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TILBURG INSTITUTE OF ECONOMICS
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A VECTOR REPRESENTATION OF MAJORITY VOTING

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0 Introduction*

In a number of articles [2,3,4,5] different conditions were presented that guarantee the consistency of the simple majority decision rule. In [5] Inada summarized these conditions. It appears that most proofs in this field are lengthy and tedious. In this note we show that by a simple vector representation of preferences between three alternatives, the proofs can be substantially facilitated, since they are reduced to the finding of hyperplanes that separate convex sets. It is also shown that the conditions for an odd number of voters can be generalised

1 Vector representation of preferences

Let R be a preference relation with derived relations P (strict preference) and I (indifference). Any ordering of three alternatives a, b and c can be represented by a three-dimensional vector $x = (x^1, x^2, x^3)$ with components that can only take on the values 0, 1 and -1, if we define

$$x^1 = \begin{cases} 1 \text{ if } a\,P\,b \\ 0 \text{ if } a\,I\,b \\ -1 \text{ if } b\,P\,a \end{cases} \quad x^2 = \begin{cases} 1 \text{ if } b\,P\,c \\ 0 \text{ if } b\,I\,c \\ -1 \text{ if } c\,P\,b \end{cases} \quad x^3 = \begin{cases} 1 \text{ if } c\,P\,a \\ 0 \text{ if } c\,I\,a \\ -1 \text{ if } a\,P\,c \end{cases} \tag{1.1}$$

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* I thank Prof. Inada for his comment which prevented an error that occured in an earlier draught of this paper.

Obviously there are different ways to represent the preferences, but the representation given above seems the most suitable one.
There exist exactly thirteen <u>transitive</u> preference orderings of a, b and c; the vector representations are denoted $x_o, x_1, x_2, \ldots, x_{12}$ and constitute the set

$$V = \{x_o, x_1, x_2, \ldots, x_{12}\} \qquad (1.2)$$

preference ordering		a	a	a	a	a	c	c	c	c	b	b	b	b
		I	P	P	P	I	P	P	P	I	P	P	P	I
		b	b	b	c	c	a	a	b	b	c	c	a	a
		I	P	I	P	P	P	I	P	P	P	I	P	P
		c	c	c	b	b	b	b	a	a	a	a	c	c
preference vector	x^1	0	1	1	1	1	1	0	-1	-1	-1	-1	-1	0
	x^2	0	1	0	-1	-1	-1	-1	-1	0	1	1	1	1
	x^3	0	-1	-1	-1	0	1	1	1	1	1	0	-1	-1
		x_o	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}

It is easily verified that for $x_i \in V$ and for k, l, m $\in$ { 1, 2, 3 }, such that $k \neq l \neq m \neq k$

$$\begin{aligned} x_i^k = -1 &\Longrightarrow 0 \leqq x_i^l + x_i^m \leqq 2 \\ x_i^k = 0 &\Longrightarrow x_i^l + x_i^m = 0 \\ x_i^k = 1 &\Longrightarrow 0 \geqq x_i^l + x_i^m \geqq -2 \end{aligned} \qquad (1.3)$$

and

$$-1 \leqq x_i^k + x_i^l + x_i^m \leqq 1 \qquad (1.4)$$

and also

$$x_i^k = 1 \qquad x_i^l = -1 \quad \text{or} \quad x_i^m = -1$$
$$x_i^k = -1 \qquad x_i^l = 1 \quad \text{or} \quad x_i^m = 1 \qquad (1.5)$$

Any of the alternatives a, b and c can take on five different positions in the preference relation: it can be the only best or worst element (strictly best or strictly worst), it can be one of two equivalent best or worst elements (weakly best or worst) or it can be medium, including the case of three equivalent alternatives. Now if we define

$$w^1 = x^1 - x^3, \; w^2 = x^2 - x^1, \; w^3 = x^3 - x^2 \qquad (1.6)$$

we have, as is easily verified, for a

if $w^1 = 2$, a is stricly best
if $w^1 = 1$, a is weakly best
if $w^1 = 0$, a is medium
if $w^1 = -1$, a is weakly worst
if $w^1 = -2$, a is strictly worst

