Identification of linear stochastic models with covariance restrictions

Bekker, P.A.; Pollock, D.S.G.

Publication date:
1984

Document Version
Publisher's PDF, also known as Version of record

Link to publication in Tilburg University Research Portal

Citation for published version (APA):
Bekker, P. A., & Pollock, D. S. G. (1984). *Identification of linear stochastic models with covariance restrictions*. (Research Memorandum FEW). Faculteit der Economische Wetenschappen.

FEW

subfaculteit der econometrie

RESEARCH MEMORANDUM

[redacted]

TILBURG UNIVERSITY
DEPARTMENT OF ECONOMICS
Postbus 90153 - 5000 LE Tilburg
Netherlands

[redacted]

[redacted]

FEW
144

IDENTIFICATION OF LINEAR STOCHASTIC MODELS WITH COVARIANCE RESTRICTIONS

by

Paul A. Bekker
D.S.G. Pollock

IDENTIFICATION OF LINEAR STOCHASTIC MODELS WITH COVARIANCE RESTRICTIONS

BY PAUL A. BEKKER[1] AND D.S.G. POLLOCK[2]

The purpose of this paper is to provide a systematic treatment of the problem of identification in systems of linear structural equations where some of the disturbances are uncorrelated.

1. INTRODUCTION

Many of the aspects of the classical linear simultaneous-equations model of econometrics have been researched in great depth, yet the problem of using restrictions on the covariances of the structural disturbances to assist in identifying the structural parameters appears to have received relatively little attention.

In his seminal book on the identification problem in econometrics, F.M. Fisher [4] did go some of the way towards presenting an overall account of the problem; but most of his results have practical applications only in the rather specialized case of block-recursive systems. It should also be mentioned that the covariance problem can be accomodated within the framework for analysing problems of identification that Wegge [12] has provided. Other authors, including Rothenberg [8] have added to the results, and more recently, the problem has been considered by Hausman and Taylor [5] in connection with limited-information estimation by instrumental variables. The latter have shown that exogeneity relationships induced by covariance restrictions may find expression in a class of models that is wider than that

---

1. Paul Bekker wishes to acknowledge the financial support of the Netherlands Organization for the Advancement of Pure Research (ZWO).

2. Stephen Pollock wishes to express his gratitude to his hosts at the University of Tilburg where he was resident from February 1983 to July 1983.

of the block-recursive models which, in its turn, is a generalization of the class of recursive models analysed by Wold [13]

In this paper, we attempt to analyse, in a systematic manner, a wide variety of relationships that may be induced by covariance restrictions. We begin our treatment of particular cases by defining the class of decomposable covariance restrictions. These are the restrictions that give rise to relationships of exogeneity; and, therefore, at this stage, we are covering much the same ground as Hausman and Taylor. However, our treatment of the problem is quite different from theirs.

By generalizing our definition of decomposability, we then proceed to introduce the wider class of recursively decomposable restrictions. An important finding is that, if all the covariance restrictions are recursively decomposable, then any set of structural parameters that are identifiable are also globally or uniquely identifiable.

In our final section, we consider covariance restrictions that are indecomposable. Such restrictions no longer afford an assurance of global identification. Nevertheless, in the case of one model which we analyse in detail, we are able to adduce a simple criterion for discriminating amongst the isolated solutions of the identifying equations.

Our attempt at providing a unified treatment of our topic rests on an analysis of the structure of the Jacobian matrix associated with the identifying equations. However, we find that, in many practical cases, our assesment of whether or not a structural equation is identified can be based on relatively simple criteria that do not require us to take account of the Jacobian matrix in its entirety. Nevertheless, there are cases where we do have to resort to a full system-wide analysis; and we shall describe the methods of such an analysis in the following section.

2. A FORMAL ANALYSIS OF THE IDENTIFICATION PROBLEM

2.1 The Model

We shall conduct our analysis in terms of a model comprising m stochastic equations in m observable variables and m unobservable disturbances. We can represent the model by writing

$$(2.1) \qquad z'\Delta = \upsilon'$$

where $z' = [z_1,\ldots,z_m]$ is an observable row vector, $\upsilon' = [\upsilon_1,\ldots,\upsilon_m]$ is an unobservable disturbance vector with an expected value of $E(\upsilon) = 0$ and $\Delta = [\delta_1,\ldots,\delta_m]$ is a nonsingular m x m matrix whose ith column contains the coefficients of the ith structural equation.

The dispersion matrices of the vectors z and υ are given by

$$(2.2) \qquad D(z) = \Sigma \quad , \quad D(\upsilon) = \Phi = [\phi_1,\ldots,\phi_m]$$

where Φ is assumed to be positive definite. It follows from (2.1) that

$$(2.3) \qquad \Delta'\Sigma\Delta = \Phi \quad ,$$

whence we see that

$$(2.4) \qquad \Sigma = \Delta'^{-1}\Phi\Delta^{-1}$$

is also positive definite.

If it is assumed that z is normally distributed, then all the information that is available from the observations is contained in Σ which is globally identified.

2.2 Restrictions on Δ and Φ

Given that a value may be attributed to Σ, we seek to identify the elements of Δ and Φ with the help of prior information represented by linear restrictions on these matrices.

We shall assume that, apart from the normalization rules which set $\delta_{ii} = 1$ for all i, Δ is subject only to exclusion restrictions of the form $\delta_{ij} = 0$. We shall also assume that Φ is subject to covariance restrictions of the form $\phi_{ij} = 0$ which are always accompanied by corresponding restrictions of the form $\phi_{ji} = 0$.

The restrictions affecting the jth equation may be written as

$$(2.5) \qquad R'_{\Delta j}\delta_j = r_j \qquad , \qquad H'_{\Phi j}\phi_j = 0$$

where $R'_{\Delta j}$ and $H'_{\Phi j}$ consist of selections of the rows of the identity matrix of order m x m. In order to separate the normalization rule from the homogeneous exclusion restrictions, we may write the restrictions on δ_j as

$$(2.6) \qquad \begin{bmatrix} e'_j \\ H'_{\Delta j} \end{bmatrix} \delta_j = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where e'_j is the jth row of the m x m identity matrix.

Taking all the equations together, we have the restrictions

$$(2.7) \qquad R'_{\Delta}\Delta^c = r$$

and

(2.8) $\quad H_{\Phi}'\Phi^{c} = 0$

where Δ^{c} and Φ^{c} are long vectors formed by a vertical arrangement of the columns of Δ and Φ respectively.

In addition to the restrictions in (2.8), we must take account of the symmetry of Φ. Let us therefore consider the operator $\textcircled{T}$, called the tensor commutator, which has the effect that $\textcircled{T} A^{c} = A'^{c}$ when A is any m x m matrix. This operator, which plays a fundamental role in the theory of matrix differential calculus, has been defined by numerous authors including Balestra [1], Magnus and Neudecker [6] and Pollock [7]. In the present context, $\textcircled{T} = \sum_{ij}(e_j e_i' \otimes e_i e_j')$ is a partitioned matrix of order $m^2 \times m^2$ whose jith block is the matrix $e_i e_j'$ of order m x m which has a unit in the ijth position and zeros elsewhere. Using the commutator, we can express the symmetry of Φ by writing the equations

(2.9) $\quad (I - \textcircled{T})\Phi^{c} = 0 \quad .$

The set of all matrices $[\Delta, \Phi]$ that obey the restrictions under (2.7), (2.8) and (2.9), in addition to the restrictions that Δ is nonsingular and that Φ is positive definite, will be called the restricted parameter set.

The symmetry of Φ can also be expressed by writing the equation $\Phi^{c} = 1/2(I + \textcircled{T})\Phi^{c}$. On substituting this into the equation under (2.8), we obtain the expression

(2.10) $\quad 1/2H_{\Phi}'(I + \textcircled{T})\Phi^{c} = 0 \quad .$

These equations are symmetric in the arguments ϕ_{ij} and ϕ_{ji}. It follows that the restriction setting ϕ_{ij} to zero is now identical to the restriction setting ϕ_{ji} to zero; and we are free to eliminate one of these.

Given that $\Phi^c = (\Delta'\Sigma\Delta)^c = (I \otimes \Delta'\Sigma)\Delta^c$, and that $\Delta'\Sigma = \Phi\Delta^{-1}$, it follows that we can rewrite the equations in (2.10) in the form

$$(2.11) \quad 1/2H'_{\Phi}(I + \textcircled{T})(I \otimes \Delta'\Sigma)\Delta^c$$

$$= 1/2H'_{\Phi}(I + \textcircled{T})(I \otimes \Phi\Delta^{-1})\Delta^c \quad .$$

Thus we see that the linear restrictions on Φ^c give rise to a set of bilinear ristrictions on Δ^c .

