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AN APPLICATION OF GAME THEORY TO A PROBLEM OF CHOICE BETWEEN PRIVATE AND PUBLIC TRANSPORT.

H.N. Weddepohl

August 1973

1. INTRODUCTION.

In this paper is discussed the choice between private and public transport, where both modes of transport use the same scarce capacity, the road system. The simplified problem is the following: a road between two places A and B is passed daily by a certain number of people, owning a car. They can use their own car, or take a bus. Their decisons determine the number of vehicles on the road, assuming that the number of buses is ajusted to the number of passengers (one bus for p passengers).
The preferences of each individual depend on three factors:
- their own choice
- the choices of the other individuals
- the costs of both modes of transport.

These costs are given besides that an authority can influence the cost of transport for private cars, by raising a toll for private cars only, and for buses by giving a subvention. These costs are given.
The problem is defined as a game in normal form, without side-paymends. The preferences of the passengers over the outcomes of the game are such that they prefer few traffic over much traffic, every one prefers his car over the bus but some people prefer a bus on a quiet road over their car on a highly used road. It is shown that the equilibrium of the game may be Pareto inefficient and that the core of the game is not empty.
In section 9 side payments are introduced and in section 10 tolls and subventions are considered.

2. THE GAME.

There are n players, the n individuals who choose to use the bus (0), or their car (1), $I = \{1,2,\ldots,n\}$.
Their set of strategies is $X_i = \{0,1\}$ for $i \in I$ and hence the set of strategy combinations is

$$X = \prod_I X_i$$

so $$x = (x_1,\ x_2,\ldots,x_n) \in X.$$

With each $x \in X$, correspond
- the number of cars used:

$$a(x) = \sum_I x_i$$

- the number of buses used: the number of passengers being $n - a(x)$,

$$b(x) = \frac{1}{p}\ n - a(x) = \frac{n}{p} - \frac{1}{p} \sum_I x_i$$

  p is the number of seats. So we assume for sake of simplicity that v can be a whole number + a fraction.
- the number of vehicles $v(x) = a(x) + b(x)$:

$$v(x) = \frac{n}{p} + \frac{p-1}{p} \sum_I x_i$$

It is also usefull to define for each $i \in I$ a number v_i, being the number of vehicles, apart from i's own-choice:

$$v_i(x) = v(x) - x_i - (1-x_i)\ \frac{1}{p} = \frac{n-1}{p} + \frac{p-1}{p} \sum_{I \setminus i} x_j$$

For each $i \in I$ we define a set of outcomes of i, containing those consequences of the combined strategies that are

relevant for i's choice. We assume that these sets only contain two factors: i's own strategy x_i and the number of vehicles, depending on the strategies of the other players, v_i.
For $i \in I$

$$Y_i = \{(x_i, v_i) \mid x_i \in X_i \text{ and } v_i \in R^+\} = X \times R^+$$

This set is larger than the set of possible outcomes for i, the outcomes that can actually occur in the game, i.e., there exists $x \in X$, such that $v_i = v_i(x)$.
Let $\tilde{Y}_i$ be this set of possible outcomes for i:

$$\tilde{Y}_i = \{(x_i, v_i) \in Y_i \mid \frac{n-1}{p} \leq v_i \leq n-1 \text{ and}$$

$$p.v_i \text{ is a whole number}\}$$

The total set of outcomes is the set

$$Y = \{(y_1, y_2, \ldots, y_n) \mid y_i = (x_i, v_i),$$

$$x_i \in X_i, v_i, = v_i(x)\}$$

Obviously to each $y \in Y$ corresponds one and only one $x \in X$.

On each set Y_i is defined a preference relation $\gtrsim_i$. So the preferences are assumed to depend only on i's strategy and on the number of vehicles. Hence it is implicitely assumed that preferences do not depend on the composition of the set of bus passengers.
So we also have preferences defined on $\tilde{Y}_i \subset Y_i$ and therefore a preference relation on X is implied:
for $x, x' \in X$, (x_i, v_i), $(x_i', v_i') \in \tilde{Y}_i$, we have

$$x \gtrsim_i x' \Leftrightarrow (x_i, v_i(x)) \gtrsim_i (x_i', v_i(x'))$$

3. PREFERENCES.

The following assumptions are made for the preferences $\succsim_i$ on Y_i.

Assumption A (preordening)

For all i $\succsim_i$ on Y_i is transitive and complete.

Hence $(x_i, v_i) \succsim_i (x_i', v_i')$ or $(x_i, v_i)_i \precsim (x_i', v_i')$ for

$(x_i, v_i), (x_i', v_i') \in Y_i$; $(x_i, v_i) \succsim_i (x_i', v_i')$ and

$(x_i', v_i') \succsim_i (x_i'', v_i'') \Rightarrow (x_i, v_i) \succsim_i (x_i'', v_i'')$

Assumption B

For all i, $x_i \in X_i$:

If $\frac{n-1}{p} \leq v_i < v_i'$, then $(x_i, v_i) \succ_i (x_i, v_i')$

This means that each individual strictly prefers a situation with less vehicles, provided that $v_i \geq \frac{n-1}{p}$, the smallest number that can actually occur in the game. This is assumed to hold both if i is a driver or a bus passenger. This may be defended by the argument, that in both means of transport the velocity is influenced negatively by the quantity of traffic. Note that is not excluded that i is indifferent between two numbers of vehicles, if $v_i < \frac{n-1}{p}$, which is impossible.

