the sets of vertices that can possibly form a maximum clique are

$$\{T \subset N \mid |T| = n - 2, \{1, 2\} \not\subset T, \{3, 4\} \not\subset T\}.$$

Therefore, a maximum clique is of the form

$$N \backslash \{i, j\},$$

with $i \in \{1, 2\}$ and $j \in \{3, 4\}$. Hence, there are only four sets of vertices that can possibly form a maximum clique, i.e., $|\Omega(G)| \leq 4$.

(b) Assume $\{1, 2\} \notin E$ and $\{2, 3\} \notin E$. Since $N \backslash \{2\}$ cannot form a clique, we know that there exists a pair of vertices $\{i, j\} \subset N \backslash \{2\}$ with $i \neq j$ and $\{i, j\} \notin E$. Hence, without loss of generality we can assume that $\{1, 3\} \notin E$, $\{3, 4\} \notin E$ or $\{4, 5\} \notin E$ (the last case is only possible if $n \geq 5$).
If $\{3, 4\} \notin E$, then we are back to case (a) and thus $|\Omega(G)| \leq 4$.

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[5]Note that this is only a sufficient condition and not a necessary condition. Consider for example the conflict graphs in Figure 2 in Section 3. All conflict graphs are perfect and have at most two maximum cliques. However, none of the conflict graphs induces a three-valued simple game because the value of the grand coalition in every induced minimum coloring game is 1.

On the other hand, if $\{4,5\} \notin E$, then the sets of vertices that can possibly form a maximum clique are

$$\{T \subset N \mid |T| = n - 2, \{1,2\} \not\subset T, \{2,3\} \not\subset T, \{4,5\} \not\subset T\}.$$

Therefore, a maximum clique is of the form

$$N \backslash \{2,j\},$$

with $j \in \{4,5\}$. Hence, there are only two sets of vertices that can possibly form a maximum clique, i.e., $|\Omega(G)| \leq 2$.

Finally, if $\{1,3\} \notin E$, then the sets of vertices that can possibly form a maximum clique are

$$\{T \subset N \mid |T| = n - 2, \{1,2\} \not\subset T, \{1,3\} \not\subset T, \{2,3\} \not\subset T\}.$$

Therefore, a maximum clique is of the form

$$N \backslash \{i,j\},$$

with $i \neq j$ and $\{i,j\} \subset \{1,2,3\}$. Hence, there are only three sets of vertices that can possibly form a maximum clique, i.e., $|\Omega(G)| \leq 3$.

As one can see, in both cases $|\Omega(G)| \leq 4$. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Due to the fact that conflict graphs are assumed to be connected on N, we may conclude from Theorem 4.1 that a conflict graph inducing a three-valued simple game has at least four vertices. Moreover, from the previous proposition we may also conclude that a perfect conflict graph $G = (N, E)$ inducing a three-valued simple game has at most four maximum cliques of size $n - 2$. Figure 3 depicts all perfect conflict graphs (up to isomorphism) on four vertices inducing a three-valued simple game. All conflict graphs have a clique number of two. The conflict graph in Figure 3(a) has four maximum cliques and the conflict graphs in Figure 3(b) and (c) have three maximum cliques.
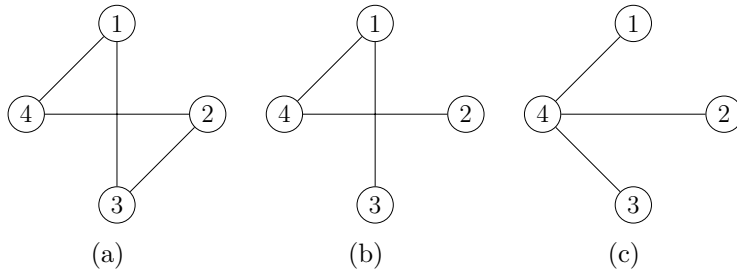

Figure 3: All perfect conflict graphs (up to isomorphism) on four vertices inducing a three-valued simple game

Musegaas et al. (2015) analyzed the core of three-valued simple games. In particular, by introducing the concepts of vital players, primary vital players and secondary vital pairs, they defined the vital core for the class of permissible three-valued simple games. The vital core was shown to be a subset of the core for every permissible three-valued simple game. We summarize the relevant notions and results from Musegaas et al. (2015),

and show that for the class of three-valued simple games induced by perfect conflict graphs, the core and the vital core coincide.

