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Research memorandum

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1. Introduction.[1)]

In general equilibrium models labour supplied by consumers is represented by some components of the consumption vector. It is assumed that consumers are free to choose the quantities of labour of different types, that they will supply. However in reality many types of labour may only be supplied in fixed quantities, because the length of the labour day is fixed. Reason for this fixation may be that the production process requires all workers to be present simultanuously, or that working hours are fixed by government of through collective action by trade unions.

In the present paper we assume that a unique labour time t has to be fixed. We consider two problems: the existence of equilibrium for fixed t and the optimum quantity of t, given that a unique t should be fixed. As was noted by Dreze (1976) it appears that t may be considered as some kind of public good. It is shown that a Lindahl-type equilibrium can be defined and the private goods equilibrium for fixed t is a local pareto optimum, provided that the mean difference between personalized wages and equilibrium wages is zero. That only a local optimum follows, is caused by the non convexity of the global set of feasible solutions. Our problem is formally similar to the problem of investment under uncertainty, as considered in Dreze (1974).

We consider an economy with a finite set of types of consumers, while there is an infinity of consumers of each type. The reason for this approach is that we wish to rule out consequences of the fixation of t in production.

We take a finite number of types of consumers rather than a contiuum of consumers, because this keeps the analysis relatively simple.

1) The author thanks B. v.d. Genugten for some valuable suggestions.

2. The model.

We consider the economy

$$E = \{\{X_i\},\{\succsim_i\},S,\{\mu_i\},\{w_i\},Y,\{\theta_i\}\}$$

2.1. Commodities and labour.

There are m commodities and n different types of labour. The commodity space is R^{m+n} and a typical element of this set is $(x,z) = (x^1,x^2,\ldots,x^m,z^1,z^2,\ldots,z^n)$. There is a labour time regulation, which prescribes that each consumer can only supply a fixed quantity of at most one type of labour. The labour time is $t \geq 0$. (Among the m commodities could figure types of labour for which the time regulation does not hold.)

$L = \{0,1,2,\ldots,n\}$ is the set of labour types; $\ell = 0$ means "not working". In the price vector $(p,q) \in R_+^{m+n}$, p is the price vector of commodites and q the price vector of types of labour.

2.2. Consumers.

There is a set S containing infinitely many consumers. There are h different types of consumers, where h is a finite number. $I = \{1,2,\ldots,h\}$ denotes the set of indices of consumer types. Consumers of the same type have identical consumption sets, identical preferences, identical resources and identical profit-rights. $S_i \subset S$ is the set of consumers of type $i \in I$; $\cup S_i = S$; if $i \neq j$, $S_i \cap S_j = \emptyset$.

μ is a measure on the measurable space $(S,\mathcal{B})$, $\mathcal{B}$ being the Borel sets of S, with $S_i \in \mathcal{B}$ and such that $\mu(S) = 1$, $\mu(S_i) = \mu_i > 0$, (hence $\Sigma\mu_i = 1$). μ_i is the proportion of consumers that are of type i.

Definition 2.2.1.: A distribution of S_i is a pair (α_i,K_i), where K_i is a finite set of indices $\{1,2,\ldots,a_i\}$, $\alpha_i = (\alpha_{i1},\alpha_{i2},\ldots,\alpha_{ia_i})$, such that $\Sigma\alpha_{ik} = 1$ and $\alpha_{ik} \geq 0$ and such that a partitioning of S_i into disjoint

subsets S_{ik} exists, for which $\mu(S_{ik}) = \alpha_{ik}\mu_i$.
$K_i^+ = \{k \in K_i | \alpha_{ik} > 0\}$.

$X_i \subset R^{m+n}$ is the consumption set of type i consumers. On X_i is defined the preference relation $\succsim_i$; $(x,z) \succ_i (x',z')$ means $(x,z) \succsim_i (x',z')$ and $(x',z') \not\succsim_i (x,z)$. $C_i : X_i \to X_i$ and $P_i : X_i \to X_i$ are called preference correspondence and strict preference correspondence respectively, where
$C_i(x,z) = \{(x',z') | (x',z') \succsim_i (x,z)\}$ and
$P_i(x,z) = \{(x',z') | (x',z') \succ_i (x,z)\}$.
Each member of S_i has a bundle of resources $(w_i,0) \in R^{m+n}$; so resources do not contain labour. The mean resources of the economy are given by the vector $(w,0) = \Sigma\mu_i(w_i,0)$. Each consumer of type i has profit rights $\theta_i(p,q)$, $\theta_i(p,q)$ being a continuous function, such that $\Sigma\mu_i\theta_i(p,q) = 1$. The mean profit of the economy is $\pi(p,q)$. Mean income in the economy is $pw + \pi(p,q)$ and individual income is $\varphi_i(p,q) = pw_i + \theta_i(p,q)\pi(p,q)$.

2.3. Production.

We only consider a single production set $Y \subset R^{m+n}$, which may be considered as a sum of production sets of individual firms.
Points $(y,v) \in Y$ represent mean production of commodities and mean labour input, w.r.t. the consumers in the economy. The (mean) profit associated with a production vector (y,v), is $py + qv = \pi(p,q)$, and we shall assume that labour may only be an input in production, hence $v \leqq 0$.
Note that the implicitely assume that the commodity vector y is related to the (mean) input of labour and does not depend on the way in which v is supplied i.e. if many workers supply a small quantity of labour of a certain type or if fewer workers supply a larger quantity. This may be unrealistic but it is certainly not more unrealistic than the traditional assumption that each consumer may freely choose the quantity of labour to supply.

2.4. Feasible solutions.

In the economy E a feasible solution consists of an $(m+n)(h+1) + 1$ - tuple$(((x_i,z_i)),(y,v),t)$, such that

(i) there exist distributions (α_i,K_i) and consumption vectors $(x_{ik},z_{ik}) \in X_i$ for all i and $k \in K_i^+$, such that $(x_i,z_i) = \Sigma\alpha_{ik}(x_{ik},z_{ik})$ and $z_{ik} = 0$ or $z_{ik}^{\ell} = -t$ and $z_{ik}^{\ell'} = 0$, for some ℓ and $\ell' \neq \ell$.

(ii) $\Sigma\mu_i(x_i,z_i) \leqq (y,v)$.

(iii) $(y,v) \in Y$.

Determination of the commodity vector x_{ik} and of the type of labour to supply, could be left to individual consumers. The determination of t however must be a result of some collective decision process.
In section 4 we consider equilibria if t is given. In section 5 we shall consider optimum values for t.

3. Assumptions.

We make the following assumptions:

on the consumption set; for all i:

A1 X_i is closed

A2 X_i is bounded below

A3 if $(x,z) \in X_i$, then $z \leqq 0$ and if $x' \geqq x$, $0 \geqq z' \geqq z$ then $(x',z') \in X_i$.

A4 X_i is convex

on preferences; for all i:

B1 $(x',z') \in C_i(x,z) \Rightarrow C_i(x',z') \subset C_i(x,z)$ (transitivity)

B2 for all $(x,z),(x',z') \in X_i : (x,z) \in C_i(x',z')$ or $(x',z') \in C_i(x,z)$ (completeness)

B3 for all $(x,z) \in X_i$: $C_i(x,z)$ is closed and $P_i(x,z)$ is open (continuity)

B4 if $(x,z),(x',z') \in X_i$ and (a) $(x',z') \geqq (x,z)$ then $(x',z') \in C_i(x,z)$ (weak monotonicity) and (b) $x' > x$ and $z' \geqq z$ then $(x',z') \in P_i(x,z)$.