On combining the equations from (2.7) and (2.11), we obtain the system

$$(2.12) \quad \begin{bmatrix} R'_{\Delta} \\ \\ 1/2H'_{\Phi}(I + \textcircled{T})(I \otimes \Delta'\Sigma) \end{bmatrix} \Delta^c = \begin{bmatrix} r \\ \\ 0 \end{bmatrix} \quad .$$

2.3 General Conditions for Identifiability

Equation (2.3) shows that, for a given value of Σ, the value of Φ is uniquely determined by that of Δ. Therefore the problem of identification rests with Δ alone.

The equations (2.12) contain all the information relating to Δ, and we shall describe them as the identifying equations. Any value of Δ^c which satisfies these equations may be termed an admissible value. The true parameter value Δ_0 is clearly an admissible value, and our object is to establish conditions under which it represents a locally isolated solution of the identifying equations such that, within the set of values of Δ obeying the restrictions in (2.7), there exists an open neighbourhood of Δ_0 containing no other admissible value.

It is well known that a sufficient condition for Δ_0 to be locally isolated is that the Jacobian matrix of the transformation in (2.12) evaluated at Δ_0 has full column rank. By differentiating the function with respect to Δ, we find that the Jacobian is

(2.13)
$$J_\Sigma(\Delta;\ \Sigma) = \begin{bmatrix} R'_\Delta \\ \\ H'_\Phi(I + \circledT)(I \otimes \Delta'\Sigma) \end{bmatrix} .$$

Let Φ_0 be the true value of Φ. Then, since $\Delta'\Sigma = \Phi\Delta^{-1}$, it follows that the matrix function

(2.14)
$$J(\Delta,\ \Phi) = \begin{bmatrix} R'_\Delta \\ \\ H'_\Phi(I + \circledT)(I \otimes \Phi\Delta^{-1}) \end{bmatrix}$$

has the same value at the point $[\Delta_0,\ \Phi_0]$ as the function $J_\Sigma(\Delta;\ \Sigma)$ has at the point Δ_0 . If it can be established that $J(\Delta,\ \Phi)$ has full rank for almost every point in the restricted parameter set, then the fulfillment of the condition on the rank of $J_\Sigma(\Delta;\ \Sigma)$ at Δ_0 is virtually assured.

The condition that $J(\Delta_0,\ \Phi_0)$ has full rank, which is sufficient for the local isolation or identification of Δ_0 , becomes a necessary condition as well if it is assumed that $[\Delta_0,\ \Phi_0]$ is a regular point of $J(\Delta,\ \Phi)$ such that there exists an open neighbourhood of $[\Delta_0,\ \Phi_0]$ in the restricted parameter set for which $J(\Delta,\ \Phi)$ has constant rank. This is an acceptable assumption since the set of irregular points is of measure zero; which can be demonstrated by using a result of Fisher [3, Th. 5.A.2] concerning the roots of analytic functions. We state the following:

ASSUMPTION 1: The true parameter point $[\Delta_0, \Phi_0]$ is a regular point of J.

In considering the identification of the parameters of a single structural equation, we will make a further assumption:

ASSUMPTION 2: $[\Delta_0, \Phi_0]$ is a regular point of J_ℓ ; $\ell = 1,\ldots,m$ where $J_\ell = J(e_\ell \otimes I)$ is the submatrix of $J = [J_1,\ldots,J_m]$ corresponding to the derivative taken with respect to the parameters of the ℓth equation.

Rothenberg [8] has used an analogous assumption in his Theorem 8 which recapitulates on a theorem by Wald [11] which is also proved by Fisher [3]. We shall restate the theorem in the form which best suits our own purposes:

PROPOSITION 1: A necessary and sufficient condition for the parameters of the first equation to be locally isolated is that $\text{Rank}(J) = \text{Rank}(J_1) + \text{Rank}(J_2,\ldots,J_m)$ and $\text{Rank}(J_1) = m$.

We can state equivalent conditions for identification in terms of any matrix that can be derived by postmultiplying J by a nonsingular matrix of order $m^2 \times m^2$. On postmultiplying J by $I \otimes \Delta$, we obtain

$$(2.15) \qquad F = \begin{bmatrix} F_0 \\ \\ F_\Phi \end{bmatrix} = \begin{bmatrix} R_\Delta'(I \otimes \Delta) \\ \\ H_\Phi'(I + \textcircled{T})(I \otimes \Phi) \end{bmatrix} = [F_1, \ldots, F_m]$$

where $F_\ell = F(e_\ell \otimes I)$. Clearly, we have $\text{Rank}(F) = \text{Rank}(J)$ and $\text{Rank}(F_\ell) = \text{Rank}(J_\ell)$. Moreover, F and F_ℓ have the same regular points as J and J_ℓ respectively.

We may note that $\text{Rank}(F_1) = m$ is a nesessary condition for the first equation to be identified; and this corresponds to Fisher's Generalized Rank Condition [3, Th 4.6.2].

The advantage of using F comes from the fact that it separates the matrices Φ and Δ, which facilitates the assesment of its rank.

2.4 The structure of the Matrix F

We shall now look more closely at the structure of the matrix F.

The submatrix F_0 corresponding to the restrictions on Δ has a relatively simple structure:

$$(2.16) \qquad F_0 = R_\Delta'(I \otimes \Delta)$$

$$= \begin{bmatrix} R_{\Delta 1}'\Delta , & 0 & , \ldots , & 0 \\ 0 & , R_{\Delta 2}'\Delta & , \ldots , & 0 \\ \vdots & \vdots & & \vdots \\ 0 & , 0 & , \ldots , & R_{\Delta m}'\Delta \end{bmatrix}$$

$$= [F_{01} , \ldots , F_{0m}] \quad .$$

If there were no covariance restrictions, then the equation $\text{Rank}(F) = \sum \text{Rank}(F_\ell)$ would always hold. However, the covariance restrictions tie together the sets of identifying equations, and this is reflected in the structure of the submatrix F_Φ .

To illustrate the structure of F_Φ, imagine that its rth row f_r' corresponds to the restriction $\phi_{ij} = (e_j' \otimes e_i')\Phi^c = 0$. Then

$$(2.17) \qquad f_r' = (e_j' \otimes e_i')(I + \bigoplus)(I \otimes \Phi)$$

$$= (e_j' \otimes e_i'\Phi) + (e_i' \otimes e_j'\Phi)$$

$$= [f_{r1}', \ldots , f_{rm}'] \quad .$$

Here we have a $1 \times m^2$ row vector consisting of subvectors $f'_{r\ell}$; $\ell = 1,\ldots,m$ of order $1 \times m$. These are zeros apart from the ith subvector $f'_{ri} = \phi_j$ and the jth subvector $f'_{rj} = \phi'_i$.

For a complete example, let us consider the case of the model specified by the matrices:

$$(2.18) \qquad \Delta = \begin{bmatrix} 1 & \delta_{12} & 0 \\ 0 & 1 & \delta_{23} \\ \delta_{31} & 0 & 1 \end{bmatrix} , \qquad \Phi = \begin{bmatrix} \phi_{11} & 0 & 0 \\ 0 & \phi_{22} & 0 \\ 0 & 0 & \phi_{33} \end{bmatrix} .$$

The matrix F is then given by

$$(2.19) \qquad [F_1, F_2, F_3] = \left[\begin{array}{ccc:ccc:ccc} 1 & \delta_{12} & 0 & & & & & & \\ 0 & 1 & \delta_{23} & & & & & & \\ \hdashline & & & 0 & 1 & \delta_{23} & & & \\ & & & \delta_{31} & 0 & 1 & & & \\ \hdashline & & & & & & 1 & \delta_{12} & 0 \\ & & & & & & \delta_{31} & 0 & 1 \\ \hdashline 0 & \phi_{22} & 0 & \phi_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi_{33} & 0 & 0 & 0 & \phi_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{33} & 0 & \phi_{22} & 0 \end{array}\right] \begin{array}{l} \delta_{11} = 1 \\ \delta_{21} = 0 \\ \delta_{22} = 1 \\ \delta_{32} = 0 \\ \delta_{13} = 0 \\ \delta_{33} = 1 \\ \phi_{12} = 0 \\ \phi_{13} = 0 \\ \phi_{23} = 0 \end{array} .$$

Here the empty blocks signify submatrices containing only zeros. The rows of F correspond to the restrictions written in the margin.

