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A VECTOR REPRESENTATION OF MAJORITY VOTING.

(REVISED VERSION)

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A VECTOR REPRESENTATION OF MAJORITY VOTING.*)

1. INTRODUCTION.

In a number of articles [2,3,4,5] different conditions were presented that guarantee the consistency of the majority decision rule, i.e. these conditions ensure that a social preference relation derived by the majority decision rule from individual transitive preferences, is also transitive. It was pointed out by Sen [8] that the treatment of triples is sufficient, since the absence of intransitivity for each triple ensures the absence of intransitivity in larger sets. The conditions were summarized by Inada in [5], and he also proved that this set of conditions was complete. In [9] Sen and Pattainak discussed conditions that only guarantee quasi-transitivity of social preference. Inada pointed out in [6] that it is quite plausible to allow individual preferences to be quasi-transitive, e.g. in the case that the difference between alternatives α and γ and γ and β is not perceptible, where-as α is considered better than β. Therefore he presented conditions guaranteeing quasi-transitivity of social preference, given quasi-transitive individual preferences. In this note we propose a new method to handle problems in the field of majority decisions, which is based on a vector representation of individual and social preferences, which was proposed by May [7]. It is shown that the conditions for transitive and quasi-transitive social preference can be derived by an application of the separation theorem for convex sets.

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* I thank prof. Inada for his comment, which prevented and error that occurred in an earlier draught of this paper.

2. VECTOR REPRESENTATION OF PREFERENCES.

Let R be a preference relation with derived relations P (strict preference) and I (indifference). Any ordering of three alternatives α, β and γ can be represented by a three-dimensional vector $x = (x^1, x^2, x^3)$ with components that can only take on the values 0, 1 and -1, if we define as in [7]:

$$x^1 = \begin{cases} 1 & \text{if } \alpha P \beta \\ 0 & \text{if } \alpha I \beta \\ -1 & \text{if } \beta P \alpha \end{cases} \quad x^2 = \begin{cases} 1 & \text{if } \beta P \gamma \\ 0 & \text{if } \beta I \gamma \\ -1 & \text{if } \gamma P \beta \end{cases} \quad x^3 = \begin{cases} 1 & \text{if } \gamma P \alpha \\ 0 & \text{if } \gamma I \alpha \\ -1 & \text{if } \alpha P \gamma \end{cases} \qquad (2.1)$$

Obviously there are different ways to represent the preferences, but the representation given above seems the most suitable one. There exist exactly thirteen <u>transitive</u> preference orderings of α, β and γ. Their vector representations are denoted $v_0, v_1, v_2, \dots, v_{12}$ and constitute the set

$$T = \{v_0, v_1, v_2, \dots, v_{12}\} \qquad (2.2)$$

Further there exist six quasi-transitive preference orderings, which are not transitive, i.e. orderings for which the relation P is transitive, but not necessarily R. If we denote the vector representations of these orderings by $v_{13}, v_{14}, \dots, v_{18}$, the set of quasi-transitive vector representations, transitive ones included, is,

$$V = \{v_0, v_1, \dots, v_{12}, v_{13}, \dots, v_{18}\} \qquad (2.3)$$

		v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}
		transitive													quasi-transitive					
preference ordering		α	α	α	α	α	γ	γ	γ	γ	β	β	β	β	α	α	γ	γ	β	β
		I	P	P	P	I	P	P	P	I	P	P	P	I	P	I	P	I	P	I
		β	β	β	γ	γ	α	α	β	β	γ	γ	α	α	β	γ	α	β	γ	α
		I	P	I	P	P	P	I	P	P	P	I	P	P	I	P	I	P	I	P
		γ	γ	γ	β	β	β	β	α	α	α	α	γ	γ	γ	β	β	α	α	γ
															I	I	I	I	I	I
															α	α	γ	γ	β	β
preference vector	x^1	0	1	1	1	1	1	0	-1	-1	-1	-1	-1	0	1	0	0	-1	0	0
	x^2	0	1	0	-1	-1	-1	-1	-1	0	1	1	1	1	0	-1	0	0	1	0
	x^3	0	-1	-1	-1	0	1	1	1	1	1	0	-1	-1	0	0	1	0	0	-1
		v_0	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}
		T													V-T					
		V																		

We shall use k, ℓ, m to denote any permutation of the numbers 1, 2, 3, hence $k \neq \ell \neq m \neq k$. Now is is easily verified that the following properties are true

If $x \in V$

$$-1 \leqq x^k + x^\ell + x^m \leqq 1 \tag{2.4}$$

and

$$\begin{aligned} x^k &= -1 \Rightarrow 0 \leqq x^\ell + x^m \leqq 2 \\ x^k &= 0 \Rightarrow -1 \leqq x^\ell + x^m \leqq 1 \\ x^k &= 1 \Rightarrow -2 \leqq x^\ell + x^m \leqq 0 \end{aligned} \tag{2.5}$$

If $x \in T$

$$x^k = 0 \Rightarrow x^\ell + x^m = 0 \tag{2.6}$$

and

$$\begin{aligned} x^k &= -1 \Rightarrow x^\ell = 1 \text{ or } x^m = 1 \\ x^k &= 1 \Rightarrow x^\ell = -1 \text{ or } x^m = -1 \end{aligned} \tag{2.7}$$

Any of the alternatives α, β and γ can take on one of five different positions in the preference ordering $x \in V$:

- if the preference is transitive it can be the only best or worst element (strictly best or strictly worst) of the set $\{\alpha, \beta, \gamma\}$
- it can be one of the best or worst elements if the preference is transitive, or the only best or worst element if the preference is quasi-transitive (weakly best or worst)
- it can be medium (including the case of three equivalent alternatives).

These concepts are different from the ones used by Sen in [8] or [9]: "a weakly best" element e.g. is both "best" and "medium" according to Sens definition.

Now we can define a vector $w = (w^1, w^2, w^3)$, which gives the positions of each alternative:

$$w^1 = x^1 - x^3,\ w^2 = x^2 - x^1,\ w^3 = x^3 - x^2 \tag{2.8}$$

We have, as is easily verified, for α

if $w^1 = 2$, α is strictly best
if $w^1 = 1$, α is weakly best
if $w^1 = 0$, α is medium
if $w^1 = -1$, α is weakly worst
if $w^1 = -2$, α is strictly worst

The same holds for β and γ with respect to w^2 and w^3.