B5 if $(x,z) \succ_i (x',z')$, there exists $\bar{x} < x$, such that $(\bar{x},z) \in X_i$

B6 for all $(x,z) \in X_i$, $C_i(x,z)$ is convex.

on the set of consumers:

C for all i: if K_i is a finite set of indices, $\alpha_{ik} \geq 0$, $\Sigma\alpha_{ik} = 1$, there exists a partitioning of S_i into disjount subsets S_{ik}, such that $\mu(S_{ik}) = \mu_i \alpha_{ik}$. (atomlessness).

on the production set:

D1 Y is closed

D2 Y is convex

D3 Y is bounded above

D4 $(x,v) \in Y \Rightarrow v \leqq 0$

D5 $0 \in Y$

on initial resources:

E $\quad \forall i = (w_i, 0) \in X_i$ (feasibility)

So in particular we assume that consumption of positive quantities of labour is impossible, for $\ell = 1,2,\dots,n$.
Assumption B5 is necessary to ensure that continuity is preserved for mean preferences, to be defined in the next section (see remark in 4.2 below). It implies that points of the lower boundary of X_i and of each preference set $C_i(x_i, z_i)$ are equivalent. Another consequence of B5 will be that any quasi-equilibrium is an equilibrium, so we need not bother about quasi-equilibria. Assumption C means that each pair (α_i, K_i) is a distribution in the sense of definition 2.2.1. It implies that the measure μ is atomless (see e.g. Hildenbrand 1974). The other assumptions are standard in equilibrium theory. Note however that the feasibility assumption E is extremely strong: it permits all consumers to survive without working. Convexity (A4 and B6) will only be required in section 5).

Proposition 3.1.: Under the assumptions A1, A2, A3, B3, B4 and B5:
(1) if $(x,z) \in X_i$ and for $x' < x, (x',z) \notin X_i$, then $X_i = C_i(x,z)$;
(2) $P_i(x,z) = \{(\tilde{x},\tilde{z}) | (\tilde{x},\tilde{z}) \in C_i(x,z)$ and $\exists \hat{x} < x : (\tilde{x},\tilde{z}) \in C_i(x,z)\}$.

> Proof: (1) Let $(x,z) \in X_i$ and $(x',z) \notin X_i$ for $x' < x$; suppose there would exist $(\bar{x},\bar{z}) \in X_i$, such that $(x,z) \succ_i (\bar{x},\bar{z})$. Then by B5, $\hat{x} < x$ would exist such that $(\hat{x},z) \in X_i$. That is a contradiction. Hence for all $(\bar{x},\bar{z}) \in X_i : (\bar{x},\bar{z}) \succsim_i (x,z)$.
> (2) Let $(x',z') \succ_i (x,z)$. By B5, there exists $\tilde{x} < x$, such that $(\tilde{x},z) \in X_i$, and by monotonicity: $(x',z') \succ_i (\tilde{x},z')$. By continuity there exists $\tilde{x} \leqq \hat{x} < x'$, such that $(\hat{x},z) \succsim_i (x,z)$.

4. Labour time given.

Let $t < 0$ be fixed beforehand. For an individual consumer the labour-time restriction is essentially a restriction on his budget set. However we shall formulate the problem in such a way that the time restriction is included in the consumption set and in the preferences. This is done by defining mean consumption and mean preferences for each consumer type, and thus deriving an economy E_t from E.

We do not assume in this section the convexity of the consumption set and of preferences.

4.1. Mean consumption set.

Let U be the set of n negative unit vectors and a vector of zero's:

$$U = \{u^0, u^1, u^2, \ldots, u^n\}$$

where $u^0 = (0,0,\ldots,0)$, $u^1 = (-1,0,\ldots,0)$, $u^2 = (0,-1,\ldots,0)$,..., $u^n = (0,0,\ldots,-1)$. We define:

$$\bar{X}_{it\ell} = X_i \cap \{(x,z) \in X_i \mid z = tu^\ell\}$$

for $\ell \in L$. So $\bar{X}_{it\ell}$ contains all possible consumption bundles where an individual consumer supplies t units of labour of type $\ell \in L$. Define

$$\bar{X}_{it} = \bigcup_L \bar{X}_{it\ell}$$

The mean consumption set of type i consumers is (Co denoting the convex hull)

$$X_{it} = \mathrm{Co}\,\bar{X}_{it}$$

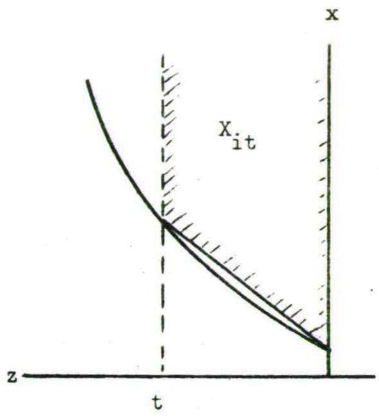

fig. 1

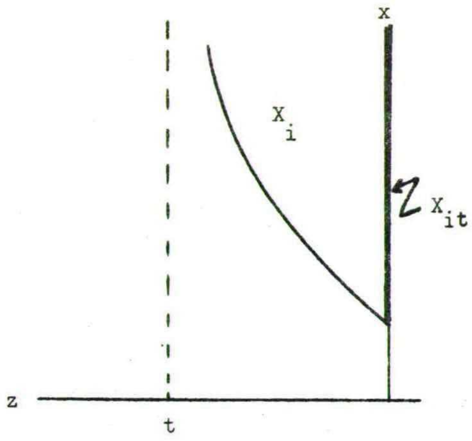

fig. 2

Note that $\bar{X}_{it\ell}$ may be empty for some $\ell \geqq 1$, if consumers of type i are not able to supply t units of labour of type ℓ. In this case for any $(x_i, z_i) \in X_i(t), z_i^\ell = 0$. $\bar{X}_{it0} \neq \emptyset$, by assumption E (see fig. 2).

If $(x_i, z_i) \in X_{it}$, then, by definition, (x_i, z_i) is a convex combination of at most m+n+1 points in $\bar{X}_{it}$. Hence by assumption C, there exist a distribution (β_{ik}, K_i), for $K_i = \{1,2,\ldots,m+n+1\}$, and vectors $(x_{ik}, z_{ik}) \in \bar{X}_{it}$, for $k \in K_i^+$, such that:

$$(x_i, z_i) = \sum_{K_i} \beta_{ik}(x_{ik}, z_{ik})$$

(x_i, z_i) is a mean consumption vector of consumers in S_i and (x_{ik}, z_{ik}) is the consumption of consumers in the associated subsets S_{ik}, while $\mu(S_{ik})/\mu_i = \beta_{ik}$. For all k: $z_{ik} \in tU$. Let $K_{i\ell} = \{k \in K_i^+ | z_{ik} = tu^\ell\}$, for $\ell \in L$. Then

$$\sum_{K_{i\ell}} \beta_{ik} = \frac{1}{t}(-z_i^\ell) \qquad \text{if } \ell \geq 1$$

$$\sum_{K_{i0}} \beta_{ik} = 1 + \frac{1}{t} \sum_{\ell} z_i^{\ell}$$

Define $\alpha_{i\ell} = \sum_{K_{i\ell}} \beta_{ik}$. We have

$$(x_i, z_i) = \sum_{\ell} \sum_{K_{i\ell}} (x_{ik}, tu^{\ell}) = \sum_{\ell} \alpha_{i\ell} \sum_{K_{i\ell}} \frac{\beta_{ik}}{\alpha_{i\ell}} (x_{ik}, tu^{\ell}) = \sum_{\ell} \alpha_{i\ell} (x_{i\ell}, tu^{\ell})$$

where $(x_{i\ell}, tu^{\ell}) = \sum_{K_{i\ell}} \frac{\beta_{ik}}{\alpha_{i\ell}} (x_{ik}, tu^{\ell})$.