The rows of F corresponding to the normalization rules $\delta_{11}, \delta_{22}, \delta_{33} = 1$ are linearly independent of all other rows; for each contains a unit which falls in a column where all the other elements are zeros. By deleting these rows and the corresponding columns, we obtain a submatrix which has full column rank if and only if F has full column rank. By permutating the rows of the submatrix in question, we obtain the matrix F^* which appears in the following equation (2.20):

(2.20)
$$F^*q = \begin{bmatrix} F_1^* \\ f_2^* \end{bmatrix} q$$

$$= \begin{bmatrix} \phi_{22}, & 0, & \phi_{11}, & 0, & 0, & 0 \\ 0, & \phi_{33}, & 0, & 0, & \delta_{11}, & 0 \\ 0, & 0, & \delta_{31}, & 1, & 0, & 0 \\ 0, & 0, & 0, & \phi_{33}, & 0, & \phi_{22} \\ 0, & 0, & 0, & 0, & 1, & \delta_{12} \\ \hdashline 1, & \delta_{23}, & 0, & 0, & 0, & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ -\phi_{11}\delta_{31}\delta_{12} \\ -\phi_{22} \\ \phi_{22}\delta_{31} \\ \phi_{33}\delta_{31}\delta_{12} \\ -\phi_{33}\delta_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \phi_{11}(1-\delta_{23}\delta_{31}\delta_{12}) \end{bmatrix} .$$

It is evident that the rows of the matrix F^*_1, which is in echelon form, are linearly independent. If $\delta_{23}\delta_{31}\delta_{12} \neq 1$, then the vector q, which is orthogonal to the rows of F_1^*, is not orthogonal to f_2^*. Therefore, f_2^* cannot be a linear combination of the rows of F_1^* and, consequently, F^* and F are of full rank. It follows that the parameters are locally identified. If $\delta_{23}\delta_{31}\delta_{12} = 1$, then the parameter point is irregular and is therefore excluded from our analysis. However, the irregular points constitute a set of measure zero.

3. DECOMPOSABILITY

A covariance term within the dispersion matrix $\Phi = \Delta'\Sigma\Delta$ is a bilinear function of the parameter vectors of two structural equations. Therefore one might expect a covariance restriction to have the effect, always, of tying together the identification problems of two equations. However, it often happens, as a result of a particular conjunction of the restrictions on Δ and Φ, that a set of covariance restrictions becomes a set of linear restrictions on the parameters of the jth equation which makes no reference to the values of other structural parameters. In such cases, we shall say that we have a set of decomposable restrictions.

When all the available covariance restrictions are decomposable, the problems of identifying individual equations are separable, and, if the parameters of an equation are identifiable, then they are uniquely or globally identifiable.

In our characterization of decomposable covariance restrictions, we shall make use of the following lemma in which the notation makes allusions to section 2.4.

LEMMA 1: Consider the matrix

$$(3.1) \qquad F = \begin{bmatrix} F_0 \\ f_1' \end{bmatrix} = \begin{bmatrix} F_{0i} , F_{0j} \\ f_{1i}' , f_{1j}' \end{bmatrix} = \begin{bmatrix} F_i , F_j \end{bmatrix}$$

wherein f_{1i}', f_{1j}' are row vectors and $\mathrm{Rank}(F_0) = \mathrm{Rank}(F_{0i}) + \mathrm{Rank}(F_{0j})$. Then $\mathrm{Rank}(F) = \mathrm{Rank}(F_i) + \mathrm{Rank}(F_j)$ if and only if f_{1i}' is linearly dependent on the rows of F_{0i} or f_{1j}' is linearly dependent on the rows of F_{0j}

The proof of this appears in the appendix.

3.1 Decomposable Covariance Restrictions

We may begin our account with the definition of a decomposable covariance restriction.

DEFINITION 1: We say that the restriction $\phi_{ij} = 0$ is decomposable if, for all points in the restricted parameter set, we have

$$\text{(3.2)} \qquad \text{Rank}\begin{bmatrix} F_{0i} , F_{0j} \\ \phi_j' , \phi_i' \end{bmatrix} = \text{Rank}\begin{bmatrix} F_{0i} \\ \phi_j' \end{bmatrix} + \text{Rank}\begin{bmatrix} F_{0j} \\ \phi_i' \end{bmatrix}$$

where F_{0i} and F_{0j} are defined in (2.16).

LEMMA 2: The restriction $\phi_{ij} = 0$ is decomposable if and only if (a) for all points in the restricted parameter set, there exists a vector λ_i such that $\phi_j = \Delta' R_{\Delta i}\lambda_i$ or (b) for all points in the restricted parameter set, there exists a vector λ_j such that $\phi_i = \Delta' R_{\Delta j}\lambda_j$.

The proof follows immediately from Lemma 1. If conditions (a) and (b) are fulfilled at the the same time, then the restriction $\phi_{ij} = 0$ is of no assistance in identifying the equations, and we say that it is redundant. If condition (a) holds together with the condition that $\text{Rank}(\Delta' R_{\Delta j}, \phi_i) = \text{Rank}(\Delta' R_{\Delta j}) + 1$ for all regular points of $[\Delta' R_{\Delta j}, \phi_i]$, then the restriction is said to be assignable to the jth equation, and we call the ith equation the instrumental equation.

To illustrate the definition and the lemma, we may consider the model specified by the matrices

$$\text{(3.3)} \qquad \Delta = \begin{bmatrix} 1 , 0 , \delta_{13}, 0 \\ 0 , 1 , 0 , \delta_{24} \\ \delta_{31}, 0 , 1 , \delta_{34} \\ 0 , \delta_{42}, 0 , 1 \end{bmatrix} , \qquad \Phi = \begin{bmatrix} \phi_{11}, \phi_{12}, \phi_{13}, 0 \\ \phi_{21}, \phi_{22}, \phi_{23}, \phi_{24} \\ \phi_{31}, \phi_{32}, \phi_{33}, 0 \\ 0 , \phi_{42}, 0 , \phi_{44} \end{bmatrix} .$$

The restriction $\phi_{14} = 0$ is decomposable. This can be seen by considering the matrix

$$(3.4)\qquad \begin{bmatrix} R'_{\Delta 1}\Delta \\ \phi'_4 \end{bmatrix} = \begin{bmatrix} e'_1\Delta \\ H'_{\Delta 1}\Delta \\ \phi'_4 \end{bmatrix} = \begin{bmatrix} 1, & 0, & \delta_{13}, & 0 \\ \hline 0, & 1, & 0, & \delta_{24} \\ 0, & \delta_{42}, & 0, & 1 \\ \hline 0, & \phi_{42}, & 0, & \phi_{44} \end{bmatrix} .$$

The vector ϕ_4 is linearly dependent on the second and third rows of $R'_{\Delta 1}\Delta$ so that $\phi_4 = \Delta' R_{\Delta 1}\lambda_1$ for some vector λ_1. On the other hand, consideration of the matrix

$$(3.5)\qquad \begin{bmatrix} R'_{\Delta 4}\Delta \\ \phi'_1 \end{bmatrix} = \begin{bmatrix} 1, & 0, & \delta_{13}, & 0 \\ 0, & \delta_{42}, & 0, & 1 \\ \hline \phi_{11}, & \phi_{12}, & \phi_{13}, & 0 \end{bmatrix}$$

shows that $\text{Rank}(\Delta' R_{\Delta 4}, \phi_1) = \text{Rank}(\Delta' R_{\Delta 4}) + 1$ for almost every point in the restricted parameter set.

As our example suggests, we may replace the matrices $R_{\Delta i} = [e_i, H_{\Delta i}]$ and $R_{\Delta j} = [e_j, H_{\Delta j}]$, wherever they occur in Definition 2 and Lemma 2, by their submatrices $H_{\Delta i}$ and $H_{\Delta j}$ respectively. To confirm this, we may refer to (2.6) which indicates that the jth column of $R'_{\Delta j}\Delta$ consists of zeros except for the unit corresponding to the normalization $e'_j\delta_j = 1$. Since ϕ_i contains a zero in the jth position corresponding to the restriction $\phi_{ij} = 0$, it is clearly independent of the row $e'_j\Delta$ in which the unit occurs. Therefore

$$(3.6)\qquad \phi_i = \Delta' R_{\Delta j}\lambda_j \qquad \text{implies} \qquad \phi_i = \Delta' H_{\Delta j}\kappa_j \quad .$$

To reveal some further consequences of the condition $\phi_j = \Delta' H_{\Delta j}\kappa_i$, let us rewrite it as $\Delta'^{-1}\phi_j = H_{\Delta i}\kappa_i$, and let $S_{\Delta i}$ consist of the rows of the identity matrix that are not included in $H_{\Delta i}$. Then

$$(3.7)\qquad \begin{bmatrix} H'_{\Delta i} \\ S'_{\Delta i} \end{bmatrix} \Delta'^{-1}\phi_j = \begin{bmatrix} H'_{\Delta i}H_{\Delta i} \\ S'_{\Delta i}H_{\Delta i} \end{bmatrix} \kappa_i = \begin{bmatrix} \kappa_i \\ 0 \end{bmatrix} \quad ;$$

and we can see that $\phi_j = \Delta' H_{\Delta i}\kappa_i$ if and only if $S_{\Delta i}\Delta'^{-1}\phi_j = 0$. At the true parameter point, or at any other admissible point where $\Delta'^{-1}\Phi = \Sigma\Delta$, the latter becomes $S'_{\Delta i}\Sigma\delta_j = 0$ which is a set of linear restrictions on δ_j. This equation can also be written as

$$(3.8)\qquad S'_{\Delta i}D(z)\delta_j = C(S'_{\Delta i}z,\ \delta'_j z)$$
$$= C(S'_{\Delta i}z,\ \upsilon_j) = 0$$

which indicates that the disturbance term υ_j is uncorrelated with all the variables entering the ith equation. We may describe this situation by saying that the variables entering the ith equation are exogenous relative to the jth equation; and these variables may be used as instruments for the identification and estimation of the jth equation.

