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Interactive operational decision making

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Interactive Operational Decision Making

purchasing situations & mutual liability problems

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Proefschrift

ter verkrijging van de graad van doctor aan Tilburg University op gezag van de rector magnificus, prof.dr. Ph. Eijlander, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op vrijdag 17 januari 2014 om 14.15 uur door

Mirjam Groote Schaarsberg

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During one of my many train trips I started with a list of people and the reasons I wanted to mention them. It became a mess instantly, so I decided to make things easier and presented the entire acknowledgements in matrix-format: rows filled with the things I would like to thank people for and the columns representing the people. I just had to tick some boxes and would be done. However, the dimensions of my pages did not allow for such a matrix. Besides, I was afraid that instead of being the most reader-friendly part, the acknowledgements would become too much math-related. There is enough of that already.

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Chapter 1

Interactive operational decision making

1.1 Interactive decision theory

Good decision making is a field of research explored from different scientific perspectives, from mathematical programming to psychological aspects in framing decisions. Many of the decision situations we face involve other, possibly antagonistic, decision makers that affect our options and the consequences of our decisions. And reversely, our decisions may affect the options and consequences of decisions of the others. In this thesis we deal with decision making in such an interactive setting. To analyze interactive decision problems we use tools from interactive decision theory.

Interactive decision theory is a more descriptive name for *game theory*: the study of mathematical models of conflict and cooperation between intelligent rational decision-makers (Myerson (1981)). It is a research discipline that considers the logical side of interactive decision making. Game theory is usually divided into two branches. In competitive, non-cooperative game theory players act individually and strategically against each other to minimize their individual costs (or maximize individual profits). For an extended introduction we refer to Fudenberg and Tirole (1991). Cooperative game theory studies situations in which players cooperate in order to reduce joint costs, and analyzes how one can fairly allocate the joint costs or the joint cost savings. A commonly used model in cooperative interactive decision theory is a transferable utility game in which one assigns a value to each subcoalition of decision makers. Allocation proposals are evaluated based on these coalitional values. For an extended introduction we refer to Sudhölter and Peleg (2007). In some cooperative situations, however, one cannot determine the costs or cost savings of a subcoalition of decision makers, but the decision makers still have to share their joint costs. In that case the model of cost sharing (see e.g. Moulin (2002)) can be an appropriate tool to find adequate allocations of the costs.

Applications of interactive decision theory can be found in, e.g., Biology (Nowak and Sigmund (2004)), Philosophy (De Bruin (2005)), Political Science (Myerson and Weber (1993)) and especially in Economics and Management Science.

1.2 Operational decision making

In this thesis we analyze interactive decision making on an operational level in an economic environment: the problems allow for a quantitative analytical approach, where information is transparent and the interactive problem has a managerial – as opposed to policy – nature. Using appropriate tools from interactive decision theory we analyze how decision makers can decide on, *e.g.*, a good strategy to minimize individual costs or on how to fairly allocate the (possible) benefits that follow from cooperative behavior.

Examples of such interactive decision problems on an operational level are, in a cooperative setting, inventory games (Meca, Timmer, García Jurado, and Borm (2004)) and in a non-cooperative setting, capacity allocation games (Cachon and Lariviere (1999)).

As its subtitle indicates, this thesis covers interactive decision making in the context of purchasing situations and in the context of mutual liability problems.

Purchasing is the formal process of buying goods and services in order to accomplish organizational goals. On an operational level, purchasing involves supplier selection and pricing agreements for frequently used tangible goods, like scribbling-pads for a university, needles for a hospital or mops for janitorial services. In interactive purchasing situations multiple buying organizations interact with similar (or possibly the same) suppliers for the procurement of the same commodity. Decisions to be made in interactive purchasing concern if and how to cooperate with other buying organizations. If so, one has to tackle the important question of how to allocate possible cost savings. And if not, how to interact with the supplier(s) on an individual strategic level, while taking into account the presence and behavior of the other purchasers.

Mutual liability problems model the interrelationship between decision makers, based on financial obligations. We will investigate the scenario where a group of agents is related by having mutual liabilities, but reaches the point in time where the agents want to cash their claims. None of the agents worry about the possible insufficient cash in the current assets, until individuals start cashing their claims. This will lead to a cascading effect and thereby will reveal the possibly insufficient cash level of agents and the agents typically might not obtain all of what they, however rightfully, claim. Here a decision has to be made regarding how the total amount of available cash can be fairly distributed among the agents.

In the next section we provide a compact overview of this thesis. An extensive introduction of each of the different topics we study can be found at the beginning of each chapter.

1.3 Overview and results

One of the most basic and probably oldest allocation problems that can be modeled in an interactive decision framework is a bankruptcy problem (cf. O'Neill (1982)). In a bankruptcy problem a (possibly) insufficient monetary amount has to be divided over a group of claimants. Aumann and Maschler (1985) have characterized one of the most well-known bankruptcy rules, the Aumann-Maschler rule, using the axiom of consistency. Chapter 2 introduces *mutual liability problems* as a generalization of bankruptcy problems, where every agent not only owns a certain amount of cash money, but has outstanding claims and debts towards the other agents as well. Assuming that the agents want to cash their claims, we analyze mutual liability rules which prescribe how the total available amount of cash can be allocated among the agents. In particular we focus on bankruptcy rule based bilateral transfer schemes. Existence of these schemes is established and it is seen that within the class of hierarchical mutual liability problems, for each bankruptcy rule such a transfer scheme is unique. Although in general a bankruptcy rule based bilateral transfer scheme need not be unique, we show that the resulting bankruptcy rule based transfer allocation is. This leads to the definition of bankruptcy rule based mutual liability rules. For hierarchical mutual liability problems an alternative characterization of such mutual liability rules is provided. Moreover, we show that the axiomatic characterization of the Aumann-Maschler bankruptcy rule on the basis of consistency can be extended to the corresponding mutual liability rule. We conclude with a discussion of alternative approaches to solve mutual liability problems.

Chapter 3 introduces a new class of interactive cooperative purchasing situations. In a *maximum cooperative purchasing (MCP) situation* the unit price of a commodity depends on the largest order quantity within a cooperating group of players. Due to quantity discounts offered by the supplier, players can obtain cost savings by purchasing cooperatively. However, to establish fruitful cooperation a decision has to be made about an adequate allocation of the corresponding cost savings. In analyzing MCP-situations from the perspective of allocation by using the model of transferable utility games, we define corresponding cooperative MCP-games. We show that if the supplier offers quantity discounts, there exists a stable and efficient allocation of the cost savings: the Direct Price solution. In the Direct Price solution each player pays the unit price that follows from maximal cooperation among the group of purchasers. However, in this allocation a player with the largest order quantity, who is decisive for the lower unit price, receives no cost savings at all. For this reason we consider two well-known solution concepts from the theory of cooperative games: the nucleolus (Schmeidler (1969)) and the Shapley value (Shapley (1953)). We show that the nucleolus of an MCP-game can be derived in polynomial time from the Direct Price solution, using a so-called nucleolus determinant. To show this result we provide an explicit general alternative characterization of the nucleolus. Moreover, using the special structure of MCP-games we find an explicit expression for the Shapley value. Interestingly, the Shapley value can be interpreted as a specific tax and subsidize system. For illustrative purposes the behavior of the three solution concepts is compared numerically.

In Chapter 4 we study *capacity restricted cooperative purchasing (CRCP) situations* from the perspective of cost sharing. A CRCP-situation is a purchasing cooperative with individual order quantities with respect to a certain commodity. Here, the sum of the order quantities determines the unit price. Instead of facing one supplier with sufficient supplies, the group faces two suppliers with (possibly) insufficient individual supplies. The combined capacity of the two suppliers, however, is sufficient. We show that to minimize joint ordering costs, the cooperative should order as much as possible at one of the two suppliers, and the remainder at the other one. To find suitable cost allocations of the total purchasing costs we model the CRCP-situation as a cost sharing problem and we find that the cost function of the corresponding cost sharing problem is piecewise concave. The domain of the cost function can be divided in separate intervals on which the function is concave; the maximally concave intervals. We introduce tailor-made and context specific cost sharing rules for cost sharing problems with piecewise concave cost functions, in which we first divide the vector of order quantities into separate vectors for the different maximally concave intervals, using a bankruptcy rule. Subsequently, for each maximally concave interval and corresponding vector we use the well-known serial cost sharing rule (Moulin and Shenker (1992)) to allocate the costs of that specific interval over the organizations. Finally, by summing these allocated costs we obtain the allocation according to the piecewise serial rule. Inspired by the context of CRCP-situations, we provide a piecewise serial rule that, on the class of cost sharing problems with piecewise concave cost functions, satisfies unit cost monotonicity, and a piecewise serial rule that satisfies monotonic vulnerability with respect to the absence of the smallest player. Furthermore we provide a characterization of the proportional piecewise serial rule. Numerical comparison with the serial cost sharing rule further supports the claim that these two piecewise serial cost sharing rules are suitable methods for allocating the purchasing costs of a CRCP-situation.

Capacity restricted strategic purchasing (CRSP) situations are the non-cooperative siblings of CRCP-situations. Here each member of a group of purchasers has an individual order quantity with respect to a commodity. The group faces two suppliers with possible insufficient individual supplies. Instead of purchasing cooperatively the purchasers act individually. In a CRSP-situation each purchaser strategically splits his order over the two suppliers in order to obtain his desired order quantity for the lowest possible cost. How an individual purchaser should place his order, depends, amongst others, on fulfillment policies of the suppliers. Such a policy prespecifies how the supplier allocates his capacity over the orders in case the total number of units ordered exceeds the capacity limit of the supplier. In Chapter 5 we study CRSP-situations by analyzing equilibrium behavior in non-cooperative games corresponding to several scenarios with respect to the fulfillment policies of the suppliers. These games are called ordering games. In the first scenario, fulfillment can be based on fixed preferences of the suppliers with respect to the identity of the players. Second, fulfillment can also be based on the order size of the purchasers, we differentiate among the policy small before large (SBL) and large before small (LBS). Third and finally, suppliers can fulfill orders proportionally. If suppliers preferences are fixed and identical, the ordering game has an equilibrium. If the policy is based on SBL, we show that there does not necessarily exists an equilibrium. On the other hand, if fulfillment is based on LBS, there exists an equilibrium in the ordering game. When orders are fulfilled proportionally, then *if* there exists an equilibrium in the ordering game, we show that it can only be found at the boundaries of the strategy space. Finally, we develop an approximation of the ordering game by means of a so-called matrification of the strategy space. Also here, we analyze equilibrium behavior in the different scenarios.

1.4 Basic notation

The set of all nonnegative integers is denoted by $\mathbb{N}$, the set of all real numbers by $\mathbb{R}$, the set of nonnegative reals by $\mathbb{R}_+$ and the set of positive reals by $\mathbb{R}_{++}$. For a finite set N, the cardinality of N is denoted by $|N|$. The collection of all subsets of N is denoted by 2^N. $\mathbb{R}^N$ denotes the set of real-valued vectors with coordinates corresponding to N. An element of $\mathbb{R}^N$ is denoted by $x = (x_i)_{i \in N}$. For every $S \in 2^N \backslash \{\emptyset\}$, the unit vector $\mathbb{1}^S \in \mathbb{R}^N$ is such that $\mathbb{1}^S_i = 1$ if $i \in S$ and $\mathbb{1}^S_i = 0$ if $i \in N \backslash S$. We denote $(y)^+ = \max\{y, 0\}$ for all $y \in \mathbb{R}$.

By $\Pi(N)$ we denote the collection of all permutations of N where $\sigma \in \Pi(N)$ is a bijection from $\{1, \ldots, |N|\}$ to N. Next $\theta : \mathbb{R}^N \to \mathbb{R}^{|N|}$ is such that for $x \in \mathbb{R}^N$ and $i \in \{1, \ldots, |N|\}$, $\theta_i(x) = x_{\sigma(i)}$, with $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \ldots \geq x_{\sigma(|N|)}$.

For any two vectors $x, y \in \mathbb{R}^t$ we have that x is lexicographically smaller than y if $x = y$ or if there exists an $s \in \{1, \ldots, t\}$ such $x_k = y_k$ for all $k \in \{1, \ldots, s-1\}$ and $x_s < y_s$. $x \leq_L y$ denotes that x is lexicographically smaller than y.

Let $A \subset \mathbb{R}^N$ be a finite collection of vectors. Then, $span(A)$ denotes the linear hull of A, *i.e.*, all linear combinations of vectors from A, and $\text{conv}\{A\}$ denotes the convex hull of A, the set of all convex combinations of vectors from A.

With $A = (a_{ij}) \in \mathbb{R}^{N \times N}$, the diagonal of A, $\text{diag}(A) \in \mathbb{R}^N$ is for all $i \in N$ defined by $\text{diag}_i(A) = a_{ii}$.

Let X be a convex subset of $\mathbb{R}^N$. A function $f : X \to \mathbb{R}$ is concave on X if for any $x^1, x^2 \in X$ and any $\lambda \in [0,1]$, $f(\lambda x^1 + (1-\lambda)x^2) \geq \lambda f(x^1) + (1-\lambda)f(x^2)$. f is strictly concave if for any $x^1, x^2 \in X$ with $x^1 \neq x^2$ and any $\lambda \in (0,1)$, $f(\lambda x^1 + (1-\lambda)x^2) > \lambda f(x^1) + (1-\lambda)f(x^2)$. Moreover $f : X \to \mathbb{R}$ is convex if $-f$ is concave.

Let $g : \mathbb{R} \to \mathbb{R}$. If g is differentiable and concave, then any $x \in \mathbb{R}$ with $g'(x) = 0$ will be a global maximum. If g is twice differentiable, then g is concave if $g''(x) \leq 0$ for all $x \in \mathbb{R}$, and g is strictly concave if if $g''(x) < 0$ for all $x \in \mathbb{R}$. If g is nondecreasing and concave, then for any $x, y \in \mathbb{R}$ with $0 < x \leq y$, it holds that $\frac{g(x)}{x} \geq \frac{g(y)}{y}$ and $g(x+z) - g(x) \geq g(y+z) - g(y)$ for any $z \in \mathbb{R}_+$.

Chapter 2

On solving mutual liability problems

2.1 Introduction

In this chapter, which is based on Groote Schaarsberg, Reijnierse, and Borm (2013), we make a side-step from the operations management context to a more financially oriented class of problems: we will consider so-called mutual liability problems as a generalization of bankruptcy problems.

The classical *bankruptcy problem*, consisting of a single estate that is not sufficient to fulfill the demands of multiple claimants, is formally introduced by O'Neill (1982). However, the problem has been known since centuries: the old Babylonian Talmud explains by means of numerical examples how one should divide a deceased's estate over his creditors. Also, in the first half of the twentieth century Benson (1935) and Kocourek (1935) studied estate division problems with multiple claimants from the perspective of law. In their work, the claimants possibly have different and circular priorities. It was Tyre (1980) that mentioned the missing link between mathematics and legal thought in such problems with so-called circular priorities.

Shortly later O'Neill (1982) considers claims problems without circularities from a mathematical perspective. A *bankruptcy rule* prescribes, for each bankruptcy problem, how to divide the estate over the claimants. In the literature one can find a wide variety of bankruptcy rules, which arise from both an axiomatic as well as a game-theoretic analysis, see for an overview Thomson (2003). One of the most renowned bankruptcy rules has been developed by Aumann and Maschler (1985). They have introduced a bankruptcy rule that explains the examples in the Talmud. Moreover, it is shown that the Aumann-Maschler (AM) rule is the unique rule satisfying two appealing properties: consistency and the Concede & Divide-principle.

The classical bankruptcy problem has been extended in different ways, *e.g.* to multi-issue allocation situations in which the estate has to be divided among a group of agents with claims stemming from different issues, see Calleja, Borm, and Hendrickx (2005), to stochastic bankruptcy games (Habis and Herings (2013)) and to allocating the losses due to financial distress within a business sector (Van Gulick (2010)). Lately, a main trending topic in this context is multiple estates. In a recent work by Bjorndal and Jornsten (2009) a bankruptcy problem with multiple banks (estates) is represented by a flow model. The banks can have separate claims on each other and there is a set of agents having separate claims on those banks. Palvolgyi, Peters, and Vermeulen (2010) consider the case of agents with non-homogeneous preferences over multiple estates. Here the agents have a single claim, but the utility per estate differs. The problem is analyzed from a non-cooperative perspective and focusses on how the agents should divide their claims into subclaims over the estates. Moulin and Sethuraman (2013) analyze bipartite rationing problems with multiple estates and agents with a single claim, but in which the agents are not necessarily compatible with all estates. These compatibilities are represented by a bipartite graph. By analyzing the flows in the graph and using a consistency axiom, bankruptcy rules are extended to this setting.

In this chapter we introduce mutual liability problems with multiple estates of a rather different nature. In financial accounting a *liability* is defined as "an obligation of an entity arising from past transactions or events, the settlement of which may result in the transfer or use of assets, provision of services or other yielding of economic benefits in the future."[1] Usually a liability is

[1] Loosely quoted from the framework of the *International Financial Reporting Standards Foundation*.

associated with an uncertainty, but this need not be the case. The more creditors an agent has, the higher the liabilities. We investigate the scenario where a group of agents is related by having mutual liabilities, but reaches the point in time where the agents want to cash their claims. None of the agents worry about the possible insufficient cash in the current assets, until, for some exogenous reason, individuals start cashing their claims. This will lead to a cascading effect and will reveal the possibly insufficient cash level of agents and therefore the agents typically might not obtain all of what they, however rightfully, claim.

This approach can be seen as a deterministic model of the monetary interrelationships between banks, governments and companies in case of a financial crisis and threatening bankruptcy of banks. Moreover mutual liabilities relate to the claims problems with circular priorities from *e.g.* Benson (1935) and Tyre (1980).
A *mutual liability problem* can be represented by a matrix in which an entry represents a claim from one agent on another agent. The diagonal entries represents the players' cash levels.

A special class of mutual liability problems is the class of *hierarchical mutual liability problems* in which the claim matrix is triangular. This implies that we can index the agents, such that every agent only claims from agents with a lower index. In this sense there is a hierarchy among the agents. For an example, think of the vertical relations in a supply-chain: insufficient cash of a buyer may lead to insufficient cash of his supplier(s).

In this chapter we analyze mutual liability problems from an allocation perspective: if in a mutual liability problem the agents reach the stage that they want to cash their claims and remove all current liabilities, how should the total amount of available cash be fairly distributed among the agents? In this setting, we implicitly assume that there is an independent authority charged with the task of fairly solving the mutual liability problem. A *mutual liability rule* prescribes for each mutual liability problem how to allocate the total cash among the agents. We assume each allocation to stem from a so-called bilateral *transfer scheme* that satisfies some basic requirements. More specifically, we consider *bankruptcy rule based transfer schemes* in which the incoming cash plus available cash of every agent is allocated among his claimants according to a specific bankruptcy rule. We show that for every bankruptcy rule there always exists a bankruptcy rule based transfer scheme, which is not necessarily unique. Interestingly, it is seen that each bankruptcy rule based transfer scheme leads to the same *bankruptcy rule based transfer allocation*, so allocation-wise a unique outcome is provided. For the subclass of hierarchical mutual liability problems, it is shown that there is also a unique bankruptcy rule based transfer scheme.

These results imply that each bankruptcy rule can be extended to a mutual liability rule: a *bankruptcy rule based mutual liability rule*. We provide an explicit characterization for the AM-based mutual liability rule, by extending the properties of consistency and the Concede & Divide-principle from the bankruptcy setting to the context of mutual liability problems.

Profiting from the special structure of hierarchical mutual liability problems, one can extend bankruptcy rules in an alternative recursive way into mutual liability rules. It is shown that for each bankruptcy rule the resulting allocation coincides with the allocation prescribed by the corresponding bankruptcy based mutual liability rule, thus providing another characterization of bankruptcy based mutual liability rules on the class of hierarchical mutual liability problems.

We conclude the chapter with a sketch of two alternative approaches to solve mutual liability problems. The first alternative involves reducing non-hierarchical problems into more tractable hierarchical mutual liability problems by bilaterally and cyclically leveling the claims. We see, however, that there is no straightforward procedure how to eliminate the cycles and that different procedures may result in different reduced problems. The second alternative is inspired by the hydraulic rationing methods for claims problems (Kaminski (2000)).

The structure of this chapter is as follows. In Section 2.2 we formally introduce mutual liability problems. Then, in Section 2.3 we give a short introduction to bankruptcy rules, define bankruptcy rule based transfer schemes and corresponding bankruptcy rule based transfer allocations. Section 2.4 studies mutual liability rules and in particular bankruptcy based mutual liability rules in a hierarchical setting, while Section 2.5 analyzes bankruptcy based mutual liability rules on the general class of mutual liability problems, including the characterization of the AM-based mutual liability rule. Section 2.6 concludes with two alternative ways to solve mutual liability problems.

2.2 Mutual liability problems and mutual liability rules

A classical bankruptcy problem involves an estate E that has to be divided among a finite group of agents N, all having a nonnegative claim d_i, $i \in N$, on the estate. We summarize these claims into a vector $d = (d_i)_{i \in N}$. The set of all bankruptcy problems (E, d) on N is denoted by $\mathcal{B}^N$.

In a mutual liability problem, a finite group of economic agents, denoted by N, have been interacting for a certain time period. Their past economic transactions have resulted in a situation in which the agents have claims on each other (think of debtors and creditors or accounts payable and receivable). As in bankruptcy problems, we assume that these claims are known, rightful and justifiable. Further, every agent has a certain nonnegative cash level or cash reserve with which he can (partially) pay his possible debtors. A *mutual liability problem* can be represented by a nonnegative matrix $C \in \mathbb{R}_+^{N \times N}$. Here each cell $c_{ij} \in C$ represents the claim of agent j on agent i, $i \neq j$, and c_{ii} represents the cash level of agent i. If

$$\sum_{i \in N} c_{ii} < \sum_{i,j \in N, i \neq j} c_{ij},$$

there is not sufficient cash to fulfill all the claims. If for some agent $i \in N$,

$$\sum_{j \in N} c_{ji} - \sum_{j \in N \setminus \{i\}} c_{ij} < 0,$$

agent i will never be able to satisfy all his claimants. We will, however, not impose any restrictions except nonnegativity on the matrix C beforehand.

The main question is how to divide $\sum_{i \in N} c_{ii}$ over the agents in N.

We denote by $\mathcal{L}^N$ the set of all mutual liability problems on N. A *mutual liability rule* $f : \mathcal{L}^N \to \mathbb{R}^N$ is such that $f(C) \geq 0$ and $\sum_{i \in N} f_i(C) = \sum_{i \in N} c_{ii}$ for all $C \in \mathcal{L}^N$.

We will distinguish a class of mutual liability problems with a special triangular structure. A mutual liability problem $C \in \mathcal{L}^N$ is called a *hierarchical mutual liability problem* if, by reordering the agents, C can be transformed into an upper triangular matrix with zeros below the diagonal. The set $\mathcal{L}^{N,\Delta}$ contains all hierarchical mutual liability problems on N. A mutual liability rule that is defined on the domain of hierarchical mutual liability problems is called a *hierarchical mutual liability rule*.

Example 2.2.1 Let $N = \{1, 2, 3\}$ and $C \in \mathcal{L}^N$ be given by

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & 1 & 4 \\ 2 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix} \end{array}.$$

The matrix should be interpreted in the following way. Agent 1 has a cash level of 3. He has a claim of 2 on agent 2 and a claim of 1 on agent 3, while agent 2 and 3 have a claim of 1 and 4 on agent 1. Agent 2 has a cash level of 2. He has no further claims, than the 1 on agent 1 we already mentioned, but agent 1 and 3 have a combined claim of 8 on him. This means in particular that agent 2 will never be able to pay off his debts. Agent 3 has a cash level of 1, agent 1 has a claim of 1 on his cash, while agent 3 has a claim of 4 on agent 1 and a claim of 6 on agent 2. ◁

Example 2.2.2 Let $N = \{1, 2, 3, 4\}$ and $C \in \mathcal{L}^N$ be given by

$$C = \begin{bmatrix} 4 & 2 & 4 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The claim matrix is upper triangular, since agent 1 only faces claims and has no claims on agents 2, 3 or 4. Furthermore, agent 2 has a claim on agent 1 but faces claims only from agents 3 and 4. Agent 3 has a claim on agent 1 and faces a claim of only agent 4, while agent 4 faces no claims at all, but he has a claim on all other three agents. ◁

Mutual liability problems can be seen as a generalization of bankruptcy problems. Each bankruptcy problem $(E, d) \in \mathcal{B}^N$ with $N = \{1, 2, \ldots, n\}$ corresponds to a hierarchical mutual liability problem $C(E, d) \in \mathcal{L}^{\bar{N}}$ with $\bar{N} = N \cup \{0\}$ given by

$$C(E,d) = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & n \end{array} \\ \begin{bmatrix} E & d_1 & \cdots & d_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}.$$

2.3 Bankruptcy rule based transfer schemes

Before elaborating on bankruptcy rule based transfer schemes, we provide some details on bankruptcy rules and the Aumann-Maschler rule in particular.

2.3.1 On the Aumann-Maschler rule

A *bankruptcy rule* $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ assigns to every bankruptcy problem $(E, d) \in \mathcal{B}^N$ a vector $\varphi(E, d) \in \mathbb{R}^N$, such that

$$\sum_{i \in N} \varphi_i(E, d) = \min\{E, \sum_{j \in N} d_j\}, \tag{2.1}$$

$0 \leq \varphi(E, d) \leq d$ and such that *monotonicity* is satisfied: for all $(E, d) \in \mathcal{B}^N$ and all $(E', d) \in \mathcal{B}^N$ with $E' \geq E$, we have $\varphi(E, d) \leq \varphi(E', d)$. Note that the class $\mathcal{B}^N$ also contains bankruptcy problems (E, d) in which E is sufficient to fulfill the claims d and in that case $\varphi(E, d) = d$ for any bankruptcy rule φ. Note that any bankruptcy rule is *continuous* in the estate (cf. Yeh (2008)):

for a sequence of nonnegative estates $E_1, E_2, \ldots$ that converges to E and for any nonnegative claim vector $d \in \mathbb{R}^N$, the sequence $\varphi(E_1, d), \varphi(E_2, d), \ldots$ converges to $\varphi(E, d)$.

For a detailed overview on bankruptcy rules we refer to Thomson (2003). Our focus will be mainly on the Aumann-Maschler rule (Aumann and Maschler (1985)), which is based on the constrained equal awards-rule.

The *constrained equal awards-rule* CEA is, for all $(E, d) \in \mathcal{B}^N$ and all $i \in N$, defined by

$$CEA_i(E, d) = \min\{\lambda, d_i\},$$

where $\lambda \in \mathbb{R}$ is such that $\sum_{i \in N} \min\{\lambda, d_i\} = \min\{E, \sum_{j \in N} d_j\}$.
The *Aumann-Maschler rule* AM is, for all $(E, d) \in \mathcal{B}^N$, defined by

$$AM(E, d) = \begin{cases} d & \text{if } \sum_{j \in N} d_j \leq E, \\ d - CEA(\sum_{j \in N} d_j - E, \frac{1}{2}d) & \text{if } E < \sum_{j \in N} d_j < 2E, \\ CEA(E, \frac{1}{2}d) & \text{if } \sum_{j \in N} d_j \geq 2E. \end{cases}$$

For bankruptcy problems involving two agents, AM satisfies the *Concede & Divide-principle* $C\&D$. This means that for $(E, d) \in \mathcal{B}^N$ with $N = \{1, 2\}$,

$$AM_1(E, d) = \begin{cases} (E - d_2)^+ + \frac{E - (E - d_1)^+ - (E - d_2)^+}{2} & \text{if } d_1 + d_2 \geq E, \\ d_1 & \text{if } d_1 + d_2 < E. \end{cases}$$

Here $(E - d_2)^+$ represents the part of the estate conceded to agent 1 by agent 2, while $\frac{E - (E - d_1)^+ - (E - d_2)^+}{2}$ indicates that the total amount of the estate that is not conceded, is divided equally.

So far, bankruptcy rules are defined on a fixed but arbitrary finite agent set N. Alternatively, bankruptcy rules can also be viewed as rules on the class $\mathcal{B}$ of bankruptcy problems with arbitrary but finite N. On the class $\mathcal{B}$, AM can be characterized by means of the $C\&D$-principle and the property of consistency.

Here, a bankruptcy rule φ on $\mathcal{B}$ is called *consistent* if for each finite agent set N, each $(E, d) \in \mathcal{B}^N$ and all $T \in 2^N \backslash \{\emptyset\}$, we have that

$$\varphi(E,d)_{|T} = \varphi(\sum_{j\in T}\varphi_j(E,d), d_{|T}).$$

Note that $(\sum_{j\in T}\varphi_j(E,d), d_{|T}) \in \mathcal{B}^T$.

Consistency of a rule requires that a possible reallocation of the total amount which has been allocated to a coalition T, on the basis of to the same bankruptcy rule, does not change the initial individual allocations within this coalition.

Example 2.3.1 Let $N = \{1,2,3\}$ and let $(E,d) \in \mathcal{B}^N$ be a bankruptcy problem with $E = 16$ and $d = (6,8,12)$.
We will allocate E according to the AM-rule. Since $\sum_{j\in N} d_j = 26 < 2E$, we have that

$$AM(E,d) = d - CEA(\sum_{j\in N} d_j - E, \frac{1}{2}d) = (6,8,12) - CEA(10,(3,4,6)).$$

Since $CEA(10,(3,4,6)) = (3,3\frac{1}{2},3\frac{1}{2})$, we find that $AM(E,d) = (6,8,12) - (3,3\frac{1}{2},3\frac{1}{2}) = (3,4\frac{1}{2},8\frac{1}{2})$.

To illustrate consistency: if we send agent 3 away with $AM_3(E,d) = 8\frac{1}{2}$ and let agents 1 and 2 reallocate the remaining amount $7\frac{1}{2}$ based on their claims of 6 and 8, we see that

$$\begin{aligned} AM(AM_1(E,d) + AM_2(E,d), d_{|N\setminus\{3\}}) &= AM(7\frac{1}{2},(6,8)) = (3,4\frac{1}{2}) \\ &= (AM_1(E,d), AM_2(E,d)). \end{aligned}$$ ◁

2.3.2 Towards transfer schemes

To devise mutual liability rules, we will explicitly consider bilateral monetary transfer schemes on which the allocations prescribed by the rule are based.

Let $C \in \mathcal{L}^N$. Then, the matrix $P = (p_{ij}) \in \mathbb{R}^{N\times N}$ is a *transfer scheme* for C, if

(i) for all $i \in N$, $p_{ii} = c_{ii}$,

(ii) for all $i, j \in N$ with $i \neq j$, $0 \leq p_{ij} \leq c_{ij}$,

(iii) for all $i \in N$, $\sum_{j \in N \setminus \{i\}} p_{ij} \leq p_{ii} + \sum_{j \in N \setminus \{i\}} p_{ji}$.

The interpretation is the following: p_{ij}, $i \neq j$, corresponds to the monetary transfer from agent i to j. For technical reasons and for computational convenience we require (i). The second condition states that the payment p_{ij} is nonnegative, but not higher than claim c_{ij} of agent j on i. The third condition requires that the sum of outgoing payments of i does not exceed his available cash plus incoming payments.

Let $\mathcal{P}(C)$ denote the set of all possible transfer schemes for the mutual liability problem $C \in \mathcal{L}^N$.

A transfer scheme directly leads to an allocation of the available cash. Let $C \in \mathcal{L}^N$ and let $P \in \mathcal{P}(C)$. Then, we define $\alpha^P \in \mathbb{R}^N$ as the *P-based transfer allocation*, *i.e.*, for all $i \in N$

$$\alpha_i^P = p_{ii} + \sum_{j \in N \setminus \{i\}} (p_{ji} - p_{ij}). \tag{2.2}$$

Note that because of (iii), $\alpha^P \geq 0$ and that

$$\begin{aligned}
\sum_{i \in N} \alpha_i^P &= \sum_{i \in N} \Big[p_{ii} + \sum_{j \in N \setminus \{i\}} (p_{ji} - p_{ij})\Big] \\
&= \sum_{i \in N} p_{ii} + \sum_{i \in N} \sum_{j \in N \setminus \{i\}} p_{ji} - \sum_{i \in N} \sum_{j \in N \setminus \{i\}} p_{ij} \\
&= \sum_{i \in N} p_{ii} = \sum_{i \in N} c_{ii}.
\end{aligned}$$

Example 2.3.2 Reconsider the mutual liability problem $C \in \mathcal{L}^N$ of Example 2.2.1 with $N = \{1, 2, 3\}$ and C given by

$$C = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix}.$$

An example of a transfer scheme for C is

$$P = \begin{bmatrix} 3 & 1 & 4 \\ 1.5 & 2 & 1.5 \\ 1 & 0 & 1 \end{bmatrix}.$$

The first two conditions (i) and (ii) can easily be checked. To verify condition (iii), observe that

$$\begin{aligned} p_{12} + p_{13} &= 5 \leq p_{11} + p_{21} + p_{31} = 5.5 \\ p_{21} + p_{23} &= 3 \leq p_{22} + p_{12} + p_{32} = 3 \\ p_{31} + p_{32} &= 1 \leq p_{33} + p_{13} + p_{23} = 6.5. \end{aligned}$$

Note that P leads to the P-based transfer allocation $\alpha^P = (0.5, 0, 5.5)$. $\triangleleft$

Next, we introduce a specific type of transfer schemes, called bankruptcy rule based transfer schemes.
Let $C \in \mathcal{L}^N$ and let φ be a bankruptcy rule. For all $i \in N$, define $d^i(C) \in \mathbb{R}^N$ by

$$d^i_j(C) = \begin{cases} c_{ij} & \text{if } j \neq i, \\ 0 & \text{if } j = i, \end{cases} \tag{2.3}$$

as the vector of claims on agent i. Then, $P = (p_{ij}) \in \mathbb{R}^{N \times N}$ is called a *φ-based transfer scheme* for C if,

(i) for all $i \in N$, $p_{ii} = c_{ii}$,

(ii) for all $i, j \in N$ with $i \neq j$,

$$p_{ij} = \varphi_j\Big(p_{ii} + \sum_{k \in N \setminus \{i\}} p_{ki}, d^i(C)\Big).$$

We denote by $\mathcal{P}^\varphi(C)$ the set of all φ-based transfer schemes.

Example 2.3.3 Let $N = \{1, 2, 3\}$ and consider the mutual liability problem $C \in \mathcal{L}^N$ given by

$$C = \begin{bmatrix} 2 & 100 & 0 \\ 100 & 4 & 12 \\ 0 & 0 & 0 \end{bmatrix}.$$

An AM-based transfer scheme is given by

$$P = \begin{bmatrix} 2 & 8 & 0 \\ 6 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

For this, observe, *e.g.*, that $d^2(C) = (100, 0, 12)$ and $p_{23} = AM_3(p_{22} + p_{12}, d^2(C)) = AM_3(12, (100, 0, 12)) = 6$. Furthermore $\alpha^P = (0, 0, 6)$.
One can check that also the matrix $\tilde{P}$ given by

$$\tilde{P} = \begin{bmatrix} 2 & 22 & 0 \\ 20 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

belongs to $\mathcal{P}^{AM}(C)$. Note that $\alpha^{\tilde{P}} = \alpha^P$. $\triangleleft$

The next lemma shows that a φ-based transfer scheme is indeed a transfer scheme.

Lemma 2.3.4 *Let $C \in \mathcal{L}^N$ and let φ be a bankruptcy rule. Then, $\mathcal{P}^{\varphi}(C) \subset \mathcal{P}(C)$.*

Proof: Take $P = (p_{ij}) \in \mathcal{P}^{\varphi}(C)$. It is sufficient to show that condition (ii) of a φ-based transfer scheme implies conditions (ii) and (iii) of a transfer scheme.
We start with showing (ii). Since φ is a bankruptcy rule, we have that for all $i, j \in N$ with $i \neq j$

$$0 \leq \varphi_j\Big(p_{ii} + \sum_{k \in N\setminus\{i\}} p_{ki}, d^i(C)\Big) \leq d^i_j(C) = c_{ij},$$

which implies that

$$0 \leq p_{ij} \leq c_{ij}.$$

Next we show condition (iii), using the basic properties of a bankruptcy rule. For all $i \in N$,

$$\begin{aligned} \sum_{j \in N\setminus\{i\}} p_{ij} &= \sum_{j \in N\setminus\{i\}} \varphi_j\Big(p_{ii} + \sum_{k \in N\setminus\{i\}} p_{ki}, d^i(C)\Big) \\ &= \sum_{j \in N} \varphi_j\Big(p_{ii} + \sum_{k \in N\setminus\{i\}} p_{ki}, d^i(C)\Big) \\ &\leq p_{ii} + \sum_{k \in N\setminus\{i\}} p_{ki}. \end{aligned}$$

$\square$

A φ-based transfer scheme P satisfies an attractive property: in the corresponding φ-based transfer allocation α^P an agent can only receive a positive amount if he paid off all his claimants.