Hence $(\alpha_{i\ell}, L)$, is a distribution of S_i and $(x_{i\ell}, tu^{\ell})$ are the <u>mean</u> consumptions of consumers in the associated subsets $S_{i\ell}$, consisting of the consumers of S_i who supply labour of type ℓ. (x_i, z_i) is the mean consumption of these mean consumptions. For all $\ell \in L_i^+$, there exists a distribution $(\gamma_{ik}, K_{i\ell})$, where $\gamma_{ik} = \beta_{ik} / \sum_{K_{i\ell}} \beta_{ik}$ and this distribution produces the mean consumptions $(x_{i\ell}, tu^{\ell})$ from (x_{ik}, tu^{ℓ}).

<u>Remark</u>: Note that if X_i is convex (assumption A4), all $(x_{i\ell}, tu^{\ell})$ are in $\bar{X}_{it}$. Note also that (x_i, z_i) could be expressed by different convex combinations, so that the distribution $(\beta_{i\ell}, K_i)$ is not unique. However $(\alpha_{i\ell}, L)$ is unique, since it is uniquely determined by z_i.

<u>Theorem</u> 4.1.: Under the assumptions A1, A2, A3, for X_i, and C, the consumption sets X_{it} fullfill assumptions A1, A2, A3 and A4.

<u>Proof</u>: see appendix.

4.2. <u>Mean preferences</u>.

For $t > 0$, we construct a new preference correspondence $\hat{C}_{it}$, representing a new preference relation $\succsim_{it}$ on X_{it}.
The correspondences $\bar{C}_{it\ell}: \bar{X}_{it} \to \bar{X}_{it\ell}$, $\bar{C}_{it}: \bar{X}_{it} \to \bar{X}_{it}$ and $C_{it}: \bar{X}_{it} \to X_{it}$ are defined:

$$\bar{C}_{it\ell}(x,z) = C_i(x,z) \cap \bar{X}_{it\ell}$$

$$\bar{C}_{it}(x,z) = \bigcup_{\ell} \bar{C}_{it\ell}(x,z)$$

$$C_{it}(x,z) = \text{Co}\ \bar{C}_{it}(x,z)$$

Define $\hat{C}_{it}: X_{it} \to X_{it}$ by:

$$A(x,z) = \{(x,0) \in \bar{X}_{it0} \mid (x,z) \in C_{it}(x,0)\}$$

$$\hat{C}_{it}(x,z) = \bigcap_{A(x,z)} C_{it}(x,0)$$

$\hat{C}_{it}$ is the preference correspondence, representing the preference relation $\succsim_{it}$, which is defined:

$$(x',z') \succsim_{it} (x,z) \Leftrightarrow (x',z') \in \hat{C}_{it}(x,z)$$

We define $\hat{P}_{it}(x,z) = \{(x',z') \mid (x',z') \in \hat{C}_{it}(x,z) \text{ and } (x,z) \notin \hat{C}_{it}(x',z')\}$.

<u>Theorem</u> 4.2.: Under assumptions A1 - A3, B1 - B5 and C, the preference relation $\succsim_{it}$ fullfills B1 - B6.

<u>Proof</u>: see appendix.

With any point $(x,z) \in X_i$ are associated a distribution (β_i, K_i) and vectors $(x_k,z_k) \in \hat{C}_{it}(x,z)$, for $k \in K_i^+$, such that $(x,z) = \Sigma\beta_{ik}(x_k,z_k)$; for $k,k' \in K_i^+$, $(x_k,z_k) \sim_i (x_{k'},z_{k'})$: since by lemma A (appendix), (x,z) is on the boundary of $\hat{C}_{it}(x,z)$, it is a convex combination of boundary points of $\bar{C}_{it}(\bar{x},0)$, for some $(\bar{x},0) \in \bar{X}_{it}$ and by proposition 3.1 these boundary points are equivalent w.r.t. $\succsim_i$. By definition they are also equivalent w.r.t. $\succsim_{it}$, i.e. $(x_k,z_k) \sim_{it} (x_{k'},z_{k'}) \sim_{it} (x,z)$.
As was shown in section 4.1, (x,z) can also be expressed as a convex combination of points $(x_{i\ell}, tu^{\ell})$. These points are also equivalent to (x,z)

w.r.t. $\sim_{it}$. Note that if $\succsim_i$ would be convex, than $(x_{i\ell}, tu^{\ell}) \in \bar{C}_{it}(\bar{x}, 0)$.

Remark: Without assumption B5 $\succsim_{it}$ needs not be continuous and the points (x_k, z_k), considered above, need not be equivalent w.r.t. $\succsim_i$. Consider fig. 3 (for m = 1, n = 1);

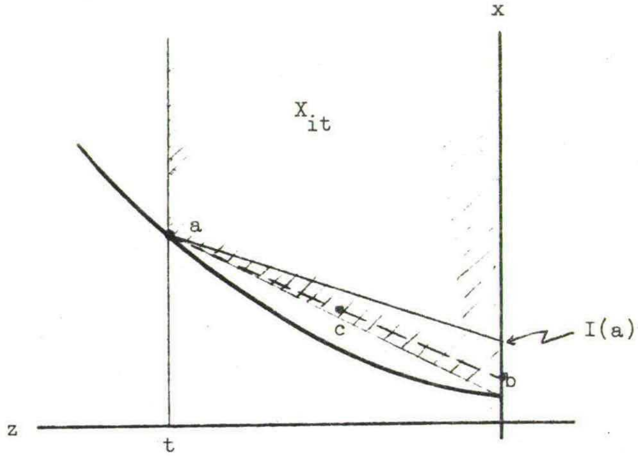

fig. 3

X_i is the set above the thick line; X_{it} is the shaded area. The indifference curves of $\succsim_i$ are lines parelled to I (a).
We have: a $\succ_i$ b; a $\in C_{it}(b) = \hat{C}_{it}(b) = \hat{C}_{it}(c)$; b $\notin C_{it}(a) = \hat{C}_{it}(a)$; c $\notin C_{it}(a)$. Hence a $\succ_{it}$ c $\sim_{it}$ b. c is a convex combination of the non-equivalent points a and b; a $\in \hat{P}_{it}(c)$, hence $\hat{P}_{it}(c)$ is not open.

4.3. Equilibrium.

In the economy

$$E_t = \{\{X_{it}\}, \{\succsim_{it}\}, S, \{\mu_i\}, \{w_i\}, Y_i\{\pi_i\}\}$$

for t > 0 and X_{it} and $\succsim_{it}$ as defined in the previous sections, we define:

<u>Definition</u> 4.3.: An equilibrium in E_t is a set of mean consumptions $(x_i,z_i) \in X_{it}$, a production vector $(y,v) \in Y$ and price vectors (p,q), such that

(1) $\Sigma\mu_i(x_i,z_i) = (y,v) + (w,0)$

(2) For all i, (x_i,z_i) is best w.r.t. $\succsim_{it}$ in the budget set $\{x_i,z_i | px_i + qz_i \leqq \varphi_i(p,q)\}$

(3) $py + qv = \max\{p\tilde{y} + q\tilde{v} | (\tilde{y},\tilde{v}) \in Y\} = \max\ (p,q)Y$.