The next proposition indicates a way of determining whether or not a particular covariance restriction is decomposable.

PROPOSITION 2: The decomposability condition $\phi_j = \Delta' H_{\Delta i}\kappa_i$, relating to the covariance restriction $\phi_{ij} = 0$, holds for every point in the restricted parameter set if and only if there exist selection matrices N_q and N_{m-q}, selecting q and m-q different columns respectively, such that

$$(3.9)\qquad N'_{m-q}H'_{\Delta i}\Delta H_{\Phi j}N_q = 0 \quad .$$

The proof appears in the appendix.

Equation (3.9) shows that the restriction $N'_{m-q}H'_{\Delta i}\delta_i = 0$ holds not just for a single equation but for q equations, indexed by $i = i_1,\ldots,i_q$, whose parameter vectors are selected from the matrix Δ by the matrix $H_{\Phi j}N_q$. Thus the equation is common to a set of q decomposability conditions $\phi_j = \Delta' H_{\Delta i}\kappa_i$; $i = 1_1,\ldots,i_q$ relating to a set of q covariance restrictions $\phi_{ij} = 0$; $i = i_1,\ldots,i_q$.

We should note that q is also the number of distinct variables entering the equations indexed by $i = i_1,\ldots,i_q$. The q variables are not necessarily present in each of these equations, and so the condition (3.8) may represent less than the full set of exogeneity relationships affecting the jth equation.

To illustrate the condition (3.9), let us consider again the model specified under (3.3). Corresponding to the restriction $\phi_{14} = 0$ which satisfies the decomposability condition $\phi_4 = \Delta' R_{\Delta 1}\lambda_1$, we have the equation

(3.10)
$$H'_{\Delta 1}\Delta H_{\Phi 4} = \begin{bmatrix} 0 , 1 , 0 , 0 \\ 0 , 0 , 0 , 1 \end{bmatrix} \begin{bmatrix} 1 , 0 , \delta_{13}, 0 \\ 0 , 1 , 0 , \delta_{24} \\ \delta_{31}, 0 , 1 , \delta_{34} \\ 0 , \delta_{42}, 0 , 1 \end{bmatrix} \begin{bmatrix} 1 , 0 \\ 0 , 0 \\ 0 , 1 \\ 0 , 0 \end{bmatrix} = \begin{bmatrix} 0 , 0 \\ 0 , 0 \end{bmatrix} .$$

The equation $H'_{\Delta 3}\Delta H_{\Phi 4} = 0$ which corresponds to the restriction $\phi_{34} = 0$ is identical to the above. Thus the two variables z_1 and z_3 entering the equations 1 and 3 are exogenous relative to equation 4; and both can be used as instruments.

In order to deal with cases where there are a number of decomposable restrictions relating to different equations and, also, to provide a basis for dealing, in the next section, with recursively decomposable restrictions, it is helpful to generalize Lemma 1 to obtain the following:

LEMMA 3: Consider the matrix

(3.11)
$$F_r = \begin{bmatrix} F_{r-1} \\ f_r' \end{bmatrix} = \begin{bmatrix} F_{r-1,1}, & \cdots & ,F_{r-1,m} \\ f_{r1}', & \cdots, & f_{rm}' \end{bmatrix}$$

wherein $\text{Rank}(F_{r-1}) = \sum_\ell \text{Rank}(F_{r-1,\ell})$, and assume that $f_{r\ell}' = 0$ if $\ell \neq i$ and $\ell \neq j$. Then $\text{Rank}(F_r) = \sum_\ell \text{Rank}(F_{r,\ell})$ if and only if f_{ri}' is dependent on the rows of $F_{r-1,i}$ or f_{rj}' is dependent on the rows of $F_{r-1,j}$.

We may apply this lemma repeatedly to show how the row vectors corresponding to n decomposable covariance restrictions may be added in any order to the matrix $F_0 = R_\Delta'(I \otimes \Delta)$ to create a matrix F_n with the property that $\text{Rank}(F_n) = \sum_\ell \text{Rank}(F_{n\ell})$. Thus

LEMMA 4: Let N_n' be the matrix which selects from H_Φ' the rows corresponding to a set of n decomposable restrictions $\phi_{ij} = 0$; $(i,j) = (i,j)_1,\ldots,(i,j)_n$. Then

(3.12)
$$\text{Rank}\begin{bmatrix} F_0 \\ N_n'F_\Phi \end{bmatrix} = \sum_\ell \text{Rank}\begin{bmatrix} F_{0\ell} \\ N_n'F_{\Phi\ell} \end{bmatrix} .$$

We now assert that, if a sufficient number of decomposable covariance restrictions are assignable to the jth equation, then this equation is globally identifiable:

PROPOSITION 3: Let N_n' be the matrix which selects from H_Φ' the rows corresponding to a set of decomposable restrictions $\phi_{ij} = 0$; $i = i_1,\ldots,i_n$ relating to the jth equation. Then a sufficient condition for the jth equation to be globally identified is that

(3.13)
$$\text{Rank}\begin{bmatrix} F_{0j} \\ N_n' F_{\Phi j} \end{bmatrix} = m \quad .$$

If every covariance restriction which references the jth equation belongs to this set of n decomposable restrictions, then this condition is necessary as well.

Here local identification follows from Proposition 1 and Lemma 4. Global identification follows from the fact that decomposable covariance restrictions give rise to linear restrictions on Δ.

We can also represent the condition (3.13) by writing

(3.14)
$$\text{Rank}\begin{bmatrix} R_{\Delta j}' \Delta \\ \bar{N}_n' H_{\Phi j}' \Phi \end{bmatrix} = m \quad .$$

where $\bar{N}_n'$ is the appropriate selection matrix operating on $H_{\Phi j}'$.

To illustrate this condition, we return again to our example under (3.3). We see that a necessary and sufficient condition for the fourth equation to be identified is that the matrix

(3.15)
$$\begin{bmatrix} R_{\Delta 4}' \Delta \\ H_{\Phi 4}' \Phi \end{bmatrix} = \begin{bmatrix} 1 , & 0 , & \delta_{13} , & 0 \\ 0 , & \delta_{42} , & 0 , & 1 \\ \hdashline \phi_{11} , & \phi_{12} , & \phi_{13} , & 0 \\ \phi_{31} , & \phi_{32} , & \phi_{33} , & 0 \end{bmatrix}$$

is nonsingular. The condition will be satisfied by every regular point in the restricted parameter set; and, according to Assumption 2, these are the only points we need consider.

4. RECURSIVE DECOMPOSABILITY

We begin with a general definition of recursively decomposable restrictions

DEFINITION 2: Let $\phi_{ij} = 0$; $(i,j) = (i,j)_1,\ldots,(i,j)_n$ be a set of covariance restrictions corresponding to the rows $f_1',\ldots,f_n'$ of the matrix F_Φ , and let us define, for $r = 1,\ldots,n$, the matrices

$$(4.1) \qquad F_r = \begin{bmatrix} F_{r-1} \\ f_r' \end{bmatrix} = \begin{bmatrix} F_{r-1,1}, & \ldots & ,F_{r-1,m} \\ f_{r1}', & \ldots & , f_{rm}' \end{bmatrix}$$

with $F_0 = R_\Delta'(I \otimes \Delta)$. Then, if, at every regular point of F_{r-1} within the restricted parameter set, we have

$$(4.2) \qquad \text{Rank}(F_r) = \textstyle\sum_\ell \text{Rank}(F_{r,\ell})$$

for all r, we say that the covariance restrictions are recursively decomposable.

It is clear that the first in a sequence of recursively decomposable restrictions must be a decomposable restriction according to Definition 2. Moreover all decomposable restrictions are also recursively decomposable.

LEMMA 6: If the first $r - 1$ restrictions are recursively decomposable, then the rth restriction $\phi_{ij} = 0$ is recursively decomposable if and only if (a) for all regular points of F_{r-1}, there exists a vector λ_i such that $\phi_j = F_{r-1,i}'\lambda_i$ or (b) for all regular points of F_{r-1}, there exists a vector λ_j such that $\phi_i = F_{r-1,j}'\lambda_j$.