Lemma 2.3.5 *Let $P \in \mathcal{P}^\varphi(C)$ for some $C \in \mathcal{L}^N$. Let $i \in N$. If $\alpha_i^P > 0$, then*

$$p_{ij} = c_{ij} \text{ for all } j \in N\backslash\{i\}.$$

Proof: Let $\alpha_i^P > 0$. Then by (2.2)

$$p_{ii} + \sum_{j\in N\backslash\{i\}} (p_{ji} - p_{ij}) > 0,$$

i.e.,

$$\sum_{j\in N\backslash\{i\}} p_{ij} < p_{ii} + \sum_{j\in N\backslash\{i\}} p_{ji}. \tag{2.4}$$

Moreover, since P is a φ-based transfer scheme, for all $j \in N\backslash\{i\}$

$$p_{ij} = \varphi_j(p_{ii} + \sum_{k\in N\backslash\{i\}} p_{ki}, d^i(C))$$

and consequently

$$\sum_{j\in N\backslash\{i\}} p_{ij} = \min\{p_{ii} + \sum_{k\in N\backslash\{i\}} p_{jk}, \sum_{j\in N\backslash\{i\}} c_{ij}\}. \tag{2.5}$$

Using (2.4) it must be that

$$\sum_{j\in N\backslash\{i\}} p_{ij} = \sum_{j\in N\backslash\{i\}} c_{ij}$$

and using (ii) of φ-based transfer schemes, it follows that $p_{ij} = c_{ij}$, for all $j \in N\backslash\{i\}$. □

The next theorem shows that one can always find at least one φ-based transfer scheme.

Theorem 2.3.6 *Let $C \in \mathcal{L}^N$ and let φ be a bankruptcy rule.*
Then, $\mathcal{P}^\varphi(C) \neq \emptyset$.

Proof: Using the following iterative procedure we construct a φ-based transfer scheme for C.
For all $i \in N$, set $d^i = d^i(C)$ and set $E^i(0) = c_{ii}$.
Then, recursively define, for all $i \in N$ and $k = 1, 2, \ldots$,

$$E^i(k+1) = c_{ii} + \sum_{j \in N\setminus\{i\}} \varphi_i(E^j(k), d^j). \tag{2.6}$$

Note that

$$E^i(1) = c_{ii} + \sum_{j \in N\setminus\{i\}} \varphi_i(c_{jj}, d^j) \geq c_{ii} = E^i(0).$$

Let $k \geq 1$ and assume that $E^i(k) \geq E^i(k-1)$. Then, by monotonicity of φ we find that

$$E^i(k+1) = c_{ii} + \sum_{j \in N\setminus\{i\}} \varphi_i(E^i(k), d^j) \geq c_{ii} + \sum_{j \in N\setminus\{i\}} \varphi_i(E^i(k-1), d^j) = E^i(k).$$

Hence, by induction, for all $i \in N$

$$E^i(0) \leq E^i(1) \leq E^i(2) \leq \ldots \tag{2.7}$$

Consider $P = (p_{ij}) \in \mathbb{R}^{N\times N}$, given by

$$p_{ij} = \begin{cases} c_{ii} & \text{for all } i, j \in N \text{ with } i = j, \\ \lim_{k\to\infty} \varphi_j\left(E^i(k), d^i\right) & \text{for all } i, j \in N \text{ with } i \neq j. \end{cases} \tag{2.8}$$

Note that the limit in (2.8) exists, because $\{E^i(k)\}_{k=0}^\infty$ is an increasing sequence, while φ is monotonic and bounded from above.

Moreover, condition (ii) of a φ-based transfer scheme is satisfied since for all $i, j \in N$ with $i \neq j$, we have that

$$\begin{aligned}
p_{ij} =& \lim_{k\to\infty} \varphi_j\left(E^i(k), d^i\right) \\
=& \varphi_j\left(\lim_{k\to\infty} E^i(k), d^i\right) \\
=& \varphi_j\left(c_{ii} + \lim_{k\to\infty} \sum_{\ell\in N\setminus\{i\}} \varphi_i(E^\ell(k), d^\ell), d^i\right) \\
=& \varphi_j\left(c_{ii} + \sum_{\ell\in N\setminus\{i\}} p_{\ell i}, d^i\right).
\end{aligned}$$

The second equality follows from continuity of φ, the third equality follows from (2.6) and the last equality follows from (2.8). □

2.4 Hierarchical mutual liability problems

As Example 2.3.3 shows, a general φ-based transfer scheme need not to be unique. For hierarchical mutual liability problems, however, there is a unique φ-based transfer scheme.

Theorem 2.4.1 *Let $C \in \mathcal{L}^{N,\Delta}$ and let φ be a bankruptcy rule. Then,* $|\mathcal{P}^\varphi(C)| = 1$.

Proof: Let $N = \{1, \ldots, n\}$. By the upper triangularity of C, we can assume, without loss of generality, that $c_{ij} = 0$ if $i > j$. Let $P = (p_{ij})$ and $\tilde{P} = (\tilde{p}_{ij})$ both be φ-based transfer schemes for C.

Clearly, if $i > j$,

$$p_{ij} = \tilde{p}_{ij} = 0. \tag{2.9}$$

And since

$$c_{11} + \sum_{j\in N\setminus\{1\}} p_{j1} = c_{11} + \sum_{j\in N\setminus\{1\}} \tilde{p}_{j1} = c_{11},$$

the fact that P and $\tilde{P}$ are φ-based transfer schemes implies for all $j \in N\backslash\{1\}$

$$p_{1j} = \tilde{p}_{1j} = \varphi_j(c_{11}, d^1(C)).$$

Now consider $i \in N$ an assume that for all $g \in \{1, \ldots, i-1\}$ and $h \in \{1, \ldots, n\}$,

$$p_{gh} = \tilde{p}_{gh}.$$

By equation (2.9),

$$\begin{aligned} c_{ii} + \sum_{g \in N\backslash\{i\}} p_{gi} &= c_{ii} + \sum_{g<i} p_{gi} \\ &= c_{ii} + \sum_{g<i} \tilde{p}_{gi} \\ &= c_{ii} + \sum_{g \in N\backslash\{i\}} \tilde{p}_{gi} \end{aligned}$$

and thus for all $j \in N\backslash\{i\}$

$$\begin{aligned} p_{ij} &= \varphi_j(c_{ii} + \sum_{g \in N\backslash\{i\}} p_{gi}, d^i(C)) \\ &= \varphi_j(c_{ii} + \sum_{g \in N\backslash\{i\}} \tilde{p}_{gi}, d^i(C)) = \tilde{p}_{ij}. \end{aligned}$$ □

This theorem implies that on $\mathcal{L}^{N,\Delta}$ a φ-based transfer allocation is uniquely defined for every bankruptcy rule φ. Hence, we can extend each bankruptcy rule to a hierarchical mutual liability rule.

Let φ be a bankruptcy rule. The corresponding *hierarchical φ-based mutual liability rule* $\rho^\varphi : \mathcal{L}^{N,\Delta} \to \mathbb{R}^N$ is for all $C \in \mathcal{L}^{N,\Delta}$ defined by

$$\rho^\varphi(C) = \alpha^P,$$

where P is the unique φ-based transfer scheme for C.

An alternative way of using a bankruptcy rule to solve hierarchical mutual liability problems, is the following recursive procedure that we first illustrate by means of an example.

Example 2.4.2 Let $N = \{1, \ldots, 4\}$ and consider $C \in \mathcal{L}^{N,\Delta}$, given by

$$C = \begin{bmatrix} 4 & 2 & 4 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

In the recursive procedure we start with agent 1, who has no claims on the other agents. His cash, $c_{11} = 4$, is divided on the basis of a bankruptcy problem with estate 4 and claims, 2, 4 and 4. Hence we treat this subproblem of the mutual liability problem as a bankruptcy problem $(4, (2, 4, 4))$. Selecting $\varphi = AM$ as an appropriate bankruptcy rule, $AM(4, (2, 4, 4)) = (1, 1.5, 1.5)$. Thus agent 2 gets 1 from agent 1's cash and agents 3 and 4 both receive 1.5. Correspondingly we can update our (partly) solved mutual liability problem into

$$C^1 = \begin{bmatrix} 4 - 1 - 1.5 - 1.5 & 0 & 0 & 0 \\ 0 & 3 + 1 & 0 & 1 \\ 0 & 0 & 2 + 1.5 & 3 \\ 0 & 0 & 0 & 2 + 1.5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 3.5 & 3 \\ 0 & 0 & 0 & 3.5 \end{bmatrix}.$$

In C^1 agent 2 has no claim on agent 1 anymore and we allocate $c^1_{22} = 4$ on the basis of the bankruptcy problem $(4, (0, 1))$. Since $AM(4, (0, 1)) = (0, 1)$, this means that 1 is transferred to agent 4 while agent 2 keeps an amount of 3. Updating leads to

$$C^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3.5 & 3 \\ 0 & 0 & 0 & 4.5 \end{bmatrix}.$$

In the next step an amount of 3 is transferred from 3 to 4, and updating gives

$$C^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 7.5 \end{bmatrix}.$$

The diagonal, i.e $(0, 3, 0.5, 7.5)$, of this matrix can be viewed as an allocation which solves this hierarchical mutual liability problem based on a recursive

application of the AM-rule.
Implicitly, we derived the following corresponding bilateral transfer scheme for C

$$P = \begin{bmatrix} 4 & 1 & 1.5 & 1.5 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Note that this is the AM-based transfer scheme with corresponding transfer allocation $\alpha^P = (0, 3, 0.5, 7.5)$. ◁

The formal definition of how to extend mutual liability rules in the recursive way as described in the previous example is provided below.

Let $C \in \mathcal{L}^{N,\Delta}$ and let φ be a bankruptcy rule. Set $N = \{1, \ldots, n\}$ and assume that $c_{ij} = 0$ for all $i, j \in N$ with $i > j$. Set $C^0 = C$. Recursively, for $j = 1, \ldots, n-1$, define $C^j \in \mathcal{L}^{N,\Delta}$ by

$$c_{ii}^j = \begin{cases} c_{ii}^{j-1} & \text{if } i < j \\ c_{ii}^{j-1} + \varphi_i\big(c_{jj}^{j-1}, (c_{jk})_{k\in\{j+1,\ldots,n\}}\big) & \text{if } i > j \end{cases}$$

$$c_{jj}^j = c_{jj}^{j-1} - \sum_{k>j} \varphi\big(c_{jj}^{j-1}, (c_{jk})_{k\in\{j+1,\ldots,n\}}\big) \tag{2.10}$$

$$c_{ik}^j = \begin{cases} 0 & \text{if } i = j \text{ and } k \neq i \\ c_{ik}^{j-1} & \text{if } i \neq j \text{ and } k \neq i. \end{cases}$$

Finally set

$$C^{rec} = C^{n-1}.$$

Correspondingly, the hierarchical *recursive φ-based mutual liability rule* $\xi^\varphi : \mathcal{L}^{N,\Delta} \to \mathbb{R}^N$ is defined by

$$\xi^\varphi(C) = \text{diag}(C^{rec})$$

for each $C \in \mathcal{L}^{N,\Delta}$.

Interestingly, for every bankruptcy rule φ, the recursive φ-based mutual liability rule ξ^φ and the hierarchical φ-based mutual liability rule ρ^φ coincide.

Theorem 2.4.3 *For all bankruptcy rules φ,*

$$\rho^\varphi = \xi^\varphi.$$

Proof: Let $C \in \mathcal{L}^{N,\Delta}$ and let φ be a bankruptcy rule. Set $N = \{1, \ldots, n\}$ and assume that $c_{ij} = 0$ for all $i, j \in N$ with $i > j$. Let $P = (p_{ij})$ be the unique φ-based transfer scheme for C. Then, we have that for all $i \in N$

$$\begin{aligned}\rho_i^\varphi(C) = \alpha_i^P &= c_{ii} + \sum_{j \in N \setminus \{i\}} p_{ji} - \sum_{j \in N \setminus \{i\}} p_{ij} \\ &= c_{ii} + \sum_{j=1}^{i-1} p_{ji} - \sum_{j=i+1}^{n} p_{ij}.\end{aligned}$$

Moreover, for all $i \in N$, $\xi_i^\varphi(C) = c_{ii}^i$, where c_{ii}^i is determined recursively using (2.10). Thus it is sufficient to show that for all $i \in N$

$$c_{ii}^i = c_{ii} + \sum_{j=1}^{i-1} p_{ji} - \sum_{j=i+1}^{n} p_{ij}. \tag{2.11}$$

For $i = 1$, (2.11) is satisfied since

$$\begin{aligned}c_{11}^1 &= c_{11} - \sum_{j=2}^{n} \varphi_j\big(c_{11}, (c_{1k})_{k \in \{2,\ldots,n\}}\big) \\ &= c_{11} - \sum_{j=2}^{n} \varphi_j\big(c_{11} + \sum_{k \in N \setminus \{1\}} p_{k1}, d^1(C)\big) \\ &= c_{11} - \sum_{j=2}^{n} p_{1j}.\end{aligned}$$

The first equality follows from (2.10), the second equality holds because $p_{k1} = 0$ for all $k \in N \setminus \{1\}$ and the last equality follows from condition (ii) of φ-based transfer schemes.

Note that, for all $j \in N \backslash \{1\}$

$$
\begin{aligned}
c^1_{jj} &= c_{jj} + \varphi_j\big(c_{11}, (c_{1k})_{k \in \{2,\ldots,n\}}\big) \\
&= c_{jj} + p_{1j}.
\end{aligned}
$$

The proof continues by means of induction. Let $i \leq n-1$ and assume that

$$
c^{i-1}_{jj} = \begin{cases} c_{jj} + \sum_{k=1}^{i-1} p_{kj} & \text{if } j > i-1 \\ c_{jj} + \sum_{k=1}^{j-1} p_{kj} - \sum_{k=j+1}^{n} p_{jk} & \text{if } j \leq i-1. \end{cases}
$$

We will prove that

$$
c^{i}_{jj} = \begin{cases} c_{jj} + \sum_{k=1}^{i} p_{kj} & \text{if } j > i \\ c_{jj} + \sum_{k=1}^{i-1} p_{kj} - \sum_{k=j+1}^{n} p_{jk} & \text{if } j = i. \end{cases}
$$

For $j \in \{i+1, \ldots, n\}$, we have that

$$
\begin{aligned}
c^i_{jj} &= c^{i-1}_{jj} + \varphi_j\big(c^{i-1}_{ii}, (c_{ik})_{k \in \{i+1,\ldots,n\}}\big) \\
&= c_{jj} + \sum_{k=1}^{i-1} p_{kj} + \varphi_j\Big(c_{ii} + \sum_{k=1}^{i-1} p_{ki}, (c_{ik})_{k \in \{i+1,\ldots,n\}}\Big) \\
&= c_{jj} + \sum_{k=1}^{i-1} p_{kj} + \varphi_j\Big(c_{ii} + \sum_{k=1}^{i-1} p_{ki}, d^i(C)\Big) \\
&= c_{jj} + \sum_{k=1}^{i-1} p_{kj} + p_{ij} \\
&= c_{jj} + \sum_{k=1}^{i} p_{kj},
\end{aligned}
$$

where the first equality follows from the definition of ξ and the second equality is based on the induction assumption. Similarly one finds

$$
\begin{aligned}
c^i_{ii} &= c^{i-1}_{ii} - \sum_{k=i+1}^{n} \varphi\big(c^{i-1}_{ii}, (c_{ik})_{k \in \{i+1,\ldots,n\}}\big) \\
&= c_{ii} + \sum_{k=1}^{i-1} p_{ki} - \sum_{k=i+1}^{n} \varphi\Big(c_{ii} + \sum_{k=1}^{i-1} p_{ki}, d^i(C)\Big) \\
&= c_{ii} + \sum_{k=1}^{i-1} p_{ki} - \sum_{k=i+1}^{n} p_{ik}.
\end{aligned}
$$

□

2.5 General mutual liability problems

As seen in Example 2.3.3, the *AM* bankruptcy rule allows for multiple *AM*-based transfer schemes for a non-hierarchical mutual liability problem. For an arbitrary bankruptcy rule, however, there is always a unique φ-based transfer allocation.

Theorem 2.5.1 *Let $C \in \mathcal{L}^N$, let φ be a bankruptcy rule and let $P, \tilde{P} \in \mathcal{P}^\varphi(C)$. Then,*

$$\alpha^P = \alpha^{\tilde{P}}.$$

Proof: On the contrary suppose that $\alpha^P \neq \alpha^{\tilde{P}}$. For notational convenience, set $\alpha^P = \alpha$ and $\alpha^{\tilde{P}} = \tilde{\alpha}$.
Let $N = \{1, \dots, n\}$. Without loss of generality we assume that, $\alpha_1 < \tilde{\alpha}_1$.
Since $0 \leq \alpha_1 < \tilde{\alpha}_1$, Lemma 2.3.5 implies that, for all $j \in N \backslash \{1\}$

$$\tilde{p}_{1j} = c_{1j}. \tag{2.12}$$

Since

$$\begin{aligned} \tilde{\alpha}_1 &= \tilde{p}_{11} + \sum_{j \in N \backslash \{1\}} (\tilde{p}_{j1} - \tilde{p}_{1j}) \\ &= c_{11} + \sum_{j \in N \backslash \{1\}} (\tilde{p}_{j1} - \tilde{p}_{1j}) > c_{11} + \sum_{j \in N \backslash \{1\}} (p_{j1} - p_{1j}) = \alpha_1, \end{aligned}$$

there must be an agent $j \in N \backslash \{1\}$ for which

$$\tilde{p}_{j1} - \tilde{p}_{1j} > p_{j1} - p_{1j}.$$

Therefore, by (2.12),

$$\tilde{p}_{j1} - p_{1j} \geq \tilde{p}_{j1} - c_{1j} = \tilde{p}_{j1} - \tilde{p}_{1j} > p_{j1} - p_{1j}$$

and hence

$$\tilde{p}_{j1} > p_{j1}.$$

Without loss of generality we assume that $j = 2$.

Note that $p_{21} < \tilde{p}_{21} \leq c_{21}$. Thus, by Lemma 2.3.5, $\alpha_2 = 0$ and therefore $\alpha_1 + \alpha_2 < \tilde{\alpha}_1 + \tilde{\alpha}_2$, *i.e.*,

$$\begin{aligned} & c_{11} + c_{22} + \sum_{j \in N\setminus\{1\}} (p_{j1} - p_{1j}) + \sum_{j \in N\setminus\{2\}} (p_{j2} - p_{2j}) \\ < & c_{11} + c_{22} + \sum_{j \in N\setminus\{1\}} (\tilde{p}_{j1} - \tilde{p}_{1j}) + \sum_{j \in N\setminus\{2\}} (\tilde{p}_{j2} - \tilde{p}_{2j}) \end{aligned}$$

and consequently

$$\begin{aligned} & \sum_{j \in N\setminus\{1,2\}} (p_{j1} - p_{1j}) + \sum_{j \in N\setminus\{1,2\}} (p_{j2} - p_{2j}) \\ < & \sum_{j \in N\setminus\{1,2\}} (\tilde{p}_{j1} - \tilde{p}_{1j}) + \sum_{j \in N\setminus\{1,2\}} (\tilde{p}_{j2} - \tilde{p}_{2j}). \end{aligned}$$

Thus there must be an agent $\ell \in N\setminus\{1, 2\}$ with

$$\tilde{p}_{\ell 1} - \tilde{p}_{1\ell} > p_{\ell 1} - p_{1\ell}$$

or

$$\tilde{p}_{\ell 2} - \tilde{p}_{2\ell} > p_{\ell 2} - p_{2\ell}.$$

Without loss of generality we assume that $\ell = 3$ and that $\tilde{p}_{31} - \tilde{p}_{13} > p_{31} - p_{13}$. Then,

$$\tilde{p}_{31} - p_{13} \geq \tilde{p}_{31} - c_{13} = \tilde{p}_{31} - \tilde{p}_{13} > p_{31} - p_{13}.$$

Thus we conclude that $\tilde{p}_{31} > p_{31}$ and, using Lemma 2.3.5, that $\alpha_3 = 0$ and $\alpha_1 + \alpha_2 + \alpha_3 < \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$.

We can continue with this reasoning with respect to agent $4, 5, \ldots, n$. As a result we will find that $\alpha_1 + \alpha_2 + \ldots + \alpha_n < \tilde{\alpha}_1 + \tilde{\alpha}_2 + \ldots + \tilde{\alpha}_n$, which is not possible because of efficiency of a φ-based transfer allocation. □

Hence, we can introduce φ-based rules for general mutual liability problems. Let φ be a bankruptcy rule. The corresponding *φ-based mutual liability rule* $\rho^\varphi : \mathcal{L}^N \to \mathbb{R}^N$ is for all $C \in \mathcal{L}^N$ defined by

$$\rho^\varphi(C) = \alpha^P,$$

where P is a φ-based transfer scheme for C.

The final part of this section will provide an axiomatic characterization of ρ^{AM} as a φ-based mutual liability rule on the class $\mathcal{L}$ of all mutual liability problems with an arbitrary but finite set of players by extending the $C\&D$-principle and the property consistency for bankruptcy rules to general mutual liability rules.

In bankruptcy problems the principle of Concede & Divide is defined for problems with two claimants. However, in a mutual liability problem with two agents, every agent faces only one (possible) claimant. For such mutual liability problems the allocation prescribed by any φ-based mutual liability rule is unique. This is shown in the following lemma.

Lemma 2.5.2 *Let $\mathcal{C} \in \mathcal{L}^N$ with $N = \{1, 2\}$. Let φ^I and φ^{II} be bankruptcy rules. Then,*

$$\rho^{\varphi^I}(C) = \rho^{\varphi^{II}}(C).$$

Proof: Note that for all $i \in N$, $d^i(C)$ has at most one positive claim. Take $P = (p_{ij}) \in \mathcal{P}^{\varphi^I}(C)$. Then, with $N = \{i, j\}$,

$$\begin{aligned} p_{ij} &= \rho_j^{\varphi^I}(p_{ii} + p_{ji}, d^i(C)) \\ &= \rho_j^{\varphi^{II}}(p_{ii} + p_{ji}, d^i(C)). \end{aligned}$$

Hence, $P \in \mathcal{P}^{\varphi^{II}}(C)$ and thus $\rho^{\varphi^I}(C) = \rho^{\varphi^{II}}(C)$. □

Instead, we will define a Concede & Divide principle for mutual liability problems with three agents, in which every agent has two (possible) claimants. A mutual liability rule f satisfies the *Concede & Divide-principle* ($C\&D$) if for each N with $|N| = 3$ and for for each $C \in \mathcal{L}^N$, there exists an underlying transfer scheme $P \in \mathcal{P}(C)$ such that $f(C) = \alpha^P$ and for each player $i \in N$, his 'estate' $e^i = c_{ii} + \sum_{\ell \neq i} p_{\ell i}$ is allocated among the remaining two players, j, k, respecting the bankruptcy Concede & Divide-principle, *i.e.*,

$$p_{ij} = \begin{cases} c_{ij} & \text{if } e^i \geq c_{ij} + c_{ik}, \\ (e^i - c_{ik})^+ + \frac{e^i - (e^i - c_{ik})^+ - (e^i - c_{ij})^+}{2} & \text{otherwise.} \end{cases} \tag{2.13}$$

Example 2.5.3 Reconsider the mutual liability problem $C \in \mathcal{L}^N$ of Example 2.2.1 with $N = \{1, 2, 3\}$ and C given by

$$C = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix}.$$

Take $P \in \mathcal{P}^{AM}(C)$ given by

$$P = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

with $\rho^{AM}(C) = \alpha^P = (0, 0, 6)$. We check that the entries in P satisfy (2.13). Here, $e^1 = p_{11} + p_{21} + p_{31} = 5$, $e^2 = 3$ and $e^3 = 7$. Both player 1's and player 3's estate are sufficient to satisfy their claimants, hence $p_{12} = c_{12} = 1$, $p_{13} = 4$ and $p_{31} = 1$. Player 2's estate is not sufficient, therefore

$$p_{21} = (e^2 - c_{23})^+ + \frac{e^2 - (e^2 - c_{21})^+ - (e^2 - c_{23})^+}{2} = 0 + \frac{3 - 1 - 0}{2} = 1$$

and $p_{23} = 2$. ◁

Next, we define the property of consistency for a mutual liability rule. This property is defined on the class $\mathcal{L}$ of mutual liability problems with arbitrary but finite N. The consistency property requires that a reallocation of the total amount which has been allocated to a coalition T, on the basis of that rule and an underlying transfer scheme, does not change the initial individual allocations within this coalition. A mutual liability rule f for $\mathcal{L}$ is called *consistent* if for all N and for all $C \in \mathcal{L}^N$ there exists a $P \in \mathcal{P}(C)$ such that $f(C) = \alpha^P$ and such that for all $T \in 2^N \backslash \{\emptyset\}$ with $C^{T,P} \in \mathcal{L}^T$,

$$f(C^{T,P}) = f(C)_{|T}, \tag{2.14}$$

where $C^{T,P} \in \mathbb{R}^{T \times T}$ is defined, for all $i, j \in T$, by

$$c_{ij}^{T,P} = \begin{cases} c_{ij} & \text{if } i \neq j, \\ c_{ii} + \sum_{k \in N \backslash T} (p_{ki} - p_{ik}) & \text{if } i = j. \end{cases} \tag{2.15}$$

Note that there is only a consistency requirement for T *if* $C^{T,P} \in \mathcal{L}^T$. As is seen in the following example, it can indeed happen that $C^{T,P} \notin \mathcal{L}^T$.

Example 2.5.4 Let $N = \{1, 2, 3, 4\}$. Reconsider the hierarchical mutual liability problem $C \in \mathcal{L}^N$ of Example 2.2.2, given by

$$C = \begin{bmatrix} 4 & 2 & 4 & 4 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The unique AM-based transfer scheme P for C is given by

$$P = \begin{bmatrix} 4 & 1 & 1.5 & 1.5 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and $\rho^{AM} = (0, 3, 0.5, 7.5)$.
With $T = \{1, 2, 4\}$ we have

$$C^{T,P} = \begin{bmatrix} 2.5 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix},$$

which is a mutual liability problem and the unique AM-based transfer scheme P^T for $C^{T,P}$ is given by

$$P^T = \begin{bmatrix} 2.5 & 1 & 1.5 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix},$$

while $\rho^{AM}(C^{T,P}) = (0, 3, 7.5)$. We see that the consistency requirement for this T is satisfied. However, with $T = \{1, 2, 3\}$, we obtain

$$C^{T,P} = \begin{bmatrix} 2.5 & 2 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

which is not a mutual liability problem and therefore does not impose a consistency requirement. ◁

The AM-based mutual liability rule satisfies consistency and $C\&D$.

Theorem 2.5.5 *ρ^{AM} is consistent and satisfies $C\&D$.*

Proof: We start with proving $C\&D$. Let $C \in \mathcal{L}^N$ with $|N| = 3$. Let $i \in N$ and set $N\backslash\{i\} = \{j,k\}$. Consider an arbitrary $P \in \mathcal{P}^{AM}(C)$. Obviously $\rho^{AM}(C) = \alpha^P$ by Theorem 2.5.1. Moreover,

$$\begin{aligned} p_{ij} &= AM_j(p_{ii} + p_{ji} + p_{ki}, d^i(C)) \\ &= AM_j(e^i, (c_{ij}, c_{ik})). \end{aligned}$$

Since the bankruptcy rule AM satisfies the $C\&D$ principle for bankruptcy problems, we find that

$$p_{ij} = \begin{cases} c_{ij} & \text{if } e^i \geq c_{ij} + c_{ik}, \\ (e^i - c_{ik})^+ + \frac{e^i - (e^i - c_{ik})^+ - (e^i - c_{ij})^+}{2} & \text{otherwise.} \end{cases}$$

Next, we show consistency. For this, let $C \in \mathcal{L}^N$, consider an arbitrary $P \in \mathcal{P}^{AM}(C)$ and let $T \in 2^N\backslash\{\emptyset\}$ be such that $C^{T,P} \in \mathcal{L}^T$. It suffices to show that $\rho^{AM}(C)_{|T} = \rho^{AM}(C^{T,P})$.

Define $P^T = (p^T_{ij}) \in \mathbb{R}^{T\times T}$ by

$$p^T_{ij} = \begin{cases} p_{ij} & \text{if } i \neq j \\ p_{ii} + \sum_{k\in N\backslash T}(p_{ki} - p_{ik}) & \text{if } i = j. \end{cases} \tag{2.16}$$

We first show that $P^T \in \mathcal{P}^{AM}(C^{T,P})$, which implies that $\alpha^{P^T} = \rho^{AM}(C^{T,P})$.

For this, note that $c^{T,P}_{ii} = p^T_{ii}$ for all $i \in T$. It remains to prove that for all $i \in T$ and $j \in T\backslash\{i\}$,

$$p^T_{ij} = AM_j\left(p^T_{ii} + \sum_{k\in T\backslash\{i\}} p^T_{ki}, d^i(C^{T,P})\right).$$

This is true because for each $i \in T$ and $j \in T\backslash\{i\}$

$$\begin{aligned} p^T_{ij} = p_{ij} = \;& AM_j\left(p_{ii} + \textstyle\sum_{k\in N\backslash\{i\}} p_{ki}, d^i(C)\right) \\ =\;& AM_j\Big(p_{ii} + \textstyle\sum_{k\in N\backslash\{i\}} p_{ki} \\ &\quad - \textstyle\sum_{k\in N\backslash T} AM_k\left(p_{ii} + \sum_{k\in N\backslash\{i\}} p_{ki}, d^i(C)\right), d^i(C)_{|T}\Big) \\ =\;& AM_j\left(p_{ii} + \textstyle\sum_{k\in N\backslash\{i\}} p_{ki} - \sum_{k\in N\backslash T} p_{ik}, d^i(C)_{|T}\right) \\ =\;& AM_j\left(p_{ii} + \textstyle\sum_{k\in N\backslash T}(p_{ki} - p_{ik}) + \sum_{k\in T\backslash\{i\}} p_{ki}, d^i(C)_{|T}\right) \\ =\;& AM_j\left(p^T_{ii} + \textstyle\sum_{k\in T\backslash\{i\}} p^T_{ki}, d^i(C^{T,P})\right), \end{aligned}$$

where the third equality follows from consistency of AM, the fourth equality follows from the fact that $P \in \mathcal{P}^{AM}(C)$, while the last equality follows from (2.16).

The proof is finished if we show that $\alpha^{P^T} = \rho^{AM}(C)_{|T}$. For this, note that with $i \in T$

$$\begin{aligned}
\alpha_i^{P^T} &= p_{ii}^T + \sum_{j \in T \setminus \{i\}} (p_{ji}^T - p_{ij}^T) \\
&= p_{ii} + \sum_{j \in N \setminus T} (p_{ji} - p_{ij}) + \sum_{j \in T \setminus \{i\}} (p_{ji} - p_{ij}) \\
&= p_{ii} + \sum_{j \in N \setminus \{i\}} (p_{ji} - p_{ij}) \\
&= \alpha_i^P = \rho_i^{AM}(C).
\end{aligned}$$

□

We conclude this section with a characterization of the AM-based mutual liability rule.

Theorem 2.5.6 *Let φ be a bankruptcy rule. Then, $\rho^\varphi = \rho^{AM}$ if and only if ρ^φ satisfies consistency and $C\&D$.*

Proof: For the "only if"-part, we refer to Theorem 2.5.5. To prove the "if"-part, let φ be a bankruptcy rule such that ρ^φ satisfies consistency and $C\&D$. As we have seen before, the class $\mathcal{B}$ of bankruptcy problems is a subclass of $\mathcal{L}$ by identifying each $(E, d) \in \mathcal{B}^N$ with $N = \{1, \ldots, n\}$, with $C(E, d) \in \mathcal{L}^{N \cup \{0\}, \Delta}$ given by

$$C(E,d) = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ n \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & n \end{array} \\ \left[\begin{array}{cccc} E & d_1 & \cdots & d_n \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{array}\right] \end{array}.$$

Let P be the unique φ-based transfer scheme for $C(E,d)$. Then,

$$P = \begin{bmatrix} E & p_{01} & \cdots & p_{0n} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{bmatrix},$$

with

$$\alpha_i^P = \begin{cases} p_{0i} & \text{if } i \in N, \\ E - \sum_{j \in N} p_{0j} & \text{if } i = 0. \end{cases}$$

Moreover, for all $i \in N$

$$\rho_i^\varphi(C(E,d)) = \alpha_i^P = p_{0i} = \varphi(c_{00}, d^0(C)) = \varphi_i(E,d). \tag{2.17}$$

Thus $\varphi(E,d) = \rho^\varphi(C(E,d))_{|N}$.

If we can show that

(i) $C\&D$ of ρ^φ on $\mathcal{L}$ implies $C\&D$ of φ on $\mathcal{B}$,

(ii) consistency of ρ^φ on $\mathcal{L}$ implies consistency of φ on $\mathcal{B}$,

then, $\varphi = AM$ (cf. Aumann and Maschler (1985)) and consequently $\rho^\varphi = \rho^{AM}$.

For this, we first show that P is the unique transfer scheme for $C(E,d)$ that leads to the transfer allocation α^P and for this reason $C\&D$ and consistency of ρ^φ can only have implications on P.

Let $\tilde{P} = (\tilde{p}_{ij}) \in \mathcal{P}(C(E,d))$ be an arbitrary transfer scheme for $C(E,d)$ with $\tilde{P} \neq P$. Then,

$$\tilde{P} = \begin{bmatrix} E & \tilde{p}_{01} & \cdots & \tilde{p}_{0n} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{bmatrix}$$

and there must be a player $i \in N$ with $\tilde{p}_{0i} \neq p_{0i}$. Hence, $\alpha^{\tilde{P}} \neq \alpha^P$.

With respect to (i), let $N = \{1,2\}$ and $(E,d) \in \mathcal{B}^N$. Let $i \in N$ and $\{j\} = N \backslash \{i\}$. We need to show that

$$\varphi_i(E,d) = \begin{cases} d_i & \text{if } E \geq d_1 + d_2, \\ (E-d_j)^+ + \frac{E-(E-d_i)^+-(E-d_j)^+}{2} & \text{otherwise.} \end{cases}$$

$C\&D$ on $\mathcal{L}$ and (2.17) imply that $\varphi_i(E,d) = \rho_i^{\varphi}(C(E,d))$ and, with $c_{0i} = C_{0i}(E,d)$ and $c_{0j} = C_{0j}(E,d)$, that

$$\begin{aligned} \rho_i^{\varphi}(C(E,d)) &= \begin{cases} c_{0i} & \text{if } e^0 \geq c_{01} + c_{02}, \\ (e^0 - c_{0j})^+ + \frac{e^0-(e^0-c_{0i})^+-(e^0-c_{0j})^+}{2} & \text{otherwise,} \end{cases} \\ &= \begin{cases} d_i & \text{if } E \geq d_1 + d_2, \\ (E-d_j)^+ + \frac{E-(E-d_i)^+-(E-d_j)^+}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

With respect to (ii), let $(E,d) \in \mathcal{B}^N$ and $T \in 2^N \backslash \{\emptyset\}$. We have to prove that

$$\varphi(E,d)_{|T} = \varphi(\sum_{j \in T} \varphi_j(E,d), d_{|T}).$$

Let $T = \{k_1, \ldots, k_t\}$. Then, using (2.15) and (2.17),

$$C^{T\cup\{0\},P}(E,d) = \begin{array}{c} \\ 0 \\ k_1 \\ \vdots \\ k_t \end{array} \begin{array}{c} \begin{array}{cccc} 0 & k_1 & \cdots & k_t \end{array} \\ \left[\begin{array}{cccc} E - \sum_{j \in N\backslash T} \varphi_j(E,d) & d_{k_1} & \cdots & d_{k_t} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{array} \right] \end{array}.$$

Clearly, $C^{T\cup\{0\},P}(E,d) \in \mathcal{L}^{T\cup\{0\},\Delta}$ and

$$C^{T\cup\{0\},P}(E,d) = C(E - \sum_{j \in N\backslash T} \varphi_j(E,d), d_{|T}).$$

Using consistency, we find for all $i \in T$ that

$$\rho_i^{\varphi}(C(E,d))_{|T\cup\{0\}} = \rho_i^{\varphi}(C(E - \sum_{j \in N\backslash T} \varphi_j(E,d), d_{|T})).$$

By equation (2.17), for all $i \in T$

$$\begin{cases} \rho_i^{\varphi}(C(E,d)) & = \varphi_i(E,d) \\ \rho_i^{\varphi}(C(E - \sum_{j\in N\setminus T} \varphi_j(E,d), d_{|T})) & = \varphi_i(E - \sum_{j\in N\setminus T} \varphi_j(E,d), d_{|T}) \end{cases}$$

and therefore,

$$\varphi((E,d))_{|T} = \varphi(E - \sum_{j\in N\setminus T} \varphi_j(E,d), d_{|T}) = \varphi(\sum_{j\in T} \varphi_j(E,d), d_{|T}),$$

where the last equality follows from (2.1). □

2.6 Alternative approaches

In this section we discuss two alternative approaches to analyze mutual liability problems: a reduction approach and a hydraulic approach.

2.6.1 Reduction approach

In the reduction approach a general mutual liability problem is reduced to a more tractable hierarchical mutual liability problem. The main difference between hierarchical and non-hierarchical mutual liability problems is the (non-)existence of cycles of claims.

In this section we show that, by eliminating these cycles, it is possible to reduce a general mutual liability problem to a hierarchical mutual liability problem, but that such a reduction is not possible without changing the nature of the mutual liability problem. There are choices to be made. Different reduction choices can result in different reduced hierarchical mutual liability problems.

The possibilities regarding reduction steps and the subsequent effects will be illustrated in the following example.

Example 2.6.1 Let $N = \{1, 2, 3, 4\}$ and let $C \in \mathcal{L}^N$ be given by

$$C = \begin{bmatrix} 4 & 5 & 8 & 7 \\ 1 & 8 & 3 & 12 \\ 9 & 6 & 6 & 2 \\ 1 & 1 & 5 & 7 \end{bmatrix},$$

with $\rho^{AM}(C) = (0, 2\frac{1}{3}, 3\frac{1}{3}, 19\frac{1}{3})$.

