Solutions for cooperative games with and without transferable utility

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CONTENTS

CHAPTER 1

INTRODUCTION

Each individual, firm, country, or any kind of economic agent in the society is making decisions on a daily basis to achieve respective goals often under some physical, technological or institutional restrictions, and intrinsically outcomes do not only depend on the decision of the agent but also on the decisions of others. Game theory is a mathematical approach to analyze the process of decision making of several agents in mutually dependent situations.

Game theory is firstly introduced by von Neumann and Morgenstern (1944) in their book "Theory of Games and Economics Behavior". They establish in this book two major approaches of the study of game theory, non-cooperative game theory and cooperative game theory. They introduce two-person zero-sum games, which is the starting point of non-cooperative game theory, and based on it they also build the foundation of n-person cooperative game theory assuming that in a situation which has more than two agents, the agents may coordinate their actions as coalitions. The distinction becomes clearer in Nash (1951), who defines that in a non-cooperative game "each participant acts independently, without collaboration or communication with any of the others", while in cooperative game they "may communicate and form coalitions which will be enforced by an umpire". While non-cooperative game theory formulates situations with possibly opposing interests and analyzes actions agents would choose in such situations, cooperative game theory is concerned with what kinds of coalitions would be formed and how much payoff every agent should receive.

A cooperative game with transferable utility, or simply a TU-game, considers a situation in which agents are able to cooperate to form coalitions and the total payoff obtained from their cooperation can be freely distributed among the agents in the coalition. More precisely, a TU-game is described by a finite set of agents, called players, and a characteristic function. A characteristic function of a TU-game assigns to each coalition the total profit, or worth, which can be maximally obtained by the coalition without cooperating with any player outside the coalition. A fundamental question of TU-games is how much payoff each player must receive.

A solution concept for TU-games assigns to each TU-game a set of allocations that satisfy certain properties, or axioms. One of the well-known solution concepts of TU-games is the core introduced by Gillies (1959), as the set of allocations that are efficient and exactly distribute the worth of the grand coalition of all players, and are stable in the sense that no group of players has the incentive to leave the grand coalition and obtain the worth of themselves. While the core of a TU-game may be empty, a single-valued solution gives precisely one allocation for each TU-game. One of the well-known single-valued solution concepts, called the Shapley value, is introduced by Shapley (1953), being the average of the marginal vectors induced from all linear orders of players. To every linear order a marginal vector corresponds, which assigns to a player the difference of the worths between the coalition which consists of all the players who are ordered before him in the linear order with and without him. The Shapley value is the only single-valued solution concept of TU-games which satisfies efficiency, additivity, the null player property, and symmetry. Other characterizations of the Shapley value are given by for example Young (1985) with a monotonicity axiom and van den Brink (2002) with a fairness axiom.

The concept of convexity for TU-games is introduced in Shapley (1971). If a TU-game is convex then the marginal contribution a player makes to a coalition increases as the coalition he joins becomes larger. It is shown that convexity of a TU-game guarantees stability of the Shapley value of the game, i.e., the Shapley value is an element of the core, since the core of a convex game equals the convex hull of all marginal vectors.

For TU-games it is assumed that any set of players can form a coalition and earn the worth of their cooperation, but in many economic situations of interest, there exist restrictions which prevent some coalitions from cooperating. To illustrate this point, consider a cooperative situation where two professors, A and B, and two students, one student of A and one of B, are writing proposals to obtain some research budget. The worth of any set of players is the budget they get if they cooperate, and assume that each professor knows (or can communicate with) his student and the other professor, while a student knows his professor and the other student. In this setting, it is not realistic to assume that the coalition of professor A and the student of B or the coalition of professor B and the student of A will be formed, since the players in such coalitions can not communicate with each other within the coalition.

In the literature, limited cooperation possibilities among the players of this kind can be represented by an undirected graph on the set of players. In this graph, the set of nodes is the set of players and a link[1] between two players means that they are able to communicate, and therefore such a graph expresses a communication structure between players. Myerson (1977) firstly introduces the idea and defines the class of TU-games with communication structure. Given a communication structure represented by an undirected graph, only players that are connected in the graph are feasible, i.e., have the opportunity to form a coalition and enjoy the resulting worth.

A single-valued solution concept on the class of TU-games with communication structure is introduced by Myerson (1977) and now it is called the Myerson value. For a TU-game with communication structure, the Myerson value is the Shapley value of the so-called Myerson restricted game, which is a TU-game derived from the original game. In a Myerson restricted game, the worth of a coalition, which is not connected in the underlying communication structure, equals the sum of the worths of the maximally connected subsets of the coalition.

In Myerson (1977) the Myerson value is characterized by component efficiency and fairness and in van den Nouweland (1993) by component efficiency, additivity, the strong superfluous link property, and point anonymity. This is an extension of a result in Borm et al. (1992) on a subclass of TU-games with communication structure. In Chapter 2 we give an alternative characterization of the Myerson value on the class of TU-games with communication structure. We use another form of fairness, called coalitional fairness, and characterize the Myerson value jointly with component efficiency, additivity, and a restricted form of the null player property. This combination of axioms is similar to the original set of axioms used for the Shapley value in Shapley (1953). It is shown by examples that the axioms are logically independent.

[1] In order to be consistent through this monograph, we call a directed edge an arc and an undirected edge a link, even if the original literature call them otherwise.

On the class of TU-games with communication structure, several other single-valued solution concepts are introduced. The position value, introduced by Meessen (1988) and Borm et al. (1992), considers the link game, another restricted TU-game derived from the original TU-game with communication structure. It defines the worth of a coalition of links, instead of that of players, and first assigns the Shapley value to each link in the link game. The position value allocates to a player the share of the Shapley value of links he belongs to. An axiomatic characterization of the position value on the class of TU-games with cycle-free communication structure is given in Borm et al. (1992) as the unique single-valued solution concept satisfying component efficiency, additivity, the superfluous link property, and link anonymity. On the class of TU-games with communication structure which may contain cycles, a characterization of the position value is provided by Slikker (2005) with component efficiency and balanced link contributions.

The average tree solution, another single-valued solution concept on the class of TU-games with communication structure, is studied in Chapter 3. The average tree solution is firstly introduced by Herings et al. (2008) on the class of TU-games with cycle-free communication structure. Instead of defining a restricted game which reflects the communication restriction between players, it considers the average of the marginal vectors corresponding to all spanning trees extracted from the underlying communication structure. Herings et al. (2008) shows that on this class of games the average tree solution is the unique single-valued solution concept satisfying component efficiency and component fairness. An alternative characterization of this solution on the class of TU-games with cycle-free communication structure is given in van den Brink (2009) with component efficiency, collusion neutrality, additivity, the communication ability property, the equal gain/loss property, and component independence. On the class of TU-games with connected cycle-free communication structure, Mishra and Talman (2010) uses efficiency, linearity, strong symmetry, the dummy property, and independence in unanimity games for a characterization of the average tree solution.

In Herings et al. (2010) the average tree solution is generalized to the class of TU-games with communication structure which may contain cycles. They give an algorithm to generate a collection of rooted spanning trees from a communication structure, and the average tree solution is defined as the average of marginal vectors corresponding to those rooted spanning trees. Baron et al. (2011) shows that this set of rooted spanning trees is the only one that satisfies the following property: In any tree, every two players who share a communication link in the underlying graph are comparable in the sense that one of them is a subordinate of the other. Compared to the axiomatic study of the average tree solution on TU-games with cycle-free communication structure, not much is done on the axiomatization of the average tree solution on TU-games with communication structure which contains cycles.

Chapter 3 studies the average tree solution on the class of TU-games with circular communication structure, where the communication structure is represented by a circle on the player set. Players could be firms or cities situated along a lake shore or a circular pipeline where players can only be connected to their two direct neighbors, one located on each side. The communication structure in the example with the two professors and two students above can be represented by a circle with four nodes. This is a new class of games to be studied by its own and we provide a characterization of the average tree solution by using efficiency, additivity, the restricted null player property, symmetry among players, and symmetry between games. The Myerson value satisfies the first four axioms, and therefore the last axiom makes a difference between the Myerson value and the average tree solution on this class of games. It is shown that these axioms are logically independent. It is also proven that on the class of TU-games with circular communication structure the average tree solution coincides with the Shapley value of Bilbao and Ordóñez (2009), which uses maximal chains on the player set instead of trees. Also, necessary and sufficient conditions on the characteristic function are given such that all marginal vectors and the average tree solution of a TU-game with circular communication structure are payoff vectors in the core of the game.

Although the class of TU-games with communication structure contains the classical TU-games as a subclass and its communication restriction is straightforward, the collection of connected sets of players in a communication structure as a collection of feasible coalitions may be restrictive. For instance, how can we implement the extra information in the example of two professors and two students above that the professors are in more dominant positions than the students? In the literature there are two approaches to express cooperation restriction between players of TU-games, beyond communication structures. One approach is to assume that the collection of feasible coalitions of players satisfies certain combinatorial structures defined as specific set systems. For example, Bilbao and Edelman (2000) considers convex geometries, Algaba et al. (2001) considers union stable structures, Algaba et al. (2003) considers antimatroids, Bilbao and Ordóñez (2009) considers augmenting systems, and Koshevoy and Talman (2014) considers building sets. All of these set systems can express a cooperation restriction which can not be expressed as a collection of connected coalitions of a communication structure, but sometimes contain the class of cooperative restriction that communication structures can represent.

Another approach to bring more flexibility into cooperation restriction between players is to explicitly introduce a kind of dominance structure on the players in a game. For instance, Faigle and Kern (1992) allows that players may be partially ordered, such as in a hierarchy, and assumes that only the coalitions that are compatible with this order may form. Gilles et al. (1992), Derks and Gilles (1995), and van den Brink and Gilles (1996) consider situations that each player has a set of predecessors in the player set induced from a permission structure, and if a player wants to cooperate with other players he must ask for permission from his predecessors in the structure. Gilles and Owen (1992) and van den Brink (1997) take another assumption that the permission from one predecessor in underlying permission structure is enough to cooperate with others. Khmelnitskaya et al. (2012) explicitly considers a directed communication graph to describe a cooperation restriction among players, where an arc represents a unilateral relation between a pair of players.

Quasi-building systems introduced in Chapter 4 unites the two approaches. A quasi-building system on a players set consists of a set system, which represents the set of feasible coalitions of players, and a choice set function, which expresses a dominance relation within each feasible coalition. A player in the choice set of a feasible coalition means that he can act as a 'boss' of the coalition, in a sense that he has the power to make the cooperation possible or to dissolve the cooperation. We call a TU-game with cooperation restriction represented by a quasi-building system a quasi-building system game, and show that the class of these games can express many classes of TU-games with restricted cooperation, including the games with set systems and the games with directed graphs mentioned above. Further, we define a single-valued solution concept called the average marginal vector value, or the AMV-value, as the average of marginal vectors induced from a set of rooted trees on the player set. Each rooted tree, a hierarchical order of players, is compatible with the underlying quasi-building system, that is, it reflects the dominance relation between players and the feasibility restriction of coalitions. We present some basic properties the AMV-value satisfies, such as efficiency and linearity. We also modify the null player property to this class, define inessential coalitions which do not play a role in affecting the resulting allocation outcome, and closed coalitions which can not do better than sharing the worth of the coalition itself between the members of it. We further define three subclasses of quasi-building system games, which are still general enough to cover all the TU-games with cooperation restriction mentioned before, and for each subclass we give a convexity-type of condition on the characteristic function which guarantees that every marginal vector considered for the AMV-value, and therefore the value itself, is stable.

Chapter 5 is on the class of cooperative games with non-transferable utility, or NTU-games, which deal with cooperative situations when benefits from cooperation are not transferable between individuals. The concept of NTU-games is introduced by Aumann and Peleg (1960) as cooperative games without side payments. In TU-games, one can think that the worth of a coalition is expressed in terms of money, and players are allowed to transfer their utilities by side payments. In NTU-games this assumption is relaxed, because there may exist not such medium of transferring utilities among players, or, if it exists, the utilities of players may not be linear in the medium. TU-games are in this sense a special case of NTU-games, and concepts defined on TU-games are often generalized to NTU-games. For example, the core is extended to NTU-games by Aumann (1961), and the balancedness condition of Bondareva (1963), a necessary and sufficient condition for TU-games to have a nonempty core, is extended in Scarf (1967) as a sufficient condition for NTU-games. A necessary and sufficient condition for the nonemptiness of the core of a NTU game is established in Predtetchinski and Herings (2004).

The concept of convexity has been extended to NTU-games in Vilkov (1977) as cardinal convexity, in Sharkey (1981) as ordinal convexity, in Hendrickx et al. (2000) as individual merge convexity, and in Masuzawa (2012) as strongly ordinal convexity. The last two conditions guarantee that every appropriately defined marginal vector of an NTU-game is stable. The first aim of Chapter 5 is to introduce a new natural condition on the payoff sets of an NTU-game such that every marginal vector of the game is stable. This condition is weaker than both individual merge convexity and strong ordinal convexity. Second, we define a multi-valued solution concept, called the solution set, which is determined by the average of all marginal vectors and is the Shapley value if the NTU-game is induced by a TU-game.

On the class of NTU-games that we consider, all marginal vectors are well defined and by construction efficient for the grand coalition as in the case of TU-games, but the average of these vectors may be not efficient, or even not be feasible for the grand coalition. To define the solution set of an NTU-game we take the average of all marginal vectors as a reference point. In case the average is feasible but not efficient for the grand coalition, the solution set consists of all efficient allocations reached from the reference point into any strictly positive direction, and if the reference point is not feasible for the grand coalition, the solution set is the set of efficient allocations reached from the reference point into any strictly negative direction. Therefore, the solution set can be seen as the set of bargaining solutions of a bargaining problem with the reference point, if it is feasible for the grand coalition and cannot be blocked by any proper subset of players, as disagreement point, and, if not feasible, as utopia point. A single-valued solution concept of this type on the class of NTU-games is the marginal based compromise value introduced by Otten et al. (1998) and this solution concept is contained in the solution set under general conditions. Also a sufficient condition is given under which the solution set is a subset of the core.

CHAPTER 2

AN AXIOMATIZATION OF THE MYERSON VALUE

2.1 Introduction

In the literature of cooperative game theory, most of the solutions proposed are characterized by axioms which state desirable properties a solution possesses. On the class of TU-games, Shapley (1953) introduces the Shapley value, the best-known single-valued solution concept, and characterizes it as the unique solution on the class of TU-games that satisfies efficiency, additivity, the null player property, and symmetry. Efficiency requires that the resulting allocation distributes to the players exactly the worth of the grand coalition. Additivity says that if there are two TU-games of the same set of players, the allocation of a new game, in which the worth of a coalition is the sum of the worths of the same coalition of the two games, is equal to the sum of the allocations of each game. The null player property gives zero payoff to a player who contributes nothing to change the worth by joining to any coalition. Symmetry says that if two players are symmetric, i.e., for any coalition which does not contain the two players, the worth of the coalition with one of the two players is equal to the worth of the coalition with the other player, then the two players should receive the same payoff. Other characterizations of the Shapley value are proposed in for example Young (1985) and van den Brink (2002).

In this chapter we study TU-games with communication structure introduced by Myerson (1977). It arises when the restriction for cooperation is represented by an undirected graph on the set of players in which a link between any two players implies that these players can communicate and only connected subsets of players are able to cooperate and obtain their worth.

One of the most well-known single-valued solutions on the class of TU-games with communication structure is the Myerson value (Myerson (1977)), defined as the Shapley value of the so-called Myerson restricted game. By Myerson (1977), the Myerson value is characterized by (component) efficiency and fairness, fair in the sense that if a link is deleted between two players, the Myerson value imposes the same loss on payoffs for each of these two players. Another characterization of the Myerson value is given by van den Nouweland (1993), which follows an axiomatization given in Borm et al. (1992) on the class of TU-games with cycle-free communication structure. In van den Brink (2009) an axiomatization of the Myerson value on this subclass is given to make comparisons between different single-valued solution concepts.

In this chapter we give an alternative axiomatization of the Myerson value for TU-games with arbitrary communication structure. Our approach is to use another form of fairness and the Myerson value is characterized by component efficiency, additivity, a restricted form of the null player property, and a different form of fairness. This combination is similar to the original characterization of the Shapley value. The fairness property we propose, called coalitional fairness, says that if the worth of one coalition changes, then the change in payoff is the same for all players within that coalition.

This chapter is organized as follows. Section 2 introduces TU-games with communication structure and the Myerson value. In Section 3 an axiomatic characterization is given. This chapter is based on Selçuk and Suzuki (2014).

2.2 TU-games with communication structure and the Myerson value

A cooperative game with transferable utility, or a TU-game, is a pair (N, v) where $N = \{1, \ldots, n\}$ is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a characteristic function with $v(\emptyset) = 0$. For a subset $S \in 2^N$, being the coalition consisting of all players in S, the real number $v(S)$ is the worth of the coalition, being the maximum payoff that the players in S can achieve by cooperation and which can be freely distributed among the players in S. Let $\mathcal{G}_N$ denote the class of TU-games with fixed player set N.

A special class of TU-games is the class of unanimity games. For $T \in 2^N$, the unanimity game $(N, u_T) \in \mathcal{G}_N$ has characteristic function $u_T : 2^N \to \mathbb{R}$ defined as

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known that any TU-game can be uniquely expressed as a linear combination of unanimity games. Let $(N, \mathbf{0}) \in \mathcal{G}_N$ denote the zero game, i.e., $\mathbf{0}(S) = 0$ for all $S \in 2^N$.

A payoff vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is an n-dimensional vector that assigns payoff x_i to player $i \in N$. A single-valued solution on $\mathcal{G}_N$ is a funstion $\xi : \mathcal{G}_N \to \mathbb{R}^n$ which assigns to every TU-game (N, v) a payoff vector $\xi(N, v)$.

The most well-known single-valued solution on the class of TU-games is the Shapley value, see Shapley (1953). It is the average of the marginal vectors induced from the collection of permutations of players. Let $\Pi(N)$ be the collection of permutations, or linear orderings, on N. Given a permutation $\sigma \in \Pi(N)$, the set of predecessors of $i \in N$ in σ is defined as

$$P_\sigma(i) = \{j \in N |\ \sigma(j) < \sigma(i)\}.$$

Here, $\sigma(i) = j$, for $i, j \in N$, means that player j is in ith position under σ. Given a TU-game $(N, v) \in \mathcal{G}_N$, for a permutation σ in $\Pi(N)$ the marginal vector $m^\sigma(N, v)$ assigns payoff

$$m_i^\sigma(N, v) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))$$

to player $i = 1, \ldots, n$. The Shapley value of (N, v), $Sh(N, v)$, is the average of all $n!$ marginal vectors, i.e.,

$$Sh(N, v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N, v).$$

A graph on N is a pair (N, L), with $N = \{1, \ldots, n\}$ a set of nodes and $L \subseteq L_N^c$, where $L_N^c = \{\{i, j\} \mid i, j \in N, i \neq j\}$ is the complete set of undirected links without loops on N, and an unordered pair $\{i, j\} \in L$ is called an link in (N, L). A subset $S \in 2^N$ is connected in (N, L) if for any $i \in S$ and $j \in S$, $j \neq i$, there is a sequence of nodes $(i_1, i_2, \ldots, i_k)$ in S such that $i_1 = i$, $i_k = j$ and $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k-1$. The collection of connected coalitions in (N, L) is denoted $C^L(N)$. By definition, the empty set $\emptyset$ and every singleton $\{i\}$, $i \in N$, are connected in (N, L). For $S \in 2^N$, the subset of links $L(S) \subseteq L$ is defined as $L(S) = \{\{i, j\} \in L |\ i, j \in S\}$, being the subset of L of links that can be established within S. The graph $(S, L(S))$ is a subgraph of (N, L). A component of a subgraph $(S, L(S))$ of (N, L) is a maximally connected coalition in $(S, L(S))$ and the collection of components of $(S, L(S))$ is denoted $\widehat{C}^L(S)$. For a graph (N, L), if $\{i, j\} \in L$, then i is called a neighbor of j and vice versa. The collection of neighbors of node $i \in N$ in (N, L) is denoted by D_i^L, that is, $D_i^L = \{j \in N \setminus \{i\} \mid \{i, j\} \in L\}$. The collection of neighbors of $S \in 2^N$ in (N, L) is defined similarly as $D_S^L = \{j \in N \setminus S \mid \exists i \in S : \{i, j\} \in L\}$.

The combination of a TU-game and an (undirected) graph on the player set is a TU-game with communication structure, introduced by Myerson (1977) and denoted by a triple (N, v, L) where (N, v) is a TU-game and (N, L) is a graph on N. A link between two players has as interpretation that the two players are able to communicate and it is assumed that only a connected set of players in the graph is able to cooperate to obtain its worth and freely transfer it as payoff among the players in the coalition. Let $\mathcal{G}_N^{cs}$ denote the class of TU-games with communication structure and fixed player set N. A single-valued solution on $\mathcal{G}_N^{cs}$ is a function $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ which assigns to every TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$ a payoff vector $\xi(N, v, L)$.

The most well-known single-valued solution on the class of TU-games with communication structure is the Myerson value. It is the Shapley value of the so-called Myerson restricted game. Following Myerson (1977), the Myerson restricted characteristic function $v^L : 2^N \to \mathbb{R}$ of $(N, v, L) \in \mathcal{G}_N^{cs}$ is defined as

$$v^L(S) = \sum_{K \in \widehat{C}^L(S)} v(K), \ S \in 2^N.$$

The pair (N, v^L) is a TU-game and is called the Myerson restricted game of (N, v, L), and the Myerson value of a game $(N, v, L) \in \mathcal{G}_N^{cs}$ is defined as

$$\mu(N, v, L) = Sh(N, v^L).$$

2.3 Axiomatic characterization

In this section we study existing characterizations and give a new axiomatization of the Myerson value on the class of TU-games with communication structure. When introducing the class of TU-games with communication structure, Myerson (1977) characterizes the Myerson value by component efficiency and fairness axioms.

Definition 2.3.1 A solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfies *component efficiency* if for any $(N, v, L) \in \mathcal{G}_N^{cs}$ it holds that $\sum_{i \in Q} \xi_i(N, v, L) = v(Q)$ for all $Q \in \widehat{C}^L(N)$.

A solution on the class of TU-games with communication structure satisfies component efficiency if the solution allocates to each component as the sum of payoff among its members the worth of the component.

Definition 2.3.2 A solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfies *fairness* if for any $(N, v, L) \in \mathcal{G}_N^{cs}$ and $\{i, j\} \in L$ it holds that

$$\xi_i(N, v, L) - \xi_i(N, v, L \setminus \{i, j\}) = \xi_j(N, v, L) - \xi_j(N, v, L \setminus \{i, j\}).$$

A solution on the class of TU-games with communication structure satisfies fairness if the deletion of a link from the graph results in the same payoff change for the two players who are endpoints of the link.

Theorem 2.3.3 ***(Myerson, 1977)*** *The Myerson value is the unique solution on $\mathcal{G}_N^{cs}$ that satisfies component efficiency and fairness.*

Another characterization of the Myerson value on the class of TU-games with communication structure is given by van den Nouweland (1993), which is in line with an earlier result of Borm et al. (1992) on the class of TU-games with cycle-free communication structure. An alternative characterization of the Myerson value on this subclass is given in van den Brink (2009). In van den Nouweland (1993) component efficiency, additivity, the strong superfluous link property, and point anonymity are used to characterize the Myerson value.

For any two TU-games (N, v) and (N, w) in $\mathcal{G}_N$, the TU-game $(N, v + w)$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \in 2^N$.

Definition 2.3.4 A solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfies *additivity* if for any (N, v, L), $(N, w, L) \in \mathcal{G}_N^{cs}$ it holds that $\xi(N, v + w, L) = \xi(N, v, L) + \xi(N, w, L)$.

Additivity of a solution means that if there are two TU-games with the same communication structure, the resulting payoff vectors coincide when applying the solution to each of the two games and adding the two vectors and when applying the solution to the game which is the sum of the two games.

Given a TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$, a link $\{i, j\} \in L$ is called strongly superfluous if $v^L = v^{L \setminus \{i,j\}}$, i.e., a link whose absence does not influence the restricted game.

Definition 2.3.5 A solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfies the *strong superfluous link property* if for any $(N, v, L) \in \mathcal{G}_N^{cs}$ and strongly superfluous link $\{i, j\} \in L$ it holds that

$$\xi(N, v, L) = \xi(N, v, L \setminus \{i, j\}).$$

For a graph (N, L), let D^L denote the set of nodes that have at least a link in (N, L), i.e., $D^L = \{i \in N \mid \{i, j\} \in L$ for some $j \in N\}$. A TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$ is called point anonymous if there is a function $f : \{0, 1, \ldots, |D^L|\} \to \mathbb{R}$ with $v^L(S) = f(|S \cap D^L|)$ for all $S \in 2^N$. For a point anonymous TU-game with communication structure, the worth of a coalition in the restricted game depends only on the number of players in the coalition who have at least a link in the communication structure.

Definition 2.3.6 A solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfies *point anonymity* if for every point anonymous TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$ it holds that $\xi_i(N, v, L) = \xi_j(N, v, L)$ for all $i, j \in D^L$ and $\xi_i(N, v, L) = 0$ for all $i \in N \setminus D^L$.

Theorem 2.3.7 ***(van den Nouweland, 1993)*** *The Myerson value is the unique solution on $\mathcal{G}_N^{cs}$ that satisfies component efficiency, additivity, the strong superfluous link property, and point anonymity.*

We give another characterization of the Myerson value by using component efficiency, additivity, a restricted form of the null player property, and another form of fairness. The combination is similar to the original characterization of the Shapley value by Shapley (1953).

A player $i \in N$ is called a restricted null player in a TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$ if this player never contributes whenever he joins to form a connected coalition, that is, $v(S \cup \{i\}) - \sum_{K \in \widehat{C}^L(S)} v(K) = 0$ for all $S \in 2^N$ such that $i \notin S$ and $S \cup \{i\} \in C^L(N)$. The restricted null player property says that this player must get zero payoff.

Definition 2.3.8 A solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfies the *restricted null player property* if for any $(N, v, L) \in \mathcal{G}_N^{cs}$ and restricted null player $i \in N$ in (N, v, L) it holds that $\xi_i(N, v, L) = 0$.

Note that a restricted null player of a TU-game with communication structure (N, v, L) is a null player of its Myerson restricted game (N, v^L), and a restricted null player of a TU-game with complete communication structure (N, v, L^c) is a null player of the TU-game (N, v). The next axiom replaces symmetry.

Definition 2.3.9 A solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfies *coalitional fairness* if for any two TU-games $(N, v, L), (N, v', L) \in \mathcal{G}_N^{cs}$ and $Q \in 2^N$ it holds that $\xi_i(N, v, L) - \xi_i(N, v', L) = \xi_j(N, v, L) - \xi_j(N, v', L)$ for all $i, j \in Q$ whenever $v(S) = v'(S)$ for all $S \in 2^N$, $S \neq Q$.

Coalitional fairness of a solution implies that given a TU-game with communication structure, if the worth of a single coalition changes, then the payoff change should be equal among all players in that coalition.

From additivity and the restricted null player property we have the following lemma.

Lemma 2.3.10 *Let a solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfy additivity and the restricted null player property. Then for any two TU-games with the same communication structure (N, v, L), $(N, v', L) \in \mathcal{G}_N^{cs}$ it holds that $\xi(N, v, L) = \xi(N, v', L)$ whenever $v(S) = v'(S)$ for all $S \in C^L(N)$.*

Proof Consider the game (N, w, L) where $w = v - v'$. Then every player is a restricted null player in this game because $w(S) = 0$ for all $S \in C^L(N)$. Therefore every player must receive zero payoff, that is, $\xi_i(N, w, L) = 0$ for all $i \in N$. From additivity and $v = w + v'$ it follows that $\xi(N, v, L) = \xi(N, w, L) + \xi(N, v', L) = \xi(N, v', L)$. □

This lemma says that the worth of an unconnected coalition does not affect the outcome of a solution that satisfies additivity and the restricted null player property, which leads to the following corollary.

Corollary 2.3.11 *If a solution $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ satisfies additivity and the restricted null player property, then $\xi(N, v, L) = \xi(N, v^L, L)$ for any $(N, v, L) \in \mathcal{G}_N^{cs}$.*

To prove that on the class of TU-games with communication structure the axioms above uniquely define the Myerson value, we consider Myerson restricted unanimity games. Given a unanimity game with communication structure $(N, u_T, L) \in \mathcal{G}_N^{cs}$ with $T \in 2^N$, the Myerson restricted unanimity game $(N, u_T^L) \in \mathcal{G}_N$ is given by

$$u_T^L(S) = \begin{cases} 1 & \text{if } T \subseteq K \text{ for some } K \in \widehat{C}^L(S), \\ 0 & \text{otherwise.} \end{cases}$$

Given a graph (N, L) and $S \in 2^N$, let $\overline{C}^L(S)$ denote the collection of connected coalitions which minimally contain S, that is,

$$\overline{C}^L(S) = \{K \in C^L(N) \mid S \subseteq K,\ K \setminus \{i\} \notin C^L(N) \ \ \forall\ i \in K \setminus S\}.$$

Note that if a graph (N, L) is not connected, the set $\overline{C}^L(S)$ may be empty for some $S \in 2^N$. On the other hand, if (N, L) is connected and cycle-free, then $\overline{C}^L(S)$ consists of one element for any $S \in 2^N$.

Lemma 2.3.12 *For a unanimity TU-game with communication structure* $(N, u_T, L) \in \mathcal{G}_N^{cs}$ *with* $T \in 2^N$*, it holds that*

$$u_T^L = \begin{cases} \sum_{J \subseteq \{1,\dots,k\}} (-1)^{|J|+1} u_{\cup_{j \in J} Q_j} & \text{if } \overline{C}^L(T) = \{Q_1, \dots, Q_k\}, \\ \mathbf{0} & \text{if } \overline{C}^L(T) = \emptyset. \end{cases}$$

Proof First consider the case when $\overline{C}^L(T) = \emptyset$. This implies that there exists no $K \in \widehat{C}^L(N)$ which contains T, and from the definition of u_T^L it follows that $u_T^L(S) = 0$ for all $S \in 2^N$. Next, let $v = \sum_{J \subseteq \{1,\dots,k\}} (-1)^{|J|+1} u_{\cup_{j \in J} Q_j}$ when $\overline{C}^L(T) \neq \emptyset$. If $T \in C^L(N)$, then $\overline{C}^L(T) = \{T\}$ and therefore it holds that $v = u_T = u_T^L$. Suppose $T \notin C^L(N)$. It is to show that $v(S) = u_T^L(S)$ holds for every $S \in 2^N$. First take $S \in 2^N$ such that there is no $K \in \widehat{C}^L(S)$ satisfying $T \subseteq K$. This implies that $Q \not\subseteq S$ for any $Q \in \overline{C}^L(T)$, and thus we have $u_{\cup_{j \in J} Q_j}(S) = 0$ for all $J \subseteq \{1, \dots, k\}$, which results in $v(S) = 0 = u_T^L(S)$. Next, take any $S \in 2^N$ such that there exists $K \in \widehat{C}^L(S)$ satisfying $T \subseteq K$. This K is unique and denote $M = \{j \in \{1, \dots, k\} \mid Q_j \subseteq K\}$. Among all $J \subseteq \{1, \dots, k\}$, it holds that $u_{\cup_{j \in J} Q_j}(S) = 1$ only when $J \subseteq M$, and otherwise $u_{\cup_{j \in J} Q_j}(S) = 0$. Let $|M| = m$. Then $v(S) = \sum_{J \subseteq M} (-1)^{|J|+1} u_{\cup_{j \in J} Q_j}(S) = \sum_{k=1}^{k=m} (-1)^{k+1} \binom{m}{k} = 1 = u_T^L(S)$, since it is known from the binominal theorem that $\sum_{k=0}^{k=m} (-1)^k \binom{m}{k} = 0$ and therefore $\sum_{k=1}^{k=m} (-1)^{k+1} \binom{m}{k} = -\sum_{k=1}^{k=m} (-1)^k \binom{m}{k} = \binom{m}{0} = 1$. □

Note that for any $J \subseteq \{1, \dots, k\}$, it holds that $\cup_{j \in J} Q_j$ is connected, since for each $j \in J$ the set Q_j itself is connected and contains T. This lemma shows that any restricted unanimity TU-game with communication structure can be uniquely expressed as a linear combination of unanimity TU-games with the same communication structure for connected coalitions, as the next example illustrates.