The same holds for b and c with respect to w^2 and w^3. The set

$$Y = \{ y \in R^3 \mid -1 \leqq y^k \leqq 1, \text{ for } k = 1,2,3 \} \qquad (1.7)$$

is the set of points that lie on or within a cube. Let X be the subset of Y containing all vectors which have components 1, 0, -1,

$$X = \{ x \in Y \mid x^k \in \{ 1, 0, -1 \} , \text{ for } k = 1,2,3 \} \qquad (1.8)$$

Now

$$V \subset X \subset Y$$

and we have

$$V = \{\ x \in X \mid x \ngeq 0 \text{ and } x \nleq 0\ \} \tag{1.9}$$

Apart from $x_o = 0$, V consists of all points of X on a closed curve on the edges of the cube Y; this curve does not intersect the positive and the negative orthants of the cube. (see fig. 1.1)

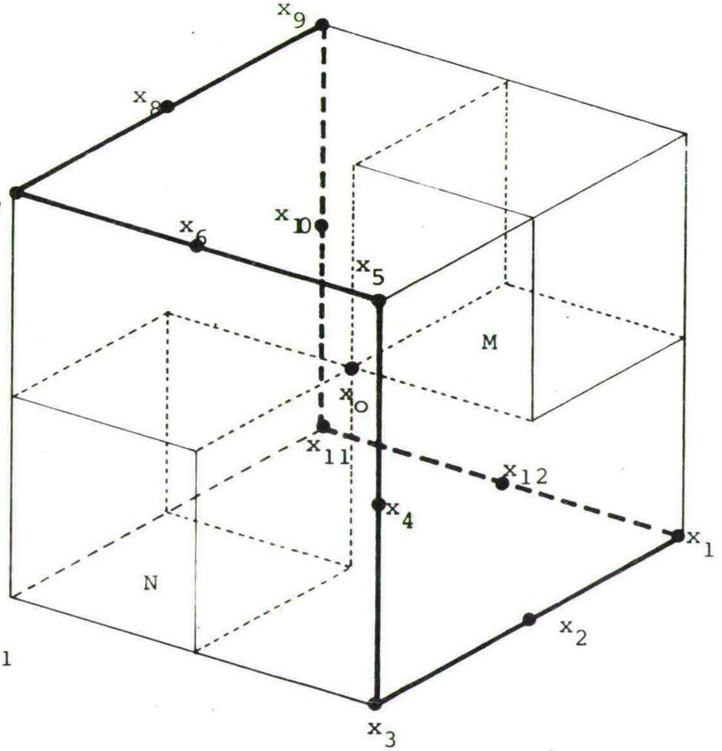


Fig. 1.1

The points of (X-V) represent preference orderings that are not transitive, e.g. $x = (1,1,1)$ means aPb, bPc and cPa, and they are all points of X that lie in the positive or negative orthants of the cube
Let M and N be

$$M = \{ y \in Y \mid y \geq 0 \}, \; N = \{ y \in Y \mid y \leq 0 \} \tag{1.10}$$

then

$$X - V \subset M \cup N \tag{1.11}$$

Note that $0 \not\in M \cup N$

2 Vector representation of voting

If every individual has a transitive preference ordering of a, b and c, voting means that every voter chooses one and only one point of V. If n is the number of voters, and n_i $(i = 0,1,2,...,12)$ is the number of voters that choose x_i, then voting can be represented by the numbers

$$\alpha_i = \frac{n_i}{n} \text{ where } \sum_{i=o}^{12} \alpha_i = 1 \tag{2.1}$$

and the result of the voting procedure is given by a vector $y \in Y$

$$y = \sum_{i=o}^{12} \alpha_i x_i \tag{2.2}$$

representing the social ordering, which obviously can be represented by a point $x \in X$, if we define

$$\begin{aligned} x^k &= 1 \quad \text{if} \quad y^k > 0 \\ x^k &= 0 \quad \text{if} \quad y^k = 0 \\ x^k &= -1 \quad \text{if} \quad y^k < 0 \end{aligned} \tag{2.3}$$