Note that for some permutation k, ℓ, m, we have either $w^k = x^k - x^\ell$ or $w^\ell = x^\ell - x^k$, depending on what kind of permutation is used.

The set

$$Y = \{y \in R^3 \mid -1 \leqq y^k \leqq 1, \quad \text{for } k = 1,2,3\} \tag{2.9}$$

is the set of points that lie on or within a cube. Let X be the subset of Y containing all vectors which have components 1, 0, -1,

$$X = \{x \in Y \mid x^k \in \{1, 0, -1\}, \text{ for } k = 1,2,3\} \tag{2.10}$$

Now

$$T \subset V \subset X \subset Y$$

and we have

$$T = \{x \in X \mid x \geqq 0 \text{ and } x \leqq 0\} \tag{2.11}$$

and

$$V = \{x \in X \mid (x^k = 1 \Rightarrow x^\ell + x^m \leqq 0) \quad \text{and} \quad (x^k = -1 \Rightarrow x^\ell + x^m \geqq 0)\} \tag{2.12}$$

Apart from $v_0 = 0$, T consists of all points of X on a closed curve on the edges of the cube Y; this curve does not intersect the positive and the negative orthant of the cube. (See fig. 1) The set V-T consists of the points that are in the center of the faces of the cube.

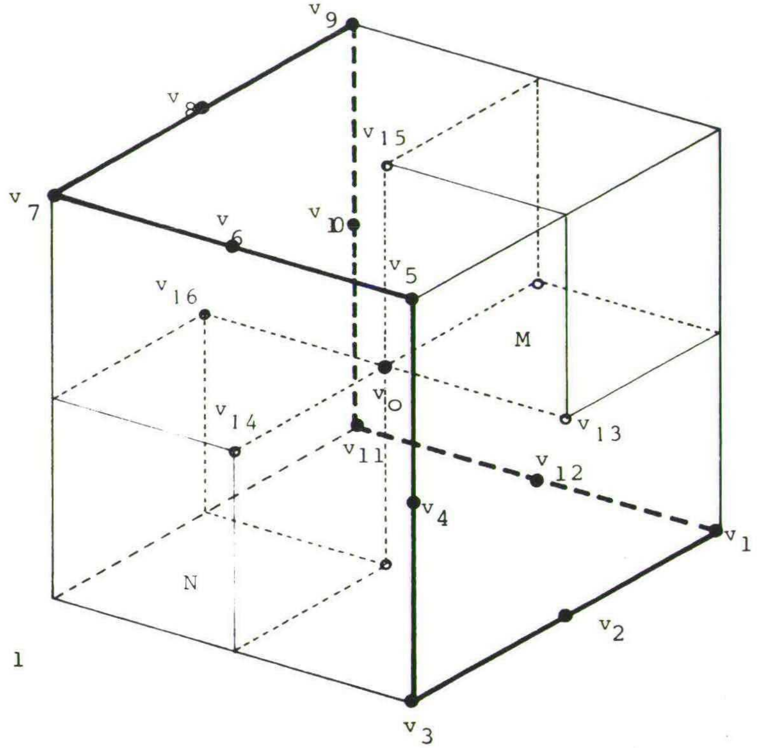


Fig. 1

The points of (X-T) represent preference orderings that are not transitive e.g. x = (1, 1, 1) means α P β, β P γ and γ P α and they are all points of x that lie in the positive or negative orthant of the cube; the points of X-V are not quasi-transitive.

3. VECTOR REPRESENTATION OF VOTING.

If every individual has a transitive or q.t. preference ordering of α, β and γ, voting means that every voter chooses one and only one point of V. If n is the number of voters, and n_i (i = 0, 1, 2,..., 18) is the number of voters that choose v_i, then voting can be represented by the numbers:

$$\lambda_i = \frac{n_i}{n} \text{, where } \sum_{i=0}^{18} \lambda_i = 1 \tag{3.1}$$

and the result of the voting procedure is given by a vector $y \in Y$

$$y = \sum_{i=0}^{18} \lambda_i v_i \tag{3.2}$$

representing the social ordering, which obviously can be represented by a point $x \in X$, if we define

$$\begin{aligned} x^k &= 1 \text{ if } y^k > 0 \\ x^k &= 0 \text{ if } y^k = 0 \\ x^k &= -1 \text{ if } y^k < 0 \end{aligned} \tag{3.3}$$

Let M_t and N_t be the positive and negative orthants of the cube Y

$$\begin{aligned} M_t &= \{ y \in Y \mid y \geq 0 \} \\ N_t &= \{ y \in Y \mid y \leq 0 \} \end{aligned} \tag{3.4}$$

If $y \in M_t \cup N_t$, the voting paradox occurs, if however $y \notin (M_t \cup N_t)$ the social ordering, represented by y, is <u>transitive</u>. Obviously the point x, derived from y by (3.3), fullfills

$$x \in (M_t \cup N_t) \Leftrightarrow y \in (M_t \cup N_t)$$

(Note that $0 \notin M_t$ and $0 \notin N_t$ and that $T \cap (M_t \cup N_t) = \emptyset$ but $V \cap (M_t \cup N_t) \neq \emptyset$)

Also we define

$$M_v = \{y \in Y \mid y \geq 0 \text{ and } y^k = 0 \Rightarrow (y^\ell > 0 \text{ and } y^m > 0)\} \subset M_t$$

$$N_v = \{y \in Y \mid y \leq 0 \text{ and } y^k = 0 \Rightarrow (y^\ell < 0 \text{ and } y^m < 0)\} \subset N_t$$

(3.5)

The points of $(M_v \cup N_v)$ are not (quasi-) transitive. Hence $V \cap (M_v \cup N_v) = \emptyset$ and if x is derived from y by (3.3):

$$x \in (M_v \cup N_v) \Leftrightarrow y \in (M_v \cup N_v)$$

Now if by imposing certain conditions it is ensured that the voting result y belongs to a set R, such that