Example 2.3.13 Consider a unanimity TU-game with communication structure $(N, u_{\{1,3,5\}}, L)$, where (N, L) is a circle graph with six nodes, that is $N = \{1,2,3,4,5,6\}$ with $L = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{1,6\}\}$. There are three connected coalitions that minimally cover $\{1,3,5\}$, $\overline{C}^L(\{1,3,5\}) = \{Q_1, Q_2, Q_3\}$ where $Q_1 = \{1,2,3,4,5\}$, $Q_2 = \{1,3,4,5,6\}$, and $Q_3 = \{1,2,3,5,6\}$. From the lemma it follows that $u_{\{1,3,5\}}^L = u_{Q_1} + u_{Q_2} + u_{Q_3} - u_{Q_1 \cup Q_2} - u_{Q_1 \cup Q_3} - u_{Q_2 \cup Q_3} + u_{Q_1 \cup Q_2 \cup Q_3} = u_{Q_1} + u_{Q_2} + u_{Q_3} - 2u_N$. Indeed, $u_{\{1,3,5\}}^L(S)$ gives the worth of 1 if S is Q_1, Q_2, Q_3 or N and the worth of 0 for any other S, and those worths are also assigned by the game $u_{Q_1} + u_{Q_2} + u_{Q_3} - 2u_N$, which is a linear combination of unanimity games for connected coalitions in (N, L).

On the class of unanimity TU-games with communication structure, we have the following expression, which is well known, see for example Mishra and Talman (2010), and we present it without proof.

Lemma 2.3.14 *For any TU-game with communication structure* $(N, cu_T, L) \in \mathcal{G}_N^{cs}$ *with* $T \in C^L(N)$, $T \neq \emptyset$, *and* $c \in \mathbb{R}$, *it holds that*

$$\mu_j(N, cu_T, L) = \begin{cases} c/|T| & \text{if } j \in T, \\ 0 & \text{if } j \notin T. \end{cases}$$

This lemma says that the Myerson value of a unanimity TU-game with communication structure with a connected coalition assigns the allocation which gives zero payoffs to the players who do not belong to the connected coalition and the worth of the connected coalition is shared equally among those who belong to it. Next, we give a characterization of the Myerson value in the following theorem.

Theorem 2.3.15 *The Myerson value is the unique solution on* $\mathcal{G}_N^{cs}$ *that satisfies component efficiency, additivity, the restricted null player property, and coalitional fairness.*

Proof First, we show that the Myerson value satisfies all properties. Component efficiency is used to characterize the value in Myerson (1977) and additivity is used in van den Nouweland (1993). If a player is a restricted null player in a TU-game with communication structure $(N, v, L) \in \mathcal{G}_N^{cs}$, then this player is a null player in the restricted game v^L and therefore the Myerson value, being the Shapley value of v^L, assigns zero to this player. Finally, suppose there are two TU-games with the same communication structure $(N, v, L), (N, v', L) \in \mathcal{G}_N^{cs}$ and $Q \in C^L(N)$ such that $v(S) = v'(S)$ for all $S \in C^L(N)$, $S \neq Q$, and take any $i \in Q$. It holds that $m_i^\sigma(N, v, L) = m_i^\sigma(N, v', L)$ for any $\sigma \in \Pi(N)$ unless $P_\sigma(i) = Q \setminus \{i\}$. There are $(|Q| - 1)!(n - |Q|)!$ permutations σ such that $P_\sigma(i) = Q \setminus \{i\}$ and for each such σ the marginal contribution of i changes by $m_i^\sigma(N, v, L) - m_i^\sigma(N, v', L) = (v^L(Q) - v^L(Q \setminus \{i\})) - (v'^L(Q) - v^L(Q \setminus \{i\})) = v^L(Q) - v'^L(Q)$, which is independent of i. Therefore every player in Q receives the same change the same number of times and so the change in the Myerson value is the same among all players in Q.

Second, let $\xi : \mathcal{G}_N^{cs} \to \mathbb{R}^n$ be a solution which satisfies all four axioms. We firstly show that for any graph (N, L) it holds that $\xi(N, cu_T, L) = \mu(N, cu_T, L)$ for any $T \in C^L(N)$ and $c \in \mathbb{R}$. Let (N, L) be any graph on N and denote $\widehat{C}^L(N) = \{Q_1, \ldots, Q_h\}$ for some $h \geq 1$. First consider the zero game

$(N, \mathbf{0}, L) \in \mathcal{G}_N^{cs}$. In this game all players are restricted null players and therefore it follows from the restricted null player property that $\xi_i(N, \mathbf{0}, L) = 0 = \mu_i(N, \mathbf{0}, L)$ for all $i \in N$. Next consider the game $(N, cu_{Q_k}, L) \in \mathcal{G}_N^{cs}$ for some $1 \leq k \leq h$. Every player outside Q_k is a restricted null player of (N, cu_{Q_k}, L) and therefore receives zero payoff. Between the two games (N, cu_{Q_k}, L) and (N, v', L), where $v'(Q_k) = 0$ and $v'(S) = cu_{Q_k}(S)$ for any other $S \in 2^N$, coalitional fairness implies that

$$\xi_i(N, cu_{Q_k}, L) - \xi_i(N, v', L) = \xi_j(N, cu_{Q_k}, L) - \xi_j(N, v', L) \;\; \forall\, i, j \in Q_k.$$

Since $v'(S) = 0$ for all $S \in C^L(N)$, by Lemma 2.3.10 it holds that $\xi_i(N, v', L) = \xi_i(N, \mathbf{0}, L) = 0$ for all $i \in N$. This means that $\xi_i(N, cu_{Q_k}, L) = \xi_j(N, cu_{Q_k}, L)$ for all $i, j \in Q_k$. Together with component efficiency and Lemma 2.3.14, we have

$$\xi_i(N, cu_{Q_k}, L) = \frac{c}{|Q_k|} = \mu_i(N, cu_{Q_k}, L) \quad \forall\, i \in Q_k,$$

and therefore $\xi(N, cu_{Q_k}, L) = \mu(N, cu_{Q_k}, L)$. Now consider a game $(N, cu_T, L) \in \mathcal{G}_N^{cs}$ with $T \in C^L(N)$, $T \subset Q_k$, and $|T| = |Q_k| - 1$. It follows from the restricted null player property that any player $i \notin T$ receives zero payoff, since this player yields zero marginal contribution when joining to any set of players to form a connected coalition. For the games (N, cu_T, L) and (N, v'', L), where $v''(T) = 0$ and $v''(S) = cu_T(S)$ for any other $S \in 2^N$, coalitional fairness then implies that

$$\xi_i(N, cu_T, L) - \xi_i(N, v'', L) = \xi_j(N, cu_T, L) - \xi_j(N, v'', L) \;\; \forall\, i, j \in T.$$

Since $v''(S) = cu_{Q_k}(S)$ for all $S \in C^L(N)$, it follows from Lemma 2.3.10 that $\xi(N, v'', L) = \xi(N, cu_{Q_k}, L)$. This means that $\xi_i(N, v'', L) = \xi_j(N, v'', L)$ for all $i, j \in T$. With component efficiency and Lemma 2.3.14, this results in

$$\xi_i(N, cu_T, L) = \frac{c}{|T|} = \mu_i(N, cu_T, L) \;\; \forall\, i \in T.$$

Next, suppose $\xi(N, cu_T, L) = \mu(N, cu_T, L)$ holds for all $T \in C^L(N)$, $T \subset Q_k$, $|T| > m > 1$. Consider $(N, cu_T, L) \in \mathcal{G}_N^{cs}$ with $T \in C^L(N)$, $T \subset Q_k$, $|T| = m$. For $i \notin T$, it follows from the restricted null player property that $\xi_i(N, cu_T, L) = 0$. Define v''' such that $v'''(T) = 0$ and $v'''(S) = cu_T(S)$ for any other $S \in 2^N$. Then coalitional fairness implies

$$\xi_i(N, cu_T, L) - \xi_i(N, v''', L) = \xi_j(N, cu_T, L) - \xi_j(N, v''', L) \;\; \forall\, i, j \in T.$$

Also define $v = \sum_{\ell \in D_T^L} cu_{T \cup \{\ell\}} - (k-1) cu_{Q_k}$ where $k = |D_T^L|$ is the number of neighbors of T in (N, L). Since $v(S) = v'''(S)$ for all $S \in C^L(N)$ it follows from Lemma 2.3.10 that $\xi(N,v,L) = \xi(N,v''',L)$. From additivity and the supposition that $\xi(N,cu_S,L) = \mu(N,cu_S,L)$ for all $S \in C^L(N)$, $S \subset Q_k$ with $|S| > m$, it follows that

$$\begin{aligned}\xi_i(N,v''',L) = \xi_i(N,v,L) &= \sum_{\ell \in D_T^L} \xi_i(N,cu_{T\cup\{\ell\}},L) - (k-1)\xi_i(N,cu_{Q_k},L)\\ &= \sum_{\ell \in D_T^L} \mu_i(N,cu_{T\cup\{\ell\}},L) - (k-1)\mu_i(N,cu_{Q_k},L)\\ &= \sum_{\ell \in D_T^L} \mu_j(N,cu_{T\cup\{\ell\}},L) - (k-1)\mu_j(N,cu_{Q_k},L)\\ &=\xi_j(N,v,L) = \xi_j(N,v''',L)\end{aligned}$$

for all $i,j \in T$, and therefore

$$\xi_i(N,cu_T,L) = \xi_j(N,cu_T,L) \ \ \forall\, i,j \in T.$$

By component efficiency it holds that $\xi_i(N,cu_T,L) = c/|T|$ for all $i \in T$, which implies $\xi(N,cu_T,L) = \mu(N,cu_T,L)$. When $|T| = 1$, component efficiency and the restricted null player property imply that ξ allocates the Myerson value to $(N,cu_T,L) \in \mathcal{G}_N^{cs}$. Therefore for a multiple of any unanimity TU-game with communication structure for a connected coalition, the four axioms uniquely give the allocation of the Myerson value. Since ξ satisfies additivity and the restricted null player property, it follows from Corollary 2.3.11 that $\xi(N,v,L) = \xi(N,v^L,L)$ for any $(N,v,L) \in \mathcal{G}_N^{cs}$. By Lemma 2.3.12 it holds that v^L can be expressed as a unique linear combination of unanimity games for connected coalitions. That is, given any $(N,v,L) \in \mathcal{G}_N^{cs}$ there exist unique numbers $c_T \in \mathbb{R}$ for $T \in C^L(N)$, $T \neq \emptyset$, such that $v^L = \sum_T c_T u_T$. The proof is completed since for any $(N,v,L) \in \mathcal{G}_N^{cs}$ it holds from additivity that

$$\begin{aligned}\xi(N,v,L) &=\xi(N,v^L,L)\\ &=\xi(N, \sum_{T\in C^L(N),T\neq\emptyset} c_T u_T, L)\\ &= \sum_{T\in C^L(N),T\neq\emptyset} \xi(N,c_T u_T,L)\\ &= \sum_{T\in C^L(N),T\neq\emptyset} \mu(N,c_T u_T,L)\\ &=\mu(N,v,L).\end{aligned}$$

□

To show the independence of the four axioms, consider the following solutions for $(N,v,L) \in \mathcal{G}_N^{cs}$.

- $\xi_i(N,v,L) = 0$ for all $i \in N$.

This solution trivially satisfies additivity, the restricted null player property, and coalitional fairness. It fails component efficiency.

- $\xi(N,v,L)$ is such that:
  - $\xi(N,v,L) = \mu(N,v,L)$ if $v(\{i\}) = 0$ for some $i \in N$.
  - $\xi_i(N,v,L) = \frac{v(Q)}{|Q|}$ for $i \in Q$, $Q \in \widehat{C}^L(N)$, otherwise.

This solution satisfies component efficiency. As for the restricted null player property, suppose player $i \in N$ is a restricted null player in (N,v,L). Then $v(\{i\}) = 0$ holds and therefore $\xi_i(N,v,L) = \mu_i(N,v,L) = 0$. Regarding coalitional fairness, consider two TU-games with the same communication structure, (N,v,L) and (N,v',L), such that for some $Q \in 2^N$ it holds that $v(Q) \neq v'(Q)$ and $v(S) = v'(S)$ for $S \in 2^N \setminus \{Q\}$. First, if $Q = \{i\}$, $i \in N$, then coalitional fairness trivially holds. Next, assume $|Q| > 1$ and $v(\{i\}) \neq 0$ for all $i \in N$. If $Q \notin \widehat{C}^L(N)$, it holds that $\xi(N,v,L) = \xi(N,v',L)$ and coalitional fairness holds. If $Q \in \widehat{C}^L(N)$, then for any $i,j \in Q$ it holds that $\xi_i(N,v,L) - \xi_i(N,v',L) = \frac{v(Q)}{|Q|} - \frac{v'(Q)}{|Q|} = \xi_j(N,v,L) - \xi_j(N,v',L)$. Therefore coalitional fairness also holds for such cases. Finally, when $|Q| > 1$ and $v(\{i\}) = 0$ for some $i \in N$, the solution gives the Myerson value which satisfies coalitional fairness. Therefore the solution satisfies coalitional fairness. The solution fails additivity. Consider $(N,u_{\{1\}},L)$ and $(N,2u_{\{2\}},L)$ with $N = \{1,2\}$ and $L = \{\{1,2\}\}$. Then it holds that $\xi(N,u_{\{1\}} + 2u_{\{2\}},L) = (\frac{3}{2},\frac{3}{2}) \neq (1,2) = (1,0) + (0,2) = \xi(N,u_{\{1\}},L) + \xi(N,2u_{\{2\}},L)$.

- $\xi_i(N,v,L) = \frac{v(Q)}{|Q|}$ for all $i \in Q$ and $Q \in \widehat{C}^L(N)$.

It is easy to check that this solution satisfies component efficiency, additivity, and coalitional fairness. This solution does not satisfy the restricted null player property, as any restricted null player of a game receives non-zero payoff if the component he belongs to has non-zero worth.

- $\xi(N,v,L) = m^\sigma(N,v^L)$ with $\sigma = (1,2,\ldots,n)$.

Since every marginal vector is component efficient, additive, and satisfies the restricted null player property, this solution satisfies these properties. It fails coalitional fairness. Consider the two TU-games with the same communication structure $(N,\mathbf{0},L)$ and (N,u_N,L), where $N \in C^L(N)$. Observe that $\mathbf{0}(S) = u_N(S)$ for every $S \in 2^N \setminus \{N\}$. Then for any $j < n$ it holds that that $\xi_n(N,u_N,L) - \xi_n(N,\mathbf{0},L) = 1 - 0 = 1 \neq 0 = 0 - 0 = \xi_j(N,u_N,L) - \xi_j(N,\mathbf{0},L)$, since $\xi(N,\mathbf{0},L) = (0,\ldots,0,0)$ and $\xi(N,u_N,L) = (0,\ldots,0,1)$.

CHAPTER 3

SOLUTION CONCEPTS FOR COOPERATIVE GAMES WITH CIRCULAR COMMUNICATION STRUCTURE

3.1 Introduction

In the previous chapter we study the Myerson value on the class of TU-games with communication structure represented by an undirected graph of which the connected sets form the collection of feasible coalitions. The Myerson value for such a game is equal to the Shapley value of the corresponding Myerson restricted game. For this class of games, several other solution concepts have been introduced in the literature. For example, the position value is introduced by Meessen (1988) and Borm et al. (1992). The position value shares the Shapley value of the induced link game, another graph restricted game which defines the worth to the power set of the set of links, among the players who own a link. It is characterized by Slikker (2005) by efficiency and balanced link contributions. The latter means that for any pair of players, the total sum of the payoff losses of one player caused by breaking each link of the other player is the same for both players.

The average tree solution is introduced by Herings et al. (2008) on the class of TU-games with cycle-free communication structure. Unlike the Myerson value and the position value, this solution is not defined via some transformation of the original game but instead it is the average of the marginal vectors deduced from a specific collection of (rooted) spanning trees on the graph. For a cycle-free graph, every player induces exactly one spanning tree with himself as the root, and hence in case of n players the average tree solution is the average of n marginal vectors, while the Myerson value is the average of $n!$ marginal vectors and the position value uses $(n-1)!$ vectors on this class of graphs. On the class of TU-games with cycle-free communication structure Herings et al. (2008) shows that the average tree solution is characterized by component efficiency and component fairness. The latter means that when a link between players is deleted the average loss of players in both resulting components is the same. Another characterization of the average tree solution on the class of TU-games with connected cycle-free communication structure is given by Mishra and Talman (2010). They show that the solution is completely characterized by efficiency, the dummy property, linearity, strong symmetry, and independence in unanimity games. The last property is not satisfied by the Myerson value and says that if a player joins to the minimum winning connected coalition of a unanimity game, then the payoff of any player in the coalition not being linked to this player does not change.

Herings et al. (2010) generalizes the average tree solution to the class of TU-games with communication structure. Given a graph, they define a collection of admissible spanning trees as the ones where each player has in each component of his subordinates one successor. This selects trees on the graph which describe how the players can be partially ordered in such a way that if there is a communication link between two players, one of them should be a subordinate of the other. When the underlying graph has cycles, and therefore more communication links, there are typically more ways for players to communicate and the number of admissible spanning trees becomes larger. Baron et al. (2008) gives an axiomatization on the class of TU-games with connected communication structure as a unique solution satisfying efficiency, linearity and $\mathcal{T}$-hierarchy. The latter property means that in a unanimity TU-game with communication structure for a connected coalition the payoff is only explained by how often a player is a root in the smallest subtree that contains the coalition under all admissible spanning trees.

We study solutions on the class of TU-games with circular communication structure where the underlying graph is assumed to be a circle. Players could be firms or cities situated along a lake shore or a circular pipeline where players can only be connected to their two direct neighbors, one located on each side. A subset of players is in such a setting only able to cooperate if it consists of consecutive nodes on the circle. As described in the previous chapter, the Shapley value for a TU-game, i.e., with full communication structure, is the average of the marginal vectors corresponding to, in case of n players, all $n!$ permutations on the player set. For a TU-game with circular communication structure we propose to take as solution the average of the marginal vectors that correspond to permutations in which each player has a communication link with the player preceding him in the permutation. The idea is that if a player is not connected to the player that is immediately preceding him in the permutation then this player is not able to cooperate with his preceding players and therefore doesn't receive his marginal contribution. It turns out that on the class of TU-games with circular communication structure the average of the marginal vectors of these admissible permutations is precisely the average tree solution introduced in Herings et al. (2010). If there are n players there are $2n$ of such admissible permutations, each yielding a different marginal vector. Instead of looking only at permutations in which every player is linked to his immediate predecessor in the permutation, one could also argue that a player may join the predecessors in the permutation if he is connected to at least one of them, not being necessarily the last one. The idea here is that if a player is linked to some of the players that precede him in the permutation, he is able to cooperate with them and get his marginal contribution. Since the starting agent can be any agent and every time one of two agents can join until the last agent is left, the number of permutations is equal to $2^{n-2}n$ in case of n players. Each such permutation leads to a different marginal vector and one may take the average of these marginal vectors as solution concept. It appears that this solution is equal to the Shapley value introduced by Bilbao and Ordóñez (2009) on the class of augmenting systems and for the class of TU-games with circular communication structure it coincides with the solution proposed before. Although the two sets of permutations and of marginal vectors differ for both solutions, the resulting payoff distribution is precisely the same.

We further give for the class of TU-games with circular communication structure an axiomatization of the solution using standard axioms. We show that it is fully characterized by efficiency, additivity, the restricted null player property, those which are used in the previous chapter, and some form of symmetry, which consists of two axioms. One of the two axioms is on anonymity of the players on a circle. If every player's position shifts along the circle in one way or the opposite way, then the entries of the solution shift accordingly. This axiom is also satisfied for the Myerson value. The second axiom is on characteristic functions of two different games. A player gets the same payoff in two TU-games with circular communication structure with the same set of players, if in both games the worth of any connected coalition to which this player is connected is the same, and also the worth of such a coalition together with this player is the same. The Myerson value does not satisfy this axiom, as it is observed in Chapter 2 that the allocation the Myerson value assigns to a player may change if the worth of a coalition to which he belongs, not necessarily the one to which he is connected to, changes.

The stability of the solution on the class of TU-games with circular communication structure is studied as well. A payoff of a game is stable if no coalition can oppose to the payoff in a sense that the players in a coalition can not do better by themselves than what they get from the payoff. The notion of core is introduced by Gillies (1959) for TU-games and for TU-games with communication structure it is defined as the set of payoff vectors that are component efficient and stable for connected coalitions. On the class of TU-games with cycle-free communication structure, Herings et al. (2008) shows that superadditivity of the Myerson restricted game is a sufficient condition under which the average tree solution is an element of the core of the game. A game is superadditive if the worth of the union of any two disjoint coalitions is at least equal to the total worth of both. This condition is further weakened by Talman and Yamamoto (2007). On the class of TU-games with communication structure which may contain cycles, Herings et al. (2010) introduces the notion of link-convexity. If the underlying communication structure is cycle-free, link-convexity is weaker than superadditivity and stronger than the condition found in Talman and Yamamoto (2007). In this chapter we introduce the notion of circular-convexity on the class of TU-games with circular communication structure. Circular-convexity is weaker than convexity but stronger than superadditivity. This convexity condition is equivalent to link-convexity on the class of TU-games with circular communication structure. It is well known that for TU-games convexity is equivalent to the property that all marginal vectors lie in the core and therefore also the Shapley value. We show that for a TU-game with circular communication structure, circular-convexity is a necessary and sufficient condition to guarantee that every admissible marginal vector for the average tree solution is an element of the core and therefore also the average tree solution. A stronger version of circular-convexity is also given as a necessary and sufficient condition for every marginal vector for the Shapley value introduced in Bilbao and Ordóñez (2009) to be in the core. We further give a necessary and sufficient condition for the solution itself to be in the core, called average-circular-convexity. This condition is not necessarily stronger than superadditivity, and weaker than circular-convexity. We also illustrate that the Myerson value may not be in the core if the game is circular-convex.

This chapter is organized as follows. Section 2 introduces TU-games with circular communication structure and the solutions. In Section 3 the axiomatic characterizations for the solutions are given. In Section 4 stability of the solution concepts is discussed. This chapter is partly based on Selçuk et al. (2013).

3.2 TU-games with circular communication structure and solutions

Consider a finite number of nodes or agents located on a circle. The nodes could for example be villages along a lake shore or around a mountain, shopping malls along a ring road of a city, or companies connected to a circular pipeline. Let the set $N = \{1, \dots, n\}$ denote the set of nodes, with $n \geq 3$. Given the location of the nodes on a circle, we assume without loss of generality that each node $i \in N$ has two neighbors, $i-1$ and $i+1$, and that there is a link between any node and each of his neighbors, where we adopt the convention and $i-1 = n$ when $i = 1$, and $i+1 = 1$ when $i = n$. Let L_N^{circle} denote the set of links between any two neighbors, that is $L_N^{circle} = \{\{i, i+1\} \mid i = 1, \dots, n\}$. A TU-game with circular communication structure is a triple (N, v, L_N^{circle}). Let $\mathcal{G}_N^{circle}$ denote the class of TU-games with circular communication structure with fixed player set N. In this chapter we fix $L = L_N^{circle}$ unless otherwise mentioned.

For $i, j \in N$, we use S_i^j to express the connected coalition containing all players from i to j in a circle graph (N, L), i.e., $S_i^j = \{i, i+1, \dots, j\}$ if $j \geq i$ and $S_i^j = \{i, i+1, \dots, n, 1, \dots, j\}$ if $i > j$. Notice that $S_i^{i-1} = N$, where $i-1 = n$ when $i = 1$, and $S_i^i = \{i\}$, $i = 1, \dots, n$. A solution on $\mathcal{G}_N^{circle}$ is a function $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ which assigns to every TU-game with circular communication structure (N, v, L) a payoff vector $\xi(N, v, L)$. The core of a TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$ is defined as

$$C(N, v, L) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N),\ \sum_{i \in S} x_i \geq v(S),\ \forall\, S \in C^L(N)\}.$$

The core is the set of allocations that are efficient, $\sum_{i=1}^{n} x_i = v(N)$, and are not opposed by any connected coalition, that is $\sum_{i \in S} x_i \geq v(S)$ for all $S \in$

$C^L(N)$. For the class of TU-games, in which all coalitions are feasible, the core is introduced by Gillies (1959) and is for a TU-game (N, v) given by

$$C(N,v) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N),\ \sum_{i \in S} x_i \geq v(S),\ \forall\, S \in 2^N\}.$$

Note that for any TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$ it holds that $C(N, v, L) = C(N, v^L)$, where (N, v^L) is the Myerson restricted game of (N, v, L) as described in Chapter 2.

The Shapley value of a TU-game, in which any coalition is connected in terms of its communication structure, can be interpreted as follows. To form the grand coalition, agents enter a room randomly one-by-one and if an agent enters he connects to the last person who entered before and he receives his marginal contribution for joining the agents who are already present in the room. In this way a permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$ is obtained in which first agent $\sigma(1)$ enters, which can be any of the n agents, and this agent receives his worth $v(\{\sigma(1)\})$, the minimum amount to let him stay in the room. Then agent $\sigma(2)$ enters, which can be any of the remaining $n-1$ agents, he receives as payoff his marginal contribution $v(\{\sigma(1), \sigma(2)\}) - v(\{\sigma(1)\})$ when joining agent $\sigma(1)$, otherwise the two agents would not stay together in the room, and agent $\sigma(2)$ connects to agent $\sigma(1)$ to form the ordering $(\sigma(1), \sigma(2))$. Then from the remaining $n-2$ agents agent $\sigma(3)$ enters, gets as payoff his marginal contribution $v(\{\sigma(1), \sigma(2), \sigma(3)\}) - v(\{\sigma(1), \sigma(2)\})$, otherwise the three agents would not stay together in the room, and connects to agent $\sigma(2)$, the last agent who joined before, to form the ordering $(\sigma(1), \sigma(2), \sigma(3))$, and so on, until the last agent, $\sigma(n)$, enters, gets his marginal contribution $v(N) - v(N \setminus \{\sigma(n)\})$, and connects to agent $\sigma(n-1)$, the last agent who joined before, to complete the ordering σ in forming the grand coalition N. In general, for $k = 2, \ldots, n$, when $k-1$ agents, $\sigma(1), \ldots, \sigma(k-1)$, have entered the room before, agent $\sigma(k)$, one of the remaining $n-k+1$ agents, enters. This agent gets as payoff $v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$, being his contribution when joining the agents in the room, otherwise they would leave the room, and connects to agent $\sigma(k-1)$, the last agent who entered before, to form the ordering $(\sigma(1), \ldots, \sigma(k))$. This process generates a marginal vector and the Shapley value is the average of all such marginal vectors.

In a circle graph it is not the case that every agent is connected to every other agent. We assume that if an agent enters the room as described above and he is not connected to the last agent that entered before, then the grand coalition cannot be formed. The idea is that if there is no link between the agent who enters and the last agent who entered before, the entering agent is not able to communicate and therefore cannot form a coalition with the agents who entered before. For example, the last agent who entered got the technological facilities to connect to the agent who entered before and is now the only agent in the room who is able to connect the next agent. In this way only orderings σ are able to form the grand coalition in which, for $k = 2, \ldots, n$, once the $k-1$ agents $\sigma(1), \ldots, \sigma(k-1)$ have entered the room in this order, agent $\sigma(k)$ will only enter and stay in the room if he is connected to the last agent that entered before, being agent $\sigma(k-1)$. In this case he receives as payoff his marginal contribution $v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$, otherwise the k agents would not stay together in the room, and agent $\sigma(k)$ connects to agent $\sigma(k-1)$ to form the ordering $(\sigma(1), \ldots, \sigma(k))$. In other words, we assume that only if for all $k = 2, \ldots, n$ node $\sigma(k)$ is linked to node $\sigma(k-1)$, then the grand coalition N can be formed through the ordering σ and every agent receives his marginal contribution. If, for at least one $k \in N$, agent $\sigma(k)$ is not connected to $\sigma(k-1)$, then we assume that the grand coalition cannot be formed through the ordering σ. In case agent $i = \sigma(1)$ enters the room first, there are just two agents, agent $i-1$ (agent n when $i = 1$) and agent $i+1$ (agent 1 when $i = n$), being connected to agent i and who therefore may enter the room to join agent i. After one of these two agents enters, there is only one of the remaining $n-2$ agents who can enter and join the agent who entered before, and so on, until the last remaining agent enters. This leads to $2n$ different orderings, or permutations, through which the grand coalition can be formed. Let us call these permutations admissible. For each node $i \in N$ there are two admissible permutations σ with $\sigma(1) = i$, denoted $\sigma_1^i = (i, i+1, \ldots, n, 1, \ldots, i-1)$ and $\sigma_2^i = (i, i-1, \ldots, 1, n, \ldots, i+1)$. The set of admissible permutations is then given by

$$\Pi^a(N) = \{\sigma_1^i \mid i = 1, \ldots, n\} \cup \{\sigma_2^i \mid i = 1, \ldots, n\}.$$

Given a TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$, to any admissible permutation $\sigma \in \Pi^a(N)$ a marginal vector $m^\sigma(N, v, L)$ corresponds and assigns payoff

$$m_i^\sigma(N, v, L) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)) \tag{3.1}$$

to agent $i = 1, \ldots, n$. As solution concept we take the average of these $2n$ marginal vectors,

$$\frac{1}{2n} \sum_{\sigma \in \Pi^a(N)} m^\sigma(N, v, L).$$

We show now that this solution coincides with the average tree solution. The average tree solution is introduced by Herings et al. (2010) on the class of TU-games with communication structure and is defined on the class of TU-games with circular communication structure as follows.

An n-tuple $B = (B_1, \dots, B_n)$ of connected coalitions in a circle graph (N, L) is admissible if there is some $r \in N$ such that $B_r = N$, for all $i \in N$ it holds that $i \in B_i$, and if $B_i \setminus \{i\} \neq \emptyset$ there exists a unique $j \in N$ satisfying $\{i, j\} \in L$ and $B_j = B_i \setminus \{i\}$. Let $\mathcal{B}^L$ denote the collection of admissible n-tuples of connected coalitions in (N, L). The next example illustrates the concept of admissible n-tuples in a circle graph.

Example 3.2.1 Consider a circle graph (N, L) with $n = 4$ and suppose $B = (B_1, B_2, B_3, B_4)$ is admissible with $B_2 = N$. Since $B_2 \setminus \{2\} = \{1, 3, 4\}$ and 2 is linked to 1 and 3, it must hold that either $B_1 = \{1, 3, 4\}$ or $B_3 = \{1, 3, 4\}$, not both. If $B_1 = \{1, 3, 4\}$, then $B_1 \setminus \{1\} = \{3, 4\}$. Since 1 is linked to both 2 and 4 but $B_2 = N$, it follows that $B_4 = \{3, 4\}$, and we obtain $B = (\{1, 3, 4\}, N, \{3\}, \{3, 4\})$. If $B_3 = \{1, 3, 4\}$, then we obtain $B = (\{1\}, N, \{1, 3, 4\}, \{3, 4\})$. Note for example that $B = (\{1, 3, 4\}, N, \{3, 4\}, \{4\})$ is not admissible because in this case $B_1 \setminus \{1\} = B_3$, where $\{1, 3\} \notin L$.