With such an equilibrium are associated distributions (α_i,L), <u>mean</u> consumptions $(x_{i\ell},tu^{\ell})$ for $k \in L_i^+$, and subsets $S_{i\ell} \subset S_i$, with $\mu(S_{i\ell}) = \mu_i\alpha_{i\ell}$; the members of $S_{i\ell}$ supply t units of labour of type ℓ and have mean consumption $x_{i\ell}$. Members of $S_{i\ell}$ may have different consumptions of commodities $x_{i\ell k}$ (see section 4.1), however all points $(x_{i\ell k},tu^{\ell})$ are elements of the boundary of the budget set.

<u>Theorem</u> 4.4.: Under assumption A1 - A3, B1 - B5, C, D and E, an equilibrium in E_t exists.

<u>Proof</u>: By theorems 4.1 and 4.2, the consumption sets X_{it} and the preferences $\succsim_{it}$, fullfill A1 - A4 and B1 - B4. Under these assumptions, existence can be proved by standard methods, applied in an economy with a finite number of consumers. In the proof summation over consumption bundles should only be replaced by weighted summation, with weights μ_i.

5. Labour time as a public good.

In the preceding section t was assumed given. In this section we consider the problem of the optimal value of t. Since t is binding for consumers with different preferences, it appears that labour time is some kind of public good, hence it seems natural to consider Lindahl equilibria.

Let the economy E be as defined in section 2. Instead of the income distribution $\varphi_i(p,q)$ of section 2.2, we apply an after transfer income distribution. Given $(\alpha_{i\ell},L)$ the income of $i\ell$-consumers is

$$\rho_{i\ell}(p,q) = \varphi_i(p,q) + \tau_{i\ell}$$

$\tau \in R^{h(n+1)}$ are the transfers paid to $i\ell$-consumers, where $\sum_{i\ell} \alpha_{i\ell}\mu_i\tau_{i\ell} = 0$, hence $\sum_{i\ell} \alpha_{i\ell}\mu_i\rho_{i\ell}(p,q) = \sum_i \mu_i\varphi_i(p,q)$

In this section we assume convexity of X_i and of $\succsim_i$ (A4 and B6).

5.1. A single type of labour.

We first consider a special case:

(a) there is only one type of labour

(b) all consumers are obliged to work.

Given (a) and (b) a feasible solution in E consists of $t > 0$, $(x_i,z_i) \in X_i$, $(y,v) \in Y$, such that (i) for all i: $z_i = -t$ and (ii) $\Sigma\mu_i(x_i,z_i) \leqq (y,v)$.

No distributions of S_i are necessary in this case to define a feasible solution because, by convexity, a mean consumption (x_i,z_i) is in the consumption set (Compare section 2.4).

We exactly have the model of an economy with private goods and one public good, (or rather with one public "bad"). For such a model a Lindahl equilibrium with personalized prices for the public good can be defined.

Definition 5.1.: A Lindahl equilibrium is the economy E given conditions (a) and (b), and with income distribution ρ_i, is an allocation (x_i), a price vector p, labour time t, personalized prices (wages) q_i, a producer's wage q and a production bundle (y,v), such that

(i) for each i, $(x_i,-t)$ is best w.r.t. $\succsim_i$ in the budget set $\{(x,z) \in X_i \mid px + q_i z \leqq \rho_i(p,q)\}$

(ii) $\Sigma\mu_i(x_i,-t) \leqq (y,v) + (w,0)$

(iii) $\Sigma\mu_i q_i = q$

(iv) $py + qv = \max\,(p,q)Y$

In the Lindahl-equilibrium, consumers of different types may get a different wage q_i^ℓ for the same type of labour. In the equilibrium in E_t all consumers get the same wage. If (a) and (b) hold, the consumption set in E_t is $X_{it} = \mathrm{Co}\bar{X}_{it1}$, and a solution $(x_i,-t)$, (y,t), (p,q) is an equilibrium if (1) $\Sigma\mu_i(x_i,-t) = (y,t) + (w,0)$; (2) $(x_i,-t)$ is best in the budget set and (3) $(y,-t)$ is profit maximizing. If $\rho_i(p,q) = \varphi_i(p,q)$, a Lindahl equilibrium in E, with solution $t = \bar{t}$, will generally not correspond to an equilibrium in $E_{\bar{t}}$, since this would require for all i: $p\bar{x}_i - q_i\bar{t} = \varphi_i(\bar{p},\bar{q}) = \bar{p}\bar{x}_i + \bar{q}\bar{t}$, which would imply $\bar{q}_i = \bar{q}$, for all i. However the equilibrium $(\bar{x}_i,\bar{z}_i),(\bar{p},\bar{q}),(\bar{y},\bar{v})$ in $E_{\bar{t}}$ could correspond to a Lindahl-equilibrium in E with income transfers: Taking into account the convexity assumptions A4 and B6, the hyperplanes $\{(x_i,-t) \mid \bar{p}x_i - \bar{q}t = \varphi_i(\bar{p},\bar{q})\}$ support the preference sets $C_{it}(\bar{x}_i,-\bar{t}) = C_i(\bar{x}_i,-\bar{t}) \cap \bar{X}_{it1}$ in $(\bar{x}_i,-\bar{t})$. There exist $\bar{q}_i$ and ρ_i, such that the hyperplanes $\{x_i,-t \mid \bar{p}x_i - \bar{q}_i t = \rho_i\}$ support the (original) preference sets $C_i(\bar{x}_i,-\bar{t})$ in $(\bar{x}_i,-\bar{t})$; so these points are best w.r.t. the original preference relation $\succsim_i$ in that budget set.

We now have simultanuously $\bar{p}\bar{x}_i - \bar{q}\bar{t} = \varphi_i(\bar{p},\bar{q})$ and $\bar{p}\bar{x}_i - \bar{q}_i\bar{t} = \rho$; hence

$$\rho_i = \varphi_i(\bar{p},\bar{q}) + (\bar{q}_i - \bar{q})\bar{t}.$$

Therefore $\{(\bar{x}_i),\bar{p},(\bar{q}_i),\bar{q},(\bar{y},\bar{v})\}$ is a Lindahl equilibrium in E for the income distribution ρ_i, <u>provided</u> that $\Sigma\mu_i\bar{q}_i = \bar{q}$, for then (iii) of definition 4.1.1 is fullfilled. Then the income distribution ρ_i could be realized from the income distribution $\varphi_i(\bar{p},\bar{q})$ by transfers $(\bar{q}_i-\bar{q})\bar{t}$, since $\Sigma\mu_i(\bar{q}_i-\bar{q})\bar{t} = 0$. Obviously this solution is efficient since a Lindahl equilibrium is efficient. On the other hand, if $(\bar{\bar{x}}_i,-\bar{\bar{t}}),\bar{\bar{p}},\bar{\bar{q}},(\bar{\bar{q}}_i),(\bar{\bar{y}},\bar{\bar{v}})$ is a Lindahl equilibrium in E for the original income distribution φ_i, then $(\bar{\bar{x}}_i,-\bar{\bar{t}}),(\bar{\bar{p}},\bar{\bar{q}}),(\bar{\bar{y}},\bar{\bar{v}})$ is an equilibrium in $E_{\bar{\bar{t}}}$ <u>for the after transfer income distribution</u> $\rho_i = \varphi_i(\bar{\bar{p}},\bar{\bar{q}}) - (\bar{\bar{q}}_i-\bar{\bar{q}})\bar{\bar{t}}$.