For $v \in \mathrm{TSI}^N$, the set of *vital players* is defined by

$$\mathrm{Vit}(v) = \bigcap \{S \subset N \mid v(S) = 2\}.$$

Hence, the vital players are those players who belong to every coalition with value 2. It is shown that only three-valued simple games that are permissible can have a non-empty core, where a game $v \in \mathrm{TSI}^N$ is called *permissible* if the following two conditions are satisfied:

(i) $\mathrm{Vit}(v) \neq \emptyset$,

(ii) $v(N \backslash \mathrm{Vit}(v)) = 0$.

For a permissible game $v \in \mathrm{TSI}^N$, the *reduced game* $v_r \in \mathrm{TSI}^{\mathrm{Vit}(v)}$ is defined by

$$v_r(S) = v(S \cup (N \backslash \mathrm{Vit}(v))),$$

for all $S \subset \mathrm{Vit}(v)$. Note that a reduced game is also a three-valued simple game and allows for only one coalition with value 2, namely the grand coalition $\mathrm{Vit}(v)$. Interestingly, the core of a permissible three-valued simple game equals the core of the corresponding reduced game, when extended with zeros for all players outside the set of vital players. For a permissible game $v \in \mathrm{TSI}^N$, the set of *primary vital players* of v is defined by

$$\mathrm{PVit}(v) = \bigcap \{S \subset \mathrm{Vit}(v) \mid v_r(S) \in \{1, 2\}\}$$

and the set of *secondary vital pairs* of v by

$$\mathrm{SVit}(v) = \{\{i, j\} \subset \mathrm{Vit}(v) \backslash \mathrm{PVit}(v) \mid i \neq j, \{i, j\} \cap S \neq \emptyset \text{ for all } S \text{ with } v_r(S) = 1\}.$$

Hence, the primary vital players are the vital players who belong to every coalition with value 1 or 2 in v_r. Similarly, the secondary vital pairs are the pairs of vital players, who are not primary vital players, such that for every coalition with value 1 or 2 in v_r, at least one player of such a pair belongs to the coalition. The *vital core* $VC(v)$ of a permissible game $v \in \mathrm{TSI}^N$ is defined by

$$\begin{aligned}
VC(v) = \mathrm{Conv}(&\{2e^{\{i\}} \mid i \in \mathrm{PVit}(v)\} \\
&\cup \{e^{\{i,j\}} \mid i \in \mathrm{PVit}(v), j \in \mathrm{Vit}(v) \backslash \mathrm{PVit}(v)\} \\
&\cup \{e^{\{i,j\}} \mid \{i, j\} \in \mathrm{SVit}(v)\}).
\end{aligned}$$

For a permissible game, the vital core is a subset of the core as is seen in the following theorem.

Theorem 4.3 (cf. Musegaas et al. (2015)). *Let $v \in TSI^N$ be permissible. Then,*

$$VC(v) \subset C(v).$$

The following theorem provides a characterization of the vital core for three-valued simple games induced by perfect conflict graphs.

Theorem 4.4. *Let $G = (N, E)$ be a perfect graph. If $v^G \in TSI^N$, then*

$$VC(v^G) = Conv\left(\left\{e^{N\backslash S} \mid S \in \Omega(G)\right\}\right).$$

Proof. Let $v^G \in \text{TSI}^N$. Then, using Theorem 4.1 and the fact that G is perfect, we have

$$\omega(G) = \chi(G) = n - 2.$$

In this proof we denote the intersection of all maximum cliques in G by Ω_G, i.e.,

$$\Omega_G = \bigcap\{S \mid S \in \Omega(G)\}.$$

We divide the proof into proving the following four statements:

(i) $\text{Vit}(v^G) = N\backslash\Omega_G$,

(ii) v^G is permissible,

(iii) $\text{PVit}(v^G) = \emptyset$,

(iv) $\text{SVit}(v^G) = \{N\backslash S \mid S \in \Omega(G)\}$.