Given a TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$, to an admissible $B \in \mathcal{B}^L$ the marginal vector $m^B(N, v, L)$ corresponds, defined by

$$m_i^B(N, v, L) = v(B_i) - v(B_i \setminus \{i\}),\ i \in N.$$

The average tree solution of a TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$ is then defined as the average of the marginal vectors corresponding to all admissible n-tuples of connected coalitions in (N, L),

$$AT(N, v, L) = \frac{1}{|\mathcal{B}^L|} \sum_{B \in \mathcal{B}^L} m^B(N, v, L).$$

Theorem 3.2.2 *For any TU-game with circular communication structure* $(N, v, L) \in \mathcal{G}_N^{circle}$ *it holds that*

$$AT(N, v, L) = \frac{1}{2n} \sum_{\sigma \in \Pi^a(N)} m^\sigma(N, v, L).$$

Proof Take any $\sigma \in \Pi^a(N)$ and suppose $\sigma = \sigma_1^i$ for some $i \in N$. Define $B_k = \{i, \dots, k\}$ for $k = i, \dots, n$, and $B_k = \{i, \dots, n, 1, \dots, k\}$ for $k = 1, \dots, i-1$. Then $B = (B_1, \dots, B_n)$ is an admissible n-tuple of connected coalitions in

(N, L), satisfying $m^B(N, v, L) = m^{\sigma_1^i}(N, v, L)$. Similarly, when $\sigma = \sigma_2^i$ for some $i \in N$, define $B_k = \{k, \ldots, i\}$ for $k = 1, \ldots, i$, and $B_k = \{k, \ldots, n, 1, \ldots, i\}$ for $k = i + 1, \ldots, n$. Then again this $B = (B_1, \ldots, B_n)$ is an admissible n-tuple of connected coalitions in (N, L), satisfying $m^B(N, v, L) = m^{\sigma_2^i}(N, v, L)$. Therefore every permutation in $\Pi^a(N)$ corresponds to a unique admissible n-tuple of connected coalitions in (N, L). Next, let $B = (B_1, \ldots, B_n)$ be an admissible n-tuple of connected coalitions in (N, L). Then there exists unique $i \in N$ such that $B_i = N$. Consider the set $B_i \setminus \{i\} = N \setminus \{i\}$. This set has two elements that are linked to i, namely $i - 1$ and $i + 1$, where $i - 1 = n$ when $i = 1$ and $i + 1 = 1$ when $i = n$. So, $B_i \setminus \{i\}$ is either B_{i+1} (B_1 when $i = n$) or B_{i-1} (B_n when $i = 1$). Suppose $B_i \setminus \{i\} = B_{i+1}$. Then, when $i < n$, $i + 2$ is the only element of $B_{i+1} \setminus \{i + 1\}$ that is linked to $i + 1$ and so $B_{i+1} \setminus \{i + 1\} = B_{i+2}$, and, when $i = n$, 2 is the only element of $B_1 \setminus \{1\}$ that is linked to 1, and so on. In every further step there is only one element in $B_k \setminus \{k\}$ that is linked to k, and that is the element $k + 1$ and so $B_k \setminus \{k\} = B_{k+1}$, for $k = i + 1, \ldots, n, 1, \ldots, i$. From this it follows that $m^B(N, v, L) = m^{\sigma_2^{i-1}}(N, v, L)$. Similarly, if $B_i \setminus \{i\} = B_{i-1}$, it holds that $m^B(N, v, L) = m^{\sigma_1^{i+1}}(N, v, L)$. Therefore every admissible n-tuple of connected coalitions in (N, L) corresponds to a unique permutation in $\Pi^a(N)$, which completes the proof. □

The theorem says that on the class of TU-games with circular communication structure the average tree solution is equal to the average of all marginal vectors that correspond to permutations on the players set in which every two consecutive players of the permutation are neighbors of each other. Only such permutations are assumed to be able to form the grand coalition, coming from the interpretation of the Shapley value described above.

The Shapley value of a TU-game can also be interpreted in a slightly different way. To form the grand coalition, agents enter a room randomly one-by-one and if an agent enters he just joins the set of agents that are already present in the room and he receives his marginal contribution. In this interpretation an entering agent does not connect to the last agent who entered before but he just joins the set of agents who entered before. The Shapley value can be seen as the average of such vectors of marginal contributions.

Under the circular communication structure with this interpretation, it holds for an ordering σ that when the $k - 1$ agents $\sigma(1), \ldots, \sigma(k - 1)$ have entered the room, the next agent, agent $\sigma(k)$, can only enter and gets his marginal contribution if he is connected to at least one of the $k - 1$ preceding agents, not necessarily being agent $\sigma(k - 1)$. The idea is that a player can only join a coalition to form a larger coalition if he is able to communicate with at least one member of that coalition. Let us call such orderings compatible. Observe that any admissible ordering is also compatible. After the first agent $\sigma(1)$, which can be any of the n agents, enters the room, the second agent who enters, agent $\sigma(2)$, can be any of the two neighbors of $\sigma(1)$, which is the same in the previous interpretation. However, agent $\sigma(3)$, who enters next, can be either the remaining neighbor of agent $\sigma(1)$, not being agent $\sigma(2)$, or the remaining neighbor of agent $\sigma(2)$, not being agent $\sigma(1)$. In general, if $\sigma(1), \ldots, \sigma(k-1)$ have entered, then agent $\sigma(k)$, who is entering the room next, is connected to one of the two end points of the induced connected coalition on the set $\{\sigma(1), \ldots, \sigma(k-1)\}$.

Given the first agent $\sigma(1)$, which can be any of the n agents, there are two choices of $\sigma(2)$ for being compatible with $\sigma(1)$. In general, for $2 \leq k \leq n-1$, there are two choices of $\sigma(k)$ for being compatible with $(\sigma(1), \sigma(2), \ldots, \sigma(k-1))$. For the case $k = n$, the last agent, $\sigma(n)$, is uniquely determined. This leads to $2^{n-2}n$ different compatible orderings, or permutations, through which the grand coalition can be formed. The set of compatible permutations can be defined as

$$\Pi^c(N) = \{\sigma \in \Pi(N) | P_\sigma(i) \in C^L(N) \ \forall \ i \in N\}.$$

As solution concept we may take the average of the marginal vectors induced by all compatible permutations,

$$\frac{1}{2^{n-2}n} \sum_{\sigma \in \Pi^c(N)} m^\sigma(N, v, L),$$

where for $\sigma \in \Pi^c(N)$ the vector $m^\sigma(N, v, L)$ is defined in the same way as above. We show now that this solution coincides with the Shapley value introduced by Bilbao and Ordóñez (2009) on the class of games with augmenting systems, which contains the class of TU-games with circular communication structure. An augmenting system on the set N is defined as a pair $(N, \mathcal{F})$ where $\mathcal{F} \subseteq 2^N$ satisfies: $\emptyset \in \mathcal{F}$; for $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{F}$; for $S, T \in \mathcal{F}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$. A TU-game on an augmenting system is a triple $(N, v, \mathcal{F})$ where $(N, \mathcal{F})$ is an augmenting system describing the set of feasible coalitions each of which is able to cooperate to earn its worth and distribute it freely among the players in it. We denote $\mathcal{G}_N^{as}$ the class of TU-games on augmenting systems with fixed player set N. As mentioned in Bilbao and Ordóñez (2009), $(N, C^L(N))$ is an augmenting system that represents the collection of feasible coalitions, i.e., the collection of connected coalitions of the communication structure (N, L), not necessarily a circle graph, and therefore $\mathcal{G}_N^{circle} \subset \mathcal{G}_N^{as}$.

Given an augmenting system $(N, \mathcal{F})$ with $N \in \mathcal{F}$, an ordering $\rho = (\rho(1), \ldots, \rho(n)) \in \Pi(N)$ is compatible on $(N, \mathcal{F})$ if $\{\rho(1), \ldots, \rho(k)\} \in \mathcal{F}$ for all $k = 1, \ldots, n$, and corresponds one-to-one to a maximal chain R in $\mathcal{F}$, being a collection of coalitions in $\mathcal{F}$ ordered with respect to set inclusion that is not contained in any larger chain in $\mathcal{F}$. Let $Ch(\mathcal{F})$ denote the set of maximal chains in $\mathcal{F}$. Given $(N, v, \mathcal{F}) \in \mathcal{G}_N^{as}$ with $N \in \mathcal{F}$, each maximal chain $R \in Ch(\mathcal{F})$ corresponding to compatible ordering ρ on $(N, \mathcal{F})$ induces a marginal vector which assigns

$$m^R_{\rho(i)}(N, v, \mathcal{F}) = v(\{\rho(1), \ldots, \rho(i)\}) - v(\{\rho(1), \ldots, \rho(i-1)\})$$

to agent $\rho(i) \in N$. Then the Shapley value of a game $(N, v, \mathcal{F}) \in \mathcal{G}_N^{as}$ with $N \in \mathcal{F}$ is defined in Bilbao and Ordóñez (2009) as the average of the marginal vectors induced from all maximal chains in $\mathcal{F}$, i.e.,

$$Sh(N, v, \mathcal{F}) = \frac{1}{|Ch(\mathcal{F})|} \sum_{R \in Ch(\mathcal{F})} m^R(N, v, \mathcal{F}).$$

Theorem 3.2.3 *For any TU-game with circular communication structure* $(N, v, L) \in \mathcal{G}_N^{circle}$ *it holds that*

$$Sh(N, v, C^L(N)) = \frac{1}{2^{n-2}n} \sum_{\sigma \in \Pi^c(N)} m^\sigma(N, v, L).$$

Proof The communication structure induced from a circle graph (N, L) is equivalent to the one induced from an augmenting system $(N, \mathcal{F})$ with the set of feasible coalitions equal to $\mathcal{F} = C^L(N)$. We show that there is a one-to-one relation between $\Pi^c(N)$ and $Ch(C^L(N))$. For any maximal chain $R \in Ch(C^L(N))$ with corresponding compatible ordering ρ on $(N, C^L(N))$, ρ is an element in $\Pi^c(N)$. Conversely, any compatible permutation $\sigma \in \Pi^c(N)$ is a compatible ordering on $(N, C^L(N))$ corresponding to some maximal chain R in $C^L(N)$. Therefore, for any maximal chain R in $Ch(C^L(N))$ with corresponding compatible permutation $\sigma \in \Pi^c(N)$ it holds for all $i \in N$ that

$$\begin{aligned} m_i^\sigma(N, v, L) &= v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)) \\ &= v(\{j \in N | \sigma(j) < \sigma(i)\} \cup \{i\}) - v(\{j \in N | \sigma(j) < \sigma(i)\}) \\ &= v(\{\sigma(1), \ldots, \sigma(\sigma^{-1}(i))\}) - v(\{\sigma(1), \ldots, \sigma(\sigma^{-1}(i) - 1)\}) \\ &= m_i^R(N, v, C^L(N)). \end{aligned}$$

□

Thus far, we have obtained two solution concepts on the class of TU-games with circular communication structure, following from two different interpretations of the Shapley value on the class of TU-games. The first one, which turns out to be the average tree solution introduced by Herings et al. (2010), is the average of $2n$ marginal vectors, while the other solution, which turns out to be the Shapley value as introduced by Bilbao and Ordóñez (2009), is the average of $2^{n-2}n$ marginal vectors. Generically, all latter marginal vectors are different and they contain the former ones. Nevertheless the two averages are the same, that is, the two solution concepts introduced above coincide on the class of TU-games with circular communication structure.

Theorem 3.2.4 *On the class of TU-games with circular communication structure the average tree solution and the Shapley value in Bilbao and Ordóñez (2009) coincide, that is, for any* $(N, v, L) \in \mathcal{G}_N^{circle}$*, it holds that* $AT(N, v, L) = Sh(N, v, C^L(N))$.

Proof For any TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$ it has to be shown that

$$\frac{1}{|\Pi^a(N)|} \sum_{\sigma \in \Pi^a(N)} m^\sigma(N, v, L) = \frac{1}{|\Pi^c(N)|} \sum_{\sigma \in \Pi^c(N)} m^\sigma(N, v, L).$$

Take any $S \in C^L(N)$ and $i \notin S$ satisfying $S \cup \{i\} \in C^L(N)$. Let $\Pi^a_{S,i}$ and $\Pi^c_{S,i}$ denote the subsets of admissible and compatible permutations σ satisfying $m_i^\sigma(N, v, L) = v(S \cup \{i\}) - v(S)$, respectively. It suffices to show that $\frac{|\Pi^a_{S,i}|}{|\Pi^a(N)|} = \frac{|\Pi^c_{S,i}|}{|\Pi^c(N)|}$. If $S = \emptyset$, then $\Pi^a_{S,i} = \{\sigma_1^i, \sigma_2^i\}$ and $\Pi^c_{S,i}$ consists of 2^{n-2} compatible permutations σ with $\sigma(1) = i$, and therefore $\frac{|\Pi^a_{S,i}|}{|\Pi^a(N)|} = \frac{1}{n} = \frac{|\Pi^c_{S,i}|}{|\Pi^c(N)|}$. If $S = N \setminus \{i\}$, then $\Pi^a_{S,i} = \{\sigma_1^{i+1}, \sigma_2^{i-1}\}$ and $\Pi^c_{S,i}$ consists of 2^{n-2} compatible permutations σ with $\sigma(n) = i$, and therefore $\frac{|\Pi^a_{S,i}|}{|\Pi^a(N)|} = \frac{1}{n} = \frac{|\Pi^c_{S,i}|}{|\Pi^c(N)|}$. Otherwise, $|\Pi^a_{S,i}| = 1$ and $|\Pi^c_{S,i}| = 2^{|S|-1} \cdot 2^{n-|S|-2} = 2^{n-3}$, the number of compatible permutations with $\sigma(|S|+1) = i$, where the first term of the product is the number of ways to fill the first $|S|$ positions and the second term is the number of ways to fill the last $n - |S| - 1$ positions. Therefore it holds that $\frac{|\Pi^a_{S,i}|}{|\Pi^a(N)|} = \frac{1}{2n} = \frac{|\Pi^c_{S,i}|}{|\Pi^c(N)|}$ for any such S, which completes the proof. □

Although both solutions differ in terms of the number of marginal vectors to take the average of, where the Shapley value in Bilbao and Ordóñez (2009) takes the average of 2^{n-3} times more different marginal vectors than the average tree solution does, they coincide on the class of TU-games with circular communication structure. This is because for any $i \in N$ and $S \in$

$C^L(N) \setminus \{N \setminus \{i\}, \emptyset\}$ such that $i \notin S$ and $S \cup \{i\} \in C^L(N)$ the marginal contribution $v(S \cup \{i\}) - v(S)$ appears only once as entry among the admissible marginal vectors and 2^{n-3} times among the compatible marginal vectors, and when $S \in \{N \setminus \{i\}, \emptyset\}$, $v(S \cup \{i\}) - v(S)$ appears twice as entry among the admissible marginal vectors and 2^{n-2} times among the compatible marginal vectors, and therefore the two averages are equal to each other.

3.3 Axiomatic characterization

In this section, we give characterizations for the average tree solution and therefore also for the Shapley value in Bilbao and Ordóñez (2009) on the class of TU-games with circular communication structure.

In the previous section it is shown that the average tree solution coincides with the Shaply value introduced by Bilbao and Ordóñez (2009) on the class of TU-games with circular communication structure. Thus on this class of games the average tree solution has the same characteristics as the latter value has. In Bilbao and Ordóñez (2009), this value is characterised with the hierarchical strength axiom. As discussed in Section 3.1, Baron et al. (2008) shows that the average tree solution is the unique solution on the class of TU-games with connected communication structure that satisfies efficiency, linearity and $\mathcal{T}$-hierarchy. Both ideas are related to the work of Faigle and Kern (1992). Given a connected coalition from a communication structure, the hierarchical strength of a player in the coalition is defined. In Bilbao and Ordóñez (2009), it is equivalent to the proportion of the maximal chains where the player joins last among the players in the coalition. In Baron et al. (2008), it is equivalent to the proportion of the admissible trees where the player is the root of the subtree which minimally contains all players in the coalition. On the class of circular communication structure, both hierarchical strengths coincide and therefore both axioms are equivalent on the class of TU-games with circular communication structure.

For the class of TU-games with cycle-free communication structure, a characterization of the average tree solution is given in Herings et al. (2008) with the component fairness axiom, in comparison with the fairness axiom for the Myerson value. Given a TU-game with cycle-free communication structure, a deletion of a link always yields two new components from its original cycle-free communication structure, and component fairness says that the average payoff change among players who are in one of the two components is equal to that in the other component. This axiom is not applicable to the solution defined on TU-games with circular communication structure because deleting a link does not create two components, and furthermore, deleting a link leads to a different class than the class of TU-games with circular communication structure. In van den Brink (2009) a characterization for the average tree solution on TU-games with cycle-free communication structure is given by using axioms on the behavior of the solution of a game when a pair of linked players can collude, which is expressed as a change in the characteristic function of the game. In Mishra and Talman (2010) another characterization of the average tree solution on this class of games is given by using some symmetry axioms defined on characteristic functions. They show that the average tree solution can be characterized by (component) efficiency, strong symmetry, the restricted null player property (they call it dummy), linearity, strong symmetry, and independence in unanimity games. Our axiomatic approach on the class of TU-games with circular communication structure is in line with the last. The first three properties are also introduced in the previous chapter for the class of TU-games with communication structure.

Definition 3.3.1 A solution $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ satisfies *efficiency* if for any $(N, v, L) \in \mathcal{G}_N^{circle}$ it holds that $\sum_{i \in N} \xi_i(N, v, L) = v(N)$.

Definition 3.3.2 A solution $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ satisfies *additivity* if for any (N, v, L), $(N, w, L) \in \mathcal{G}_N^{circle}$ it holds that $\xi(N, v + w, L) = \xi(N, v, L) + \xi(N, w, L)$.

Definition 3.3.3 A solution $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ satisfies the *restricted null player property* if for any $(N, v, L) \in \mathcal{G}_N^{circle}$ and restricted null player $i \in N$ in (N, v, L) it holds that $\xi_i(N, v, L) = 0$.

As a consequence of Lemma 2.3.10, we have the following.

Corollary 3.3.4 *If a solution $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ satisfies additivity and the restricted null player property, then $\xi(N, v, L) = \xi(N, v^L, L)$ for any $(N, v, L) \in \mathcal{G}_N^{circle}$.*

The next two axioms together form a kind of symmetry with respect to the circle.

Definition 3.3.5 A solution $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ satisfies *symmetry among players* if for any admissible permutation $\pi \in \Pi^a(N)$ and game $(N, v, L) \in \mathcal{G}_N^{circle}$ it holds that $\xi_i(N, v', L) = \xi_{\pi(i)}(N, v, L)$, where $v'(S) = v(\pi(S))$ for all $S \in 2^N$.

Symmetry among players of a solution means that if two TU-games with circular communication structure differ only by a shift of the players along the circle in one way or the reverse way, i.e., a reordering of the players through an admissible permutation, then the entries of the solution differ only by this shift.

Definition 3.3.6 A solution $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ satisfies *symmetry between games* if for any (N, v, L), $(N, v', L) \in \mathcal{G}_N^{circle}$ and $i \in N$, it holds that $\xi_i(N, v, L) = \xi_i(N, v', L)$ when $v(S) = v'(S)$ and $v(S \cup \{i\}) = v'(S \cup \{i\})$ for all $S \in C^L(N)$ satisfying $i \notin S$ and $S \cup \{i\} \in C^L(N)$.

Symmetry between games of a solution implies that in two different TU-games with circular communication with the same player set, a player gets the same payoff if in both games the worth of any connected coalition to which this player is connected is the same and also the worth of such a coalition together with this player is the same.

Given a connected coalition $T \in C^L(N)$, player $j \in T$ is an end player of T if $T \setminus \{j\} \in C^L(N)$. Let $E(T)$ denote the set of end players of connected coalition T, i.e., $E(T) = \{i \in T \mid T \setminus \{i\} \in C^L(N)\}$. Notice that $E(N) = N$ and $E(\{j\}) = \{j\}$ for all $j \in N$. First we give in the following lemma an expression of the average tree solution on the class of unanimity TU-games with circular communication structure for a connected coalition.

Lemma 3.3.7 *For any unanimity TU-game with circular communication structure $(N, cu_T, L) \in \mathcal{G}_N^{circle}$ with $T \in C^L(N)$ and $c \in \mathbb{R}$, it holds that*

$$AT_j(N, cu_T, L) = \begin{cases} 0 & \text{if } j \notin T, \\ c & \text{if } j \in T \text{ and } |T| = 1, \\ (n - |T| + 2)c/(2n) & \text{if } j \in E(T) \text{ and } |T| > 1, \\ c/n & \text{if } j \in T \setminus E(T). \end{cases}$$

Proof If $j \notin T$, then for any $\sigma \in \Pi^a(N)$ it holds that $m_j^\sigma(N, cu_T, L) = 0$ and therefore the average tree solution assigns zero. If $T = \{j\}$, then $m_j^\sigma(N, cu_T, L) = c$ for all admissible permutations and the average tree solution assigns c to this player. If $|T| > 1$ and $j \in E(T)$, consider any admissible permutation σ such that $\sigma(k) = j$, $1 \leq k \leq n$. It holds that $m_j^\sigma(N, cu_T, L) = 0$ for $k < |T|$. For each k, $|T| \leq k \leq n - 1$, there is one admissible permutation σ with $\sigma(k) = j$ such that $T \setminus \{j\} \subseteq P_\sigma(j)$ and therefore $m_j^\sigma(N, cu_T, L) = c$. When $k = n$, both admissible permutations σ with $\sigma(n) = j$ yield $m_j^\sigma(N, cu_T, L) = c$. In total, out of $2n$ marginal vectors induced from admissible permutations, j yields a marginal contribution of c in $(n - |T|) + 2$ vectors and zero in the others. Thus $AT_j(N, cu_T, L) = (n - |T| + 2)c/(2n)$ holds. Finally, if $j \in T \setminus E(T)$, among $\sigma \in \Pi^a(N)$, $m_j^\sigma(N, cu_T, L) = c$ occurs for the two permutations σ with

$\sigma(n) = j$, and otherwise $m_j^\sigma(N, cu_T, L) = 0$. This gives $AT_j(N, cu_T, L) = c/n$, which completes the proof. □

Note in the proof that when $T = N$ it follows that $(n - |T| + 2)c/(2n) = c/n$.

Next, we give a characterization of the average tree solution on the class of TU-games with circular communication structure.

Theorem 3.3.8 *On the class of TU-games with circular communication structure, the average tree solution is the unique solution satisfying efficiency, additivity, the restricted null player property, symmetry among players, and symmetry between games.*

Proof First, we show that the average tree solution satisfies all properties. Efficiency follows from the fact that all marginal vectors are efficient by construction. Since all admissible marginal vectors of a TU-game with circular communication structure are linear in the worths of the connected coalitions and the average tree solution is the average of these vectors, the average tree solution satisfies additivity. If a player is a restricted null player, this player has marginal contribution equal to zero at any admissible permutation and therefore the average is also zero. If players are shifted or reversely shifted along the circle, the entries of the marginal vectors corresponding to admissible permutations shift accordingly and therefore also their average. Finally, if in two different games a player has the same marginal contribution to any connected coalition he is connected to, each admissible marginal vector assigns to that player in both games the same payoff and therefore also their average.

Second, let $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ be a solution which satisfies all five axioms. The proof is first done for the class of unanimity TU-games with circular communication structure for connected coalitions. Consider the game $(N, cu_N, L) \in \mathcal{G}_N^{circle}$. Take $\pi = \sigma_1^k$ for some $k \neq 1$. Then $\pi \in \Pi^a(N)$ and $cu_N(\pi(S)) = cu_N(S)$ for all $S \in C^L(N)$. From symmetry among players it follows that $\xi_1(N, cu_N, L) = \xi_k(N, cu_N, L)$, which implies that in the game (N, cu_N, L) all players receive the same payoff. By efficiency this yields

$$\xi_k(N, cu_N, L) = \frac{c}{n} = AT_k(N, cu_N, L) \quad \forall\, k \in N.$$

For the game $(N, cu_T, L) \in \mathcal{G}_N^{circle}$ with $T = \{i\}$ it follows from efficiency and the restricted null player property that player i receives c and all other players 0 as in the average tree solution. Now, take any $T \in C^L(N)$ with $1 < |T| < n$ and consider $(N, cu_T, L) \in \mathcal{G}_N^{circle}$. Then each $i \notin T$ is a restricted null player in (N, cu_T, L) and therefore this player receives zero payoff as in the average tree solution. Next, let $i \in T \setminus E(T)$, then for all $S \in C^L(N)$ such that $i \notin S$ and $S \cup \{i\} \in C^L(N)$ it holds that $cu_T(S \cup \{i\}) = cu_N(S \cup \{i\})$. From symmetry between games it follows that

$$\xi_i(N, cu_T, L) = \xi_i(N, cu_N, L) = \frac{c}{n} = AT_i(N, cu_T, L).$$

Finally, let $i \in E(T)$. Because of symmetry among players and since $1 < |T| < n$, we may assume that $i = 1$ and $T = \{1, \ldots, j\}$ for some $1 < j < n$. Let $\pi = \sigma_2^j$, then $\pi \in \Pi^a(N)$, $\pi(1) = j$, and $\pi(T) = T$. Define the game $(N, v', L) \in \mathcal{G}_N^{circle}$ by $v'(S) = cu_T(\pi(S))$ for all $S \in 2^N$. Because of symmetry among players and since $\pi(1) = j$, it holds that $\xi_j(N, v', L) = \xi_1(N, cu_T, L)$. Moreover, $v'(S \cup \{j\}) = c = cu_T(S \cup \{j\})$ if $S \in C^L(N)$ and $S \supseteq T \setminus \{j\}$, and $v'(S \cup \{j\}) = 0 = cu_T(S \cup \{j\})$ for all other $S \in C^L(N)$. From symmetry between games it then follows that $\xi_j(N, v', L) = \xi_j(N, cu_T, L)$. Together this implies $\xi_j(N, cu_T, L) = \xi_1(N, cu_T, L)$, and so the two end players of T receive the same payoff in the game (N, cu_T, L). From efficiency and the facts that all other players in T receive payoff $\frac{c}{n}$ and all players outside T receive payoff zero, the two end players 1 and j of T receive both payoff equal to

$$\xi_1(N, cu_T, L) = \xi_j(N, cu_T, L) = \frac{c}{2}\Big(1 - \frac{|T| - 2}{n}\Big) = \frac{n - |T| + 2}{2n} c,$$

which is the same as both players receive at the average tree solution. Thus $\xi(N, cu_T, L) = AT(N, cu_T, L)$ holds for any $T \in C^L(N)$.

Since ξ satisfies additivity and the restricted null player property, it follows from Corollary 2.3.11 that $\xi(N, v, L) = \xi(N, v^L, L)$ for any $(N, v, L) \in \mathcal{G}_N^{circle}$. By Lemma 2.3.12 it holds that v^L can be expressed as a unique linear combination of unanimity games for connected coalitions, i.e., given any $(N, v, L) \in \mathcal{G}_N^{circle}$ there exist unique numbers $c_T \in \mathbb{R}$ for $T \in C^L(N)$, $T \neq \emptyset$, such that $v^L = \sum_T c_T u_T$. The proof is completed since

$$\begin{aligned}
\xi(N, v, L) &= \xi(N, v^L, L) \\
&= \xi(N, \sum_{T \in C^L(N), T \neq \emptyset} c_T u_T, L) \\
&= \sum_{T \in C^L(N), T \neq \emptyset} \xi(N, c_T u_T, L) \\
&= \sum_{T \in C^L(N), T \neq \emptyset} AT(N, c_T u_T, L) \\
&= AT(N, v, L).
\end{aligned}$$

□

To show the independence of the five axioms, consider the following solutions for $(N, v, L) \in \mathcal{G}_N^{circle}$.

- $\xi_i(N, v, L) = 0$ for all $i \in N$.

This solution trivially satisfies additivity, the restricted null player property, symmetry among players, and symmetry between games. It fails efficiency.

- $\xi(N, v, L)$ is such that:
  - $\xi_i(N, v, L) = 0$ for all $i \in N$ if $v(N) = 0$.
  - $\xi(N, v, L) = AT(N, v, L)$, otherwise.

This solution satisfies efficiency, the restricted null player property, and symmetry among players. Regarding symmetry between games, consider two TU-games with circular communication structure on the same set of players (N, v, L) and (N, v', L), where there exists $i \in N$ such that $v(S) = v'(S)$ and $v(S \cup \{i\}) = v'(S \cup \{i\})$ for all $S \in C^L(N)$ satisfying $i \notin S$ and $S \cup \{i\} \in C^L(N)$. First assume that $v(N) = 0$. Since $N \in C^L(N)$ and $N \setminus \{i\} \in C^L(N)$ for all $i \in N$, it holds that $v'(N) = v(N) = 0$. Then it follows that $\xi_i(N, v, L) = \xi_i(N, v', L) = 0$. Next, assume $v(N) \neq 0$. For the same reason, it holds that $v'(N) = v(N)$. In this case the solution assigns the average tree solution and thus $\xi_i(N, v, L) = \xi_i(N, v', L)$ holds as well. Therefore this solution satisfies symmetry between games. It fails additivity. Consider $(N, u_{\{1,2\}} - u_N, L)$ and (N, u_N, L) where $N = \{1, 2, 3\}$. It holds that $\xi(N, u_{\{1,2\}} - u_N + u_N, L) = \xi(N, u_{\{1,2\}}, L) = (\frac{1}{2}, \frac{1}{2}, 0) \neq (0, 0, 0) + (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \xi(N, u_{\{1,2\}} - u_N, L) + \xi(N, u_N, L)$.

- $\xi_i(N, v, L) = \frac{v(N)}{n}$ for all $i \in N$.

This solution satisfies efficiency, additivity, symmetry among players, and symmetry between games. However, this solution does not meet the restricted null player property, as a restricted null player of any TU-game with circular communication structure receives non-zero payoff if the worth of the grand coalition is non-zero.

- $\xi(N, v, L) = m^\sigma(N, v, L)$ with $\sigma = (1, 2, \ldots, n)$.

Since every marginal vector is efficient, additive, and satisfies the restricted null player property, this solution satisfies these properties. It can be easily verified that the solution also satisfies symmetry between games, since it allocates to $i \in N$ that $\xi_i(N, v, L) = v(S \cup \{i\}) - v(S)$ with some $S \in C^L(N)$, $i \notin S$, and $S \cup \{i\} \in C^L(N)$. It fails symmetry among players. For example, consider $(N, v, L) = (N, u_N + u_{\{1,2,3\}}, L)$ where $N = \{1, 2, 3, 4\}$. Then, $\xi(N, v, L) = (0, 0, 1, 1)$. Now take the admissible permutation $\pi = (2, 3, 4, 1)$ and let $v'(S) = v(\pi(S))$ for all $S \in 2^N$, then $\xi(N, v', L) = (v(\{2\}), v(\{2, 3\}) - v(\{2\}), v(\{2, 3, 4\}) - v(\{2, 3\}), v(N) - v(\{2, 3, 4\})) = (0, 0, 0, 2)$, which can not be obtained by shifting the entries of $(0, 0, 1, 1)$.

- $\xi(N, v, L) = \mu(N, v, L)$.

From the results of Chapter 2, the Myerson value on the class of TU-games with circular communication structure, which is contained by the class of TU-games with communication structure, satisfies efficiency, the restricted null player property, and additivity. It satisfies symmetry among players, because the Myerson value is also an average of marginal vectors and a shift induced by an admissible permutation just shifts the entries of the marginal vectors. It fails symmetry between games. Consider (N, u_N, L) and $(N, u_{\{1,2,3\}}, L)$ where $N = \{1, 2, 3, 4\}$. Observe that $u_N(S) = u_{\{1,2,3\}}(S)$ except for $S = \{1, 2, 3\}$. Symmetry between games would then imply $\xi_2(N, u_N, L) = \xi_2(N, u_{\{1,2,3\}}, L)$, but $\mu_2(N, u_N, L) = \frac{1}{4} \neq \frac{1}{3} = \mu_2(N, u_{\{1,2,3\}}, L)$.

The characterization we obtain in Chapter 2 for the Myerson value differs in the symmetry axioms regarding the behavior of a solution between two games. Coalitional fairness, which is satisfied by the Myerson value, says that if two games are the same except the worth of only one connected coalition, then the payoff change should equally occur for each member of the coalition. For the average tree solution, if two games are the same except the worth of one connected coalition, then symmetry between games implies that a payoff change may occur only for the end players of the coalition and the end players of its complement, and thus the average tree solution does not satisfy coalitional fairness. With the other axioms the payoff change must be equal between the end players of the coalition, and between the end players of its complement. Symmetry between games also implies that the payoff of a player should not change between two games if his marginal contributions induced from the collection of admissible permutations are the same in both games. On the other hand, the Myerson value gives the same payoff to a player if his marginal contributions induced from all permutations are the same. If, however, there are two games in which any player has the same marginal contributions induced from all permutations, then the two games correspond to the same Myerson restricted game. The axiom regarding marginal contribution is called marginality and introduced in Young (1985) on the class of TU-games to characterize the Shapley value. We remark that the restricted null player property and symmetry between games can be replaced by the following axiom.

Definition 3.3.9 A solution $\xi : \mathcal{G}_N^{circle} \to \mathbb{R}^n$ satisfies *restricted marginality* if for any (N, v, L), $(N, v', L) \in \mathcal{G}_N^{circle}$ and $i \in N$ it holds that $\xi_i(N, v, L) = \xi_i(N, v', L)$ when $v(S \cup \{i\}) - v(S) = v'(S \cup \{i\}) - v'(S)$ for all $S \in C^L(N)$ satisfying $i \notin S$ and $S \cup \{i\} \in C^L(N)$.