A natural first step in reducing a general mutual liability problem is to assume that on a bilateral level the claims are already settled. Thus for all pairs $i, j \in N$ with $i \neq j$, $c_{ij}c_{ji} = 0$. The bilaterally leveled claim matrix $\bar{C} = (\bar{c}_{ij}) \in \mathcal{L}^N$ is obtained from C in the following way

$$\bar{c}_{ij} = \begin{cases} [c_{ij} - c_{ji}]^+ & \text{if } j \neq i \\ c_{ii} & \text{if } j = i. \end{cases}$$

Thus, we eliminate cycles of length 2 and obtain

$$\bar{C} = \begin{bmatrix} 4 & 4 & 0 & 6 \\ 0 & 8 & 0 & 11 \\ 1 & 3 & 6 & 0 \\ 0 & 0 & 3 & 7 \end{bmatrix},$$

which is still a non-hierarchical mutual liability problem.

Not only can we level claims bilaterally, we can also do this for longer cycles. In the matrix $\bar{C}$ we can find multiple cycles of claims. The longest one, with length 4, goes from player 1 to player 2, then from player 2 to player 4, from player 4 to player 3 and from player 3 back to player 1, see the bold entries in $\bar{C}$ below:

$$\bar{C} = \begin{bmatrix} 4 & \mathbf{4} & 0 & 6 \\ 0 & 8 & 0 & \mathbf{11} \\ \mathbf{1} & 3 & 6 & 0 \\ 0 & 0 & \mathbf{3} & 7 \end{bmatrix}.$$

Since the lowest claim in this cycle is 1 ($\bar{c}_{31} = 1$), we can reduce the cycle by 1, which results in the following non-hierarchical mutual liability problem

C^1:

$$C^1 = \begin{bmatrix} 4 & 3 & 0 & 6 \\ 0 & 8 & 0 & \mathbf{10} \\ 0 & \mathbf{3} & 6 & 0 \\ 0 & 0 & \mathbf{2} & 7 \end{bmatrix}.$$

In C^1 we detect another cycle: from 2 to 4, to 3 and back to 2. We can reduce the claims by an amount of 2, with the hierarchical mutual liability problem $C^{1,\Delta}$ as a result. Here,

$$C^{1,\Delta} = \begin{bmatrix} 4 & 3 & 0 & 6 \\ 0 & 8 & 0 & 8 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

is a hierarchical mutual liability problem; if we rearrange the rows and columns in the order $(1, 3, 2, 4)$, the matrix is upper triangular. We have that $\rho^{AM}(C^{1,\Delta}) = (0, 2.5, 5, 17.5) \neq \rho^{AM}(C)$.

In the mutual liability problem $\bar{C}$, we can also start with the cycle: from 2 to 4, to 3 and back to 2 as shown by the bold entries in $\bar{C}$ below:

$$\bar{C} = \begin{bmatrix} 4 & 4 & 0 & 6 \\ 0 & 8 & 0 & \mathbf{11} \\ 1 & \mathbf{3} & 6 & 0 \\ 0 & 0 & \mathbf{3} & 7 \end{bmatrix}.$$

In this case we can reduce all claims with an amount of 3 and we would immediately end up with the hierarchical mutual liability problem $C^{2,\Delta}$ given by

$$C^{2,\Delta} = \begin{bmatrix} 4 & 4 & 0 & 6 \\ 0 & 8 & 0 & 8 \\ 1 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

If we rearrange the players in the order $(3, 1, 2, 4)$, then the matrix is upper triangular. Note that $\rho^{AM}(C^{2,\Delta}) = (0, 2, 5, 18)$ which is different from both $\rho^{AM}(C)$ and $\rho^{AM}(C^{1,\Delta})$. $\triangleleft$

2.6.2 Hydraulic approach

Kaminski (2000) states that a bankruptcy rule is called hydraulic if it can be represented as a system of connected vessels. The vessels represent the claim of a player and the liquid is the estate available. Inspired by hydraulic representation of bankruptcy rules, one could search for hydraulic solutions for hierarchical mutual liability problems. We will, however, not introduce a general hydraulic framework for liability problems. The aim of this subsection is to give an idea of a possible alternative approach. First, we describe a hydraulic method by means of an example.

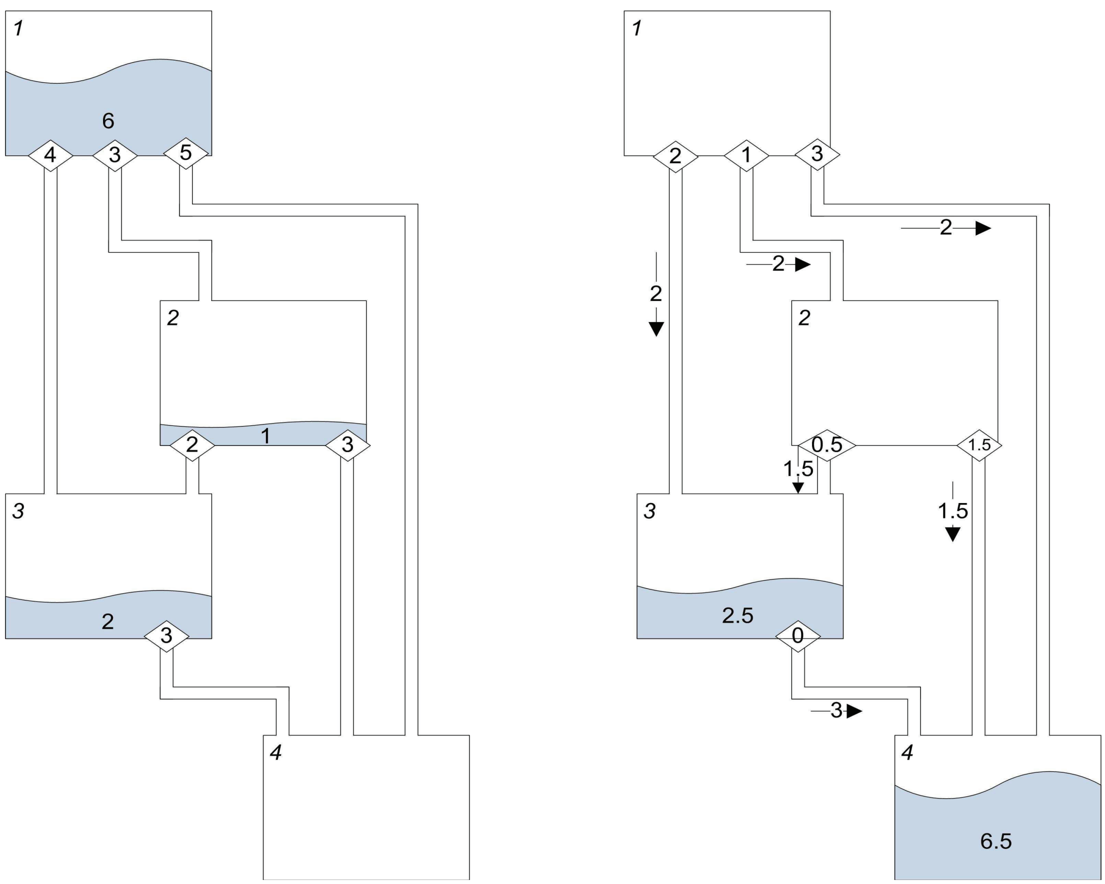

Figure 2.6.1: A hydraulic solution for Example 2.6.2

Example 2.6.2 Let $N = \{1, 2, 3, 4\}$. Consider the mutual liability problem

$C \in \mathcal{L}^{N,\Delta}$, given by

$$C = \begin{bmatrix} 6 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Every agent's cash can be seen as a vessel of liquid that is connected to its claimants by a system of equally wide tubes. The tube from 1 to 3 can only transfer $c_{13} = 4$ units of liquid, after this amount the tube is closed or disconnected. Hence c_{13} is the capacity of tube$_{13}$. The same holds for all other entries in C. This is represented in Figure 2.6.1, in the left system of connected vessels. The digit in a shaded area indicates the content level of the vessel and the digit in a rhombus represent the capacity of a tube. We open all the vessels simultaneously and let the liquid flow, until no flow is possible anymore.

In this way, for agent 1, through every outgoing tube, an amount of 2 will be transferred to vessels 2, 3 and 4. The remaining capacity of the tubes are 2, 1 and 3, respectively.

The initial content of the second vessel, 1, is divided equally among 3 and 4, but at the same time an extra amount of 2 flows into his vessel via tube$_{12}$ and this amount is also divided among 3 and 4. In this way we can continue with vessel 3 and 4. The final result is shown in Figure 2.6.1, in the righthand side system of connected vessels. The final allocation equals $(0, 0, 2.5, 6.5)$.

This hydraulic scheme fits with the CEA idea and in fact it can be shown that for this example $\rho^{CEA}(C) = (0, 0, 2.5, 6.5)$. ◁

Example 2.6.2 shows that one can model a hierarchical mutual liability problem as a system of connected vessels. We choose for a rationing method that, in this example, coincides with the CEA-based mutual liability rule. We presume, but did not prove, that this is the case in general. There are, however, many other hydraulic paths to follow.

Chapter 3

Game theoretic analysis of maximum cooperative purchasing situations

3.1 Introduction

This chapter, which is based on Groote Schaarsberg, Borm, Hamers, and Reijnierse (2013), introduces and analyzes a new class of interactive cooperative purchasing situations: maximum cooperative purchasing situations. Before introducing this new class of problems, we first describe the rationale behind interactive cooperative purchasing.

A purchasing cooperative consists of organizations that collaborate in their purchasing process, *e.g.*, in sharing information, bundling order quantities or sharing transportation services, in order to obtain benefits. If we focus on bundling order quantities, cooperative purchasing[2] seems an easy and rational solution for a group of purchasers facing quantity discounts. However, it is not as simple as that. In this chapter we analyze interactive purchasing situations where the unit price depends on the largest order quantity within a group of players. According to Tella and Virolainen (2005) the main motives of organizations to become member of a cooperative are to obtain information and — in the long run — to obtain cost savings, due to increasing returns

[2]Cooperative purchasing is also referred to as group buying or group purchasing.

to scale. Increasing returns to scale are analyzed in many micro-economic situations, *e.g.*, in the production industry (Hsu and Li (2009)) and health care industry (Schneider, Miller, Ohsfeldt, Morrisey, Zelner, and Li (2008)). In purchasing practise the increasing returns to scale of production often translates in a quantity discount for the buyers or customers (*cf.* Monahan (1984)). Naturally purchasers want to exploit these discounts, but they cannot simply increase their order quantities. So organizations try to cooperate in purchasing. But why do all hospitals in the US *not* unite themselves to suppress pharmacists' prices of medicines? How come all high-schools in the Netherlands do *not* purchase computers cooperatively to obtain a substantial price reduction?

The reasons are simple. Firstly, it is quite hard to manage and combine the purchasing processes of a large group of organizations. But even if the groups would be manageable, they would not easily form because of a second reason: the group members should agree on how to divide the obtained cost savings beforehand. Too simple cost savings allocation schemes may cause some members of the cooperative to feel better off alone, to be better off in sub-cooperatives or simply to think that the allocation method is not fair. According to Schotanus (2007) the fair allocation of cost savings is one of the main critical success factors for the stability of purchasing groups. Also in a large-scale survey among logistic service providers in Flanders, it was found by Cruijssen, Cools, and Dullaert (2007) that organizations believe in the potential of horizontal cooperation, but consider the allocation of the actual cost savings as one of the most important impediments of the cooperative.

A useful tool for finding fair allocations of cost savings that follow from cooperative behavior is provided by *cooperative game theory*. The frequently used model of transferable utility games fits the nature of cooperative purchasing situations. In a *transferable utility game* (or TU-game) each coalition of players is associated with a certain monetary value (transferable utility), which corresponds to the benefits this coalition can obtain by optimal cooperation amongst themselves, and without help from players outside the coalition. These coalitional values can be used as a benchmark for dividing the cost savings of the grand coalition (coalition of all players). Based on different ideas of fairness, multiple solution concepts for general cooperative TU-games have been developed. A generally accepted concept of fairness is the combination of efficiency and stability. Efficiency implies that we do not allocate more than the value of the grand coalition, nor do we award a positive allocation to entities outside the players of the game. Stability means that no coalition has an incentive to split off. The set of all efficient and stable allocations is called the *core* of a game. This set can, however, be empty. While the core prescribes a set of possible allocations, one can also look for single-valued solutions. Several single-valued or one-point solution concepts for TU-games have been introduced in the literature. Due to their attractive properties the *Shapley value* (Shapley (1953)) and the *nucleolus* (Schmeidler (1969)) are the most commonly studied and applied solution concepts. In this chapter we will show that these two solution concepts result in suitable allocation methods for a specific class of interactive purchasing situations.

We are not the first to investigate cooperative purchasing (CP)-situations. Anand and Aron (2003) are pioneers in studying cooperative purchasing using analytical models. Amongst others, they derive optimal pricing schedules for the supplier facing purchasing cooperatives. Also, from the purchasers perspective analytical models have been developed, *e.g.*, on the development of purchasing groups in the health care industry (Nollet and Beaulieu (2003)) and the formation of coalitions by means of the internet (Granot and Sošić (2005)). There have appeared qualitative considerations in the literature linking game theory and cooperative purchasing, but more in an explanatory sense than in analyzing its exact implications, *e.g.*, Blomqvist, Kylheiko, and Virolainen (2002) or Tella and Virolainen (2005). Only recently, CP-situations have been formally modeled as a game. Keskinocak and Savasaneril (2008) analyze the situation where purchasers are possible competitors. In Heijboer (2003), Schotanus (2007) and Nagarajan, Sošić, and Zhang (2010) a purchasing cooperative is modeled as a cooperative transferable utility game. The unit price depends on the sum of the order quantities; the higher the total order quantity, the lower the unit price.

In this chapter we analyze situations that are not covered by the cooperative models considered in the literature above. Consider the situation where a general practitioner needs daily supplies like sterile needles, bandages, compresses and drugs. He can buy these supplies at a pharmaceutical company. The pharmaceutical industry and its prices are not transparent, so one can imagine that the general practitioner ends up with a high unit price. A large hospital buys its supplies at the same pharmaceutical companies, but it has greater knowledge of the market and a better bargaining position towards its possible suppliers. As a result the hospital can negotiate for a lower unit price. The general practitioner could try to set up some cooperative with the hospital to decrease his own unit prices. Adding the two order quantities, however, will not result in a lower unit price for both organizations. If the single practitioner can use the terms and contract of the hospital, he would be willing to pay a (small) fee to the hospital for this 'riding along'. Both the hospital and the general practitioner would then be better off in this small cooperative.

This form of cooperative purchasing fits in a framework developed by Schotanus (2007): a typology of organizational forms of cooperative purchasing. One of these types involves *piggy-backing* groups, informal purchasing cooperatives that wish to keep cooperation as simple as possible. Mostly it enhances a relatively large organization that negotiates with the supplier on its own and the resulting contract may be used by some smaller organizations. The example the author provides is a consortium of local governmental institutions in the North of the Netherlands. These institutions have a piggy-backing group that has existed for more than 20 years.

Now, let us describe this subclass of cooperative purchasing situations more formally. In this chapter, we consider horizontal cooperation between two or more organizations that find themselves at the same position in the supply chain. We consider a group of organizations all having individual order quantities with respect to a certain commodity. The involved organizations might be competitors in the end market but it is not likely that they will influence the cooperation, since they are better off within a cooperative. The Dutch *Superunie* is a good example. It is a purchasing cooperative consisting of small competing supermarket chains, who must cooperate to remain competitive towards large organizations like *Albert Heijn*. For this reason we assume that the fact that the organizations are possible competitors will not influence the cooperation.

Like in previous work, we consider the bundling and sharing of purchasing volumes and focus on the main motive on the long run: quantity discounts. These discounts imply direct cost savings for the members of a cooperative. Contrary to Schotanus (2007) and Nagarajan et al. (2010) we consider purchasing situations where the unit price does not depend on the *sum* of those order quantities, but on the *maximum* of the individual order quantities. Each group of organizations can negotiate for their own terms and unit prices separately. The outcome of each negotiation depends on the organization with largest order quantity and is independent of the size of order quantities of other group members. The larger the largest order quantity, the lower the unit price. Hence, by cooperating the organizations can obtain a smaller unit price and obtain cost savings. Within a group of purchasing organizations, the smaller organizations simply let the largest organization add their order quantities to the total order. They use the terms and contract of the larger organization and its individually negotiated unit price. As explained by Schotanus (2007), the coordination costs for this form of cooperative purchasing can be assumed to be relatively low.

We explicitly address the problem of finding suitable allocation methods for the cost savings in this class of cooperative purchasing situations. To this end we define a cooperative transferable utility game corresponding to a *maximum cooperative purchasing (MCP) situation*, *i.e.*, a CP-situation with a max unit price function. In general, quantity discounts are a sufficient condition for a nonempty core of an associated *MCP-game* as we can always find a stable allocation of the cost savings. One of the core-elements is a marginal vector of the MCP-game and can be obtained via the Direct Price solution method in which every organization pays the price that follows from the grand coalition. For the organization with largest order quantity this implies that he receives no price reductions at all. This solution method is such that the payoffs to organizations increase as the group size increases (population monotonic). In terms of piggy-backing, however, this method leads to a cooperation fee equal to zero. Hence, the organization with largest order quantity will not easily agree with the Direct Price solution as allocation method. Therefore we propose two alternative allocation methods: the nucleolus (Schmeidler (1969)) and the Shapley value (Shapley (1953)) of the MCP-game.

The nucleolus of a TU-game minimizes the maximal unhappiness over all coalitions, where the unhappiness of a coalition with respect to an allocation is measured by the excess; the difference between the worth of the coalition and what they together obtain in the allocation. If the core of a TU-game is nonempty, the nucleolus is an element of this set.

In general, finding the nucleolus of a cooperative game is a hard task. Kohlberg (1971) developed a criterion to check whether an allocation equals the nucleolus of that game. Based on this criterion, several algorithms have been developed to compute the nucleolus, all not of polynomial time. For a compact overview see Leng and Parlar (2010). Reijnierse and Potters (1998) explain which collections of coalitions are essential to determine the nucleolus and show that these collections may differ from the ones of Kohlberg. Inspired by the results of Potters, Reijnierse, and Ansing (1996) and Reijnierse and Potters (1998), this chapter provides an alternative and explicit characterization of the nucleolus for general cooperative games with a nonempty core. To its advantage this characterization is more constructive in nature than the Kohlberg criterion.

Using this new criterion, the nucleolus of an MCP-game can be found via a so-called nucleolus-determinant: a collection of disjoint coalitions and their corresponding excesses. It is shown that these excesses can be interpreted as the fee the organizations in the coalition have to pay. Moreover we show how to find a nucleolus-determinant recursively with an algorithm of polynomial time.

The second single-valued solution concept we consider is the Shapley value. The Shapley value incorporates all possible marginal contributions from a player to a coalition and averages them over all coalitions, with a correction to the size of the coalitions. However, computing all marginal contributions is quite time consuming in general. An alternative way of finding the Shapley value of a TU-game is by using the decomposition of the game into a linear combination of unanimity games. Due to the specific structure of MCP-games, the decomposition into a linear combination of unanimity games can be easily determined. Using this decomposition we derive an explicit expression for the Shapley value, which can be nicely interpreted as stemming from a tax and subsidize system in which an individual organization receives or pays a certain percentage of the cost savings of all two-player coalitions.

Both the nucleolus and the Shapley value are attractive solution concepts from a general game theoretic point of view. We conclude the chapter with a numerical comparison between the behavior of the Shapley value and the nucleolus in MCP-situations and for illustrative reasons we also compare the two game theoretic solution concepts with the Direct Price solution. We see that for MCP-situations, the Shapley value and the nucleolus prescribe rather similar allocation proposals and that the differences between the prescribed proposals are relatively small. Generally speaking, the difference is that organizations with order quantities close to the order quantity of the largest player are expected to be better off in the Shapley value, while players with smaller order quantities are expected to better off in the nucleolus. The difference between the two game theoretic solutions on the one hand and the Direct Price solution on the other hand is, however, relatively large.

The structure of this chapter is as follows. In Section 3.2 we formally introduce MCP-situations, define corresponding MCP-games, discuss some appealing properties of these games and explain their specific structure. Then, in Section 3.3 we analyze the Direct Price solution and its relation to the core of an MCP-game. In Section 3.4 an explicit alternative characterization is provided for the nucleolus of an arbitrary cooperative game with a nonempty core, and in Section 3.5 we calculate the nucleolus of an MCP-game, based on this alternative characterization. Section 3.6 focusses on the Shapley value of an MCP-game and Section 3.7 provides a numerical comparison between the various allocation proposals.

3.2 MCP-situations and corresponding games

This section provides the formal description of maximum cooperative purchasing (MCP)-situations and defines corresponding cooperative MCP-games.

Formally, we have a player set $N = \{1, \ldots, n\}$, $n \geq 2$, with a vector of order quantities $q \in \mathbb{R}^N_+$. There is a commonly known unit price function $p : [0, \infty) \to [0, \infty)$ that maps an order quantity to some unit price. We assume the unit price function to be weakly decreasing. For the remainder of this chapter we assume, without loss of generality, that the order quantities are arranged in nondecreasing order, *i.e.*, $0 < q_1 \leq q_2 \leq \ldots \leq q_n$. The class of all MCP-situations is denoted by $\mathcal{M}$ and a single MCP-situation is given by the triple $(N, q, p) \in \mathcal{M}$. Note that n, the number of players, is variable.

To analyze the allocation aspects of an MCP-situation we will construct a corresponding cooperative TU-game. A cooperative game (N, v) is defined by a finite set N of players and a function v on the set 2^N of all subsets (coalitions) of N. This function v is called the characteristic function and assigns to each coalition $S \in 2^N$ a value $v(S) \in \mathbb{R}$ such that $v(\emptyset) = 0$. The value $v(S)$ represents the joint monetary rewards a coalition S can accomplish or realize by optimal cooperation among themselves.

Consider a subgroup $S \in 2^N$ of purchasing organizations. The unit price corresponding to that coalition of purchasers is determined by its member with maximal order quantity. By assumption this is the player with highest index. Hence S pays $p(q_s)$ per unit, with $s = \max\{i : i \in S\}$. Without cooperation a player $i \in S$ would have paid $p(q_i)$ per unit. Looking at the corresponding cost savings as monetary revenues, the characteristic function of the cooperative MCP-game (N, w) corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$ is defined for all $S \in 2^N$ by

$$w(S) = \sum_{j \in S} [p(q_i)q_i] - p(q_s) \sum_{j \in S} q_j \tag{3.1}$$

and it reflects the maximum cost savings a coalition S can establish. For ease of notation we set $p_i = p(q_i)$ for all $i \in N$. Specifically, from (3.1) it readily follows that the value of an arbitrary two-player coalition with $i < j$ equals

$$w(\{i, j\}) = (p_i - p_j)q_i. \tag{3.2}$$

Since we assume the unit price function to be weakly decreasing we find a specific order in the two-player coalitions. If $j < k$ then for all $i < j$ it holds that

$$w(\{i, j\}) \leq w(\{i, k\}). \tag{3.3}$$

By the nature of cost savings, each MCP-game is nonnegative with all single player coalitions having value 0. The following example illustrates an MCP-game.

Example 3.2.1 Consider an MCP-situation $(N, q, p) \in \mathcal{M}$ with $N = \{1, 2, 3, 4\}$, $q = (2, 4, 8, 12)$ and unit price function $p : [0, \infty) \to [0, \infty)$ with $p(t) = 10+\frac{12}{t}$. Hence the individual ordering costs of player 1 are $p(2)\cdot 2 = 16\cdot 2 = 32$. For players 2, 3 and 4, the individual ordering costs are 52, 92 and 132, respectively. The following table represents the corresponding MCP-game.

S	{1}	{2}	{3}	{4}	{1,2}
$w(S)$	0	0	0	0	6
S	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}
$w(S)$	9	10	6	8	4
S	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}
$w(S)$	15	18	14	12	22

Note that $w(\{1, 4\})+w(\{2, 4\})+w(\{3, 4\}) = w(\{1, 2, 3, 4\})$, while $w(\{1, 3\})+ w(\{2, 3\}) = w(\{1, 2, 3\})$. ◁

The final observations we made in Example 3.2.1 are true in general. Since the price depends on the maximum order quantity, the largest player s in a coalition S solely determines the unit price. Every other player in coalition S profits from the unit price decrease, independent of the order quantities of the players in $S\backslash\{s\}$. Thus the value of a coalition S consists of what every individual player in S can accomplish with the largest player in S. This leads to the following theorem.

Theorem 3.2.2 *Let (N, w) be the MCP-game corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$. Let $S \subset N$ with $|S| \geq 2$ and $s \in S$ with $s = \max\{i : i \in S\}$. Then,*

$$w(S) = \sum_{j \in S} w(\{j, s\}). \tag{3.4}$$

Proof: By (3.1) and (3.2) we have

$$w(S) = \sum_{j \in S} p_j q_j - \sum_{j \in S} p_s q_j = \sum_{j \in S} (p_j - p_s) q_j = \sum_{j \in S} w(\{j, s\}). \qquad \square$$

The special structure of MCP-games fits in a framework of cost-coalitional problems as provided by Meca and Sošić (2013). In such problems there is a player or a group of players, called the benefactors, whose participation in a cooperative always contributes to the savings of all members. In MCP-situations the player or players with largest order quantity are the benefactors. Furthermore, note that an MCP-game is not a k-game as described in Van den Nouweland, Borm, Golstein Brouwers van, Groot Bruinderink, and Tijs (1996)
A game (N, v) is *monotonic* if for all $S, T \in 2^N$, with $S \subset T$, $v(S) \leq v(T)$. The game is *superadditive* if for all $S, T \in 2^N$ with $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$. These two properties imply that if a coalition adapts more players, its value increases as well, and if a coalition breaks up in smaller coalitions they cannot increase the cost savings. Hence, any player $i \in N$ would like to join the largest coalition possible, *i.e.*, $N \backslash \{i\}$.

Using Theorem 3.2.2, we can verify that each MCP-game is monotonic and superadditive.

Corollary 3.2.3 *Let $(N, q, p) \in \mathcal{M}$ and let (N, w) be the corresponding MCP-game. Then, (N, w) is monotonic and superadditive.*

Proof: Since (N, w) is non-negative, it is sufficient to show that (N, w) is superadditive. Take $S, T \in 2^N$ with $s = \max\{i : i \in S\}$ and $t = \max\{t : t \in T\}$. Then, for all $S, T \in 2^N$ with $S \cap T = \emptyset$ and without loss of generality $s < t$,

$$\begin{aligned} w(S) + w(T) &= \sum_{j \in S} w(\{j, s\}) + \sum_{j \in T} w(\{j, t\}) \\ &\leq \sum_{j \in S \cup T} w(\{j, t\}) = w(S \cup T). \end{aligned}$$

□

3.3 The core and the Direct Price solution

As explained in the introduction of this chapter, within the game theoretic literature one can find several ways or policies to allocate the value of the grand coalition. The coalitional values $v(S)$ in a game (N, v) form a natural benchmark to evaluate an allocation method. We mentioned two basic evaluation properties, efficiency and stability. An allocation $x \in \mathbb{R}^N$ is *efficient* if $\sum_{i \in N} x_i = v(N)$. An allocation x is *stable* if for all $S \in 2^N$, $\sum_{i \in S} x_i \geq v(S)$. The set of efficient and stable allocations is called the core of a TU-game, which is denoted by $\mathcal{C}(v)$. In general, this set can be empty.

A frequently studied property for a game is *convexity*. Convexity requires that for all $j \in N$ and $S, T \in 2^N$ with $S \subset T \subset N \backslash \{j\}$, $v(S \cup \{j\}) - v(S) \leq v(T \cup \{j\}) - v(T)$. Hence, joining a large coalition T will lead to a larger marginal contribution than joining a subset of this group of players $S \subset T$. It is shown by Shapley (1967) that if (N, v) is convex, then $\mathcal{C}(v) \neq \emptyset$. However, an MCP-game is not necessarily convex.

Example 3.3.1 Reconsider the game of Example 3.2.1. This game is not convex. If player 3 joins player 1, the extra cost savings are $w(\{1, 3\}) - w(\{1\}) = 9$. If, however, player 3 joins the larger coalition $\{1, 4\}$, the extra cost savings are $w(\{1, 3, 4\}) - w(\{1, 4\}) = 4$, which is lower than 9, contradicting convexity.
The core of this game is nonempty and is given by

$$\begin{aligned} \mathcal{C}(w) = &\text{conv}\{(10, 8, 4, 0), (10, 8, 0, 4), (10, 6, 0, 6), (10, 2, 4, 6), \\ &(9, 8, 0, 5), (5, 8, 4, 5), (9, 2, 4, 7), (5, 6, 4, 7), (9, 6, 0, 7)\}. \end{aligned}$$

A special core element is the allocation $(10, 8, 4, 0)$. This allocation can be seen as the result of applying a direct pricing principle: every player pays the price that can be negotiated for the grand coalition. In this case that price is 11. Thus player 1 obtains cost savings for his own order quantity of $q_1(p_1 - p_4) = 2(16 - 11) = 10$, player 2 of $4(13 - 11) = 8$, player 3 of $8(11.5 - 11) = 4$, and player 4 of 0. ◁

The direct pricing principle illustrated in Example 2.3.1 can be formalized.

Definition The *Direct Price solution DP* on $\mathcal{M}$ is such that for all $(N, q, p) \in \mathcal{M}$ and for all $i \in N$,

$$DP_i(N, q, p) = (p_i - p_n)q_i.$$

Note that according to (3.2), $DP_i(N, q, p) = w(\{i, n\})$ for all $i \in N$ and all $(N, q, p) \in \mathcal{M}$, with (N, w) the corresponding MCP-game. For an arbitrary MCP-situation, the allocation resulting from the Direct Price solution belongs to the core of the corresponding MCP-game. Hence, any MCP-game has a nonempty core.

Theorem 3.3.2 *Let (N, w) be the MCP-game corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$. Then,*

$$DP(N, q, p) \in \mathcal{C}(w).$$

Proof: Using Theorem 3.2.2,

$$\sum_{i \in N} DP_i(N, q, p) = \sum_{i \in N} w(\{i, n\}) = w(N).$$

Let $S \in 2^N$. Using (3.3) we find that

$$w(S) = \sum_{i \in S} w(\{i, s\}) \leq \sum_{i \in S} w(\{i, n\}) = \sum_{i \in S} DP_i(N, q, p),$$

where $s = \max\{i : i \in S\}$. □

Let (N, v) be a TU-game and let $\sigma \in \Pi(N)$, where $\sigma(k)$ is interpreted as the player in position k. For $\sigma \in \Pi(N)$, the *marginal vector* $m^{\sigma}(v) \in \mathbb{R}^N$ corresponding to σ is defined by

$$m^{\sigma}_{\sigma(k)}(v) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\}) \tag{3.5}$$

for all $k \in \{1, \ldots, |N|\}$. Note that for all $\sigma \in \Pi(N)$, $m^{\sigma}_{\sigma(1)}(v) = v(\{\sigma(1)\})$. A game (N, v) is convex if and only if all marginal vectors of that game belong to the core. An MCP-game is in general not convex, but we will point out several specific marginal vectors of an MCP-game (N, w) that are elements of $\mathcal{C}(w)$.

Lemma 3.3.3 *Let (N, w) be the MCP-game corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$.*

(i) The marginal vector corresponding to any $\sigma^1 \in \Pi(N)$ with $\sigma^1(1) = n$, is a core element.

(ii) The marginal vector corresponding to any $\sigma^2 \in \Pi(N)$ with $\sigma^2(1) = n-1$ and $\sigma^2(2) = n$, is a core element.

Proof: (i) Using (3.4) the marginal vector corresponding to σ^1 is given by

$$m^{\sigma^1}(w) = (w(\{1, n\}), w(\{2, n\}), \ldots, w(\{n-1, n\}), 0) = DP(N, q, p).$$

(ii) Using (3.4) the marginal vector corresponding to σ^2 is given by

$$m^{\sigma^2}(w) = (w(\{1, n\}), w(\{2, n\}), \ldots, w(\{n-2, n\}), 0, w(\{n-1, n\})).$$

Thus for all $S \in 2^N$

$$\sum_{i \in S} m^{\sigma^2}_i(w) = \begin{cases} w(\{n-1, n\}) + \sum_{i \in S \setminus \{n-1\}} w(\{i, n\}) & \text{if } n \in S, \\ \sum_{i \in S \setminus \{n-1\}} w(\{i, n\}) & \text{otherwise.} \end{cases}$$

If $n \in S$, then $w(\{n-1, n\}) + \sum_{i \in S \setminus \{n-1\}} w(i, n) \geq \sum_{i \in S} w(i, n) \geq w(S)$. If $n \notin S$, then $\sum_{i \in S \setminus \{n-1\}} w(\{i, n\}) \geq \sum_{i \in S \setminus \{n-1\}} w(\{i, s\}) = w(S)$, with $s = \max\{i : i \in S\}$. Hence, we have that $\sum_{i \in S} m^{\sigma^2}_i(w) \geq w(S)$ for all $S \in 2^N$. □

The next theorem shows that if there are two players with lowest unit price, the core of the MCP-game consists of one single point, the allocation prescribed by the Direct Price solution. Moreover this is the only class of MCP-situations for which the core of the MCP-game consists of one point, this is shown in the next theorem.

Theorem 3.3.4 *Let* $(N, q, p) \in \mathcal{M}$ *and let* (N, w) *be the corresponding MCP-game. Then*

$$\mathcal{C}(w) = \{DP(N, q, p)\} \text{ if and only if } p_{n-1} = p_n.$$

Proof: Let $\mathcal{C}(w) = \{DP(N, q, p)\}$. Then, according to Lemma 3.3.3, $m^{\sigma_1}(w) = m^{\sigma_2}(w)$ with σ_1 and σ_2 as defined there. Hence $0 = w(\{n-1, n\}) = (p_{n-1} - p_n)q_{n-1}$ and thus $p_{n-1} = p_n$.

Let $p_{n-1} = p_n$. This implies that for all $i \in \{1, \ldots, n-2\}$, $w(\{i, n-1\}) = w(\{i, n\})$ and $w(\{n-1, n\}) = 0$. By Theorem 3.3.2 it is sufficient to show that $x \in \mathcal{C}(w)$ implies that $x = DP(N, q, p)$. Let $x \in \mathcal{C}(w)$. Then

$$\begin{aligned} 0 = w(\{n\}) \leq x_n &\leq w(N) - w(N \backslash \{n\}) \\ &= \sum_{j=1}^{n-1} w(\{j, n\}) - \sum_{j=1}^{n-2} w(\{j, n-1\}) \\ &= w(\{n-1, n\}) \\ &= 0. \end{aligned}$$

Hence, $x_n = 0 = DP_n(N, q, p)$.
Similarly we have for all $i \in N \backslash \{n\}$,

$$\begin{aligned} x_i \leq w(N) - w(N \backslash \{i\}) &= \sum_{j \in N} w(\{j, n\}) - \sum_{j \in N \backslash \{i\}} w(\{j, n\}) \\ &= w(\{i, n\}) \end{aligned}$$

and hence

$$w(N) = \sum_{i=1}^{n-1} x_i \leq \sum_{i=1}^{n-1} w(\{i, n\}) = w(N).$$

Hence, for all $i \in N\backslash\{n\}$, $x_i = w(\{i,n\}) = DP_i(N,q,p)$. $\square$

The Direct Price solution coincides with the altruistic allocation of Meca and Sošić (2013), where it is also used for describing the size of the core of the game corresponding to cost-coalitional problems.
The Direct Price solution has the appealing property that the players in N do not have a monetary incentive to reject a new agent j who wants to join N.

Theorem 3.3.5 *Let $(N,q,p) \in \mathcal{M}$ and consider a player $j \notin N$ with demand q_j. Then for all $i \in N$*

$$DP_i(N,q,p) \leq DP_i(N \cup \{j\}, \bar{q}, p),$$

where $\bar{q} = (q_\ell)_{\ell \in N \cup \{j\}}$.

Proof: Let $i \in N$. There are two possibilities: either $q_j > q_n$ or $q_j \leq q_n$. In the first case $p_j \leq p_n$ and $DP_i(N \cup \{j\}, \bar{q}, p) = (p_i - p_j)q_i \geq (p_i - p_n)q_i = DP_i(N,q,p)$. In the second case, obviously $DP_i(N \cup \{j\}, \bar{q}, p) = DP_i(N,q,p)$. $\square$

In fact, from Theorem 3.3.5 one readily derives that the Direct Price solution is *population monotonic*. Combining this with Theorem 3.3.4, we can conclude that the Direct Price solution leads to a stable allocation scheme in which each player's payoff increases (non-decreases) as the grand coalition of cooperative purchasers grows larger. This is also known as a *population monotonic allocation scheme* (PMAS) as introduced by Sprumont (1990).