$R \cap (M_t \cup N_t) = \emptyset$ or $R \cap (M_v \cup N_v) = \emptyset$

respectively, than the voting paradox is excluded or the social ordering is quasi-transitive. If there is no restriction on the votes λ_i, this is certainly not true, since in this case the set of all possible results is given by the convex hull of V:

$$\text{Conv } V = \{ y \in Y \mid y = \Sigma \lambda_i v_i \quad \text{for} \quad \lambda_i \geqq 0$$
$$\text{and } \Sigma \lambda_i = 1 \} \tag{3.6}$$

and

$$\text{Conv } V \cap (M_v \cup N_v) \neq \emptyset$$

also

$$\text{Conv } T \cap (M_t \cup N_t) \neq \emptyset \tag{3.7}$$

Obviously only rational vectors in Y are possible, if the number of voters is finite, but for sake of simplicity we permit all real vectors. If some of the λ_i are known to be zero, the voting results must be in the convex hull of the points that may have positive weights. As Inada [5], we call a set of preference vectors v_i that may have nonzero votes, a <u>list</u> $L \subset V$.
Hence

$$v_i \notin L \Rightarrow \lambda_i = 0 \tag{3.8}$$

Note that this does <u>not</u> mean that $\lambda_i > 0$ for all $v_i \in L$. If the set of possible results of a voting process is denoted R(L), R(L) is the convex hull of L, provided that there are no other conditions than (3.8)

$$R(L) = \text{Conv } L =$$

$$\{ y \in Y \mid \Sigma \lambda_i v_i = y, \quad \text{for} \quad \lambda_i \geqq 0, \lambda_i = 0 \text{ for}$$
$$v_i \notin L \quad \text{and} \quad \Sigma \lambda_i = 1 \} \tag{3.9}$$

We shall construct all list for the following cases

1) $L \subset T$, such that the voting must result in a transitive social ordering (see [5])

$$R(L) \cap (M_t \cup N_t) = \emptyset$$

2) $L \subset T$, such that the voting must result in a (quasi-) transitive social ordering (see [9])

$$R(L) \cap (M_v \cup N_v) = \emptyset$$

3) $L \subset V$, such that the voting must result in a (quasi-) transitive social ordering (see [6])

$$R(L) \cap (M_v \cup N_v) = \emptyset$$

4) Finally we shall introduce additional conditions, such that the quasitransitive points of $R(L) = \text{Conv } L$ in case 2 above are excluded. It appears that this can be done by requiring that at least one of the following conditions is fullfilled.

1) some λ_i, which will be defined in theorem 2, are positive
2) the votes for nonzero preferences cannot be divided into two equal groups. This conditions is fullfilled if the number of voters is odd.

If we denote the set of all voting results, that fullfill one of these conditions, by $R'(L)$, it appears that $R'(L) \cap (M_t \cup N_t) = \emptyset$ for all lists defined by Inada [5] for an odd number of voters.

4. THEOREMS.

In this section we shall present three theorems. These theorems provide a simple procedure to construct all lists for the cases discussed in the preceding section. They are essentially based on the separation theorem for convex sets. We first introduce some new concepts.
Let

$$P = \{p \in R^3 \mid p^1 + p^2 + p^3 = 1 \text{ and } p \geqq 0 \text{ and } p^k \neq 1\} \quad (4.1)$$

whereas

$$P^+ = \{p \in P \mid p > 0\} \quad (4.2)$$

(Note that the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) are not in P)

If we define

$$px = \sum_{k=1}^{3} p^k x^k \quad (4.3)$$

the set

$$F(p) = \{y \in Y \mid py = 0\} \quad (4.4)$$

divides the cube Y into two subsets (half-cubes)

$$G(p) = \{y \in Y \mid py \leqq 0\} \quad (4.5)$$

and

$$H(p) = \{y \in Y \mid py \geqq 0\} \quad (4.6)$$

where

$$F(p) = G(p) \cap H(p)$$

we have for $p \in P$

$$y \in M_v \Rightarrow py > 0 \quad \text{and} \quad y \in N_v \Rightarrow py < 0 \qquad (4.7)$$

If p is strictly positive ($p \in P^+$)

$$y \in M_t \Rightarrow py > 0 \quad \text{and} \quad y \in N_t \Rightarrow py < 0 \qquad (4.8)$$

Now if a set R(L) is strictly separated from M_v by one hyperplane and from N_v by another hyperplane, it cannot intersect M_v of N_v.

If p, q $\in$ P, and if

$$y \in R(L) \Rightarrow py \leqq 0 \text{ and } qy \geqq 0 \qquad (4.9)$$

we have

$$R(L) \cap (M_v \cup N_v) = \emptyset$$

In this case the voting leads to a <u>(quasi-) transitive</u> result. Thus we have the following lemma

<u>Lemma 1</u>

Let $L \subset V$, then

$$\exists\, p, q \in P\colon L \subset G(p) \cap H(q) \Rightarrow \text{Conv } L \cap (M_v \cup N_v) = \emptyset$$

Proof.

Let $x \in M_v$, hence $x^k > 0$, $x^\ell > 0$ and $x^m \geqq 0$.

Now we must have $p\,x \geqq 0$, since $p \geq 0$ and $x \geqq 0$.

Suppose $px = 0$, then $p^k x^k = p^\ell x^\ell = p^m x^m = 0$ and $p^k = p^\ell = 0$.
But then $p \in P$.

Hence $x \notin G(p)$ and $G(p) \cap M_v = \emptyset$.

In the same way it can be shown that $H(q) \cap N_v = \emptyset$.

Since $G(p) \cap H(q)$ is convex, conv $L \subset G(p) \cap H(q)$.

We have

$$\text{Conv } L \cap (M_v \cup N_v) \subset G(p) \cap H(q) \cap (M_v \cup N_v)$$

$$= (G(p) \cap H(q) \cap M_v) \cup (G(p) \cap H(q) \cap N_v) = \emptyset$$

If the vectors p and q are strictly positive the voting result must be transitive. Obviously this is possible only if $L \subset T$.

Lemma 2.

Let $L \subset V$, then

$$\exists\, p,\, q \in P^+ : L \subset G(p) \cap H(q) \Rightarrow \text{Conv } L \cap (M_t \cup N_t) = \emptyset$$

Proof.

Let $x \in M_t$, hence $x^k > 0$, $x^\ell \geqq 0$ and $x^m \geqq 0$.

If $p \in P^+$, $p > 0$ and therefore $px > 0$. So $x \notin G(p)$ and

$G(p) \cap M_t = \emptyset$.

The rest of the proof parallels the proof of Lemma 1.

The converse of Lemma 1 is also true. That means, that if some list cannot give a result which is not quasi-transitive, the points of this list can be separated from M_v and N_v by two hyperplanes of P.

Lemma 3.