Corollary 3.3.10 *On the class of TU-games with circular communication structure, the average tree solution is the unique solution satisfying efficiency, additivity, symmetry among players, and restricted marginality.*

3.4 Stability of the solutions

In this section, we study on the class of TU-games with circular communication structure the relationship between the average tree solution and the core. Especially, we provide necessary and sufficient conditions for each admissible marginal vector, for each compatible marginal vector, and the average tree solution, to be in the core. Shapley (1971) introduces the notion of convex TU-game and shows that if a game is convex, then all marginal vectors, as well as the Shapley value, are in the core. Ichiishi (1981) further shows that all marginal vectors of a TU-game are in the core if and only if the game is convex. A TU-game (N, v) is convex if for all $S \subset T$, $T \in 2^N$, and $i \in S$ it holds that $v(T) - v(T \setminus \{i\}) \geq v(S) - v(S \setminus \{i\})$, or equivalently, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ holds for every $S, T \subset N$. If the latter inequality holds for every S, T with $S \cap T = \emptyset$, the game is called superadditive. A TU-game with circular communication structure (N, v, L) is said to be convex if its restricted game (N, v^L) is convex, and is called superadditive if (N, v^L) is superadditive. The Myerson value of a TU-game with circular communication structure is an element of the core of the game if the game is convex, because the Myerson value is the Shapley value of the restricted game. A convex game ensures that the core of the game is not empty, while this may not always hold if the game is superadditive. We introduce a weaker form of convexity on the class of TU-games with circular communication structure, which is on this class equivalent to link-convexity introduced by Herings et al. (2010).

Definition 3.4.1 A TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$ is *circular-convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for any $S, T \in C^L(N)$ that satisfy at least one of the following conditions:

(1) $S \cup T = N$ and $S \cap T \in C^L(N)$;

(2) $S \cup T \in C^L(N)$ and $S \cap T = \emptyset$.

Condition (2) requires superadditivity of the restricted game and therefore circular-convexity requires more than what is needed for a game $(N, v, L) \in \mathcal{G}_N^{circle}$ to be superadditive. If $n = 3$ then circular-convexity coincides with convexity of (N, v), because in that case a circle graph is the complete graph. For $n > 3$, circular-convexity is weaker than convexity of (N, v, L), because it does not take the convex relationship into account between two non-disjoint connected coalitions S and T if $S \cup T \neq N$ or if $S \cap T$ consists of two components, i.e., when S and T overlap each other at both their ends. For a connected coalition S the number of connected coalitions T satisfying the conditions (1) and (2) does not depend on the size of S as far as $1 \leq |S| < n$. The total number of different connected coalitions T with which circular-convexity has to be satisfied for S is equal to $2n - 2$, since there are n connected coalitions in a circle graph which connect to S from one side of the graph, and also n connected coalitions which connect to S from the other side, with N and $N \setminus S$ being counted twice. This number is the same for convexity of (N, v, L) only if $|S| = 1$. If $|S| > 1$, this number is larger for convexity than $2n - 1$ and also depends on the size of S. Therefore, on the class of TU-games with circular communication structure, circular-convexity is stronger than superadditivity, but weaker than convexity.

We show that circular-convexity of a TU-game with circular communication structure is a necessary and sufficient condition for all admissible marginal vectors to be in the core.

Theorem 3.4.2 *For a TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$, every admissible marginal vector $m^\sigma(N, v, L)$, $\sigma \in \Pi^a(N)$, is in the core if and only if the game is circular-convex.*

Proof Suppose the TU-game with circular communication structure (N, v, L) is circular-convex. Take any connected coalition $S \in C^L(N)$. We show that for every $\sigma \in \Pi^a(N)$, $m^\sigma(S) \geq v(S)$, where $m^\sigma(S) = \sum_{j \in S} m_j^\sigma(N, v, L)$. Since

$S \in C^L(N)$, we have $S = S_a^b$ for some $1 \leq a, b \leq n$. Without loss of generality, we assume $a \leq b$ and $\sigma = \sigma_1^i$ for some $i \in N$. Recall that σ_1^i is the admissible permutation with $\sigma(1) = i$ and $\sigma(n) = i - 1$. First suppose $i \notin S$, that is, either $1 \leq i < a$ or $b < i \leq n$. Then it holds that

$$\begin{aligned} m^{\sigma_1^i}(S_a^b) &= v(S_i^b) - v(S_i^{a-1}) \\ &= v(S_i^{a-1} \cup S_a^b) - v(S_i^{a-1}) \\ &\geq v(S_i^{a-1}) + v(S_a^b) - v(S_i^{a-1}) \\ &= v(S_a^b), \end{aligned}$$

where the inequality follows from condition (2) of Definition 3.4.1. Next, suppose $i \in S$. If $i = a$, then

$$m^{\sigma_1^i}(S_a^b) = v(S_a^b) - v(\emptyset) = v(S_a^b).$$

If $a < i \leq b$, then

$$\begin{aligned} m^{\sigma_1^i}(S_a^b) &= m^{\sigma_1^i}(S_i^b) + m^{\sigma_1^i}(S_a^{i-1}) \\ &= v(S_i^b) - v(\emptyset) + v(S_i^{i-1}) - v(S_i^{a-1}) \\ &= v(S_i^b) + v(N) - v(S_i^{a-1}) \\ &= v(S_i^{a-1} \cap S_a^b) + v(S_i^{a-1} \cup S_a^b) - v(S_i^{a-1}) \\ &\geq v(S_a^b), \end{aligned}$$

where the inequality follows from condition (1) of Definition 3.4.1. Therefore circular-convexity is a sufficient condition.

Suppose that $m^\sigma(N, v, L) \in C(N, v, L)$ holds for every $\sigma \in \Pi^a(N)$, but (N, v, L) is not satisfying circular-convexity. Then there are two distinct connected coalitions S and T which satisfy at least one of the conditions of Definition 3.4.1 while $v(S) + v(T) > v(S \cup T) + v(S \cap T)$. First, consider the case when Condition (2) of Definition 3.4.1 holds, i.e., $S \cup T \in C^L(N)$ and $S \cap T = \emptyset$. Without loss of generality, let $S = S_a^b$ and $T = S_{b+1}^c$ with $1 \leq a \leq b < c \leq n$. Then it holds for the marginal vector $m^{\sigma_2^c}(N, v, L)$ that

$$\begin{aligned} m^{\sigma_2^c}(S_a^b) &= v(S_a^c) - v(S_{b+1}^c) \\ &= v(S_a^b \cup S_{b+1}^c) - v(S_{b+1}^c) \\ &= v(S \cup T) - v(T) \\ &< v(S) = v(S_a^b). \end{aligned}$$

This contradicts that $m^{\sigma_2^c}(N, v, L) \in C(N, v, L)$. Next, consider the case when Condition (1) of Definition 3.4.1 holds, i.e., $S \cup T = N$ and $S \cap T \in C^L(N)$.

Without loss of generality, let $S = S_a^b$ and $T = S_c^{a-1}$ with $1 \leq a < c \leq b \leq n$. Observe that $S \cap T = S_c^b$. Then for the marginal vector $m^{\sigma_1^c}(N, v, L)$ it holds that

$$\begin{aligned} m^{\sigma_1^c}(S_a^b) &= m^{\sigma_1^c}(S_c^b) + m^{\sigma_1^c}(S_a^{c-1}) \\ &= v(S_c^b) - v(\emptyset) + v(S_c^{c-1}) - v(S_c^{a-1}) \\ &= v(S_c^b) + v(N) - v(S_c^{a-1}) \\ &= v(S_c^{a-1} \cap S_a^b) + v(S_c^{a-1} \cup S_a^b) - v(S_c^{a-1}) \\ &= v(S \cap T) + v(S \cup T) - v(T) \\ &< v(S) = v(S_a^b). \end{aligned}$$

This contradicts that $m^{\sigma_1^c}(N, v, L) \in C(N, v, L)$. It is shown that whenever there is a violation for circular-convexity, there is an admissible marginal vector outside the core. This concludes that circular-convexity is also a necessary condition. □

From the theorem it immediately follows that the convex hull of all admissible marginal vectors is a subset of the core if and only if the game is circular-convex.

Corollary 3.4.3 *For any TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$ it holds that the set $Conv\,\{m^\sigma(N, v, L) | \sigma \in \Pi^a(N)\}$ is a subset of the core $C(N, v, L)$ if and only if the game is circular-convex.*

Since the average tree solution is the average of all admissible marginal vectors, which are in the core under circular-convexity of a game, we have the following corollary.

Corollary 3.4.4 *For any circular-convex TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$ it holds that $AT(N, v, L) \in C(N, v, L)$.*

A condition for the stability of all compatible marginal vectors requires more than what circular-convexity requires, since there are more marginal vectors which are compatible than admissible, while all admissible marginal vectors are compatible, and all marginal vectors are different from each other.

Definition 3.4.5 A TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$ is *strongly circular-convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for any $S, T \in C^L(N)$ such that $S \cup T \in C^L(N)$ and $S \cap T \in C^L(N)$.

Note that for $n = 3$ strong circular-convexity is equivalent to convexity, as is the case for circular-convexity. For $n > 3$ the condition is weaker than convexity but stronger than circular-convexity, since it also requires a convex relationship between some but not all intersecting pairs of connected coalitions S and T with $S \cup T \neq N$.

Theorem 3.4.6 *For a TU-game with circular communication structure $(N, v, L) \in \mathcal{G}_N^{circle}$, every compatible marginal vector $m^\sigma(N, v, L)$, $\sigma \in \Pi^c(N)$, is in the core if and only if the game is strongly circular-convex.*

Proof Suppose the TU-game with circular communication structure (N, v, L) is strongly circular-convex. Take any $S \in C^L(N)$ and $\sigma \in \Pi^c(N)$, and let $m^\sigma(S) = \sum_{j \in S} m_j^\sigma(N, v, L)$. Without loss of generality, let us order the players in S on σ as $i_1, \ldots, i_{|S|}$ such that $j < k$ implies $P_\sigma(i_j) \subset P_\sigma(i_k)$. Note that this ordering is uniquely determined given σ and S. Then, from equation (3.1), we have

$$\begin{aligned}
m^\sigma(S) - v(S) &= \sum_{k=1}^{|S|} \Big(v\big(P_\sigma(i_k) \cup \{i_k\}\big) - v\big(P_\sigma(i_k)\big) \Big) - v(S) \\
&= \sum_{k=1}^{|S|-1} \Big(v\big(P_\sigma(i_k) \cup \{i_k\}\big) - v\big(P_\sigma(i_k)\big) \Big) \\
&\quad + v\big(P_\sigma(i_{|S|}) \cup \{i_{|S|}\}\big) - v\big(P_\sigma(i_{|S|})\big) - v(S) \\
&= \sum_{k=1}^{|S|-1} \Big(v\big(P_\sigma(i_k) \cup \{i_k\}\big) - v\big(P_\sigma(i_k)\big) \Big) \\
&\quad + v\big(P_\sigma(i_{|S|}) \cup S\big) - v\big(P_\sigma(i_{|S|})\big) - v(S) \\
&\geq \sum_{k=1}^{|S|-1} \Big(v\big(P_\sigma(i_k) \cup \{i_k\}\big) - v\big(P_\sigma(i_k)\big) \Big) - v\big(P_\sigma(i_{|S|}) \cap S\big) \\
&= \sum_{k=1}^{|S|-1} \Big(v\big(P_\sigma(i_k) \cup \{i_k\}\big) - v\big(P_\sigma(i_k)\big) \Big) - v(S \setminus \{i_{|S|}\}),
\end{aligned}$$

where the inequality follows from the strong circular-convexity condition for connected coalitions S and $P_\sigma(i_{|S|})$. Notice that $P_\sigma(i_{|S|}) \cup S = P_\sigma(i_{|S|}) \cup \{i_{|S|}\}$ and $P_\sigma(i_{|S|}) \cap S = S \setminus \{i_{|S|}\}$. Repeating the procedure gives

$$\begin{aligned}
m^\sigma(S) - v(S) &\geq \sum_{k=1}^{|S|-1} \Big(v\big(P_\sigma(i_k) \cup \{i_k\}\big) - v\big(P_\sigma(i_k)\big) \Big) - v(S \setminus \{i_{|S|}\}) \\
&\geq \sum_{k=1}^{|S|-2} \Big(v\big(P_\sigma(i_k) \cup \{i_k\}\big) - v\big(P_\sigma(i_k)\big) \Big) - v(S \setminus \{i_{|S|-1}, i_{|S|}\})
\end{aligned}$$

$$
\begin{aligned}
&\cdots \\
&\geq \Big(v\big(P_\sigma(i_1) \cup \{i_1\}\big) - v\big(P_\sigma(i_1)\big) \Big) - v(S \setminus \{i_2, \dots, i_{|S|}\}) \\
&= v\big(P_\sigma(i_1) \cup \{i_1\}\big) - v\big(P_\sigma(i_1)\big) - v(\{i_1\}) \\
&\geq 0.
\end{aligned}
$$

Therefore strong circular-convexity is a sufficient condition.

Suppose that $m^\sigma(N,v,L) \in C(N,v,L)$ holds for every $\sigma \in \Pi^c(N)$, but (N,v,L) is not satisfying strong circular-convexity. Then there are two distinct connected coalitions S and T which satisfy the conditions of Definition 3.4.5 while $v(S) + v(T) > v(S \cup T) + v(S \cap T)$. From the proof of Theorem 3.4.2, if $S \cup T \in C^L(N)$ and $S \cap T = \emptyset$ or if $S \cup T = N$ and $S \cap T \in C^L(N)$, there is an admissible, therefore compatible, permutation σ satisfying $m^\sigma(N,v,L) \notin C(N,v,L)$, which is a contradiction. Thus it suffices to find a compatible permutation σ with $m^\sigma(N,v,L) \notin C(N,v,L)$ whenever $S \cap T \in C^L(N) \setminus \{\emptyset\}$ and $S \cup T \neq N$. Take any compatible permutation σ with the first $|S \cap T|$ positions occupied by the elements of $S \cap T$, the next $|T \setminus S|$ positions occupied by the elements of $T \setminus S$, and the next $|S \setminus T|$ positions occupied by the elements of $S \setminus T$. Such a compatible permutation exists, because $S \cap T$, $T \setminus S$, $S \setminus T$ and $N \setminus (S \cup T)$ are nonempty connected coalitions while $S \cap T$ and $T \setminus S$, also $(S \cap T) \cup (T \setminus S) = T$ and $S \setminus T$, and also $(S \cap T) \cup (T \setminus S) \cup (S \setminus T) = S \cup T$ and $N \setminus (S \cup T)$ are disjoint but connected. Observe that σ cannot be admissible. It holds that $m^\sigma(S \cap T) = v(S \cap T)$ and $m^\sigma(S \setminus T) = m^\sigma(S \cup T) - m^\sigma(T) = v(S \cup T) - v(T)$. Then it follows that $m^\sigma(S) = m^\sigma(S \cap T) + m^\sigma(S \setminus T) = v(S \cap T) + v(S \cup T) - v(T) < v(S)$ and therefore $m^\sigma(N,v,L) \notin C(N,v,L)$. □

Corollary 3.4.7 *For any TU-game with circular communication structure $(N,v,L) \in \mathcal{G}_N^{circle}$ it holds that the set $Conv\,\{m^\sigma(N,v,L) | \sigma \in \Pi^c(N)\}$ is a subset of the core $C(N,v,L)$ if and only if the game is strongly circular-convex.*

In general, for a (strongly) circular-convex TU-game with circular communication structure $(N,v,L) \in \mathcal{G}_N^{circle}$ with $n > 3$ it does not hold that the set $Conv\,\{m^\sigma(N,v,L) | \sigma \in \Pi^a(N)\}$ ($Conv\,\{m^\sigma(N,v,L) | \sigma \in \Pi^c(N)\}$) is equal to the core $C(N,v,L)$. In particular, when (N,v,L) is a convex TU-game with circular communication structure, $C(N,v,L)$ is equal to the convex hull of all marginal vectors $m^\sigma(v^L)$, $\sigma \in \Pi(N)$, defined in Chapter 2, which are all extreme points of the core, whereas $Conv\,\{m^\sigma(N,v,L) | \sigma \in \Pi^c(N)\}$ is the convex hull of less (different) marginal vectors and therefore is a proper subset of

$C(N,v,L)$. The set $Conv\,\{m^\sigma(N,v,L)|\sigma \in \Pi^a(N)\}$ is the convex hull of even less (different) marginal vectors, which holds true for any TU-game with circular communication structure. Also in case a (strongly) circular-convex TU-game with circular communication structure $(N,v,L) \in \mathcal{G}_N^{circle}$ with $n > 3$ is not convex, the core $C(N,v,L)$ is in general not equal to $Conv\{m^\sigma(N,v,L)|\,\sigma \in \Pi^a(N)\}$ ($Conv\{m^\sigma(N,v,L)|\sigma \in \Pi^c(N)\}$) and contains these sets as proper subsets.

The next examples show that strong circular-convexity is weaker than convexity and that circular-convexity is weaker than strong circular-convexity.

Example 3.4.8 Consider the 4-person TU-game with circular communication structure (N,v,L) with characteristic function

$$v(S) = \begin{cases} 2 & \text{if } \ S = N, \{1,3,4\}, \\ 1 & \text{if } \ S = \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \\ 0 & \text{otherwise.} \end{cases}$$

This game is strongly circular-convex but not convex (take $S = \{1,2,3\}$ and $T = \{1,3,4\}$). From Theorem 3.4.6, it follows that $m^\sigma(N,v,L) \in C(N,v,L)$ for all $\sigma \in \Pi^c(N)$. Player 2 is a restricted null player and therefore a stable allocation assigns to this player zero payoff. Since player 2 is not a null player in the Myerson restricted game (N,v^L) $\big(v^L(\{1,2,3\}) - v^L(\{1,3\}) = 1 > 0\big)$, the Myerson value allocates some positive value to this player, yielding $\mu(N,v,L) = (\frac{7}{12}, \frac{1}{12}, \frac{7}{12}, \frac{9}{12}) \notin C(N,v,L) = C(N,v^L)$. The average tree solution equals $AT(N,v,L) = (\frac{5}{8}, 0, \frac{5}{8}, \frac{6}{8}) \in C(N,v,L)$.

Example 3.4.9 Consider the 4-person TU-game with circular communication structure (N,v,L) with characteristic function

$$v(S) = \begin{cases} 2 & \text{if } \ S = N, \\ 2-\epsilon & \text{if } \ S = \{1,3,4\}, \\ 1 & \text{if } \ S = \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \\ 0 & \text{otherwise,} \end{cases}$$

for some $0 \leq \epsilon \leq 1$. For $\epsilon = 0$ this is Example 3.4.8. For $0 < \epsilon \leq 1$, the game is circular-convex but not strongly-circular-convex (take $S = \{1,4\}$ and $T = \{3,4\}$). From Theorem 3.4.2 it follows that $m^\sigma(N,v,L) \in C(N,v,L)$ for all $\sigma \in \Pi^a(N)$. For the permutation $\sigma = (4,3,1,2) \in \Pi^c(N) \setminus \Pi^a(N)$, however, it holds that $m^\sigma(N,v,L) = (1-\epsilon, \epsilon, 1, 0) \notin C(N,v,L)$, since $m_1^\sigma(N,v,L) + m_4^\sigma(N,v,L) < v(\{1,4\})$. The average tree solution of this game is $AT(N,v,L) = (\frac{5-\epsilon}{8}, \frac{2\epsilon}{8}, \frac{5-\epsilon}{8}, \frac{6}{8}) \in C(N,v,L)$ and the Myerson value of the game equals to

$\mu(N,v,L) = (\frac{7-\epsilon}{12}, \frac{1+3\epsilon}{12}, \frac{7-\epsilon}{12}, \frac{9-\epsilon}{12})$. Note that $\mu(N,v,L) \notin C(N,v,L)$ for $0 \leq \epsilon < \frac{1}{9}$.

Next, we proceed to find a necessary and sufficient condition for the average tree solution on the class of TU-games with circular communication structure to be stable. The value is the average of admissible marginal vectors that are stable under circular-convexity, and the desired condition can be seen as the one under which circular-convexity is, on average, satisfied. For a nonempty connected coalition $S \in C^L(N), S \neq N$, of a circle graph (N,L), let $\mathcal{CC}(S)$ be the collection of connected coalitions that are considered for the circular-convexity condition with respect to S, i.e.,

$$\mathcal{CC}(S) = \left\{ T \in C^L(N) \,\middle|\, \begin{array}{l} S \cup T = N \text{ and } S \cap T \in C^L(N), \text{ or} \\ S \cup T \in C^L(N) \text{ and } S \cap T = \emptyset \end{array} \right\}.$$

Definition 3.4.10 A TU-game with circular communication structure $(N,v,L) \in \mathcal{G}_N^{circle}$ is *average-circular-convex* if for any $S \in C^L(N)$, $S \neq N$, it holds that

$$\sum_{T \in \mathcal{CC}(S)} (v(S \cup T) + v(S \cap T) - v(S) - v(T)) \geq v(S) + v(N \setminus S) - v(N).$$

Iñarra and Usategui (1993) introduces the notion of average-convexity on the class of TU-games. A TU-game is average-convex if for every coalition $S \in 2^N$ the sum of the marginal contributions of every agent in S to any coalition which is a superset of S is larger than the sum of the marginal contributions of these agents when joining to S. It is well-known that if a TU-game is convex, then the marginal contribution of a player becomes larger if he joins to a bigger set of players, and average-convexity of TU-games is related to this property. On the other hand, the average-circular-convex condition of a TU-game with circular communication structure requires that the sum of surpluses $(v(S \cup T) + v(S \cap T) - v(S) - v(T))$ a connected coalition (S) generates with all other connected coalitions (T) to which the coalition is connected and has a nonempty connected intersection only if the union is the grand coalition, is at least equal to the (possibly negative) loss the coalition generates with its complement. For a circular-convex game all terms in the left hand side are non-negative and the term in the right-hand side is non-positive, and therefore a circular-convex TU-game with circular communication structure is average-circular-convex. However, the condition does not imply superadditivity, because some of the surpluses can be negative. In the latter case not all admissible marginal vectors will be elements of the core. We show that average-circular-convexity of a TU-game with circular communication structure is a necessary and sufficient condition that the average tree solution of the game is stable.

Theorem 3.4.11 *For any TU-game with circular communication structure* $(N, v, L) \in \mathcal{G}_N^{circle}$ *it holds that* $AT(N, v, L) \in C(N, v, L)$ *if and only if the game is average-circular-convex.*

Proof Due to efficiency of the average tree solution, the grand coalition N does not block $AT(N, v, L)$. Take any connected coalition S with $|S| < n$ and without loss of generality let $S = S_a^b$ for some $1 \leq a \leq b \leq n$. The sum of payoffs allocated to S by the average tree solution equals

$$\begin{aligned}
\sum_{i \in S} AT_j(N, v, L) &= \sum_{j \in S_a^b} AT_j(N, v, L) = \sum_{j=a}^{b} AT_j(N, v, L) \\
&= \sum_{j=a}^{b} \frac{1}{2n} \sum_{i \in N} \left(m_j^{\sigma_1^i}(N, v, L) + m_j^{\sigma_2^i}(N, v, L)\right) \\
&= \sum_{j=a}^{b} \frac{1}{2n} \sum_{i \in N} \Big(v(S_i^j) - v(S_i^j \setminus \{j\}) + v(S_j^i) - v(S_j^i \setminus \{j\})\Big) \\
&= \sum_{j=a}^{b} \Big(\frac{1}{n} v(N) + \frac{1}{2n} \sum_{i \in N} \left(v(S_j^i) - v(S_{j+1}^i) + v(S_i^j) - v(S_i^{j-1})\right)\Big) \\
&= \frac{|S_a^b|}{n} v(N) + \frac{1}{2n} \sum_{i \in N} \Big(v(S_a^i) - v(S_{b+1}^i) + v(S_i^b) - v(S_i^{a-1})\Big) \\
&= \frac{1}{2n} \Big[\sum_{i \notin S} \Big(v(S_a^i) - v(S_{b+1}^i) + v(S_i^b) - v(S_i^{a-1})\Big) \\
&\quad + \sum_{i \in S} \Big(v(N) + v(S_a^i) - v(S_{b+1}^i) + v(N) + v(S_i^b) - v(S_i^{a-1})\Big)\Big] \\
&= \frac{1}{2n} \Big[\sum_{i \notin S} \Big(v(S_a^b \cup S_{b+1}^i) - v(S_{b+1}^i) + v(S_a^b \cup S_i^{a-1}) - v(S_i^{a-1})\Big) \\
&\quad + \sum_{i \in S} \Big(v(S_a^b \cup S_{b+1}^i) + v(S_a^i) - v(S_{b+1}^i) + v(S_a^b \cup S_i^{a-1}) \\
&\quad + v(S_i^b) - v(S_i^{a-1})\Big)\Big] \\
&= \frac{1}{2n} \Big[\sum_{i \in N} \Big(v(S_a^b \cup S_{b+1}^i) + v(S_a^b \cap S_{b+1}^i) - v(S_{b+1}^i)\Big) \\
&\quad + \sum_{i \in N} \Big(v(S_a^b \cup S_i^{a-1}) + v(S_a^b \cap S_i^{a-1}) - v(S_i^{a-1})\Big)\Big].
\end{aligned}$$

It holds that $\mathcal{CC}(S) = \{S_i^{a-1} \mid i \in N\} \cup \{S_{b+1}^i \mid i \in N\}$. It also holds that $\{S_i^{a-1} \mid i \in N\} \cap \{S_{b+1}^i \mid i \in N\} = \{N, S_{b+1}^{a-1}\}$ and $S_{b+1}^{a-1} = N \setminus S_a^b$. Therefore it follows that

$$\sum_{j \in S} AT_j(N,v,L) = \frac{1}{2n}\Big[\sum_{T \in \mathcal{CC}(S)} \Big(v(S \cup T) + v(S \cap T) - v(T)\Big) + v(N) - v(N \setminus S) + v(S)\Big].$$

Then,

$$\sum_{j \in S} AT_j(N,v,L) - v(S) = \frac{1}{2n}\Big[\sum_{T \in \mathcal{CC}(S)} \Big(v(S \cup T) + v(S \cap T) - v(S) - v(T)\Big) + v(N) - v(S) - v(N \setminus S)\Big],$$

which is nonnegative for all $S \in C^L(N)$, $S \neq N$, if and only if (N,v,L) is average-circular-convex. □

Since average-circular-convexity implies that the average tree solution is stable, we obtain the following corollary.

Corollary 3.4.12 *If a TU-game with circular communication structure $(N,v,L) \in \mathcal{G}_N^{circle}$ is average-circular-convex, then the core $C(N,v,L)$ is nonempty.*

The next example shows that average-circular-convexity does not imply superadditivity.

Example 3.4.13 Consider the 4-person TU-game with circular communication structure (N,v,L) with characteristic function

$$v(S) = \begin{cases} 2 & \text{if } S = N, \{1,3,4\}, \\ 1+\epsilon & \text{if } S = \{3,4\}, \{1,4\}, \{1,2,3\}, \\ 1 & \text{if } S = \{1,2,4\}, \{2,3,4\}, \\ \epsilon & \text{if } S = \{1,2\}, \{2,3\}, \\ 0 & \text{otherwise,} \end{cases}$$

for some $0 \leq \epsilon \leq \frac{1}{3}$. For $\epsilon = 0$ this is Example 3.4.8. For any $\epsilon > 0$, this game is not superadditive (take $S = \{2\}$ and $T = \{3,4\}$) and in particular, it can be checked that every marginal vector is outside the core, i.e., $m^\sigma(N,v,L) \notin C(N,v,L)$ for all $\sigma \in \Pi(N)$. For $0 \leq \epsilon \leq \frac{1}{3}$, this game is average-circular-convex and the average tree solution equals $AT(N,v,L) = (\frac{5+\epsilon}{8}, 0, \frac{5+\epsilon}{8}, \frac{6-2\epsilon}{8}) \in C(N,v,L)$, while the Myerson value is never in the core, $\mu(N,v,L) = (\frac{7+\epsilon}{12}, \frac{1+\epsilon}{12}, \frac{7+\epsilon}{12}, \frac{9-3\epsilon}{12}) \notin C(N,v,L)$.

From Theorem 3.4.11 it follows that the average tree solution of a TU-game with circular communication structure is in the core if for each connected coalition the surpluses are on average non-negative and also its surplus with its complement is non-negative.

Corollary 3.4.14 *For a TU-game with circular communication structure* $(N, v, L) \in \mathcal{G}_N^{circle}$, $AT(N, v, L) \in C(N, v, L)$ *if for any* $S \in C^L(N)$, $S \neq N$, *it holds that*

$$\sum_{T \in \mathcal{CC}(S)} (v(S \cup T) + v(S \cap T) - v(S) - v(T)) \geq 0 \text{ and } v(N) \geq v(S) + v(N \setminus S).$$

In the case where $n = 3$, the average tree solution of a TU-game with circular communication structure coincides with the Shapley value, and therefore the average-circular-convexity condition is a necessary and sufficient condtion for the Shapley value to be stable.

Corollary 3.4.15 *For a TU-game* $(N, v) \in \mathcal{G}_N$ *with* $n = 3$, *it holds that* $Sh(N, v) \in C(N, v)$ *if and only if*

$$\sum_{T \in 2^N \setminus \{\emptyset\}} (v(S \cup T) + v(S \cap T) - v(S) - v(T)) \geq v(S) + v(\{1,2,3\} \setminus S) - v(\{1,2,3\})$$

holds for all $S \in 2^N$, $S \neq \{1, 2, 3\}$.

CHAPTER 4

QUASI-BUILDING SYSTEM: A NEW COOPERATIVE RESTRICTION

4.1 Introduction

In the previous chapters we study the class of TU-games with communication structure, for which only any connected set of players is assumed to be able to form a coalition to obtain its worth. Although this class of games contains the classical TU-games as a subclass, the collection of connected sets of players in a communication structure, as a collection of feasible coalitions of the structure, is still subject to some conditions. For example, for a communication structure every coalition consisting of one player is feasible to form. However, consider a situation where an employer hires a professional to generate a joint profit and both parties are deciding how to share the profit. There is a hierarchical structure in this case and it may be natural to assume that the factors that are brought on the table of negotiation is the outside option of the employee and the profits they jointly generate, not the outside option of the employer working as an individual. Another implicit assumption that underlies a communication structure is that if there are feasible coalitions which contain a player in common, then the union of such coalitions is also feasible. There are cases where one may find it difficult to take this assumption as granted. Especially when there are two coalitions whose intersection is a player who is not influential in both coalitions. There is no reason to assume a priori the union of such two coalitions to be feasible. TU-games with communication structure treat players asymmetrically in terms of the connectivity in the underlying graph. These two examples are cases in which players are more asymmetric, in terms of power of each player in the structure, than a communication structure allows.

In the literature of cooperative games many studies go beyond a communication structure. One way to go further is to explicitly introduce a kind of dominance structure on the players in a game. Faigle and Kern (1992) introduces games on a partially ordered set. In such a game there is a precedence relation on the players set, expressed by a partial order, and a coalition is feasible if for every player in the coalition, all of his preceding players, coming from the precedence relation, are in the coalition as well. Another example of this kind is introduced by Gilles et al. (1992) to assume an underlying permission structure. A permission structure is described by a directed graph representing a hierarchical structure among the players, which defines for each player its set of predecessors, and if a player wants to cooperate with other players he must ask for permission from all his predecessors (conjunctive approach). In the same setting, Gilles and Owen (1992) takes another approach, called disjunctive approach, by assuming that a player needs at least one of its predecessors to give permission to cooperate with other players. Given a TU-game with permission structure, each approach yields different TU-game, a permission restricted game. van den Brink and Gilles (1996) and van den Brink (1997) study the Shapley value of such restricted games. Recently, van den Brink et al. (2015) characterizes the average tree solution of the permission restricted games on the class of TU-games with permission structure when a permission structure is represented by a tree. On the other hand, Khmelnitskaya et al. (2012) introduces TU-games with dominance structure and interpret a directed graph on the player set as subordination relation among the players. They define a collection of feasible hierarchical orderings of players given the structure, and as solution concept they consider the average of all the marginal vectors induced from such hierarchical orderings. Another way to generalize communication structures is to introduce a class of set systems which is more general than the collection of connected coalitions obtained from a communication structure. For example, augmenting systems introduced in Bilbao (2003) do not require every singleton to be feasible. It is known that the collection of feasible coalitions that arise from a game on a poset as well as from a game with a permission structure forms an augmenting system. On the other hand, convex geometries considered in Bilbao and Edelman (2000) do not have to be union stable. Other set systems of feasible coalitions for cooperative games with restricted coalitions that have been considered in the literature are union stable structures (Algaba et al. (2001)), antimatroids (Algaba et al. (2004)), and partition systems (Algaba et al. (2000)), or building sets (Koshevoy and Talman (2014)). In all these models, marginal vectors are defined, although using different methods, and Shapley-type values are studied as solutions, that is, the average of marginal vectors is taken as solution concept to determine how much payoff every player will get.