Note that if the statements (i) - (iv) all hold, then it immediately follows from the definition of the vital core that

$$VC(v^G) = \text{Conv}\left(\left\{e^{N\backslash S} \mid S \in \Omega(G)\right\}\right).$$

Part (i): $[\text{Vit}(v^G) = N\backslash\Omega_G]$
("$\subset$") Let $i \in \Omega_G$, i.e., i belongs to every maximum clique of G. This implies that if vertex i is removed, then the clique number decreases with one. Therefore,

$$v^G(N\backslash\{i\}) = n - 1 - \chi(G[N\backslash\{i\}]) = n - 1 - \omega(G[N\backslash\{i\}])$$
$$= n - 1 - (\omega(G) - 1) = n - 1 - (\chi(G) - 1) = n - 1 - (n - 3) = 2,$$

where the second and the fourth equalities follow from the fact that G is perfect. Hence, there exists an $S \subset N\backslash\{i\}$ such that $v^G(S) = 2$, so $i \notin \text{Vit}(v^G)$.
("$\supset$") Let $i \in N\backslash\Omega_G$, i.e., there exists a maximum clique of G to which i does not belong. This implies that if vertex i is removed, then the clique number does not change. Therefore,

$$v^G(N\backslash\{i\}) = n - 1 - \chi(G[N\backslash\{i\}]) = n - 1 - \omega(G[N\backslash\{i\}])$$
$$= n - 1 - \omega(G) = n - 1 - \chi(G) = n - 1 - (n - 2) = 1,$$

where the second and the fourth equalities follow from the fact that G is perfect. Moreover, from monotonicity of v^G it follows that $v^G(S) \leq 1$ for all $S \subset N\backslash\{i\}$, so there does not exist an $S \subset N\backslash\{i\}$ with $v(S) = 2$. As a consequence, for all $S \subset N$ with $v(S) = 2$ we have $i \in S$ and thus $i \in \text{Vit}(v^G)$.

Part (ii): $[v^G$ is permissible$]$
Since $\omega(G) = n - 2$, we know that N cannot be a maxium clique and thus $\Omega_G \neq N$. Hence,

$$\text{Vit}(v^G) = N\backslash\Omega_G \neq \emptyset.$$

Moreover, since Ω_G is the intersection of all maximum cliques in G, we know that Ω_G forms a clique as well and thus $\chi(G[\Omega_G]) = |\Omega_G|$. As a consequence,

$$v^G(N\backslash\text{Vit}(v^G)) = v^G(\Omega_G) = 0.$$

This implies that v^G is permissible.

Part (iii): $[\text{PVit}(v^G) = \emptyset]$
Since $\text{Vit}(v^G) = N\backslash\Omega_G$, we know that for every vital player there exists a maximum clique of G to which this player does not belong. As a consequence, for $i \in \text{Vit}(v^G)$, we have

$$\chi(G[N\backslash\{i\}]) = \omega(G[N\backslash\{i\}]) = \omega(G) = n - 2,$$

where the first equality follows from the fact that G is perfect and the second equality follows from the fact that there exists a maximum clique of G to which i does not belong. Therefore,

$$\begin{aligned} v_r^G(\text{Vit}(v^G)\backslash\{i\}) &= v^G(\text{Vit}(v^G)\backslash\{i\} \cup (N\backslash\text{Vit}(v^G))) = v^G(N\backslash\{i\}) \\ &= n - 1 - \chi(G[N\backslash\{i\}]) = n - 1 - (n - 2) = 1, \end{aligned}$$

for all $i \in \text{Vit}(v^G)$. As a consequence,

$$\text{PVit}(v^G) = \bigcap \left\{ S \subset \text{Vit}(v) \mid v_r^G(S) \in \{1, 2\} \right\} \subset \bigcap \left\{ \text{Vit}(v^G)\backslash\{i\} \mid i \in \text{Vit}(v^G) \right\} = \emptyset.$$