If $L \subset V$,

$$\text{Conv } L \cap (M_v \cup N_v) = \emptyset \Rightarrow \exists p, q \in P : L \subset G(p) \cap H(q)$$

Proof.

a) Let $L \subset L' = L \cup \{0\}$. Now $\text{Conv } L' \cap (M_v \cup N_v) = \emptyset$.

For suppose $y' \in \text{Conv } L' \cap M_v$, where

$$y' = \sum_{v_i \in L} \mu_i v_i + \mu_o \cdot 0, \text{ then } y = \frac{1}{\sum \mu_i} (\sum \mu_i v_i) \in \text{Conv } L \cap M_v$$

and that is a contradiction.

Since both M_v and Conv L' are convex sets, by the separation theorem, there exists a vector $r \in R^3$ and a constant ϕ such that

$$x \in \text{Conv } L \Rightarrow r x \leqq \phi$$

and

$$x \in M_v \Rightarrow r x \geqq \phi$$

Since $v_o = 0$ is in the boundary of both sets, we have $\phi = 0$

and $r \geqq 0$, for otherwise we would have $rx = 0$ for some $0 \lneq x \in M_v$.

Now

$$p = \frac{1}{r^1 + r^2 + r^3} r, \text{ hence } p^1 + p^2 + p^3 = 1$$

In the same way we can find $q \in R^3$, such that $q \geq 0$ and $q^1 + q^2 + q^3 = 1$.

Hence

$$L \subset \text{Conv } L \subset \text{Conv } L' \subset G(p) \cap H(q)$$

b) However we cannot be sure that $p, q \in P$, since it is not excluded that $p^k = 1$, $p^\ell = p^m = 0$. Now suppose without loss of generality, that $p = (1, 0, 0)$. We show that

$$L \subset G\ (1, 0, 0) \Rightarrow \exists p' \in P : L \subset G(p')$$

There are three candidates for p', namely $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2}, 0)$.

Now it can easily be chequed that

$$G\ ((1, 0, 0)) \cap V - G\ ((\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4})) \cap V = \{(0,0,1),(0,1,0)\} = A$$

$$G\ ((1, 0, 0)) \cap V - G\ ((\tfrac{1}{2}, \tfrac{1}{2}, 0)) \cap V = \{(0,1,0),(0,1,-1)\} = B$$

$$G\ ((1, 0, 0)) \cap V - G\ ((\tfrac{1}{2}, 0, \tfrac{1}{2})) \cap V = \{(0,0,1),(0,-1,1)\} = C$$

and now

$$A \cap L = \emptyset \Rightarrow L \subset G\ ((\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}))$$

$$B \cap L = \emptyset \Rightarrow L \subset G\ ((\tfrac{1}{2}, \tfrac{1}{2}, 0))$$

$$C \cap L = \emptyset \Rightarrow L \subset G\ ((\tfrac{1}{2}, 0, \tfrac{1}{2}))$$

At least one of these intersections must be empty, for suppose $A \cap L \neq \emptyset$. If $(0, 1, 0) \in L$, we have $C \cap L = \emptyset$, since

$$\text{Conv}\ \{(0, 1, 0), (0, 0, 1)\} \cap M_v \neq \emptyset \quad \text{and}$$

$$\text{Conv}\ \{(0, 1, 0), (0, -1, 1)\} \cap M_v \neq \emptyset$$

If $(0, 0, 1) \in L$, we must have $B \cap L = \emptyset$.

By applying lemma's 1 and 3 we cannot yet construct all lists for case 1, since P is an infinite set. Therefore we define a new set $Q \subset P$, consisting of the seven points of the table below (see fig. 2)

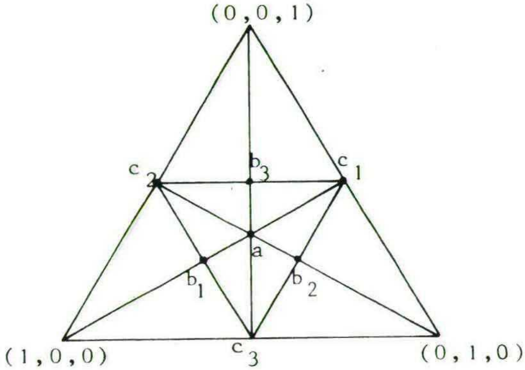
p^1	1/3	1/2	1/4	1/4	0	1/2	1/2
p^2	1/3	1/4	1/2	1/4	1/2	0	1/2
p^3	1/3	1/4	1/4	1/2	1/2	1/2	0
	a	b_1	b_2	b_3	c_1	c_2	c_3

$$Q = \{a, b_1, b_2, b_3, c_1, c_2, c_3\} \subset P \tag{4.10}$$

$$Q^+ = Q \cap P^+ = \{a, b_1, b_2, b_3\} \tag{4.11}$$

If some half-cube contains a set of points of V, there is some $q \in Q$, such that the half-cubes G(p) or H(q) also contain these points.

(fig. 2)*



Lemma 4.

$\forall\, p \in P,\ \exists\, q \in Q : G(p) \cap V \subset G(q) \cap V$ and $H(p) \cap V \subset H(q) \cap V$

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* Fig. 2. represents the set P', being the positive section of a plane in R^3 : $P' = \{ p \in R^3 \mid p \geq 0$ and $p^1 + p^2 + p^3 = 1 \}$.
The lines in this figure are the intersections of P' with the planes $\{ p \in R^3 \mid p\, v_i = 0 \}$ for $v_i \in V$.

The points of Q are those points of P in which the greatest number of planes intersect.

<u>Proof (for G(p)).</u>

If $p \in P$, there is some permutation such that one of the following three cases occurs:

a) $p^k = p^\ell = p^m > 0$

b) $p^k > p^\ell = p^m > 0$

c) $p^k \geqq p^\ell > p^m \geqq 0$

(a) Now $p = (1/3,\ 1/3,\ 1/3) \in Q$

(b) Choose $q^k = \frac{1}{2}$, $q^\ell = q^m = \frac{1}{4}$

Let $x \in G(p) \cap V$, hence $p^k x^k + p^\ell x^\ell + p^m x^m \leqq 0$

There are three possibilities

i. $x^k = -1$; since by (2.5) $0 \leqq x^\ell + x^m \leqq 2$, we have

$x^k + \frac{1}{2}(x^\ell + x^m) \leqq -1 + \frac{1}{2}.\ 0 \leqq 0$; so $x \in G(q)$

ii. $x^k = 0$; hence $\frac{p^\ell}{p^k}(x^\ell + x^m) \leqq 0$ and therefore

$x^k + \frac{1}{2}(x^\ell + x^m) \leqq 0$

iii. $x^k = 1$; hence $\frac{p^\ell}{p^k}(x + x^m) \leqq -1$. By applying (2.5) we have

$-2 \leqq x^\ell + x^m \leqq -\frac{p^k}{p^\ell} < -1$ and since $x \in V$, we must have

$x^k + x^\ell = -2$ and now $qx \leqq 0$.