In this chapter we go one step further and not only assume that some set system of feasible coalitions is given but also that every feasible coalition may contain players that are not able to form or leave the coalition. Only players that are able to form or leave the coalition are assumed to obtain marginal contribution. To get an idea of a possible distinction between players, let us consider a TU-game with communication structure. If we consider how well a player is connected in a subgraph reflects his communication power in the underlying coalition, it may be assumed that given a feasible coalition only the players who have the maximal communication power within the coalition, i.e., who share the maximal number of edges within the subgraph, are able to form or leave the underlying coalition. Another example is a communication game on a hierarchy or more general a directed or mixed graph with directed and undirected edges, where in any connected subgraph only players who are not dominated by any other player in the subgraph are able to contribute to the underlying feasible coalition. Undominated players could be the members of the management team in a coalition or in case the players are jobs those tasks that can be performed after the other tasks in the coalition have been performed.

We model the distinction between players in a coalition who are able to contribute or not by assigning to every feasible coalition a nonempty subset of players, called its choice set, consisting of the players who are able to form or leave the coalition. Specifically, we introduce the notion of a quasi-building system. A quasi-building system on a set of players consists of a set system of feasible coalitions of the players set and a choice function that assigns to each feasible coalition a choice set. Only a player that is in the choice set of a coalition is able to form or leave the coalition, while the remaining players form a unique maximal partition[1] into feasible subcoalitions satisfying that no union of subsets of two or more of them is feasible. In case the choice set of

[1]A set of players can be uniquely maximal partitioned into feasible coalitions if there is a collection of disjoint feasible coalitions whose union is equal to the players set and any other collection of disjoint feasible coalitions whose union is the players set is a proper subset of it.

each feasible coalition is the coalition itself, a building set system is obtained.

A quasi-building system for the examples above is the following. The set of feasible coalitions is the set of connected subgraphs, and the choice function of any connected subgraph is the set of nodes with maximal degree in the subgraph or the set of undominated players.

Every quasi-building system induces a nonempty collection of compatible rooted trees on the players set. In each rooted tree every node is a player and has a unique predecessor, except the root itself, and a possibly empty set of successors. For each node, the player, which labels this node, forms together with the players, which label the successors of this node in the tree, a feasible coalition, and this player belongs to the choice set of this coalition. Such a rooted tree represents a hierarchical structure on the set of players compatible with the restrictions to form and leave coalitions according to the quasi-building system.

For a TU game on a quasi-building system we propose a solution as follows. For each compatible rooted tree arising from the quasi-building system we define a marginal vector. A marginal vector corresponding to a rooted tree assigns as payoff to any player how much this player contributes in worth when joining his successor players in the tree. Only if a player belongs to the choice set of a coalition this player is able to receive payoff, which reflects that only such a player is able to form or leave the coalition. As solution we take the average of all such marginal vectors. This solution we call the average marginal vector value (AMV-value). The AMV-value satisfies efficiency, linearity, a restricted null player property, the inessential coalition property, and the closed coalition property. The second last property says that the solution should not change if the worth of an inessential coalition changes. The last property says that any non-inessential feasible coalition whose players are not in the choice set of any non-inessential coalition that contains the coalition should obtain its own worth.

We consider several subclasses of quasi-building systems and give for each of them convexity-type of conditions under which the AMV-value is an element of the core and therefore cannot be blocked by any feasible coalition. We first study union stable quasi-building systems for which it holds that the union of any two non-disjoint feasible coalitions is also feasible if at least one of the intersecting players is in the choice set of one or both coalitions. For such a quasi-building system the convexity condition only has to hold for any pair of union-closed coalitions. For undirected, directed, or mixed graphs (Ambec and Sprumont (2002), Khmelnitskaya et al. (2012), Myerson (1977)), the collection of connected sets of players with as choice set the set of undominated players within the set is a union stable quasi-building system. The AMV-value for TU-games on this subclass is equal to the average covering tree solution introduced in Khmelnitskaya et al. (2012). Set systems like partition systems (Algaba et al. (2000)), or building sets (Koshevoy and Talman (2014)), augmenting systems (Bilbao (2003)), posets (Faigle and Kern (1992)), and antimatroids (Dilworth (1940)) can be defined as union stable quasi-building systems, in which the choice set of a feasible coalition is equal to the set of players for which the set of remaining players can be uniquely maximal partitioned. For TU-games on building sets the AMV-value is equal to the GC-solution introduced in Koshevoy and Talman (2014). From our core stability results several existing results for these cases follow.

Another class we study is the class of intersection-closed quasi-building systems. For an intersection-closed quasi-building system it holds that if the intersection of two feasible coalitions is nonempty, then this intersection is also a feasible coalition and, moreover, if a player is in the choice set of some feasible coalition then this player must be in the choice set of any feasible subcoalition that contains this player. The latter property is known as independence of irrelevant alternatives in bargaining solutions and is called the heredity property, Chernoff property, or α-axiom of Sen (Chernoff (1954), Sen (1971)). For this class of quasi-building systems the convexity condition only has to hold for any pair of strongly union-closed coalitions. For convex geometries as class of set systems, which contains posets as a special case, it holds that the induced quasi-building system defined in the same way as for augmenting systems is an intersection-closed quasi-building system. If for augmenting systems and convex geometries the choice set of a feasible coalition is restricted to those players for which the remaining players in the set form a single feasible coalition, a chain quasi-building system is obtained, the third class of quasi-building systems we study. In general, for a chain quasi-building system, for every player in the choice set of a feasible coalition it holds that the set of remaining players is a feasible coalition. All rooted trees induced from a chain quasi-building system are line-trees. Posets are a special case of all three quasi-building systems being studied. In case a convex geometry or an augmenting system is described by such a chain quasi-building system, the AMV-value coincides with the Shapley value in Bilbao and Edelman (2000) and Bilbao and Ordóñez (2009), respectively, and in case of a poset the AMV-value coincides with the Shapley value in Faigle and Kern (1992). If cooperation is universal and the choice set of every coalition is the coalition itself, then the AMV-value is the Shapley value (Shapley (1953)).

This chapter is organized as follows. In Section 2 quasi-building systems are introduced and the AMV-value is defined. Section 3 studies special cases of quasi-building systems. Section 4 discusses properties of the AMV-value. Core stability and convexity are studied in Section 5.

This chapter is based on Koshevoy et al. (2013).

4.2 Quasi-building system games and the average marginal vector value

In cooperative game theory, the Shapley value, the average of all marginal vectors of a TU-game, is one of the most well-known solution concepts. When cooperation between players is restricted, a Shapley-type solution concept concerns how the restriction limits the way a player receives his marginal contribution.

First of all, in order for a player to receive his marginal contribution when joining a set of players to form a coalition, the resulting coalition must be feasible. Otherwise, not only that player, but any player who is a member of it cannot receive a marginal contribution from it. This occurs for example if the feasible coalitions are the connected subsets of an undirected graph on the set of players as we study in the previous chapters.

Second, even if a coalition is feasible, not all of its members might be able to leave or join to the coalition and obtain their marginal contribution. This may occur if some players dominate each other as in a hierarchy, a directed graph with or without cycles, or the presence of precedence constraints. For example, if player 1 dominates player 2, then we can interpret this as that player 1 can join player 2 to form the coalition consisting of players 1 and 2, but player 2 cannot join player 1 to form this coalition. Only player 1 is then able to receive a marginal contribution from the coalition consisting of players 1 and 2, but player 2 is not.

In this chapter we assume that for a set of players a collection of feasible coalitions is given, including the grand coalition of all players[2] and that every feasible coalition contains a nonempty subset of players, called the choice set

[2]In case the grand coalition of all players is not feasible, we assume that it has a unique maximal partition into feasible coalitions satisfying that every feasible coalition is a subset of one of the partition members. The analysis can then be applied separately to every partition member.

of the coalition.

If a player in the choice set of a feasible coalition leaves the coalition, we assume that the set of remaining players of the coalition has a unique maximal partition into feasible coalitions, of which subsets are not able to form a feasible coalition. This implies that each player in the choice set of a feasible coalition is able to form that coalition in a unique way. The combination of a set system of feasible coalitions and a choice function that assigns to every feasible coalition a choice set is called a quasi-building system.

Definition 4.2.1 A pair $\mathcal{Q} = (\mathcal{H}, U)$ is a *quasi-building system* on N if it satisfies the following conditions:

(Q1) $\mathcal{H} \subseteq 2^N$ is a set system on N containing both $\emptyset$ and N and $U : \mathcal{H} \to 2^N$ is a choice function, that is, $U(\emptyset) = \emptyset$ and for every nonempty $H \in \mathcal{H}$ it holds that $U(H) \neq \emptyset$ and $U(H) \subseteq H$.

(Q2) For every $H \in \mathcal{H}$ and $h \in U(H)$, there exists a unique maximal partition of $H \setminus \{h\}$ into elements of $\mathcal{H}$, $P(H \setminus \{h\}) = \{S_1, \ldots, S_k\}$, satisfying that for any $J \subseteq \{1, \ldots, k\}$ with $|J| \geq 2$ and nonempty $T_j \subseteq S_j$, $j \in J$, it holds that $\cup_{j \in J} T_j \notin \mathcal{H}$.

Condition (Q1) says that the grand coalition of all players is feasible and that every nonempty feasible coalition contains a nonempty choice set. Condition (Q2) gives some structure to the collection of feasible coalitions and requires that when a player in the choice set of a feasible coalition is removed then there exists a unique maximal partition of the remaining players into feasible subcoalitions. Moreover, the members of this partition are components in the sense that no subsets of different members can form a feasible coalition together. It means that these components or any subsets of them are not able to cooperate with each other. Note that condition (Q2) is weaker than union stable condition, see Algaba et al. (2001) which says that the union of any two intersecting feasible coalitions is also feasible. In the next section we show that quasi-building systems can express cooperative situations described by undirected, directed, or mixed graphs, or set systems like partition systems, or building sets, augmenting systems, posets, convex geometries, and antimatroids.

Example 4.2.2 Consider a cooperative situation between agents that is represented by an undirected graph among the agents, as communication structure discussed in the previous chapters. A link between two agents is a communication possibility between these agents due to for example a political, geographical, technological, or social relation, and only connected sets of agents in the graph are able to form a team to cooperate. Given a connected subset of agents, those who may form or leave the team are the ones who are connected to at least one agent outside the team. These players are the ones that are able to extend the coalition further and may get credit for it. This situation can be expressed by a quasi-building system, constituted from the collection of connected subsets of nodes in the graph, and as choice set of a connected subset of nodes the agents that are connected to agents outside the coalition.

Example 4.2.3 Consider a set of tasks to be performed to produce some product, for example an airplane, which consists of several components including right wing, left wing, and aircraft body. Given the set of all tasks, which yields an airplane, all tasks that can be performed only at the end of the production belong to the choice set of the set of all tasks, e.g. painting the exterior and furnishing the interior. Without the task of painting (furnishing) the remaining tasks still form a feasible set of tasks and furnishing (painting) should be then in the choice set of these tasks. Suppose before painting the exterior and furnishing the interior the airplane is being assembled, then the choice set of the remaining tasks without painting and furnishing consists of assembling the airplane. Suppose at this job the right wing, the left wing, and the aircraft body are assembled together, then the remaining tasks without assembling the airplane can be split into three independent groups of tasks, one for the right wing with assembling the right wing as its choice set, one for the left wing with assembling the left wing as its choice set, and one for the aircraft body with assembling the aircraft body as its choice set, and so on, until no tasks are left. Tasks from different groups are unrelated and performed independently from each other. In this way the whole production procedure can be described by a quasi-building system. The choice set of a feasible group of related tasks consists of all the tasks that might be performed after all other tasks in the group have been undertaken. There can be more than one such task like painting the exterior and furnishing the interior in the set of all tasks.

For a given set system there may exist different choice functions which satisfy condition (Q2).

Example 4.2.4 Consider the complete set system with three players, i.e., there is no restriction on the feasibility of coalitions. Note that $U(\{i\}) = \{i\}$ for $i = 1, 2, 3$.

1. If in any coalition every player is in the choice set, i.e., $U(H) = H$ for all $H \subseteq \{1, 2, 3\}$, then in any coalition each player is able to leave or join the coalition.

2. If $U(\{1,2,3\}) = \{1,2,3\}$, $U(\{1,2\}) = \{1\}$, $U(\{1,3\}) = \{3\}$, and $U(\{2,3\}) = \{2\}$, then the quasi-building system $(\mathcal{H}, U)$ corresponds to the situation at which there is no domination in the grand coalition $\{1,2,3\}$, player 1 dominates player 2 in coalition $\{1,2\}$, player 2 dominates player 3 in coalition $\{2,3\}$, and player 3 dominates player 1 in coalition $\{1,3\}$. This quasi-building system corresponds to a directed circle in which player 2 is an immediate successor of player 1 but not of player 3, player 3 is an immediate successor of player 2 but not of player 1, and player 1 is an immediate successor of player 3 but not of player 2.

3. If $U(\{1,2,3\}) = \{1\}$, $U(\{1,2\}) = \{1\}$, $U(\{1,3\}) = \{1\}$, and $U(\{2,3\}) = \{2,3\}$, then player 1 dominates the other players in any coalition he is a member of, while there is no domination between the players 2 and 3. This quasi-building system corresponds to a graph in which players 2 and 3 are immediate successors of player 1 and players 2 and 3 are linked to each other.

4. If $U(\{1,2,3\}) = \{1,2\}$, $U(\{1,2\}) = \{1,2\}$, $U(\{1,3\}) = \{1\}$, and $U(\{2,3\}) = \{2\}$, then players 1 and 2 always dominate player 3, while there is no domination between players 1 and 2. This quasi-building system corresponds to a graph in which player 3 is an immediate successor of players 1 and 2 and players 1 and 2 are linked to each other.

5. If $U(\{1,2,3\}) = \{1,2,3\}$, $U(\{1,2\}) = \{1\}$, $U(\{1,3\}) = \{1\}$, and $U(\{2,3\}) = \{2\}$, then there is no domination in the grand coalition, player 1 dominates player 2 in coalition $\{1,2\}$, player 1 dominates player 3 in coalition $\{1,3\}$, and player 2 dominates player 3 in coalition $\{2,3\}$. This quasi-building system cannot be represented by any graph.

Now we explain how to construct a solution for cooperative games with restrictions to form and leave feasible coalitions described by a quasi-building system. The solution is the average of the marginal vectors that correspond to the rooted trees that are compatible with the quasi-building system. In this way only players who belong to the choice set of a feasible coalition are able to receive a marginal contribution.

A rooted tree on N is a set $T \subseteq \{(i,j) \mid i,j \in N,\ i \neq j\}$ satisfying that there is a unique node, labeled by $r(T)$, called the root of the tree, such that $(i, r(T)) \notin T$ for all $i \neq r(T)$ and from $r(T)$ to any other node there is a unique directed path in T. Given a set $D \subseteq \{(i,j) \mid i,j \in N,\ i \neq j\}$ on N, a node labeled by $j \in N$ is a predecessor of a node labeled by $i \in N$, or i is an immediate successor of j, if $(j, i) \in D$. The set of immediate successors of j in D is denoted by $S^D(j)$. An element $j \in N$ is a successor of $i \in N$ in D if there is a directed path in D from the node labeled by i to the node labeled by j. The set of all successors of node labelled by i in D is denoted by $F^D(i)$ and $\bar{F}^D(i) = F^D(i) \cup \{i\}$.

Definition 4.2.5 Given a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on N, a rooted tree T on N is *compatible with* $\mathcal{Q}$ if for all $i \in N$ it holds that $\bar{F}^T(i) \in \mathcal{H}$, $i \in U(\bar{F}^T(i))$, and $P(F^T(i)) = \{\bar{F}^T(j) \mid j \in S^T(i)\}$.

A rooted tree T on N is compatible with a quasi-building system on N if for every node labeled by $i \in N$ it holds that the coalition consisting of i and all his successors in T is a feasible coalition and i is in the choice set of this coalition. Moreover, each member of the unique maximal partition of the set of successors of i consists of an immediate successor of i and his successors in T. A rooted tree represents a hierarchy structure on the set of individual players (see, for example, Demange (2004)). Thus, a rooted tree compatible with a quasi-building system represents a hierarchy structure on the set of players that is compatible with the restrictions to form and leave coalitions according to the quasi-building system.

An important property of a quasi-building system $\mathcal{Q}$ is that the set of rooted trees compatible with $\mathcal{Q}$, denoted by $\mathcal{T}(\mathcal{Q})$, is nonempty.

Theorem 4.2.6 *Let $\mathcal{Q}$ be a quasi-building system on N, then $\mathcal{T}(\mathcal{Q}) \neq \emptyset$.*

Proof Let $\mathcal{Q} = (\mathcal{H}, U)$. We construct a compatible tree by induction on n. For $n = 1$ the result is trivial. Take $n \geq 2$. From (Q1) it follows that $N \in \mathcal{H}$ and $U(N) \neq \emptyset$. We label the root of T by any element $r \in U(N)$. According to (Q2) there exists a unique maximal partition $S_1, \ldots, S_k$ of $N \setminus \{r\}$ for some $k \geq 1$ such that $S_j \in \mathcal{H}$ for all $j = 1, \ldots, k$. In each S_j, $j = 1, \ldots, k$, there exists according to (Q1) an element $r_j \in U(S_j)$. For $j = 1, \ldots, k$, we take in T each node labeled by r_j as an immediate successor of r. By induction, for each $j = 1, \ldots, k$, there exists a rooted tree T_j for the restriction of the quasi-building system $\mathcal{Q}$ to S_j such that the root of T_j is the node labeled by r_j. By this construction we obtain by induction a rooted tree T compatible with $\mathcal{Q}$. □

The next example shows that two quasi-building systems with the same set system but different choice functions may lead to different collections of compatible rooted trees.

Example 4.2.7 Consider the quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $N = \{1, 2, 3\}$, where $\mathcal{H} = 2^N$ and $U(H) = H$ for all $H \in \mathcal{H}$. All six line-trees on N are compatible with this system.

Next Consider the quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $N = \{1, 2, 3\}$, where $\mathcal{H} = 2^N$ and $U(\{1, 2, 3\}) = \{1, 2, 3\}$, $U(\{1, 2\}) = \{1\}$, $U(\{1, 3\}) = \{3\}$, and $U(\{2, 3\}) = \{2\}$. There are three line-trees compatible with this system. One line-tree has as root the node labeled by 1 and has $(1, 2)$ and $(2, 3)$ as edges. Another line-tree has as root the node labeled by 2 and has $(2, 3)$ and $(3, 1)$ as edges. The third line-tree has as root the node labeled by 3 and has $(3, 1)$ and $(1, 2)$ as edges.

Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building system on N and $v : \mathcal{H} \to \mathbb{R}$ a function such that $v(\emptyset) = 0$. We consider $\mathcal{H}$ as a coalition structure on a set of n players and v as a characteristic function of a cooperative game with $v(H)$, $H \in \mathcal{H}$, the worth of feasible coalition H. The triple $(N, v, \mathcal{Q})$ is a quasi-building system game on the player set N. The collection of all quasi-building system games on N is denoted by $\mathcal{G}_N^{qbs}$. A single-valued solution on $\mathcal{G}_N^{qbs}$ is a mapping $\xi : \mathcal{G}_N^{qbs} \to \mathbb{R}^n$, assigning a payoff vector $\xi(N, v, \mathcal{Q}) \in \mathbb{R}^n$ to any quasi-building system game $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$.

As solution concept for a quasi-building system game, we propose the average of the marginal vectors corresponding to all compatible trees. For a quasi-building system game, at a marginal vector corresponding to a compatible tree every player receives as payoff what he contributes in worth when he joins his successors in the tree, i.e., given a quasi-building system game $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$, the marginal vector $m^T(N, v, \mathcal{Q}) \in \mathbb{R}^n$ corresponding to rooted tree $T \in \mathcal{T}(\mathcal{Q})$ is defined by

$$m_i^T(N, v, \mathcal{Q}) = v(\bar{F}^T(i)) - \sum_{j \in S^T(i)} v(\bar{F}^T(j)),\ i \in N.$$

Definition 4.2.8 On the class of quasi-building system games $\mathcal{G}_N^{qbs}$ the *Average Marginal Vector value*, or *AMV-value*, assigns to every quasi-building system game $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ the payoff vector

$$AMV(N, v, \mathcal{Q}) = \frac{1}{|\mathcal{T}(\mathcal{Q})|} \sum_{T \in \mathcal{T}(\mathcal{Q})} m^T(N, v, \mathcal{Q}).$$

The AMV-value of a quasi-building system game is the average of the marginal vectors induced by all trees compatible with the quasi-building system. The AMV-value is well-defined on the class of quasi-building system games, since according to Theorem 4.2.6 every quasi-building system has at least one tree compatible with it. The AMV-value takes into account that only players that are in the choice set of a coalition can receive a marginal contribution to form the coalition.

4.3 Special cases for quasi-building system

In this section we discuss how a quasi-building system is induced when the underlying structure for cooperation has some specific properties, such as a collection of connected subsets in a (mixed) graph, or a combinatorial structure such as augmenting system, antimatroid, poset, partition system, or convex geometry. When the cooperation structure is a communication situation represented by a graph, the choice function is used to represent the underlying dominance relations between the players. In case the cooperative structure is expressed by some set system, the choice set of a feasible coalition consists precisely of the players in the coalition who satisfy condition (Q2).

4.3.1 Graphical quasi-building systems

A mixed graph $G = (V, E)$ on N with $E \subseteq \{(i, j) \mid i, j \in N,\ i \neq j\}$ consists of a set of nodes V equal to the set N and a set of edges E which is constituted from a set of links $L = \{(i, j) \in E \mid (j, i) \in E)\}$, being undirected edges, and a set of arcs $A = \{(i, j) \in E \mid (j, i) \notin E)\}$, being directed edges. A mixed graph without arcs is an undirected graph and a mixed graph without links is a directed graph. A graph $G = (V, E)$ on N is complete if $E = \{(i, j) \mid i, j \in N, i \neq j\}$.

Definition 4.3.1 Given a mixed graph $G = (V, E)$ on N with $E = L \cup A$, the pair $\mathcal{Q}(G) = (\mathcal{H}, U)$ consists of a set system $\mathcal{H}$ and mapping $U : \mathcal{H} \to 2^N$ given by the following conditions:

- $\mathcal{H}$ consists of all subsets H of N such that H is a connected set in G and if there is a directed path in G from some node in H to some node outside H, then there exists also a directed path in G from the latter node to some node in H.

- U assigns to any $H \in \mathcal{H}$ a set of nodes which are undominated in the subgraph $G(H)$ of G on H, i.e., $U(H)$ is the set of nodes in H from which there exists a directed path in the subgraph $G(H)$ to any of its predecessors in $G(H)$.

Note that in case of an undirected graph each connected set is a feasible coalition and that its choice set consists of all its nodes.

Lemma 4.3.2 *For any connected mixed graph G on N, $\mathcal{Q}(G)$ is a quasi-building system on N.*

Proof Let $\mathcal{Q}(G) = (\mathcal{H}, U)$. Since G is connected, it holds that $N \in \mathcal{H}$. For any nonempty $H \in \mathcal{H}$, H is connected in G and since H is finite there exists an undominated node in $G(H)$. This implies that $U(H)$ is a nonempty subset of H, which proves condition (Q1). Since G is a graph, for every $h \in U(H)$ there exists a unique maximal partition of $H \setminus \{h\}$ into elements of $\mathcal{H}$. Also because G is a graph, any union of nonempty subsets of at least two sets in such a maximal partition is not connected in G and is therefore not an element of the set system $\mathcal{H}$. Consequently, condition (Q2) is also fulfilled. □

For a connected mixed graph G, $\mathcal{Q}(G)$ is called the graphical quasi-building system corresponding to G.

In Example 4.2.7 two different quasi-building systems are presented. The first one in the example corresponds to the graphical quasi-building system induced from the complete graph of three nodes, or an undirected circle. The second one corresponds to the graphical quasi-building system induced from a directed circle with three nodes. The example shows that different connected mixed graphs may have the same set system of connected coalitions. The differences in dominance between nodes within the graphs is expressed in the choice function since the choice sets of the feasible coalitions may differ.

On the class of mixed graph games, the average covering tree value is introduced in Khmelnitskaya et al. (2012) as the average of the marginal vectors that correspond to the set of all covering trees induced from the graph. The AMV-value coincides with the average covering tree solution since the set of trees being compatible with a graphical quasi-building system coincides with the collection of covering trees on the underlying graph.

4.3.2 Set systems

In this subsection we discuss quasi-building systems that are induced by set systems of feasible coalitions like partition systems, or building sets, augmenting systems, antimatroids, posets, and convex geometries.

Definition 4.3.3 Given a set system $\mathcal{F}$ on N, the pair $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$ consists of a set system $\mathcal{H}$ and mapping $U : \mathcal{H} \to 2^N$ satisfying the following conditions:

- $\mathcal{H} = \mathcal{F}$.
- $U(H) = \{h \in H \mid$ there exists a unique maximal partition of $H \setminus \{h\}$ satisfying (Q2)$\}$ for all $H \in \mathcal{H}$.

For a set system $\mathcal{F}$, the pair $\mathcal{Q}(\mathcal{F}) = (\mathcal{F}, U)$ is a quasi-building system if $\emptyset$ and N belong to $\mathcal{F}$ and $U(H) \neq \emptyset$ for all nonempty $H \in \mathcal{F}$.

A quasi-building system is a generalization of a partition system, introduced in Algaba et al. (2000), or building set (Koshevoy and Talman (2014)). A set system $\mathcal{H}$ is a building set on N if the following conditions are satisfied:

(B1) $\mathcal{H}$ is a set system on N containing both $\emptyset$ and N.

(B2) If $S, T \in \mathcal{H}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{H}$.

(B3) For all $i \in N$, $\{i\} \in \mathcal{H}$.

A building set is a set system containing the grand coalition ((B1)) and all singletons ((B3)). A set system satisfying condition (B2) is called union stable, see Algaba et al. (2001).

Proposition 4.3.4 *For a set system $\mathcal{H}$ on N and function $U : \mathcal{H} \to 2^N$ satisfying $U(H) = H$ for all $H \in \mathcal{H}$, it holds that $(\mathcal{H}, U)$ is a quasi-building system if and only if $\mathcal{H}$ is a building set.*

Proof. Suppose $(\mathcal{H}, U)$ is a quasi-building system on N with $U(H) = H$ for all $H \in \mathcal{H}$. Condition (B1) obviously holds. Let $S, T \in \mathcal{H}$ with $S \cap T \neq \emptyset$. If $S \cup T = N$, then (B2) is verified by (Q1). Suppose $S \cup T \neq N$. Take any $j \in N \setminus (S \cup T)$, then $j \in U(N)$ since $U(N) = N$. Because of (Q2) and since $S \cap T \neq \emptyset$, $S \cup T$ is contained in some single member of the partition $P(N \setminus \{j\})$. Let R be this set, then $R \in \mathcal{H}$. If $S \cup T = R$ then (B2) is verified. Otherwise, take any $j' \in R \setminus (S \cup T)$. Again, by (Q2) and, since $U(R) = R$ and $S \cap T \neq \emptyset$, we get that $S \cup T$ belongs to a single member of the partition $P(R \setminus \{j'\})$, and so on. At some step, we get $S \cup T \in \mathcal{H}$ and (B2) is verified. For verifying (B3), take any $i \in N$ and $j \in N \setminus \{i\}$. Then there is a unique $S \in P(N \setminus \{j\})$ in $\mathcal{H}$ containing i and take any $j' \in S \setminus \{i\}$. Then take the unique member of $P(S \setminus \{j'\})$ in $\mathcal{H}$ containing i, and so on, until we get $\{i\} \in \mathcal{H}$.

For the reverse implication, let the set system $\mathcal{H}$ be a building set and consider $(\mathcal{H}, U)$ where $U(H) = H$ for all $H \in \mathcal{H}$. Condition (Q1) follows from condition (B1) and the supposition $U(H) = H$ for all $H \in \mathcal{H}$. For condition (Q2), it is to show that there exists a unique maximal feasible partition of $H \setminus \{h\}$ for every $H \in \mathcal{H}$ and $h \in H$. Take any $H \in \mathcal{H}$ and $h \in H$. Due to (B3), there exists a feasible partition of $H \setminus \{h\}$, and therefore there is at least one maximal partition $\mathcal{S}$. Suppose there is another maximal feasible partition of $H \setminus \{h\}$, say, $\mathcal{T}$. Since $\mathcal{S} \neq \mathcal{T}$, there exists $S \in \mathcal{S}$ such that $S \nsubseteq T$ for all $T \in \mathcal{T}$, otherwise $\mathcal{S}$ cannot be a maximal partition of $H \setminus \{h\}$. Now consider $\mathcal{T}^S = \{T \in \mathcal{T} \mid T \cap S \neq \emptyset\}$. Then $|\mathcal{T}^S| \geq 2$ and from (B2) it follows that $\bigcup_{T \in \mathcal{T}^S}(T \cup S) = \bigcup_{T \in \mathcal{T}^S} T \in \mathcal{H}$, which contradicts that $\mathcal{T}$ is a maximal partition of $H \setminus \{h\}$. Finally, to show the second part of condition (Q2), take any $H \in \mathcal{H}$ and $h \in H$. Let $\{S_1, \ldots, S_k\}$ be the unique maximal partition of $H \setminus \{h\}$, and suppose there exists $J \subseteq \{1, \ldots, k\}$ with $|J| \geq 2$ and some nonempty $T_j \subseteq S_j$, $j \in J$, such that $T = \cup_{j \in J} T_j \in \mathcal{H}$. From (B2) it follows that $\bigcup_{j \in J}(T \cup S_j) = \bigcup_{j \in J} S_j \in \mathcal{H}$, since $T \cap S_j \neq \emptyset$ for all $j \in J$, which contradicts that $\{S_1, \ldots, S_k\}$ is a maximal partition. □

The proposition implies that if a set system is a building set, then we obtain a quasi-building system if we take as choice set of any feasible coalition the coalition itself. Therefore, $\mathcal{Q}(\mathcal{F})$ is a quasi-building system if $\mathcal{F}$ is a building set. For a building set $\mathcal{F}$, the AMV-value of a game $(N, v, \mathcal{Q}(\mathcal{F}))$ is equal to the gravity center solution of the building set game $(N, v, \mathcal{F})$ introduced in Koshevoy and Talman (2014), because the collection of maximal strictly nested sets of a building set $\mathcal{F}$ defined in Koshevoy and Talman (2014) corresponds one-to-one to the collection of rooted trees compatible with $\mathcal{Q}(\mathcal{F})$. In Koshevoy and Talman (2014) it is shown that, for a building set, the gravity center solution coincides with the Shapley value defined in Faigle et al. (2010) using the Monge algorithm. Furthermore, if the set system is the collection of all coalitions of players, then the AMV-value of the corresponding quasi-building system game $(N, v, (2^N, U))$ with $U(H) = H$ for all $H \in 2^N$ is the Shapley value (Shapley (1953)) of the TU-game v.

In Bilbao (2003) augmenting systems are introduced as set systems in cooperative games. A set system $\mathcal{F}$ on N is an augmenting system on N if it satisfies the following conditions:

(S1) $\emptyset \in \mathcal{F}$.

(S2) If $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{F}$.