Assumption C

For all $i \in I$

$$(1, v_i) \succ_i (0, v_i)$$

For any given quantity v_i, all i prefer the car over the bus.

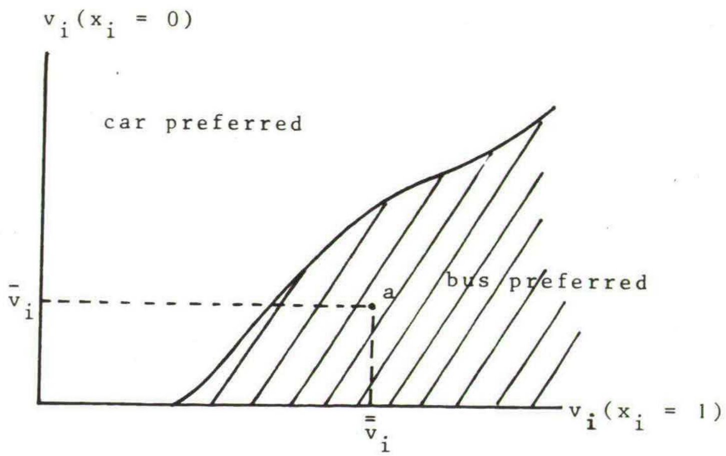
This means that the cost structure of both modes of transport is such that the bus is not very cheap, nor the car is very expensive.
This might be a plausible assumption if both modes of transport are managed at cost-price. It seems ressonable to argue that anyhow there exists some cost structure (including toll, taxes and subventions) such that the assumption holds and we take this structure as a point of departure.

<u>Assumption D</u>

For all v_i, there exists v_i', such that

$$(0,\ v_i) \succ_i (1,\ v_i')$$

So each individual prefers to be a bus passenger at some v_i above being a driver at some larger v_i'.
(that $v_i' > v_i$ directly follows from assumption D).
The preference relation is depicted in figure 1, which represents the graphe of $\succsim_i$: the horizontal axis represents the number of vehicles, for $x_i = 1$, the vertical axis gives v_i, with $x_i = 0$. So in point a, we have $(0,\ \bar{v}_i) \succ_i (1,\ \bar{\bar{v}}_i)$.



<u>Fig. 1</u>

From the preference relation a utility function $u_i : Y_i \to R$, can be derived, where

$$u_i(x_i, v_i) \geq u_i(x_i', v_i') \Leftrightarrow (x_i, v_i) \gtrsim_i (x_i', v_i')$$

This function is depicted in figure 2

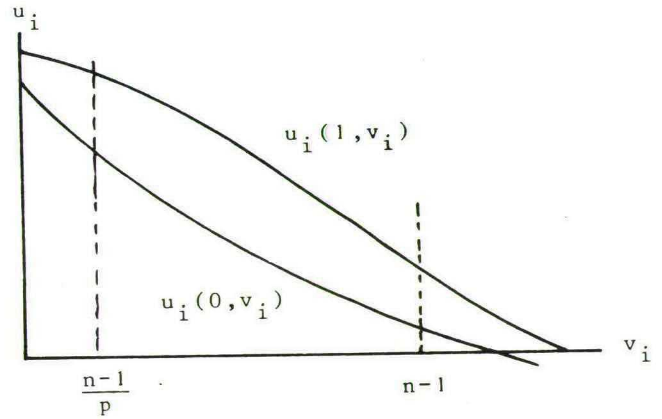


Fig. 2

Utility increases with decreasing v_i; for the same v_i an outcome with $x_i = 1$ is preferred to an outcome with $x_i = 0$.

4. SOLUTION CONCEPTS.

In this part of the paper we will consider two solution concepts, the equilibrium and the core with respect to the game defined above. Besides this we consider Pareto efficiency.

Definition 4.1.

A strategy combination $\bar{x} \in X$ is an equilibrium, if for all i

$$(\bar{x}_i, v_i(\bar{x})) \gtrsim_i (x_i, v_i(\bar{x}))$$

i.e. every i prefers the mode of transport of the solution above the other transport mode, given $\bar{v}_i$.

Definition 4.2.

An outcome $y \in Y$ is Pareto efficient, if there exists no other outcome $y' \in Y$, such that

$$\text{for all } i \quad y_i \gtrsim_i y'_i \qquad (\text{for } y_i = (x_i, v_i))$$
$$\text{for some } i \quad y_i \succ_i y'_i$$

Definition 4.3.