Example 3.3.6 Reconsider Example 3.2.1 with $DP(N,q,p) = (10,8,4,0)$. This allocation is quite extreme as all bilateral profits go to the 'smaller' players. A less extreme allocation x would be to let the two players i and n share the obtained cost savings equally, leading to

$$\begin{aligned} x =&\Big(\frac{1}{2}w(\{1,4\}), \frac{1}{2}w(\{2,4\}), \frac{1}{2}w(\{3,4\}), \\ &\frac{1}{2}w(\{1,4\}) + \frac{1}{2}w(\{2,4\}) + \frac{1}{2}w(\{3,4\})\Big) \\ =&(5,4,2,11). \end{aligned}$$

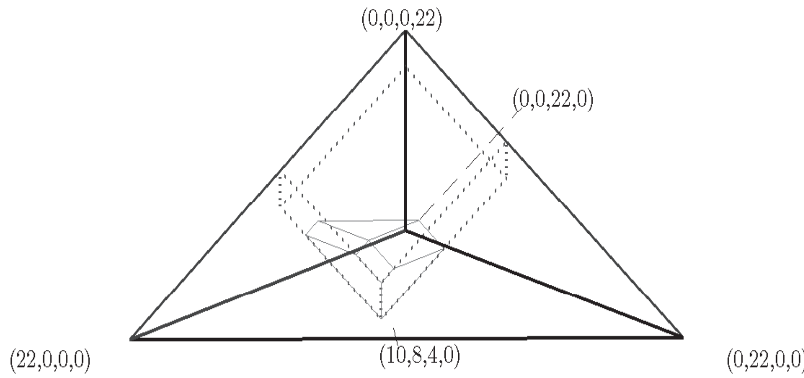

Figure 3.3.1: Core and Shared Set of Example 3.2.1

This allocation is not a core-element. It can be seen, however, that all core-elements can be obtained if we allow differentiation with respect to the sharing level among pairs of players. For example for the barycenter $y = (8\frac{5}{9}, 6, 2\frac{2}{9}, 5\frac{2}{9})$ of the core, we have that

$$\begin{aligned} y = \Big(& \frac{77}{90} w(\{1,4\}), \frac{3}{4} w(\{2,4\}), \frac{5}{9} w(\{3,4\}), \\ & \frac{13}{90} w(\{1,4\}) + \frac{1}{4} w(\{2,4\}) + \frac{4}{9} w(\{3,4\}) \Big). \end{aligned}$$

In the tetrahedron of Figure 3.3.1, the dotted box represents all allocations obtained via sharing methods. The polyhedron inside that box is the core of the game. ◁

Based on the ideas of Example 3.3.6 we define a *sharing rule* δ^λ on $\mathcal{M}$. Define Δ^N as the set of all share vectors $\lambda \in \mathbb{R}^N$ with for all $i \in N$, $\lambda_i \in [0,1]$. Here λ_i can be interpreted as the share player i obtains of his bilateral profits with n.

Definition For all $(N,q,p) \in \mathcal{M}$ and all $\lambda \in \Delta^N$

$$\delta_i^\lambda(N,q,p) = \begin{cases} \lambda_i (p_i - p_n) q_i & \text{if } i \in N \backslash \{n\}, \\ \sum_{j \in N} (1-\lambda_j)(p_j - p_n) q_j & \text{if } i = n. \end{cases}$$

Note that the choice of λ_n does not affect the allocation proposed by δ^λ. Also note that $\delta^{\lambda^*} = DP$, for $\lambda^* \in \Delta^N$ such that $\lambda_i^* = 1$ for all $i \in N\backslash\{n\}$. Next, consider the *Shared Set*, defined by

$$\mathcal{S}(N,q,p) = \{\delta^\lambda(N,q,p)|\lambda \in \Delta^N\},$$

as all possible allocations generated by a sharing rule. The next proposition shows that all core elements can be generated via sharing rules.

Proposition 3.3.7 *Let $(N,q,p) \in \mathcal{M}$ and let (N,w) be the corresponding MCP-game. Then*

$$\mathcal{C}(w) \subset \mathcal{S}(N,q,p).$$

Proof: Let $x \in \mathcal{C}(w)$. Define $\lambda \in \mathbb{R}^N$ by $\lambda_i = \frac{x_i}{(p_i - p_n)q_i}$, $i \in \{1,\ldots,n-1\}$ and $\lambda_n = 1$. Since, for all $i \in N\backslash\{n\}$, $0 \leq x_i \leq w(N) - w(N\backslash\{i\}) \leq w(\{i,n\})$, we have $\lambda \in \Delta^N$. Obviously, $\delta_i^\lambda(N,q,p) = x_i$ for all $i \in N$ and hence $x \in \mathcal{S}(N,q,p)$. □

3.4 The nucleolus of a game with a nonempty core

In this section we derive an alternative characterization of the nucleolus of a TU-game with nonempty core, which will be used in Section 3.5 to find the nucleolus of an MCP-game.

Let (N,v) be a TU-game. We define $C^N = 2^N\backslash\{\emptyset, N\}$ as the collection of proper subsets of N. The imputation set is given by all individually rational and efficient vectors, *i.e.*,

$$I(v) = \left\{x \in \mathbb{R}^N \middle| \sum_{j\in N} x_j = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N\right\}.$$

An element $x \in I(v)$ is called an *imputation*. Let $E(S,x) = v(S) - \sum_{j\in S} x_j$ be the *excess* of coalition $S \in 2^N$ with respect to an imputation $x \in I(v)$.

Note that $\theta(x) \in \mathbb{R}^{|2^N|}$ is the vector of excesses arranged in non-increasing order. The nucleolus $nuc(v)$ (Schmeidler (1969)) of a TU-game (N, v) with $I(v) \neq \emptyset$, is the unique imputation which lexicographically minimizes θ. Thus for all $x \in I(v)$,

$$\theta(nuc(v)) \leq_L \theta(x).$$

The nucleolus minimizes the maximum dissatisfaction level over all coalitions and it is a stable solution concept, *i.e.*, if $\mathcal{C}(v) \neq \emptyset$, then $nuc(v) \in \mathcal{C}(v)$.

Finding the nucleolus of a TU-game is not easy, in general it takes $\mathcal{O}(|N| \times 2^{|N|})$ steps[3]. To check whether a certain imputation $x \in I(v)$ is the nucleolus of the game, one can use the following criterion, due to Kohlberg (1971). Let $\mathcal{B}_1(x) = \{T \in C^N | E(T, x) \geq E(S, x)$ for all $S \in C^N\}$ be the collection of coalitions with highest excess. Recursively define for $r = 2, 3, \ldots$

$$\mathcal{B}_r(x) = \Big\{ T \in C^N \Big| T \notin \bigcup_{k=1}^{r-1} \mathcal{B}_k(x), E(T, x) \geq E(S, x) \text{ for all } S \in C^N \text{ with } S \notin \bigcup_{k=1}^{r-1} \mathcal{B}_k(x) \Big\}.$$

Let $t \in \mathbb{N}$ be such that $\mathcal{B}_t(x) \neq \emptyset$ and $\mathcal{B}_{t+1}(x) = \emptyset$.
For $r \in \{1, \ldots, t\}$ define

$$\overline{\mathcal{B}}_r(x) = \bigcup_{k=1}^{r} \mathcal{B}_k(x).$$

Theorem 3.4.1 *(Kohlberg (1971)) Let (N, v) be a TU-game with $\mathcal{C}(v) \neq \emptyset$ and let $x \in I(v)$. Then, $x = nuc(v)$ if and only if $\overline{\mathcal{B}}_r(x)$ is balanced for all $r \in \{1, \ldots, t\}$.*

Here a collection $\mathcal{B} \subset C^N$ is *balanced* if there exists a vector $\lambda \in \mathbb{R}^{C^N}$ satisfying $\lambda_S > 0$ for all $S \in \mathcal{B}$ and $\lambda_S = 0$ for all $S \notin \mathcal{B}$, such that $\sum_{S \in \mathcal{B}} \lambda_S \mathbb{1}^S = \mathbb{1}^N$.

[3]Let $g : X \to \mathbb{R}$. We say $f(x)$ is of the order $g(x)$, or $\mathcal{O}(g(x))$, if there exists $M \in \mathbb{R}_+$ and $x_o \in \mathbb{R}$ such that $|f(x)| \leq M|g(x)|$ for all $x \in X$ with $x > x_o$.

From this theorem Kohlberg (1972) derived a procedure to calculate the nucleolus of a cooperative game, by solving one large linear program. Due to Potters et al. (1996) there is a faster method to determine the nucleolus. The rough idea is based on efficiency of the nucleolus. If one has found two disjoint coalitions S and T that belong to $\mathcal{B}_1(nuc(v))$ and one also knows their corresponding excesses, then one also knows the excesses that belong to $N\backslash S$, $N\backslash T$, $S \cup T$ and $N\backslash(S \cup T)$. As a follow up Reijnierse and Potters (1998) provide a sufficient condition for a collection of coalitions to determine the nucleolus of a game. This condition can be used to formulate a new alternative criterion to check whether a certain imputation is the nucleolus of the game.

For a collection $\mathcal{B} \subset C^N$ define

$$H(\mathcal{B}) = \{S \in 2^N | \mathbb{1}^S \in span\{\mathbb{1}^N, [\mathbb{1}^T]_{T\in\mathcal{B}}\}\}.$$

An alternative way of finding $H(\mathcal{B})$ is using $\mathcal{H}$-closed sets. A collection of coalitions $\mathcal{W} \subset 2^N$ is $\mathcal{H}$-closed, if

(i) $N \in \mathcal{W}$

(ii) for all $R \in \mathcal{W}$, $N\backslash R \in \mathcal{W}$

(iii) for all $R, U \in \mathcal{W}$ with $R \cap U = \emptyset$, $R \cup U \in \mathcal{W}$.

Then, $H(\mathcal{B})$ is the smallest $\mathcal{H}$-closed set containing $\mathcal{B}$.

The following theorem gives an explicit alternative characterization of the nucleolus of a game with a nonempty core.

Theorem 3.4.2 *Let (N, v) be a TU-game with $\mathcal{C}(v) \neq \emptyset$. Let $x \in I(v)$. Then $x = nuc(v)$ if and only if there exists a sequence $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_\tau$ of nonempty subcollections of C^N with the following properties*

(i) for al $r \in \{1, \ldots, \tau\}$ the collection $\overline{\mathcal{D}}_r = \bigcup_{k=1}^{r} \mathcal{D}_k$ is balanced,

(ii) *there exists a sequence of real numbers* $\gamma_1, \gamma_2, \ldots, \gamma_\tau$ *such that* $E(S,x) = \gamma_r$ *for every* $S \in \mathcal{D}_r$ *and* $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_\tau$.

(iii) *for all* $S \in C^N \backslash \overline{\mathcal{D}}_\tau$ *we have* $S \in H(\{T \in \overline{\mathcal{D}}_\tau : E(T,x) \geq E(S,x)\})$.

Proof: "only if"-part: Let $x = nuc(v)$ and $t \in \mathbb{N}$ is such that $\mathcal{B}_t(nuc(v)) \neq \emptyset$ and $\mathcal{B}_{t+1}(nuc(v)) = \emptyset$. Set $\tau = t$ and $\mathcal{D}_r = \mathcal{B}_r(nuc(v))$ for all $r \in \{1, \ldots, \tau\}$. Then, by Theorem 3.4.1 it is obvious that the sequence satisfies Properties (i) and (ii) and since $\overline{\mathcal{D}}_t = C^N$ also Property (iii) is satisfied.

"if"-part: Let $x \in I(v)$ and let $t \in \mathbb{N}$ be such that $\mathcal{B}_t(x) \neq \emptyset$ and $\mathcal{B}_{t+1}(x) = \emptyset$. Let the collections $\mathcal{D}_1, \ldots, \mathcal{D}_\tau$ and the numbers $\gamma_1, \gamma_2, \ldots, \gamma_\tau$ be such that Properties (i), (ii) and (iii) are satisfied. For all $r \in \{1, \ldots, t\}$, define $\kappa_r = E(S,x)$ for $S \in \mathcal{B}_r(x)$. Obviously $\kappa_1 > \kappa_2 > \ldots > \kappa_t$. We will show that x meets the Kohlberg criterion, proceeding by means of induction.

If $\kappa_1 = \gamma_\tau$, define $\ell = \tau$. Otherwise, $\gamma_r = \kappa_1 > \gamma_{r+1}$ for some $r \in \{1, \ldots, \tau - 1\}$, define $\ell = r$. Then, $\kappa_1 = \gamma_1 = \ldots = \gamma_\ell$ and $\overline{\mathcal{D}}_\ell \subset \mathcal{B}_1(x)$. Hence

$$\mathcal{B}_1(x) = \overline{\mathcal{D}}_\ell \cup \{S \notin \overline{\mathcal{D}}_\tau | E(S,x) = \kappa_1\}.$$

For any $S \in \mathcal{B}_1(x)$ with $S \notin \overline{\mathcal{D}}_\tau$

$$\{T \in \overline{\mathcal{D}}_\tau : E(T,x) \geq E(S,x)\} = \overline{\mathcal{D}}_\ell.$$

Hence, by Property (iii), $S \in H(\overline{\mathcal{D}}_\ell)$.
In order to prove that $\mathcal{B}_1(x)$ is balanced, it is sufficient to show that for any $S \in \mathcal{B}_1(x)$ with $S \notin \overline{\mathcal{D}}_\tau$, $\overline{\mathcal{D}}_\ell \cup \{S\}$ is balanced. Since $S \in H(\overline{\mathcal{D}}_\ell)$,

$$\mathbb{1}^S = \mu_N \mathbb{1}^N + \sum_{T \in \overline{\mathcal{D}}_\ell} \mu_T \mathbb{1}^T \text{ for some } \mu \in \mathbb{R}^{2^N \backslash \{\emptyset\}}.$$

Moreover, since $\overline{\mathcal{D}}_\ell$ is balanced there exists a vector $\lambda \in \mathbb{R}^{C^N}$ with $\lambda_T > 0$ for all $T \in \overline{\mathcal{D}}_\ell$ such that

$$\sum_{T \in \overline{\mathcal{D}}_\ell} \lambda_T \mathbb{1}^T = \mathbb{1}^N.$$

We have that

$$\mathbb{1}^S = \sum_{T \in \overline{\mathcal{D}}_\ell} (\mu_N \lambda_T + \mu_T) \mathbb{1}^T$$

and for any $\alpha \in \mathbb{R}$,

$$\sum_{T \in \overline{\mathcal{D}}_\ell} \lambda_T \mathbb{1}^T + \alpha(\mathbb{1}^S - \sum_{T \in \overline{\mathcal{D}}_\ell} (\mu_N \lambda_T + \mu_T) \mathbb{1}^T) = \mathbb{1}^N,$$

i.e.,

$$\alpha \mathbb{1}^S + \sum_{T \in \overline{\mathcal{D}}_\ell} (\lambda_T - \alpha[\mu_N \lambda_T + \mu_T]) \mathbb{1}^T = \mathbb{1}^N.$$

Choosing $\alpha > 0$ small enough, we derive that $\overline{\mathcal{D}}_\ell \cup \{S\}$ is balanced.

Proceeding by induction we assume that $\overline{\mathcal{B}}_k(x)$ is balanced for some $k \in \{2, \ldots, \tau - 1\}$. We will show that $\overline{\mathcal{B}}_{k+1}(x)$ is balanced too. If $\kappa_{k+1} \leq \gamma_\tau$, define $\ell = \tau$. Otherwise, $\gamma_r \geq \kappa_{k+1} > \gamma_{r+1}$ for some $r \in \{1, \ldots, \tau - 1\}$. In this case, define $\ell = r$. Then, $\overline{\mathcal{D}}_\ell \subset \overline{\mathcal{B}}_{k+1}(x)$. Hence,

$$\overline{\mathcal{B}}_{k+1}(x) = \overline{\mathcal{D}}_\ell \cup \{S \notin \overline{\mathcal{D}}_\tau | E(S, x) = \kappa_{k+1}\}.$$

It is sufficient to prove that for any $S \in \mathcal{B}_{k+1}(x)$ with $S \notin \overline{\mathcal{D}}_\tau$, $\overline{\mathcal{D}}_\ell \cup \{S\}$ is balanced. Since $S \in H(\overline{\mathcal{D}}_\ell)$ and $\overline{\mathcal{D}}_\ell$ is balanced, we can use the same argument as for the induction base. There exists a vector $\mu \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$, and a vector $\lambda \in \mathbb{R}^{C^N}$ with $\lambda_T > 0$ for all $T \in \overline{\mathcal{D}}_\ell$ such that for any $\alpha \in \mathbb{R}$

$$\alpha \mathbb{1}^S + \sum_{T \in \overline{\mathcal{D}}_\ell} (\lambda_T - \alpha[\mu_N \lambda_T + \mu_T]) \mathbb{1}^T = \mathbb{1}^N.$$

Again, choosing $\alpha > 0$ small enough, we derive that $\overline{\mathcal{D}}_\ell \cup \{S\}$ is balanced. $\square$

3.5 The nucleolus of MCP-games

This section explains how one can compute the nucleolus of an MCP-game. We start with the following easy observation.

Proposition 3.5.1 *Let (N, w) be an MCP-game corresponding to MCP-situation $(N, q, p) \in \mathcal{M}$. Then,*

$$nuc(w) = DP(N, q, p) \text{ if and only if } p_{n-1} = p_n.$$

Proof: If $p_{n-1} = p_n$, then by Theorem 3.3.4, $\mathcal{C}(w) = \{DP(N, q, p)\}$ and hence $nuc(w) = DP(N, q, p)$.

Let $nuc(w) = DP(N, q, p)$. Then, for all $S \in C^N$ such that $n \in S$,

$$E(S, nuc(w)) = w(S) - \sum_{i \in S} DP_i(N, q, p) = \sum_{i \in S} q_i(p_i - p_n) - \sum_{i \in S} q_i(p_i - p_n) = 0.$$

Since $\{S \in 2^N \backslash \{N, \emptyset\} | n \in S\}$ is not balanced, the Kohlberg criterion (Theorem 3.4.1) implies that there is a coalition $S \subset N \backslash \{n\}$ with the same excess as a coalition containing n, *i.e.*

$$\begin{aligned} 0 = E(S, nuc(w)) &= w(S) - \sum_{j \in S} w(\{j, n\}) \\ &= \sum_{j \in S} w(\{j, s\}) - \sum_{j \in S} w(\{j, n\}) \\ &= \sum_{j \in S} q_j(p_j - p_s) - \sum_{j \in S} q_j(p_j - p_n), \end{aligned}$$

with $s = \max\{i : i \in S\}$. Hence, $p_s = p_n$ and consequently $p_{n-1} = p_n$. □

Next we provide an explicit relation between the Direct Price solution of an MCP-situation and the nucleolus of the corresponding MCP-game. For this we use a so-called *nucleolus-determinant*. First, we simply state the recursion to find such a nucleolus-determinant. This recursion is rather technical. Then, an explicit expression for the nucleolus is provided. The interpretation of this nucleolus-determinant and the nucleolus itself are given after the proof. Finally, by means of an example, we explain how one can use this recursion. Furthermore we show that this recursion leads to an algorithm of polynomial time to find the nucleolus of an MCP-game.
Denote $p_S = p_s$ if $s = \max\{i : i \in S\}$.

Nucleolus-Determinant-recursion

Input: $(N, q, p) \in \mathcal{M}$.

Initialization:

$$\begin{array}{lll} \text{Set} & \mathcal{F}_1 & = \{S | S \subset N \backslash \{n\}, S \neq \emptyset\}, \\ & \tilde{\mathcal{F}}_1 & = \left\{ T \in \mathcal{F}_1 \middle| \frac{(p_n - p_T)\sum_{j \in T} q_j}{1+|T|} \geq \frac{(p_n - p_S)\sum_{j \in S} q_j}{1+|S|} \text{ for all } S \in \mathcal{F}_1 \right\}, \\ \text{choose} & T_1 & \in \tilde{\mathcal{F}}_1, \\ \text{set} & e_1 & = \frac{(p_n - p_{T_1})\sum_{j \in T_1} q_j}{1+|T_1|}, \\ & S_1 & = T_1 \\ \text{and} & r & = 2. \end{array}$$

Recursion:

$$\begin{array}{lll} \text{If} & \bigcup_{k=1}^{r-1} S_k & \neq N \backslash \{n\}, \\ \text{define} & f_r : & \mathcal{F}_r \to (-\infty, 0] \\ \text{by} & f_r(S) & = \dfrac{(p_n - p_S)\sum_{j \in S} q_j - \sum_{k=1}^{r-1} |S \cap S_k| e_k}{1 + \left| S \backslash \bigcup_{k=1}^{r-1} S_k \right|}, S \in \mathcal{F}_r. \\ \text{Set} & \mathcal{F}_r & = \{S | S \subset N \backslash \{n\}, S \backslash \bigcup_{k=1}^{r-1} S_k \neq \emptyset\}, \\ & \tilde{\mathcal{F}}_r & = \{T \in \mathcal{F}_r \,|\, f_r(T) \geq f_r(S) \text{ for all } S \in \mathcal{F}_r\}. \end{array}$$

$$\begin{array}{lll} \text{Choose} & T_r & \in \tilde{\mathcal{F}}_r, \\ \text{set} & e_r & = f_r(T_r), \\ & S_r & = T_r \backslash \bigcup_{k=1}^{r-1} S_k \\ \text{and} & r & = r + 1. \\ \text{Otherwise: set} & \tau & = r - 1. \text{ STOP} \end{array}$$

Output: A nucleolus-determinant $\{(S_1, e_1), (S_2, e_2), \ldots, (S_\tau, e_\tau)\}$, with $(S_1, \ldots, S_\tau)$ a partition of $N \backslash \{n\}$.

Theorem 3.5.2 *Let $(N, q, p) \in \mathcal{M}$ and let (N, w) be the corresponding MCP-game.*
Let $\{(S_1, e_1), (S_2, e_2), \ldots, (S_\tau, e_\tau)\}$ be a nucleolus-determinant. Then, for all

$r \in \{1, \ldots, \tau\}$ *and all* $i \in S_r$

$$nuc_i(w) = q_i(p_i - p_n) + e_r,$$
$$nuc_n(w) = \sum_{k=1}^{\tau}(-|S_k|e_k).$$

In the proof of Theorem 3.5.2 we use the following lemma.

Lemma 3.5.3 *Let* $(N, q, p) \in \mathcal{M}$ *and let* (N, w) *be the corresponding MCP-game. Let* $i \in N$ *and* $S \in 2^N$ *be such that* $n \in S \subset N\backslash\{i\}$ *and let* $x \in I(w)$ *be such that* $x_i \leq w(N) - w(N\backslash\{i\})$. *Then,*

$$E(S \cup \{i\}, x) \geq E(S, x).$$

Proof: Since $x_i \leq w(N) - w(N\backslash\{i\})$,

$$\begin{aligned} E(S \cup \{i\}, x) - E(S, x) &= w(S \cup \{i\}) - w(S) - x_i \\ &= q_i(p_i - p_n) - x_i \\ &\geq q_i(p_i - p_n) - w(N) + w(N\backslash\{i\}) \\ &= 0. \end{aligned}$$

□

Proof of Theorem 3.5.2: For all $r = 1, \ldots, \tau$, and for all $i \in S_r$ we set

$$\begin{cases} x_i &= q_i(p_i - p_n) + e_r \\ x_n &= \sum_{k=1}^{\tau}(-|S_k|e_k). \end{cases}$$

For all $r \in \{1, \ldots, \tau\}$, let $T_r \in \tilde{\mathcal{F}}_r$ be the coalition the recursion chose. Define for all $r \in \{1, \ldots, \tau\}$

$$\mathcal{D}_r = T_r \cup \{N\backslash\{i\}\}_{i \in S_r}.$$

We will show that the sequence $\mathcal{D}_1, \ldots, \mathcal{D}_\tau$ satisfies the three properties of Theorem 3.4.2.

Property (i)

Clearly, we have that $\overline{\mathcal{D}}_r$ is balanced for all $r \in \{1, \ldots, \tau\}$. Hence the sequence $\mathcal{D}_1, \ldots, \mathcal{D}_\tau$ satisfies Property (i) of Theorem 3.4.2.

Property (ii)

Regarding Property (ii), we first prove that for all $r \in \{1, \ldots, \tau\}$, $E(S, x) = e_r$ for all $S \in \mathcal{D}_r$.

Take $r \in \{1, \ldots, \tau\}$. Then for all $i \in S_r$

$$\begin{aligned}
E(N\backslash\{i\}, x) &= w(N\backslash\{i\}) - \sum_{j \in N\backslash\{i\}} x_j \\
&= \sum_{j \in N\backslash\{i\}} q_j(p_j - p_n) - (w(N) - x_i) \\
&= \sum_{j \in N\backslash\{i\}} q_j(p_j - p_n) - \sum_{j \in N} q_i(p_i - p_n) + x_i \\
&= -q_i(p_i - p_n) + q_i(p_i - p_n) + e_r \\
&= e_r.
\end{aligned}$$

For T_r itself it holds that

$$\begin{aligned}
E(T_r, x) &= w(T_r) - \sum_{j \in T_r} x_j \\
&= \sum_{j \in T_r} q_j(p_j - p_{T_r}) - \sum_{k=1}^{r} \sum_{j \in T_r \cap S_k} [q_j(p_j - p_n) + e_k] \\
&= \sum_{j \in T_r} q_j(p_n - p_{T_r}) - \sum_{k=1}^{r} |T_r \cap S_k| e_k \\
&= f_r(T_r)(1 + |T_r \cap S_r|) - |T_r \cap S_r| e_r \\
&= e_r(1 + |S \cap S_r|) - |S \cap S_r| e_r \\
&= e_r.
\end{aligned}$$

To finish the proof of Property (ii) it remains to show that $e_1 \geq e_2 \geq \ldots \geq e_\tau$. Note that $T_{r+1} \in \mathcal{F}_r$. Suppose that for some $r \leq \tau - 1$, $e_r < e_{r+1}$. Then by definition of e_r,

$$f_{r+1}(T_{r+1}) = e_{r+1} > e_r \geq f_r(T_{r+1}).$$

Hence

$$\frac{(p_n - p_{T_{r+1}}) \sum_{j \in T_{r+1}} q_j - \sum_{k=1}^{r} |T_{r+1} \cap S_k| e_k}{1 + \left| T_{r+1} \backslash \bigcup_{k=1}^{r} S_k \right|} >$$

$$\frac{(p_n - p_{T_{r+1}}) \sum_{j \in T_{r+1}} q_j - \sum_{k=1}^{r-1} |T_{r+1} \cap S_k| e_k}{1 + \left| T_{r+1} \backslash \bigcup_{k=1}^{r-1} S_k \right|}. \tag{3.6}$$

For (3.6) to hold, we must have that $T_{r+1} \cap S_r \neq \emptyset$. Denote

$$\alpha = (p_n - p_{T_{r+1}}) \sum_{j \in T_{r+1}} q_j - \sum_{k=1}^{r-1} |T_{r+1} \cap S_k| e_k.$$

Then, however, Inequality (3.6) would imply that

$$\frac{\alpha - |T_{r+1} \cap S_r| e_r}{1 + \left| T_r \backslash \bigcup_{k=1}^{r} S_k \right|} - \frac{\alpha}{1 + \left| T_{r+1} \backslash \bigcup_{k=1}^{r-1} S_k \right|} > 0,$$

and thus that

$$\Big(1 + \Big| T_{r+1} \backslash \bigcup_{k=1}^{r-1} S_k \Big|\Big)\Big(\alpha - |T_{r+1} \cap S_r| e_r\Big) - \Big(1 + \Big| T_{r+1} \backslash \bigcup_{k=1}^{r} S_k \Big|\Big)\alpha > 0,$$

resulting in

$$\Big(1 + \Big| T_{r+1} \backslash \bigcup_{k=1}^{r-1} S_k \Big|\Big)\alpha - \Big(1 + \Big| T_{r+1} \backslash \bigcup_{k=1}^{r-1} S_k \Big|\Big)|T_{r+1} \cap S_r| e_r >$$

$$\Big(1 + \Big| T_{r+1} \backslash \bigcup_{k=1}^{r} S_k \Big|\Big)\alpha,$$

and the following sequence of inequalities

$$|T_{r+1} \cap S_r|\alpha - \Big(1 + \Big| T_{r+1} \backslash \bigcup_{k=1}^{r-1} S_k \Big|\Big)|T_{r+1} \cap S_r| e_r > 0,$$

$$\alpha - \Big(1 + \Big|T_{r+1} \backslash \bigcup_{k=1}^{r-1} S_k\Big|\Big) e_r > 0,$$

$$\frac{\alpha}{1 + \Big|T_{r+1} \backslash \bigcup_{k=1}^{r-1} S_k\Big|} > e_r,$$

$$f_r(T_{r+1}) > e_r,$$

that establishes a contradiction.

Property (iii)

Regarding Property (iii). Let $S \in C^N \backslash \overline{\mathcal{D}}_\tau$.
We will prove that $S \in H(\{T \in \overline{\mathcal{D}}_\tau | E(T,x) \geq E(S,x)\})$.

Case 1: $n \in S$

Let $r \in \{1,\ldots,\tau\}$ be such that

$$\begin{cases} N\backslash\{j\} \in \mathcal{D}_r \text{ for some } j \in N\backslash S, \\ N\backslash\{i\} \in \mathcal{D}_s \text{ with } r \geq s \text{ for all } i \in N\backslash S. \end{cases}$$

Then, $N\backslash\{i\} \in \overline{\mathcal{D}}_r$ for all $i \in N\backslash S$ and by Lemma 3.5.3[4], for all $i \in N\backslash S$, $E(S,x) \leq E(N\backslash\{j\},x) \leq E(N\backslash\{i\},x)$.
Since $\mathbb{1}^S = (|S| - n + 1)\mathbb{1}^N + \sum_{i \in N\backslash S} \mathbb{1}^{N\backslash\{i\}}$, we have that $S \in H(\{T \in \overline{\mathcal{D}}_\tau : E(T,x) \geq E(S,x)\})$.

Case 2: $n \notin S$.

Let $S \in \mathcal{F}_\tau$. Since $f_\tau(S) \leq e_\tau$, we have

$$\begin{aligned} E(S,x) &= \sum_{j \in S} q_j(p_n - p_S) - \sum_{k=1}^{\tau} |S \cap S_k| e_k \\ &= f_\tau(S)(1 + |S \cap S_\tau|) - |S \cap S_\tau| e_\tau \\ &\leq e_\tau(1 + |S \cap S_\tau|) - |S \cap S_\tau| e_\tau \\ &= e_\tau. \end{aligned}$$

[4]Note that since $e_1 \leq 0$, Property (ii) implies that $e_r \leq 0$ for all $r \in \{2,\ldots,\tau\}$. Hence, $x_i \leq DP_i(N,q,p)$, and thus $x_i \leq w(N) - w(N\backslash\{i\})$ for all $i \in N\backslash S$. The condition in Lemma 3.5.3 is satisfied.

Let $S \notin \mathcal{F}_\tau$ and let $r \in \{1, \ldots, \tau - 1\}$ be such that $S \in \mathcal{F}_r \backslash \mathcal{F}_{r+1}$. Then $S \subset \bigcup_{k=1}^{r} S_k$ and since $f_r(S) \leq e_r$ we have

$$\begin{aligned} E(S, x) &= \sum_{j \in S} q_j(p_n - p_S) - \sum_{k=1}^{r} |S \cap S_k| e_k \\ &= f_r(S)(1 + |S \cap S_r|) - |S \cap S_r| e_r \\ &\leq e_r(1 + |S \cap S_r|) - |S \cap S_r| e_r \\ &= e_r. \end{aligned}$$

Let $i \in S$. Since $S \subset \bigcup_{k=1}^{r} S_k$, there is an S_k such that $i \in S_k$ and hence $N \backslash \{i\} \in \mathcal{D}_k$, with $k \leq r$. Moreover $\mathbb{1}^S = |S| \mathbb{1}^N - \sum_{i \in S} \mathbb{1}^{N \backslash \{i\}}$. Hence $S \in H(\{T \in \overline{\mathcal{D}}_\tau : E(T, x) \geq E(S, x)\})$. □

In the proof of Theorem 3.5.2 we showed that $e_r \leq 0$ for all $r \in \{1, \ldots, \tau\}$. Here, e_r is the excess of coalition T_r. Thus the nucleolus of an MCP-game modifies the Direct Price solution. It can be interpreted in the following way. Every player in S_j should pay a fee $-e_j$ to player n for using his discounted unit price. Thus a nucleolus-determinant puts every player in a certain fee-class and determines the heights of those fees. From Proposition 3.5.1 it is clear that if $p_{n-1} = p_n$, the fee of all players equals zero.
Before showing that the nucleolus of an MCP-game can be found using an algorithm with polynomial time-complexity, we want to make a short remark.

Remark
We sketch how a similar algorithm for determining the nucleolus of an MCP-game can be derived from the results of Arin and Feltkamp (1997). That paper introduces an algorithm for computing the nucleolus of a veto-rich game: games in which for all coalitions one of the players is needed in order to obtain a positive payoff. The algorithm they develop is exponential in the number of players. Formally, we can transform an MCP-game (N, w) into a veto-rich game $(N, \bar{w})$ with veto-player n, by defining for all $S \in 2^N \backslash \{\emptyset\}$

$$\bar{w}(S) = \sum_{j \in S} w(\{i, n\}) - (w(N) - w(N \backslash S)).$$

One can show that for every game with a nonempty core (N,v), $nuc(N,v) = -nuc(N,-v^*)$ where (N,v^*) represents the dual game of (N,v), *i.e.*, for all $S \in 2^N$

$$v^*(S) = v(N) - v(N \backslash S).$$

Also for any additive game $a \in \mathbb{R}^N$, $nuc(v^a) = nuc(v) + a$, where for all $S \in 2^N$

$$v^a(S) = v(S) + \sum_{j \in S} a_j.$$

Since $(N,\bar{w})$ is an additive game minus the dual game of (N,w), the nucleolus of (N,w) can be directly obtained from the nucleolus of $(N,\bar{w})$.

However, when applying the general algorithm of Arin and Feltkamp (1997), one does not obtain a direct interpretation of the allocations proposed by the nucleolus in terms of the parameters of the underlying MCP-situation. Therefore, we have chosen to develop a situation-specific algorithm, that explicitly depends on the prices p and the quantities q. Moreover, the algorithm allows for a specific acceleration step to make it polynomial, as is shown below.

Example 3.5.4 Consider the following MCP-situation (N,q,p), with $N = \{1,2,3,4\}$, $q = (10,45,100,250)$ and $p = (10,8,7,5)$. Let (N,w) be the corresponding MCP-game. We are going to compute the nucleolus of the corresponding MCP-game (N,w), using the nucleolus-determinant recursion. We have $\mathcal{F}_1 = \{S | S \subset \{1,2,3\}\}$ and the values for $f_1(S)$, $S \in \mathcal{F}_1$ are

Table 3.5.1: Values of $f_r(S)$

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$f_1(S)$	$\mathbf{-25}$	-67.5	-100	-55	$-73\frac{1}{3}$	$-96\frac{2}{3}$	$-77\frac{1}{2}$
$f_2(S)$		$\mathbf{-67\frac{1}{2}}$	-100	-70	$-97\frac{1}{2}$	$-96\frac{2}{3}$	-95
$f_3(S)$			-100		$\mathbf{-97\frac{1}{2}}$	$-114\frac{1}{4}$	$-108\frac{3}{4}$

presented in Table 3.5.1, in the second row. From this table we can conclude that $\tilde{\mathcal{F}}_1 = \{1\}$, hence $T_1 = \{1\}$, $S_1 = \{1\}$ and $e_1 = -25$. From this step in the recursion one can conclude that: $nuc_1(w) = 50 - 25 = 25$ and

$$\begin{cases} E(\{1\}, nuc(w)) & = -25 \\ E(N\backslash\{1\}, nuc(w)) & = -25. \end{cases}$$

The next step in the nucleolus-determinant recursion is to find $T_2 \in \tilde{\mathcal{F}}_2$, where $\mathcal{F}_2$ equals $\{\{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. The values for $f_2(S)$, $S \in \mathcal{F}_2$, are also given in Table 3.5.1, in the third row. Thus $T_2 = S_2 = \{2\}$ with $e_2 = -67.5$. Hence $nuc_2(w) = 67.5$ and

$$\begin{cases} E(\{2\}, nuc(w)) & = -67.5 \\ E(N\backslash\{2\}, nuc(w)) & = -67.5. \end{cases}$$

Since the nucleolus is efficient, we also know the excesses of $\{1,2\}$ and $\{3,4\}$,

$$\begin{cases} E(\{1,2\}, nuc(w)) & = w(\{1,2\}) - (nuc_1(w) + nuc_2(w)) = -72.5 \\ E(N\backslash\{1,2\}, nuc(w)) & = w(\{3,4\}) - (nuc_3(w) + nuc_4(w)) \\ & = w(\{3,4\}) - (w(N) - nuc_1(w) - nuc_2(w)) \\ & = -92.5 \end{cases}$$

Hence, $\mathcal{F}_3 = \{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$, and $f_3(S)$, $S \in \mathcal{F}_3$, can be found in the last row of Table 3.5.1. In this step we find that $T_3 = \{1,3\}$ thus $S_3 = \{3\}$ with $e_3 = -97.5$. Thus for player 3, $nuc_3(w) = 102.5$. Now $S_1 \cup S_2 \cup S_3 = N\backslash\{4\}$, hence we can stop and determine $nuc_4(w) = -(-25 - 67.5 - 97.5) = 190$.

This procedure is faster than the general way of computing the nucleolus, but it is still exponential in the number of players. In the next paragraphs we again calculate the nucleolus of this example, but use a polynomial method. If we look closer at the function $f_1(S)$, we can find a coalition T_1 belonging to $\tilde{\mathcal{F}}_1$ more efficiently. To do so, we need to maximize

$$\frac{\sum_{j \in S} q_j (p_4 - p_S)}{1 + |S|},$$

over all coalitions in $\mathcal{F}_1$. Since p_4 is always less than the unit prices of players 1, 2 or 3, the fraction is negative. Thus it is wise to have p_S and $|S|$ large, but $\sum_{j \in S} q_j$ low. So we need to determine the price setter of S, and then add smaller players to increase $|S|$ and keep $\sum_{j\in S} q_j$ small. If we have that player 3 is the price setter and it is beneficial to add player 2, then adding player 1 must also increase the fraction. Thus T_1 has the following structure,

$$T_1 = \{1, \ldots, m, z\},$$

where z is the largest player in the coalition, the price setter, and we add smaller players 1 up to m. If $m = 0$, then $T_1 = \{z\}$. For this example we can make the following combinations:

(m, z)	$(0, 1)$	$(1, 2)$	$(1, 3)$	$(0, 2)$	$(2, 3)$	$(0, 3)$
S	$\{1\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2\}$	$\{1, 2, 3\}$	$\{3\}$

In Table 3.5.1 we can see that $f_1(\{1\})$ is maximal, thus $z = 1$.