If $y \in M \cup N$, the voting paradox occurs, if however $y \notin (M \cup N)$ the social ordering, represented by y, is transitive. Obviously the point x, derived from y by (2.3), fullfills

$$x \in (M \cup N) \Longleftrightarrow y \in (M \cup N)$$

Now if by imposing certain conditions it is ensured that the voting result y belongs to a set R, such that

$$R \cap (M \cup N) = \emptyset \tag{2.4}$$

then the voting paradox is excluded. If the votes α_i are not restricted, this is not true, since in this case the set of all possible results is given by the convex hull of V:

$$\text{Conv } V = \{ y \in V \mid y = \Sigma \alpha_i x_i \quad \text{for} \quad \alpha_i \geqq 0 \quad \text{and } \Sigma \alpha_i = 1 \} \tag{2.5}$$

and

$$\text{Conv } V \cap (M \cup N) \neq \emptyset. \tag{2.6}$$

Obviously only rational vectors in Y are possible, if the number of voters is finite, but for sake of simplicity we permit all real vectors.

If some of the α_i are known to be zero, the voting result must be in the convex hull of the points

that may have positive weights. As Inada, we call a set of preference vectors x_i that may have nonzero votes, a list $L \subset V$.
Hence

$$x_i \notin L \Longrightarrow \alpha_i = 0 \qquad (2.7)$$

Note that this does <u>not</u> mean that $\alpha_i > 0$ for all $x_i \in L$.

If the set of possible results of a voting process is denoted $R(L)$, $R(L)$ is the convex hull of L, provided that there are no other conditions than (2.7)

$$R(L) = \text{Conv } L =$$

$$\{ y \in Y \mid \Sigma\, \alpha_i\, x_i = y, \text{ for } \alpha_i \geqq 0,\ \alpha_i = 0 \text{ for } x_i \notin L \text{ and } \Sigma\, \alpha_i = 1 \} \qquad (2.8)$$

Now the voting paradox cannot occur, if and only if

$$R(L) \cap (M \cup N) = \emptyset$$

In section 4 the lists of this type will be given. They will be called <u>unrestricted</u> lists.

Now suppose that the convex hull of some list intersects the positive and negative orthants of $(M \cup N)$, but that this intersection only contains boundary points of both sets, e.g. $y = (\frac{1}{2}, \frac{1}{2}, 0)$, i.e. aPb, bPc and aIc. Hence for $y \in \text{Conv } L$, we have $y \not> 0$ and $y \not< 0$. Now

$$\begin{aligned} &\text{Conv } L \cap (M \cup N) \neq \emptyset \\ &\text{Conv } L \cap \text{Int } (M \cup N) = \emptyset \end{aligned} \qquad (2.9)$$

By excluding the boundary points in the set of voting results, we also exclude the voting paradox. It appears that this can be done by requiring that at least one of the following conditions is fullfilled.

1) some α_i, which will be defined in theorem 2, are positive

2) the votes for nonzero preferences cannot be divided into two equal groups. This condition is fullfilled if the number of voters is odd.

If we denote the set of all voting results, that fullfill one of these conditions, by R'(L), it appears that $R'(L) \cap (M \cup N) = \emptyset$ for all lists defined by Inada for an odd number of voters.

The above results will be given in two theorems, by means of separating hyperplanes.

3 Two theorems

Let

$$P = \{ p \in R^3 \mid p^1 + p^2 + p^3 = 1 \text{ and } p \geqq 0 \} \tag{3.1}$$

whereas

$$P^+ = \{ p \in P \mid p > 0 \} \tag{3.2}$$

If we define

$$px = \sum_{k=1}^{3} p^k x^k \tag{3.3}$$

the set

$$F(p) = \{ y \in Y \mid py = 0 \} \tag{3.4}$$

is a hyperplane that separates the cube Y into two subsets and we have

$$y \in M \Longrightarrow py \geqq 0 \quad \text{and} \quad y \in N \Longrightarrow py \leqq 0 \qquad (3.5)$$

If p is strictly positive ($p \in P^+$),

$$y \in M \Longrightarrow py > 0 \quad \text{and} \quad y \in N \Longrightarrow py < 0 \qquad (3.6)$$