An outcome $y \in Y$ is blocked via a coalition $S \subset I$ if there exist strategies x_i for all $i \in S$, such that for all strategies x_j for $j \notin S$

$$\forall\ i \in S: (x_i, v_i\ (x_S, x_{I/S})) \gtrsim_i y_i \quad (x_S = (x_i) \text{ for } i \in S)$$

$$\exists\ i \in S: (x_i, v_i\ (x_S, x_{I/S})) \succ_i y_i$$

The outcome is blocked, if the coalition has a strategy combination, which gives a better outcome, against every strategy of the other players. Given the preferences as fixed by the assumptions, the most unfavorable strategy of the others for all $i \in S$, is that all $j \notin S$ play $x_i = 1$.
So the possibility of blocking requires that there exists a strategy combination x, where $x_j = 1$ for $j \notin S$, such that

$$\forall\ i \in S \qquad (x, v_i(x)) \gtrsim_i y_i$$

$$\exists\ i \in S \qquad (x, v_i(x)) \succ_i y_i$$

Definition 4.4.

The core is the set of unblocked outcomes $y \in Y$.

5. EQUILIBRIUM.

The equilibrium in this game is a very simple one: everybody takes his car. In fact this follows directly from assumption C: for given v_i, everybody prefers $x_i = 1$. Hence

Theorem 5.

The strategy combination $x^e = (1,1,\dots,1)$ is an equilibrium and it is the only one.

Proof: Let $i \in I$. If for all $j \neq i$, $x_j^e = 1$, then $v_i = n-1$ and now $(1,n-1) \succ_i (0,n-1)$.
Suppose $x \neq x^e$ would also be an equilibrium. Then for some i, $x_i = 0$. But this would imply $(0, v_i(x)) \succsim_i (1, v_i(x))$, which contradicts assumption C.

The equilibrium outcome $\bar{y}^e$ (where $\bar{y}_i = (1,n-1)$, needs not be Pareto efficient. This can easily be seen from the following example.
Let all i have the same utility function, linear in v_i:

$$u_i(x_i, v_i) = A + Cx_i - Bv_i \quad (A > 0,\ B > 0,\ C > 0)$$

Now $u(1,n-1) = A + C - B(n-1)$ and $u(0, \frac{n-1}{p}) = A - B\frac{n-1}{p}$
Provided that $C < B\frac{p-1}{p}(n-1)$, $u_i(0, \frac{n-1}{p}) > u_i(1, (n-1))$

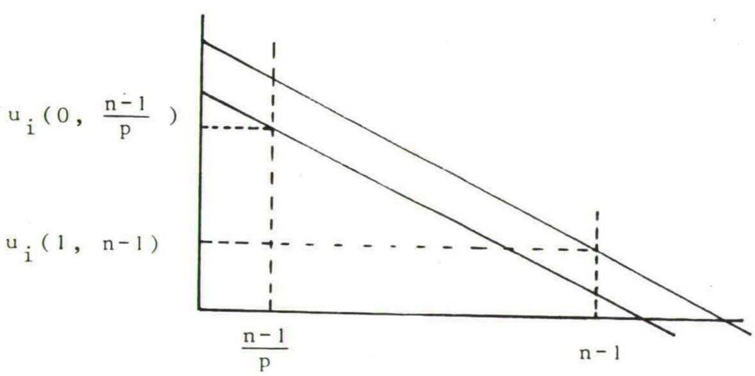


Fig. 3

This example shows that the equilibrium needs not be Pareto efficient. When does this occur?
Some individuals will never prefer the bus above y^e (the case that $n-1 < n_0$ in figure 4), other individuals will prefer $(0, v_i)$ over the equilibrium outcome $(1, n-1)$, provided that v_i is sufficiently small.

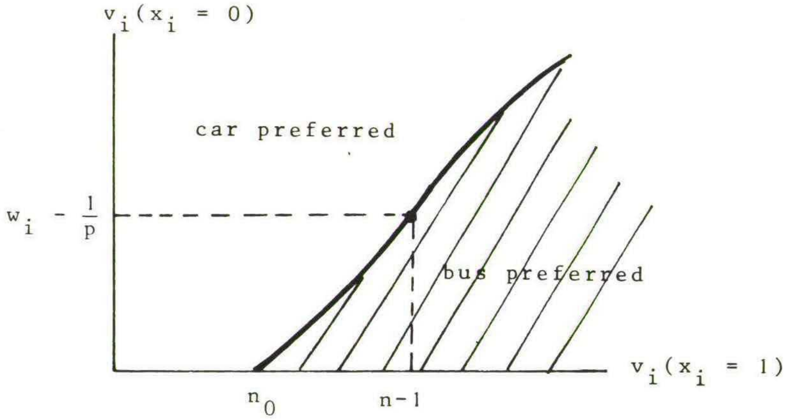


Fig. 4

Let $w_i = \max \{w/(0, w-\frac{1}{p})\} \gtrsim_i (1, n-1)$, now $w_i - \frac{1}{p}$ is the maximal number of vehicles of other players, such has i prefers the bus over the equilibrium outcome y^e.
Let

$$J(w) = \{i \mid w \leqq w_i\}$$

be the set of all players who prefer the bus at $w - \frac{1}{p}$ and let $\alpha(w)$ be their number: $\alpha(w) = |J(w)|$