Part (iv): $[\text{SVit}(v^G) = \{N\backslash S \mid S \in \Omega(G)\}]$
("$\supset$") Let $S \in \Omega(G)$. Since $\omega(G) = n - 2$, we can denote $N\backslash S = \{i, j\}$ with $i \neq j$. Note $\{i, j\} \cap \Omega_G = \emptyset$ and thus $\{i, j\} \subset \text{Vit}(v^G)$. Moreover, since $\text{PVit}(v^G) = \emptyset$, we have $\{i, j\} \subset \text{Vit}(v^G)\backslash\text{PVit}(v^G)$. Suppose $\{i, j\} \notin \text{SVit}(v^G)$, then it follows from the definition of secondary vital pairs that there exists a $T \subset \text{Vit}(v^G)\backslash\{i, j\}$ with $v_r^G(T) = 1$. Since $T \subset \text{Vit}(v^G)\backslash\{i, j\}$, we have

$$(T \cup \Omega_G) \subset ((\text{Vit}(v^G)\backslash\{i, j\}) \cup \Omega_G) = N\backslash\{i, j\} = S.$$

Moreover, since S forms a maximum clique in G, we have $\chi(G[T \cup \Omega_G]) = |T \cup \Omega_G|$ and thus

$$\begin{aligned} v_r^G(T) &= v^G(T \cup (N\backslash\text{Vit}(v^G))) = v^G(T \cup \Omega_G) = |T \cup \Omega_G| - \chi(G[T \cup \Omega_G]) \\ &= |T \cup \Omega_G| - |T \cup \Omega_G| = 0, \end{aligned}$$

which contradicts $v_r^G(T) = 1$. Hence, $\{i, j\} \in \text{SVit}(v^G)$.
("$\subset$") Let $\{i, j\} \in \text{SVit}(v^G)$ and suppose $N\backslash\{i, j\}$ does not form a maximum clique in G, i.e., $\omega(G[N\backslash\{i, j\}]) < n - 2$. Then,

$$\begin{aligned} v_r^G(\text{Vit}(v^G)\backslash\{i, j\}) &= v^G((\text{Vit}(v^G)\backslash\{i, j\}) \cup (N\backslash\text{Vit}(v^G))) = v^G(N\backslash\{i, j\}) \\ &= (n - 2) - \chi(G[N\backslash\{i, j\}]) = (n - 2) - \omega(G[N\backslash\{i, j\}]) \\ &> (n - 2) - (n - 2) = 0, \end{aligned}$$

where the penultimate equality again follows from the fact that G is perfect. Consequently, using the fact that a reduced game allows for only one coalition with value 2, namely the grand coalition $\text{Vit}(v^G)$, we have $v_r^G(\text{Vit}(v^G)\backslash\{i, j\}) = 1$ which contradicts the assumption that $\{i, j\} \in \text{SVit}(v^G)$. $\qquad\square$

Example 4.1. Consider the perfect conflict graph $G = (N, E)$ with $N = \{1, \ldots, 6\}$ as depicted in Figure 4. Since $\Omega(G) = \{\{1, 3, 5, 6\}, \{1, 4, 5, 6\}\}$ and $v^G \in TSI^N$, it follows from Theorem 4.4 that

$$VC(v^G) = \text{Conv}\left(\{(0, 1, 0, 1, 0, 0), (0, 1, 1, 0, 0, 0)\}\right). \qquad \triangle$$
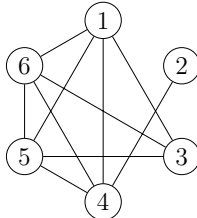

Figure 4: The conflict graph of Example 4.1

Combining Theorem 2.2 and Theorem 4.4, it immediately follows that for three-valued simple minimum coloring games induced by perfect conflict graphs the core equals the vital core.