5.2. <u>Different types of labour</u>.

Things become more complex, if there are different types of labour, including the case of one type of labour, where consumers are allowed not to work. The reason is, like in Dréze (1974), that the set of feasible solutions in E is not convex. This can be shown by the following counter-example:

Let $n = 1$; $((x_i,z_i),t,(y,z))$ and $((\bar{x}_i,\bar{z}_i),\bar{t},(\bar{y},\bar{z}))$ are two feasible solutions. Assume that consumers of type 1 are not able to supply much more labour than t, i.e. for some $\varepsilon > 0$: if $\tilde{t} > t+\varepsilon$, then $\bar{X}_{1\tilde{t}1} = \emptyset$. Let $\bar{t} > t+\varepsilon$, hence $\bar{z}_1 = 0$. Let $z_1 < 0$, hence there exist x_{10} and x_{11}, such that for $\alpha_{10} = 1 + \frac{1}{t} z_1$ and $\alpha_{11} = -\frac{1}{t} z_1$, $(x_1,z_1) = \alpha_{10}(x_{10},0) + \alpha_{11}(x_{11},-t)$. Consider the convex combination produced by $0 < \gamma < 1$, such that $\tilde{t} = \gamma t + (1-\gamma)\bar{t} > t+\varepsilon$. Hence $X_{1\tilde{t}1} = \emptyset$. However $(\tilde{x}_1,\tilde{z}_1) = \gamma(x_1,z_1) + (1-\gamma)(\bar{x}_1,\bar{z}_1)$ and $\tilde{z}_1 < 0$. So we must have, for $\tilde{\alpha}_{10} = 1 + \frac{1}{t}\tilde{z}_1$ and $\tilde{\alpha}_{11} = -\frac{1}{t}\tilde{z}_1 > 0$: $(\tilde{x}_1,\tilde{z}_1) = \alpha_{10}(\tilde{x}_{10},0) + \alpha_{11}(\tilde{x}_{11},-\tilde{t})$, which is impossible, since $(\tilde{x}_{11},-\tilde{t}) \notin \bar{X}_{1t1} = \emptyset$.

However, if the distribution of consumers over types of labour is fixed, the set of feasible solutions is convex. Particularly let (α_i,L) be given and (x_i,z_i) and $(\bar{x}_i,\bar{z}_i)$ are feasible consumptions of type i consumers, where labour time is t and $\bar{t}$ respectively, then $\alpha_{i\ell} = -\frac{1}{t} z_i^\ell = -\frac{1}{\bar{t}} \bar{z}_i^\ell$ (for $\ell \geq 1$) and there exist $(x_{i\ell},tu^\ell)$ and $(\bar{x}_{i\ell},\bar{t}u^\ell)$ in X_i, for $\ell \in L_i$, such that $(x_i,z_i) = \Sigma_{L_i^+} \alpha_{i\ell}(x_{i\ell},tu^\ell)$ and $(\bar{x}_i,\bar{z}_i) \in \Sigma_{L_i^+} \alpha_{i\ell}(\bar{x}_{i\ell},\bar{t}u^\ell)$. Then for $1 \geq \gamma \geq 0$, we have $(\tilde{x}_i,\tilde{z}_i) = \gamma(x_i,z_i) + (1-\gamma)(\bar{x}_i,\bar{z}_i) = \Sigma\alpha_{i\ell}(\gamma(x_{i\ell},z_{i\ell})+(1-\gamma)(\bar{x}_{i\ell},\bar{z}_{i\ell}) \in X_i$ by assumption A4.

For (α_i,L) given, we reduce E to an economy E_α with a <u>single</u> labour variable t. This is possible since the distribution determines the type of labour to supply by each consumer and because the composition of the total supply of labour is fixed: $v = t(\Sigma\alpha_{i\ell}\mu_i u^\ell)$.
For a set of distributions (α_i,L), we define the economy

$$E_\alpha = \{\{\hat{X}_{i\ell}\},\{\succsim_{i\ell}\},S_{i\ell},\{\mu_i\alpha_{i\ell}\},\{w_i\},\hat{Y},\{\theta_i\},\tau_{11}\}$$

for $i\ell \in \hat{I} = \{i,\ell | i \in I \text{ and } \ell \in L_i^+\}$.
For $\ell \geq 1$:

$$\hat{X}_{i\ell} = \{(x,-t) | (x,tu^\ell) \in X_i\} \subset R^{m+1}$$

$$(x,-t) \succsim_{i\ell} (x',-t') \Leftrightarrow (x\ ,tu^\ell) \succsim_i (x',t'u^\ell)$$

For $\ell = 0$:

$$\hat{X}_{i0} = \{(x,-t) | (x,0) \in X_i \text{ and } \bar{t} \geq t \geq 0\}$$

$$(x,-t) \succsim_{i0} (x',-t') \Leftrightarrow (x,0) \succsim_i (x',0)$$

where $\bar{t}$ is such that, for all $i \in I$: $\bar{X}_{i\bar{t}} = \emptyset$ (see section 4.1). Such a $\bar{t}$ exists, since all X_i are bounded below by A2.

$$\hat{Y} = \{(y,-t) | (y,\Sigma\mu_i\alpha_{i\ell}tu^\ell) \in Y\}$$

and $$\rho_{i\ell} = \varphi_i(p,\hat{q}) + \tau_{i\ell} = pw_i + \theta_i(p,\hat{q})\pi(p,\hat{q}) + \tau_{i\ell}.$$

<u>Definition</u> 5.2.: A Lindahl equilibrium in E_α is an allocation $(x_{i\ell})$,

a labour time t, a price vector $p \in R^m$, a producer's composite wage $\hat{q} \in R_+$, personalized wages $\hat{q}_{i\ell} \in R_+$, such that:

(1) for all $(i,\ell) \in \hat{I}$: $(x_{i\ell},-t)$ is best w.r.t. $\succsim_{i\ell}$ in the budget set $\{(x,-t)|px - \hat{q}_{i\ell}t \leq \rho_{i\ell}(p,q)$

(2) $\Sigma\alpha_{i\ell}\mu_i x_{i\ell} = y$

(3) $\Sigma\alpha_{i\ell}\mu_i\hat{q}_{i\ell} = \hat{q}$

(4) $py - \hat{q}t = \max (p,\hat{q})\hat{Y}$

The personalized wages $\hat{q}_{i\ell}$ are the shadow prices of labour of type ℓ supplied by consumers of type $i\ell$. The wage $\hat{q}$ is a composite wage; the profit maximizing wage of the producer for a bundle $\Sigma\mu_i\alpha_{i\ell}tu^\ell$.

<u>Proposition</u> 5.3.: $(x_{i\ell}),y,t,p,\hat{q},\hat{q}_{i\ell}$ is a Lindahl equilibrium in E_α, if and only if there exist $q \in R^n$ and $q_{i\ell} \in R^n$, such that

(a) for all $(i\ell) \in \hat{I}$: $(x_{i\ell},tu^\ell)$ is best in the budget set $\{ x,z|px + q_{i\ell}z \leq \rho_{i\ell}\}$ w.r.t. $\succsim_i$

(b) $\sum_{i\ell} \alpha_{i\ell}\mu_i(x_{i\ell},tu^\ell) = (y,v)$

(c) $\sum_{i\ell} \alpha_{i\ell}\mu_i(q_{i\ell}u^\ell-qu^\ell) = 0$

(d) $py + qv = \max (p,q)Y$

(e) $\hat{q}_{i\ell} = q^\ell_{i\ell}$ and $\hat{q} = q(\Sigma\alpha_{i\ell}\mu_{i\ell}u^\ell)$

<u>Proof</u>: Define $H(p,q,\rho) = \{x,z|px + qz = \rho\} \subset R^{m+n}$ and $H(p,\hat{q},\rho) = \{x,-t|px - \hat{q}t = \rho\} \subset R^{m+n}$.