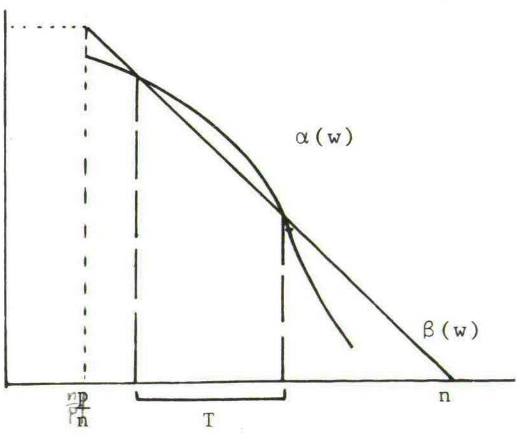


Fig. 5

$\alpha(w)$ is a non decreasing function defined on $[0,n]$: the more vehicles, the less people prefer $(0, w-\frac{1}{p})$ over $(1, n-1)$ In order to realise w vehicles,

$$\beta(w) = \frac{p}{p-1} \quad (n-w) \qquad \left(\text{for } \frac{n-1}{p} \leq w \leq n-1\right)$$

players must take the bus.
Let W be the set all values of w, such that the number of potential buspassengers is not lower than the number of necessary buspassengers.

$$W = \{w \mid \alpha(w) \geq \beta(w), \frac{n}{p}\cancel{\frac{p}{n}} \leq w \leq n\}$$

Now for any $w \in W$, there exists a subset

$$J'(w) \subset J(w) \text{ such that } |J'(w)| = \beta(w).$$

Now if $y_i = (0, w-\frac{1}{p})$ for $i \in J'(t)$ and

$y_i = (0, w-1)$, then $y = (y_1, y_2, \ldots, y_n) \in Y$ and we have

$$i \in J'(t) \qquad (0, w-\frac{1}{p}) \succsim_i (1, n-1)$$

$$i \notin J'(t) \qquad (1, w-1) \succ_i (1, n-1)$$

Both bus passengers and drivers are better off in the outcomes of W. Obviously the outcome y^e is blocked via the coalition $J'(w)$.
Only if $W = \emptyset$, i.e. if at no w, a sufficient number of players can be found who prefer the bus over the equilibrium, then the equilibrium is Pareto efficient.

6. THE CORE.

In section 5 we defined the set w, if W is empty, then x^e is Pareto optimal and the only element in the core:

6.1. Theorem

If $W = \emptyset$ then x^e is the only element in the core.

Proof: a. x^e is in the core: since W is empty, we have for all $\frac{n}{p} \leq w \leq n$: $\alpha(w) < \beta(w)$, hence no coalition can contain a sufficient number of individuals to block x^e.

b. Let $x \neq x^e$. Now x is blocked: $\alpha(v(x)) < \beta(v(x))$ by assumption. So there must exist at least one player for whom $(1, n-1) \succ_i (0, v(x) - \frac{1}{p})$ and $x_i = 0$. So the coalition $\{i\}$ blocks x.

However if $W \neq \emptyset$, x^e is not Pareto efficient and therefore certainly not in the core. Not all outcomes corresponding to elements $w \in W$ are in the core, since some of these can be blocked via coalitions containing more bus passengers. The core is not empty since it certainly contains the outcome corresponding to the smallest value of W.

Let

$$w^o = \min\{w \in W\} = \min\{w \mid \alpha(w) \geqq \beta(w)\}$$

This minimum exists for some $w^o = n - \beta(w_0)\,\frac{p-1}{p}$, with $\beta(w^o)$ being a whole number. Now

$$\alpha(w^o) = |J(w^o)| = \beta(w^o)$$

(if $\alpha(w^o) > \beta(w^o)$, then $\alpha(w^o - \frac{1}{p}) \geq \alpha(w^o) \geq \beta(w^o) + 1 \gtreqqless \beta(w^o - \frac{1}{p})$)

$= \beta(w^o) + \frac{1}{1-p}$

Then $(x^o, w^o) \in Y$, such that $x_i^o = 0$ for $i \in J(w^o)$ and $x_i^o = 1$ for $i \notin J(w^o)$ is in the core.

Theorem 6.2.

(x^o, w^o) is in the core.