The proof of this follows directly from Lemma 3. If the condition (a) holds together with the condition that $\text{Rank}(F_{r,j}) = \text{Rank}(F_{r-1,j}) + 1$ for all regular points of $F_{r,j}$, then the restriction is said to be assignable to the jth equation, and the ith equation is said to be the instrumental equation.

For an example, let us consider the model specified by the matrices

$$\text{(4.3)} \qquad \Delta = \begin{bmatrix} 1, & 0, & \delta_{13}, & 0 \\ 0, & 1, & 0, & \delta_{24} \\ \delta_{31}, & 0, & 1, & \delta_{34} \\ 0, & \delta_{42}, & 0, & 1 \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \phi_{11}, & 0, & \phi_{13}, & 0 \\ 0, & \phi_{22}, & \phi_{23}, & 0 \\ \phi_{31}, & \phi_{32}, & \phi_{33}, & 0 \\ 0, & 0, & 0, & \phi_{44} \end{bmatrix}.$$

which may be obtained from the model specified in (3.3) by setting $\phi_{42}, \phi_{21} = 0$. The corresponding matrix F is given by

$$\text{(4.4)} \qquad \begin{bmatrix} F_0 \\ F_\Phi \end{bmatrix} = \left[\begin{array}{cccc} R'_{\Delta 1}\Delta, & 0, & 0, & 0 \\ 0, & R'_{\Delta 2}\Delta, & 0, & 0 \\ 0, & 0, & R'_{\Delta 3}\Delta, & 0 \\ 0, & 0, & 0, & R'_{\Delta 4}\Delta \\ \hdashline \phi'_4, & 0, & 0, & \phi'_1 \\ 0, & 0, & \phi'_4, & \phi'_3 \\ 0, & \phi'_4, & 0, & \phi'_2 \\ \phi'_2, & \phi'_1, & 0, & 0 \end{array}\right] \quad \begin{array}{l} \\ \\ \\ \\ \\ f'_1 : \phi_{14} = 0 \\ f'_2 : \phi_{34} = 0 \\ f'_3 : \phi_{42} = 0 \\ f'_4 : \phi_{21} = 0 \end{array}.$$

We already know, from the analysis of section 3, that the restrictions $\phi_{14}, \phi_{34} = 0$ are decomposable; and in the present example, they represent the first two restrictions in a recursively decomposable sequence. The restriction $\phi_{42} = 0$ is the third in the sequence; for consideration of the matrix

$$(4.5)\qquad \begin{bmatrix} R'_{\Delta 4}\Delta \\ \phi'_1 \\ \phi'_3 \\ \phi'_2 \end{bmatrix} = \begin{bmatrix} 1, & 0, & \delta_{13}, & 0 \\ 0, & \delta_{42}, & 0, & 1 \\ \phi_{11}, & 0, & \phi_{13}, & 0 \\ \phi_{31}, & \phi_{32}, & \phi_{33}, & 0 \\ 0, & \phi_{22}, & \phi_{23}, & 0 \end{bmatrix}$$

shows that $\phi_2 = F'_{2,4}\lambda_4$ for all regular points of $F_{2,4}$ and of F_2.

Finally, the restriction $\phi_{21} = 0$ is also recursively decomposable; for consideration of the matrix

$$(4.6)\qquad \begin{bmatrix} R'_{\Delta 2}\Delta \\ \phi'_4 \\ \phi'_1 \end{bmatrix} = \begin{bmatrix} 1, & 0, & \delta_{13}, & 0 \\ 0, & 1, & 0, & \delta_{24} \\ \delta_{31}, & 0, & 1, & \delta_{34} \\ 0, & 0, & 0, & \phi_{44} \\ \phi_{11}, & 0, & \phi_{13}, & 0 \end{bmatrix}$$

shows that $\phi_1 = F'_{3,2}\lambda_2$ for all regular points of $F_{3,2}$ and F_3.

It is easy to see that, as was the case with decomposable restrictions, we can rewrite Definition 3 and Lemma 6 in terms of the matrix $F^*_0 = H'_\Delta(I \otimes \Delta)$ which omits the rows of F_0 corresponding to the normalization rules. In place of the matrices $F_{r-1,i}$ and $F_{r-1,j}$ of Lemma 6, we then have $F^*_{r-1,i}$ and $F^*_{r-1,j}$. Let us also note that, for the true parameter point, or for any other admissible point, we can write $F^*_{r-1,\ell} = J^*_{r-1,\ell}\Delta$ where $J^*_{r-1,\ell}$ is a submatrix of $J_\Sigma(\Delta, \Phi)$ from which the row corresponding to the normalization rule has been deleted. At such points, the decomposability condition (a) in Lemma 6 can be written in the form

$$(4.7)\qquad \Sigma\delta'_j = J^{*\prime}_{r-1,i}\kappa_i \quad .$$

In section 3.1, we showed that each decomposable covariance restriction corresponds to a set of linear restrictions on structural parameters. We shall now show, more generally, that each recursively decomposable restriction corresponds to a set of linear restrictions.

Let us therefore consider a full system of restrictions where the first p covariance restrictions form a recursively decomposable sequence. Then we have the following conditions:

$$\text{(4.8)}\qquad H'_{\Delta\ell}\delta_\ell = 0 \quad ; \ell = 1,\dots,m \quad ,$$

$$\delta'_{i_r}\Sigma\delta_{j_r} = 0 \quad ; r = 1,\dots,n \quad ,$$

$$\Sigma\delta_{j_r} = J^{*\prime}_{r-1,i_r}\kappa_{i_r} \quad ; r = 1,\dots,p \quad .$$

We can demonstrate that these are equivalent to the conditions

$$\text{(4.9)}\qquad H'_{\Delta\ell}\delta_\ell = 0 \quad ; \ell = 1,\dots,m \quad ,$$

$$B'_{r,j_r}\Sigma\delta_{j_r} = 0 \quad ; r = 1,\dots,p \quad ,$$

$$\delta'_{i_r}\Sigma\delta_{j_r} = 0 \quad ; r = p+1,\dots,n \quad ,$$

wherein the matrices B_{r,j_r} depend only on Σ.

The matrices B_{r,j_r} may be defined recursively. When r = 1, B_{1,j_1} is a matrix of linearly independent columns that are orthogonal and complementary to those of $H_{\Delta i_1}$ such that $B'_{1,j_1}H_{\Delta i_1} = 0$ whilst $[H_{\Delta i_1}, B_{1,j_1}]$ is nonsingular. In addition, we define a set of matrices $B_{1,\ell}$; $\ell = 1,\dots,m$ of the same order as B_{1,j_1} but consisting of zeros when $\ell \neq j$. For other values of r not exceeding p, we define B_{r,j_r} to be a matrix whose columns are orthogonal and complementary to those of the matrix

$$[H_{\Delta i_r}, \Sigma(B_{1,i_r}, B_{2,i_r},\dots,B_{r-1,i_r})];$$

and we also define the matrices $B_{r,\ell} = 0$ for $\ell \neq j_r$ which are of the same order.

We use the principle of induction to demonstrate the equivalence of (4.8) and (4.9). Let r = 1. Then the restriction $\delta_{i_1}' \Sigma\delta_{j_1} = 0$ is decomposable such that $\Sigma\delta_{j_1} = H_{\Delta i_1}\kappa_{i_1}$ and, consequently, $B_{1,j_1}' \Sigma\delta_{j_1} = 0$. Conversely, as δ_{i_1} is a vector in the null space of $H_{\Delta i_1}'$, and as B_{1,j_1} forms a basis of that space, it follows that $\delta_{i_1} = B_{1,j_1}\mu_1$ for some μ_1; and, therefore, the condition $B_{1,j_1}' \Sigma\delta_{j_1} = 0$ implies the condition $\delta_{i_1}' \Sigma\delta_{j_1} = 0$. Since $H_{\Delta i_1}$ forms a basis of the null space of B_{1,j_1}', the condition $B_{1,j_1}' \Sigma\delta_{j_1} = 0$ also implies that $\Sigma\delta_{j_1} = H_{\Delta i_1}\kappa_{i_1}$ for some κ_{i_1}, which is the decomposability condition. Thus the first covariance restriction and its associated decomposability condition are equivalent to the linear restrictions $B_{1,j_1}' \Sigma\delta_{j_1} = 0$ and may be replaced by the latter.

Now assume that the replacement is valid for the first r - 1 recursively decomposable restrictions. Then, for the rth restriction, the decomposability condition $\Sigma\delta_{j_r} = J_{r-1,i_r}^{*\prime}\kappa_{i_r}$ may be written as

$$\Sigma\delta_{j_r} = [H_{\Delta i_r}, \Sigma(B_{1,i_r}\mu_1, \ldots, B_{r-1,i_r}\mu_{r-1})]\kappa_{i_r}$$

so that $B_{r,j_r}' \Sigma\delta_{j_r} = 0$. Conversely, since $\delta_{i_r} = B_{r,j_r}\mu_r$ for some μ_r, it follows that the linear restrictions $B_{r,j_r} \Sigma\delta_{j_r} = 0$ imply $\delta_{i_r}' \Sigma\delta_{j_r} = 0$ together with its associated decomposability condition. Thus the rth covariance restriction may be replaced by a set of linear restrictions.