Corollary 4.5. *Let $G = (N, E)$ be a perfect graph. If $v^G \in TSI^N$, then*

$$C(v^G) = VC(v^G).$$

4.2 Three-valued simple minimum coloring games induced by imperfect conflict graphs

In this section we consider three-valued simple minimum coloring games induced by imperfect conflict graphs. First, we characterize this type of conflict graphs, after that we show that the induced minimum coloring games always have an empty core.

Note that all imperfect graphs have at least five vertices, because it must contain an odd cycle graph of length at least five, or a complement of such graph as an induced subgraph (cf. Theorem 2.3). The following theorem provides a necessary and sufficient condition for an imperfect conflict graph to induce a three-valued simple game. For this theorem, we use the notion of a dominating vertex. A vertex $i \in N$ is called *dominating* in G if $\{j \mid j \in N, \{i, j\} \in E\} = N \setminus \{i\}$, i.e., if i is adjacent to every other vertex.

Theorem 4.6. *Let $G = (N, E)$ be an imperfect graph. Then, $v^G \in TSI^N$ if and only if there exists an $S \subset N$ such that $G[S] = C_5$ and all vertices outside S are dominating.*

Proof. ("$\Leftarrow$") Let $S \subset N$ be such that $G[S] = C_5$ and let all vertices outside S be dominating. Since all vertices in $N \setminus S$ are dominating, we have $G[N \setminus S] = K_{|N \setminus S|}$ and thus $\chi(G[N \setminus S]) = n - 5$. Moreover, since each vertex in S is adjacent to each vertex in $N \setminus S$, we have

$$\chi(G) = \chi(G[N \setminus S]) + \chi(G[S]) = n - 5 + 3 = n - 2,$$

where the second equality follows from Lemma 3.2. Consequently, using Theorem 4.1, we have $v^G \in TSI^N$.

("$\Rightarrow$") Let $v^G \in TSI^N$. Since G is not perfect, we know from Theorem 2.3 that there exists an $S \subset N$ such that $G[S] = C_{2k+1}$ or $G[S] = \bar{C}_{2k+1}$ with $k \geq 2$. Suppose $k > 2$. Then, using Lemma 3.2, we have

$$v^G(S) = |S| - \chi(G[S]) = 2k + 1 - 3 = 2k - 2 > 2,$$

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in case $G[S] = C_{2k+1}$, or

$$v^G(S) = |S| - \chi(G[S]) = 2k + 1 - (k + 1) = k > 2,$$

in case $G[S] = \bar{C}_{2k+1}$, which both contradict v^G being a three-valued simple game. Hence, $k = 2$ and thus $G[S] = C_5$ or $G[S] = \bar{C}_5$. Since C_5 and $\bar{C}_5$ are isomorphic to each other[6], and thus both graphs have the same clique and chromatic number, we can conclude that $G[S] = C_5$.

Now, suppose $\chi(G[N\backslash S]) < |N\backslash S| = n - 5$. Then

$$v^G(N) = n - \chi(G) \geq n - (\chi(G[S]) + \chi(G[N\backslash S])) > n - (3 + n - 5) = 2,$$

which contradicts v^G being a three-valued simple game. Hence, we may assume $\chi(G[N\backslash S]) = n - 5$ and thus $G[N\backslash S] = K_{|N\backslash S|}$. As a consequence, all players in $N\backslash S$ are mutually adjacent, i.e.,

$$\{j \mid j \in N\backslash S, \{i, j\} \in E\} = (N\backslash S)\backslash\{i\},$$

for all $i \in N\backslash S$.