Proof: Assume x^o is blocked via a coalition S, by some solution z, where $z_j = 1$ for $j \notin S$.
Assume first $v(z) > w^o$. It is certain, that $z \neq x^e$, since $x_i^o \geq_i x_i^e$ for all i. So S contains at least one i, for whom $z_i = 0$ and $(0, w^o - \frac{1}{p}) \succ_i (0, v(z) - 1)$.
So let $v(z) < w^o$. Since $\alpha(v(z)) < \beta(v(z))$, there is $i \in S$, such that $x_i^o = 1$, $z_i = 0$ and $(1, n-1) \succ_i (0, v(z) - \frac{1}{p})$.
From $(1, w^o-1) \succ_i (1, n-1)$, it follows however that $(1, w^o-1) \succ_i (0, v(z) - 1)$. So i cannot be in S, which is a contradiction.

The core will contain more solutions than x^o. Some outcomes corresponding to elements $w \in W$ are in the core.
Let $S \subset J(w) \subset J(w^o)$ and $x_i = 0$ for $i \in S$ and $x_i = 1$ for $i \notin S$. If there are not sufficiently many people from $J(w^o) \setminus S$ for whome some reduction of the number of vehicles outweighs the change from the car to the bus, then x is in the core.
This is illustrated in fig. 6 where all individuals have the same preferences and $w^o = n-1/p$.

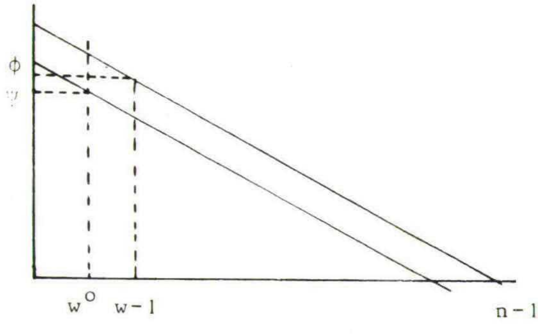


Fig. 6

Now if $|S| = \beta(w)$ and $x_i = 0$ for $i \in S'$ and $x_i = 1$ for $i \in S$, then this solution is in the core: for $i \notin S$, $u_i(1, w-1) = \phi$ and $u(0, w^o-1) = \Psi < \phi$.
So no member of $I \setminus S$ will be a member of a coalition blocking x.

7. PARETO EFFICIENCY.

The outcomes of the core are Pareto efficient. Let us assume that x^e is not in the core, hence $W \neq \emptyset$. There may exist efficient solutions outside the core. This is certainly true if $w^o > \frac{n}{p}$, i.e. if $J(w^o) \neq I$. Since there are people who so strongly prefer the car, that they can never be convinced to take the bus. As an illustration, we take the case of two groups of players. Members of the same group have the same utility function. Members of I_o have a weak preference for the car, members of I_1 have a strong preference for the car.

Let $u_o = A - B\, v_i + C_0\, x_i \quad C_o < C_1$

$u_1 = A - B\, v_i + C_1\, x_i$

and for $v = n_1 + \frac{n_2}{p}$

$$x_i = 1 \text{ and } u_i(1, n-1) > u_i(0, v-1) \quad \text{for } i \in I_1$$

$$x_i = 0 \text{ and } u_i(0, v-\tfrac{1}{p}) > u_i(1, n-1) \quad \text{for } i \in I_0$$

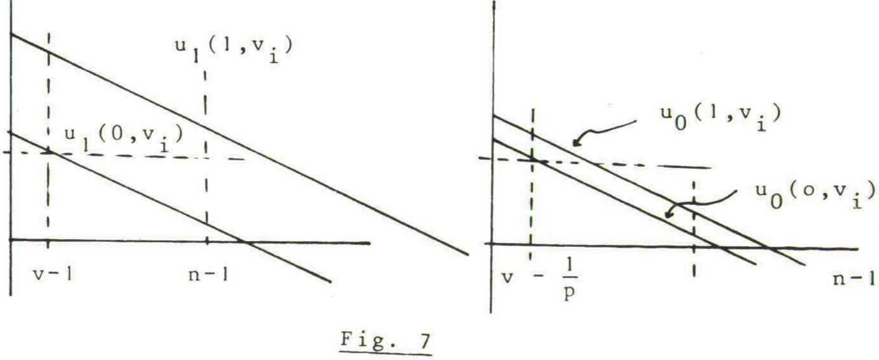


Fig. 7

This does however not exclude that the strategy combination $x = (0,0,...,0)$ is efficient.

Theorem 7.1.

If x^e is not Pareto efficient, then $x = 0$ is Pareto efficient.

Proof: It is to be shown that no strategy combination z gives an outcome preferred by all over the outcome $(0, \frac{n-1}{p})$.

a Suppose $z = x^e$. Since x^e is not efficient, there is a player for whom $(0, \frac{n-1}{p}) \succ_i (1, n-1)$

b If $z \neq x^e$, for some i, $z^i = 0$ and $(0, \frac{n-1}{p}) \succ_i (0, v(z)-1)$.

8. REMARKS.

1 We have shown, that in the present model, two cases can occur.

a. the equilibrium outcome y^e is efficient and is the only element in the core.

b. the equilibrium outcome is not Pareto efficient, better solutions exist some of these being in the core.