(S3) If $S, T \in \mathcal{F}$ and $S \subset T$, then there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

An augmenting system is a set system which is union stable ((S2)) and satisfying one-point extension ((S3)). Although the collection of connected subsets of an undirected graph is an augmenting system, this does not hold in general for a mixed graph with cycles. Antimatroids, another class of set systems, introduced in Dilworth (1940), form a subclass of the class of augmenting systems. In Faigle and Kern (1992), a restriction among players is expressed as precedence constraints, represented by a poset. Given a poset on a player set, a subset of the player set is feasible as a coalition if for every player in the subset, all players who are ordered below this player are also in the subset. The collection of feasible coalitions of a poset forms an antimatroid, see Algaba et al. (2004). An augmenting system on N may not contain N as a feasible coalition.[3]

Lemma 4.3.5 *For any augmenting system $\mathcal{F}$ on N with $N \in \mathcal{F}$, $\mathcal{Q}(\mathcal{F})$ is a quasi-building system on N.*

Proof Let $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$, then $\mathcal{H} = \mathcal{F}$. The empty set belongs to $\mathcal{H}$ by condition (S1) and N belongs to $\mathcal{H}$ by assumption. By definition $U(H) \subseteq H$ for all $H \in \mathcal{H}$. Regarding the non-emptiness of the set $U(H)$, it follows from condition (S1) and repeated application of condition (S3) starting with $S = \emptyset$, that for any $H \in \mathcal{H}$ there exists $h \in H$ such that $H \setminus \{h\} \in \mathcal{H}$, which implies that $U(H)$ is nonempty. Condition (Q1) is therefore satisfied. Condition (Q2) is satisfied by construction. □

The notion of convex geometry is introduced in Edelman and Jamison (1985). A set system $\mathcal{F}$ is a convex geometry on N if it satisfies the following conditions:

(C1) $\emptyset \in \mathcal{F}$.

(C2) If $S, T \in \mathcal{F}$, then $S \cap T \in \mathcal{F}$.

(C3) If $S \in \mathcal{F}, S \neq N$, then there exists $i \in N \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

A convex geometry is a set system satisfying intersection-closedness ((C2)), see Bilbao and Edelman (2000), and another form of one-point extension ((C3)). Note that a mixed graph may not be expressed as a convex geometry, since the collection of connected subsets of a graph may not satisfy intersection-closedness. Different from augmenting systems, in a convex geometry the grand coalition N is necessarily feasible. Since union-closedness does not imply intersection-closedness and vice versa, there is no inclusion relationship between the class of convex geometries and the class of augmenting systems.

[3] If for an augmenting system on N it holds that N is not a member, but every $i \in N$ belongs to at least one member, then there exists a unique maximal partition of N into feasible coalitions satisfying that every feasible coalition is a subset of one of the partition members. The analysis can then be applied separately to every partition member.

Lemma 4.3.6 *For any convex geometry $\mathcal{F}$ on N, $\mathcal{Q}(\mathcal{F})$ is a quasi-building system on N.*

Proof Let $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$, then $\mathcal{H} = \mathcal{F}$. The empty set and N belong to $\mathcal{H}$ by (C1) and (C3). Suppose that for some nonempty $H \in \mathcal{H}$ there is no $h \in H$ such that $H \setminus \{h\} \in \mathcal{H}$. By (C1), H cannot be a singleton and therefore $|H| \geq 2$. From (C1) and (C3) it follows that there exists a sequence of n sets $S_1, \ldots, S_n$, with $|S_k| = k$, $S_k \in \mathcal{H}$, $k = 1, \ldots, n$, and $S_1 \subset S_2 \subset \cdots \subset S_n = N$. Consider S_{n-1} and denote it as $N \setminus \{i_1\}$. From (C2) it follows that $i_1 \notin H$, otherwise $H \cap (N \setminus \{i_1\}) = H \setminus \{i_1\} \in \mathcal{H}$, which contradicts the supposition. Next, consider S_{n-2} and denote it as $N \setminus \{i_1, i_2\}$. Similarly, it holds that $i_2 \notin H$, and so on. Now consider $S_{|H|}$ and let $T = N \setminus S_{|H|}$. $H \cap T = \emptyset$ and therefore $S_{|H|} = H$. Then $S_{|H|-1} \in \mathcal{H}$ and there exists $h \in H$ such that $S_{|H|-1} = H \setminus \{h\}$ is feasible, which again is a contradiction. This proves condition (Q1). Condition (Q2) is satisfied by construction. □

From the prrof of the lemma it follows that the collection of trees compatible with a quasi-building system induced by a convex geometry always contains at least one line-tree.

Solutions introduced in the literature for the class of games on augmenting systems containing also the grand coalition and on convex geometries are the Shapley value in Bilbao and Ordóñez (2009), which we study in the previous chapter, and the Shapley value in Bilbao and Edelman (2000), respectively. Both these values are defined as the average of the marginal vectors that correspond to all maximal chains in the underlying set system. The AMV-value for games on the induced quasi-building system of these set systems differs from those two values because not all compatible trees are necessarily line-trees. This means that for these structures the AMV-value is the average of typically more marginal vectors than the two Shapley values are. Notice that for both kinds of set systems $\mathcal{F}$ these Shapley values are equal to the AMV-value defined on a quasi-building system $(\mathcal{H}, U)$ given by

- $\mathcal{H} = \mathcal{F}$,
- $U(H) = \{h \in H \mid H \setminus \{h\} \in \mathcal{F}\}$ for all $H \in \mathcal{F}$.

For both cases the one-point extension conditions (S3) and (C3) ensure that $U(H) \neq \emptyset$ for every nonempty $H \in \mathcal{F}$, and we see in Section 5 that such a pair $(\mathcal{H}, U)$ is what we call a chain quasi-building system.

Another solution on the class of cooperative games on augmenting system is introduced in Bilbao (2003) and characterized in Algaba et al. (2010).

They define a restricted TU-game, like the Myerson restricted game from a TU-game with communication structure, and propose as solution concept the Shapley value of the restricted game.

4.4 Properties of the average marginal vector value

In this section we discuss some properties of the AMV-value on the class of quasi-building system games. The first two properties are standard.

Definition 4.4.1 A solution $\xi : \mathcal{G}_N^{qbs} \to \mathbb{R}^n$ satisfies *efficiency* if for all $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ it holds that

$$\sum_{i \in N} \xi_i(N, v, \mathcal{Q}) = v(N).$$

An efficient value generates for every quasi-building system game a payoff vector which allocates the worth of the grand coalition among the players.

Proposition 4.4.2 *The AMV-value satisfies efficiency.*

Proof Let $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ be a quasi-building system game. It suffices to show that the marginal vector induced from any tree compatible with $\mathcal{Q}$ is efficient, since the AMV-value is the average of all such vectors. Take any tree $T \in \mathcal{T}(\mathcal{Q})$. For all $i \in N$ it holds that

$$\begin{aligned}\sum_{j \in \bar{F}^T(i)} m_j^T(N, v, \mathcal{Q}) &= \sum_{j \in \bar{F}^T(i)} v(\bar{F}^T(j)) - \sum_{j \in \bar{F}^T(i)} \sum_{k \in S^T(j)} v(\bar{F}^T(k)) \\ &= \sum_{j \in \bar{F}^T(i)} v(\bar{F}^T(j)) - \sum_{k \in F^T(i)} v(\bar{F}^T(k)) \\ &= v(\bar{F}^T(i)),\end{aligned}$$

since for each $k \in F^T(i)$, there is a unique $j \in \bar{F}^T(i)$ such that k is an immediate successor of j in T. Let $r \in N$ be such that $\bar{F}^T(r) = N$, i.e., r is the root of T. Then it holds that

$$\sum_{i \in N} m_i^T(N, v, \mathcal{Q}) = \sum_{i \in \bar{F}^T(r)} m_i^T(N, v, \mathcal{Q}) = v(\bar{F}^T(r)) = v(N).$$

□

Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building system on N. For any two quasi-building system games $(N, v, \mathcal{Q})$ and $(N, w, \mathcal{Q})$ in $\mathcal{G}_N^{qbs}$ and $a, b \in \mathbb{R}$ the quasi-building system game $(N, av + bw, \mathcal{Q})$ in $\mathcal{G}_N^{qbs}$ is defined by $(av + bw)(H) = av(H) + bw(H)$ for all $H \in \mathcal{H}$.

Definition 4.4.3 A solution $\xi : \mathcal{G}_N^{qbs} \to \mathbb{R}^n$ satisfies *linearity* if for any quasi-building system games $(N, v, \mathcal{Q})$ and $(N, w, \mathcal{Q})$ in $\mathcal{G}_N^{qbs}$ and $a, b \in \mathbb{R}$ it holds that

$$\xi(N, av + bw, \mathcal{Q}) = a\xi(N, v, \mathcal{Q}) + b\xi(N, w, \mathcal{Q}).$$

Proposition 4.4.4 *The AMV-value satisfies linearity.*

Proof Let $(N, v, \mathcal{Q})$ and $(N, w, \mathcal{Q})$ be two games on a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ and let $a, b \in \mathbb{R}$. It suffices to show that the marginal vector corresponding to any tree compatible with $\mathcal{Q}$ satisfies linearity, since the AMV-value is the average of all such vectors. Consider the quasi-building system game $(N, av + bw, \mathcal{Q})$. All three games have the same collection $\mathcal{T}(\mathcal{Q})$ of compatible trees. For each $T \in \mathcal{T}(\mathcal{Q})$ and $i \in N$, it holds that

$$\begin{aligned} m_i^T(N, av + bw, \mathcal{Q}) =& (av + bw)(\bar{F}^T(i)) - \sum_{j \in S^T(i)} (av + bw)(\bar{F}^T(j)) \\ =& a(v(\bar{F}^T(i)) - \sum_{j \in S^T(i)} v(\bar{F}^T(j))) \\ &+ b(w(\bar{F}^T(i)) - \sum_{j \in S^T(i)} w(\bar{F}^T(j))) \\ =& am_i^T(N, v, \mathcal{Q}) + bm_i^T(N, w, \mathcal{Q}), \end{aligned}$$

which implies that each marginal vector corresponding to a tree compatible with $\mathcal{Q}$ satisfies linearity. □

The null player property is a widely known concept. In a standard TU-game, a player is a null player if he never contributes to the worth of any coalition he joins. A restricted null player is defined in the previous chapters on the class of TU-games with communication structure as the player who never contributes to the worth when he joins to a set of players to form a connected coalition. For a quasi-building system game, however, a player is only able to receive a marginal contribution if he is a member of the choice set of that coalition, and therefore a restricted null player is defined as follows.

Definition 4.4.5 For a quasi-building system game $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ with $\mathcal{Q} = (\mathcal{H}, U)$, a player $i \in N$ is a *restricted null player* if for all $H \in \mathcal{H}$ such that $i \in U(H)$ it holds that

$$v(H) - \sum_{K \in P(H \setminus \{i\})} v(K) = 0.$$

The definition of a restricted null player for a quasi-building system game depends not only on the set system but also on the choice function.

Definition 4.4.6 A solution $\xi : \mathcal{G}_N^{qbs} \to \mathbb{R}^n$ satisfies the *restricted null player property* if for every $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ it holds that $\xi_i(N, v, \mathcal{Q}) = 0$ whenever $i \in N$ is a restricted null player for $(N, v, \mathcal{Q})$.

The restricted null player property says that a player who never contributes any worth to a feasible coalition he is able to form should receive zero payoff.

Proposition 4.4.7 *The AMV-value satisfies the restricted null player property.*

Proof Let $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ be a quasi-building system game with $\mathcal{Q} = (\mathcal{H}, U)$ and let $i \in N$ be a restricted null player for $(N, v, \mathcal{Q})$. Since the AMV-value is the average of the marginal vectors corresponding to all rooted trees compatible with $\mathcal{Q}$, it suffices to show that $m_i^T(N, v, \mathcal{Q}) = 0$ for all $T \in \mathcal{T}(\mathcal{Q})$. Take any $T \in \mathcal{T}(\mathcal{Q})$ and let $H = \bar{F}^T(i)$, then $i \in U(H)$. Since i is a restricted null player for $(N, v, \mathcal{Q})$, it follows that

$$\begin{aligned} m_i^T(N, v, \mathcal{Q}) &= v(\bar{F}^T(i)) - \sum_{j \in S^T(i)} v(\bar{F}^T(j)) \\ &= v(H) - \sum_{K \in P(H \setminus \{i\})} v(K) = 0. \end{aligned}$$

□

For a quasi-building system a feasible coalition of players can be inessential. A feasible coalition of a given quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ is inessential if it is not the grand coalition and is either not a member of a maximal partition $P(H \setminus \{h\})$ for any $H \in \mathcal{H}$, $h \in U(H)$, or if it is a member of some maximal partition $P(H \setminus \{h\})$, then the coalition H itself is also inessential. This recursively defines the set $\mathcal{I}(\mathcal{Q})$ of inessential coalitions of a quasi-building system $\mathcal{Q}$.

Example 4.4.8 Consider the quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $N = \{1, 2, 3\}$ where $\mathcal{H} = 2^N \setminus \{2, 3\}$ and $U(\{1, 2, 3\}) = \{1\}$, $U(\{1, 2\}) = \{1\}$, and $U(\{1, 3\}) = \{1\}$. Player 1 dominates players 2 and 3. This situation can be expressed by a directed graph where player 1 is the unique predecessor of player 2 and player 3. The coalitions $\{1, 2\}$, $\{1, 3\}$ and $\{1\}$ are inessential, because those coalitions do not belong to any maximal partition. Notice that $U(\{1, 2, 3\}) = \{1\}$ and $P(\{1, 2, 3\} \setminus \{1\}) = \{\{2\}, \{3\}\}$, which implies that both $\{2\}$ and $\{3\}$ are not inessential coalitions.

The main property of an inessential coalition is that it does not show up as $\bar{F}^T(i)$ for any $i \in N$ and compatible tree T, as we see in the next lemma. It is also shown that inessential coalitions are the only feasible coalitions that have this property.

Lemma 4.4.9 *Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building system on N and let $H \in \mathcal{H}$. Then $H \neq \bar{F}^T(i)$ for any tree $T \in \mathcal{T}(\mathcal{Q})$ and $i \in N$ if and only if $H \in \mathcal{I}(\mathcal{Q})$.*

Proof Suppose $H \in \mathcal{I}(\mathcal{Q})$ and $H = \bar{F}^T(i)$ for some tree $T \in \mathcal{T}(\mathcal{Q})$ and $i \in N$. Since $H \in \mathcal{I}(\mathcal{Q})$, it holds that $H \neq N$. Therefore there exists H_1 and $h_1 \in U(H_1)$ such that $H_1 = \bar{F}^T(h_1)$ and $H \in P(H_1 \setminus \{h_1\})$, which would be a contradiction unless $H_1 \in \mathcal{I}(\mathcal{Q})$. If $H_1 \in \mathcal{I}(\mathcal{Q})$, by following the same argument there exists $H_2 = \bar{F}^T(h_2)$ for some $h_2 \in U(H_2)$ satisfying $H_1 \in P(H_2 \setminus \{h_2\})$. Then it must hold that $H_2 \in \mathcal{I}(\mathcal{Q})$ to avoid a contradiction, and so on. Since the player set is finite and $N \notin \mathcal{I}(\mathcal{Q})$, we obtain a finite sequence of feasible coalitions $(H_1, \ldots, H_m)$ for some $m < n$ satisfying $H_1 \subset \cdots \subset H_m$, $H_{m-1} \in \mathcal{I}(\mathcal{Q})$ and $H_m \notin \mathcal{I}(\mathcal{Q})$, whereas $H_{m-1} \in P(H_m \setminus \{h_m\})$ and $H_m = \bar{F}^T(h_m)$ for some $h_m \in U(H_m)$. This implies $H_{m-1} \notin \mathcal{I}(\mathcal{Q})$, which is a contradiction.

Next, suppose $H \notin \mathcal{I}(\mathcal{Q})$. If $H = N$, then $H = \bar{F}^T(r)$ for some $r \in U(N)$ and tree $T \in \mathcal{T}(\mathcal{Q})$. If $H \neq N$, then there exists $H_1 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ such that $H \in P(H_1 \setminus \{h_1\})$ for some $h_1 \in U(H_1)$. Since H_1 is not inessential, there exists $H_2 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ such that $H_1 \in P(H_2 \setminus \{h_2\})$ for some $h_2 \in U(H_2)$, and so on. Since the player set is finite, there is a finite sequence of players $(h_1, \ldots, h_m)$ and feasible sets $(H_0, H_1, \ldots, H_m)$ for some $m < n$ such that $H_m = N$, $H_0 = H$, and $h_j \in U(H_j)$ and $H_{j-1} \in P(H_j \setminus \{h_j\})$ for $j = 1, \ldots, m$. Then as in the proof of Theorem 4.2.6 we can construct a tree T on N having player h_m as root and containing (h_j, h_{j-1}) for $j = 2, \ldots, m$ as arcs. Then T corresponds to a rooted tree compatible with $\mathcal{Q}$ and $H = \bar{F}^T(j)$ for some $j \in N$. □

Definition 4.4.10 A solution $\xi : \mathcal{G}_N^{qbs} \to \mathbb{R}^n$ satisfies the *inessential coalition property* if for every $(N, v, \mathcal{Q}), (N, w, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ such that $v(H) = w(H)$ for all $H \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, it holds that $\xi(N, v, \mathcal{Q}) = \xi(N, w, \mathcal{Q})$.

The inessential coalition property says that the solution is independent of the worth of inessential coalitions. In Example 4.4.8, where player 1 is the only player who can form or leave the grand coalition, the worths of smaller feasible coalitions containing player 1 ($v(\{1\}), v(\{1,2\})$ and $v(\{1,3\})$) will not be brought to the negotiation table because he is in a dominant position. The solutions we study in the previous chapters are independent of the worths of not feasible, i.e. not connected, coalitions in underlying communication structure, cf. Lemma 2.3.11. With respect to this point, the inessential coalition property of a solution on the class of quasi-building system game takes into account, not only the feasibility of coalitions, but also the choice function which underlies in the system.

The AMV-value on the class of quasi-building system games satisfies this property, since according to Lemma 4.4.9 the collection of compatible trees and the marginal vectors compatible to those trees do not change by changing the worths of inessential coalitions.

Proposition 4.4.11 *The AMV-value satisfies the inessential coalition property.*

While an inessential coalition cannot be a maximal subset of successors of a node in any compatible tree, there might also exist feasible coalitions that are in every compatible tree maximal subsets of successors of some node. Such a coalition is called a closed coalition of the quasi-building system.

Definition 4.4.12 Given a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$, a coalition $H \in \mathcal{H}$ is a *closed coalition* if for every $T \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ satisfying $H \subset T$ it holds that $H \cap U(T) = \emptyset$.

A closed coalition of a quasi-building system is a feasible coalition, of which no player belongs to the choice set of any non-inessential feasible coalition that contains the coalition. Notice that the grand coalition N is by definition a closed coalition. In a hierarchical structure, any coalition consisting of a player together with all his successors in the hierarchy is a closed coalition. Since members of a closed coalition can never contribute outside their own coalition, as none of them can form or leave a coalition which contains them, their total payoff should be equal to the worth of the coalition itself. In Example 4.4.8, where players 2 and 3 are dominated and not able to form a feasible coalition by themselves, a solution that satisfies the closed coalition property will allocate to each of these players the worth of himself. This property incorporates the concept of choice sets, and it may be seen as a generalization of component efficiency to the class of quasi-building system games. In Chapter 2, on the class of TU-games with communication structure the Myerson value satisfies component efficiency, saying that the solution allocates to each component in the underlying communication graph its worth. The closed coalition property does the same for all coalitions that can not benefit from cooperation with the players outside.

Definition 4.4.13 A solution $\xi : \mathcal{G}_N^{qbs} \to \mathbb{R}^n$ satisfies the *closed coalition property* if for every $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ with $\mathcal{Q} = (\mathcal{H}, U)$ and closed coalition $H \in \mathcal{H}$ it holds that $\sum_{i \in H} \xi_i(N, v, \mathcal{Q}) = v(H)$.

A solution that satisfies the closed coalition property allocates as total payoff to the players who form a closed coalition exactly the worth of that coalition. Since the grand coalition is a closed coalition, the closed coalition property implies efficiency.

Proposition 4.4.14 *The AMV-value satisfies the closed coalition property.*

Proof Let $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ be a quasi-building system game with $\mathcal{Q} = (\mathcal{H}, U)$ and $H \in \mathcal{H}$ a closed coalition. We first show that for all $T \in \mathcal{T}(Q)$ it holds that $H = \bar{F}^T(i)$ for some $i \in N$. Suppose there exists $T \in \mathcal{T}(Q)$ such that $H \neq \bar{F}^T(j)$ for all $j \in N$. Then there exists $i \in H$ such that $H \subsetneq \bar{F}^T(i)$. This implies that $i \in H \cap U(\bar{F}^T(i))$, whereas $\bar{F}^T(i) \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, which contradicts that H is a closed coaltion. Since the AMV-value is the average of the marginal vectors corresponding to all rooted trees compatible to $\mathcal{Q}$, it suffices to show that $\sum_{j \in H} m_j^T(N, v, \mathcal{Q}) = v(H)$ for all $T \in \mathcal{T}(\mathcal{Q})$. Take any $T \in \mathcal{T}(\mathcal{Q})$. Let $i \in H$ be such that $\bar{F}^T(i) = H$, then it follows that

$$\sum_{j \in H} m_j^T(N, v, \mathcal{Q}) = \sum_{j \in \bar{F}^T(i)} m_j^T(N, v, \mathcal{Q}) = v(\bar{F}^T(i)) = v(H).$$

□

4.5 Stability of the average marginal vector value

In this section we discuss the stability of the AMV-value for several subclasses of quasi-building system games. For each subclass, a convexity type of condition, under which the AMV-value of the game in this subclass lies in the core, is given.

The core of a cooperative game is the set of efficient and stable payoff vectors. On the class of quasi-building system games the core is defined as follows.

Definition 4.5.1 Let $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ be a quasi-building system game, where $\mathcal{Q} = (\mathcal{H}, U)$. The *core* of $(N, v, \mathcal{Q})$ is given by the set

$$C(N, v, \mathcal{Q}) = \{x \in \mathbb{R}^n | \sum_{i=1}^{n} x_i = v(N), \sum_{i \in H} x_i \geq v(H) \text{ for all } H \in \mathcal{H}\}.$$

The core reflects the property that only coalitions that are feasible are able to block a payoff vector, see for example Bilbao et al. (1999).

4.5.1 Union stable quasi-building systems

In this subsection we consider the subclass of union stable quasi-building systems.

Definition 4.5.2 A quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on N is *union stable* if the following condition holds:

(Q3) For any $H_1 \in \mathcal{H}$ and $H_2 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ satisfying $H_1 \cap U(H_2) \neq \emptyset$ it holds that $H_1 \cup H_2 \in \mathcal{H}$.

Condition (Q3) says that the union of two feasible coalitions, of which at least one is non-inessential, is also feasible if their intersection contains an element in the choice set of the non-inessential coalition. Note that this condition is weaker than the union stable condition for set systems.

Union stable quasi-building system games cover games on communication graphs (undirected, directed, or mixed), partition systems or building sets, augmenting systems, antimatroids, and posets.

Definition 4.5.3 Given a union stable quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$, (A, B) is *union-closed* if $A \in \mathcal{H}$, $B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, and $A \cap U(B) \neq \emptyset$.

A feasible coalition A and a non-inessential feasible coalition B of a union stable quasi-building system form a union-closed pair of coalitions if there exists a player in A which is in the choice set of B. Note that the union $A \cup B$ is in $\mathcal{H}$ because of (Q3).

Definition 4.5.4 Let $\mathcal{Q} = (\mathcal{H}, U)$ be a union stable quasi-building system on N. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-supermodular* if for any union-closed pair (A, B), $i \in A \cap U(B)$ and maximal partition $\mathcal{D}$ of $A \cap B \setminus \{i\}$ into elements of $\mathcal{H}$ it holds that

$$f(A) + \sum_{K \in P(B \setminus \{i\})} f(K) \leq f(A \cup B) + \sum_{K \in \mathcal{D}} f(K).$$

Notice that condition (Q2) implies that the set $B \setminus \{i\}$ has a unique maximal partition $P(B \setminus \{i\})$. The maximal partition of $A \cap B \setminus \{i\}$ into feasible coalitions may not be unique, and the condition is required to hold for all such maximal partitions. The next example shows that, for a union-closed pair, a maximal partition of its intersection into feasible coalitions might not be unique.

Example 4.5.5 Consider a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $N = \{1,2,3,4,5\}$, where $\mathcal{H} = \{N, \{1,2,3,4\},\{1,2,3,5\},\{1,2\},\{2,3\},\{1\},\{2\},\{3\},\{4\},\{5\}\}$ and its choice function U is such that $U(N) = \{4,5\}$, $U(\{1,2,3,4\}) = \{1\}$, $U(\{1,2,3,5\}) = \{3\}$, $U(\{1,2\}) = \{1\}$, $U(\{2,3\}) = \{3\}$, and $U(\{i\}) = \{i\}$, $i \in N$. The system $\mathcal{Q}$ is union stable and has two compatible trees $T_1 = \{(4,3),(3,1),(1,2),(3,5)\}$ and $T_2 = \{(5,1),(1,3),(3,2),(1,4)\}$. The pair (A, B) with $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5\}$ is union-closed with $A \cap U(B) = \{4\}$. The intersection $A \cap B \setminus \{4\} = \{1,2,3\}$ has two maximal partitions into feasible coalitions, namely $\{\{1\},\{2,3\}\}$ and $\{\{1,2\},\{3\}\}$. Both $\{1\}$ and $\{3\}$ are inessential coalitions.

In the next theorem it is shown that $\mathcal{Q}$-supermodularity is a sufficient condition for the stability of the AMV-value.

Theorem 4.5.6 *Let $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ be a union stable quasi-building system game. If v is $\mathcal{Q}$-supermodular, then $AMV(N, v, \mathcal{Q}) \in C(N, v, \mathcal{Q})$.*

Proof Since the AMV-value is efficient and the solution is the average of the marginal vectors corresponding to all compatible rooted trees, it suffices to show that for all $T \in \mathcal{T}(Q)$ and $H \in \mathcal{H}$ it holds that

$$\sum_{j \in H} m_j^T(N, v, \mathcal{Q}) \geq v(H).$$

Take any $T \in \mathcal{T}(\mathcal{Q})$ and $H \in \mathcal{H}$. Let $H_1, \ldots, H_s$ be the maximally connected subsets of H in T. For $k = 1, \ldots, s$ denote $H_k = \{i_1^k, \ldots, i_{t_k}^k\}$ and let $h < l$ if $i_h^k \in F^T(i_l^k)$. For $k = 1, \ldots, s$ denote $r_k = i_{t_k}^k$ and let $h < l$ if $r_h \in F^T(r_l)$. Since T is a tree, r_k is the root of the subtree of T on $\bar{F}^T(r_k)$ containing the set H_k, $k = 1, \ldots, s$. Moreover, $\bar{F}^T(r_s)$ contains the set H, otherwise there exists $r \in N \setminus H$ such that H is split into (nonempty subsets of) more than one element of $P(F^T(r))$, which violates (Q2). The set $\bar{F}^T(r_s)$ therefore also contains $\bar{F}^T(r_k)$ for $k = 1, \ldots, s-1$. For $k = 1, \ldots, s$ it holds that

$$H_k = \bar{F}^T(r_k) \setminus \Big(\bigcup_{h=1}^{t_k} (\bigcup_{j \in S^T(i_h^k) \setminus H_k} \bar{F}^T(j)) \Big),$$

which implies that

$$\sum_{j \in H} m_j^T(N, v, \mathcal{Q}) = \sum_{k=1}^{s} \Big(v(\bar{F}^T(r_k)) - \sum_{h=1}^{t_k} \sum_{j \in S^T(i_h^k) \setminus H_k} v(\bar{F}^T(j)) \Big).$$

To show that the latter expression is at least equal to $v(H)$, let $I^k = H \cup (\cup_{h=1}^{k} \bar{F}^T(r_h))$ for $k = 0, \dots, s$ and $I_h^k = I^{k-1} \cup (\cup_{j=1}^{h} \bar{F}^T(i_j^k))$ for $h = 0, \dots, t_k$, $k = 1, \dots, s$. Notice that $I^0 = H$ and $I^s = \bar{F}^T(r_s)$ and that for $k = 1, \dots, s$ it holds that $I_0^k = I^{k-1}$ and $I_{t_k}^k = I^k$. We first show by induction that $I_h^k \in \mathcal{H}$ for all $h = 0, \dots, t_k$, $k = 1, \dots, s$. Since $I_0^1 = I^0 = H$ and $H \in \mathcal{H}$, it holds that $I_0^1 \in \mathcal{H}$. Suppose $I_h^1 \in \mathcal{H}$ for some $h < t_1$. Since $\bar{F}^T(i_h^1) \in \mathcal{H}$ and $i_h^1 \in H \cap U(\bar{F}^T(i_h^1)) \subseteq I_h^1 \cap U(\bar{F}^T(i_h^1))$, it follows from (Q3) that the union I_{h+1}^1 of the sets I_h^1 and $\bar{F}^T(i_{h+1}^1)$ is in $\mathcal{H}$. In particular, this implies for $h = t_1 - 1$ that $I_{t_1}^1$ is in $\mathcal{H}$. Since $I_{t_1}^1 = I^1 = I_0^2$, it also holds that $I_0^2 \in \mathcal{H}$. Continuing the same argument, we obtain by induction that $I_h^k \in \mathcal{H}$ for all k and h.

Let $A = I_{h-1}^k$ and $B = \bar{F}^T(i_h^k)$ for some $h = 1, \dots, t_k$, $k = 1, \dots, s$. Then $A \in \mathcal{H}$, $B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, $i_h^k \in A \cap U(B)$, and $A \cup B = I_h^k \in \mathcal{H}$. Hence, the pair (A, B) is union-closed in $\mathcal{Q}$. Concerning the intersection of A and B without i_h^k, for $j \in S^T(i_h^k) \setminus H_k$ define

$$D_h^k(j) = \{r \mid \bar{F}^T(r) \subset \bar{F}^T(j),\ \nexists\, l < k \text{ with } \bar{F}^T(r) \subset \bar{F}^T(r_l) \subset \bar{F}^T(j)\}.$$

Then $A \cap B \setminus \{i_h^k\}$ is maximally partitioned into elements of $\mathcal{H}$ by the collection

$$\mathcal{D} = \{\bar{F}^T(r) \mid r \in D_h^k(j),\ j \in S^T(i_h^k) \setminus H_k\} \cup \{\bar{F}^T(j) \mid j \in S^T(i_h^k) \cap H_k\}.$$

Since v is $\mathcal{Q}$-supermodular, $A = I_{h-1}^k$, and $A \cup B = I_h^k$, this implies that

$$v(I_{h-1}^k) + \sum_{K \in P(B \setminus \{i_h^k\})} v(K) \leq v(I_h^k) + \sum_{K \in \mathcal{D}} v(K).$$

Since $P(B \setminus \{i_h^k\}) \cap \mathcal{D} = \{\bar{F}^T(j) \mid j \in S^T(i_h^k) \cap H_k\}$, the terms indexed by these sets cancel on both sides and we obtain

$$v(I_{h-1}^k) + \sum_{j \in S^T(i_h^k) \setminus H_k} v(\bar{F}^T(j)) \leq v(I_h^k) + \sum_{j \in S^T(i_h^k) \setminus H_k} \sum_{r \in D_h^k(j)} v(\bar{F}^T(r)).$$

Applying this inequality successively for $h = 1, \dots, t_k$, $k = 1, \dots, s$, we obtain that

$$v(I_0^1) + \sum_{k=1}^{s} \sum_{h=1}^{t_k} \sum_{j \in S^T(i_h^k) \setminus H_k} v(\bar{F}^T(j)) \leq v(I_{t_s}^s) + \sum_{k=1}^{s} \sum_{h=1}^{t_k} \sum_{j \in S^T(i_h^k) \setminus H_k} \sum_{r \in D_h^k(j)} v(\bar{F}^T(r)).$$

Since $I_0^1 = H$, $I_{t_s}^s = \bar{F}^T(r_s)$ and each r_i, $i = 1, \dots, s-1$, belongs to precisely one $D_h^k(j)$ for some $j \in S^T(i_h^k) \setminus H_k$, $h \in \{1, \dots, t_k\}$, $k \in \{2, \dots, s\}$, it follows that

$$\sum_{j \in H} m_j^T(v, \mathcal{Q}) = \sum_{k=1}^{s} \Big(v(\bar{F}^T(r_k)) - \sum_{h=1}^{t_k} \sum_{j \in S^T(i_h^k) \setminus H_k} v(\bar{F}^T(j)) \Big) \geq v(H).$$

□

Note that $\mathcal{Q}$-supermodular is a sufficient condition for the stability of the AMV-value. In the next subsection, Example 4.5.11 shows that the AMV-value may be in the core of a game which is not $\mathcal{Q}$-supermodular.

4.5.2 Intersection-closed quasi-building systems

In this subsection the subclass of intersection-closed quasi-building systems is considered.

Definition 4.5.7 A quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on N is *intersection-closed* if the following conditions hold:

(Q4) If $H_1, H_2 \in \mathcal{H}$, then $H_1 \cap H_2 \in \mathcal{H}$.