Now if an unrestricted list R(L) is strictly separated from M by one hyperplane and from N by another hyperplane, it cannot intersect M or N.
If $p, q \in P^+$, and if

$$y \in R(L) \Longrightarrow py \leqq 0 \quad \text{and} \quad qy \geqq 0 \qquad (3.7)$$

we have

$$R(L) \cap (M \cup N) = \emptyset$$

Hence the voting paradox cannot occur, provided every voter chooses a vector of L.
This result leads to the following theorem

<u>THEOREM 1</u>

If $L \subset V$ and there exist $p,\ q \in P^+$, such that

$$x_i \in L \Longrightarrow px_i \leqq 0 \quad \text{and} \quad qx_i \geqq 0$$

then

$$R(L) \cap (M \cup N) = \emptyset$$

Proof

Since

$$x_i \in L \Longrightarrow px_i \leqq 0$$

we have for $y \in R(L) = \text{Conv } L$, $y = \sum_{x_i \in L} \alpha_i x_i$ and hence $py \leqq 0$. Since $z \in M \Longrightarrow pz > 0$ we have

$$\text{Conv } L \cap M = \emptyset$$

In the same way it follows, applying $x_i \in L \Longrightarrow qx_i \geqq 0$,

$$\text{Conv } L \cap N = \emptyset$$

hence

$$R(L) \cap (M \cup N) = \emptyset$$

If (3.6) holds for some points p, q $\in$ P, the hyperplanes F(p) and F(q) do not necessarily strictly separate R(L) from M and N. Therefore we need the conditions 1 or 2 to guarantee that

$$R'(L) \cap (M \cup N) = \emptyset$$

THEOREM 2

Let $L \subset V$ and there exist p, q $\in$ P, such that

$$x_i \in L \Longrightarrow px_i \leqq 0 \quad \text{and} \quad qx_i \geqq 0$$

Let

$$R'(L) = \{y \in \text{Conv } L \mid \text{condition 1 or 2 holds}\}$$

condition 1: $\exists\, x_i \in L: \alpha_i p x_i < 0$ and $\exists\, x_j \in L: \alpha_j q x_j > 0$

condition 2: $\nexists\, K \subset L: 0 \notin K$ and $\sum_{x_i \in K} \alpha_i = \sum_{x_j \in L-K-\{0\}} \alpha_j$

then

$$R'(L) \cap (M \cup N) = \emptyset$$

Proof

1 Let condition 1 hold
Hence for some $x_i \in L$, we have $\alpha_i p x_i < 0$, therefore $\alpha_i > 0$ and $p_i x_i < 0$
Now for $y \in R'(L)$ holds

$$y = \Sigma \alpha_i\, x_i$$

and

$$py = \Sigma\, \alpha_i p x_i < 0$$

and since $y \in M \Longrightarrow py \geqq 0$
we have $y \notin M$.

In the same way it follows, applying $\alpha_i q x_i > 0$ that

$$y \notin N.$$

2 Let condition 2 hold.

Suppose $y \in R'(L) \cap M$

Withput loss of generality we may assume

$$y^1 > 0,\ y^2 \geqq 0,\ y^3 = 0$$

Since $y \in$ Conv L, we have $py \leqq 0$ and since $y \in M$, we have $py \geqq 0$, hence $py = 0$ and this implies

$$\alpha_i > 0 \longrightarrow px_i = 0 \qquad \text{(i)}$$

and

$$p^1 = 0,\ p^2 \geqq 0,\ p^3 \geqq 0 \qquad \text{(ii)}$$

a) Now suppose first that for some $x_{i_o} \in L$, we have

$$\alpha_{i_o} > 0 \text{ and } x^3_{i_o} = 0 \text{ and } x_{i_o} \neq 0$$

Hence $x^2_{i_o} \neq 0$

and since by (i)

$$p^1x^1_{i_o} + p^2x^2_{i_o} + p^3x^3_{i_o} = 0,$$

we must have $p^2 = 0$, and since $p \in P$, $p^3 > 0$ and now for x_i such that $\alpha_i > 0$ we have

$$px_i = p^3x^3_i = 0,$$

hence $\alpha_i > 0 \longrightarrow x^3_i = 0.$

But

$$x^3_i = 0 \Longrightarrow x^1_i + x^2_i = 0$$

and therefore

$$\Sigma\alpha_i(x_i^1 + x_i^2) = 0$$

but this is a contradiction, since $y^1 = \Sigma\alpha_i x_i^1 > 0$.