(c) Choose $q^k = q^\ell = \frac{1}{2}$, $q^m = 0$

Let $x \in G(p) \cap V$: there are three possibilities:

i. $x^k = -1$; since $x^\ell \leqq 1$, we have $x^k + x^\ell \leqq 0$

ii. $x^k = 0$; hence $x^\ell \leqq -\frac{p^m}{p^\ell} < 1$ and since

$x \in V$, $x^\ell \leqq 0$ and therefore $x^k + x^\ell \leqq 0$

iii. $x^k = 1$; now $\frac{p^\ell}{p^k} x^\ell \leqq -1 - \frac{p^m}{p^k} x^m \leqq -1 + \frac{p^m}{p^k} < 0$;

since $x \in V$, $x^\ell = -1$ and therefore $x^k + x^1 = 0$.

Now we can prove our main theorem; if and only if some list gives (quasi-) transitive results only, it must be in the intersection of two half cubes generated by points of Q.

<u>Theorem 1.</u>

For $L \subset V$,

$$\text{Conv } L \cap (M_V \cup N_V) = \emptyset \iff \exists\, p, q \in Q: L \subset G(p) \cap H(q)$$

<u>Proof.</u>

$\Rightarrow$ By lemma 3, p' q' $\in$ P exist, and by lemma 4

$L \subset G(p') \cap V \cap H(p') \subset G(p) \cap H(q) \cap V$ for $p, q \in Q$.

$\Rightarrow$ Since $Q \subset P$, this follows from lemma 1.

Our second theorem shows that a list gives transitive results if and only if it is in the intersection of half-cubes generated by points of Q^+.

Theorem 2.

For $L \subset T$

$$\text{Conv } L \cap (M_t \cup N_t) \iff \exists\, p, q \in Q^+ : L \subset G(p) \cap H(q)$$

Proof.

$\Leftarrow$ Follows directly from lemma 2

$\Rightarrow$ By theorem 1, $p, q \in Q$ can be found such that $L \subset G(p) \cap H(q)$.
Now suppose without loss of generality, that $p = (\frac{1}{2}, \frac{1}{2}, 0)$.
There are three candidates for another p.

$$G((\tfrac{1}{2},\tfrac{1}{2},0)) \cap T - G((\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4})) \cap T = \{(1,-1,1),(1,-1,0))\} = A$$

$$G((\tfrac{1}{2},\tfrac{1}{2},0)) \cap T - G((\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{4})) \cap T = \{(-1,1,1),(-1,1,0))\} = B$$

$$G((\tfrac{1}{2},\tfrac{1}{2},0)) \cap T - G((\tfrac{1}{3}\,\tfrac{1}{3}\,\tfrac{1}{3})) \cap T = \{(1,-1,1),(-1,1,1))\} = C$$

Now $A \cap L = \emptyset \Rightarrow L \subset G\,(\frac{1}{2},\frac{1}{4},\frac{1}{4})$ etc.

And at least one of the intersections must be empty:

Suppose $C \cap L \neq \emptyset$

If $(1, -1, 1) \in L$, $\check{B} \cap L = \emptyset$, since

$(0, 0, 1) \in \text{Conv } \{(1, -1, 1),(-1, 1, 1) \cap M_t \neq \emptyset$

and $\text{Conv } \{(1, -1, 1),(-1, 1, 0)\} \cap M_t \neq \emptyset$

If $(1, -1, 0) \in L$, $C \cap L = \emptyset$

Finally we show that by introducing an additional condition, we can guarantee that the voting result is transitive, for lists of which only quasi-transitivity is ensured by theorem 1.

We can define for $L \subset T$ and $p, q \in P$, such that $L \subset G(p) \cap H(q)$.

$$R'(L) = \{y \in \text{Conv } L \mid \text{condition 1 or 2 holds}\}$$

where

condition 1: $\exists v_i \in L: \lambda_i p v_i < 0$ and $\exists v_j \in L: \lambda_j \, q v_j > 0$

condition 2: $\nexists K \subset L: 0 \notin K$ and $\sum_{v_i \in K} \lambda_i = \sum_{v_j \in L-K-\{0\}} \lambda_j$

Theorem 3.

If $L \subset T$ and $p, q \in P$

$$L \subset G(p) \cap H(q) \Rightarrow R'(L) \cap (M_t \cup N_t) = \emptyset$$

Proof.

1. Let condition 1 hold. Hence for some $v_i \in L$, we have $\lambda_i \, p v_i < 0$, therefore $\lambda_i > 0$ and $p \, v_i < 0$. Now for $y \in R'(L)$ holds

$$y = \sum \lambda_i v_i \text{ and } py = \sum \lambda_i p v_i < 0$$

and since $y \in M_t \Rightarrow py \geqq 0$, we have $y \notin M_t$.

In the same way it follows, applying $\lambda_i \, q \, v_i > 0$ that

$y \notin N_t$

2. Let condition 2 hold and suppose that $y \in R'(L) \cap M_t$

Since $R(L) \cap M_v = \emptyset$, we must have

$$y^k > 0, \; y^\ell = y^m = 0$$

Since $y \in \text{Conv } L$, we have $py \leqq 0$ and since $y \in M_t$, $py \geqq 0$,

hence $py = 0$ and this implies

$\lambda_i > 0 \Rightarrow p\ v_i = 0$

and since $p \varepsilon P$

$p^k = 0,\ p^\ell > 0,\ p^m > 0$

Suppose that for some $v_{io} \varepsilon L$, we have

$\lambda_{io} > 0$ and $v_{io}^{\ \ell} = 0$ and $v_{io} \neq 0$

Since $v_{io} \varepsilon T$, $v_{io}^{\ m} \neq 0$, but then

$pv_{io} = p^m\ v_{io}^{\ m} \neq 0$

and that is a contradiction. Hence we must have for $v_i \neq 0$,

$\alpha_i > 0 \Rightarrow v_i^\ell \neq 0$

Let $K = \{x_i \varepsilon L \mid v_i^\ell = 1\}$ and $L\text{-}K\text{-}\ \{0\} = \{x_i \varepsilon L \mid v_i^\ell = -1\}$

Now

$$\sum_{x_i \varepsilon L} \lambda_i\ v_i^\ell = 0$$

hence

$$\sum_{v_i \varepsilon K} \lambda_i = \sum_{v_i \not\varepsilon L-K-\{0\}} \lambda_i$$ but this is excluded by condition 2.