However for no individual it is possible to know if the case a or case b occurs, since preferences are not revealed by choices, apart from the preference for the car at a given behaviour of the others.
In order to realise another solution, the players should reveal their preferences, and if it appears, that case b occurs, cooperate.
They could form a group of people who take the bus. However the formation of such a group is difficult for two reasons: the number of players is large and any player will try to make the group of bus passengers as large as possible, without being a member of it himself.

2 The present problem has the same characteristics as the well known prisoners dilemma or its n-person analogies, (see Luce and Raiffa, p. 97). For these cases however it can be argued that in a wider context the equilibrium solution is Pareto efficient. However in the present case an inefficient solution seems not be desirable from the viewpoint of society.

3 Our problem is a very restricted one. However a nearly related problem is similar. Let there again go a road from A to B and suppose that there is also a train. The frequency of the train depends on the number of passengers. Car users prefer a small number of cars over a large number of cars, train passengers prefer high frequency over low frequency. At a given number of cars and the frequency derived from this, everybody prefers the train. Now cases a end b can occur as above.

4 The problem remains the same if there exist players who have only one strategy, e.g. they can only take the bus, because they have no car. Let m be their number and n the number of players who can choose.
In this case the number of vehicles is between $\frac{m}{p} + \frac{n}{p}$ and $\frac{m}{p} + n$.

9. SIDE PAYMENTS.

We can extend the model of the previous sections by the introduction of payments. Now the set of outcomes of each individual does not only contain his own strategy and the total number of vehicles, but also an amount of money to be paid (< 0) or to be received (> 0). Let $M_i = R$.
Then for $M = \prod_i M_i$,

$$Z = Y \times M \qquad \text{and} \qquad \tilde{Z} = \tilde{Y} \times M$$

is the set of feasible outcomes, where $z \in Z$, $z = (x_i, v_i, m_i)$
On this set a preordening $\succsim_i$ is defined, which is the preordening on Y for the case that $m_i = 0$.
We making the following assumptions:

Assumption A'

$\succsim_i$ on Z is a preordening.

Assumption B' (see ass. B)

For all i, x_i, m_i:

if $\quad \frac{n-1}{p} \leq v_i < v_i'$ then $(x_i, v_i, m_i) \succ_i (x_i, v_i', m_i)$

Assumption C' (= ass. C)

For all i

$$(1, v_i, m_i) \succ_i (0, v_i, m_i)$$

Assumption D

For all m_i, v_i, there exists v_i', such that

$$(0, v_i, m_i) \succ_i (1, v_i', m_i)$$

Assumption E

$$(x, v_i, m_i) \succsim_i (x_i', v_i', m_i) \Leftrightarrow (x_i, v_i, m_i') \succsim_i (x_i, v_i, m_i')$$

Assumption F

$$(x_i, v_i, m_i) \succ (x_i, v_i, m_i') \quad \text{if} \quad m_i > m_i'$$

A solution is preferred if the amount of money to be received is larger.

Assumption G

For all i, and $m > 0$ and $m' < 0$ there exist $\bar{m} < 0$ and $\bar{m}' > 0$ such that

$$(0, v_i, m_i) \sim (1, v_i, \bar{m})$$

$$(0, v_i, \bar{m}_i') \sim (1, v_i, m')$$

We construct a function $f : Y \rightarrow R$, where $f_i(x_i, v_i)$ represents the amount of money, that each player is willing to pay, or wants to receive, such that the outcome including the transfer of money is equivalent to the equilibrium solution:

$$f_i(x_i, v_i) = -m_i \quad \text{if} \quad (x_i, v_i, m_i) \sim_i (1, n-1, 0)$$

So if the combined strategy is x and $v_i = v_i(x)$ for each player, and each player pays or receives the amounts $-m_i$, then, everybody is just as good off as in the equilibrium x^e.

If there is a solution x prefered by all players, then

$$f_i(x_i, v_i(x)) > 0 \text{ for all } i \text{ and all players pay.}$$

Hence $\sum_I f_i(x_i, v_i(x)) > 0$ and the amount $\sum f_i$ could be divided among the players so that everybody is better off. If each individual now receives g_i ($\Sigma g_i = \Sigma f_i$) then the outcome is $(x_i, v_i(x), g_i - f_i(x))$.
This is also true if for some $f_i(x_i, v_i) < 0$, but $\sum f_i(x_i, v_i(x)) > 0$. Now this sum is left after everybody has paid. Some people are compensated. The residual $\sum f_i$ can be divided.
There certainly exists a strategy combination $\bar{x}$ such that

$$\Sigma f_i(\bar{x}_i, v_i(\bar{x})) = \max_{x \in X} f_i(x_i, v_i(x)),$$ since X has a finite number

of elements and $\Sigma f_i(\bar{x}_i, v_i(\bar{x}))$ is the maximal amount that can be divided. $\bar{x}$ is not necessarily Pareto efficient, but it is efficient if the utility of money is linear.
The function $f_i(x_i, v_i(x))$ can be considered as a utility function on $\tilde{Y}$. However its not a utility function on Z.
To $\bar{x}$, and the imputation g_i corresponds the outcome $(\bar{x}_i, v_i(\bar{x}), g_i - f_i)$. If we make the additional assumption.