b) Hence we must have for $x_i \neq 0$

$$\alpha_i > 0 \Longrightarrow x_i^3 \neq 0$$

Let $K = \{x_i \mid x_i^3 = 1\}$ and $L - K - \{0\} = \{x_i \mid x_i^3 = -1\}$
Now

$$\sum_{x_i \varepsilon L} \alpha_i x_i^3 = 0$$

hence

$$\sum_{x_i \varepsilon K} \alpha_i = \sum_{x_i \varepsilon L-K-\{0\}} \alpha_i$$

but this is excluded by condition 2.
Therefore

$$R'(L) \cap M = \emptyset$$

In the same way we can show that

$$R'(L) \cap N = \emptyset$$

<u>Corrollory</u>

If the number of voters choosing $x_i \neq 0$ is odd, condition 2 of theorem 2 is satisfied

<u>Proof</u>

$$\sum_{i=1}^{n} \alpha_i = \alpha$$

then α n is an odd number, hence it is impossible that

$$\gamma = \sum_{x_i \in K} \alpha_i = \sum_{L-K-0} \alpha_i = \beta$$

since

$$\alpha = \gamma + \beta \text{ and } \alpha n = \gamma n + \beta n = 2\gamma n$$

hence $\gamma n = \frac{1}{2} \alpha n$ is not a whole number.

4 Conditions and lists

We shall now consider a series of lists of both the restricted and the unrestricted type. These are the same as those given by Inada. It will be shown that the proof is now very easy applying theorems 1 and 2 respectively. We have only to give the vectors p and q.

A Conditions of the unrestricted type

Condition I (condition A in [5])

Each voter considers at least two of the three alternatives equivalent.

There is only one list satisfying this condition:

$$L = \{x_i \in V \mid \exists k \in \{1,2,3\}: x^k = 0\}$$

Since $x_i \in V$, we have $x^k = 0$, $x^l = 1$, $x^m = -1$ for k, l, m $\in \{1,2,3\}$

Proof

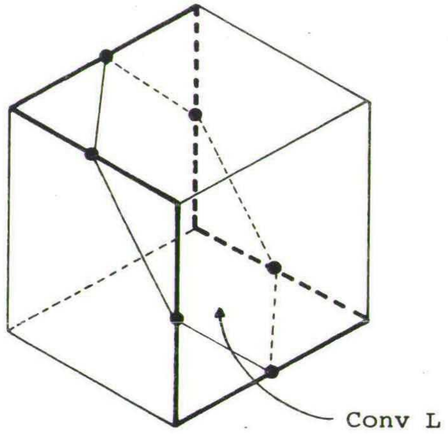
Choose $p = q = (1/3, 1/3, 1/3)$
Now for $x_i \in L$, we have

$$px_i = qx_i = 1/3\ (x_i^k + x_i^l + x_i^m) = 1/3(0 + 1 - 1) = 0$$

Hence by theorem 1

$$R(L) \cap (M \cup N) = \emptyset$$

fig. 4.1



Condition I

Condition II (condition C in [5])

All voters either consider two of the three alternatives equivalent or one of them is strictly best and the other is strictly worst.

We may have e.g.

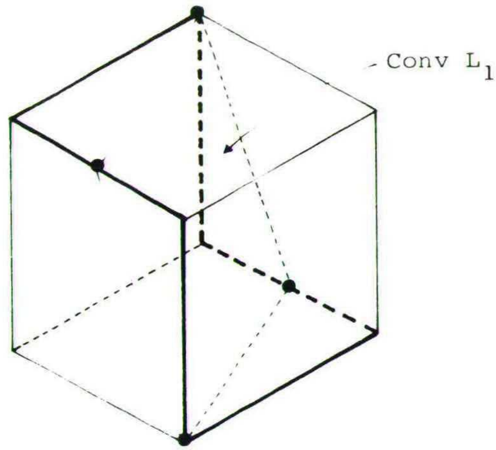
aPbPc or cPbPa or aIb

There are three different lists of this type

$$L_k = \{x_i \in V \mid x_i^k = 0 \quad \text{or} \quad x_i^k = -x_i^l = -x_i^m \quad \text{for} \quad l,m \in \{1,2,3\}\}$$

for k = 1,2,3

fig. 4.2.