(a) For $\ell \geq 1$, $H(p,\hat{q}_{i\ell},\rho_{i\ell})$ supports the set $\hat{C}_{i\ell}(x_{i\ell},-t) = \{(x_{i\ell},-t)|(x',-t') \succsim_{i\ell} (x_{i\ell},-t)\}$ in $(x_{i\ell},-t)$.

Hence the set $S = \{(x,z) | px + \hat{q}_{i\ell} z^{\ell} = \rho_{i\ell}\}$ supports $C_{i\ell}(x_{i\ell}, tu^{\ell})$ in $(x_{i\ell}, tu^{\ell})$. These two sets are convex and therefore can be separated by a hyperplane $H(\tilde{p}, \tilde{q}, \rho_{i\ell})$ and this hyperplane contains S, hence $\tilde{p} = p$ and $\tilde{q}^{\ell} = \hat{q}_{i\ell}$.
Conversely, if $H(p, q_{i\ell}, \rho_{i\ell})$ supports $C_{i\ell}(x_{i\ell}, tu^{\ell})$ in $(x_{i\ell}, tu^{\ell})$, then $H(p, q^{\ell}_{i\ell}, \rho_{i\ell})$ supports $\hat{C}_{i\ell}(x_{i\ell}, -t)$ in $(x_{i\ell}, -t)$.
For $\ell = 0$, $\hat{q}_{i0} = 0$, by the definition of $\succsim_{i0}$.
(d) Let $H(p, \hat{q}, \pi)$ support $\hat{Y}$ in $(y, -t)$. The set $S = \{y', v' | py' - \hat{q}t = \pi \text{ and } v' = \Sigma\alpha_{i\ell}\mu_i tu^{\ell}\}$ supports Y in (y, v). Hence S and Y are separated by a hyperplane $H(\tilde{p}, \tilde{q}, \pi)$ which contains S, so $\tilde{p} = p$ and $\hat{q}t = \tilde{q}v' = \tilde{q}(\Sigma\alpha_{i\ell}\mu_i u^{\ell})t$ and $\pi = \max(\tilde{p}, \tilde{q})Y$.
Conversely, if $py + qv = \max(p,q)Y$, $v = \Sigma\alpha_{i\ell}\mu_i tu^{\ell}$ and $\hat{q} = \Sigma\alpha_{i\ell}\mu_i u^{\ell}$, then for all $(y', t') \in \hat{Y}$,
$py' + q(\alpha_{i\ell}\mu_i u^{\ell})t' = py' - \hat{q}t' \leq py - \hat{q}t = py + qv$, hence
$py - \hat{q}t = \max(p, \hat{q})Y$.

<u>Theorem</u> 5.4.: Under the assumptions A1 - A4, B1 - B6, C, D and E there exists a Lindahl equilibrium in E_{α}.

<u>Proof</u>: It is easy to prove, that $X_{i\ell}$, $\succsim_{i\ell}$ and $\hat{Y}$ fullfill all assumptions A1 - A4, B1 - B6 and D1 - D5.
These assumptions are sufficient to prove the existence of a Lindahl equilibrium. In the proof summation over individuals should be replaced by weighted summation, using weights $\alpha_{i\ell}\mu_i$. (See e.g. Milleron (1972), Ruys (1974).)

Let $(\bar{x}_i, \bar{z}_i), (\bar{p}, \bar{q}), (\bar{y}, \bar{v}), \bar{t}$ be an equilibrium in $E_{\bar{t}}$ for $\bar{\alpha}_{i\ell} = -\frac{1}{t}\bar{z}_{i\ell}$ and $(\bar{x}_{i\ell})$, such that $\bar{x}_i = \Sigma\bar{\alpha}_{i\ell}\bar{x}_{i\ell}$. This equilibrium corresponds to a Lindahl equilibrium, if and only if their exist prices $\bar{q}_{i\ell}$ and incomes $\rho_{i\ell}$, such that $(\bar{x}_{i\ell}, \bar{t}u^{\ell})$ is best in $\{x, z | \bar{p}x + \bar{q}_{i\ell}z \leq \rho_{i\ell}\}$ and $\Sigma\alpha_{i\ell}\mu_i(\bar{q}_{i\ell} - \bar{q})\bar{t}u^{\ell} = 0$. Since $\rho_{i\ell} = \bar{p}\bar{w}_i + \theta_i(\bar{p}, \bar{q})\pi(\bar{p}, \bar{q}) + \tau_{i\ell} = \bar{p}\bar{x}_{i\ell} + \bar{q}_{i\ell}\bar{t}u^{\ell}$ and $\varphi_{i\ell} = \bar{p}\bar{w}_i + \bar{\theta}_i(\bar{p}, \bar{q})\pi(\bar{p}, \bar{q}) = \bar{p}\bar{x}_{i\ell} + \bar{q}\bar{t}u^{\ell}$, this implies $\tau_{i\ell} = (\bar{q}_{i\ell} - \bar{q})\bar{t}u^{\ell}$ and $\Sigma\tau_{i\ell} = \Sigma\alpha_{i\ell}\mu_i(q_{i\ell} - q)\bar{t}u^{\ell} = 0$. So the transfers are only conceptual and need not effectively be paid.

6. Equilibria and Pareto optima.

Since the feasible solutions in E are not a convex set, it is not clear if equilibria and optimal solutions in E exist and what are their properties (compare Dreze (1974), section III).
A natural definition for a "second best" optimum seems to be: a solution that is: (1) a Pareto optimum for fixed labour time and (2) a Pareto optimum for given distributions over types of labour. Similarly an equilibrium in E could be defined as a solution that is: (1) an equilibrium in E_t and (2) a Lindahl equilibrium in E_α for the after transfer income distribution $\rho_{i\ell} = (q^\ell_{i\ell} - q^\ell)t$. (An alternative definition: a Lindahl equilibrium for the original income distribution and a equilibrium for some after transfer income distribution, seems to be less reasonable, since this would require that effectively different wages are paid for the same labour).
A natural proceedure for finding such an equilibrium, would run as follows:

(1) Fix some arbitrary t and find an equilibrium in E_t;

(2) Find the shadow prices $q^\ell_{i\ell}$ at the equilibrium consumptions and compute $\gamma = \Sigma\alpha_{i\ell}\mu_i(q^\ell_{i\ell} - q^\ell)$. If $\gamma = 0$, the equilibrium of the first step is also a Lindahl equilibrium in E_α. If $\gamma > 0$, increase t, it $\gamma < 0$, decrease t;

(3) For the new labour time t', find a new equilibrium in $E_{t'}$;

And so on, until $\gamma = 0$.
It is however not known if this procedure would converge, since cycling is not excluded.
A (purely conceptual) procedure for finding a second best Pareto optimum, could consist of a sequence of steps 1,2,3,..., where in the odd steps an equilibrium (or a Pareto optimum) is determined in E_t and in the even steps a Lindahl equilibrium (or a Pareto optimum) in E_α. In each step the resources of consumers consist of the consumption bundle of the preceeding step. These resources are to be understood as claims on commodities and obligations to work. The procedure is described in Appendix B. To prove convergence one needs additional assumptions, as e.g. in Dreze (1974) or in Chamsaur, Dreze, Henry (1977).