(Q5) If $H_1, H_2 \in \mathcal{H}$, $H_1 \subset H_2$, and $i \in U(H_2) \cap H_1$, then $i \in U(H_1)$.

Intersection-closedness condition (Q4) reflects the name of this subclass. It says that the (nonempty) intersection of two feasible coalitions is also feasible. Condition (Q5) states that if a player is in the choice set of a feasible coalition, then he must also be in the choice set of any feasible subcoalition that contains this player. This is in line with the property called independence of irrelevant alternatives (IIA), the α-axiom of Sen, or the heredity axiom, saying that a choice in a set remains a choice in any subset that it contains. This property may not be compatible with union-closed quasi-building systems. For example, the second example of Example 4.2.7, the graphical quasi-building system induced from a directed circle is a union-closed quasi-building system while condition (Q5) is not satisfied for $H_1 = \{1, 2\}$ and $H_2 = N$. Intersection-closed quasi-building system games cover games with convex geometries and cycle-free graphical quasi-building systems.

Lemma 4.5.8 *For a convex geometry $\mathcal{F}$ on N, $\mathcal{Q}(\mathcal{F})$ is an intersection-closed quasi-building system on N.*

Proof Take any $S, T \in \mathcal{F}, T \subset S$, where $P(S \setminus \{i\})$ exists for some $i \in T$. From $P(S \setminus \{i\})$, take the minimum set of feasible coalitions $S_1, \ldots, S_l$ which covers $T \setminus \{i\}$, i.e., $S_k \cap (T \setminus \{i\}) \neq \emptyset$ for any $k = 1, \ldots, l$ and $T \setminus \{i\} \subset \bigcup_{k=1}^{l} S_k$. From intersection-closedness between T and $S_1, \ldots, S_l$, there is a feasible partition $\{T_1, \ldots, T_l\}$ of $T \setminus \{i\}$, where $T_k = T \cap S_k$, $k = 1, \ldots, l$. Further, $\{T_1, \ldots, T_l\}$ is the unique maximal partition of $T \setminus \{i\}$ since $P(S \setminus \{i\})$ is the unique maximal partition of $S \setminus \{i\}$ satisfying (Q2). Therefore $P(T \setminus \{i\})$ exists and satisfies

(Q2), which implies that $i \in U(T)$. This shows that condition (Q5) holds for the choice function of a quasi-building system induced by a convex geometry. Condition (Q4) is implied by condition (C2). □

For the class of intersection-closed quasi-building system games, a convexity condition is defined as follows.

Definition 4.5.9 Let $\mathcal{Q} = (\mathcal{H}, U)$ be an intersection-closed quasi-building system on N. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-convex* if

$$f(T) - \sum_{K \in P(T \setminus \{i\})} f(K) \leq f(S) - \sum_{K \in P(S \setminus \{i\})} f(K)$$

for all $S \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q}), T \in \mathcal{H}, T \subset S$, and $i \in U(S) \cap T$.

A game on an intersection-closed quasi-building system is $\mathcal{Q}$-convex if the marginal loss caused by a player is greater whenever he is removed from a larger feasible and non-inessential coalition. This condition is similar to a convexity condition introduced in Bilbao et al. (1999) on the class of games on convex geometries to guarantee stability of marginal vectors induced from all maximal chains.

Theorem 4.5.10 *Let $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ be an intersection-closed quasi-building system game. If v is $\mathcal{Q}$-convex, then $AMV(N, v, \mathcal{Q}) \in C(N, v, \mathcal{Q})$.*

Proof Since the AMV-value is efficient and the solution is the average of the marginal vectors corresponding to all compatible rooted trees, it suffices to show that for all $T \in \mathcal{T}(Q)$ and $H \in \mathcal{H}$ it holds that

$$\sum_{j \in H} m_j^T(N, v, \mathcal{Q}) \geq v(H).$$

Take any $T \in \mathcal{T}(Q)$ and $H \in \mathcal{H}$. Denote $H = \{i_1, \ldots, i_s\}$ and let $h < l$ if $i_h \in F^T(i_l)$. Since $F^T(i)$ satisfies (Q2) for any $i \in N$, i_s is uniquely determined. This implies that

$$\sum_{j \in H} m_j^T(N, v, \mathcal{Q}) = \sum_{k=1}^{s} \Big(v(\bar{F}^T(i_k)) - \sum_{K \in P(F^T(i_k))} v(K) \Big).$$

Let $Q_k = \bar{F}^T(i_k) \cap H$, $k = 1, \ldots, s$, then $Q_s = H$ and $Q_k \in \mathcal{H}$ for $k = 1, \ldots, s-1$, since $\bar{F}^T(i_k) \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ for $k = 1, \ldots, s-1$ and $H \in \mathcal{H}$. For $k = 1, \ldots, s$, since $i_k \in U(\bar{F}^T(i_k)) \cap Q_k$ and $Q_k \subseteq \bar{F}^T(i_k)$, it follows from (Q5) that $i_k \in U(Q_k)$ and thus $Q_k \setminus \{i_k\}$ has a unique maximal partition $P(Q_k \setminus \{i_k\})$.

Clearly, each member of $P(Q_k \setminus \{i_k\})$, $k \in \{1, \ldots, s\}$, is equal to Q_j for some $j < k$. Since the game is $\mathcal{Q}$-convex, we obtain

$$m^T_{i_k}(N, v, \mathcal{Q}) = v(\bar{F}^T(i_k)) - \sum_{K \in P(F^T(i_k))} v(K) \geq v(Q_k) - \sum_{K \in P(Q_k \setminus \{i_k\})} v(K),$$

for $k = 1, \ldots, s$. Adding up this inequality for $k = 1, \ldots, s$, we get

$$\sum_{j \in H} m^T_j(N, v, \mathcal{Q}) \geq \sum_{k=1}^{s} \Big(v(Q_k) - \sum_{K \in P(Q_k \setminus \{i_k\})} v(K) \Big).$$

Since $Q_s = H$, this inequality becomes

$$\sum_{j \in H} m^T_j(N, v, \mathcal{Q}) \geq v(H) + \sum_{k=1}^{s-1} v(Q_k) - \sum_{k=1}^{s} \sum_{K \in P(Q_k \setminus \{i_k\})} v(K).$$

The last two terms cancel out since $\bigcup_{k=1}^{s} P(Q_k \setminus \{i_k\}) = \{Q_1, \ldots, Q_{s-1}\}$, and the desired result follows. □

The next example is a quasi-building system game with underlying quasi-building system both union stable and intersection-closed. The game is $\mathcal{Q}$-convex, but not $\mathcal{Q}$-supermodular.

Example 4.5.11 Consider a quasi-building system game $(N, v, \mathcal{Q})$ with $\mathcal{Q} = (\mathcal{H}, U)$ on $N = \{1, 2, 3, 4, 5\}$, where $\mathcal{H} = \{N, \{1, 2, 3, 4\}, \{1, 2, 3\}, \{2, 4\}, \{1\}, \{2\}, \{3\}, \{5\}\}$, U is such that $U(N) = \{4\}$, $U(\{1, 2, 3, 4\}) = \{4\}$, $U(\{1, 2, 3\}) = \{2\}$, $U(\{2, 4\}) = \{4\}$, $U(\{1\}) = \{1\}$, $U(\{2\}) = \{2\}$, $U(\{3\}) = \{3\}$, $U(\{5\}) = \{5\}$, and v is such that $v(N) = 5$, $v(\{1, 2, 3, 4\}) = 2$, $v(\{1, 2, 3\}) = 2$, $v(\{2, 4\}) = 3$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{5\}) = 0$. $\mathcal{Q}$ is both union stable and intersection-closed, with $\{1, 2, 3, 4\}$, $\{2, 4\}$ and $\{2\}$ being inessential coalitions. There is one compatible tree, $T = \{(4, 2), (4, 5), (2, 1), (2, 3)\}$, and the corresponding marginal vector $m^T(N, v, \mathcal{Q}) = (0, 2, 0, 3, 0)$ is in the core of the game. This game is not $\mathcal{Q}$-supermodular (take the pair (A, B) with $A = \{2, 4\}$ and $B = \{1, 2, 3\}$), but the game is $\mathcal{Q}$-convex.

With properly defined pairs of sets, strongly union-closed pairs, there is an equivalent expression to $\mathcal{Q}$-convexity. On the class of intersection-closed quasi-building system, a strongly union-closed pair is defined as follows.

Definition 4.5.12 Given an intersection-closed quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$, a pair (A, B) of subsets of N is *strongly union-closed* if $A \in \mathcal{H}$, $B, A \cup B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, and $U(A \cup B) \cap A \cap B \neq \emptyset$.

Note that the definition is different from Definition 4.5.3 of a union-closed pair on the class of union stable quasi-building system. Now the condition requires that $U(A \cup B) \cap A \cap B \neq \emptyset$. It follows from (Q4) that $A \cap B \in \mathcal{H}$. It also follows from (Q5) that for any $i \in U(A \cup B)$, there exist $P(A \setminus \{i\})$, $P(B \setminus \{i\})$, and $P(A \cap B \setminus \{i\})$. It also excludes situations where the union $A \cup B$ is an inessential coalition. Since $i \in U(B)$ does not imply $i \in U(A \cup B)$ nor $A \cup B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, for a quasi-building system which is both union stable and intersection-closed, a strongly union-closed pair is union-closed, but a union-closed pair is not necessarily strongly union-closed.

Theorem 4.5.13 *Let $\mathcal{Q} = (\mathcal{H}, U)$ be an intersection-closed quasi-building system on N. A function $v : \mathcal{H} \to \mathbb{R}$ is $\mathcal{Q}$-convex if and only if*

$$v(A) + \sum_{K \in P(B \setminus \{i\})} v(K) \leq v(A \cup B) + \sum_{K \in P(A \cap B \setminus \{i\})} v(K) \tag{4.1}$$

holds for any strongly union-closed pair (A, B) and $i \in U(A \cup B) \cap A \cap B$.

Proof For the sufficiency, suppose the condition holds for v. Take any $S \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q}), T \in \mathcal{H}$, $T \subset S$, and $i \in U(S) \cap T$. With $A = T$ and $B = S$, the pair (T, S) is strongly union-closed. It then follows from the condition that

$$v(T) + \sum_{K \in P(S \setminus \{i\})} v(K) \leq v(S) + \sum_{K \in P(T \setminus \{i\})} v(K),$$

because of the fact that $S \cup T = S$ and $S \cap T \setminus \{i\} = T \setminus \{i\}$.

For the necessity, suppose v is $\mathcal{Q}$-convex. Take any strongly union-closed pair (A, B). It is to show that (4.1) holds for any $i \in U(A \cup B) \cap A \cap B$. Take $S = A \cup B$ and $T = A$. Since $S \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, $T \in \mathcal{H}$ and $T \subset S$, from $\mathcal{Q}$-convexity it holds that for every $i \in U(A \cup B) \cap A$

$$v(A \cup B) - \sum_{K \in P(A \cup B \setminus \{i\})} v(K) \geq v(A) - \sum_{K \in P(A \setminus \{i\})} v(K).$$

From (4.1), it is to show that

$$\sum_{K \in P(A \cup B \setminus \{i\})} v(K) - \sum_{K \in P(A \setminus \{i\})} v(K) \geq \sum_{K \in P(B \setminus \{i\})} v(K) - \sum_{K \in P(A \cap B \setminus \{i\})} v(K). \tag{4.2}$$

To this end, we use the following lemma.

Lemma 4.5.14 *Given an intersection-closed quasi-building set $\mathcal{Q} = (\mathcal{H}, U)$ with $S, T \in \mathcal{H}$ with $T \subset S$ and $i \in U(S) \cap T$, it holds for any $X \in P(S \setminus \{i\})$ and $Y \in P(T \setminus \{i\})$ with $X \cap Y \neq \emptyset$ that $Y \subset X$.*

The proof of this lemma follows immediately from Condition (Q2), which prohibits a feasible coalition Y from splitting into more than one element in $P(S \setminus \{i\})$. This lemma, together with (Q2), implies that it suffices to show that

$$v(X) - \sum_{K \in P(A \setminus \{i\}), K \subset X} v(K) \geq \sum_{K \in P(B \setminus \{i\}), K \subset X} v(K) - \sum_{K \in P(A \cap B \setminus \{i\}), K \subset X} v(K) \tag{4.3}$$

holds for any $X \in P(A \cup B \setminus \{i\})$, since each element in $P(A \setminus \{i\})$, $P(B \setminus \{i\})$ and $P(A \cap B \setminus \{i\})$ is a subset of one element in the set $P(A \cup B \setminus \{i\})$, and (4.2) is obtained by adding up the inequalities of (4.3) for all $X \in P(A \cup B \setminus \{i\})$.

First, we divide $P(A \cup B \setminus \{i\})$ into three mutually exclusive sets of feasible coalitions.

- $\mathcal{X} = \{K \in P(A \cup B \setminus \{i\}) \mid K \subset A \setminus B\}$,
- $\mathcal{Y} = \{K \in P(A \cup B \setminus \{i\}) \mid K \subset B\}$,
- $\mathcal{Z} = \{K \in P(A \cup B \setminus \{i\}) \mid K \not\subset A \setminus B, K \not\subset B\}$.

Suppose $X \in \mathcal{X}$ and take $K \in P(A \setminus \{i\})$ such that $K \subset X$, which exists due to Lemma 4.5.14. If $K \subsetneq X$, then a feasible coalition X splits into more than one element in $P(A \setminus \{i\})$, which violates (Q2), and therefore $X = K$ and (4.3) holds trivially.

Suppose $X \in \mathcal{Y}$ and take $K \in P(A \setminus \{i\})$ such that $K \subset X$ and $K' \in P(A \cap B \setminus \{i\})$ such that $K' \subset X$. Since $K \subset A \cap B \setminus \{i\}$, $A \cap B \setminus \{i\} \subset A \setminus \{i\}$ and $K \in \mathcal{H}$, it must hold from Lemma 4.5.14 and (Q2) that $K' = K$. For $K'' \in P(B \setminus \{i\})$ such that $K'' \subset X$, it must hold that $K'' = X$ for the similar reason. Therefore (4.3) holds with equality.

Finally, suppose $X \in \mathcal{Z}$. From (Q4) it holds that $X \cap A$, $X \cap B$, and $X \cap A \cap B$ are elements of $\mathcal{H}$. This, together with (Q2) implies that $X \cap A \in P(A \setminus \{i\})$, $X \cap B \in P(B \setminus \{i\})$ and $X \cap A \cap B \in P(A \cap B \setminus \{i\})$, and (4.3) becomes

$$v(X) - v(X \cap A) \geq v(X \cap B) - v(X \cap A \cap B). \tag{4.4}$$

First we show that $U(X) \setminus B \neq \emptyset$. Because $B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $i \in U(B)$, there exists a compatible tree $T \in \mathcal{T}(\mathcal{Q})$, such that for the set of feasible coalitions $\bar{\mathcal{F}}^T := \{\bar{F}^T(j) \mid j \in N\}$ it holds that $B \in \bar{\mathcal{F}}^T$ and $P(B \setminus \{i\}) \subset \bar{\mathcal{F}}^T$. There also exists unique $\bar{X} \subset \bar{\mathcal{F}}^T$ which minimally covers X, i.e.,

$$\bar{X} = \{K \in \bar{\mathcal{F}}^T \mid X \subset K, \nexists K' \in \bar{\mathcal{F}}^T, X \subset K' \subset K\}.$$

It follows from $B \in \bar{\mathcal{F}}^T$, $X \setminus B \neq \emptyset$ and $i \notin X$ that $B \subsetneq \bar{X}$. Since $\bar{X}$ and B are elements of $\bar{\mathcal{F}}^T$, there exists $j \in U(\bar{X}) \setminus B$ with $P(\bar{X} \setminus \{j\}) \subset \bar{\mathcal{F}}^T$. With Condition (Q2) and the fact that $\bar{X}$ minimally covers X in T, it holds that $j \in X$ and $j \in X \setminus B$. From (Q5) it then follows that $j \in U(X)$ since $\bar{X}, X \in \mathcal{H}$, $\bar{X} \supset X$ and $j \in U(\bar{X}) \cap X$, and we conclude that $U(X) \setminus B \neq \emptyset$. Particularly, $j \in X \setminus B$ means $j \in X \cap A$. Condition (Q5) is applied again for X and $X \cap A$, leading that $j \in U(X \cap A)$ and therefore $P(X \cap A \setminus \{j\})$ exists. From $\mathcal{Q}$-convexity it follows that

$$v(X) - \sum_{K \in P(X \setminus \{j\})} v(K) \geq v(X \cap A) - \sum_{K \in P(X \cap A \setminus \{j\})} v(K),$$

since $X \cap A \subset X$, $X \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $j \in U(X) \cap A$. The inequality (4.4) is satisfied if

$$\sum_{K \in P(X \setminus \{j\})} v(K) - \sum_{K \in P(X \cap A \setminus \{j\})} v(K) \geq v(X \cap B) - v(X \cap A \cap B). \tag{4.5}$$

Due to (Q2), among elements in $P(X \setminus \{j\})$, there is unique X' such that $X \cap B \subset X'$. Similarly, the feasible coalition $X' \cap A$ must be an element of $P(X \cap A \setminus \{j\})$. The condition also implies that $P(X \setminus \{j\}) \setminus \{X'\} = P(X \cap A \cap B \setminus \{j\}) \setminus \{X' \cap A\}$, and (4.5) is equivalent to

$$v(X') - v(X' \cap A) \geq v(X \cap B) - v(X \cap A \cap B). \tag{4.6}$$

If $X' \subset B$, then the inequality holds with equality and the proof ends. If not, then $X' \setminus B \neq \emptyset$. Since $X' \in P(X \setminus \{j\})$ where $X \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, it holds that $X' \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and thus exactly the same argument follows to conclude that there is $k \in X' \setminus B$ such that $k \in U(X')$. From $\mathcal{Q}$-convexity it then holds that

$$v(X') - \sum_{K \in P(X' \setminus \{k\})} v(K) \geq v(X' \cap A) - \sum_{K \in P(X' \cap A \setminus \{k\})} v(K),$$

since $X' \cap A \subset X'$, $X' \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and $k \in U(X') \cap A$. Therefore (4.6) holds if

$$\sum_{K \in P(X' \setminus \{k\})} v(K) - \sum_{K \in P(X' \cap A \setminus \{k\})} v(K) \geq v(X \cap B) - v(X \cap A \cap B)$$

holds, and so on. To conclude, given $X_1 \in P(A \cup B \setminus \{i\})$, there is a finite sequence of coalitions $X_1, \ldots, X_k$ in $\mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ and a finite sequence of players $i_1, \ldots, i_k$ with $i_h \in U(X_h) \cap A \setminus B$ such that $X_h \in P(X_{h-1} \setminus \{i_{h-1}\})$ for $h = 1, \ldots, k$ with $X_1 \cap B \in P(X_k \setminus \{i_k\})$. We apply $\mathcal{Q}$-convexity on X_h and $X_h \cap A$ with $i_h \in U(X_h) \cap A \setminus B$ to obtain

$$v(X_h) - v(X_h \cap A) \geq \sum_{K \in P(X_h \setminus \{i_h\})} v(K) - \sum_{K \in P(X_h \cap A \setminus \{i_h\})} v(K)$$

$$= v(X_{h+1}) - v(X_{h+1} \cap A),$$

for $h = 1, \dots, k-1$, and therefore we have

$$v(X_1) - v(X_1 \cap A) \geq \cdots \geq v(X_k) - v(X_k \cap A) \geq v(X_1 \cap B) - v(X_1 \cap A \cap B),$$

which is the desired result. □

4.5.3 Chain quasi-building systems

In this subsection we consider the subclass of chain quasi-building systems. This subclass contains the sets of feasible coalitions induced by a poset on the player set.

Definition 4.5.15 A pair $\mathcal{Q} = (\mathcal{H}, U)$ is a *chain quasi-building system* on N if it satisfies the following conditions:

(Q1) $\mathcal{H} \subseteq 2^N$ is a set system on N containing both $\emptyset$ and N and $U : \mathcal{H} \to 2^N$ is a choice function, that is, $U(\emptyset) = \emptyset$ and for every nonempty $H \in \mathcal{H}$ it holds that $U(H) \neq \emptyset$ and $U(H) \subseteq H$.

(Q2)′ For every $H \in \mathcal{H}$ and $h \in U(H)$, $H \setminus \{h\} \in \mathcal{H}$.

Condition (Q2)′ is a combination of condition (Q2) and the one-point extension property, i.e., if $H \in \mathcal{H} \setminus \{N\}$, then there exists $i \in N \setminus H$ such that $H \cup \{i\} \in \mathcal{H}$ and $i \in U(H \cup \{i\})$. The next lemma immediately follows from condition (Q2)′. Hence a chain quasi-building system is a quasi-building system.

Lemma 4.5.16 *Let $\mathcal{Q} = (\mathcal{H}, U)$ be a chain quasi-building system on N. Then every rooted tree compatible with $\mathcal{Q}$ is a line-tree.*

For the stability of the AMV-value on the class of chain quasi-building system games, we introduce the following condition. Since every compatible tree is a line-tree, the condition involves chains.

Definition 4.5.17 Let $\mathcal{Q} = (\mathcal{H}, U)$ be a chain quasi-building system on N. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-increasing* if

$$\sum_{i=1}^{k} \Big(f(S^i) - f(S^i \setminus H_i) \Big) \geq f(H)$$

holds for any $H \in \mathcal{H}$ and $S^1, \dots, S^k \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ satisfying $S^1 \subset \cdots \subset S^k$, $H \subset S^k$ and $S^i \setminus H_i \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, where $H_i = (S^i \setminus S^{i-1}) \cap H \neq \emptyset$, for all $i = 1, \dots, k$.

This condition states that the worth of a feasible coalition is less than or equal to the sum of its marginal contributions to any increasing sequence of non-inessential feasible coalitions.

Theorem 4.5.18 *Let $(N, v, \mathcal{Q}) \in \mathcal{G}_N^{qbs}$ be a chain quasi-building system game on N. If v is $\mathcal{Q}$-increasing, then $AMV(v, \mathcal{Q}) \in C(N, v, \mathcal{Q})$.*

Proof Let $\mathcal{Q} = (\mathcal{H}, U)$. Since the AMV-value is efficient and the solution is the average of the marginal vectors corresponding to all compatible rooted trees, it suffices to show that for all $T \in \mathcal{T}(Q)$ and $H \in \mathcal{H}$ it holds that

$$\sum_{j \in H} m_j^T(N, v, \mathcal{Q}) \geq v(H).$$

Take any $T \in \mathcal{T}(Q)$ and $H \in \mathcal{H}$. Let $H_1, \ldots, H_k$ be the maximal connected subsets of H in the tree T. By Lemma 4.5.16, T is a line-tree. Then, in $\{\bar{F}^T(h) \mid h \in N\}$, for $i = 1, \ldots, k$, there exists unique minimal $\bar{S}^i = \bar{F}^T(\bar{r}_i)$ such that $H_i \subseteq \bar{S}^i$ and there exists unique maximal $\underline{S}^i = \bar{F}^T(\underline{r}_i)$ such that $H_i \cap \underline{S}^i = \emptyset$. Note that $\bar{r}_i \in H_i$ and $\bar{S}^i \setminus H_i = \underline{S}^i$ for each $i = 1, \ldots, k$. The sets $\bar{S}^i, i = 1, \ldots, k$, can be ordered such that $\bar{S}^1 \subset \cdots \subset \bar{S}^k$. Notice that $H \subset \bar{S}^k$. For $i = 1, \ldots, k$ it holds that $\bar{S}^i \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ since $\bar{S}^i = \bar{F}^T(\bar{r}_i)$ and also that $\bar{S}^i \setminus H_i \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ since $\bar{S}^i \setminus H_i = \underline{S}^i$ and $\underline{S}^i = \bar{F}^T(\underline{r}_i)$. Therefore, $\bar{S}^1, \ldots, \bar{S}^k$ and H satisfy the condition for $\mathcal{Q}$-increasing. Because T is a line-tree and $H_i = F^T(\bar{r}_i) \setminus F^T(\underline{r}_i)$ for $i = 1, \ldots, k$, we have

$$\begin{aligned}\sum_{j \in H} m_j^T(N, v, \mathcal{Q}) = \sum_{i=1}^{k} \sum_{j \in H_i} m_j^T(N, v, \mathcal{Q}) &= \sum_{i=1}^{k} \Big(v(\bar{F}^T(\bar{r}_i)) - v(\bar{F}^T(\underline{r}_i)) \Big) \\ &= \sum_{i=1}^{k} \Big(v(\bar{S}^i) - v(\bar{S}^i \setminus H_i) \Big),\end{aligned}$$

which is greater or equal to $v(H)$ since v is $\mathcal{Q}$-increasing. □

CHAPTER 5

SUPERMODULAR NTU-GAMES

5.1 Introduction

In the previous chapters we study several single-valued solution concepts on classes of TU-games with some cooperation restriction. Those solution concepts, the Myerson value, the average tree solution, and the AMV-value, are the averages of appropriately defined marginal vectors of a game. In Chapter 3 and 4 some stability conditions of the solution concepts are derived. The approach we take is to show that under those conditions every admissible marginal vector is in the core and therefore the average as well, since the core of TU-game is a convex set. It is well-known that all the marginal vectors of a standard TU-game are elements of the core of the game if and only if the game is convex, which means that the characteristic function underlying the game is supermodular, see Ichiishi (1981).

In this chapter we introduce a multi-valued solution concept and study its core stability for the class of cooperative games with non-transferable utility. A cooperative game with non-transferable utility (NTU-game) consists of a finite number of players and a mapping that assigns to each coalition a feasible set of payoff vectors. The core of an NTU-game (Aumann (1961)) consists of all payoff vectors that are feasible for the grand coalition of all players and cannot be blocked by any coalition of players. Two type of convexity conditions on NTU-games, individual merge convexity (Hendrickx et al. (2002)) and strong ordinal convexity (Masuzawa (2012)), concern the core stability of marginal vectors of an NTU-game. Both conditions are sufficient to guarantee the core stability of all marginal vectors. Other types of convexity conditions, such as ordinal convexity (Vilkov (1977)) and cardinal convexity (Sharkey (1981)), are studied in relation with the von-Neumann-Morgenstern solution of NTU-games. For an overview of these convexity conditions, one may refer Csóka et al. (2011).

On the class of NTU-games, we consider a solution concept that utilizes the average of marginal vectors. We start with introducing the concept of supermodularity for NTU-games. For TU-games this concept is equivalent to convexity. If an NTU-game is supermodular all the appropriately defined marginal vectors are elements of the core of the game. The class of supermodular NTU-games contains the classes of individual merge convex NTU-games and strong ordinal convex NTU-games.

As solution concept for NTU-games we propose a set of solutions that is determined by the average of all marginal vectors of the game. The solution set is never empty. In case the NTU-game is induced by a TU-game, the solution set coincides with this average and is the Shapley value of the TU-game. In general, if the average of all marginal vectors is an efficient allocation for the grand coalition, as is always the case if the set of efficient payoff vectors for the grand coalition is a hyperplane, the solution set is a singleton, being this average. If the average of all marginal vectors is a feasible but not efficient allocation for the grand coalition, then the payoffs at the average are increased in any strictly positive direction until an efficient allocation for the grand coalition is obtained. And, if the average is not a feasible allocation for the grand coalition, then the payoffs are decreased in any strictly negative direction until an efficient allocation for the grand coalition is reached.

For a TU-game supermodularity of the characteristic function does not only guarantee that all marginal vectors of the game are elements of the core, but also that their average, the Shapley value, is in the core. As mentioned, this is because the core of a TU-game is a convex set. However, even for supermodular NTU-games the core is typically not a convex set and therefore the average of the marginal vectors may not be an efficient allocation or not be a feasible allocation for the grand coalition, and, moreover, the average may be blocked by a proper subcoalition, although the marginal vectors are not. To guarantee that the solution set is a subset of the core, we introduce a convexity condition that roughly says that the payoff set of the grand coalition is not less convex-shaped or its complement is not more convex-shaped than for the payoff sets of any subcoalition holds.

Being determined by the average of all marginal vectors of the game, the solution takes a similar approach as the marginal based compromise value, or MC-value (Otten et al. (1998)). It turns out that the MC-value lies in the solution set if all payoff sets satisfy non-levelness. There are several solution concepts on the class of NTU-games that generalize the Nash bargaining solution of a pure bargaining problem (Nash (1950)), such as the Harsanyi value (Harsanyi (1963)), the Shapley NTU-value (Shapley (1969)), and the consistent Shapley value (Maschler and Owen (1989), Maschler and Owen (1992)). From a viewpoint of bargaining problems, the solution set we propose can be seen as the set of bargaining solutions of an induced bargaining problem in which the average of the marginal vectors is the disagreement point. In order for the allocation at the average of the marginal vectors to be the disagreement point, the allocation should be feasible for the grand coalition and not be blocked by any proper subcoalition. On the other hand, if the average of the marginal vectors is not feasible, then one may see the average as a utopia point. We also discuss a specific solution in the solution set, an egalitarian type of solution.

This chapter is organized as follows. In Section 2 the concept of supermodularity is introduced for NTU-games. In Section 3 the solution set is introduced and core stability is studied. Also a comparison with other solutions is made in that section.

This chapter is based on Koshevoy et al. (2014).

5.2 Supermodularity

We consider cooperative games without side-payment. A non-transferable utility game (or NTU-game) is a pair (N, V) which consists of a finite set $N = \{1, \ldots, n\}$ of $n \geq 2$ players and a mapping $V(\cdot)$ assigning to every subset S of N a subset $V(S)$ of $\mathbb{R}^S$ with $V(\emptyset) = \{0\}$. An element $x = (x_i)_{i \in S}$ in $V(S)$ is an allocation for coalition S that can be realized by the players within S and at which player $i \in S$ receives payoff x_i. Let $\mathcal{G}_N^{ntu}$ denote the class of NTU-games with fixed player set N. For a vector $x \in \mathbb{R}^T$ and $S \subseteq T$, $T \in 2^N$, x_S denotes the vector $(x_i)_{i \in S}$ in $\mathbb{R}^S$, with $x_S = 0$ if $S = \emptyset$. We often write $(x_i, x_{S \setminus \{i\}})$ for $x \in \mathbb{R}^S$ and $i \in S$. For $S \in 2^N$, we denote $\mathbb{R}_+^S = \{x \in \mathbb{R}^S \mid x_i \geq 0 \text{ for all } i \in S\}$ and $\mathbb{R}_{++}^S = \{x \in \mathbb{R}^S \mid x_i > 0 \text{ for all } i \in S\}$.

For an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ we make the standard assumptions on V that for all $S \in 2^N$, $S \neq \emptyset$, the set $V(S)$ is closed and comprehensive in $\mathbb{R}^S$ and that the game is zero-normalized, i.e., $V(\{i\}) = (-\infty, 0]$ for all $i \in N$. We also assume that for all $S \in 2^N$, $S \neq \emptyset$, and $b \in \mathbb{R}^S$ it holds that the set

$\{x \in V(S) \mid x_i \geq b_i$ for all $i \in S\}$ is either empty or, if not empty, a bounded set. Moreover, we assume that V is monotone, i.e., for any $S \subsetneq T$, $T \in 2^N$, and $x \in V(S)$ there exists $y \in V(T)$ such that $y_S \geq x$.

Given an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ and coalition $S \in 2^N$, let $D^V(S) = \{x \in \mathbb{R}^S \mid \nexists y \in V(S),\ y \gg x\}$, $E^V(S) = V(S) \cap D^V(S)$, and $I^V(S) = V(S) \setminus E^V(S)$. Then a payoff vector $x \in \mathbb{R}^T$, with $S \subseteq T$, is blocked by coalition S if $x_S \in I^V(S)$, i.e., there exists $y \in V(S)$ such that $y \gg x_S$, x is not blocked by coalition S if $x_S \in D^V(S)$, i.e., there exists no $y \in V(S)$ such that $y \gg x_S$, and x is weakly Pareto-optimal, or efficient, for coalition S if $x_S \in E^V(S)$, i.e., $x_S \in V(S)$ and there is no $y \in V(S)$ such that $y \gg x_S$. Notice that $D^V(\emptyset) = E^V(\emptyset) = \{0\}$ and $I^V(\emptyset) = \emptyset$.