Also we can speed up the search for a coalition T_2 belonging to $\tilde{\mathcal{F}}_2$. We need to maximize the following fraction

$$\frac{\sum_{j\in S} q_j(p_4 - p_S) - |S \cap S_1|e_1}{1 + |S \backslash S_1|},$$

over all $S \in \mathcal{F}_2$. Either $p_S = p_2$ or $p_S = p_3$, since $\{1\} \notin \mathcal{F}_2$. If adding player k not yet in S_1 increases the fraction, then also adding player $j < k$ with $j \notin S_1$ increases the fraction. We only add player 1 if $q_1(p_4 - p_z) < e_1$. Thus T_2 has the following structure

$$T_2 = \big(\{1, \ldots, m, z\} \cap N\backslash S_1\big) \cup \{i \in \{1\} : q_i(p_4 - p_z) > e_i\}.$$

Since $q_1(p_4 - p_3) = -20 > -25$, it is always beneficial to add player 1 to a coalition with player 3 being the price setter. Hence we start by comparing the values of $f_2(\{1, 3\})$ and $f_2(\{1, 2, 3\})$ and find that $f_2(\{1, 2, 3\})$ is the highest. Since $q_1(p_4 - p_2) = -30 < -25$, we do not want to add player 1 to a coalition with player 2 being the price setter. Now we need to compare $f_2(\{1, 2, 3\})$ with $f_2(\{2\})$ and we find that $f_2(\{2\})$ is the highest, hence $T_2 = \{2\}$.

Following the same reasoning as for step 2 in the recursion we know that $p_S = p_3$ and T_3 has the following structure

$$T_3 = \{3\} \cup \{i \in \{1, 2\} : q_i(p_4 - p_3) > e_i\}.$$

We only add player 1 if $q_1(p_4 - p_3) > e_1$ and player 2 if $q_2(p_4 - p_3) > e_2$. In the previous round we found that $q_1(p_4 - p_3) > e_1$, hence we add player 1 to player 3 and since $q_2(p_4 - p_3) = -90 < -67.5$ we do not add player 2. Thus $T_3 = \{1, 3\}$. ◁

Example 3.5.4 shows that one can speed up the Nucleolus-determinant recursion at the point of determining a coalition T_r belonging to $\tilde{\mathcal{F}}_r$. T_1 has the structure $\{1, \ldots, m, z\}$ for some $0 \leq m < z \leq n - 1$, thus $\sum_{z=1}^{n-1} z = \frac{1}{2}(n-1)(n-2)$ numbers need to be compared to find T_1. Furthermore T_r for $r \in \{2, \ldots, \tau\}$ has the structure

$$\{z\} \cup \left(\{1, \ldots, m\} \cap \left[N \backslash \Big(\bigcup_{k=1}^{r-1} S_k \cup \{n\} \Big) \right] \right) \\ \cup \left\{ j \in S_k : q_j(p_n - p_z) - e_k > 0, j < z \right\}_{k=1}^{r-1}.$$

Fix z and look which m maximizes

$$f_r \left(\{z\} \cup \left(\{1, \ldots, m\} \cap \left[N \backslash \Big(\bigcup_{k=1}^{r-1} S_k \cup \{n\} \Big) \right] \right) \right).$$

Then, we add players from $\bigcup_{k=1}^{r-1} S_k$ if that further increases f_r. Hence, by comparing at most $(n-1)^2$ numbers, we can find $T_r \in \tilde{\mathcal{F}}_r$ for all $r \in \{2, \ldots, \tau\}$ with $\tau \leq n - 1$. From these observations we can readily derive that the nucleolus of an MCP-game can be found in polynomial time.

Theorem 3.5.5 *Let $(N, q, p) \in \mathcal{M}$ be an MCP-situation with (N, w) the corresponding MCP-game. Then, $nuc(w)$ can be determined in $\mathcal{O}(n^3)$ time.*

3.6 The Shapley value of MCP-games

In this section we analyze the Shapley value (Shapley (1953)) of an MCP-game and show that it is a suitable allocation method for an MCP-situation. An explicit context-specific expression for the Shapley value is provided.

Originally the Shapley value is introduced as the average of all $|N|!$ marginal vectors of a TU-game. We introduced marginal vectors in Section 3.3 in Equation (3.5). Let (N, v) be a TU-game. Then, the Shapley value $\phi(v) \in \mathbb{R}^N$ is given by

$$\phi(v) = \sum_{\sigma \in \Pi(N)} \frac{m^\sigma(v)}{|N|!}.$$

There is another method for calculating the Shapley value of a TU-game, which we will use in determining the Shapley value of an MCP-game. For this we need the notion of unanimity games. For $T \in 2^N \backslash \{\emptyset\}$ the unanimity game (N, u_T) is defined by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $S \in 2^N$. Thus the unanimity game (N, u_T) states that without all players in T, a coalition $S \in 2^N$ has value zero.

Every TU-game (N, v) can be written in a unique way as a linear combination of unanimity games, *i.e.* there is a unique vector $c \in \mathbb{R}^{2^N \backslash \{\emptyset\}}$, such that

$$v = \sum_{T \in 2^N \backslash \{\emptyset\}} c_T u_T.$$

Example 3.6.1 We consider the MCP-situation of Example 3.2.1 without player 4. Hence $N = \{1, 2, 3\}$, $q = (2, 4, 8)$ and $p(t) = 10 + \frac{12}{t}$, for $t \in \mathbb{R}_+$. We are going to decompose the corresponding MCP-game into a linear combination of unanimity games. The following table represents the corresponding MCP-game, and for illustrative purposes also the unanimity games $u_{\{1\}}$, $u_{\{1,2\}}$ and u_N.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$w(S)$	0	0	0	6	9	6	15
$u_{\{1\}}(S)$	1	0	0	1	1	0	1
$u_{\{1,2\}}(S)$	0	0	0	1	0	0	1
$u_N(S)$	0	0	0	0	0	0	1

To find a vector $c \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$ such that $w = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T$, we start with the singletons. We have that $c_{\{1\}} u_{\{1\}} = w(\{1\})$, hence $c_{\{1\}} = 0$. Similarly $c_{\{2\}} = c_{\{3\}} = 0$. Secondly, $c_{\{1,2\}} u_{\{1,2\}} + c_{\{1\}} u_{\{1\}} + c_{\{2\}} u_{\{2\}} = w(\{1,2\})$. Hence, $c_{\{1,2\}} = w(\{1,2\}) = 6$ and similarly $c_{\{1,3\}} = w(\{1,3\}) = 9$ and $c_{\{2,3\}} = w(\{2,3\}) = 6$. Finally, $c_{\{1,2,3\}} = w(\{1,2,3\}) - \sum_{T \in 2^N \setminus N} c_T = 15 - 6 - 9 - 6 = -6$. ◁

In case of an MCP-game, the coefficients for a linear decomposition into unanimity games, can be determined quite easily using Theorem 3.2.2. For all $T \in 2^N$ with $|T| \geq 2$ define $a_T, b_T \in T$ such that $T \cap \{1, \ldots, b_T\} = \{a_T, b_T\}$, *i.e.*, player a_T is the player within T with lowest index and player b_T is the player within T with second lowest index.

Theorem 3.6.2 *Let (N, w) be an MCP-game corresponding to an MCP-situation $(N, q, p) \in \mathcal{M}$. Let $c \in \mathbb{R}^{2^N \setminus \{\emptyset\}}$ be such that*

$$w = \sum_{T \in 2^N \setminus \{\emptyset\}} c_T u_T.$$

Then, for $|T| = 1$, $c_T = 0$ and for $|T| \geq 2$, we have that

$$c_T = (-1)^{|T|} w(\{a_T, b_T\}). \tag{3.7}$$

In the proof of Theorem 3.6.2, we use the following lemma.

Lemma 3.6.3 *Let $a \in \mathbb{N}, a \geq 1$. Then*

(i) $\sum_{j=0}^{a} \binom{a}{j} (-1)^j = 0,$

(ii) $\sum_{j=0}^{a-1} \binom{a}{j} (-1)^j = \begin{cases} -1 & \text{if } a \text{ is even,} \\ 1 & \text{if } a \text{ is odd.} \end{cases}$

Proof: (i) Clearly,

$$\sum_{j=0}^{a} \binom{a}{j} (-1)^j = \sum_{j=0}^{a} \binom{a}{j} (-1)^j (1)^{a-j} = (1-1)^a = 0,$$

where the second equality follows from the binomium of Newton.
(ii) This follows immediately from (i). □

Proof of Theorem 3.6.2: First note that for all $T \in 2^N \backslash \{\emptyset\}$ the coefficients in the decomposition of w can be determined recursively in the following way:

$$c_T = w(T) - \sum_{S \in 2^N \backslash \{\emptyset\}, S \subsetneq T} c_T. \tag{3.8}$$

Hence, for $|T| = 1$, $c_T = w(\{i\}) = 0$. Let $|T| = 2$. According to (3.8)

$$c_T = w(T) = (-1)^{|T|} w(T) = (-1)^{|T|} w(\{a_T, b_T\}).$$

Thus for $|T| = 2$, (3.7) is valid. To prove (3.7), we will use induction on the number of players in T.

Assume that for all $|T| \leq k - 1$

$$c_T = (-1)^{|T|} w(\{a_T, b_T\}).$$

Let $|T| = k$ and for ease of notation we set $T = \{t_1, t_2, \ldots, t_k\}$ such that $q_{t_1} \leq q_{t_2} \leq \ldots \leq q_{t_k}$. Clearly $a_T = t_1$ and $b_T = t_2$. Then, by (3.8) and by Theorem 3.2.2, we have

$$c_T = \sum_{h=1}^{k} w(\{t_h, t_k\}) - \sum_{S \in 2^N \backslash \{\emptyset\}, S \subsetneq T} c_S. \tag{3.9}$$

The last sum in (3.9) can be rewritten by counting the number of proper sub-coalitions S containing player t_h and t_i with $h < i$ such that $S \cap \{1, 2, \ldots, t_i\} = \{t_h, t_i\}$. For $\{t_h, t_i\}$ with $i \geq 3$, this number is $\sum_{j=0}^{k-i} \binom{k-i}{j}$

and for $\{t_1, t_2\}$ this number is $\sum_{j=0}^{k-3} \binom{k-2}{j}$. Hence

$$\begin{aligned}
\sum_{\substack{S\in 2^N\setminus\{\emptyset\}\\ S\subsetneq T}} c_S = & \sum_{i=3}^{k}\sum_{h=1}^{i-1} w(\{t_h,t_i\}) \sum_{j=0}^{k-i}\binom{k-i}{j}(-1)^{j+2} \\
& + w(\{t_1,t_2\})\sum_{j=0}^{k-3}\binom{k-2}{j}(-1)^{j+2} \\
= & \sum_{h=1}^{k-1} w(\{t_h,t_k\}) + \sum_{i=3}^{k-1}\sum_{h=1}^{i-1} w(\{t_h,t_i\}) \sum_{j=0}^{k-i}\binom{k-i}{j}(-1)^{j} \\
& + w(\{t_1,t_2\})\sum_{j=0}^{k-3}\binom{k-2}{j}(-1)^{j} \\
= & \sum_{h=1}^{k-1} w(\{t_h,t_k\}) + w(\{t_1,t_2\})(-1)^{k-1}.
\end{aligned}$$

The third equality follows from Lemma 3.6.3(i) and 3.6.3(ii). Hence

$$\begin{aligned}
c_T = & \sum_{h=1}^{k-1} w(\{t_h,t_k\}) - \sum_{h=1}^{k-1} w(\{t_h,t_k\}) \quad - w(\{t_1,t_2\})(-1)^{k-1} \\
= & -w(\{t_1,t_2\})(-1)^{k-1} \\
= & \; w(\{a_T,b_T\})(-1)^{|T|}.
\end{aligned}$$

□

For any TU-game (N, v) with $v = \sum_{T\in 2^N\setminus\{\emptyset\}} c_T u_T$, the unique decomposition into unanimity games, the Shapley value is for all $i \in N$ given by,

$$\phi_i(v) = \sum_{T\in 2^N\setminus\{\emptyset\}, i\in T} \frac{c_T}{|T|}. \tag{3.10}$$

Hence, using the decomposition in Theorem 3.6.2, we can compute the Shapley value of an MCP-game. For all $i \in N$, define $PP(i) = \{(k,\ell)|k < \ell, \ell < i, k \in N, \ell \in N\}$ as the collection of preceding pairs, pairs of players with index smaller than i.

Theorem 3.6.4 *Let (N, w) be an MCP-game corresponding to $(N, q, p) \in$*

$\mathcal{M}$. *Then for all $i \in N$, the Shapley value is given by*

$$\begin{aligned}\phi_i(w) = &\sum_{k=1}^{i-1} \frac{1}{(n-i+2)(n-i+1)} w(\{k,i\}) \\ &+ \sum_{m=i+1}^{n} \frac{1}{(n-m+2)(n-m+1)} w(\{i,m\}) \\ &- \sum_{(k,\ell)\in PP(i)} \frac{2}{(n-\ell+2)(n-\ell+1)(n-\ell)} w(\{k,\ell\}). \qquad (3.11)\end{aligned}$$

In the proof of Theorem 3.6.4 we use the following lemma.

Lemma 3.6.5 (Poblete, Munro, and Papadakis (2006) , Table 1) *For $t \in \mathbb{N}$ let $a_t = \dfrac{1}{x+t}$ for some fixed $x \in \mathbb{R}_+$. Then, for all $m \in \mathbb{N}$*

$$\sum_{t=0}^{m} \binom{m}{t} a_t (-1)^t = \frac{(x-1)!m!}{(x+m)!}.$$

Proof of Theorem 3.6.4: Let $i \in N$ and let $\mathcal{S}$ be the collection of all coalitions T containing i with $|T| \geq 2$. By Theorem 3.6.2 and (3.10) we get

$$\begin{aligned}\phi_i(w) = &\sum_{T\in\mathcal{S}} \frac{1}{|T|}(-1)^{|T|} w(\{a_T, b_T\}) \\ = &\sum_{T\in\mathcal{S}_1} \frac{(-1)^{|T|} w(\{a_T, b_T\})}{|T|} + \sum_{T\in\mathcal{S}_2} \frac{(-1)^{|T|} w(\{a_T, b_T\})}{|T|} \\ &+ \sum_{T\in\mathcal{S}_3} \frac{(-1)^{|T|} w(\{a_T, b_T\})}{|T|}, \qquad (3.12)\end{aligned}$$

where

(i) $\mathcal{S}_1 = \{T \in \mathcal{S} | i = b_T\}$,

(ii) $\mathcal{S}_2 = \{T \in \mathcal{S} | i = a_T\}$,

(iii) $\mathcal{S}_3 = \{T \in \mathcal{S} | i > b_T\}$.

For $T \in \mathcal{S}_1$, setting $|T| = 2 + t$ with $t \in \{0, \dots, n-2\}$ and for one particular $k < i$ at place a_T we have

$$\begin{aligned}
\sum_{T\in\mathcal{S}_1:k=a_T} \frac{c_T}{|T|} &= \sum_{t=0}^{n-i} \frac{1}{t+2}\binom{n-i}{t}(-1)^{t+2}w(\{k,i\}) \\
&= w(\{k,i\})\sum_{t=0}^{n-i}\binom{n-i}{t}(-1)^t\frac{1}{t+2} \\
&= w(\{k,i\})\frac{(2-1)!(n-i)!}{(n-i+2)!} \\
&= w(\{k,i\})\frac{1}{(n-i+2)(n-i+1)},
\end{aligned}$$

where the third equality follows from Lemma 3.6.5. Hence

$$\sum_{T\in\mathcal{S}_1} \frac{c_T}{|T|} = \sum_{k=1}^{i-1}\sum_{T\in\mathcal{S}_1:k=a_T}\frac{c_T}{|T|} = \sum_{k=1}^{i-1}\frac{1}{(n-i+2)(n-i+1)}w(\{k,i\}). \quad (3.13)$$

For $T \in \mathcal{S}_2$, setting $|T| = 2 + t$ with $t \in \{0, \dots, n-2\}$ and for one particular $m > i$ at place b_T we have

$$\begin{aligned}
\sum_{T\in\mathcal{S}_2:m=b_T} \frac{c_T}{|T|} &= \sum_{t=0}^{n-m} \frac{1}{t+2}\binom{n-m}{t}(-1)^{t+2}w(\{i,m\}) \\
&= w(\{i,m\})\frac{1}{(n-m+2)(n-m+1)}.
\end{aligned}$$

Hence

$$\sum_{T\in\mathcal{S}_2} \frac{c_T}{|T|} = \sum_{m=i+1}^{n}\frac{1}{(n-m+2)(n-m+1)}w(\{i,m\}). \quad (3.14)$$

For $T \in \mathcal{S}_3$, setting $|T| = 3 + t$ with $t \in \{0, \dots, n-3\}$ and for one particular $\ell < i$ and $\ell > k$ at place b_T and k at place a_T we have

$$\begin{aligned}
\sum_{\substack{T\in\mathcal{S}_3:\\ a_T=k,b_T=\ell}} \frac{c_T}{|T|} &= \sum_{t=0}^{n-\ell-1} \frac{1}{t+3}\binom{n-\ell-1}{t}(-1)^{t+3}w(\{k,\ell\}) \\
&= (-1)w(\{k,\ell\})\frac{(3-1)!(n-\ell)!}{(n-\ell+3)!} \\
&= w(\{k,\ell\})\frac{-2}{(n-\ell+2)(n-\ell+1)(n-\ell)}.
\end{aligned}$$

Hence

$$\sum_{T\in\mathcal{S}_3}\frac{c_T}{|T|}=\sum_{(k,\ell)\in PP(i)}\frac{-2}{(n-\ell+2)(n-\ell+1)(n-\ell)}w(\{k,\ell\}). \tag{3.15}$$

Filling (3.13), (3.14) and (3.15) in (3.12) results in (3.11) for all $i\in N$. □

Hence, instead of computing all marginal vectors, one can determine the Shapley value of an MCP-game in a single step. The following example explains how the Shapley value of an MCP-game can be interpreted.

Example 3.6.6 Consider the following 5-player MCP-game corresponding to a 5-player MCP-situation with $q=(1,2,3,4,10)$ and $p_1=12$, $p_2=11$, $p_3=9.5$, $p_4=7.5$ and $p_5=4$. The table below provides the values of the 2-player coalitions only. The other coalitional values can be easily determined using Theorem 3.2.2, *e.g.* $w(N)=w(\{1,5\})+w(\{2,5\})+w(\{3,5\})+w(\{4,5\})=8+14+16.5+14=52.5$.

S	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}
$w(S)$	1	2.5	4.5	8	3
S	{2,4}	{2,5}	{3,4}	{3,5}	{4,5}
$w(S)$	7	14	6	16.5	14

Expression (3.11) can be split in three parts:

- a positive part due to cost savings with players with a lower index,
- a positive part due to cost savings with players with a larger index,
- a negative part, due to paybacks to pairs of players with a smaller index (the preceding pairs).

For this example, the three parts are represented in the Tables 3.6.1, 3.6.2 and 3.6.3, respectively.
Table 3.6.1 indicates that player 3 gets $\frac{1}{12}$ of the cost savings he can obtain with player 1 and also $\frac{1}{12}$ of the cost savings he can make with player 2, while player 5 gets $\frac{1}{2}$ of the cost savings he can make with any of the other players. Note that all fractions are the same per row, *i.e.*, per player under

consideration.
Table 3.6.2 indicates that player 3 gets $\frac{1}{6}$ of the cost savings he can make with player 4 and $\frac{1}{2}$ of the cost savings he can make with player 5. Note that in Table 3.6.2 all fractions are the same per column, *i.e.*, per player with larger index. Moreover, the matrix in Table 3.6.2 is the transpose of the matrix in Table 3.6.1. These two tables represent what the players receive, but in

Table 3.6.1: Cost savings with players with lower index

Smaller	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
Player 1					
Player 2	$\frac{1}{20}w(\{1,2\})$				
Player 3	$\frac{1}{12}w(\{1,3\})$	$\frac{1}{12}w(\{2,3\})$			
Player 4	$\frac{1}{6}w(\{1,4\})$	$\frac{1}{6}w(\{2,4\})$	$\frac{1}{6}w(\{3,4\})$		
Player 5	$\frac{1}{2}w(\{1,5\})$	$\frac{1}{2}w(\{2,5\})$	$\frac{1}{2}w(\{3,5\})$	$\frac{1}{2}w(\{4,5\})$	

Table 3.6.2: Cost savings with players with higher index

Larger	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
Player 1		$\frac{1}{20}w(\{1,2\})$	$\frac{1}{12}w(\{1,3\})$	$\frac{1}{6}w(\{1,4\})$	$\frac{1}{2}w(\{1,5\})$
Player 2			$\frac{1}{12}w(\{2,3\})$	$\frac{1}{6}w(\{2,4\})$	$\frac{1}{2}w(\{2,5\})$
Player 3				$\frac{1}{6}w(\{3,4\})$	$\frac{1}{2}w(\{3,5\})$
Player 4					$\frac{1}{2}w(\{4,5\})$
Player 5					

Table 3.6.3: Paybacks

Negative	{1,2}	{1,3}	{1,4}	{2,3}	{2,4}	{3,4}
Player 1						
Player 2						
Player 3	$-\frac{1}{30}w(\{1,2\})$					
Player 4	$-\frac{1}{30}w(\{1,2\})$	$-\frac{1}{12}w(\{1,3\})$		$-\frac{1}{12}w(\{2,3\})$		
Player 5	$-\frac{1}{30}w(\{1,2\})$	$-\frac{1}{12}w(\{1,3\})$	$-\frac{1}{3}w(\{1,4\})$	$-\frac{1}{12}w(\{2,3\})$	$-\frac{1}{3}w(\{2,4\})$	$-\frac{1}{3}w(\{3,4\})$

total we allocate too much this way. If we sum all entries in the last row in Table 3.6.1 and the last column in Table 3.6.2, we already obtain $w(N)$. So we need to get back the sum of all remaining entries, given by

$$\frac{1}{10}w(\{1,2\})+\frac{1}{6}w(\{1,3\})+\frac{1}{6}w(\{2,3\})+\frac{1}{3}w(\{1,4\})+\frac{1}{3}w(\{2,4\})+\frac{1}{3}w(\{3,4\}).$$

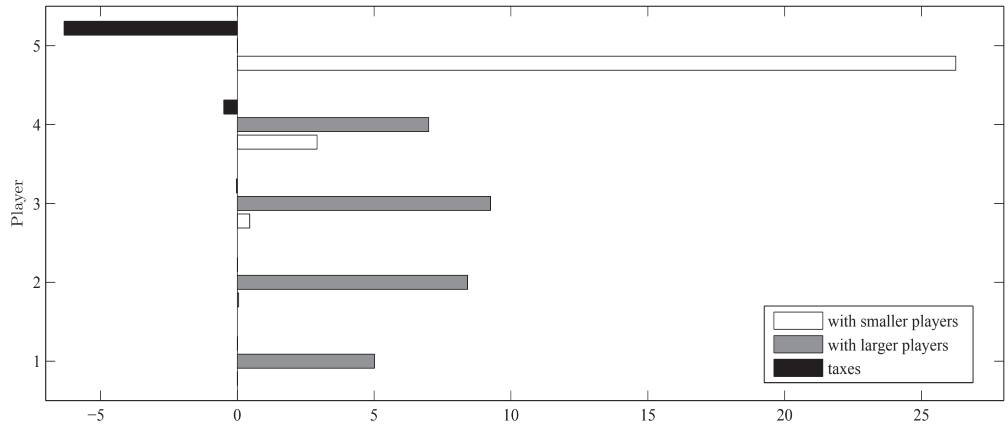

Figure 3.6.1: Composition of the Shapley value of Example 2.6.6

(3.16)

According to the remaining negative part of (3.11), each two-player component of (3.16) is paid by all players with higher index than the player with the highest index in the two-person coalition at hand. One can think of this payback as a taxation on the basis of the relative size of the order quantity. Table 3.6.3 indicates that $\frac{1}{6}w(\{1,3\})$ is paid back equally by players 4 and 5. Then, the Shapley value for player 3 equals

$$\begin{aligned}\phi_3(w) &= \left[\frac{1}{12}w(\{1,3\}) + \frac{1}{12}w(\{2,3\})\right] \\ &\quad + \left[\frac{1}{6}w(\{3,4\}) + \frac{1}{2}w(\{3,5\})\right] \\ &\quad - \left[\frac{1}{30}w(\{1,2\})\right] = 9\frac{27}{40}.\end{aligned}$$

In fact

$$\phi(w) = \left(5\frac{1}{120}, 8\frac{7}{15}, 9\frac{27}{40}, 9\frac{17}{40}, 19\frac{37}{40}\right).$$

In Figure 3.6.1 the three building blocks of the Shapley value of this example are visualized. The white bars represent the cost savings with players with lower index, the grey bars are the cost savings with players with higher index and the black bars represent paybacks. ◁

In fact, Theorem 3.6.2 and Theorem 3.6.4 can be generalized to all zero-normalized and nonnegative TU-games with an ordering on the players such that (3.3) and (3.4) are satisfied. A game, however, that satisfies (3.3) and (3.4) does not need to be an MCP-game. This is illustrated in the following example.

Example 3.6.7 Consider the 5-person TU-game (N, z) that satisfies (3.4) and the values of all two-player coalitions given by the following table.

S	{1,2}	{1,3}	{2,3}	{1,4}	{2,4}
$z(S)$	4	7	5	9	7
S	{3,4}	{1,5}	{2,5}	{3,5}	{4,5}
$z(S)$	6	11	9	7	5

Clearly, (3.3) is satisfied. Suppose (N, z) is an MCP-game. Thus there are vectors q and p such that (3.2) holds for all $i, j \in N$ with $i < j$. In particular we find that $p_2 = p_1 - \frac{4}{q_1}$, $p_3 = p_1 - \frac{7}{q_1}$ and $p_4 = p_1 - \frac{9}{q_1}$.
Then, however, $z(\{2,3\}) = q_2(p_2 - p_3) = \frac{3q_2}{q_1}$, $z(\{2,4\}) = \frac{5q_2}{q_1}$ and hence $\frac{z(\{2,4\})}{z(\{2,3\})} = \frac{5}{3}$, which contradicts the fact that $z(\{2,4\}) = 7$ and $z(\{2,3\}) = 5$. $\triangleleft$

3.7 Numerical examples

This section takes a numerical look at the Direct Price solution, the nucleolus and the Shapley value as allocation rules for MCP-situations. In the previous sections we have discussed the three solution concepts from an analytical point of view. The Direct Price solution lets every player pay the lowest available unit price, provided by the largest player. The nucleolus lets every player pay a fee for using the low unit price of the largest player. These fees correspond to the determining excesses of the nucleolus. Whereas the nucleolus only looks at price reductions due to the presence of the largest player, the Shapley value considers all pairwise cost savings of the players. Depending on a player's relative size, he can gain more cost savings with another player. In order to prevent too skewed cost allocations, small players are subsidized for the fact that they could obtain some cost savings without

the presence of larger players. These subsidies are paid by taxing players with higher cost savings.
In practice, organizations that join a purchasing cooperative might not know the exact details of their fellow cooperation members, except the quantity discount of the largest player. Naturally, they like to know what their share in the total cost savings would be. Therefore we will simulate for several instances the expected allocation of cost savings to an organization that joins an MCP-situation, according to these three solution concepts.

As input for the simulation we take 5-player MCP-situations with integer-valued order quantities and with fixed q_1 and q_5. The order quantities q_2, q_3 and q_4 are unknown, but in between q_1 and q_5. We restrict to cases where $q_{n-1} < q_n$ and we use the following unit price function:

$$p(t) = 10 + \frac{12}{t^x},$$

with $t \in (0, \infty)$ and $x \in [0.25, 1.25]$. This type of unit price functions adequately represents most quantity discount schemes seen in practice, as explained by Schotanus (2007). We determine by simulation what a random player with order quantity $q_1 \leq t < q_5$ can expect as his share in the cost savings — according to the Direct Price solution, the nucleolus or the Shapley value — in such an MCP-situation. For MCP-situations with a larger group of players, similar results can be obtained.
One step of the simulation is executed as follows. For fixed q_1, q_5 and x, the order quantities q_2, q_3 and q_4 are randomly and simultaneously drawn from a discrete uniform distribution. Then, the Direct Price solution, the nucleolus and the Shapley value of the corresponding MCP-game are calculated. We are interested in the share in cost savings for a player with order quantity $t \in \{q_1, q_1 + 1, q_1 + 2, \ldots, q_5 - 1\}$, regardless of the fact that he is player 2, 3 or 4. Hence, we store these allocations per different value of q_i, independent of the index i. This run is repeated successively. Then, for every $t \in \{q_1, q_1 + 1, \ldots, q_5 - 1\}$ we average the stored Direct Price solutions, nucleoli and Shapley values over the number of times they have appeared. This simulation is executed for twelve different instances. First, for $q_1 = 1$ and $q_5 = 20$ and for each of the discount parameters $x = 0.3$, $x = 0.5$, $x = 0.8$ and

$x = 1.1$ we derived approximately 1500 MCP-situations. Second, for $q_1 = 1$, $q_5 = 40$ and for each of the discount parameters $x = 0.3$, $x = 0.5$, $x = 0.8$ and $x = 1.1$ we derived approximately 11000 MCP-situations. And third, for $q_1 = 10$ and $q_5 = 50$ we derived approximately 11000 MCP-situations, for $x = 0.3$, $x = 0.5$, $x = 0.8$ and $x = 1.1$. For larger values of q_1 and q_5 the attainable quantity discounts for the several players become very small.

In Figure 3.7.1 one can find the results of the simulations with respect to the MCP-situations with $q_1 = 1$, $q_5 = 20$ and the four different discount parameters x. For fixed x, we have 19 different values of t, $1, 2, \ldots, 19$, each having an expected share in cost savings according to the Direct Price solution, the nucleolus and the Shapley value. The 19 points belonging to the

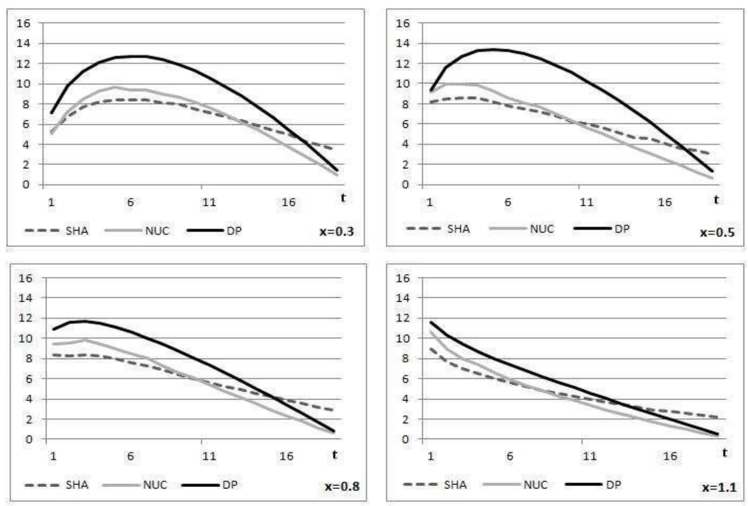

Figure 3.7.1: Expected payoffs in a 5-player MCP-game with $q_1 = 1$ and $q_5 = 20$

nucleolus and the 19 points belonging to the Shapley value are connected by the light grey line and dotted grey line, respectively. The black line connects the 19 points belonging to the Direct Price solution. *E.g.* a point on the light grey line represents the expected share of cost savings for a player with order quantity t in such an MCP-situation according to the nucleolus.
In Figure 3.7.2 one can find the results of the same situation, only in this case $q_5 = 40$ and in Figure 3.7.3 one can find the results of a situation with $q_1 = 10$ and $q_5 = 50$.

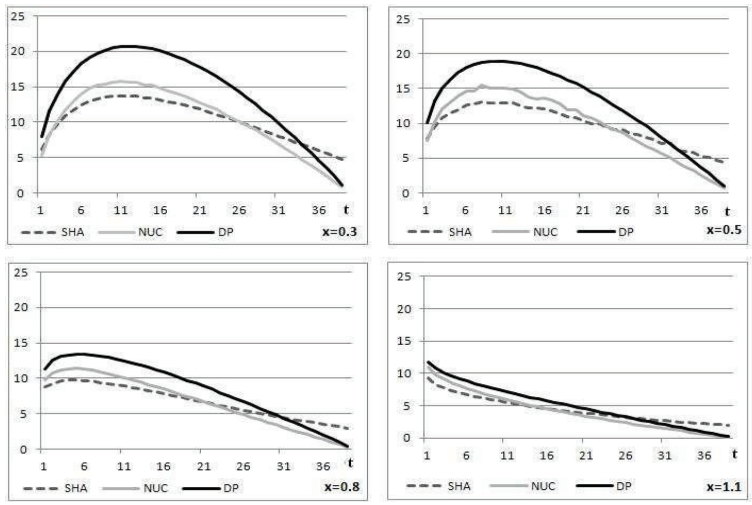

Figure 3.7.2: Expected payoffs in a random 5-player MCP-game with $q_1 = 1$ and $q_5 = 40$

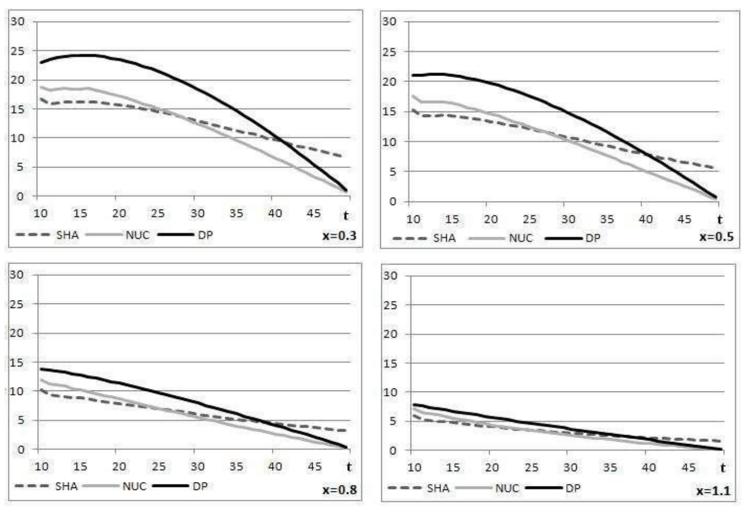

Figure 3.7.3: Expected payoffs in a 5-player MCP-game with $q_1 = 10$ and $q_5 = 50$

For the situation with $q_1 = 1$ and $q_5 = 40$ we also compared the expected shares in cost savings to players 1 and 5 for 9 different values of x, they can be found in Table 3.7.1.

We can make the following observations. The curve of the nucleolus always lies below the curve of the Direct Price. This is a confirmation of the results from Section 3.5. The Direct Price solution gives all bilateral profits to

Table 3.7.1: Expected payoffs to player 1 and 5 for $q_1 = 1$ and $q_5 = 40$

x	$\bar{\phi}_1$	$n\bar{u}c_1$	$\bar{DP}_1$	$\bar{\phi}_5$	$n\bar{u}c_5$	$\bar{DP}_5$
0.3	6.1073	5.3565	8.0321	13.9026	14.1642	0
0.4	7.0825	6.5056	9.2562	13.3979	13.3889	0
0.5	7.7681	7.5305	10.1026	12.1327	11.7777	0
0.6	8.2865	8.3917	10.6879	10.6323	10.2442	0
0.7	8.6152	9.1919	11.0927	9.0905	8.4306	0
0.8	8.8631	9.8080	11.3726	7.7002	6.9363	0
0.9	9.0455	10.3163	11.5662	6.4195	5.5601	0
1.0	9.1663	10.7434	11.7000	5.2371	4.2668	0
1.1	9.2664	11.0067	11.7925	4.4524	3.5057	0

players 1, 2, 3 and 4, while within the nucleolus players 1 up to 4 have to pay player 5 for joining the purchasing cooperative. The curve of the Shapley value does not always lie below the curve of the Direct price. This is due to the fact that the Shapley value need not lie in the core of the game. In all situations the behavior of the Shapley value is less volatile as the nucleolus and the nucleolus is less volatile as the direct price solution. We can see this from the figures by looking at the range of the curves. The Shapley value curve has the smallest range while the Direct Price solution has the largest.

We continue by comparing the nucleolus with the Shapley value. In all three situations, the players with order quantities close to q_5 are better off with the Shapley value and the players with somewhat smaller order quantities are better off with the nucleolus. In general we can also conclude that, although the nucleolus and the Shapley value are different game theoretic solution concepts, for MCP-situations their behavior with respect to the input parameters of the model, is quite similar and the expected differences for a player are small.

If we compare Figure 3.7.1 with Figure 3.7.2 we can see that the size of player 5 does not have much effect on the differences between the allocations of the nucleolus and the Shapley value.

In all three figures we can see that increasing the discount parameter x, makes the three solution concepts come closer to each other. This is partly due to the fact that the total savings obtained decreases as x increases. Also, for larger x, the threshold for players to prefer the Shapley value over the nucleolus tends to decrease. For players with order quantities close to q_1 the nucleolus becomes more attractive as x increases.

From Table 3.7.1 we can draw a similar conclusion as above. For larger x, the smallest player has a better position than player 5. In case of $x = 1.1$ ($q_1 = 1$ and $q_5 = 40$), the expected share in total cost savings of player 1 equals 36 and 43 percent for the Shapley value and nucleolus, respectively. For player 5, these expected shares equal 16 percent (Shapley) and 12 percent (nucleolus). While for the instance with $x = 0.3$, player 1's expected shares are 12 percent (Shapley) and 11 percent (nucleolus), and for player 5 the expected shares for both the Shapley value and the nucleolus equal 26 percent.