Therefore

$R'(L) \cap M_t = \emptyset$

In the same way we can show that

$R'(L) \cap N_t = \emptyset$

Corrollary.

If the number of voters choosing $v_i \neq 0$ is odd, condition 2 of theorem 2 is satisfied.

Proof.

$$\sum_{i=1}^{n} \lambda_i = \lambda \leqq 1$$

Then λn is an odd number, hence it is impossible that

$$\mu = \sum_{v_i \varepsilon K} \lambda_i = \sum_{v_i \varepsilon\ L-K-\{0\}} \lambda_i = \phi$$

since

$$\lambda = \mu + \phi \text{ and } \lambda n = \mu n + \phi n = 2\mu n$$

hence

$\mu n = \frac{1}{2} \lambda n$ is not a whole number.

5. LISTS AND CONDITIONS.

The theorems 1, 2 and 3 permit to construct the lists for the cases 1 - 4. Let for p, q ε Q

$$L(p,q) = G(p) \cap H(q) \cap V \qquad (5.1)$$

be the list associated with any combination of points of P. Any subset of L(p,q) is a quasi-transitive or transitive list. Obviously we are only interested in the maximal lists, i.e. lists such that they are not a proper subset of some other list. These maximal lists are found by defining all lists L(p, q) for p, q ε Q and by dropping the ones that are not maximal.

Case 1.

By theorem 2 lists are transitive if and only if they are generated by points of Q^+. There exist exactly 16 different combinations p, q ε Q^+ and it appears that these actually result in 16 different maximal lists of 4 types (I, II, III, IV below).

Cases 2 and 3.

By theorem 1, any L(p, q) for p, q ε Q is a quasi-transitive list. There can be at most 49 of these. However only 19 of them are maximal and different. These lists are of 5 different types, including the first 2 types of case 1. (I, II, V, VI, VII). The only difference between cases 2 and 3 is that for case 2 the points of V-T are dropped hence L(p, q) $\cap$ T is a list for case 2, if L(p, q) is a list for case 3.

So a list is quasi-transitive if and only if it is of one of these five types.

Case 4.

All lists of case 2 give transitive results if one of the conditions of theorem 3 is fullfilled.
The lists which are constructed are the same as those given by Inada [5], [6] and Sen and Pattainak [9].
They are derived in the rest of this section and summarised below.

	p	q			number of lists	case 1	case 2/3/4
I	a	a	Dichotomous preferences	Extremal restriction	1	x	x
II	b_k	b_k	Antagonistic preferences		3	x	x
III	b_k	b_ℓ	Connected echoic preferences		6	x	
IV	a / b_k	b_k / a	Disconnected echoic prefenrences		6	x	
V	c_k	c_k	Separated into two groups	Value restriction	3		x
VI	c_k	c_ℓ	Single peaked and single caved preferences		6		x
VII	c_k / b_k	b_k / c_k	Limited agreement		6		x

It will be shown that the remaining combinations (b_k, c_ℓ) and (a, c_k) do not generate maximal lists.

I. Dichotomous preferences.

$$L(a,\ a) = G(a) \cap H(a) \cap V$$

$$= \{x \in V \mid 1/3\ (x^1+x^2+x^3) \leqq 0 \text{ and } 1/3\ (x^1+x^2+x^3) \geqq 0\}$$

$$= \{x \in V \mid x^1+x^2+x^3 = 0\}$$

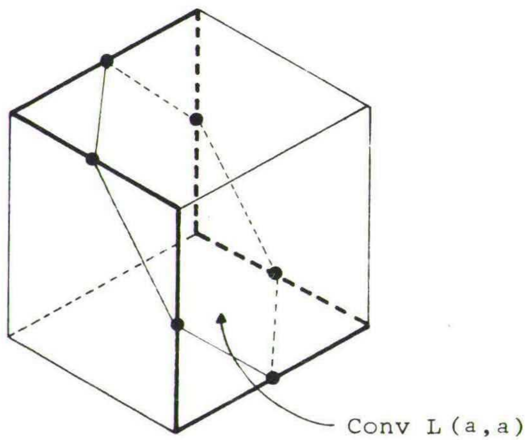
$$= \{x \in T \mid \exists k : x^k = 0\}$$

Hence we can state for this list the following condition:

Each voter has transitive preferences and considers at least two of the alternatives equivalent.

This is called the condition of dichotomous preferences, since each voter classifies the three alternatives in two groups such that he is indifferent between the alternatives within the group. See Inanda [5]. There is only onelist of this type (see fig. 3)

Fig. 3



II. Antagonistic preferences.

$$L(b_k, b_k) = \{x \in T \mid b_k x \leq 0 \text{ and } b_k x \geq 0\}$$

$$L(b_k, b_k) = \{x \in T \mid 2\, x^k + x^\ell + x^m = 0\}$$

$$= \{x \in T \mid x^k = 0 \text{ or } x^k = -x^\ell = -x^m\}$$

Now depending on the kind of permutation, $w^k = x^k - x^\ell$ or $w^k = x^k - x^m$. Hence we also have

$$L(b_k, b_k) = \{x \in T \mid x^k = 0 \text{ or } (w^k = 2 \text{ and } w^\ell = -2) \text{ or } (w^k = -2 \text{ and } w^\ell = 2)\}$$

and we can state that a list is of this type, if the following condition is satisfied:

<u>All voters have transitive preferences and they either consider two of the three alternatives equivalent or one of them is strictly best and the other is strictly worst.</u>

There are three different lists of this type (k = 1, 2, 3)