Condition II

<u>Proof</u>

Choose $p = q$ and $p^k = \frac{1}{2}$, $p^l = p^m = \frac{1}{4}$.

For $x_i^k = 0$

$$\tfrac{1}{4}\, x_i^l + \tfrac{1}{4}\, x_i^m = 0$$

for $x_i^k \neq 0$

$$\tfrac{1}{2}\, x_i^k + \tfrac{1}{4}\, x_i^l + \tfrac{1}{4}\, x_i^m = \tfrac{1}{2}\,(x_i^k - x_i^k) = 0$$

<u>Condition III</u> a (first part of Inada's condition B)

<u>There is one alternative that all voters consider at least as good as the other two or that all voters consider not better than the other two</u>.

e.g. aRc and aRb for all voters.

There are 6 different lists satisfying this condition

$$L_{k,l} = \{x_i \in V \mid x_i^k \geqq 0 \text{ and } x_i^l \leqq 0\}$$

These lists can also be expressed in terms of the variable w (1.6) and now we have two groups:

$$\{x_i \in V \mid w^k \geqq 1\}. \qquad \text{for } k = 1,2,3$$

and

$$\{x_i \in V \mid w^k \leqq -1\}. \qquad \text{for } k = 1,2,3$$

Hence one of the alternatives must be weakly or strictly best (worst) for all voters.

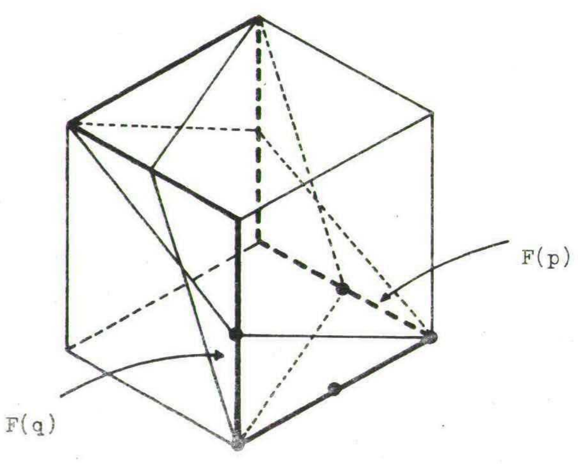


fig. 4.3.

Condition III a: L_{13}

Proof

Choose $p^k = \frac{1}{4},\ p^l = \frac{1}{2},\ p^m = \frac{1}{4}$

$q^k = \frac{1}{2},\ q^l = \frac{1}{4},\ q^m = \frac{1}{2}$

if $x_i^l = -1,\ x_i^k + x_i^m \leqq 2$, hence $px_i \leqq 0$

if $x_i^l = 0,\ x_i^k + x_i^m = 0$, hence $px_i = 0$

if $x_i^k = 1,\ x_i^l + x_i^m \geqq -2$, hence $qx_i \geqq 0$

if $x_i^k = 0,\ x_i^l + x_i^m = 0$, hence $qx_i = 0$

Condition III b (This is the second part of Inada's condition B)

Either one alternative is strictly best and a second is strictly worst, or there are at least two equivalent alternatives, while the second is not strictly best or the first is not strictly worst.

e.g. aPbPc, bPaIc, aIbPc, aPbIc, aIcPb, aIcIb

There are 6 different lists of this type

$$L_{K+} = \{x_i \in V \mid x_i^k \geqq 0 \text{ and } x_i^k + x_i^l + x_i^m \leqq 0\}$$

$$L_{K-} = \{x_i \in V \mid x_i^k \leqq 0 \text{ and } x_i^k + x_i^l + x_i^m \geqq 0\}$$