7. Final remark.

It was assumed that there is to be a single labour time for all types of labour. The preceeding model could be generalized to allow for different labour times for different types of labour, and to allow for a small number of different labour times for the same type of labour.

Appendix.

We denote by A and B the assumptions of section 3 w.r.t. X_i and $\succsim_i$, and by $\bar{A}$ and $\bar{B}$ the same assumptions w.r.t. X_{it} and $\succsim_{it}$.

Proof of theorem 4.1.:

By A1 and A2, X_i is closed and unbounded, hence also $\bar{X}_{it}$ is closed and unbounded and also its convex hull X_{it}, which is convex by definition. This proves $\bar{A}1$, $\bar{A}2$ and $\bar{A}4$.

If $(x,tu^\ell) \in X_{it}$ and $x' \geqq x$, then $(x',tu^\ell) \in X_{it}$: there exist (β_k,K) and $(x_k,tu^\ell) \in \bar{X}_{it}$, for $k \in K^+$, such that $(x,tu^\ell) = \Sigma\beta_k(x_k,tu^\ell)$. By A2: $(x_k + (x'-x),tu^\ell) \in \bar{X}_{it}$, hence $(x',tu^\ell) = \Sigma\beta_k(x_k + (x'-x),tu^\ell) \in X_{it}$.

By the argument used in section 4.1, for $(x,z) \in X_{it}$, there exist (α_ℓ,L) and (x_ℓ,tu^ℓ) for $\ell \in L^+$ such that $(x,z) = \Sigma\alpha_\ell(x_\ell,tu^\ell)$, and where $\alpha_\ell = -\frac{1}{t} z^\ell$ (if $\ell \geq 1$) and $\alpha_0 = 1 + \frac{1}{t}\Sigma z^\ell$. Let $(\bar{x},\bar{z}) \geqq (x,z)$; we may write:

$$\alpha_0(x_0 + (\bar{x}-x),0) + \sum_{\ell\geq 1} -\frac{1}{t}\bar{z}^\ell(x_\ell + (\bar{x}-x),tu^\ell) +$$

$$\sum_{\ell\geq 1} -\frac{1}{t}(z^\ell - \bar{z}^\ell)(x_\ell + (\bar{x}-x),0) =$$

$$(\Sigma\alpha_\ell\bar{x}, -\frac{1}{t}\Sigma\bar{z}^\ell tu^\ell) = (\bar{x},\bar{z})$$

Since by A2: $(x_\ell + (\bar{x}-x),tu^\ell)$ and $(x_\ell + (\bar{x}-x),0)$ are in X_{it}, whereas $\alpha_0 + \sum_{\ell\geq 1}(-\frac{1}{t}\bar{z}^\ell) + \sum_{\ell\geq 1}(-\frac{1}{t}(z^\ell - \bar{z}^\ell)) = \Sigma\alpha_\ell = 1$ and $-\frac{1}{t}\bar{z}^\ell \geq 0$ and $-\frac{1}{t}(z^\ell - \bar{z}^\ell) \geq 0$ (since $t \geq -z^\ell \geq -\bar{z}^\ell$), $(\bar{x},\bar{z})$ is a convex combination of points in X_{it}, so $(\bar{x},z) \in X_{it}$. This proves $\bar{A}3$.

Lemma A: For all $(x,z) \in X_{it}$, there exists $(\bar{x},0) \in X_i$, such that $\hat{C}_{it}(x,z) = C_{it}(\bar{x},0)$ and $x' < x$ implies $(x',z) \notin \hat{C}_{it}(x,z)$.

Proof:

(a) $A(x,z) \neq \emptyset$: Choose $(\bar{x},0) \in X_i$ such that, if $\bar{x}' < \bar{x}$, $(\bar{x},0) \in X_i$. By proposition 3.1 $C_i(\bar{x},0) = X_i$, hence $C_{it}(\bar{x},0) = X_{it}$, so $(\bar{x},0) \in A(x,z)$ for any $(x,z) \in X_{it}$.

(b) $A(x,z) \neq X_{it0}$: Let $(p,q) \in R^{L+M}$, $(p,q) > 0$, $px + qz < 1$, and $B = \{(\tilde{x},\tilde{z}) \in X_i | px + qz \leq 1\}$. B is compact and contains a maximal element $(\hat{x},\hat{z})$ w.r.t. $\succsim_i$ (by continuity and transivity). Hence $(x,z) \notin \text{Co}\ C_i(\hat{x},\hat{z}) \supset C_{it}(\hat{x},\hat{z})$, and by monotonicity: $C_{it}(\hat{x},\hat{z}) \supset C_{it}(\hat{x},0)$. Hence $(\hat{x},0) \notin A(x,z)$.

(c) $A(x,z)$ is closed: by continuity and monotonicity C_i has a closed graphe. The graphe of $\bar{C}_{it}$, being the intersection of the graphe of C_i and the closed set $\bar{X}_{it} \times \bar{X}_{it}$, is also closed. C_{it}, being the convex hull of a closed correspondence, is also closed. This implies that $A(x,z)$ is a closed set, since $A(x,z) = C_{it}^{-1}(x,z) \cap \bar{X}_{it0}$.

(d) Let $(x',0) \in A(x,z)$ and $(x'',0) \in \bar{X}_{it} \setminus A(x,z)$, and $x'' > x'$. By (a), (b), A3 and B4, these points exist. Choose $(\bar{x},0) = \lambda(x',0) + (1-\lambda)(x'',0)$ such that $(\bar{x},0) \in \text{Bnd}\ A(x,z)$. By (c), $(\bar{x},0) \in A(x,z)$.
Suppose for some $(\hat{x},0) \in A(x,z), (\hat{x},0) \succ_i (\bar{x},0)$. By transitivity, $A(x,z) \supset \{(\tilde{x},\tilde{z}) | (\hat{x},0) \succ_i (\tilde{x},\tilde{z})\} \cap \bar{X}_{it0}$, which is open by continuity. Hence an open neighbourhood of $(\bar{x},0)$ is also in $A(x,z)$, which is a contradiction. Hence $(\bar{x},0)$ is a maximal element of $A(x,z)$ w.r.t. $\succsim_i$.
For all $(\tilde{x},0) \in A(x,z)$: $C_{it}(\tilde{x},0) \supset C_{it}(\bar{x},0)$. Therefore $\hat{C}_{it}(x,z) = C_{it}(\bar{x},0)$.

(e) Suppose $x' < x$ and $(x',z) \in \hat{C}_{it}(x,z) = C_{it}(\bar{x},0)$. There exist (β_k, K) and $(x_k,z_k) \in \bar{C}_{it}(\bar{x},0)$, for $k \in K^+$, such that $(x',z) = \Sigma\beta_k(x_k,z_k)$ and $(x,z) = \Sigma\beta_k(x_k + (x-x'),z_k)$. By (strong) monotonicity:
$(x_k + (x-x') \succ_i (x_k,z_k) \succsim_i (\bar{x},0)$, for all $k \in K^+$. By continuity $(\bar{x}',0)$ exists, such that for $k \in K^+$: $(x_k + (x-x'),z_k) \succ_i (\bar{x}',0) \succ_i (\bar{x},0)$. Hence $(\bar{x}',0) \in A(x,z)$. But in (d) it was shown that $(\bar{x},0)$ is a maximal element of $A(x,z)$. So we have a contradiction.