Next, suppose $\{i, j\} \notin E$ for some $i \in S$ and $j \in N\backslash S$. Then, since there exists a vertex in S that is not adjacent to a vertex in $N\backslash S$, we have

$$v^G(N) = n - \chi(G) > n - (\chi(G[S]) + \chi(G[N\backslash S])) = n - (3 + n - 5) = 2,$$

which again contradicts v^G being a three-valued simple game. Hence, we may assume that every player in $N\backslash S$ is adjacent to all players in S, i.e.,

$$\{j \mid j \in S, \{i, j\} \in E\} = S,$$

for all $i \in N\backslash S$. Consequently,

$$\{j \mid j \in N, \{i, j\} \in E\} = N\backslash\{i\},$$

for all $i \in N\backslash S$, i.e., all vertices outside S are dominating. $\square$

The previous theorem implies that, for given $n \geq 5$, the class of imperfect conflict graphs on n vertices inducing a three-valued simple game consists of a unique conflict graph (up to isomorphism), namely the conflict graph that contains C_5 as an induced subgraph and all other vertices being dominated. Figure 5 depicts all imperfect conflict graphs (up to isomorphism) on five, six and seven vertices inducing a three-valued simple game. For all these conflict graphs, the subgraph induced by $\{1, 2, \ldots, 5\}$ is the cycle graph C_5 and all other vertices are dominating.

Using Theorem 4.6, the clique number and the number of maxiumum cliques for imperfect conflict graphs inducing three-valued simple games immediately follows, as is stated in the following corollary.

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[6]To see that C_5 and $\bar{C}_5$ are isomorphic to each other, consider the graphs $G = (N, E)$ and $G' = (N', E')$ with $N = N' = \{1, \ldots, 5\}$,

$$E = \{\{1, 2\}, \{1, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\},$$

and

$$E' = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 5\}.$$

If one takes the bijection $f : N \to N'$ with $f(1) = 1$, $f(2) = 3$, $f(3) = 5$, $f(4) = 2$ and $f(5) = 4$, then $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$.
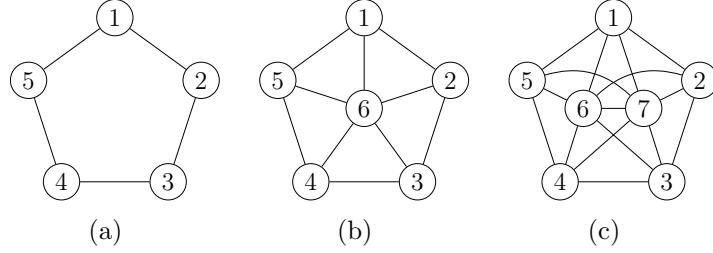
Figure 5: All imperfect conflict graphs (up to isomorphism) on five, six or seven vertices inducing a three-valued simple game

Corollary 4.7. *Let $G = (N, E)$ be an imperfect graph. If $v^G \in TSI^N$, then $\omega(G) = n - 3$ and $|\Omega(G)| = 5$.*

For three-valued simple minimum coloring games induced by imperfect conflict graphs, the core is empty as is seen in the following theorem.

Theorem 4.8. *Let $G = (N, E)$ be an imperfect graph. If $v^G \in TSI^N$, then v^G is not permissible and thus $C(v^G) = \emptyset$.*

Proof. Let $v^G \in \mathrm{TSI}^N$. Then, Theorem 4.6 implies that there exists an $S \subset N$ such that $G[S] = C_5$. For $i \in S$, we have

$$v^G(S \backslash \{i\}) = 4 - \chi(G[S \backslash \{i\}]) = 4 - 2 = 2.$$

As a consequence,

$$\mathrm{Vit}(v^G) = \bigcap \{S \subset N \mid v^G(S) = 2\} \subset \bigcap \{S \backslash \{i\} \mid i \in S\} = \emptyset,$$

so v^G is not permissible. Since only three-valued simple games that are permissible can have a non-empty core, we have $C(v^G) = \emptyset$. $\qquad\qquad\square$

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