Proof

Choose for L_{k+}: $p = (1/3, 1/3, 1/3)$, $q^k = 1/2$, $q^l = q^m = 1/4$

for L_k : $p^k = 1/2$, $p^l = p^m = 1/4$, $q = (1/3, 1/3, 1/3)$

Now for L_{k-} we have either

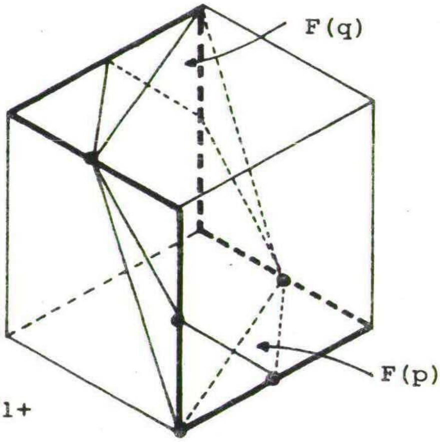
$x_i^k = -1$, hence $1 \leqq x^1 + x^m \leqq 2$ and

$$px_i = -1/2 + 1/4(x_i^1 + x_i^m) \leqq -1/2 + 1/4.2 = 0$$

$$qx_i = -1/3 + 1/3(x_i^1 + x_i^m) \geqq -1/3 + 1/3 = 0$$

$x_i^k = 0$, hence $(x_i^1 + x_i^m) = 0$ and $px_i = qx_i = 0$

fig. 4.4



Condition III b: L_{1+}

B Conditions of the restricted type

Condition IV (condition F in [5])

There is one alternative that all voters consider strictly best or strictly worst or all three are equivalent

e.g. aPb and aPc or bPa and cPa.

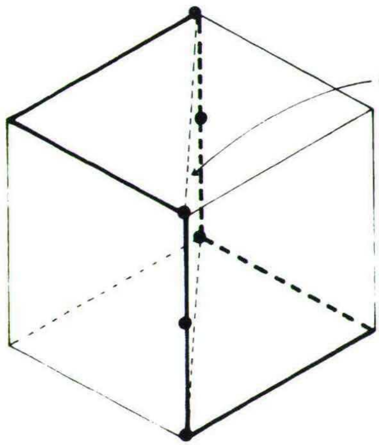
There are three different lists of this type

$$L_{k,1} = \{x_i \in V \mid x_i^k = -x_i^1\} \quad \text{for } k,1 = 1,2,3,\ k < 1$$

In terms of w, these list can be defined for each k

$$\{x_i \varepsilon V \mid x_i = 0 \text{ or } w^k = 2 \text{ or } w^k = -2\}$$

Fig. 4.5



Condition IV

Proof

Choose p = q, where

$$p^k = p^l = \tfrac{1}{2} \text{ and } p^m = 0$$

Now for $x_i \varepsilon L_{k,l}$

$$px_i = p^k x_i^k + p^l x_i^l = \tfrac{1}{2}(x_i^k + x_i^l = = 0 = qx_i$$

Obviously $R'(L) = \{y \varepsilon V \mid py = 0\}$
Only condition 2 of theorem 2 is relevant since py = 0 for all $y \varepsilon R'(L)$.

Condition V (Inada's conditions D and E)

One of the alternatives is considered not best by all voters or not worst, by all voters, or all are equivalent.

e.g. cPa or cPb

This is the case of single peakedness or of single cavedness. There are six lists of this type

$$L_{k,l} = \{x_i \in V \mid x_i^k = 1 \quad \text{or} \quad x_i^l = -1 \quad \text{or} \quad x_i = 0\}$$
$$\text{for } k,l = 1,2,3;\ k \neq l$$

In terms of w, we have for single peaked lists

$$\{x_i \in V \mid w^k \geqq 0\}. \qquad \text{for } k = 1,2,3$$

and for single caved lists

$$\{x_i \in V \mid w^k \leqq 0\} \qquad \text{for } k = 1,2,3.$$

Note that any type V list contains some type III list.