Chapter 4

Cost sharing methods for capacity restricted cooperative purchasing situations

4.1 Introduction

In the previous chapter we dealt with a special class of interactive purchasing situations. One of the main underlying assumptions in Chapter 3 and in cooperative purchasing in general is that the capacity of the supplier is sufficient to fulfill the total order of the group of purchasers. Although commonly assumed, one should realize that in practice the capacity of a supplier is limited. In particular, while a purchasing cooperative gets larger, the supplier's capacity might be exceeded and the cooperative has to use a second supplier. Capacity restrictions in cooperative purchasing situations will be the main topic of this chapter.

Not much literature can be found on capacity restrictions within cooperative purchasing. Supplier selection and order quantity allocation for a single purchaser has been studied from different perspectives. Berger, Gerstenfeld, and Zeng (2004) argue that maintaining a relationship with multiple suppliers can be a good strategy to decrease supply chain risks. Jayaraman and Srivastava (1999) developed a mixed integer programming model for selecting suppliers and for allocating the total order quantity among the selected suppliers. Ghodsypour and O'Brien (1998) incorporated an analytical hierarchy process for allocating orders among suppliers based on both quantitative and qualitative criteria. Ghodsypour and O'Brien (2001) provide an algorithm for quantity allocation where the possible suppliers have limited capacities. Suppliers' optimal pricing strategies in a multiple supplier environment have been discussed in, *e.g.*, Marvel and Yang (2008) and Hsieh and Kuo (2011). More precisely, Marvel and Yang (2008) discuss pricing strategies when two suppliers face a purchasing cooperative. From a purchasers' perspective, however, a purchasing cooperative with capacity restricted suppliers has not yet been studied.

In this chapter we consider a purchasing cooperative with individual order quantities with respect to a certain commodity. Here, the sum of the order quantities determines the unit price. Instead of facing one supplier with sufficient supplies, as in the classical CP-situations described by Schotanus (2007) and Nagarajan et al. (2010), the group faces two suppliers with (possibly) insufficient individual supplies. The combined capacity of the two suppliers is however sufficient. Like in regular cooperative purchasing situations, the unit price of a supplier weakly decreases with the size of the total order, that is, however, up to his capacity bound. These unit prices or quantity discount schemes are not necessarily the same for both suppliers. We show that in these *capacity restricted cooperative purchasing (CRCP) situations* individual cost savings are not guaranteed. Nevertheless, the group of purchasers is assumed to cooperate. Think of a group of departments, a group of ministries or a group of municipalities with a joined purchasing programme.

We are interested in finding the answers to two questions. Firstly, how to split the total order over the two suppliers such that the total purchasing costs are minimized? Secondly, how to adequately divide the total purchasing costs over the group of purchasers?

For the first question, we show that there is a straightforward solution by solving a minimization problem. We will show that it is optimal to order as much as possible at one supplier and the possible remainder at the other.

The second problem is more involved. To find suitable cost allocations we model the CRCP-situation as a cost sharing problem.

Generally, a cost sharing problem involves a set of users of a certain 'technology' and each of the users has an individual level of demanded output. To produce the total demanded output a certain level of input or costs is needed. The relationship between input and output, is represented by a cost function, where the function describes for each level of output the needed input (costs). How to fairly distribute the needed input, based on the desired output and the cost function is the central theme in cost sharing literature. In Moulin (2002) one can find an overview of different types of cost sharing problems and multiple cost allocation mechanisms.

In our setting the input needed can be represented by a monetary value: purchasing costs. The output is the sum of the individual order quantities. The cost function of the cost sharing problem corresponding to a CRCP-situation provides for each level of order quantities, the minimal purchasing costs. These minimal purchasing costs follow from dividing the order quantities optimally over the two suppliers. The resulting cost sharing problem corresponding to a capacity restricted cooperative purchasing situation then falls within the class of so-called *one-input-one-output-technologies*, such as airport problems (cf. Littlechild and Thomson (1977)) or single-product inventory problems. In this class the output is a single homogeneous divisible good.

We show that the cost function of a cost sharing problem corresponding to a CRCP-situation is piecewise concave and that the concave intervals are determined by so-called involuntary switches from one supplier to the other supplier. The switches are called involuntary because the restricted capacity of one supplier forces the purchasing group to also place an order at the second supplier. A concave cost function implies unlimited increasing returns to scale, whereas the piecewise concave cost function implies limited increasing returns to scale: after a certain output level, new investments are needed. According to Swoveland (1975) piecewise concave cost functions are a realistic representation of returns to scale in a production environment. For this reason, we broaden our view to general piecewise concave cost functions in the search for allocation methods for CRCP-situations.

In Chapter 3 we argued that finding a fair cost allocation method is one of the critical success factors for cooperative purchasing. Especially in the presence of differences in order quantity size, organizations with a large order quantity could get the feeling that organizations with a small order quantity profit from their size, without making any further contributions.
As Moulin (2002) points out, when there are no quantity discounts, the fair distribution of purchasing costs should simply follow Aristotle's proportionality: *Equals should be treated equally, and unequals, unequally in proportion to relevant similarities and differences.* However, since both suppliers have a decreasing unit price function, quantity discounts will be present and we need to look for a more sophisticated allocation method.
For cost allocation methods of the purchasing costs in a CRCP-situation, there are two desirable properties. Firstly, the quantity discounts should be incorporated in the cost allocation, *i.e*, organizations with large order quantities do not pay a higher unit price than organizations with smaller order quantities. A second desirable effect of an allocation method is that organizations with large order quantities do profit, in terms of cost allocations, from the presence of players with smaller order quantities. Loosely formulated: the smaller players are not considered as profiteers.

In the process of finding a suitable cost allocation method for CRCP-situations we start by considering the three main cost sharing rules: the Shapley-Shubik formula (Shubik (1962)), Aumann-Shapley pricing (Aumann and Shapley (1974)) and the serial cost sharing rule (Moulin and Shenker (1992)). There are two main arguments for Friedman and Moulin (1999) to conclude that from the three main cost sharing methods, the serial cost sharing rule is most appropriate for one-input-one-output-technologies such as the cost sharing problems we consider. First, since the cost sharing problem corresponding to a CRCP-situation concerns only homogeneous inputs and outputs, the Aumann-Shapley pricing boils down to average cost pricing, *i.e.* dividing the total purchasing costs proportionally (based on order quantities) over the purchasers. As argued before, this method neglects the quantity discounts that are present in cooperative purchasing. Second, one of the main properties of the Shapley-Shubik formula is, that it is invariant to the scale in which the output is measured, which is not of any relevance in a situation in which order quantities are for a single good.

Furthermore, there are arguments in favor of the serial cost sharing rule. For concave cost functions, the serial cost sharing rule satisfies properties that are attractive from the perspective of CRCP-situations. Firstly, the serial rule satisfies *unit cost monotonicity*: when organization 2 has a higher demanded output than organization 1, organization 2 does not pay a higher cost per unit than organization 1. Secondly, the serial rule satisfies monotonic weakness for the absence of the smallest player (MOWASP). This property implies that when the player with smallest order quantity is absent, every remaining player's cost allocation increases. More precisely, this increase in cost allocation (weakness) is monotonic in the size of the order quantity.

However, these rather compelling properties are lost when we add 'piecewise' to the cost function's recipe. We explicitly show that for cost sharing problems with piecewise concave cost functions, the serial rule in general does not satisfy unit cost monotonicity or MOWASP.

Therefore, we introduce a new context specific class of cost sharing rules for cost sharing problems with piecewise concave cost functions, in which we first divide the vector of order quantities into separate vectors for the different concave intervals, using a bankruptcy rule. Subsequently, for each concave interval and corresponding vector we use the serial rule to allocate the costs of that specific interval over the organizations. Finally, by summing these allocated costs we obtain the allocation according to the piecewise serial rule.

In particular, we consider the piecewise serial rule where we divide the vector of order quantities into separate vectors, using the proportional rule and the constrained equal losses-rule. It will be shown that the proportional rule is the only bankruptcy rule for which the piecewise serial rule satisfies unit cost monotonicity. For the constrained equal losses piecewise serial rule we will show that when the organization with smallest order quantity is not present in the cooperation, the group of remaining organizations can be split in a group of smaller organizations for which the allocated costs decrease and a

group of larger organizations for which the allocated costs increase. Also, here there is a monotonic relation: the larger the order quantity, the higher the increase (or the smaller the decrease) in cost allocation. This property is a weaker variant of MOWASP and is called monotonic vulnerability for the absence of the smallest player (MOVASP).
Both the properties unit cost monotonicity and MOVASP are inspired by the CRCP context. Unit cost monotonicity implies that in the cost allocation a purchaser with a higher order quantity obtains a lower unit price, *i.e.*, a higher quantity discount. MOVASP implies that the organization with the largest order quantity has either the least decrease or the highest increase in cost allocation when the smallest player is absent. Hence, it creates a group cohesiveness in which the organization with smallest order quantity can contribute to lower cost allocations of organizations with larger order quantities.

For illustrative purposes we conclude the chapter with a numerical comparison of the cost allocations of CRCP-situations according to the two piecewise serial rules and the serial cost sharing rule. These examples further support the claim that the two piecewise serial rules are appropriate allocation methods for CRCP-situations.

The structure of the chapter is as follows. Section 4.2 formally describes a capacity restricted cooperative purchasing situation. In Section 4.3 we model a CRCP-situation as a cost sharing problem and show that the cost function is piecewise concave. As an alternative to the serial cost sharing rule, we introduce the piecewise serial rules in Section 4.4 and we derive characterizing properties of the proportional and constrained equal losses-variants. Section 4.5 briefly discusses the differences in cost allocations of CRCP-situations according to the two piecewise serial rules and the serial cost sharing rule, on the basis of a numerical analysis.

4.2 Capacity restrictions in cooperative purchasing

After a brief description of regular cooperative purchasing situations, this section will provide the formal description of capacity restricted cooperative purchasing situations.

A *cooperative purchasing situation* (cf. Schotanus (2007)) is given by a finite set of players N, with a vector of individual order quantities $q \in \mathbb{R}^N_+$. There is a commonly known unit price function $p : \mathbb{R}_+ \to \mathbb{R}_+$ that maps an order quantity to some unit price. In a cooperative purchasing situation, the sum of the individual order quantities determines the unit price. It is assumed that p is non-increasing and that the turnover function $p(t)t$ is increasing and concave on $[0, \sum_{j\in N} q_j]$. This latter assumption results in the corresponding cooperative purchasing TU-game (N, z), for all $S \in 2^N$ defined by $z(S) = \sum_{j\in S} p(q_j)q_j - p(\sum_{j\in S} q_j)(\sum_{j\in S} q_j)$, to be convex. Hence, in a cooperative purchasing situation there exist stable and efficient allocations of the cost savings gained by purchasing cooperatively.

In a *capacity restricted cooperative purchasing (CRCP) situation*, there is a finite player set $N = \{1, \ldots, n\}$, with $n \geq 2$ and again a vector of individual order quantities $q \in \mathbb{R}^N_+$. There are two suppliers providing this commodity: A and B. Both suppliers have a limited capacity $Q_A, Q_B \in \mathbb{R}_{++}$. The combined capacity is, however, sufficient, $\sum_{j\in N} q_j \leq Q_A + Q_B$. Both suppliers have a linearly decreasing unit price function. For A, $p_A : [0, Q_A] \to \mathbb{R}_+$ and for B, $p_B : [0, Q_B] \to \mathbb{R}_+$, are given by

$$p_A(t) = \alpha_1 - \alpha_2 t, \quad t \in [0, Q_A] \tag{4.1}$$

and

$$p_B(t) = \beta_1 - \beta_2 t \quad t \in [0, Q_B], \tag{4.2}$$

respectively, where t denotes the order size and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}_+$.
Note that for all $t \in [0, Q_A]$, $p'_A(t) \leq 0$ and that $p''_A(t) = 0$.

It is a natural assumption that the revenue of a supplier does not decrease if t increases. For supplier A this is the case if α_1 and α_2 are such that for all $t \in [0, Q_A]$, the revenue of A, $c_A(t) = p_A(t)t$, is nondecreasing. Hence for all $t \leq Q_A$, we assume

$$c'_A(t) = \alpha_1 - 2\alpha_2 t \geq 0.$$

i.e.,

$$\frac{\alpha_1}{2\alpha_2} \geq Q_A.$$

Similarly for supplier B we assume

$$\frac{\beta_1}{2\beta_2} \geq Q_B.$$

Note that both c_A and c_B are differentiable and concave on $[0, Q_A]$ and $[0, Q_B]$ respectively.

For the remainder of this chapter, we assume, without loss of generality that

$$Q_A \leq Q_B$$

and that the order quantities are arranged in nondecreasing order, *i.e.*,

$$0 < q_1 \leq q_2 \leq \ldots \leq q_n.$$

When we refer to a smaller player, we refer to a player with smaller order quantity and thus smaller index. When we refer to a larger or bigger player, we refer to a player with a larger order quantity or larger index.

A CRCP-situation on player set N, is given by $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B])$. We denote the set of all CRCP-situations on N by $\mathcal{Z}^N$.

The main assumption of this chapter is that the players are purchasing cooperatively. The next example shows that, contrary to regular cooperative purchasing situations and maximum cooperative purchasing situations, in CRCP-situations we cannot easily determine whether cooperation leads to cost savings.

Example 4.2.1 Let $N = \{1, 2, 3\}$ and let $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$ be given by $q = (8, 9, 15)$ and

$$\begin{cases} p_A(t) & = 18 - \frac{1}{3}t, \quad t \in [0, 16] \\ p_B(t) & = 25 - \frac{1}{2}t, \quad t \in [0, 20]. \end{cases}$$

If player 1 would be on his own, he would like to order 8 at A with purchasing costs $c_A(8) = p_A(8)8 = 122\frac{2}{3}$. Similarly player 2 would like to order 9 at A with purchasing costs $c_A(9) = 135$. Also, player 3 prefers ordering 15 at supplier A with purchasing costs $c_A(15) = 195$. This is not a feasible set of individual orders, since $8 + 9 + 15 > 16 = Q_A$. Also, if the three players purchase cooperatively they cannot simply sum their individual 'desired' orders at the two suppliers. In this example the lowest ordering costs for order quantity $8 + 9 + 15 = 32$ can be obtained by ordering 20 at B and 12 at A.$\triangleleft$

If the suppliers' capacities would have been unlimited or both $Q_A \geq \sum_{i \in N} q_i$ and $Q_B \geq \sum_{i \in N} q_i$, then the problem boils down to a regular cooperative purchasing situation, simply by taking the minimum of the two functions as unit price function p.

4.3 Cost sharing problems corresponding to CRCP-situations

CRCP-situations can be modeled and analyzed by using the concept of cost sharing.

4.3.1 One-input-one-output cost sharing problems

The nature of our CRCP-situation matches a special class of cost sharing problems: *one-input-one-output-technologies*. In this class, the order quantities are scalars and enter additively in the continuous cost function. Such a *cost sharing problem* on $N = \{1, \dots, n\}$ is represented by a pair (C, q), with $q \in \mathbb{R}^N_+$ such that $q_1 \leq \ldots \leq q_n$, and $C : [0, Q] \to \mathbb{R}_+$ with $Q \geq \sum_{i \in N} q_i$ is such that C is continuous and nondecreasing, and with $C(0) = 0$. Here the argument t in $C(t)$ represents the total demanded output. We denote by

$\mathcal{CS}^N$ the set of all such cost sharing problems on N.
A *cost sharing rule* f is a mapping $f : \mathcal{CS}^N \to \mathbb{R}^N$, such that $\sum_{i\in N} f_i(C,q) = C(\sum_{i\in N} q_i)$ and $f(C,q) \geq 0$.

4.3.2 Optimal ordering policy

Let $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$. In the corresponding cost sharing problem (C^Z, q), $C^Z(t)$ gives for each $t \in [0, Q_A+Q_B]$ the corresponding ordering costs. These ordering costs follow from an optimal splitting of t over the suppliers A and B. An *ordering policy* for t is a pair (t_A, t_B) such that $0 \leq t_A \leq Q_A$, $0 \leq t_B \leq Q_B$ and $t_A + t_B = t$. Here, t_A represents the total order at supplier A and t_B the total order at supplier B. For ordering policy (t_A, t_B) the ordering costs are

$$c_A(t_A) + c_B(t_B) = p_A(t_A)t_A + p_B(t_B)t_B.$$

An ordering policy is optimal if the associated ordering costs are minimal. Thus the minimal ordering costs $C^Z(t)$ for t are determined by

$$C^Z(t) = \min\{c_A(t_A) + c_B(t_B) | t_A + t_B = t, 0 \leq t_A \leq Q_A, 0 \leq t_B \leq Q_B\}.$$

Theorem 4.3.1 *Let $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$ and let $(C^Z, q) \in \mathcal{CS}^N$ be the corresponding cost sharing problem. Then,*

$$C^Z(t) = \begin{cases} \min\{c_B(t), c_A(t)\} & t \in [0, Q_A] \\ \min\{c_B(t), c_A(Q_A) + c_B(t - Q_A)\} & t \in (Q_A, Q_B] \\ \min\{c_A(t - Q_B) + c_B(Q_B), & \\ \qquad c_A(Q_A) + c_B(t - Q_A)\} & t \in (Q_B, Q_A + Q_B]. \end{cases} \tag{4.3}$$

Proof: Take $t \in [0, Q_A + Q_B]$. Then,

$$\begin{aligned} C^Z(t) &= \min\{c_A(t_A) + c_B(t_B) | t_A + t_B = t, 0 \leq t_A \leq Q_A, 0 \leq t_B \leq Q_B\} \\ &= \min\{c_A(t_A) + c_B(t - t_A) | t_A \in [(t - Q_B)^+, \min\{Q_A, t\}]\}. \end{aligned}$$

Let $g : [(t - Q_B)^+, \min\{Q_A, t\}] \to \mathbb{R}_+$ be defined in the following way,

$$g(t_A) = c_A(t_A) + c_B(t - t_A).$$

The interval $[(t-Q_B)^+, \min\{Q_A, t\}]$ is nonempty since $Q_A > 0$ and $Q_B > 0$.

Note that

$$\begin{aligned} g'(t_A) &= c'_A(t_A) + c'_B(t-t_A) \\ &= p'_A(t_A)t + p_A(t_A) - p'_B(t-t_A)(t-t_A) - p_B(t-t_A) \end{aligned}$$

and that

$$\begin{aligned} g''(t_A) &= p''_A(t_A)t_A + 2p'_A(t_A) + p''_B(t-t_A)(t-t_A) + 2p'_B(t-t_A) \\ &= 0 \qquad + 2p'_A(t_A) + 0 \qquad + 2p'_B(t-t_A) \\ &\leq 0. \end{aligned}$$

Hence g is concave and thus the minimum of g can be found at the boundaries of the domain of g: either $t_A = (t-Q_B)^+$ or $t_A = \min\{Q_A, t\}$.
The proof is complete if we can show that we can separate the cases as in (4.3).
If $t \leq Q_A$, then $t_A = 0$ or $t_A = t$ and consequently $t_B = t$ or $t_B = 0$. Thus

$$C^Z(t) = \min\{c_B(t), c_A(t)\}.$$

If $Q_A < t \leq Q_B$, then $t_A = 0$ or $t_A = Q_A$ and consequently $t_B = t$ or $t_B = t - Q_A$. Thus

$$C^Z(t) = \min\{c_B(t), c_A(Q_A) + c_B(t-Q_A)\}.$$

If $t > Q_B$, then $t_A = t - Q_B$ or $t_A = Q_A$ and consequently $t_B = Q_B$ or $t_B = t - Q_A$. Thus

$$C^Z(t) = \min\{c_A(t-Q_B) + c_B(Q_B), c_A(Q_A) + c_B(t-Q_A)\}. \qquad \square$$

This theorem implies that the cost function of the cost sharing problem corresponding to a CRCP-situation follows from the minimum of two policies.

Corollary 4.3.2 *Let $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$ and let $(C^Z, q) \in \mathcal{CS}^N$ be the corresponding cost sharing problem. Let $t \in [0, Q_A + Q_B]$. Then,*

$$\begin{aligned} C^Z(t) = \min\{&c_A(\min\{Q_A, t\}) + c_B((t-Q_A)^+), \\ &c_A((t-Q_B)^+) + c_B(\min\{Q_B, t\})\}. \end{aligned} \tag{4.4}$$

To minimize ordering costs, one has to compare two extreme policies: order as much as possible at one of the two suppliers and the remaining part at the other one. Depending on the unit price functions and the total order quantity t one might prefer A over B or B over A.

The following two examples show how one can use Theorem 4.3.1 and Corollary 4.3.2 in finding the cost function of the cost sharing problem corresponding to a CRCP-situation.

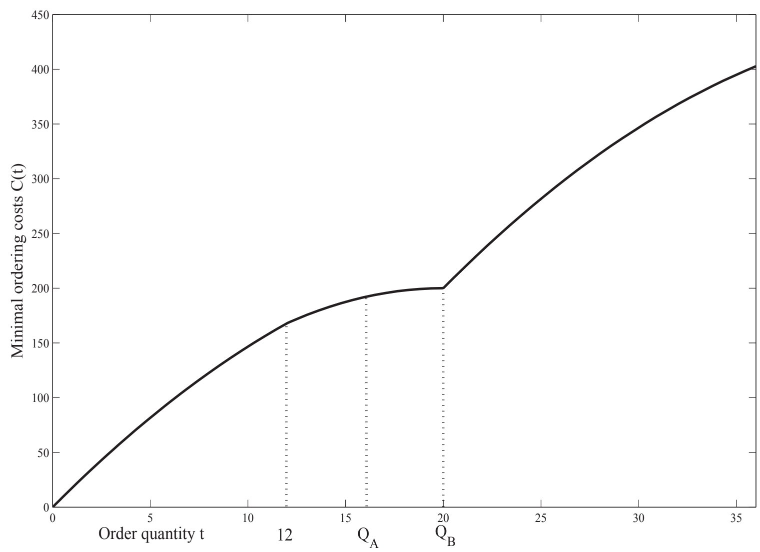

Figure 4.3.1: Cost function of Example 4.3.3

Example 4.3.3 Let $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$ be such that

$$\begin{cases} p_A(t) & = 18 - \frac{1}{3}t, \ t \in [0, 16] \\ p_B(t) & = 20 - \frac{1}{2}t, \ t \in [0, 20] \end{cases}$$

and let $(C^Z, q) \in \mathcal{CS}^N$ be the corresponding cost sharing problem.
Using Theorem 4.3.1 we can find the exact expression for C^Z: c_A and c_B intersect at $t = 12$, $c_A(Q_A) + c_B(t - Q_A) \geq c_B(t)$ on $[16, 20]$ and on $[20, 36]$ $c_A(Q_A) + c_B(t - Q_A) \geq c_A(t - Q_B) + c_B(Q_B)$. Thus we find the following

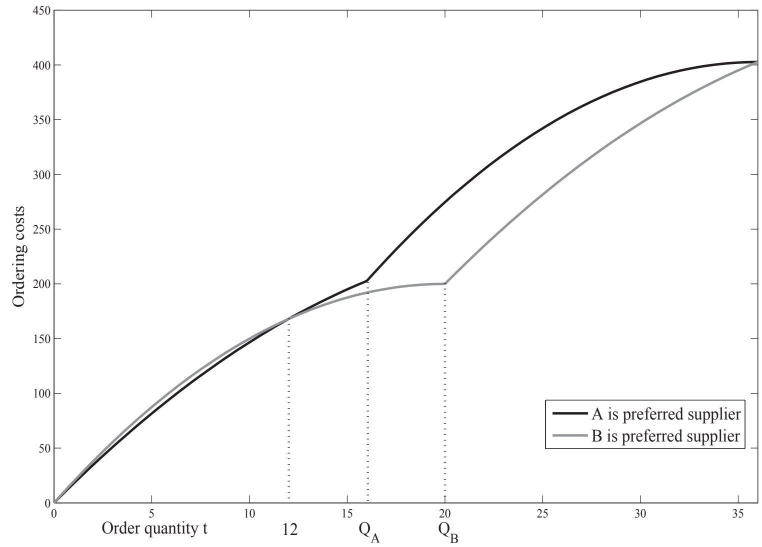

Figure 4.3.2: Ordering costs for the two extreme policies of Example 4.3.3

cost function,

$$C^Z(t) = \begin{cases} \min\{c_A(t), c_B(t)\} & \text{if } t \in [0,16], \\ \min\{c_A(Q_A) + c_B(t-Q_A), c_B(t)\} & \text{if } t \in [16,20], \\ \min\{c_A(Q_A) + c_B(t-Q_A), \\ \qquad c_A(t-Q_B) + c_B(Q_B)\} & \text{if } t \in [20,36], \end{cases}$$

$$= \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0,12], \\ 20t - \frac{1}{2}t^2 & \text{if } t \in [12,16], \\ 20t - \frac{1}{2}t^2 & \text{if } t \in [16,20], \\ 20 \cdot 10 + 18(t-20) - \frac{1}{3}(t-20)^2 & \text{if } t \in [20,36], \end{cases}$$

$$= \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0,12], \\ 20t - \frac{1}{2}t^2 & \text{if } t \in [12,20], \\ 200 + 18(t-20) - \frac{1}{3}(t-20)^2 & \text{if } t \in [20,36]. \end{cases}$$

As mentioned in Corollary 4.3.2, the cost function C is the minimum of the following two policies: order as much as possible at A and then go to B (1) or order as much as possible at B and then go to A (2), the cost functions

of these two policies are shown in Figure 4.3.2.
The minimum of these two cost functions coincides with the cost function of Figure 4.3.1. Note that the cost function is piecewise concave with two maximally concave intervals $[0, 20]$ and $[20, 36]$. ◁

Example 4.3.4 Let $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$ be such that

$$\begin{cases} p_A(t) & = 20 - \frac{1}{2}t,\ t \in [0, 16], \\ p_B(t) & = 18 - \frac{1}{3}t,\ t \in [0, 20]. \end{cases}$$

Let $(C^Z, q) \in \mathcal{CS}^N$ be the corresponding cost sharing problem. Then, in Figure 4.3.3 one can find the cost functions corresponding to the two extreme policies (A or B). In this situation we see more switches between policy A or B than in Example 4.3.3. This can also be seen from the explicit expression of C^Z, *i.e.*,

$$C^Z(t) = \begin{cases} c_B(t) & \text{if } t \in [0, 12], \\ c_A(t) & \text{if } t \in (12, 16], \\ c_A(16) + c_B(t - 16) & \text{if } t \in (16, 17], \\ c_B(t) & \text{if } t \in (17, 20], \\ c_B(20) + c_A(t - 20) & \text{if } t \in (20, 34 - \sqrt{2}], \\ c_A(16) + c_B(t - 16) & \text{if } t \in (34 - \sqrt{2}, 34 + \sqrt{2}], \\ c_B(20) + c_A(t - 20) & \text{if } t \in (34 + \sqrt{2}, 36]. \end{cases}$$

Note that also this cost function is piecewise concave. It has, however, 3 maximally concave intervals: $[0, 16]$ $[16, 20]$ and $[20, 36]$. ◁

We can generalize the observations we made in Example 4.3.3 and 4.3.4. In Theorem 4.3.1 we have shown that the cost function is piecewise defined on three separate intervals[5]. On each of the three intervals, $[0, Q_A]$, $[Q_A, Q_B]$, $[Q_B, Q_A + Q_B]$, C^Z takes the minimum of two continuous concave functions. Thus the cost function C^Z of the cost sharing problem corresponding to CRCP-situation $Z \in \mathcal{Z}^N$ is continuous piecewise concave. In most cases C^Z

[5]With slight abuse of notation we use the notion of maximally concave intervals.

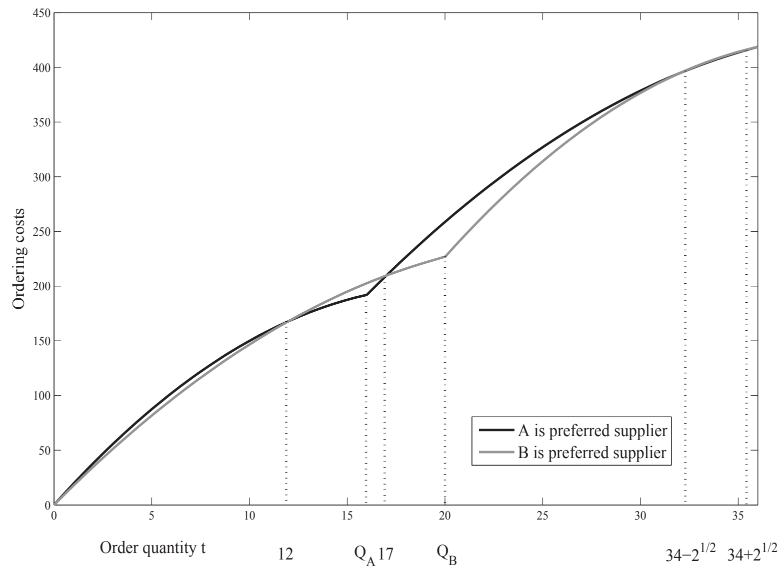

Figure 4.3.3: Two extreme policies of Example 4.3.4

will have two or three maximally concave intervals. Only if $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2 = 0$, C^Z will be concave on $[0, Q_A + Q_B]$.

For all CRCP-situations $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$ with $\alpha_2 = \beta_2 = 0$, the corresponding cost sharing problem $(C^z, q) \in \mathcal{CS}^N$ has a convex piecewise linear cost function. *E.g.*, if $\alpha_1 \leq \beta_1$, then

$$C^Z(t) = \begin{cases} \alpha_1 t & \text{if } t \in [0, Q_A], \\ \alpha_1 Q_A + \beta_1 (t - Q_A) & \text{if } t \in (Q_A, Q_A + Q_B]. \end{cases}$$

The piecewise concavity of C^Z is directly caused by the limitations of the suppliers. Due to the capacity restrictions, at some point, the purchasers are forced to start buying at the other supplier. These points are called *involuntary switches*. So the number of involuntary switches is either 0, 1 or 2. In Example 4.3.3 there is one involuntary switch at the point $t = Q_B$ and in Example 4.3.4 there are two involuntary switches, at $t = Q_A$ and at $t = Q_B$. In the latter example, the cost function also switches between preferred supplier, without violating concavity, on 4 different points. These switches are called *voluntary switches* and they are caused by the minimization of the two ordering policies. Hence, they can occur on the entire domain of C^Z.

Next, we investigate where these voluntary switches might take place. Let $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$ and let (C^Z, q) be the corresponding cost sharing problem.
Consider the interval $[0, Q_A]$. Note that $c_A(0) = c_B(0) = 0$. A voluntary switch occurs if there exists a $t \in (0, Q_A]$ with $c_A(t) = c_B(t)$, *i.e.*, with

$$t = \frac{\beta_1 - \alpha_1}{\beta_2 - \alpha_2}. \tag{4.5}$$

Consider the interval $(Q_A, Q_B]$. Then, a voluntary switch occurs if there exists a $t \in (Q_A, Q_B]$ with $c_A(Q_A) + c_B(t - Q_A) = c_B(t)$, *i.e.*, with

$$\alpha_1 Q_A - \alpha_2 Q_A^2 - \beta_1 Q_A - \beta_2 Q_A^2 + 2\beta_2 t Q_A + \beta_1 t - \beta_2 t^2 = \beta_1 t - \beta_2 t^2,$$

by rewriting we obtain

$$2\beta_2 t = \alpha_2 Q_A + \beta_1 - \alpha_1 + \beta_2 Q_A,$$

thus if

$$t = \frac{\beta_1 - \alpha_1 + (\beta_2 + \alpha_2) Q_A}{2\beta_2}. \tag{4.6}$$

Consider the interval $(Q_B, Q_A + Q_B]$. Note that if $t = Q_A + Q_B$, then

$$c_A(t - Q_B) + c_B(Q_B) = c_A(Q_A) + c_B(t - Q_A) = c_A(Q_A) + c_B(Q_B).$$

A voluntary switch occurs if there exists a $t \in (Q_B, Q_A + Q_B)$ with $c_A(t - Q_B) + c_B(Q_B) = c_A(Q_A) + c_B(t - Q_A)$, *i.e.*, with

$$\begin{aligned} &\alpha_1 Q_A - \alpha_2 Q_A^2 - \beta_1 Q_A - \beta_2 Q_A^2 + 2\beta_2 t Q_A + \beta_1 t - \beta_2 t^2 \\ =&\beta_1 Q_B - \beta_2 Q_B^2 - \alpha_1 Q_B - \alpha_2 Q_B^2 + 2\alpha_2 t Q_B + \alpha_1 t - \alpha_2 t^2, \end{aligned}$$

or equivalently with

$$\begin{aligned} &-(\alpha_2 + \beta_2) t^2 + (\beta_1 - \alpha_1 + 2(\beta_2 Q_A - \alpha_2 Q_B)) t \\ &+ (\alpha_1 - \beta_1)(Q_A - Q_B) - (\alpha_2 + \beta_2)(Q_A^2 + Q_B^2) = 0 \end{aligned} \tag{4.7}$$

Since (4.7) is a quadratic equation, there can be at most two voluntary switches on $(Q_B, Q_A + Q_B)$.

In total there can be maximally four voluntary switches. As we have seen in Example 4.3.4 the cost function corresponding to a CRCP-situation can make 2 involuntary switches as well as 4 voluntary switches.

4.4 Cost sharing rules for piecewise concave cost functions

In this section we device a new class of cost sharing rules that are suitable for cost sharing problems with piecewise concave cost functions. We focus on this class of cost sharing problems since we are looking for allocation methods for CRCP-situations and also since piecewise concave cost functions are an accurate representation of limited economies of scale in production environments. The examples and motivation for the rules will come from the application of allocating costs in CRCP-situations. The new class of rules is based on the serial cost sharing rule.

4.4.1 The serial cost sharing rule

In the introduction we argued that, of the traditional cost sharing rules, the serial cost sharing rule is the most suitable rule for the class of cost sharing problems under consideration, *i.e.*, one-input-one-output-technologies.

The serial cost sharing rule is based on the requirement that a player's costs should not depend on the size of the order quantity of larger players. For a concave cost function, this requirement implies that smaller players profit less from the economies of scale than the larger players. If we think of CRCP-situations in which large players generally account for more quantity discounts, this seems a suitable solution method for dividing costs that follow from purchasing cooperatively. The *serial cost sharing rule* (Moulin and Shenker (1992)) , Ser, allocates the costs in the following way.

Definition: The *serial cost sharing rule*, *Ser*, on $\mathcal{CS}^N$ is such that for all $(C,q) \in \mathcal{CS}^N$ and for all $i \in N$,

$$Ser_i(C,q) = \sum_{j=1}^{i} \frac{C(s_j) - C(s_{j-1})}{n-j+1}, \tag{4.8}$$

with $s_0 = 0$ and for all $i \in N$,

$$s_i = \sum_{j=1}^{i-1} q_j + (n-i+1)q_i.$$

Furthermore, for each $i \in N$, $Ser_i(C,q)$ rewrites to the expression below, which we will apply several times.

Lemma 4.4.1 *Let $(C,q) \in \mathcal{CS}^N$. Then, for all $i \in N$,*

$$Ser_i(C,q) = \frac{C(s_i)}{n-i+1} - \sum_{j=1}^{i-1} \frac{C(s_j)}{(n-j+1)(n-j)}.$$

Proof: Take $i \in N$, we have that,

$$\begin{aligned} Ser_i(C,q) &= \sum_{j=1}^{i} \frac{C(s_j) - C(s_{j-1})}{n-j+1} \\ &= \frac{C(s_i)}{n-i+1} + \sum_{j=1}^{i-1} \frac{C(s_j)}{n-j+1} - \sum_{j=1}^{i-1} \frac{C(s_j)}{n-j} \\ &= \frac{C(s_i)}{n-i+1} + \sum_{j=1}^{i-1} \frac{(n-j)C(s_j) - (n-j+1)C(s_j)}{(n-j+1)(n-j)} \\ &= \frac{C(s_i)}{n-i+1} - \sum_{j=1}^{i-1} \frac{C(s_j)}{(n-j+1)(n-j)}. \end{aligned}$$

Here, the second equality follows from the fact that $C(s_0) = C(0) = 0$. □

The property that characterizes the serial rule (Moulin and Shenker (1992)) is *independence of the size of larger demands (ISLAD)*. Take $(C,q) \in \mathcal{CS}^N$.

ISLAD implies that for all $i, j \in N$ with $q_i \leq q_j$, and for all $(C, \bar{q}) \in \mathcal{CS}^N$ with $\bar{q} = ((q_k)_{k \in N \setminus \{j\}}, r)$, with $r \geq q_j$,

$$Ser_i(C, q) = Ser_i(C, \bar{q}).$$

The serial cost sharing rule also satisfies basic properties as *demand monotonicity*, *i.e.*, for all $(C, q) \in \mathcal{CS}^N$ and all $i \in N \setminus \{n\}$, $s_i \leq s_{i+1}$ and thus

$$Ser_i(C, q) \leq Ser_{i+1}(C, q),$$

and *symmetry*, *i.e.* for all $(C, q) \in \mathcal{CS}^N$ and all $i, j \in N$ with $q_i = q_j$, $s_i = s_j$ and thus

$$Ser_i(C, q) = Ser_j(C, q).$$

For concave C the serial cost sharing rule obeys two favorable properties. First, a player with a higher demand obtains a weakly lower cost per unit than a player with a smaller demand.