Our argument has served to demonstrate the following:

PROPOSITION 4: If the first p covariance restrictions are recursively decomposable, then a sufficient condition for the jth equation to be globally identifiable is that

(4.9) $\quad \text{Rank}(F_{p,j}) = m \quad .$

If all the covariance restrictions which reference the jth equation are in this set of p restrictions, then the condition is necessary as well.

5. THE CLASSICAL MODEL

The classical simultaneous-equations system of econometrics is a special case of our model which can be written in the form of (2.1) as a block-recursive system:

$$(5.1)\qquad [y', x']\begin{bmatrix}\Gamma , & 0\\ B , & A\end{bmatrix} = [\varepsilon', \xi'] \quad .$$

The vector x contains K variables that are exogenous relative to the G jointly dependent variables in y . The dispersion matrices for $z' = [y', x']$ and $\upsilon' = [\varepsilon', \xi']$ take the forms of

$$(5.2)\qquad \Sigma = \begin{bmatrix}\Sigma_{yy}, & \Sigma_{yx}\\ \Sigma_{xy}, & \Sigma_{xx}\end{bmatrix} \quad , \quad \Phi = \begin{bmatrix}\Psi , & 0\\ 0 , & \Theta\end{bmatrix} \quad .$$

The nondiagonal blocks of Φ are set to zero by the covariance restrictions $\phi_{ij}, \phi_{ji} = 0$ where $j = 1,\ldots,G$ and $i = G+1,\ldots,G+K$. These restrictions are all decomposable as can be seen by considering

$$(5.3)\qquad \begin{bmatrix}H'_{\Delta i}\Delta\\ \phi'_j\end{bmatrix} = \begin{bmatrix}\Gamma , & 0\\ \psi'_j, & 0\end{bmatrix}$$

where $H'_{\Delta i}$ is the matrix associated with the restrictions $\delta_{1i},\ldots,\delta_{Gi} = 0$ relating to any column of the zero matrix in (5.1). Since Γ is nonsingular, (5.3) shows that $\phi'_j = [\psi'_j, 0]$ is dependent on the rows of $H'_{\Delta i}\Delta = [\Gamma, 0]$. In fact, the zero block of $[\Gamma, 0]$ corresponds to the matrix selected in (3.9).

In treating the identification problem of the classical system, we shall confine our attention to cases where the restrictions on Ψ are recursively decomposable. The identification of the jth equation, where $j \leq G$, may then be assessed by considering the matrix consisting of the nonzero row vectors of F_j:

(5.4)
$$\begin{bmatrix} R'_{\Delta j}\Delta \\ H'_{\Phi j}\Phi \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} R'_{\Gamma j}, & 0 \\ 0, & H'_{Bj} \end{bmatrix} \begin{bmatrix} \Gamma, & 0 \\ B, & A \end{bmatrix} \\ \cdots\cdots\cdots\cdots \\ \begin{bmatrix} H'_{\Psi j}, & 0 \\ 0, & I \end{bmatrix} \begin{bmatrix} \Psi, & 0 \\ 0, & \Theta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} R'_{\Gamma j}\Gamma, & 0 \\ H'_{Bj}B, & H'_{Bj}A \\ \cdots\cdots & \cdots \\ H'_{\Psi j}\Psi, & 0 \\ 0, & \Theta \end{bmatrix} .$$

A necessary and sufficient conditon for the identifiability of the jth equation is that this matrix has a rank of $m = G + K$. However, since the $K \times K$ matrix $\Theta = A'\Sigma_{xx}A$ is nonsingular, this is equivalent to the condition that

(5.5)
$$\text{Rank}\begin{bmatrix} R'_{\Gamma j}\Gamma \\ H'_{Bj}B \\ H'_{\Psi j}\Psi \end{bmatrix} = G \qquad .$$

In the absence of restrictions on Ψ , the term $H'_{\psi j}\Psi$ in (5.5) is suppressed, and we obtain the conventional rank condition which is stated by Schmidt [9, p.134] amongst others.

As a corollary to Lemma 6, we have the following statement which is analogous to Lemma 2:

COROLLARY 1: In the classical simultaneous-equations system, the restrictions on Ψ are recursively decomposable only if there exists a covariance restriction $\psi_{ij} = 0$ such that (a) at every regular point of $[\Gamma' R_{\Gamma i}, B' H_{Bi}]$, we have $\psi_j = [\Gamma' R_{\Gamma i}, B' H_{Bi}]\lambda_i$ for some vector λ_i or (b) at every regular point of $[\Gamma' R_{\Gamma j}, B' H_{Bj}]$, we have $\psi_i = [\Gamma' R_{\Gamma j}, B' H_{Bj}]\lambda_j$ for some λ_j.

Of course, we can replace the matrices $R_{\Gamma i} = [e_i, H_{\Gamma i}]$ and $R_{\Gamma j} = [e_j, H_{\Gamma j}]$ wherever they occur in this statement by their submatrices $H_{\Gamma i}$ and $H_{\Gamma j}$ respectively.

We should note that, if ψ_i' or ψ_j' are linear combinations of the rows of Γ alone, then the covariance restriction $\psi_{ij} = 0$ is, in fact, decomposable; and we are back in the world of section 3. This corresponds to the context in which Hausman and Taylor [5] have derived their propositions; for they have confined their attention to cases where the covariance restrictions on Ψ are conjoined only with restrictions on Γ (which they denote by B).

The following is analogous to Proposition 2:

PROPOSITION 6. The condition $\psi_j = [\Gamma' H_{\Gamma i}, B' H_{Bi}]\kappa_i$ relating to the recursively decomposable restriction $\psi_{ij} = 0$ holds for all regular points of $[\Gamma' H_{\Gamma i}, B' H_{Bi}]$ in the restricted parameter set if and only if there exist selection matrices N_q and N_{G-q} such that

$$(5.6) \qquad N_{G-q}' \begin{bmatrix} H_{\Gamma i}'\Gamma \\ H_{Bi}'B \end{bmatrix} H_{\Psi j} N_q = 0 \qquad \text{and} \qquad \text{Rank}\left\{ N_{G-q}' \begin{bmatrix} H_{\Gamma i}'\Gamma \\ H_{Bi}'B \end{bmatrix} \right\} = G - q \quad .$$

The proof, which is analogous to that of Proposition 2, is given in the appendix

For an example, let us consider the model specified by the matrices

(5.7)
$$\begin{bmatrix} \Gamma \\ B \end{bmatrix} = \begin{bmatrix} 1 & \gamma_{12} & 0 & 0 \\ 0 & 1 & \gamma_{23} & 0 \\ 0 & \gamma_{32} & 1 & \gamma_{34} \\ 0 & 0 & \gamma_{43} & 1 \\ \hdashline \beta_{11} & \beta_{12} & 0 & 0 \end{bmatrix} , \qquad \Psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_{22} & 0 & 0 \\ \psi_{31} & 0 & \psi_{33} & \psi_{34} \\ \psi_{41} & 0 & \psi_{43} & \psi_{44} \end{bmatrix} .$$

The conventional rank condition shows that the first and fourth equations are identified in the absence of any restrictions on Ψ.

Given that the fourth equation is identified, it follows that the covariance restriction $\psi_{42} = 0$ is recursively decomposable. Although the third equation is not identified, the restriction $\psi_{32} = 0$ is also recursively decomposable; for the matrix

(5.8)
$$\begin{bmatrix} H'_{\Gamma 3}\Gamma \\ H'_{B3}B \end{bmatrix} H'_{\Psi 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

fulfills the condition in (5.6).

Given the recursively decomposable nature of the restrictions ψ_{32}, $\psi_{42} = 0$, it follows that a necessary and sufficient condition for the global identifiability of the second equation is that the matrix

(5.9)
$$\begin{bmatrix} R'_{\Gamma 2} \\ H'_{B2} \\ \psi'_3 \\ \psi'_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \gamma_{23} & 0 \\ 0 & 0 & \gamma_{43} & 1 \\ \psi_{31} & 0 & \psi_{33} & \psi_{34} \\ \psi_{41} & 0 & \psi_{43} & \psi_{44} \end{bmatrix}$$

is nonsingular. The condition will be satisfied at every regular point in the restricted parameter set.

What is notable about this example is that the identification of the second equation is achieved with the assistance of two covariance restrictions, neither of which is decomposable and one of which, namely $\psi_{32} = 0$, relates to an instrumental equation which is unidentified.