<u>Proof</u>

Choose

$$p^k = 0,\ p^l = p^m = \tfrac{1}{2}$$
$$q^k = q^m = \tfrac{1}{2},\ q^l = 0$$

if $x_i^k = 1$, $x_i^l + x_i^m \leqq 0$, hence $px_i = \tfrac{1}{2}(x_i^l + x_i^m) \leqq 0$

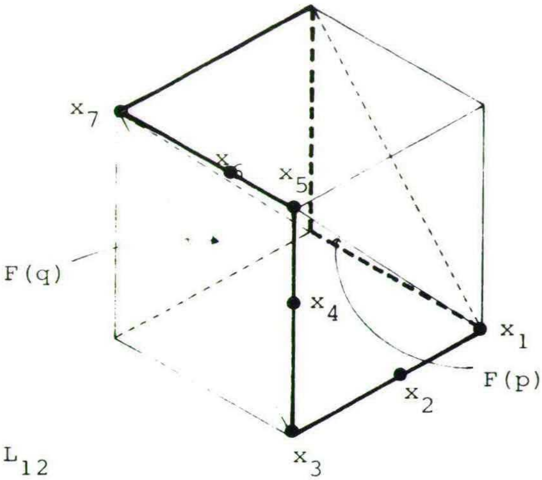
if $x_i^l = -1$, $x_i^m \leqq 1$, hence $px_i = \tfrac{1}{2}(x_i^l + x_i^m) \leqq 0$

and

if $x_i^l = -1$, $(x_i^k + x_i^m) \geqq 0$, hence $qx_i \geqq 0$

if $x_i^k = 1$, $x_i^l \geqq -1$, hence $qx_i \geqq 0$

fig. 4.6



Condition V: L_{12}

Obviously for k = 1, l = 2 (fig. 4.5) condition 1 of theorem 2 is satisfied if

$$\alpha_2 + \alpha_3 + \alpha_4 > 0 \quad \text{and} \quad \alpha_4 + \alpha_5 + \alpha_6 > 0$$

since

$$px_i > 0 \quad \text{for} \quad i = 2,3,4$$

and $$qx_i > 0 \quad \text{for} \quad i = 4,5,6$$

<u>Condition VI</u> (Inada's condition B)

<u>There are two alternatives, such that no voter strictly prefers the first one to the second one</u>

e.g. aRb

There are six lists of this type

$$L_{k+} = \{x_i \in L \mid x_i^k \geqq 0\}$$

$$L_{k-} = \{x_i \in L \mid x_i^k \leqq 0\}$$

Proof

Choose for $x_i^k \geqq 0$

$$p^k = 0,\ p^l = p^m = \tfrac{1}{2}$$
$$q^k = 1,\ q^l = q^m = 0$$

Now

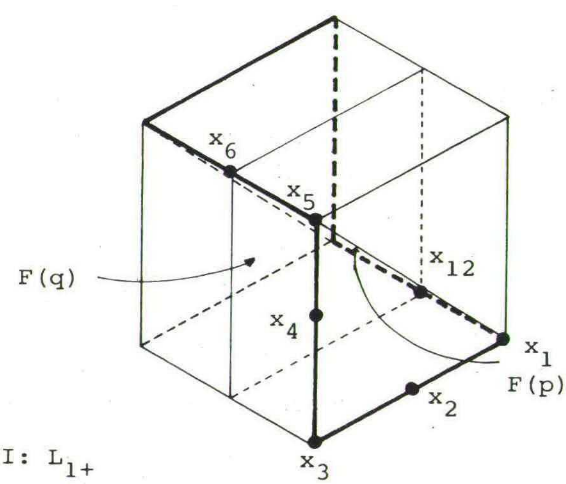
if $x_i^k = 1$, $(x_i^l + x_i^m) \leqq 0$, hence $px_i = \tfrac{1}{2}(x_i^l + x_i^m) \leqq 0$
if $x_i^k = 0$, $(x_i^l + x_i^m) = 0$, hence $px_i = 0$

and $qx_i = x_i^k \geqq 0$

Choose for $x_i^k \leqq 0$

$$p^k = 1,\ p^l = p^m = 0$$
$$q^k = 0,\ q^l = q^m = \tfrac{1}{2}$$

Fig. 4.7



Condition VI: L_{1+}

Obviously for $L_{1,+}$, condition 1 of theorem 2 is satisfied, if

$$\alpha_2 + \alpha_3 + \alpha_4 < 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 > 0$$

since $px_i < 0$ for $i = 2,3,4$

$qx_i > 0$ for $i = 1,2,3,4,5$

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