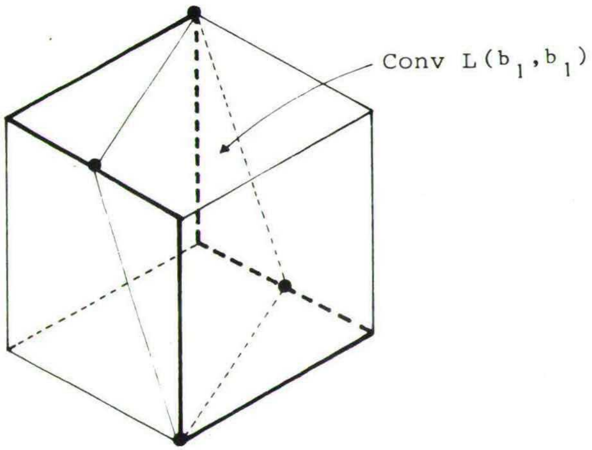


Fig. 4

III. Connected echoic preferences.

$$L(b_k, b_\ell) = \{x \in T \mid b_k\, x \leqq 0 \text{ and } b_\ell\, x \geqq 0\}$$

$$= \{x \in T \mid x^k + \tfrac{1}{2}(x^\ell + x^m) \leqq 0 \text{ and } x^\ell + \tfrac{1}{2}(x^k+x^m) \geqq 0\} \quad \text{(i)}$$

$$= \{x \in T \mid x^k \leqq 0 \quad \text{and} \quad x^\ell \geqq 0\} \quad \text{(ii)}$$

It is easily proved that (i) and (ii) are equivalent:

(i) ⇒ (ii) : from $b_k\, x \leqq 0$ and $-b_\ell\, x \leqq 0$, it follows:

$3/4\ x^k + 1/4\ x^m \leqq 0$, hence $x^k \leqq -1/3\ x^m \leqq 1/3$ and this

implies $x^k \leqq 0$.

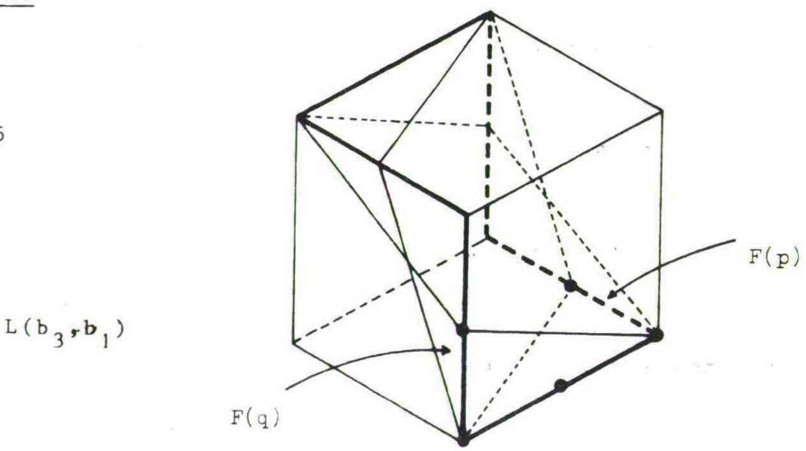
(ii) ⇒ (i): by (2.6), $x^k = 0 \Rightarrow x^\ell + x^m = 0$, hence $b_k\, x = 0$;

and, $x^k = 1 \Rightarrow x^\ell + x^m \leqq 0$, hence $b_k\, x \leqq 0$.

If $k = 1$ and $\ell = 2$, we have $\beta\ R\ \alpha$ and $\beta\ R\ \gamma$. So for these lists the following condition holds:

<u>All voters have transitive preferences and there is one alternative that all voters consider at least as good as the other two or that all voters consider not better than the other two.</u>

Fig. 5



There are six different lists of this type.

This condition, together with IV below, was called "the case of echoic preferences" by Inada. To discriminate III and IV we added "connected" and "disconnected". The reason for this terminology can be understood by comparing the figures 5 and 6.

The conditions in (ii) can also be written

$$x^k = x^\ell = 0 \quad \text{or} \quad x^k - x^\ell < 0$$

and since we have either $w^k = x^k - x^\ell$ or $w^\ell = x^\ell - x^k$, we have

$$L(b_k, b_\ell) = \{x \in T \mid w^k \leqq 1 \quad \text{or} \quad x = 0\}$$

or

$$L(b_k, b_\ell) = \{x \in T \mid w^k \geqq 1 \quad \text{or} \quad x = 0\}$$

IV. Disconnected echoic preferences.

$$L(a, b_k) \quad \text{or} \quad L(b_k, a)$$

where

$$L(a, b_k) = \{x \in T \mid x^k + x^\ell + x^m \leqq 0 \text{ and } x^k + \tfrac{1}{2}(x^\ell + x^m) \geqq 0\}$$

$$= \{x \in T \mid x^k + x^\ell + x^m \leqq 0 \text{ and } x^k \geqq 0$$

$$= \{x \in T \mid x^k = 0 \quad \text{or} \quad (x^k - x^\ell \geqq \ell \text{ and } x^k - x^m \geqq 1)\}$$

and this means that, depending on the permutation, we have

$$L(a,b_k) = \{x \in T \mid x^k = 0 \text{ or } (w^k \geq 1 \text{ and } w^m \leq 1)\}$$

or

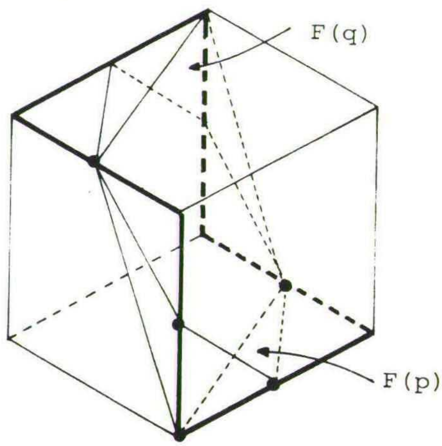
$$L(a,b_k) = \{x \in T \mid x^k = 0 \text{ or } (w^k \geq 1 \text{ and } w^\ell \leq 1)\}$$

and we can state that must hold:

<u>All voters have transitive preferences and of two alternatives for all voters either the first is best and the second is worst, or both are equivalent.</u>

There are six lists of this type.