6. INDECOMPOSABLE RESTRICTIONS

We define an indecomposable covariance restriction to be any covariance restriction that cannot be subsumed under Definition 2 of recursively decomposable restrictions.

The effectiveness of an indecomposable restriction in assisting the identification of a particular equation depends crucially on the way in which other indecomposable restrictions are distributed throughout the system. In the appendix, we prove

PROPOSITION 7: An indecomposable covariance restriction can be of assistance in identifying equations which it references only if it belongs to a set of s indecomposable restrictions which reference no more than s equations.($\phi_{ij} = 0$ is said to reference the ith and jth equations).

As a corollary to the proposition we have

COROLLARY 2: In a classical system with exogenous variables where all the restrictions on Ψ reference the jth equation - that is, where the restrictions are confined to the jth row and column of Ψ - the restriction $\psi_{ij} = 0$ can be useful for identification if and only if it is recursively decomposable.

6.1 An Indecomposable System

The simplest case of an indecomposable system which is locally identified is provided by the model in section 2.4 which is specified by the matrices

(6.1)
$$\Delta = \begin{bmatrix} 1, & \delta_{12}, & 0 \\ 0, & 1, & \delta_{23} \\ \delta_{31}, & 0, & 1 \end{bmatrix} \quad , \quad \Phi = \begin{bmatrix} \phi_{11}, & 0, & 0 \\ 0, & \phi_{22}, & 0 \\ 0, & 0, & \phi_{33} \end{bmatrix} .$$

We may recall that $\Phi = \Delta'\Sigma\Delta$, whilst $\Sigma = [\sigma_{ij}]$ has a fixed value. By analysing the associated F matrix, we have established that the identifying equations yield isolated solutions; but this does not exclude the possibility of multiple solutions.

Consider, for example, the matrix

(6.2)
$$\Sigma = \begin{bmatrix} 171, & -11, & 16 \\ -11, & 1, & -1 \\ 16, & -1, & 2 \end{bmatrix} .$$

Two isolated solutions for Δ and Φ are given by

(6.3)
$$\Delta^{(1)} = \begin{bmatrix} 1, & \frac{2}{27}, & 0 \\ 0, & 1, & -1 \\ -9, & 0, & 1 \end{bmatrix} , \quad \Phi^{(1)} = \begin{bmatrix} 45, & 0, & 0 \\ 0, & \frac{25}{81}, & 0 \\ 0, & 0, & 5 \end{bmatrix} ,$$

(6.4)
$$\Delta^{(2)} = \begin{bmatrix} 1, & \frac{3}{38}, & 0 \\ 0, & 1, & -2 \\ -\frac{19}{2}, & 0, & 1 \end{bmatrix} , \quad \Phi^{(2)} = \begin{bmatrix} \frac{95}{2}, & 0, & 0 \\ 0, & \frac{25}{76}, & 0 \\ 0, & 0, & 10 \end{bmatrix} .$$

In order to analyse the model further, we may consider the equations

(6.5)
$$\Sigma^{-1} = \Delta\Phi^{-1}\Delta' .$$

From these, we may derive the expression

$$(6.6) \qquad (\Sigma^{-1})_{31} = \delta_{31}\phi_{11}^{-1} \quad .$$

For any two solutions, we must have

$$(6.7) \qquad (\delta_{31}\phi_{11}^{-1})^{(1)} = (\delta_{31}\phi_{11}^{-1})^{(2)} \quad .$$

We also have $\delta_1'\Sigma\delta_1 = \phi_{11}$, or

$$(6.8) \qquad \sigma_{11} + 2\sigma_{31}\delta_{31} + \sigma_{33}\delta_{31}^2 = \phi_{11} \quad .$$

Together, (6.7) and (6.8) yield

$$(6.9) \qquad \sigma_{11}(\delta_{31}^{(2)} - \delta_{31}^{(1)}) = \delta_{31}^{(1)}\delta_{31}^{(2)}\sigma_{33}(\delta_{31}^{(2)} - \delta_{31}^{(1)}) \quad .$$

Therefore, if $\delta_{31}^{(1)} \neq \delta_{31}^{(2)}$, we have

$$(6.10) \qquad \delta_{31}^{(1)}\delta_{31}^{(2)} = \frac{\sigma_{11}}{\sigma_{33}} \; ;$$

from which it follows that there can be no more than two solutions. Analogous expressions hold for the other parameters; and thus we find that

$$(6.11) \qquad (\delta_{31}\delta_{12}\delta_{23})^{(1)} = \frac{1}{(\delta_{31}\delta_{12}\delta_{23})^{(2)}} \quad .$$

In interpreting this result, we may recall the opinion that has been stated by F.M. Fisher [4] and, more recently, by Bentler and Freeman [2] that simultaneous-equations models should be regarded as limiting approximations to dynamic non-simultaneous models in which certain time lags approach zero. The requirement that the non-simultaneous model should be dynamically stable leads to restrictions on the matrix of parameters associated with the endogenous variables. In particular, it is required of the classical model in (5.1) that the matrix $(I-\Gamma)$ should be convergent in the sense that

(6.12) $(I-\Gamma)^n \to 0$ as $n \to \infty$.

A necessary and sufficient condition for (6.12) to hold is that the absolute value of the largest latent root of $(I-\Gamma)$ is less than one. In the present case, we have

(6.13)
$$(I-\Delta)^3 = -\begin{bmatrix} \delta_{31}\delta_{12}\delta_{23}, & 0, & 0 \\ 0, & \delta_{31}\delta_{12}\delta_{23}, & 0 \\ 0, & 0, & \delta_{31}\delta_{12}\delta_{23} \end{bmatrix}$$

and thus, for the system in (6.1) to be stable, we require that

(6.14) $|\delta_{31}\delta_{12}\delta_{23}| < 1$.

This result is readily intelligible since it concerns the product of the coefficients that describe a circular path linking the variables of our model.

According to (6.11), the condition (6.14) can only hold for one of the two solutions of the identifying equations; and, in this sense, the model is globally identified. In our numerical example, it is the solution under (6.3) that satisfies the criterion of stability.

It is an attractive speculation to suppose that similar criteria of stability may be available for discriminating amongst the solutions of the identifying equations of other more complicated models that give rise to indecomposable problems of identification.

APPENDIX

LEMMA 1: Consider the matrix

(i)
$$F = \begin{bmatrix} F_0 \\ f'_1 \end{bmatrix} = \begin{bmatrix} F_{0i} , F_{0j} \\ f'_{1i} , f'_{1j} \end{bmatrix} = \begin{bmatrix} F_i , F_j \end{bmatrix}$$

wherein f'_{1i}, f'_{1j} are row vectors and $\text{Rank}(F_0) = \text{Rank}(F_{0i}) + \text{Rank}(F_{0j})$. Then we have

(ii) $\text{Rank}(F) = \text{Rank}(F_i) + \text{Rank}(F_j)$

if and only if f'_{1i} is linearly dependent on the rows of F_{0i} or f'_{ij} is linearly dependent on the rows of F_{0j} .

PROOF: (Necessity). Imagine that, contrary to the condition, f'_{1i} and f'_{1j} are linearly independent of the rows of F_{0i} and F_{0j} respectively. Then we have $\text{Rank}(F) = \text{Rank}(F_0) + 1$, $\text{Rank}(F_i) = \text{Rank}(F_{0i}) + 1$ and $\text{Rank}(F_j) = \text{Rank}(F_{0j}) + 1$; and so the equality under (ii) cannot hold.

(Sufficiency). Now imagine that f'_{1i} is linearly dependent on the rows of of F_{0i} such that $f'_{1i} = p'F_{0i}$ for some vector p. For there to be linear dependence between the columns of F_i and the columns of F_j , there must exist vectors b, c such that

$$\begin{bmatrix} F_{0i}b \\ f'_{1i}b \end{bmatrix} = \begin{bmatrix} F_{0j}c \\ f'_{1j}c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

But, by assumption, we must have $F_{0i}b = F_{0j}c = 0$ and, therefore, we must also have $p'F_{0i}b = f_{1i}b = f_{1j}c = 0$. Therefore there exist no vectors b, c satisfying the requirements; and so the columns of F_i and F_j are mutually independent. Hence the equality under (ii) must hold. We may repeat this analysis after interchanging F_{0j} and f'_{1j} with F_{0i} and f'_{1i} respectively.

PROPOSITION 2: The condition $\phi_j = \Delta' H_{\Delta i}\kappa_i$ holds for every point in the restricted parameter set if and only if there exist selection matrices N_q and N_{m-q} such that $N'_{m-q}H'_{\Delta i}\Delta H_{\phi j}N_q = 0$.