The core of an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$, denoted $C(N, V)$, is the set of weakly Pareto-optimal allocations for the grand coalition N of all players, that are not blocked by any coalition (Aumann (1961)), i.e.,

$$C(N, V) = \{x \in V(N) \mid x_S \notin I^V(S),\ \forall\, S \in 2^N\}.$$

Let $\Pi(N)$ be the set of permutations on N. Given an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ and a permutation $\sigma \in \Pi(N)$, the marginal vector $m^\sigma(N, V) \in \mathbb{R}^N$ is defined by

$$m^\sigma_{\sigma(k)}(N, V) = \max\{y_{\sigma(k)} \mid y \in V(\{\sigma(1), \ldots, \sigma(k)\}),\ y_{\sigma(i)} = m^\sigma_{\sigma(i)}(N, V),\ \forall\, i < k\},$$

for $k = 1, 2, \ldots, n$. Notice that $m^\sigma(N, V)$ always exists and is uniquely defined.

Next we introduce a condition that guarantees that all marginal vectors of an NTU-game are elements of the core.

Definition 5.2.1 An NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ is *supermodular* if for any $A \in 2^N$, $j \in A$, and $x \in E^V(A)$ satisfying $x_{A\setminus\{j\}} \in E^V(A \setminus \{j\})$ and $x_j = \max\{y \mid (y, x_{A\setminus\{j\}}) \in V(A)\}$ it holds that for all $B \subset A$ such that $j \in B$

$$x_{B\setminus\{j\}} \in D^V(B \setminus \{j\}) \Rightarrow x_B \in D^V(B).$$

Supermodularity of an NTU-game means that if for a player in a coalition it holds that a payoff vector is efficient with and without him and a subcoalition without him cannot block this payoff vector, then this payoff vector can also not be blocked by this subcoalition together with him. Notice that if $B = \{j\}$ for some $j \in N$ then $D^V(B \setminus \{j\}) = \{0\}$ and $x_B \in D^V(B)$ means $x_j \geq 0$. In the following theorem we show that all marginal vectors of a supermodular NTU-game are in the core of the game.

Theorem 5.2.2 *Let $(N,V) \in \mathcal{G}_N^{ntu}$ be a supermodular NTU-game, then $m^\sigma(N,V) \in C(N,V)$ for all $\sigma \in \Pi(N)$.*

Proof We prove the result by induction. The theorem clearly holds for $n = 2$. Suppose for $n = k$, $k \geq 2$, the theorem is true. For $n = k+1$ we may assume without loss of generality that $\sigma(k+1) = k+1$. From the construction of $m^\sigma(N,V)$ and the induction argument it follows that $m^\sigma(N,V)$ cannot be blocked by any coalition $S \subseteq \{1,\ldots,k\}$. Take any coalition $S \cup \{k+1\}$ where $S \subseteq \{1,\ldots,k\}$. Then for $A = \{1,\ldots,k+1\}$, $B = S \cup \{k+1\}$ and $j = k+1$, we have that $m^\sigma(N,V) \in E^V(A)$, $m_j^\sigma(N,V) = \max\{y \mid (y, m^\sigma_{A\setminus\{j\}}(N,V)) \in V(A)\}$ and $m^\sigma_{A\setminus\{j\}}(N,V) \in E^V(A \setminus \{j\})$ since $m^\sigma(N,V)$ is a marginal vector and $A \setminus \{j\} = \{1,\ldots,k\}$. If $S = \emptyset$, then supermodularity of V implies that $m_j^\sigma(N,V) \geq 0$ and therefore singleton coalition $\{k+1\}$ can not block $m^\sigma(N,V)$. Suppose $S \neq \emptyset$. Then it follows from the induction argument that $m^\sigma_{B\setminus\{j\}}(N,V) \in D^V(B \setminus \{j\})$. Since (N,V) is supermodular and $S \neq \emptyset$, this implies that $m_B^\sigma(N,V) \in D^V(B)$, i.e., $m^\sigma(N,V)$ is not blocked by coalition $S \cup \{k+1\}$. □

Next, we show that for a TU-game convexity is equivalent to supermodularity of the corresponding NTU-game. A (zero-normalized) TU-game (N,v) induces an NTU-game (N,V), where $V(S) = \{x \in \mathbb{R}^S \mid \sum_{i\in S} x_i \leq v(S)\}$ for all $S \in 2^N$.

Proposition 5.2.3 *A TU-game (N,v) is convex if and only if the induced NTU-game (N,V) is supermodular.*

Proof Suppose that the NTU-game (N,V) induced from a TU-game (N,v) is supermodular. Given $T \in 2^N$, $S \subset T$, and $i \in S$, take any $x \in \mathbb{R}^T$ such that $\sum_{h\in S\setminus\{i\}} x_h = v(S \setminus \{i\})$, $\sum_{h\in T\setminus\{i\}} x_h = v(T \setminus \{i\})$, and $\sum_{h\in T} x_h = v(T)$. It holds for V that $x \in E^V(T)$, $x_{T\setminus\{i\}} \in E^V(T \setminus \{i\})$, $x_i = v(T) - v(T \setminus \{i\}) = \max\{y \mid (y, x_{T\setminus\{i\}}) \in V(T)\}$, and $x_{S\setminus\{i\}} \in E^V(S \setminus \{i\}) \subset D^V(S \setminus \{i\})$ if $S \setminus \{i\} \neq \emptyset$. Since (N,V) is supermodular, it follows that $x_i \geq 0 = v(\{i\})$. Thus if $S \setminus \{i\} = \emptyset$, then $v(T) - v(T \setminus \{i\}) \geq v(\{i\})$ is satisfied. If $S \setminus \{i\} \neq \emptyset$, then it follows from supermodularity of (N,V) that $x_S \in D^V(S)$, i.e., $\sum_{h\in S} x_h \geq v(S)$, and therefore

$$v(T) - v(T \setminus \{i\}) = x_i = \sum_{h\in S} x_h - \sum_{h\in S\setminus\{i\}} x_h \geq v(S) - v(S \setminus \{i\}).$$

Next, suppose that a TU-game (N,v) is convex and let $(N,V) \in \mathcal{G}_N^{ntu}$ be the NTU-game induced by (N,v). Given $A \in 2^N$, $B \subset A$, and $j \in B$, take any

$x \in \mathbb{R}^N$ such that $x_A \in E^V(A)$, $x_{A\setminus\{j\}} \in E^V(A \setminus \{j\})$, and $x_{B\setminus\{j\}} \in D^V(B \setminus \{j\})$. Then $\sum_{h\in A} x_h = v(A)$ and $\sum_{h\in A\setminus\{j\}} x_h = v(A \setminus \{j\})$, which implies $x_j = v(A) - v(A \setminus \{j\}) = \max\{y \mid (y, x_{A\setminus\{j\}}) \in V(A)\}$, and $\sum_{h\in B\setminus\{j\}} x_h \geq v(B \setminus \{j\})$. Since v is convex and zero-normalized, it follows that

$$\sum_{h\in B} x_h = \sum_{h\in B\setminus\{j\}} x_h + x_j \geq v(B \setminus \{j\}) + v(A) - v(A \setminus \{j\}) \geq v(B),$$

which implies $x_B \in D^V(B)$. □

In the literature several convexity conditions of NTU-games are introduced under which every marginal vector of an NTU-game is in the core of the game and which are equivalent to convexity of TU-games. One of such conditions is introduced by Milgrom and Shannon (1996) as strong ordinal convexity.

Definition 5.2.4 An NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ is *strong ordinal convex* if for any $S, T \in 2^N$ and $x \in \mathbb{R}^N$ it holds that

$$x_S \in V(S),\ x_{S\cap T} \in D^V(S \cap T) \text{ and } x_T \in V(T) \Rightarrow x_{S\cup T} \in V(S \cup T).$$

Milgrom and Shannon (1996) shows that every marginal vector of a strong ordinal convex NTU-game is a core element and that the NTU-game induced from a TU-game is strong ordinal convex if the TU-game itself is convex, while Masuzawa (2012) shows that if the NTU-game induced from a TU-game is strong ordinal convex then the TU-game is convex. Therefore on the class of TU-games supermodularity and strong ordinal convexity are equivalent. The next proposition shows that strong ordinal convexity implies supermodularity. An NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ is superadditive if for any $S, T \in 2^N$, $S \cap T = \emptyset$, and $x \in \mathbb{R}^{S\cup T}$, it holds that $x_S \in V(S)$ and $x_T \in V(T)$ imply $x \in V(S \cup T)$. Note that a strong ordinal convex game is superadditive.

Proposition 5.2.5 *Let $(N, V) \in \mathcal{G}_N^{ntu}$ be a strong ordinal convex NTU-game. Then (N, V) is supermodular.*

Proof Suppose (N, V) is not supermodular. For any $A \in 2^N$, $j \in A$, and $x \in E^V(A)$ satisfying $x_{A\setminus\{j\}} \in E^V(A \setminus \{j\})$. and $x_j = \max\{y \mid (y, x_{A\setminus\{j\}}) \in V(A)\}$ it follows from superadditivity of (N, V) and since $0 \in V(\{j\})$ that $x_j \geq 0$. Then there exists $A \in 2^N, B \subset A$, $j \in B$ with $B \setminus \{j\} \neq \emptyset$, and $x \in \mathbb{R}^N$ such that $x_A \in E^V(A)$, $x_{A\setminus\{j\}} \in E^V(A \setminus \{j\})$, $x_j = \max\{y \mid (y, x_{A\setminus\{j\}}) \in V(A)\}$, $x_{B\setminus\{j\}} \in D^V(B \setminus \{j\})$, and $x_B \in I^V(B)$. Let $S = A \setminus \{j\}$ and $T = B$. Since $x_T \in I^V(T)$, there exists $x' \in \mathbb{R}^N$ such that $x'_S = x_S$, $x'_j > x_j$, and $x'_T \in V(T)$.

Then $x'_j > x_j$, $x'_S = x_S$, and $x_j = \max\{y \mid (y, x_S) \in V(S \cup \{j\})\}$ imply $x'_{S\cup\{j\}} \notin V(S \cup \{j\})$. Therefore, $x'_S \in V(S)$, $x'_{S\cap T} \in D^V(S \cap T)$, and $x'_T \in V(T)$, whereas $x'_{S\cup T} \notin V(S \cup T)$, which contradicts that (N, V) is strong ordinal convex. □

The following example shows that supermodularity is weaker than strong ordinal convexity.

Example 5.2.6 Consider the 4-person NTU-game (N, V) with

$$V(\{i\}) = (-\infty, 0] \text{ for } i = 1, 2, 3, 4,$$
$$V(S) = \{x \in \mathbb{R}^S \mid \sum_{i\in S} x_i \leq |S|^2\} \text{ if } |S| \geq 2 \text{ and } S \neq \{2, 3, 4\},$$
$$V(\{2, 3, 4\})$$
$$= \{x \in \mathbb{R}^{\{2,3,4\}} \mid x_2 + x_3 + x_4 \leq 9\} \cup \{x \in \mathbb{R}^{\{2,3,4\}} \mid x_2, x_3 \leq 0, x_4 \leq 17\}.$$

To show that (N, V) is not strong ordinal convex, consider the payoff vector $x = (0, 0, 0, 17)$ and take $S = \{1, 2\}$ and $T = \{2, 3, 4\}$. It holds that $x_S = (0, 0) \in V(S), x_{S\cap T} = 0 \in D^V(S \cap T)$ and $x_T = (0, 0, 17) \in V(T)$, while $x \notin V(\{1, 2, 3, 4\})$. Clearly, (N, V) is supermodular.

Hendrickx et al. (2002) introduces another notion of convexity, called individual merge convexity, and shows that for an NTU-game satisfying this condition all marginal vectors are core elements.

Definition 5.2.7 An NTU-game $(N, V) \in \mathcal{G}^{ntu}_N$ is *individual merge convex* if it is superadditive and for any $i \in N$, $T \subseteq N \setminus \{i\}$, and nonempty $S \subset T$ it holds that for any $p \in E^V(S)$, $q \in V(T)$, and $r \in V(S \cup \{i\})$ such that $r_S \geq p$ there exists $s \in V(T \cup \{i\})$ satisfying $s_T \geq q$ and $s_i \geq r_i$.

Hendrickx et al. (2002) proves that for a TU-game individual merge convexity of the induced NTU-game is equivalent to convexity. The next proposition shows that individual merge convexity implies supermodularity.

Proposition 5.2.8 *Let $(N, V) \in \mathcal{G}^{ntu}_N$ be an individual merge convex NTU-game. Then (N, V) is supermodular.*

Proof Suppose (N, V) is not supermodular. For any $A \in 2^N$, $j \in A$, and $x \in E^V(A)$ satisfying $x_{A\setminus\{j\}} \in E^V(A \setminus \{j\})$ and $x_j = \max\{y \mid (y, x_{A\setminus\{j\}}) \in V(A)\}$ it follows from superadditivity of (N, V) and $0 \in V(\{j\})$ that $x_j \geq 0$. Then there exist $A \in 2^N$, $B \subset A$, $j \in B$ with $B \setminus \{j\} \neq \emptyset$, and $x \in \mathbb{R}^N$ such that $x_A \in E^V(A)$, $x_{A\setminus\{j\}} \in E^V(A \setminus \{j\})$, $x_j = \max\{y \mid (y, x_{A\setminus\{j\}}) \in V(A)\}$,

$x_{B\setminus\{j\}} \in D^V(B \setminus \{j\})$, and $x_B \in I^V(B)$. Let $S = B \setminus \{j\}$, $T = A \setminus \{j\}$, and $i = j$. Since $x_{S\cup\{i\}} \in I^V(S \cup \{i\})$, there exists $x' \in \mathbb{R}^N$ such that $x'_T = x_T$, $x'_i > x_i$, and $x'_{S\cup\{i\}} \in V(S \cup \{i\})$. Then $x'_i > x_i$, $x'_T = x_T$, and $x_i = \max\{y \mid (y, x_T) \in V(T \cup \{i\})\}$ imply $x'_{T\cup\{i\}} \notin V(T \cup \{i\})$. Since $x'_S = x_S$ and $x_S \in D^V(S)$, there exists $z \leq x'_S$ such that $z \in E^V(S)$. Take $p = z$, $q = x_T$, and $r = x'_{S\cup\{i\}}$, then $p \in E^V(S)$, $q \in V(T)$, and $r \in V(S \cup \{i\})$. Moreover, $r_S \geq p$, but there exists no $s \in V(T \cup \{i\})$ such that $s_T \geq q$ and $s_i \geq r_i$, since $x'_{T\cup\{i\}} \notin V(T \cup \{i\})$, which contradicts that (N, V) is individual merge convex. □

The next example shows that supermodularity is weaker than individual merge convexity.

Example 5.2.9 (Example 4.5 in Hendrickx et al. (2000)) Consider the 3-person NTU-game (N, V) with

$$
\begin{aligned}
&V(\{i\}) = (-\infty, 0] \text{ for } i = 1, 2, 3,\\
&V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \mid x_1 + x_2 \leq 3\},\\
&V(\{1,3\}) = \{x \in \mathbb{R}^{\{1,3\}} \mid x_1 + x_3 \leq 2\},\\
&V(\{2,3\}) = \{x \in \mathbb{R}^{\{2,3\}} \mid x_2 + x_3 \leq 6\},\\
&V(N) = \{x \in \mathbb{R}^N \mid \frac{x_1}{6} + \frac{x_2}{10} + \frac{x_3}{14} \leq 1\}.
\end{aligned}
$$

To show that (N, V) is not individual-merge convex, take $p = 0 \in E^V(\{2\})$, $q = (6, 0) \in V(\{2,3\})$, and $r = (3, 0) \in V(\{1,2\})$, then $r_2 \geq 0$, but there exists no $s \in V(\{1,2,3\})$ such that $s_{\{2,3\}} \geq (6, 0)$ and $s_1 \geq 3$. However, (N, V) is supermodular. Take for example $A = \{1,2,3\}$ and $j = 3$. Then $x = (x_1, 3 - x_1, \frac{49}{5} - \frac{14}{15}x_1)$ satisfies $x \in E^V(A)$, $x_{A\setminus\{j\}} \in E^V(A \setminus \{j\})$, and $x_j = \max\{y \mid (y, x_{A\setminus\{j\}}) \in V(A)\}$. When $B = \{1,3\}$ and $x_{B\setminus\{j\}} \in D^V(B \setminus \{j\})$, i.e., $x_1 \geq 0$, then $x_1 + x_3 = \frac{49}{5} + \frac{1}{15}x_1 > 2$ and therefore $x_B \in D^V(B)$. When $B = \{2,3\}$ and $x_{B\setminus\{j\}} \in D^V(B \setminus \{j\})$, i.e., $x_2 \geq 0$ and therefore $x_1 \leq 3$, then $x_2 + x_3 = 3 - x_1 + \frac{49}{5} - \frac{14}{15}x_1 = \frac{64}{5} - \frac{29}{15}x_1 > 6$ and therefore $x_B \in D^V(B)$, and so on.

5.3 Solution concept

In this section we propose a solution concept for NTU-games and discuss its core stability. For an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$, let $a(N, V)$ be the average of the marginal vectors over all permutations, i.e.,

$$a(N, V) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N, V).$$

If (N, V) is induced by a TU-game (N, v), then $a(N, V)$ is the Shapley value of (N, v). In general, the vector $a(N, V)$ may not be an allocation for the grand coalition and if it is an allocation for the grand coalition, it may not be efficient. For this reason we propose as solution concept the following set.

Definition 5.3.1 The *solution set* of an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ is the closure, $S(N, V)$, of the set

$$S^o(N, V) = \{x \in E^V(N) \mid x = a(N, V) + \lambda d \text{ for some } \lambda \in \mathbb{R} \text{ and } d \in \mathbb{R}^n_{++}\}.$$

If $(N, V) \in \mathcal{G}_N^{ntu}$ satisfies the non-levelness condition,[1] then $S(N, V)$ can also be defined by taking in the definition of $S^o(N, V)$ any $d \in \mathbb{R}^n_+$ instead of taking the closure of $S^o(N, V)$. In case the NTU-game (N, V) is induced by a TU-game (N, v), then the solution set $S(N, V)$ of (N, V) is a singleton, the point $a(N, V)$, being the Shapley value of (N, v). For an arbitrary NTU-game (N, V), the solution set $S(N, V)$ is a nonempty set and contains all payoff vectors that are efficient allocations for the grand coalition and are obtained by if necessary either increasing or decreasing the payoffs of the average $a(N, V)$ of all marginal vectors. More precisely, if $a(N, V)$ is an efficient allocation for the grand coalition, as is always the case for an NTU-game induced by a TU-game or more general for an NTU-game (N, V) for which the set $E^V(N)$ of efficient allocations for the grand coalition is a hyperplane, the solution set $S(N, V)$ consists of only the singleton $a(N, V)$.

If the average of all marginal vectors $a(N, V)$ of an NTU-game (N, V) is an inefficient allocation for the grand coalition, e.g., when $V(N)$ is a strictly convex set, then the solution set $S(N, V)$ of (N, V) is obtained by increasing the payoffs of $a(N, V)$ in any strictly positive direction until the payoffs become efficient for the grand coalition. When in this case $a(N, V)$ is not blocked by any proper subcoalition, $S(N, V)$ is precisely equal to the set of bargaining solutions of the bargaining problem $B(V(N), a(N, V))$ with disagreement point $a(N, V)$ and bargaining set $V(N)$. If the average $a(N, V)$ of marginal vectors is not a feasible allocation for the grand coalition, e.g., when $D^V(N)$ is a strictly convex set, then the solution set $S(N, V)$ is obtained by decreasing the payoffs of $a(N, V)$ in any strictly negative direction until the payoffs become efficient for the grand coalition. For this case one may see $a(N, V)$ as a utopia point. In general, the solution set $S(N, V)$ of an NTU-game (N, V) consists of all efficient allocations for the grand coalition that are "close" to $a(N, V)$.

[1] An NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ satisfies the non-levelness condition if for all $S \in 2^N$, $S \neq \emptyset$, $x, y \in E^V(S)$ and $y \geq x$ imply $y = x$.

In Otten et al. (1998) the marginal based compromise value is introduced. The marginal based compromise value, or MC-value, of an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ is given by

$$MC(N, V) = a(N, V) \max\{\lambda \in \mathbb{R} \mid \lambda a(N, V) \in V(N)\}.$$

In case the non-levelness condition is satisfied or if $a(N, V)$ is a strictly positive payoff vector, then the MC-value of an NTU-game (N, V) belongs to the solution set $S(N, V)$.

Example 5.3.2 Consider the 2-person NTU-game (N, V) with $V(\{i\}) = (-\infty, 0]$, $i = 1, 2$, and $V(N) = \{x \in \mathbb{R}^2 \mid x_2 \leq \epsilon - \epsilon x_1, x_1 \leq 1\} \cup \{x \in \mathbb{R}^2 \mid x_1 \leq 1 - \epsilon x_2, x_1 \geq 1\}$ for some $\epsilon \geq 0$. For $\epsilon > 0$, (N, V) satisfies the non-levelness condition and $S(N, V)$ consists of the average $a(N, V) = (\frac{1}{2}, \frac{1}{2}\epsilon)$ of the two marginal vectors $m^1(N, V) = (1, 0)$ and $m^2(N, V) = (0, \epsilon)$. For $\epsilon = 0$, (N, V) does not satisfy the non-levelness condition and $S(N, V)$ is still a singleton, being the average $a(N, V) = (\frac{1}{2}, 0)$ of the two marginal vectors $m^1(N, V) = (1, 0)$ and $m^2(N, V) = (0, 0)$. For $\epsilon > 0$ it holds that $MC(N, V) = (\frac{1}{2}, \frac{1}{2}\epsilon) = a(N, V)$, while for $\epsilon = 0$, $MC(N, V) = (1, 0)$, being the marginal vector $m^1(N, V)$. For any $\epsilon \geq 0$, both vectors $a(N, V)$ and $MC(N, V)$ are elements of the core $C(N, V)$ of (N, V), also for $\epsilon = 0$. Notice that the MC-value is not continuous in the parameter ϵ when ϵ converges to zero.

Example 5.3.3 Consider the 2-person NTU-game (N, V) with $V(\{i\}) = (-\infty, 0]$, $i = 1, 2$, and $V(N) = \{x \in \mathbb{R}^2 \mid x_1 \leq 1, x_2 \leq \epsilon\}$ for some $\epsilon \geq 0$. (N, V) does not satisfy the non-levelness condition for any $\epsilon \geq 0$. For $\epsilon > 0$, $S(N, V) = \{x \in \mathbb{R}^2 \mid \max\{\epsilon x_1, x_2\} = \epsilon, x_1 \geq \frac{1}{2}, x_2 \geq \frac{1}{2}\epsilon\}$. This set contains the MC-value of (N, V), $MC(N, V) = (1, \epsilon)$. As in the previous example, for $\epsilon = 0$, $S(N, V)$ consists of the singleton payoff vector $a(N, V) = (\frac{1}{2}, 0)$, being the average of the two marginal vectors $m^1(N, V) = (1, 0)$ and $m^2(N, V) = (0, 0)$, and $MC(N, V) = (1, 0)$, being the marginal vector $m^1(N, V)$. For any $\epsilon \geq 0$, every payoff vector in $S(N, V)$ and also $MC(N, V)$ are elements of the core of (N, V), also for $\epsilon = 0$. Notice that in this game the MC-value is continuous in the parameter ϵ.

Above we showed that under supermodularity it holds that all marginal vectors of an NTU-game belong to the core and therefore are feasible allocations for the grand coalition and cannot be blocked by any coalition. However, because for an NTU-game the core may not be a convex set, it is not guaranteed that under supermodularity the average of all marginal vectors is also feasible for the grand coalition or cannot be blocked by any coalition.

For an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$, let

$$V^*(N) := \{x \in V(N) \mid x_S \notin I^V(S),\ \forall\, S \in 2^N \setminus \{N\}\}$$

be the set of allocations for the grand coalition that cannot be blocked by any proper subcoalition. If (N, V) is supermodular, then every marginal vector $m^\sigma(N, V)$ belongs to $V^*(N)$.

Assumption 5.3.4 For an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$, it holds that for every $x, y \in V^*(N)$, $\alpha \in [0, 1]$, and $d \in \mathbb{R}^N_{++}$ there exists $\lambda \in \mathbb{R}$ such that $\alpha x + (1 - \alpha)y + \lambda d \in V^*(N)$.

Notice that this assumption is automatically satisfied if the NTU-game (N, V) is induced by a TU-game (N, v) because $V^*(N)$ is in that case a convex set. Assumption 5.3.4 means that if a convex combination of two allocations for the grand coalition that cannot be blocked by any proper subcoalition, is also an allocation for the grand coalition and is blocked by some proper subcoalition, then the payoffs can be increased in any strictly positive direction to obtain an allocation for the grand coalition that cannot be blocked by any proper subcoalition. And, if the convex combination is not an allocation for the grand coalition, then the payoffs can be decreased in any strictly negative direction to obtain an allocation for the grand coalition that cannot be blocked by any proper subcoalition.

Roughly the assumption requires that the payoff set $V(N)$ for the grand coalition is "not less" convex, or its complement $\mathbb{R}^N \setminus V(N)$ is "not more" convex, than the payoff set $V(S)$ (or its complement) of any proper subcoalition S.

Lemma 5.3.5 *Suppose that for an NTU-game $(N, V) \in \mathcal{G}_N^{ntu}$ it holds that $V(N)$ is a convex set and that for every $S \in 2^N \setminus \{N\}$ the complement of $V(S)$, the set $\mathbb{R}^S \setminus V(S)$, is a convex set. Then (N, V) satisfies Assumption 5.3.4.*

Proof Take any x and y in $V^*(N)$ and α, $0 < \alpha < 1$. Since x and y are in $V(N)$ and $V(N)$ is convex, it holds that $z = \alpha x + (1 - \alpha)y$ is also in $V(N)$. Take any $S \in 2^N \setminus \{N\}$. Both x_S and y_S belong to $D^V(S)$. Since $D^V(S)$ is the closure of $\mathbb{R}^S \setminus V(S)$ and the latter set is convex, it holds that the vector z_S belongs also to $D^V(S)$, and therefore $z_S \notin I^V(S)$. Consequently, $z \in V^*(N)$. □

Notice that the condition in the lemma is automatically satisfied if the NTU-game (N, V) is induced by a TU-game (N, v). In fact, in the lemma we prove that under that condition the set $V^*(N)$ of an NTU-game (N, V) is a convex set. If (N,V) is also supermodular, then this implies that the payoff vector $a(N,V)$ belongs to $V^*(N)$. Then $a(N,V)$ is either an element of the core or an inefficient allocation for the grand coalition that is not blocked by any proper subcoalition. In the latter case there exists into any strictly positive direction from $a(N,V)$ a unique allocation which is efficient for the grand coalition and is therefore an element of the core.

An example of the condition in Lemma 5.3.5 is when there exists negative externality between players except for the grand coalition, which is reflected by the convexity of the complements of the payoff sets of the proper subcoalitions and the convexity of the payoff set of the grand coalition.

Assumption 5.3.4 implies that the projection of the core on any hyperplane with strictly positive normal vector is a convex set. To show this, define for any vector $d \in \mathbb{R}^N_{++}$ and real number c the projection mapping $P_{d,c}$ on the hyperplane $H_{d,c} = \{x \in \mathbb{R}^N |\ \sum_{i=1}^n x_i d_i = c\}$, i.e., for any $S \subseteq \mathbb{R}^N$

$$P_{d,c}(S) = \{x \in H_{d,c} |\ x = y + \lambda d \text{ for some } \lambda \in \mathbb{R} \text{ and } y \in S\}.$$

Lemma 5.3.6 *For any NTU-game (N,V) which satisfies Assumption 5.3.4 it holds that $P_{d,c}(C(N,V))$ is a convex set for all $d \in \mathbb{R}^N_{++}$ and $c \in \mathbb{R}$.*

Proof Take any $d \in \mathbb{R}^N_{++}$, $c \in \mathbb{R}$, and $u, w \in P_{d,c}(C(N,V))$. It is to show that for all $\alpha \in [0,1]$ it holds that $z = \alpha u + (1-\alpha)w \in P_{d,c}(C(N,V))$. It follows that $z \in H_{d,c}$ because

$$\sum_{i=1}^n z_i d_i = \alpha(\sum_{i=1}^n u_i d_i) + (1-\alpha)(\sum_{i=1}^n w_i d_i) = \alpha c + (1-\alpha)c = c.$$

Since $u, w \in P_{d,c}(C(N,V))$, there exist $x, y \in C(N,V)$ and $\mu, \nu \in \mathbb{R}$ such that $u = x + \mu d$ and $w = y + \nu d$. Since $x, y \in V^*(N)$ and $d \in \mathbb{R}^N_{++}$, by Assumption 5.3.4 there exists $\lambda \in \mathbb{R}$ such that $s = \alpha x + (1-\alpha)y + \lambda d \in V^*(N)$. Therefore, there exists $\beta \geq \lambda$ such that $t = \alpha x + (1-\alpha)y + \beta d \in C(N,V)$. Hence, $t = z + \gamma d$, where $\gamma = \beta - \alpha\mu - (1-\alpha)\nu$. This implies that $z \in P_{d,c}(C(N,V))$. □

Theorem 5.3.7 *For any supermodular NTU-game $(N,V) \in \mathcal{G}^{ntu}_N$ satisfying Assumption 5.3.4, the solution set $S(N,V)$ is a nonempty subset of the core $C(N,V)$.*

Proof For an NTU-game (N,V) it holds that $S(N,V)$ is nonempty. Take first any $x \in S^o(N,V)$. Then $x \in E^V(N)$ and $x = a(N,V) + \lambda d$ for some $\lambda \in \mathbb{R}$ and $d \in \mathbb{R}^N_{++}$. Take $c = d^\top a(N,V)$, then $a(N,V)$ is the projection of x on $H_{d,c}$.

For $\sigma \in \Pi(N)$, let $p^\sigma(N,V) = m^\sigma(N,V) - \lambda^\sigma d$ be the projection of $m^\sigma(N,V)$ on $H_{d,c}$, then

$$a(N,V) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)}(p^\sigma(N,V)+\lambda^\sigma d) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} p^\sigma(N,V) + \frac{1}{n!}\sum_{\sigma\in\Pi(N)}\lambda^\sigma d.$$

Pre-multiplying by the vector d yields $\sum_{\sigma\in\Pi(N)}\lambda^\sigma = 0$ because it holds that $d^\top d > 0$, $d^\top a(N,V) = c$, and $d^\top p^\sigma(N,V) = c$ for all $\sigma \in \Pi(N)$. This implies that

$$a(N,V) = \frac{1}{n!}\sum_{\sigma\in\Pi(N)} p^\sigma(N,V).$$

Since for all $\sigma \in \Pi(N)$ it holds that $p^\sigma(N,V) \in P_{d,c}(C(N,V))$, it follows from Lemma 5.3.6 that $a(N,V)$ is an element of $P_{d,c}(C(N,V))$. Hence, there exists $y \in C(N,V)$ such that $y = a(N,V) + \mu d$ for some $\mu \in \mathbb{R}$. Since d is a strictly positive vector and $x = a(V) + \lambda d \in E^V(N)$, it holds that $\mu = \lambda$ and therefore $x \in C(N,V)$. This implies $S^o(N,V) \subseteq C(N,V)$. Since $S(N,V)$ is the closure of $S^o(N,V)$ and the core $C(N,V)$ is a closed set, it follows that $S(N,V) \subseteq C(N,V)$. □

A particular element of the solution set $S(N,V)$ of an NTU-game $(N,V) \in \mathcal{G}_N^{ntu}$ is the payoff vector $e(N,V) = a(N,V) + \lambda^*(1,\ldots,1)^\top$, where λ^* is the unique real number such that $a(N,V) + \lambda^*(1,\ldots,1)^\top$ is an element of $E^V(N)$. If $\lambda^* \neq 0$, then every player receives the same amount of payoff more (or less) than at $a((N,V)$ and the solution can be considered as a kind of egalitarian solution.

In case the average of all marginal vectors of an NTU-game (N,V) is an inefficient allocation for the grand coalition and is not blocked by any proper subcoalition, i.e., the solution set $S(N,V)$ is the bargaining set of the bargaining problem $B(V(N), a(N,V))$, one may also consider the Nash bargaining solution of the bargaining problem as a single-valued solution, see Nash (1950). The Nash bargaining solution is always an element of the solution set $S(N,V)$ and is therefore also an element of the core. Also any other solution of the bargaining problem $B(V(N), a(N,V))$ is an element of the solution set $S(N,V)$, like the Raiffa-Kalai-Smorodinsky bargaining solution (Raiffa (1953) and Kalai and Smorodinsky (1975)).

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