<u>Assumption H</u>

$$(x_i, v_i, m_i) \sim (x_i', v_i', m_i') \Leftrightarrow (x_i, v_i, m_i+d) \sim (x_i', v_i', m_i'+d)$$

then there exists a utility function, which represents the preferences, such that

$$u(x_i, v_i, m) = m + f_i(x_i v_i)$$

Without loss of generality we can define

$$u(1, n-1, d_i) = d_i$$

Now for any (x_i, v_i, m_i), we have

$$(x_i, v_i, -f_i(x_i, v_i)) \sim (1, n-1, 0)$$

so $(x_i, v_i, m_i + f_i - f_i) \sim (1, n-1, m_i + f_i)$

and $u(x_i, v_i, m) = m_i + f_i(x_i, v_i)$

In this case $\bar{x}$ is the optimal strategy combination and the problem that is left is to find a suitable imputation.
The problem with solutions of this type is however, that they require (in general) different payments for each individual (i.e. total payment = compensation payment + imputation) which hardly seems a practical solution.

10. TOLLS AND SUBVENTIONS.

In this section we shall consider the question if an authority who has the power to raise a toll for the use of private cars and to pay subventions to bus passengers (by reducing the fare below its cost), in such a way that the amount of subventions paid is not larger than the amount of toll received, could generate a solution which is better then the equilibrium, as defined above, if case b occurs.
The answer to this question is negative; by means of a toll and a subvestion, the number of vehicles can be reduced, but the outcomes, taking payments into account, need not be better for every player, then the equilibrium outcome without payments.

Let t be the toll rate and s the subvention rate.
Now an outcome for a driver is represented by a point $(1, v_i, t)$ and an outcome for a bus passenger is $(0, v_i, s)$, where $t \leq 0$ and $s \geq 0$.

Now we extend the definition of an equilibrium:

Definition:

An equilibrium is a strategy combination $x \in X$ and real numbers s and t, such that for all i, either

$$(0,\ v(x) - \frac{1}{p},\ s) \succsim_i (1,\ v(x) - 1,\ t)$$

or

$$(1,\ v(x) - 1,\ t) \succsim_i (0,\ v(x) - \frac{1}{p},\ s)$$

and

$$t \sum x_i \geq s(n - \sum x_i)$$

Obviously x^e as defined in section 4 is an equilibrium in this sense for $s = t = 0$. The last condition of the definition requires that

$$\frac{t}{s} \geq \frac{n - \sum x_i}{\sum x_i} = \frac{np - pv}{pv - n}$$

If we make the additional assumption.

Assumption H'.

$$(0,\ v_i,\ s) \succsim_i (1,\ v_i,\ t) \Rightarrow (0,\ v_i',\ s) \succsim_i (1,\ v_i',\ t) \text{ if } v_i' > v_i$$

then

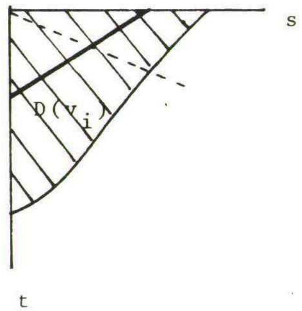
Theorem

For all v there exists an equilibrium $x \in X$ and $t \leq 0$ and $s \geq 0$.

<u>Proof</u>: By assumption F and G the set

$$D(v_i) = \{s,t \mid (1,\ v_i,\ s) \succsim_i (0,\ v_i,\ t)\}$$

is closed and if $(s,t) \in D$ and $s' \leq s$ and $t' \geq t$, then $(s',t') \in D$



<u>Fig. 8</u>

Let v be given $\frac{n-1}{p} \leq v \leq$ and pv a whole number)
Choose τ and δ, such that

$$\frac{\tau}{\delta} = \frac{np - pv}{pv - n} \quad \text{and} \quad \tau + \delta = 1.$$

We construct λ such that $t = \lambda\tau$ and $\delta = \lambda\delta$.
For each i there exists λ, such that $(1,v_i-1,\lambda\tau) \sim_i (0,v_i-1,\lambda\delta)$.
We choose $J(\lambda) \subset I$ so that for $i \in J(\lambda)$

$$(1,v_i-1,\lambda\tau) \succsim_i (0,v_i-1,\lambda\delta)$$

and $|J(\lambda)| \geq \frac{\tau}{\delta}$ and $|J(\lambda)'| < \frac{\tau}{\delta}$ if $\lambda' < \lambda$.
Now choose $\bar{J}(\lambda) \subset J(\lambda)$ such that for $i \in J(\lambda) \setminus \bar{J}(\lambda)$, $(1,v_i-1,\lambda\delta) \sim_i (0,v_i-1,\lambda\tau)$.
Now let $\bar{x}$ be such that $\bar{x}_i = 1$ if $i \in \bar{J}(\lambda)$ and $x_i = 0$ if $i \notin \bar{J}(\lambda)$.