PROOF: (Sufficiency). The m x (m-q) matrix $A = \Delta' H_{\Delta i}N_{m-q}$ and the m x q matrix $B = H_{\phi j}N_q$ both have full column rank. Imagine that they obey the condition $A'B = N'_{m-q}H'_{\Delta i}\Delta H_{\phi j}N_q = 0$. Then, since by assumption, ϕ_j obeys the condition $\phi'_j B = \phi'_j H_{\phi j}N_q = 0$, it follows that we must have $\phi_j = A\mu_i = \Delta' H_{\Delta i}N_{m-q}\mu_i$ for some vector μ_i. That is to say, we must have $\phi_j = \Delta' H_{\Delta i}\kappa_i$ where $\kappa_i = N_{m-q}\mu_i$.

(Necessity). For the converse, let (Δ_0, Φ_0) be any point in the restricted parameter set, and let $A = \Delta' H_{\Delta i}N_{m-q}$ consist of the fewest columns of Δ for which $\phi_j = A\mu_i$ for all points in an open neighbourhood of (Δ_0, Φ_0). Since the columns of $A_0 = \Delta_0 H_{\Delta i}N_{m-q}$ are linearly independent, there exists a selection matrix P'_{m-q}, comprising m-q rows of the identity matrix, such that $P'_{m-q}A_0$ is nonsingular. Consequently (cf. Shapiro [10]), $P'_{m-q}A$ is nonsingular in an open neighbourhood O_0 of (Δ_0, Φ_0); and, for each point in O_0, the value of μ_i is completely determined by the equation

(i) $$P'_{m-q}\phi_j = P'_{m-q}A\mu_i \quad .$$

Let P'_q comprise the rows of the identity matrix not included in P'_{m-q}. Then, given that the equation (i) determines μ_i, the condition $\phi_j = A\mu_i$ can hold for every point in O_0 only if

(ii) $$P'_q\phi_j = P'_qA\mu_i = 0$$

(For, otherwise, we could find another point in O_0, differing only with respect to the elements in $P'_q\phi_j$, for which there exists no μ_i such that $\phi_j = A\mu_i$). By the same token, the condition $P'_qA\mu_i = 0$ can hold for every point only if $P_q'A = 0$ identically (For, otherwise, we could always find a point in O_0 generating a nonzero column corresponding to a nonzero element of μ_i).

Finally, since $H_{\phi j}'\phi_j = 0$ comprises all the zero restrictions on ϕ_j, we must have $P_q' = N_q'H_{\phi j}'$; and so the condition $P_q'A = 0$ is the condition that $N_q'H_{\phi j}'\Delta'H_{\Delta i}M_{m-q} = 0$.

PROPOSITION 6: The condition $\psi_j = [\Gamma'H_{\Gamma i}, B'H_{Bi}]\kappa_i$ relating to the recursively decomposable restriction $\psi_{ij} = 0$ holds for all regular points of $[\Gamma'H_{\Gamma i}, B'H_{Bi}]$ in the restricted parameter set if and only if there exist selection matrices N_q and N_{G-q} such that

$$N_{G-q}'\begin{bmatrix} H_{\Gamma i}'\Gamma \\ H_{Bi}'B \end{bmatrix} H_{\Psi j}N_q = 0 \quad \text{and} \quad \text{Rank}\left\{N_{G-q}'\begin{bmatrix} H_{\Gamma i}'\Gamma \\ H_{Bi}'B \end{bmatrix}\right\} = G - q$$

PROOF: (Sufficiency). The proof is analogous to that of Proposition 2.

(Necessity). Let (Δ_0, Φ_0) be any regular point of $[\Gamma'H_{\Gamma i}, B'H_{Bi}]$ within the restricted parameter set, and let $A = [\Gamma'H_{\Gamma i}, B'H_{Bi}]N_{G-q}$ consist of the fewest columns for which $\psi_j = A\mu_i$ holds in an open neighbourhood of (Δ_0, Φ_0) in which A has constant rank. Then, at the point (Δ_0, Φ_0), the matrix $A = A_0$ must have full column rank. For let us write $A = [a_1, A_2]$ and let us assume, to the contrary, that a_1 is dependent on A_2. Then, in each neighbourhood of (Δ_0, Φ_0), we could find a point (Δ_*, Φ_*) for which a_1 is independent of A_2 — for otherwise the matrix A would not consist of the fewest columns. But this would imply that

$$\text{Rank}(A_2)_0 = \text{Rank}(A)_0 = \text{Rank}(A)_* = \text{Rank}(A_2)_* + 1$$

or simply $\text{Rank}(A_2)_0 > \text{Rank}(A_2)_*$, which cannot be true since, by virtue of the semicontinuity of rank (cf. Shapiro [10]), a matrix sufficiently close to $(A_2)_0$ must have $\text{Rank}(A_2)_* \geq \text{Rank}(A_2)_0$. Thus it is established that A has full column rank at (Δ_0, Φ_0); and the proof may now proceed along the lines of the proof of Proposition 2.

PROPOSITION 7: An indecomposable covariance restriction can be of assistance in identifying equations only if it belongs to a set of s indecomposable restrictions which reference no more that s equations. ($\phi_{ij} = 0$ is said to reference the ith and jth equations).

PROOF: Imagine that, for every subset of the indecomposable restrictions, the number t of equations that are referenced is greater than the number s of restrictions. Select any restriction $\phi_{pq} = 0$, and let the pth and qth equations be the first and second in a renumbered sequence of equations indexed by $j = 1,\dots,t$. We can construct this sequence in such a way that, if all the restrictions are written in the form $\phi_{ij} = 0$ with $i < j$, then there is no more than one such restriction for every j.

Now consider the matrix

$$F = \begin{bmatrix} F_r \\ F_s \end{bmatrix} = \begin{bmatrix} F_{r1}, & \dots & ,F_{rt},F_{r,t+1}, & \dots & ,F_{rm} \\ F_{s1}, & \dots & ,F_{st}, \ 0 & , \dots , & 0 \end{bmatrix}$$

$$= \left[F_1 , \dots , F_t, F_{t+1} , \dots , F_m \right]$$

where F_r comprises $F_0 = R'_\Delta(I \otimes \Delta)$ and the rows corresponding to r recursively decomposable restrictions, whilst F_s is a matrix in lower echelon form comprising the rows corresponding to the indecomposable restrictions $\phi_{ij} = 0$; $i < j$ ordered according the index j. Then, for $j = 1,\dots,t$,we can find vectors λ_j such $F_{rj}\lambda_j = 0$ whilst $F_{sj}\lambda_j \neq 0$. The matrix $[F_{s2}\lambda_2,\dots,F_{st}\lambda_t]$ is in lower echelon form and is, consequently, of full row rank. Hence there exists a vector μ such that $F_{s1}\lambda_1 = [F_{s2},\dots,F_{st}]\mu$ whilst $F_{r1}\lambda_1 = [F_{r2},\dots,F_{rt}]\mu = 0$; and, therefore, the condition $\text{Rank}(F) = \text{Rank}(F_1) + \text{Rank}(F_2,\dots,F_m)$, which is necessary for the identification of the first equation, cannot hold.

REFERENCES

[1] BALESTRA, P.: La Derivation Matricielle. Paris: Sirey, 1976.

[2] BENTLER, P. M. and E. H. FREEMAN: "Tests for Stability in Linear Structural Equation Systems," Psychometrica, 48(1983), 143-145.

[3] FISHER, F. M.: The Identification Problem in Econometrics. New York: McGraw-Hill, 1966.

[4] FISHER, F. M.: "A Correspondence Principle for Simultaneous Models," Econometrica, 38(1970), 73-145.

[5] HAUSMAN, J. A. and W. E. TAYLOR: "Identification in Linear Simultaneous Equations Models with Covariance Restrictions: An Instrumental Variables Interpretation," Econometrica 51(1983), 1527-1549.

[6] MAGNUS, J. R. and H. NEUDECKER: "The Commutation Matrix: Some Properties and Applications," Annals of Mathematical Statistics, 7(1979), 381-394.

[7] POLLOCK, D. S. G.: The Algebra of Econometrics. Chichester: John Wiley and Sons, 1979.

[8] ROTHENBERG, T. J.: "Identification in Parametric Models," Econometrica, 39(1971), 577-579.

[9] SCHMIDT, P.: Econometrics. New York: Marcel Dekker, 1976.

[10] SHAPIRO, A.: "On Local Identifiability in the Analysis of Moment Structures," unpublished paper. The Department of Mathematics of the University of South Africa.

[11] WALD, A.: "Note on the Identification of Economic Relations," in Statistical Inference in Dynamic Economic Models. New York: John Wiley and Sons, 1950.

[12] WEGGE, L.: "Identifiability Criteria for a System of Equations as a Whole," The Australian Journal of Statistics, 7(1965), 67-77.

[13] WOLD, H., in association with L. Jureen: Demand Analysis: A Study in Econometrics. New York: John Wiley and Sons, 1953.