<u>Proof of theorem</u> 4.2.

$\bar{B}1$ (transitivity): if $(x',z') \in \hat{C}_{it}(x,z)$, then $\hat{C}_{it}(x',z') \subset \hat{C}_{it}(x,z)$: for since $(x',z') \in C(\hat{x},0)$ for all $(\hat{x},0) \in A(x,z)$, $A(x',z') \subset A(x,z)$, by B1. Let $(x'',z'') \succsim_{it} (x',z') \succsim_{it} (x,z)$; then $(x'',z'') \in \hat{C}_{it}(x',z')$ and $(x',z') \in \hat{C}_{it}(x,z)$, hence $(x'',z'') \in \hat{C}_{it}(x,z)$.

$\bar{B}2$ (completeness): let $\hat{C}_{it}(x,z) = C_{it}(\bar{x},0)$ and $\hat{C}_{it}(x',z') = C_{it}(\bar{x}',0)$, applying lemma A. By B1 and B2, $C_{it}(\bar{x}',0) \subset C_{it}(\bar{x},0)$ and/or $C_{it}(\bar{x}',0) \supset C_{it}(\bar{x},0)$

$\bar{B}3$ (weak monotonicity) can be proved by applying the argument used to prove $\bar{A}3$, on each set $C_{it}(\bar{x},0)$.
(strong monotonicity): let $(x',z') \succsim_{it} (x,z)$ and $x'' > x'$.
By weak monotonicity of $\succsim_{it}$: $(x'',z') \in \hat{C}_{it}(x',z')$ and by lemma A: $(x'',z') \in \hat{C}_{it}(x',z')$. Hence by the definition of $\succsim_{it}$: $(x'',z') \succ_{it} (x',z')$, which implies by $\bar{B}1$: $(x'',z') \succ_{it} (x,z)$.

$\bar{B}4$ (continuity): $C_{it}(x,z)$ is closed since it is the convex hull of a closed set, bounded below.
Let $\hat{P}_{it}(x,z) = \{(\tilde{x},\tilde{z}) | (\tilde{x},\tilde{z}) \succ_{it} (x,z)\}$. It is to be proved that this set is open.
Let $Q = \{(\tilde{x},\tilde{z}) | (\tilde{x},\tilde{z}) \in \hat{C}_{it}(x,z)$ and $\exists \bar{x} < \tilde{x} : (\bar{x},\tilde{z}) \in \hat{C}_{it}(x,z)\}$.
Clearly Q is open. By $\bar{B}3$, we have $Q \subset \hat{P}_{it}(x,z)$.
Let $(x',z') \in \hat{P}_{it}(x,z)$. By lemma A, there exist $(\bar{x},0)$ and $(\bar{x}',0)$ such that $\hat{C}_{it}(x,z) = C_{it}(\bar{x},0)$ and $\hat{C}_{it}(x',z') = C_{it}(\bar{x}',0)$, where $(\bar{x}',0) \succ_i (\bar{x},0)$.
There exist (α,K) and $(x_k',z_k) \in C_{it}(x',z')$, for $k \in K^+$, such that $(x',z') = \Sigma\alpha_k(x_k',z_k')$. For $k \in K^+ : (x_k',z_k') \succ_i (\bar{x},0)$. By B5 there exist $(\tilde{x}_k,z_k') \in \bar{X}_{it}$, such that $\tilde{x}_k < x_k$ and by continuity (B4) there exist $\hat{x}_k$, such that $\tilde{x}_k \leq \hat{x}_k \leq x_k'$ and $(\hat{x}_k,z_k') \succ_i (\bar{x},0)$.
Hence $(\hat{x}_k,z_k') \in C_{it}(\bar{x},0)$ and $(\hat{x},z) = \Sigma\alpha_k(\hat{x}_k,z_k') \in \hat{C}_{it}(x,z)$.
Hence $(x',z') \in Q_i$. So $\hat{P}_{it}(x,z) = Q_i$ and is open

$\bar{B}5$ if $(x',z') \succsim_{it} (x,z)$, then $(x',z') \in \hat{P}_{it}(x,z)$, and it was shown above that $P_{it}(x,z) = Q_i$. Hence $(x',z) \in \hat{C}_{it}(x,z) \subset X_{it}$ with $x' < x$ exists.

$\bar{B}6$ (convexity) by definition.

B. Procedure for finding an Pareto-optimum in E:

Step 1: Fix an arbitrary value of t^0 and find an equilibrium in E_{t^0}. Determine $(x_{i\ell}^{(1)}, t^0, u^\ell)$ and $\alpha_i^{(1)} = -\frac{1}{t^0} z_i^{\ell(1)}$.

Step 2: Define an economy E_{α^1} (as in section 5.2) where the primary resources however are given by $(x_{i\ell}^{(1)}, t^{(1)}u^\ell)$, i.e. this bundle consists of claims on commodities and obligations to work $t^{(1)}$ units of labour of type ℓ, and the production set $\hat{Y}$ is determined from $\tilde{Y} = Y - \{y^{(1)}, v^{(1)}\}$ whereas the profit distribution $\tilde{\theta}_i(p,q)$ is adapted from $\theta_i(p,q)$, taking into account the change of the production set.
In the resulting Lindahl equilibrium the consumption $(x_{i\ell}^{(2)}, t^{(2)}u^\ell)$ is a best element from the budget set
$\{x,z \mid p^{(2)}x + q_{i\ell}^{(2)}z \leq p^{(2)}x_{i\ell}^{(1)} - q_{i\ell}^{(2)}t^{(1)} + \tilde{\theta}\,(p,q)\,\tilde{\pi}\,(p,q)$.
$(t^{(2)} - t^{(1)})$ is the increase or decrease of each consumer's obligation to work. The equilibrium production $(\tilde{y},\tilde{v})$ is the change in production, starting from $(y^{(1)}, v^{(1)})$, hence $(y^{(2)}, v^{(2)}) = (y^{(1)}, v^{(1)}) + (\tilde{y},\tilde{v})$.

Step 3: Define an economy $E_{t^{(2)}}$ (as in section 4.3) where mean resources of type i consumers consist of claims on commodities and obligations to work $(x_i^{(2)}, z_i^{(2)}) = \Sigma\alpha_{i\ell}^{(1)}\mu_i(x_{i\ell}^{(2)}, t^2u^\ell)$. The production set is $\tilde{Y} = Y - (y^{(2)}, v^{(2)})$ and $\tilde{\theta}_i(p,q)$ is adapted from θ_i.
The resulting equilibrium consumption $(x_i^{(3)}, z_i^{(3)})$ is a best element from $\{x,z \mid p^{(3)}x_i + q^{(3)}z_i \leq p^{(3)}x_i^{(3)} + q^{(3)}z_i^{(2)} + \tilde{\theta}_i(p,q)\,\tilde{\pi}\,(p,q)\}$, and its composition $(x_{i\ell}^{(3)}, t^{(3)}u^\ell)$ makes each consumer better off.
Type i consumers may change their labour supply, i.e. buy or sell obligations to work. The new distribution is $\alpha_{i\ell} = -\frac{1}{t^{(2)}} z_i^\ell$. The change in production is $(\tilde{y},\tilde{v})$ and $(y^{(3)}, v^{(3)}) = (y^{(2)}, v^{(2)}) + (\tilde{y},\tilde{v})$.

And so on untill the variables do not change any more, then a Pareto optimum is reached.

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