For $i \notin \bar{J}(\lambda)$, we have

$$(0, v_i - 1, \lambda\delta) \succsim_i (1, v_i - 1, \lambda\tau) \text{ and hence by assumption } H'$$

$$(0, v_i - \frac{1}{p}, \lambda\delta) \succsim_i (1, v_i - \frac{1}{p}, \lambda\tau)$$

There may be exist more equilibria then the ones constructed in the proof, namely those where $t\,\Sigma x_i > s\,(n - \Sigma x_i)$. It is not true in general that among the equilibria there is one which gives a set of strategies which is in the core of the orginal game.
Assume that $\bar{x}$ is in the core and $\bar{x}, \bar{t}, \bar{s}$ is an equilibrium such that $v(\bar{x}) = (v(\tilde{x})$. Then $\bar{x} = \tilde{x}$ if

$$\text{for } i, \text{ such that } \tilde{x}_i = 0\text{: } (0, v(\bar{x}) - \frac{1}{p}, s) \succsim_i (1, v(\bar{x}) - \frac{1}{p}, t)$$

$$\text{for } i, \text{ such that } \tilde{x}_i = 1\text{: } (1, v(\bar{x}) - 1, t) \succsim_i (1, v(\bar{x}) - 1, s)$$

If these relations hold it is however not ensured that

$$(1, v(\bar{x}) - 1, t) \succsim_i (1,\ n-1,\ 0)$$

<u>Some examples</u>.

1) Let $x = 0$ be in the core (of the orginal game). Hence for all i: $(0, \frac{n-1}{p}, 0) \succsim_i (1, n-1, 0)$. Now by assumption F, there exists some toll rate t such that for all i

$$(0, \frac{n-1}{p}, 0) \succsim_i (1, \frac{n-1}{p}, t)$$

So $x = 0$ is an equilibrium for t and $s = 0$: everybody takes the bus and nobody pays the toll. Everybody is better off.

2) Assume that there are two equal groups I_0 and I_1 of players, members of the same group having identical utility functions u_0 and u_1, respectively.
Let $v = \frac{1}{2}n + \frac{1}{n}\frac{n-1}{p}$ and

$$u_0(0, v-\frac{1}{p}, 0) > u_0(1, n-1, 0)$$

$$u_1(1, n-1, 0) > u_1(0, v_i', 0) \text{ for every } v_i' > \frac{n-1}{p}$$

Then obviously the solution $\tilde{x}$ such that

$\tilde{x}_i = 0$ for $i \in I_0$ and $\tilde{x}_i = 1$ for $i \in I_1$ is in the core of the original game (see section 6).
It is possible that, if we find the equilibrium solution for $v = v(\tilde{x})$, $\bar{x} \in X$ and s, t, that we have

$$u_0(1, v-1, t) \geq u_0(0, v-1, s)$$

$$u_1(1, n-1, 0) > u_1(0, v-\frac{1}{p}, s) \geq u_1(1, v-\frac{1}{p}, t)$$

i.e. for $\tilde{x}_i = 0$, $\bar{x}_i = 1$ and $\tilde{x}_i = 1$, $\bar{x}_i = 0$, because those who have a "weak preference" for the car, also have a low utility of money and those with "strong preference" for the car have a large utility of money. (fig. 9)

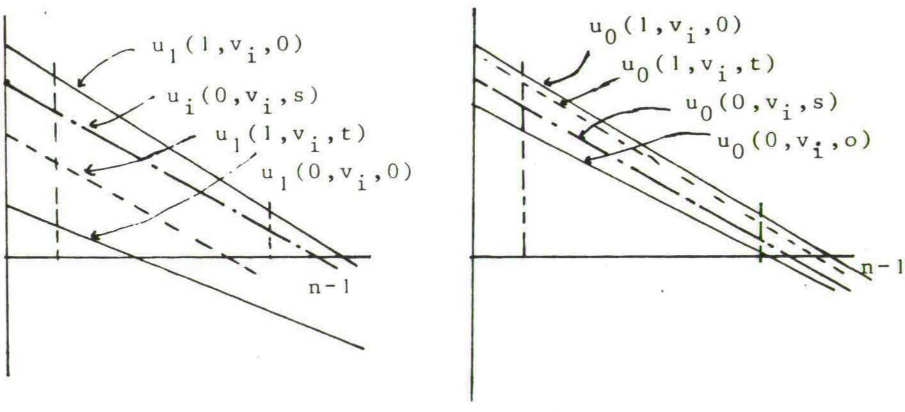


Fig. 9

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