Fig. 6



$L(a,b_1)$

V. Separated into two groups.

$$L(c_k,c_k) = \{x \in V \mid c_k\, x \leq 0 \text{ and } c_k\, x \geq 0\}$$

$$= \{x \in V \mid x^\ell + x^m = 0\}$$

$$= \{x \in V \mid x^\ell = -x^m\}$$

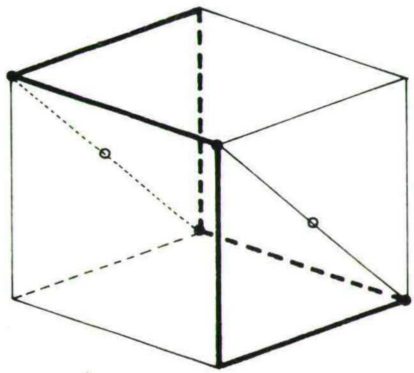
$$= \{x \in T \mid x^\ell = -x^m\} \cup \{x \in V\text{-}T \mid w^m = 0\}$$

and therefore a list of this type must satisfy the following condition:

<u>Each voter either considers all three alternatives equivalent, or there is one alternative which is strictly best or strictly worst for voters with transitive preferences and which is medium for voters with quasi-transitive preferences.</u>

This condition is generally called "not medium" which obviously within our definition of this concept is only true for transitive preferences. Within our definition of worst on best also weakly worst and weakly best as excluded for transitive preferences.

Fig. 7



$L(c_1, c_1)$

VI. Single peaked and single caved preferences.

$$L(c_k, c_\ell) = \{x \in V \mid c_k\, x \leq 0 \text{ and } c_\ell\, x \geq 0\}$$

$$= \{x \in V \mid x^\ell + x^m \leq 0 \text{ and } x^k + x^m \geq 0\} \qquad \text{(i)}$$

$$= \{x \in V \mid x^\ell = -1 \text{ or } x^k = 1 \text{ or } x = 0\} \qquad \text{(ii)}$$

The two last expressions are equivalent:
(i) ⇒ (ii): we have $x^\ell \leq -x^m \leq x^k$. Hence if $x^\ell = 1$, it follows

$x^k = 1$ and if $x^\ell = 0$, also $x^k = 0$, unless $x = 0$

(ii) ⇒ (i): if $x^{\ell} = -1$, $x^{\ell} + x^{m} \leq 0$ and by (2.5) $x^{k} + x^{m} \geq 0$

if $x^{k} = 1$, $x^{k} + x^{m} \geq 0$ and by (2.5) $x \; + x^{m} \leq 0$

There exist six lists of this type.

Now

$$L(c_k, c_\ell) = \{x \in T \mid x^{\ell} - x^{k} \leq 0\} \cup \{x \in V-T \mid x^{\ell} - x^{k} = -1\}$$

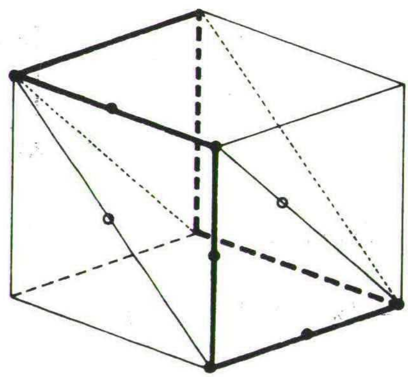
Suppose that $w^{\ell} = x^{\ell} - x^{k}$, then

$$L(c_k, c_\ell) = \{x \in T \mid w^{\ell} \leq 0\} \cup \{x \in V-T \mid w^{\ell} = -1\}$$

and we have

<u>All voters with transitive preference considers one alternative not best (worst) and all voters with q.t.preferences consider this alternative worst (best)</u>

Fig. 8



$L(c_1, c_2)$

<u>VII. Limited agreement.</u>

$L(b_k, c_k)$ and $L(c_k, b_k)$

$$L(b_k, c_k) = \{x \in V \mid b_k\, x \leq 0 \text{ and } c_k\, x \geq 0\}$$

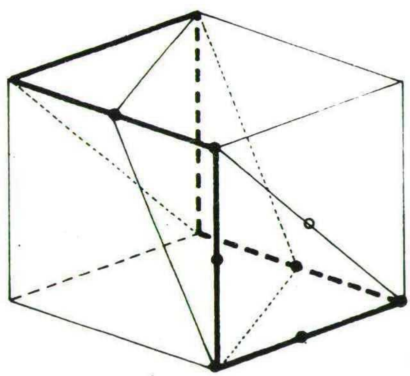
$$= \{x \in V \mid x^k + \tfrac{1}{2}(x^\ell + x^m) \leq 0 \text{ and } x^\ell + x^m \geq 0\}$$

$$= \{x \in T \mid x^k \leq 0\} \cup \{x \in V-T \mid x^k = -1\}$$

Hence it is required that:
<u>All voters with transitive preferences consider one alternative not better than a second, whereas voters with q.t.preferences do prefer the second to the first.</u>

There are six lists of this type.

Fig. 9



$L(c_1, b_1)$

It remains to prove that

1) $L(b_k, c_\ell) = \{x \in V \mid x^k + \tfrac{1}{2}(x^\ell + x^m) \leq 0 \text{ and } x^k + x^m \geq 0\}$

$$\subset \{x \in V \mid x^k + x^\ell \leq 0 \text{ and } x^k + x^m \geq 0\}$$

$$= L(c_m, c_\ell)$$

since

$$x^k + x^\ell = 2x^k + x^\ell + x^m - x^k - x^m \leq 0$$

2) $L(a,c_k) = \{x \in V \mid x^k + x^\ell + x^m \leq 0 \text{ and } x^\ell + x^m \geq 0\}$

$\subset \{x \in V \mid x^k + \frac{1}{2}(x^\ell + x^m) \leq 0 \text{ and } x^\ell + x^m \geq 0\}$

$= L(b_k, c_k)$

since

$$x^k + \tfrac{1}{2}(x^\ell + x^m) = x^k + x^\ell + x^m - \tfrac{1}{2}(x^\ell + x^m) \leq 0$$

Note also that

$$L(b_k, b_k) \subset L(b_k, c_k)$$

and

$$L(a, b_k) \subset L(c_k, b_k)$$

Finally we note that condition 2 of theorem 3 can be applied to the lists of type V, VI and VII and condition 1 to type VI and VII.

Let e.g.

$$L(c_1, c_2) \cap T = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_o\}$$

then the condition is satisfied if

$$\lambda_2 + \lambda_3 + \lambda_4 > 0 \text{ and } \lambda_4 + \lambda_5 + \lambda_6 > 0$$

since

$$c_\ell v_i < 0 \text{ for } i = 2, 3, 4$$

$$c_2 v_i > 0 \text{ for } i = 4, 5, 6.$$

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A.K.

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