Definition: A cost sharing rule f satisfies *unit cost monotonicity* if for all $(C, q) \in \mathcal{CS}^N$ and for all $i \in N \setminus \{n\}$

$$\frac{f_i(C, q)}{q_i} \geq \frac{f_{i+1}(C, q)}{q_{i+1}}.$$

Proposition 4.4.2 *Ser satisfies unit cost monotonicity on the class of cost sharing problems with continuous, nondecreasing and concave cost functions.*

Proof: Let $(C, q) \in \mathcal{CS}^N$ with C concave on $[0, \sum_{j \in N} q_j]$. Take $i \in N \setminus \{n\}$. We have that

$$\begin{aligned} \frac{Ser_i(C, q)}{q_i} - \frac{Ser_{i+1}(C, q)}{q_{i+1}} &= \frac{Ser_i(C, q)}{q_i} - \frac{Ser_i(C, q)}{q_{i+1}} - \frac{C(s_{i+1}) - C(s_i)}{(n-i)q_{i+1}} \\ &= \frac{(q_{i+1} - q_i)Ser_i(C, q)}{q_i q_{i+1}} - \frac{C(s_{i+1}) - C(s_i)}{(n-i)q_{i+1}}. \end{aligned}$$

Hence it is sufficient to show that

$$\frac{C(s_{i+1}) - C(s_i)}{(n-i)} \leq \frac{(q_{i+1} - q_i)Ser_i(C, q)}{q_i}.$$

First note that, since C is concave and

$$Ser_i(C,q) = \sum_{j=1}^{i} \frac{C(s_j) - C(s_{j-1})}{n-j+1},$$

with $s_j \leq s_i$ for all $j \in \{1, \ldots, i\}$, we have that

$$\frac{Ser_i(C,q)}{q_i} \geq \frac{C(s_i)}{s_i}.$$

Furthermore, by concavity of C,

$$\begin{aligned}
\frac{C(s_{i+1}) - C(s_i)}{(n-i)} &= \frac{C(s_i + (n-i)(q_{i+1} - q_i)) - C(s_i)}{(n-i)} \\
&\leq C(s_i + (q_{i+1} - q_i)) - C(s_i) \\
&\leq (q_{i+1} - q_i)\frac{C(s_i)}{s_i} \\
&\leq (q_{i+1} - q_i)\frac{Ser_i(C,q)}{q_i}.
\end{aligned}$$

□

Second, if the smallest player would not have been present in the cooperation, the costs for every remaining player increase. More precisely, the increase in ordering costs is larger for a player with a higher demand. So, although a small player's costs are independent of the size of larger demands, the larger players do profit from cooperating with smaller players. In absolute terms, the largest player profits the most.

A cost sharing rule satisfies *monotonic weakness for the absence of the smallest player* (MOWASP) if for all $(C,q) \in \mathcal{CS}^N$ with $|N| \geq 2$,

$$0 \geq f_2(C,q) - f_2(C, q_{|N\backslash\{1\}}) \geq \ldots \geq f_n(C,q) - f_n(C, q_{|N\backslash\{1\}}). \qquad (4.9)$$

Proposition 4.4.3 *Ser satisfies MOWASP on the class of cost sharing problems with continuous, nondecreasing and concave cost functions.*

Proof: Let $(C,q) \in \mathcal{CS}^N$ be a cost sharing problem, with C concave on $[0, \sum_{j\in N} q_j]$. Let $\Delta Ser_j = Ser_j(C,q) - Ser_j(C, q_{|N\backslash\{1\}})$ for all $j \in N\backslash\{1\}$. We have

$$Ser_2(C,q) = \frac{C(s_2)}{n-1} - \frac{C(s_1)}{n(n-1)}$$

and

$$Ser_2(C, q_{|N\setminus\{1\}}) = \frac{C(s_2 - q_1)}{n-1}.$$

Hence

$$\Delta Ser_2 = \frac{C(s_2) - C(s_2 - q_1)}{n-1} - \frac{C(s_1)}{n(n-1)}.$$

By concavity of C

$$\frac{C(s_2) - C(s_2 - q_1)}{n-1} \leq \frac{C(q_1)}{n-1} \leq \frac{C(nq_1)}{n(n-1)} = \frac{C(s_1)}{n(n-1)},$$

thus $\Delta Ser_2 \leq 0$.

Let $i \in \{3, \ldots, n\}$. According to Lemma 4.4.1 we have

$$Ser_i(C, q) = \frac{C(s_i)}{n-i+1} - \sum_{j=1}^{i-1} \frac{C(s_j)}{(n-j+1)(n-j)}$$

and

$$Ser_i(C, q_{|N\setminus\{1\}}) = \frac{C(s_i - q_1)}{n-i+1} - \sum_{j=2}^{i-1} \frac{C(s_j - q_1)}{(n-j+1)(n-j)}.$$

Hence,

$$\Delta Ser_i = \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \sum_{j=2}^{i-1} \frac{C(s_j) - C(s_j - q_1)}{(n-j)(n-j+1)} - \frac{C(s_1)}{n(n-1)}.$$

We conclude the proof by showing that $\Delta Ser_i - \Delta Ser_{i-1} \leq 0$. We have

$$\begin{aligned}
\Delta Ser_i - \Delta Ser_{i-1} &= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \sum_{j=2}^{i-1} \frac{C(s_j) - C(s_j - q_1)}{(n-j)(n-j+1)} \\
&- \frac{C(s_1)}{n(n-1)} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+2} \\
&+ \sum_{j=2}^{i-2} \frac{C(s_j) - C(s_j - q_1)}{(n-j)(n-j+1)} + \frac{C(s_1)}{n(n-1)} \\
&= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{(n-i+2)(n-i+1)} \\
&- \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+2} \\
&= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} \\
&- \frac{(n-i+1+1)\big(C(s_{i-1}) - C(s_{i-1} - q_1)\big)}{(n-i+2)(n-i+1)} \\
&= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+1} \\
&\leq 0,
\end{aligned}$$

where the last inequality follows from the concavity of C and the fact that $s_i \geq s_{i-1}$. □

For piecewise concave functions, however, these two properties are lost.

Example 4.4.4 Let $N = \{1, 2, 3\}$ and let $(C, q) \in \mathcal{CS}^N$ be a cost sharing problem with $q = (8, 9, 15)$ and with cost function

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0, 12], \\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12, 20], \\ 200 + 18(t-20) - \frac{1}{3}(t-20)^2 & \text{if } t \in (20, 36]. \end{cases}$$

Note that this cost sharing problem corresponds to the CRCP-situation from Example 4.3.3.

C is piecewise concave with two maximal concave intervals $[0, 20]$ and $[20, 36]$.

The serial cost sharing rule prescribes the following allocation of the total ordering costs $C(32) = 368$:

$$\begin{aligned} Ser_1(C,q) =& \frac{C(3\cdot 8)}{3} = 88\frac{8}{9},\\ Ser_2(C,q) =& \frac{C(3\cdot 8)}{3} + \frac{C(2\cdot 9+8) - C(3\cdot 8)}{2} = 103\frac{5}{9},\\ Ser_3(C,q) =& \frac{C(3\cdot 8)}{3} + \frac{C(2\cdot 9+8) - C(3\cdot 8)}{2}\\ &+ C(32) - C(2\cdot 9+8) = 175\frac{5}{9}. \end{aligned}$$

In this allocation player 1 pays $\frac{88\frac{8}{9}}{8} \approx 11.11$ per unit, while player 2 pays 11.51 per unit and player 3 pays 11.70 per unit. Hence, in this example the cost per unit are increasing rather than decreasing. $\triangleleft$

Example 4.4.5 Let $N = \{1,2,3,4\}$ and consider cost sharing problem $(C,q) \in \mathcal{CS}^N$ with $q = (2,4,9,15)$ and with cost function

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0,12],\\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12,16],\\ 192 + 18(t-16) - \frac{1}{3}(t-16)^2 & \text{if } t \in (16,17],\\ 20t - \frac{1}{2}t^2 & \text{if } t \in (17,20],\\ 226\frac{2}{3} + 20(t-20) - \frac{1}{2}(t-20)^2 & \text{if } t \in (20, 34-\sqrt{2}],\\ 192 + 18(t-16) - \frac{1}{3}(t-16)^2 & \text{if } t \in (34-\sqrt{2}, 34+\sqrt{2}],\\ 226\frac{2}{3} + +20(t-20) - \frac{1}{2}(t-20)^2 & \text{if } t \in (34+\sqrt{2}, 36]. \end{cases}$$

Note that this cost sharing problem corresponds to the CRCP-situation from Example 4.3.4. Here C has three maximally concave intervals $[0,16]$, $[16,20]$ and $[20,36]$. We have,

$$Ser(C,q) = \left(30\frac{2}{3}, 50\frac{4}{9}, 108\frac{7}{9}, 186\frac{7}{9}\right).$$

If player 1 would not have been present, however,

$$Ser(C, q_{|N\setminus\{1\}}) = (56, 104\frac{1}{3}, 194\frac{1}{3}).$$

Thus cost allocations of player 2 and player 4 increase, while player 3's cost allocation decreases. $\triangleleft$

4.4.2 Piecewise serial rules

In this subsection we modify the serial cost sharing rule into piecewise serial rules that are suitable for cost sharing problems with piecewise concave cost functions.
We will pinpoint a specific rule that satisfies unit cost monotonicity and a specific rule that satisfies a weaker variant of MOWASP: monotonic *vulnerability* for the absence of the smallest player. These two properties have nice interpretations for the application we have in mind.Furthermore we characterize one of the two cost sharing rules using the property of unit cost monotonicity.

Let $\mathcal{CS}^{N,m} \subset \mathcal{CS}^N$ with $m \in \mathbb{N}_+$ denote the set of cost sharing problems where the cost function is piecewise concave with m maximally concave intervals. The j-th concave interval is denoted by $[t_{j-1}, t_j]$, $j \in \{1, \dots, m\}$.
Using this notation, the results of the previous paragraph, Proposition 4.4.2 and Proposition 4.4.3, can be read as results on the class $\mathcal{CS}^{N,1}$.

Next, we explain the idea for the piecewise serial rule by means of an example.

Example 4.4.6 Let $N = \{1, 2, 3\}$ and let $(C, q) \in \mathcal{CS}^{N,2}$ be a cost sharing problem with $q = (8, 9, 15)$ and in which the cost function C is given by

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0, 12], \\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12, 20], \\ 200 + 18(t - 20) - \frac{1}{3}(t - 20)^2 & \text{if } t \in (20, 36]. \end{cases}$$

This is the cost sharing problem corresponding to the CRCP-situation of Example 4.3.3.

C has two maximally concave intervals: $[0, 20]$ and $[20, 36]$. If we can divide the vector q over these two intervals, *i.e.*, find a suitable vector $x^1 \in \mathbb{R}^N$ with $\sum_{j \in N} x_j^1 = 20$ for the first interval and a suitable vector $x^2 = q - x^1$ for the second interval, we can apply the serial cost sharing rule on each of these two cost sharing problems.
The first interval has length 20 and in this interval the returns to scale are

larger than in the second interval, see Figure 4.3.3. Hence, the players prefer the first interval. The demands q can be considered as claims on the interval $[0, 20]$. Thus to find a suitable vector x^1 we can use a bankruptcy rule φ, for bankruptcy problem $(20, q) \in \mathcal{B}^N$.

Arguing that large players should obtain a lower cost per unit than small players, we can opt for a bankruptcy rule that allocates relatively more to large claims than to smaller claims. The constrained equal losses-rule is such a bankruptcy rule. In this case it gives $x^1 = (8, 9, 15) - (\frac{12}{3}, \frac{12}{3}, \frac{12}{3}) = (4, 5, 11)$. Hence for the second interval we have $x^2 = q - x^1 = (4, 4, 4)$, *i.e.*, we use only the first twelve units of the second interval.
On interval $[0, 20]$ we have cost sharing problem (C^1, x^1) with $C^1(t) = C(t)$ for $t \in [0, 20]$ and on the second interval we have cost sharing problem (C^2, x^2) where $C^2(t) = C(t) - C(20) = C(t) - 200$ for all $t \in [20 - 20, 36 - 20]$. Both C^1 and C^2 are nondecreasing and concave. We have

$$Ser_1(C^1, x^1) = \frac{C^1(12)}{3} = 56,$$
$$Ser_2(C^1, x^1) = \frac{C^1(12)}{3} + \frac{C^1(14) - C^1(12)}{2} = 63,$$

$$Ser_3(C^1, x^1) = \frac{C^1(12)}{3} + \frac{C^1(14) - C(12)}{2} + \frac{C^1(20) - C^1(14)}{1} = 81$$

and by symmetry of Ser

$$Ser_1(C^2, x^2) = Ser_2(C^2, x^2) = Ser_3(C^2, x^2) = \frac{C^2(12)}{3} = \frac{C(32) - C(20)}{3} = 56.$$

Thus the cost allocation according to the combination of constrained equal loss and the serial rule is $(56, 63, 81) + (56, 56, 56) = (112, 119, 137)$.

However, we could also argue that the players should have relatively equal rights to all of the intervals, which can be realized by dividing q proportionally over the intervals.
Then, $x^1 = \frac{20}{32}(8, 9, 15) = (5, 5\frac{5}{8}, 9\frac{3}{8})$ and thus $x^2 = (3, 3\frac{3}{8}, 5\frac{5}{8})$. And the

allocation is based on

$$\begin{aligned}
Ser_1(C^1, x^1) &= \frac{C^1(15)}{3} = 62.5,\\
Ser_2(C^1, x^1) &= \frac{C^1(15)}{3} + \frac{C^1(16\frac{1}{4}) - C^1(15)}{2} \approx 65.23,\\
Ser_3(C^1, x^1) &= \frac{C^1(15)}{3} + \frac{C^1(16\frac{1}{4}) - C^1(15)}{2} + \frac{C^1(20) - C^1(16\frac{1}{4})}{1}\\
&\approx 72.27
\end{aligned}$$

and

$$\begin{aligned}
Ser_1(C^2, x^2) &= \frac{C^2(9)}{3} = 45,\\
Ser_2(C^2, x^2) &= \frac{C^2(9)}{3} + \frac{C^2(9\frac{3}{4}) - C^2(9)}{2} \approx 49.41,\\
Ser_3(C^2, x^2) &= \frac{C^2(9)}{3} + \frac{C^2(9\frac{1}{4}) - C^2(9)}{2} + \frac{C^2(12) - C^2(9\frac{3}{4})}{1} \approx 73.59.
\end{aligned}$$

So, when dividing q proportionally over the two intervals, we obtain a different allocation of the costs, $(107.5, 114.64, 145.86)$. ◁

In Example 4.4.6 we used two symmetric and continuous bankruptcy rules that are suitable for allocating q over the maximally concave intervals of C: the proportional rule and the constrained equal losses-rule. The *proportional rule*, $PROP$, divides the estate proportionally over the claimants, *i.e.*, for $(E, d) \in \mathcal{B}^N$ and $i \in N$,

$$PROP_i(E, d) = \min\left\{ d_i \frac{E}{\sum_{j\in N} d_j}, d_i \right\}.$$

The *constrained equal losses-rule*, CEL, is based on the opposite principle of CEA, *i.e.*, for $(E, d) \in \mathcal{B}^N$ and $i \in N$,

$$CEL_i(E, d) = (d_i - \lambda)^+,$$

where $\lambda \in \mathbb{R}_+$ is such that $\sum_{j\in N}(d_j - \lambda)^+ = E$.

The idea we presented in the Example 4.4.6, can be formalized for any bankruptcy rule φ.

Definition: The *φ-piecewise serial rule* $\Psi^\varphi : \mathcal{CS}^{N,m} \to \mathbb{R}^N$ is determined in the following way.

Let $(C, q) \in \mathcal{CS}^{N,m}$ be a cost sharing problem, where $[t_{j-1}, t_j]$ describes the j-th maximally concave interval of C, $j \in \{1, \ldots, m\}$.

Let $q^j \in \mathbb{R}^N_+$ denote the vector of remaining order quantities for interval j and let $x^j \in \mathbb{R}^N_+$ denote the vector of allocated order quantities to interval j.
With $q^1 = q$, recursively compute the vectors q^j and x^j for $j = 1, \ldots, m$

$$\begin{cases} x^j &= \varphi(t_j - t_{j-1}, q^j), \\ q^{j+1} &= q^j - x^j. \end{cases} \tag{4.10}$$

$C^j : [0, t_j - t_{j-1}] \to \mathbb{R}_+$ denotes the translated cost function for interval $j \in \{1, \ldots, m\}$, *i.e.*,

$$C^j(t) = C(t + t_{j-1}) - C(t_{j-1}). \tag{4.11}$$

Then,

$$\Psi^\varphi(C, q) = \sum_{j=1}^{m} Ser(C^j, x^j). \tag{4.12}$$

Note that for all $j \in \{1, \ldots, m\}$, $(C^j, x^j) \in \mathcal{CS}^{N,1}$. For arbitrary φ, the piecewise serial rule is efficient and satisfies demand monotonicity. For symmetric bankruptcy rules, the piecewise serial rule is symmetric as well.

We will focus on Ψ^{PROP} and on Ψ^{CEL} as allocation methods for cost sharing problems with piecewise concave cost functions.

With respect to the proportional rule, let $(E, d) \in \mathcal{B}^N$ be a bankruptcy problem. One can easily observe that for all $i, k \in N$,

$$\frac{PROP_i(E, d)}{d_i} = \frac{PROP_k(E, d)}{d_k}.$$

Furthermore, PROP obeys the property of *order preservation*: for $(E,d) \in \mathcal{B}^N$ with for some pair $i,j \in N$, $d_i \leq d_j$, the following two inequalities hold

$$\begin{cases} PROP_i(E,d) & \leq PROP_j(E,d) \\ d_i - PROP_i(E,d) & \leq d_j - PROP_j(E,d). \end{cases} \tag{4.13}$$

These properties of $PROP$ result in the fact that Ψ^{PROP} satisfies unit cost monotonicity on $\mathcal{CS}^{N,m}$.
In the context of CRCP-situations, unit cost monotonicity implies discount monotonicity: quantity discounts are translated in a monotonic way to the players. Players with larger order quantities obtain a higher quantity discount than players with a smaller order quantity.

Theorem 4.4.7 Ψ^{PROP} *satisfies unit cost monotonicity on* $\mathcal{CS}^{N,m}$.

Proof: For $m = 1$, by Proposition 4.4.2, unit cost monotonicity is satisfied.

Let $m \geq 2$ and let $(C,q) \in \mathcal{CS}^{N,m}$ be a cost sharing problem. Take $j \in \{1,\ldots,m\}$. Because $PROP$ obeys order preservation, we have

$$\begin{cases} x_1^j \leq x_2^j \leq \ldots \leq x_n^j, \\ q_1^j \leq q_2^j \leq \ldots \leq q_n^j. \end{cases}$$

We have that $(C^j, x^j) \in \mathcal{CS}^{N,1}$. Thus by Proposition 4.4.2 for all $i \in N\backslash\{n\}$,

$$\frac{Ser_i(C^j,x^j)}{x_i^j} \geq \frac{Ser_{i+1}(C^j,x^j)}{x_{i+1}^j}. \tag{4.14}$$

We continue the proof, using induction on the concave intervals. Take $i \in N\backslash\{n\}$. Then,

$$\frac{PROP_i(t_1,q)}{q_i} = \frac{PROP_{i+1}(t_1,q)}{q_{i+1}}$$

and thus,

$$\frac{x_i^1}{q_i} = \frac{x_{i+1}^1}{q_{i+1}}.$$

Take $j \in \{2, \ldots, m-1\}$ and assume that for all $k \in \{1, \ldots, j-1\}$

$$\frac{x_i^k}{q_i} = \frac{x_{i+1}^k}{q_{i+1}}.$$

Then,

$$\frac{q_i^j}{q_i} = \left(\frac{q_i - \sum_{k=1}^{j-1} x_i^k}{q_i}\right) = 1 - \sum_{k=1}^{j-1} \frac{x_i^k}{q_i} = 1 - \sum_{k=1}^{j-1} \frac{x_{i+1}^k}{q_{i+1}} = \frac{q_{i+1}^j}{q_{i+1}}$$

and hence

$$\frac{x_i^j}{q_i} = \frac{x_i^j}{q_i^j}\frac{q_i^j}{q_i} = \frac{x_{i+1}^j}{q_{i+1}^j}\frac{q_{i+1}^j}{q_{i+1}} = \frac{x_{i+1}^j}{q_{i+1}}. \tag{4.15}$$

Thus, combining (4.14) and (4.15), we find that

$$\begin{aligned}\frac{\Psi_i^{PROP}(C,q)}{q_i} &= \sum_{j=1}^m \frac{Ser_i(C^j,q^j)}{x_i^j}\frac{x_i^j}{q_i} = \sum_{j=1}^m \frac{Ser_i(C^j,q^j)}{x_i^j}\frac{x_{i+1}^j}{q_{i+1}} \\ &\geq \sum_{j=1}^m \frac{Ser_{i+1}(C^j,q^j)}{x_{i+1}^j}\frac{x_{i+1}^j}{q_{i+1}} = \frac{\Psi_{i+1}^{PROP}(C,q)}{q_{i+1}}.\end{aligned}$$ □

Next, we show that $PROP$ is the unique bankruptcy rule for which the piecewise serial rule satisfies unit cost monotonicity.

Theorem 4.4.8 *Let φ be a bankruptcy rule. Then, $\Psi^\varphi = \Psi^{PROP}$ if and only if Ψ^φ satisfies unit cost monotonicity on $\mathcal{CS}^{N,m}$.*

Proof: For the "only if"-part we refer to Theorem 4.4.7. To prove the "if"-part, let φ be a bankruptcy rule such that Ψ^φ satisfies unit cost monotonicity. To show $\Psi^\varphi = \Psi^{PROP}$ we show that $\varphi = PROP$. Let us assume that $\varphi \neq PROP$. Then, there exists $(E,d) \in \mathcal{B}^N$ such that for some pair $i,j \in N$ with $d_i < d_j$ either

$$(i)\ \frac{\varphi_i(E,d)}{d_i} > \frac{\varphi_j(E,d)}{d_j}$$

or

$$(ii)\ \frac{\varphi_i(E,d)}{d_i} < \frac{\varphi_j(E,d)}{d_j}$$

For both cases we show that there is at least one cost sharing problem $(C,d) \in \mathcal{CS}^{N,m}$ for which $\frac{\Psi_i^{\varphi}(C,d)}{d_i} < \frac{\Psi_j^{\varphi}(C,d)}{d_j}$, establishing a contradiction.

Case (i)
Take $(C,d) \in \mathcal{CS}^{N,m}$ with $m \geq 2$ such that $t_1 = E$ and $t_2 \leq \sum_{j \in N} d_j$. Furthermore for $t \in [0, t_2]$, let C be given by

$$C(t) = \begin{cases} a_1 t & \text{if } t \in [0, t_1] \\ a_2 t & \text{if } t \in (t_1, t_2], \end{cases}$$

where $0 < a_1 < a_2$.

Then,

$$\begin{aligned} \Psi_i^{\varphi}(C,d) &= a_1 \varphi_i(E,d) + a_2(d_i - \varphi_i(E,d)) \\ \Psi_j^{\varphi}(C,d) &= a_1 \varphi_j(E,d) + a_2(d_j - \varphi_j(E,d)) \end{aligned}$$

and since both $a_1 < a_2$ and $\frac{\varphi_i(E,d)}{d_i} > \frac{\varphi_j(E,d)}{d_j}$ we find that

$$\begin{aligned} \frac{\Psi_i^{\varphi}(C,d)}{d_i} &= a_1 \frac{\varphi_i(E,d)}{d_i} + a_2 (1 - \frac{\varphi_i(E,d)}{d_i}) \\ &< a_1 \frac{\varphi_j(E,d)}{d_j} + a_2 (1 - \frac{\varphi_j(E,d)}{d_j}) = \frac{\Psi_j^{\varphi}(C,d)}{d_j}. \end{aligned}$$

Case (ii)
Take $(C^{\varepsilon},d) \in \mathcal{CS}^{N,m}$ with $m \geq 2$ such that $t_1 = E$ and $t_2 \leq \sum_{j \in N} d_j$. Furthermore for $t \in [0, t_2]$, let C^{ε} be given by

$$C^{\varepsilon}(t) = \begin{cases} a_2 t & \text{if } t \in [0, t_1] \\ a_3 t & \text{if } t \in (t_1, t_1 + \varepsilon] \\ a_1 t & \text{if } t \in (t_1 + \varepsilon, t_2], \end{cases}$$

where $0 < a_1 < a_2 < a_3$ and $\varepsilon > 0$ such that

$$\varepsilon < \frac{(a_2 - a_1)}{(a_3 - a_1)} (\frac{\Psi_j^{\varphi}(C,d)}{d_j} - \frac{\Psi_i^{\varphi}(C,d)}{d_i}). \tag{4.16}$$

Note that since we are in case (ii), the righthand side of the above equation is positive.

Then,

$$\frac{\Psi_i^{\varphi}(C^{\varepsilon},d)}{d_i} \leq a_2\frac{\varphi_i(E,d)}{d_i} + a_3\varepsilon + a_1(1-\frac{\varphi_i(E,d)}{d_i} - \varepsilon)$$
$$\frac{\Psi_j^{\varphi}(C^{\varepsilon},d)}{d_j} \geq a_2\frac{\varphi_j(E,d)}{d_j} + 0 \quad + a_1(1-\frac{\varphi_j(E,d)}{d_j} - 0).$$

Subtracting these two inequalities leads to

$$\begin{aligned}\frac{\Psi_i^{\varphi}(C^{\varepsilon},d)}{d_i} - \frac{\Psi_j^{\varphi}(C^{\varepsilon},d)}{d_j} &\leq (a_2-a_1)(\frac{\varphi_i(E,d)}{d_i} - \frac{\varphi_j(E,d)}{d_j}) + \varepsilon(a_3-a_1)\\ &< 0,\end{aligned}$$

where the last inequality follows from (4.16). □

Although Ψ^{CEL} does not satisfy unit cost monotonicity, we can show that if the smallest player is absent, there is a monotonic relation in the effect on the cost allocations of the remaining players. Here, the largest player is most vulnerable for the absence of the smallest player.

Definition: A cost sharing rule f satisfies *monotonic vulnerability* for the *absence* of the *smallest player* (MOVASP) if for all $(C,q) \in \mathcal{CS}^N$ and all $i \in N\backslash\{1,n\}$

$$f_i(C,q) - f_i(C,q_{|N\backslash\{1\}}) \geq f_{i+1}(C,q) - f_{i+1}(C,q_{|N\backslash\{1\}}). \tag{4.17}$$

The property MOVASP is a weaker variant of MOWASP. In the latter case, cost allocations increase if the smallest player is absent and the larger the order quantity, the higher the cost increase, *i.e.*, the weaker the player is for the absence of the smallest player. MOVASP, on the other hand, implies that increased cost allocations are more likely for larger players, *i.e.*, the larger the order quantity, the more vulnerable a player is for the absence of the smallest player.

If we think of applications in CRCP-situations, an allocation method that satisfies MOVASP can create group cohesiveness in the sense that the smallest player can contribute to smaller cost allocations of the largest player. Note that if $Z = (q,[\alpha_1,\alpha_2,Q_A],[\beta_1,\beta_2,Q_B]) \in \mathcal{Z}^N$ and $(C^Z,q) \in \mathcal{CS}^{N,m}$ is the corresponding cost sharing problem, then $(C^Z, q_{|N\backslash\{1\}}) \in \mathcal{CS}^{N\backslash\{1\},m}$ is the cost sharing problem corresponding to $(q_{|N\backslash\{1\}}, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^{N\backslash\{1\}}$.

Theorem 4.4.9 *Ψ^{CEL} satisfies MOVASP on $\mathcal{CS}^{N,m}$.*

To prove this theorem we first show a property of CEL as a bankruptcy rule in the context of bankruptcy problems.

Proposition 4.4.10 *Let $(E, d) \in \mathcal{B}^N$ with $N = \{1, 2, \ldots, n\}$ such that $d_1 \leq d_2 \leq \ldots \leq d_n$ and such that $\sum_{j\in N\backslash\{1\}} d_j \geq E$. Then, for all $j \in N\backslash\{1\}$*

$$CEL_j(E, d_{|N\backslash\{1\}}) = CEL_j(E, d) + \frac{CEL_1(E, d)}{n-1}.$$

Proof: Let $\lambda \in \mathbb{R}_+$ be such that

$$\sum_{i\in N} \max\{0, d_i - \lambda\} = E.$$

Thus for all $j \in N$

$$CEL_j(E, d) = \max\{0, d_j - \lambda\}.$$

If $CEL_1(E, d) = 0$, then by efficiency of CEL

$$\sum_{i\in N} \max\{0, d_i - \lambda\} = \sum_{i\in N\backslash\{1\}} \max\{0, d_i - \lambda\} = E.$$

Thus for all $j \in N\backslash\{1\}$

$$\begin{aligned} CEL_j(E, d_{|N\backslash\{1\}}) &= CEL_j(E, d) + 0 \\ &= CEL_j(E, d) + \frac{CEL_1(E, d)}{n-1}. \end{aligned}$$

If $CEL_1(E, d) > 0$, then also $CEL_j(E, d) > 0$ for any $j \in N$. Thus

$$\sum_{i\in N} \max\{0, d_i - \lambda\} = \sum_{i\in N} (d_i - \lambda) = E.$$

Take

$$\mu = \lambda - \frac{d_1 - \lambda}{n-1}.$$

Then, for all $j \in N\backslash\{1\}$,

$$\max\{0, d_j - \mu\} = d_j - \mu$$

and thus

$$\begin{aligned}\sum_{i \in N\backslash\{1\}} d_i - \mu &= \sum_{i \in N\backslash\{1\}} d_i - (n-1)\left(\lambda - \frac{d_1 - \lambda}{n-1}\right)\\ &= \sum_{i \in N\backslash\{1\}} d_i - n\lambda + d_1 = \sum_{i \in N}(d_i - \lambda) = E.\end{aligned}$$

Thus μ is such that

$$\sum_{i \in N\backslash\{1\}} \max\{0, d_i - \mu\} = E.$$

Hence, for all $j \in N\backslash\{1\}$

$$\begin{aligned}CEL_j(E, d_{|N\backslash\{1\}}) &= \max\{0, d_j - \mu\} = d_j - \mu\\ &= d_j - \lambda + \frac{d_1 - \lambda}{n-1}\\ &= CEL_j(E, d) + \frac{CEL_1(E, d)}{n-1}.\end{aligned}$$

□

Using this result we can show MOVASP for Ψ^{CEL}.

Proof of Theorem 4.4.9: Let $(C, q) \in \mathcal{CS}^{N,m}$ be a cost sharing problem. For all $j \in \{1, \ldots, m\}$, let $q^j \in \mathbb{R}^N$ be the remaining vectors of order quantities and let $x^j \in \mathbb{R}^N$ be the allocated vectors of order quantities for interval $[t_{j-1}, t_j]$ according to (4.10).
Let $(C, q_{|N\backslash\{1\}}) \in \mathcal{CS}^{N,m}$ and set $\bar{q} = q_{|N\backslash\{1\}}$. With $\bar{q}^1 = \bar{q}$, for all $j \in \{1, \ldots, m\}$, let $\bar{q}^j \in \mathbb{R}^{N\backslash\{1\}}$ be the remaining vectors of order quantities, and let $\bar{x}^j \in \mathbb{R}^{N\backslash\{1\}}$ be the allocated vectors of order quantities for interval $[t_{j-1}, t_j]$ according to (4.10).
Let C^j be the cost function for interval $j \in \{1, \ldots, m\}$, as in (4.11).

Equation (4.12) tells that $\Psi(C,q) = \sum_{j=1}^{m} Ser(C^j, x^j)$, so it is sufficient to show that for each $i \in \{3,\ldots,n\}$ and each $j \in \{1,\ldots,m\}$

$$Ser_i(C^j, x^j) - Ser_i(C^j, \bar{x}^j) \leq Ser_{i-1}(C^j, x^j) - Ser_{i-1}(C^j, \bar{x}^j). \quad (4.18)$$

Let $i \in \{3,\ldots,n\}$ and $j \in \{1,\ldots,m\}$. We distinguish between four cases:

$$\text{(I)} \qquad q^j_{N\backslash\{1\}} = \bar{q}^j \text{ and } \sum_{k\in N} q^j_k \leq t_j - t_{j-1},$$

$$\text{(II)} \qquad q^j_{N\backslash\{1\}} = \bar{q}^j \text{ and } \sum_{k\in N} q^j_k > t_j - t_{j-1} \geq \sum_{k\in N\backslash\{1\}} \bar{q}^j_k,$$

$$\text{(III)} \qquad q^j_{N\backslash\{1\}} = \bar{q}^j \text{ and } \sum_{k\in N\backslash\{1\}} \bar{q}^j_k > t_j - t_{j-1},$$

$$\text{(IV)} \qquad q^j_{N\backslash\{1\}} \neq \bar{q}^j.$$

Let (C^j, x^j) and $(C^j, \bar{x}^j)$ be the two cost sharing problems for interval j. Then, for all $\ell \in N$ we set

$$s_\ell = \sum_{j=1}^{\ell-1} x_j + (n-\ell+1)x_\ell$$

and for all $\ell \in N\backslash\{1\}$ we set

$$\bar{s}_\ell = \sum_{j=2}^{\ell-1} \bar{x}_j + (n-\ell+1)\bar{x}_\ell.$$

Case I

In this case (C^j, x^j) is a cost sharing problem with a concave increasing cost function. Since $t_j - t_{j-1}$ is sufficient to fulfill all orders, we have that $x^j = q^j$ and $\bar{x}^j = \bar{q}^j = q^j_{|N\backslash\{1\}}$. Hence, by Proposition 4.4.3, inequality (4.18) holds.

Case II

If $\sum_{k\in N} q^j_k > t_j - t_{j-1} \geq \sum_{k\in N\backslash\{1\}} \bar{q}^j_k$. Then, for all $\ell \in \{2,\ldots,n\}$,

$$\bar{x}^j_\ell = CEL(t_j - t_{j-1}, \bar{q}) = \bar{q}^j_\ell. \quad (4.19)$$

Moreover,

$$\varepsilon = q^j_1 - (\sum_{k\in N} q^j_k - (t_j - t_{j-1})) > 0$$

and

$$x_1^j = CEL_1(t_j - t_{j-1}, q^j) > 0.$$

Hence, for all $\ell \in \{2, \dots, n\}$ we have that

$$x_\ell^j = CEL_\ell(t_j - t_{j-1}, q^j) = q_\ell^j - \frac{\sum_{h \in N} q_h^j - (t_j - t_{j-1})}{n}.$$

Thus

$$\begin{aligned}
s_\ell =& (n-\ell+1)x_\ell^j + \sum_{k=1}^{\ell-1} x_k^j \\
=& (n-\ell+1)\left(q_\ell^j - \frac{\sum_{h \in N} q_h^j - (t_j - t_{j-1})}{n}\right) \\
&+ \sum_{k=1}^{\ell-1}\left(q_k^j - \frac{\sum_{h \in N} q_h^j - (t_j - t_{j-1})}{n}\right) \\
=& (n-\ell+1)q_\ell^j + \sum_{k=1}^{\ell-1} q_k^j \\
&- [(n-\ell+1)+(\ell-1)]\left(\frac{\sum_{h \in N} q_h^j - (t_j - t_{j-1})}{n}\right) \\
=& (n-\ell+1)q_\ell^j + \sum_{k=1}^{\ell-1} q_k^j - \left(\sum_{h \in N} q_h^j - (t_j - t_{j-1})\right) \\
=& (n-\ell+1)q_\ell^j + \sum_{k=2}^{\ell-1} q_k^j + q_1^j - \left(\sum_{h \in N} q_h^j - (t_j - t_{j-1})\right) \\
=& (n-\ell+1)\bar{x}_\ell^j + \sum_{k=2}^{\ell-1} \bar{x}_k^j + \varepsilon \\
=& \bar{s}_\ell + \varepsilon.
\end{aligned}$$

Here, the last equality follows from (4.19).

Using (4.8),

$$\begin{aligned}
Ser_i(C^j, x^j) - Ser_{i-1}(C^j, x^j) =& \frac{C(s_i) - C(s_{i-1})}{n-i+1} \\
=& \frac{C(\bar{s}_i + \varepsilon) - C(\bar{s}_{i-1} + \varepsilon)}{n-i+1},
\end{aligned}$$

while

$$Ser_i(C^j, \bar{x}^j) - Ser_{i-1}(C^j, \bar{x}^j) = \frac{C\big(\bar{s}_i\big) - C\big(\bar{s}_{i-1}\big)}{n-i+1}.$$

Hence,

$$\begin{aligned}
&\Big(Ser_i(C^j, x^j) - Ser_i(C^j, \bar{x}^j)\Big) - \Big(Ser_{i-1}(C^j, x^j) - Ser_{i-1}(C^j, \bar{x}^j)\Big)\\
=&\Big(Ser_i(C^j, x^j) - Ser_{i-1}(C^j, x^j)\Big) - \Big(Ser_i(C^j, \bar{x}^j) - Ser_{i-1}(C^j, \bar{x}^j)\Big)\\
=&\frac{C\big(\bar{s}_i + \varepsilon\big) - C\big(\bar{s}_{i-1} + \varepsilon\big)}{n-i+1} - \frac{C\big(\bar{s}_i\big) - C\big(\bar{s}_{i-1}\big)}{n-i+1}\\
\leq&0,
\end{aligned}$$

where the last equality follows from the fact that $\varepsilon > 0$ and that C^j is concave on $[0, t_j - t_{j-1}]$. Thus

$$Ser_i(C^j, x^j) - Ser_i(C^j, \bar{x}^j) \leq Ser_{i-1}(C^j, x^j) - Ser_{i-1}(C^j, \bar{x}^j).$$

Case III

In this case we can show that

$$Ser_i(C^j, x^j) - Ser_i(C^j, \bar{x}^j) = Ser_{i-1}(C^j, x^j) - Ser_{i-1}(C^j, \bar{x}^j).$$

Since $\sum_{k \in N\setminus\{1\}} \bar{q}^j_k > t_j - t_{j-1}$, we have that for $\ell \in \{2, \ldots, n\}$

$$\begin{aligned}
\bar{x}^j_\ell &= CEL_\ell(t_j - t_{j-1}, q^j_{|N\setminus\{1\}})\\
&= CEL_\ell(t_j - t_{j-1}, q^j) + \frac{CEL_1(t_j - t_{j-1}, q^j)}{n-1}\\
&= x^j_\ell + \frac{x^j_1}{n-1},
\end{aligned}$$

where the second equality follows from Proposition 4.4.10. Next, note that

$$\begin{aligned}
\bar{s}_2 &= (n-1)\bar{x}^j_2 = (n-1)\left(x^j_2 + \frac{x^j_1}{n-1}\right) = (n-1)x^j_2 + x^j_1\\
&= s_2 \qquad (4.20)
\end{aligned}$$

and that for $\ell \in \{3, \dots, n\}$

$$\begin{aligned}
\bar{s}_\ell =& (n-\ell+1)\bar{x}_\ell^j + \sum_{k=2}^{\ell-1} \bar{x}_k^j \\
=& (n-\ell+1)\left(x_\ell^j + \frac{x_1^j}{n-1}\right) + \sum_{k=2}^{\ell-1}\left(x_k^j + \frac{x_1^j}{n-1}\right) \\
=& (n-\ell+1)x_\ell^j + \sum_{k=2}^{\ell-1} x_k^j + [(n-\ell+1)+(\ell-2)]\left(\frac{x_1^j}{n-1}\right) \\
=& (n-\ell+1)x_\ell^j + \sum_{k=1}^{\ell-1} x_k^j \\
=& s_\ell.
\end{aligned} \tag{4.21}$$

Combining (4.20) and (4.21), we have that for all $\ell \in \{2, \dots, n\}$

$$\bar{s}_\ell = (n-\ell+1)\bar{x}_\ell^j + \sum_{k=2}^{\ell-1} \bar{x}_k^j = (n-\ell+1)x_\ell^j + \sum_{k=1}^{\ell-1} x_k^j = s_\ell.$$

Thus for all $\ell \in \{2, \dots, n\}$,

$$\begin{aligned}
Ser_\ell(C^j, \bar{x}^j) &= \frac{C((n-1)\bar{x}_2^j)}{n-1} + \sum_{k=3}^{\ell} \frac{C(\bar{s}_k) - C(\bar{s}_{k-1})}{n-\ell+1} \\
&= \frac{C((n-1)\bar{x}_2^j)}{n-1} + \sum_{k=3}^{\ell} \frac{C(s_k) - C(s_{k-1})}{n-\ell+1}
\end{aligned}$$

and hence

$$\begin{aligned}
Ser_\ell(C^j, x^j) - Ser_\ell(C^j, \bar{x}^j) =& \frac{C(nx_1^j)}{n} + \sum_{k=3}^{\ell} \frac{C(s_k) - C(s_{k-1})}{n-\ell+1} \\
& - \frac{C((n-1)\bar{x}_2^j)}{n-1} - \sum_{k=3}^{\ell} \frac{C(s_k) - C(s_{k-1})}{n-\ell+1} \\
=& \frac{C(nx_1^j)}{n} - \frac{C(nx_1^j)}{n-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
Ser_i(C^j, x^j) - Ser_i(C^j, \bar{x}^j) &= Ser_{i-1}(C^j, x^j) - Ser_{i-1}(C^j, \bar{x}^j) \\
&= \frac{C(nx_1^j)}{n} - \frac{C(nx_1^j)}{n-1}.
\end{aligned}$$

Case IV

Also, for this final case we show that

$$Ser_i(C^j, x^j) - Ser_i(C^j, \bar{x}^j) = Ser_{i-1}(C^j, x^j) - Ser_{i-1}(C^j, \bar{x}^j).$$

If $q^j_\ell \neq \bar{q}^j_\ell$ for some $\ell \in \{2, \ldots, n\}$, then $j > 1$ and for some $h \in \{1, \ldots, j-1\}$,

$$x^h_1 > 0.$$

Hence, for all $\ell \in \{h+1, \ldots, m\}$

$$\begin{cases} q^\ell_i &= q^\ell_{i-1} \\ \bar{q}^\ell_i &= \bar{q}^\ell_{i-1}. \end{cases}$$

By symmetry of CEL also $x^j_i = x^j_{i-1}$ and $\bar{x}^j_i = \bar{x}^j_{i-1}$. Hence,

$$Ser_i(C^j, x^j) - Ser_i(C^j, \bar{x}^j) = Ser_{i-1}(C^j, x^j) - Ser_{i-1}(C^j, \bar{x}^j). \qquad \square$$

The next example shows that Ψ^{PROP} does not satisfy MOVASP.

Example 4.4.11 Let $N = \{1, 2, 3, 4\}$ and let $(C, q) \in \mathcal{CS}^{N,3}$ be a cost sharing problem given by $q = (2, 5, 6, 9)$ and

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0, 12], \\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12, 16], \\ 192 + 18(t-16) - \frac{1}{3}(t-16)^2 & \text{if } t \in (16, 17], \\ 18t - \frac{1}{3}t^2 & \text{if } t \in (17, 18], \\ 216 + 20(t-18) - \frac{1}{2}(t-18)^2 & \text{if } t \in (18, 22], \\ 192 + 18(t-16) - \frac{1}{3}(t-16)^2 & \text{if } t \in (22, 34]. \end{cases}$$

Note that this cost sharing problem corresponds to CRCP-situation $Z = (q, [\alpha_1, \alpha_2, Q_A], [\beta_1, \beta_2, Q_B]) \in \mathcal{Z}^N$ with unit price functions

$$\begin{cases} p_A(t) = 20 - \frac{1}{2}t \text{ for } t \in [0, 16], \\ p_B(t) = 18 - \frac{1}{3}t \text{ for } t \in [0, 18]. \end{cases}$$

The maximal concave intervals of C are $[0,16]$, $[16,18]$ and $[18,34]$. Hence,

$$\begin{aligned}\Psi^{PROP}(C,q) &= Ser(C^1,(\frac{16}{11},\frac{40}{11},\frac{48}{11},\frac{72}{11})) + Ser(C^2,(\frac{2}{11},\frac{5}{11},\frac{6}{11},\frac{9}{11}))\\ &+ Ser(C^3,(\frac{4}{11},\frac{10}{11},\frac{12}{11},\frac{18}{11}))\\ &\approx (33.59, 71.83, 80.46, 102.10),\end{aligned}$$

while

$$\Psi^{PROP}(C, q_{|N\backslash\{1\}}) \approx (72.59, 80.82, 100.57).$$

Thus, without player 1, player 2's costs increase with 0.76, player 3's costs increase with 0.36 and player 4's costs decrease with 1.53. In this example player 4 benefits the most from the absence of player 1. ◁

On the other hand, it need not be the case that if, due to the absence of the smallest player, the cost allocation to player 2 decreases, all other players' cost allocations decrease as well.

Example 4.4.12 Let $N = \{1,2,3,4\}$ and let $(C,q) \in \mathcal{CS}^{N,2}$ be a cost sharing problem with $q = (0.5, 0.8, 1.5, 2.4)$ and with

$$C(t) = \begin{cases} 20t - 2t^2 & \text{if } t \in [0,5],\\ 50 + 36(t-5) - 3(t-5)^2 & \text{if } t \in (5, 10.5].\end{cases}$$

The maximal concave intervals of C are $[0,5]$ and $[5,10.5]$. Note that this cost sharing problem corresponds to CRCP-situation $(q, [\alpha_1,\alpha_2,Q_A],[\beta_1,\beta_2,Q_B]) \in \mathcal{Z}^N$ with

$$\begin{cases} p_A(t) = 20 - 2t & \text{for } t \in [0,5],\\ p_B(t) = 36 - 3t & \text{for } t \in [0,5.5].\end{cases}$$

We have that,

$$\begin{aligned}\Psi_2^{CEL}(C,q) &\approx 10.68 + 1.77 = 12.45,\\ \Psi_3^{CEL}(C,q) &\approx 10.68 + 4.48 + 1.77 = 16.93,\\ \Psi_4^{CEL}(C,q) &\approx 10.68 + 4.48 + 1.62 + 1.77 = 18.55,\end{aligned}$$

and

$$\Psi_2^{CEL}(C, q_{|N\setminus\{1\}}) \approx 12.16,$$
$$\Psi_3^{CEL}(C, q_{|N\setminus\{1\}}) \approx 12.16 + 5.32 = 17.48,$$
$$\Psi_4^{CEL}(C, q_{|N\setminus\{1\}}) \approx 12.16 + 5.32 + 2.7 = 20.18.$$

Hence $\Psi_2^{CEL}(C, q) - \Psi_2^{CEL}(C, q_{|N\setminus\{1\}}) \approx 0.29$ is positive, while $\Psi_3^{CEL}(C, q) - \Psi_3^{CEL}(C, q_{|N\setminus\{1\}}) \approx -0.55$ and $\Psi_4^{CEL}(C, q) - \Psi_4^{CEL}(C, q_{|N\setminus\{1\}}) \approx -1.63$ are negative. $\triangleleft$

4.5 CRCP-situations: comparing cost sharing rules

In the previous section we developed a class of cost sharing rules for cost sharing problems with a piecewise concave cost function. The CEL-piecewise serial rule and $PROP$-piecewise serial rule seem particularly appropriate for allocating the cooperative purchasing costs of a CRCP-situation. The $PROP$-piecewise serial rule is appropriate since it satisfies unit cost monotonicity, implying that in the purchasing cooperative larger players obtain a larger quantity discount. The CEL-piecewise serial rule satisfies MOWASP, which can create a group cohesiveness in the purchasing cooperative, since in the allocation the smallest player can directly contribute to cost savings of larger players. In this section we illustrate numerical differences and similarities between the two piecewise serial rules and the classic serial cost sharing rule applied to cost sharing problems arising from CRCP-situations.

In the Figures 4.5.2 and 4.5.3 the results of some simulations can be found. The same type of set-up as in Section 3.7 has been used. As input we take CRCP-situations where $\sum_{i\in N} q_i = 52$ and with unit price functions

$$\begin{cases} p_A(t) = 60 - \frac{1}{2}t, \text{ with} t \in [0, 29], \\ p_B(t) = 140 - 2t, \text{ with } t \in [0, 35]. \end{cases}$$

The cost function C of the corresponding cost sharing problem can be found in Figure 4.5.1. It can be seen that C has 2 maximally concave intervals.

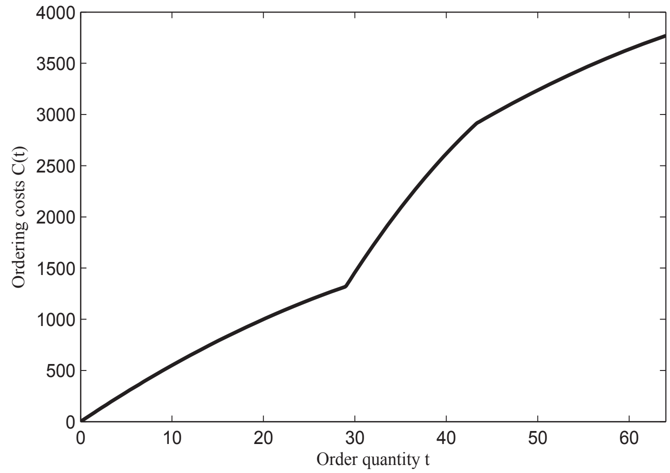

Figure 4.5.1: Cost function for comparing allocation methods

To create a CRCP-situation we randomly generate a vector of discrete order quantities such that the sum of the order quantities equals 52. Then, the cost allocations of $C(52)$ according to Ser, Ψ^{CEL} and Ψ^{PROP} are calculated. As in Section 3.7, we store the allocations per different value of q_i independent of the index i and we repeat this step successively. For every possible value of the order quantity, we average the stored cost allocations over the number of times they have appeared. We compare two instances: $|N| = 5$ with

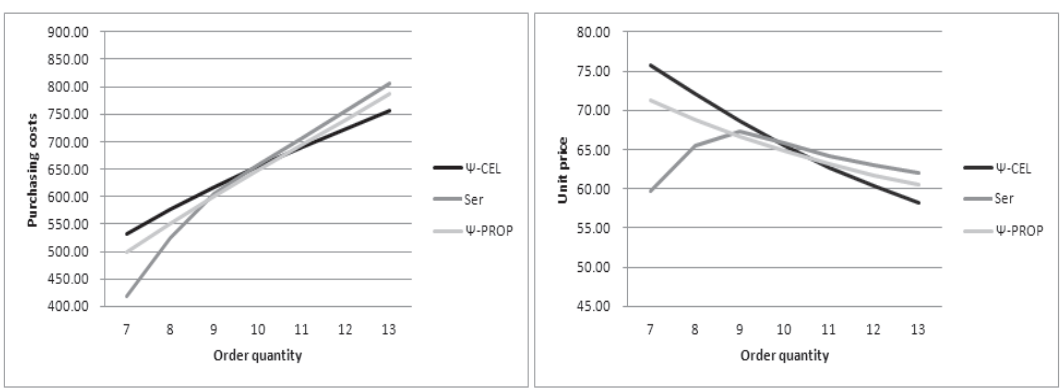

Figure 4.5.2: Cost allocations and costs per unit for 'small' differences

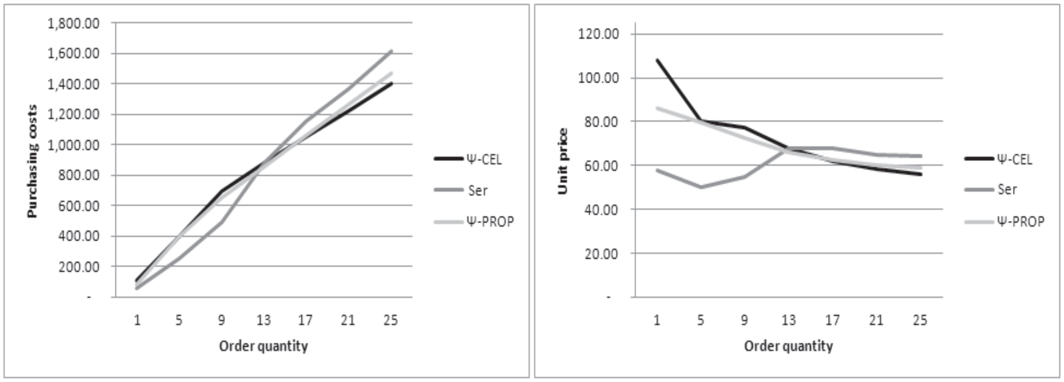

Figure 4.5.3: Cost allocations and costs per unit for 'big' differences

for all $i \in N$, $q_i \in \{7, 8, 9, 10, 11, 12, 13\}$ and $|N| = 4$ with for all $i \in N$, $q_i \in \{1, 5, 9, 13, 17, 21, 25\}$, *i.e.*, instances with 'small' and 'big' differences, respectively, between the possible order quantities.

In Figure 4.5.2 we plotted the average cost allocations according to the three different solution concepts and we plotted the average costs per unit, *i.e.*, the cost allocation divided by order quantity, for a player with order quantity $q_i \in \{7, 8, 9, 10, 11, 12, 13\}$, according to the three different cost allocation methods. The cost allocations and costs per unit according to Ser are connected by the dark grey line, Ψ^{CEL} by the black line and Ψ^{PROP} by the light grey line. Figure 4.5.3 shows the average cost allocations and costs per unit for a player with $q_i \in \{1, 5, 9, 13, 17, 21, 25\}$.

We can make the following observations. On average, the serial cost sharing rule allocates more costs to the largest player than the two piecewise serial rules. For middle players the cost allocations are almost the same, while for smaller players the serial cost sharing rule allocates the least costs to smaller players. The differences between Ψ^{PROP} and Ψ^{CEL} are smaller. In Figure 4.5.1 we can see that the first concave interval of the cost function gives higher quantity discounts than the second concave interval (up to $t = 52$). As a result, large players prefer Ψ^{CEL} over Ψ^{PROP}.

Generally, in the first (two) interval(s) of a cost function of a cost sharing problem corresponding to a CRCP-situation, the quantity discounts will be larger than in the last interval. This occurs because when having a smaller order quantity, one is more likely to buy at a single supplier. Using CEL as a mechanism for allocating q over the intervals will result in larger players having relatively bigger shares in the first interval than in later ones. This does not necessarily imply unit cost monotonicity, but in many cases it will. As we can see in the righthand sided figures, the costs per unit belonging to Ψ^{PROP} and also Ψ^{CEL} are decreasing, whereas for Ser this is not the case. For Ψ^{PROP} the decrease in costs per unit seems to follow a more constant pattern than the decrease in costs per unit for Ψ^{CEL}.

Chapter 5

Ordering strategies for capacity restricted strategic purchasing situations

5.1 Introduction

In this chapter we continue the analysis of capacity restrictions in interactive purchasing situations from Chapter 4, but we take a different perspective. Instead of a capacity restricted *cooperative* purchasing situation, we consider a *capacity restricted strategic purchasing (CRSP) situation*. In a CRSP-situation each purchaser strategically splits his order over the suppliers in order to obtain his desired order quantity for the lowest possible cost. How an individual purchaser should place his order depends, amongst others, on fulfillment policies of the suppliers, *i.e.*, how a supplier allocates his restricted capacity over a set of orders.

The literature on the allocation of scarce capacity from a strategic perspective is richer than the literature on cooperative purchasing with limited supplies. Capacity allocation has been considered in various contexts, *e.g.*, in allocating MRI scanner time in a hospital (Zonderland and Timmer (2012)) or the allocation of capacity in semiconductor manufacturing (Mallika and Harker (2004)). Cachon and Lariviere (1998) analyze capacity allocation games in which a single supplier allocates a scarce commodity among two retailers. They compare the effect of different allocation policies of the supplier. It is explained that the possible order inflation by the retailers is a major drawback of a proportional fulfillment policy. Since capacity is scarce, retailers will ask for more than they need, which might result in an allocation of the scarce commodity in which some retailers obtain more than they need. Cachon and Lariviere (1999) analyze capacity fulfillment policies for a single capacity restricted supplier. It is also considered how the supplier could determine an optimal capacity level.

A common aspect of the literature on capacity allocation is the fact that there is a single supplier and a group of retailers with *private* information: the retailers only know their own individual demand. The main topics of analysis are: whether the fulfillment policy of the scarce commodity induces truth telling of the retailers, whether in the final allocation of the scarce commodity no retailer obtains more than he needs and whether the fulfillment policy of the supplier supports maximizing the supplier's profit or the combined utility of the supplier and the retailers.

The current setup of CRSP-situations is different. The starting point is the same: a group of purchasers with individual order quantities with respect to one commodity. The group faces two suppliers with (possibly) insufficient individual supplies. The combined capacity of the two suppliers is assumed to be sufficient and both suppliers offer quantity discounts. The more a purchaser obtains from one specific supplier, the lower the unit price. In a CRSP-situation, however, every purchaser strategically places an order at both or at one of the suppliers in order to obtain his order quantity. Each supplier has a fulfillment policy: in case the total ordered units exceed the capacity of the supplier it is prespecified how the supplier allocates his capacity over the orders. The suppliers are not considered to be interactive decision makers: their pricing, quantity discounts and fulfillment policies are fixed before the purchasers make any decisions with respect to their orders. Furthermore, there is no private information, all suppliers' and purchasers' characteristics are publicly known by all purchasers. The reason to choose for public information is twofold. First, CRSP-situations are the non-cooperative siblings of CRPC-situations, in which all information is public. Second, private information can make the purchasing situation intractable. Moreover, tractable results with the assumption of public information can be useful in future research for understanding CRSP-situations with private information. The most important difference between our and previous capacity allocation models, is that in our model purchasers are assumed to obtain at least their order quantity, because the total available global capacity is sufficient. If the orders are such that there is local scarcity at one of the two suppliers, purchasers are forced to reorder at the other supplier. Although seemingly so, this model and its approach are not the same as non-cooperative estate division problems from Atlamaz, Berden, Peters, and Vermeulen (2011), where the estate is seen as an interval and where players have to specify exactly which part of the 'estate-interval' they would like to receive. In CRSP-situations, organizations only need to split their order over the two suppliers. Our main question is: what is the effect of suppliers' fulfillment policies on the ordering strategies of the purchasers? We are not looking for a 'best' fulfillment policy, we want to describe possible equilibrium behavior of individualistic purchasers in different scenarios.

For the analysis of CRSP-situations we use non-cooperative game theory. In a non-cooperative game, players are considered to be individual cost minimizers. Every player's costs may depend not only on his own strategy but also on the strategies of the other players. A central notion in the literature on non-cooperative games is formulated by Nash (1951): in a Nash equilibrium every player minimizes his costs given the equilibrium strategies of the other players. In a non-cooperative game where players have an infinite number of strategies available, the existence of a Nash equilibrium is not guaranteed. In a game in which each player has a finite set of actions and in which one allows for mixed strategies, there exists at least one Nash equilibrium in mixed strategies.

In modeling CRSP-situations one can make multiple reasonable assumptions with respect to the behavior of and limitations set by the suppliers. Since all information — including the prespecified fulfillment policies of the suppliers — is publicly available, it seems reasonable to assume that each purchaser can only order such that his combined order at the two suppliers equals his individual order quantity. Moreover, an order cannot exceed the capacity restrictions of the suppliers. In other words: the purchasers' orders are restricted by feasibility. Nevertheless, the feasibility restrictions still give each purchaser an infinite number of possibilities to split his order quantities over the two suppliers.

For analyzing specific CRSP-situations, in this chapter, we differentiate among two dimensions: (1) the available ordering options for the purchasers and (2) the fulfillment policies of the suppliers. Specific choices with respect to these dimensions lead to scenarios.

With respect to the first dimension, we separate two cases. First we study infinite ordering games, in which purchasers can place any feasible order. Second, in order to decrease the number of options for the purchasers we will also consider ordering games in which the purchasers are allowed to choose from only a finite but representative set of ordering possibilities, but in which we allow for mixed strategies.

These two cases along the first dimension result in two different sets of possible strategies for the purchasers. Hence, we develop two different ordering games corresponding to a CRSP-situation: an *infinite ordering game* and a so-called *matrified ordering game*. The matrified ordering game can also be seen as an approximation of the infinite ordering game. The feasibility restriction on the orders ensures that purchasers are not able to inflate their orders in order to obtain a higher fulfillment level. On the other hand, depending on the fulfillment policy, purchasers might not obtain enough. Therefore we allow purchasers to reorder after announcement of the fulfillment levels. This specific timing of the ordering game is incorporated in the general cost function of both ordering games. In both ordering games the specific cost function further depends on the fulfillment policies of the suppliers.

With respect to the second dimension, we consider four different types of fulfillment policies. Each of these fulfillment policies changes the cost function of both the infinite as well as the matrified ordering game.

On the one hand, we consider three different types of fulfillment policies that are based on a certain *preference* order over the purchasers. On the other hand we consider suppliers that allocate their capacity proportionally over the orders (PROP). Proportional fulfillment is, according to Cachon and Lariviere (1998), the most intuitive fulfillment policy, although it usually does not induce truth-telling by the purchasers.

For the fulfillment policies that are based on preference orders, first, we analyze situations in which suppliers' fulfillment is based on preferences that are fixed and identical (FID), *i.e.*, the preference order does not depend on the orders of the purchasers, but solely on the identity of the purchasers. Second, one could argue that, in order to maximize profit, suppliers might prefer to fulfill small orders before large orders (SBL). A third reasonable fulfillment policy is based on the idea that suppliers might value the purchaser-supplier relationship with purchasers with large order quantities over the relationship with smaller purchasers, hence suppliers prefer fulfilling large orders before smaller orders (LBS).

Next, we summarize the results on infinite ordering games. For an infinite ordering game we show that if both suppliers' fulfillment policies are based on FID, there exists a profile of orders that corresponds to a Nash equilibrium in which there is no need to correct orders after announcement of the fulfillment level by the suppliers. On the other hand, if the fixed preferences are not identical, it remains an open problem whether the infinite ordering game has an equilibrium.

If both suppliers' fulfillment policies are based on LBS, the infinite ordering game has a Nash equilibrium which can be found at the boundaries of the strategy space. On the other hand, if the fulfillment policy of both suppliers is based on (SBL), the infinite ordering game does not need to have a Nash equilibrium.

In an infinite ordering game in which suppliers use a proportional fulfillment policy, we show that *if* there is an equilibrium, the equilibria of the game can be found at the boundaries of the strategy space. General existence of equilibria in this scenario, however, is an open problem.

As explained, in the *matrified* ordering game purchasers are only allowed to choose from a limited number of actions. In order to decrease the number of ordering possibilities we use the knowledge that in an infinite ordering game, equilibria can often be found at the boundaries of purchasers' strategy spaces. Purchasers have two extreme options: order as much as possible at supplier A or order as much as possible at supplier B. Here, we allow for mixed strategies.
Also in the matrified ordering game we differentiate with respect to the fulfillment policies of the suppliers. We show that, if suppliers' fulfillment is based on LBS, the matrified ordering game has a pure Nash equilibrium. This equilibrium corresponds to a Nash equilibrium in the infinite ordering game. Not every equilibrium of the infinite ordering game corresponds, however, to a pure equilibrium in the matrified ordering game.
On the other hand, if both suppliers' fulfillment policies are based on SBL, the matrified ordering game also has a pure Nash equilibrium, whereas the infinite ordering game does not necessarily have an equilibrium.
And, in case suppliers fulfill orders proportionally, the existence of a pure equilibrium in the matrified game corresponds to the existence of an equilibrium in the original infinite ordering game.

We conclude the chapter by a few remarks on some of the assumptions we have made. Furthermore we summarize the remaining open problems.

The organization of this chapter is as follows. In Section 5.2 we formally introduce capacity restricted strategic purchasing situations and in Section 5.3 we explain some important notions on non-cooperative cost games. Section 5.4 analyzes infinite ordering games, while Section 5.5 analyzes matrified ordering games. In Section 5.6 we state some concluding remarks.

5.2 Capacity restricted strategic purchasing situations

A *capacity restricted strategic purchasing (CRSP) situation* can be described by similar parameters as its cooperative counterpart from Chapter 4. There

is a set of players $N = \{1, \ldots, n\}$ with order quantities $q \in \mathbb{R}^N_{++}$. There are two suppliers, A and B, with respective supply capacities $Q_A, Q_B \in \mathbb{R}_{++}$ such that $Q_A + Q_B \geq \sum_{j \in N} q_j$. Supplier A has unit price function $p_A : [0, Q_A] \to \mathbb{R}_+$ and B has unit price function $p_B : [0, Q_B] \to \mathbb{R}_+$.
Without loss of generality we assume that the order quantities are arranged in nondecreasing order, *i.e.*, $q_1 \leq q_2 \leq \ldots \leq q_n$, and that $Q_A \leq Q_B$.
Furthermore, we assume that the unit price functions of the suppliers are decreasing and twice differentiable. The unit price function of supplier A is such that the turnover function of A, for all $t \in [0, Q_A]$ defined by $c_A(t) = p_A(t)t$, is increasing and strictly concave on $[0, Q_A]$. Similarly, we assume that the unit price function of B is such that the turnover function, c_B, is increasing and strictly concave on $[0, Q_B]$.

Instead of purchasing cooperatively, the players in N act strategically. Each player $i \in N$ places an order $0 \leq x_i^A \leq Q_A$ at A and an order $0 \leq x_i^B \leq Q_B$ at B in order to obtain his order quantity q_i with the restriction that

$$x_i^A + x_i^B = q_i.$$

Both suppliers have a certain order *fulfillment policy*. Let $x^A \in \mathbb{R}^N_+$ be the vector of orders of the players in N at supplier A. Then A uses policy $\pi^A : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ to allocate Q_A over the orders at A. Similarly $\pi^B : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ gives for each set of orders $x^B \in \mathbb{R}^N_+$, an allocation of Q_B.
We assume that the fulfillment policies of the suppliers are such that for A,

$$\sum_{j \in N} \pi_j^A(x^A) = \min\{Q_A, \sum_{j \in N} x_j^A\}$$

and $\pi^A(x^A) \leq x^A$, *i.e.*, fulfillment is efficient and bounded by the orders. Similarly, for B

$$\sum_{j \in N} \pi_j^B(x^B) = \min\{Q_B, \sum_{j \in N} x_j^B\}$$

and $\pi^B(x^B) \leq x^B$.

We have that c_A is increasing if for all $t \in [0, Q_A]$, $c_A^{'}(t) = p_A^{'}(t)t + p_A(t) > 0$, thus if

$$p_A'(t) > -\frac{p_A(t)}{t}. \tag{5.1}$$

Furthermore, c_A is strictly concave on $[0, Q_A]$ if for all $t \in [0, Q_A]$, $c_A''(t) = p_A''(t)t + 2p_A'(t) < 0$, thus if

$$p_A''(t) < -\frac{2p_A'(t)}{t}. \tag{5.2}$$

Similarly c_B is increasing if for all $t \in [0, Q_B]$

$$p_B'(t) > -\frac{p_B(t)}{t} \tag{5.3}$$

and c_B is strictly concave on $[0, Q_B]$ if for all $t \in [0, Q_B]$

$$p_B''(t) < -\frac{2p_B'(t)}{t}. \tag{5.4}$$

Let W be the CRSP-situation given by q, supplier A's characteristics, p_A, Q_A, and π^A, and supplier B's characteristics, p_B, Q_B, and π^B. The set $\mathcal{W}^N$ contains all CRSP-situations on N.

Due to the assumptions we make with respect to the unit price functions of A and B, we can generalize the results of Theorem 4.3.1; it is optimal to either order as much as possible at A or as much as possible at B.

Proposition 5.2.1 *Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^N$ be a CRSP-situation. Then, for all $t \in [0, Q_A + Q_B]$*

$$\begin{aligned} &\min\left\{c_A(t_A) + c_B(t - t_A) | 0 \le t_A \le Q_A, 0 \le t_B \le Q_B\right\} \\ =&\min\left\{c_A\left(\min\{Q_A, t\}\right) + c_B((t - Q_A)^+)\ ,\right. \\ &\left.\quad c_B(\min\{Q_B, t\}) + c_A((t - Q_B)^+)\right\}. \end{aligned} \tag{5.5}$$

Proof: Let $t \in [0, Q_A + Q_B]$. First, we observe that

$$\begin{aligned} &\min\left\{c_A(t_A) + c_B(t - t_A) | 0 \le t_A \le Q_A, 0 \le t_B \le Q_B\right\} \\ =&\min\left\{c_A(t_A) + c_B(t - t_A) | t_A \in [(t - Q_B)^+, \min\{Q_A, t\}]\right\}. \end{aligned}$$

The interval $[(t-Q_B)^+, \min\{Q_A, t\}]$ is nonempty since $Q_A > 0$ and $Q_B > 0$ and $t \leq Q_A + Q_B$.
Let $h : [(t-Q_B)^+, \min\{Q_A, t\}] \to \mathbb{R}_+$ be defined in the following way,

$$h(t_A) = c_A(t_A) + c_B(t - t_A).$$

Then, for $t_A \in [(t-Q_B)^+, \min\{Q_A, t\}]$

$$\begin{aligned} h^{'}(t_A) =& c_A^{'}(t_A) - c_B^{'}(t - t_A) \\ =& p_A^{'}(t_A)t_A + p_A(t_A) - p_B^{'}(t-t_A)(t-t_A) - p_B(t-t_A) \end{aligned}$$

and

$$\begin{aligned} h^{''}(t_A) =& c_A^{''}(t_A) + c_B^{''}(t - t_A) \\ =& p_A^{''}(t_A)t_A + 2p'_A(t_A) + p_B^{''}(t-t_A)(t-t_A) + 2p_B^{'}(t-t_A) < 0, \end{aligned}$$

where the inequality follows from the assumptions we made on the unit price functions of A and B.

The objective function $h(t_A)$ is strictly concave. Hence the ordering costs are minimized by choosing t_A at one of the boundaries of the domain, *i.e.*, $t_A = (t-Q_B)^+$ or $t_A = \min\{Q_A, t\}$. Consequently $t - t_A = \min\{Q_B, t\}$ or $t - t_A = (t - Q_A)^+$. Hence,

$$\begin{aligned} & \min\{c_A(t_A) + c_B(t-t_A) | 0 \leq t_A \leq Q_A, t - t_A \leq Q_B\} \\ =& \min\left\{c_A\left(\min\{Q_A, t\}\right) + c_B((t-Q_A)^+)\right., \\ & \left. c_B(\min\{Q_B, t\}) + c_A((t-Q_B)^+)\right\}. \end{aligned}$$

□

In the next example we show the various assumptions we can make with respect to the possible order fulfillment policies of the suppliers and the consequences on reordering possibilities, as they will be formally defined and analyzed in the upcoming sections.

Example 5.2.2 Let $N = \{1, 2, 3\}$. Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^N$ be a CRSP-situation given by $q = (8, 9, 15)$, $Q_A = 16$, $Q_B = 20$, and unit price functions

$$\begin{cases} p_A(t) = 18\frac{1}{4} - \frac{1}{3}t, \\ p_B(t) = 20\frac{1}{4} - \frac{1}{2}t. \end{cases}$$

First, we verify that both c_A and c_B are increasing and strictly concave. Note that for all $t \in [0, 16]$,

$$p_A^{'}(t) = -\frac{1}{3} > -\frac{18\frac{1}{4}}{t} - \frac{1}{3}$$

and that

$$p_A^{''}(t) = 0 < \frac{2}{3t}.$$

Hence, the conditions in (5.1) and (5.2) are satisfied. Similarly, we find that for B the conditions (5.3) and (5.4) are satisfied as well. For all $t \in [0, 20]$, we have that

$$p_B^{'}(t) = -\frac{1}{2} > -\frac{20\frac{1}{4}}{t} - \frac{1}{2}$$

and that

$$p_B^{''}(t) = 0 < \frac{1}{t}.$$

Based on Proposition 5.2.1, since $c_A(8) = 124\frac{2}{3} < 130 = c_B(8)$, player 1 would like to obtain all 8 units from supplier A. For player 2, since $c_A(9) = 137\frac{1}{4} < 141\frac{3}{4} = c_B(9)$, it would be best to obtain all 9 units from supplier A, and for player 3, since $c_A(15) = 198\frac{3}{4} > 191\frac{1}{4} = c_B(15)$, it would be best to obtain all 15 units from supplier B.
Let us assume that $x_1^A = 8$ and $x_1^B = 0$, that $x_2^A = 9$ and $x_2^B = 0$ and that $x_3^A = 0$ and $x_3^B = 15$. We can immediately see that $\sum_{j \in N} x_j^A = 17 > 16 = Q_A$.

Next, we make various assumptions on how supplier A and B react upon the orders x^A and x^B.

Let us assume that π^A and π^B are such that large orders are fulfilled before the smaller orders (LBS). Then, $\pi^A(x^A) = (7, 9, 0)$ and $\pi^B(x^B) = (0, 0, 15)$. For player 1, however, $\pi_1^A(x^A) + \pi_1^B(x^B) = 7$, while $q_1 = 8$. Since player 1 can only place orders such that $x_1^A + x_1^B = 8$, we allow him to change his order at B.

For these fulfillment policies and under the assumption of order adjustment, the ordering costs for player 1 are $c_A(7) + c_B(1) = 131\frac{1}{6}$, while the ordering costs for player 2 are $c_A(9) = 137\frac{1}{4}$ and those for player 3 are $c_B(15) = 191\frac{1}{4}$.

On the other hand, one can also assume that π^A and π^B are such that both suppliers first fulfill small orders before fulfilling large orders (SBL). Hence, $\pi^A(x^A) = ((8, 8, 0))$ and $\pi^B(x^B) = (0, 0, 15)$. Here, player 2 needs to reorder an extra unit at supplier B. For these fulfillment policies, the ordering costs for player 1 are $c_A(8) = 124\frac{2}{3}$, the ordering costs for player 2 are $c_A(8) + c_B(1) = 144\frac{5}{12}$, and the ordering costs for player 3 remain $c_B(15) = 191\frac{1}{4}$.

Note that for the fulfillment policy based on SBL, the turnover of supplier A is larger than in the case of a fulfillment policy based on LBS, $c_A(8) + c_A(8) = 249\frac{1}{3} > 248\frac{2}{3} = c_A(7) + c_A(9)$.

Another reasonable assumption is that supplier A is indifferent with respect to the identity of the players, and distributes Q_A proportionally over the orders, *i.e.*, $\pi^A(x^A) = \frac{16}{17}(8, 9, 0)$. Here, both player 1 and player 2 need to reorder at B. The ordering costs for player 3 remain the same, $c_B(15) = 191\frac{1}{4}$. Player 1's ordering costs are $c_A(\frac{128}{17}) + c_B(\frac{8}{17}) \approx 127.93$ and player 2's ordering costs are $c_A(\frac{144}{17}) + c_B(\frac{9}{17}) \approx 141.25$. $\triangleleft$

To analyze the effect of fulfillment policies on purchaser's behavior in CRSP-situations we will model these situations as non-cooperative cost games.

5.3 Non-cooperative cost games

A *non-cooperative cost game* G in strategic form with players $N = \{1, \ldots, n\}$ is given by

$$G = ((X_j)_{j \in N}, (f_j)_{j \in N}),$$

where X_i denotes the strategy space and $f_i : \Pi_{j \in N} X_j \to \mathbb{R}$ the cost function of player $i \in N$. It is assumed that all information is publicly available and that players choose their strategies simultaneously and independently. Let $G = ((X_j)_{j\in N}, (f_j)_{j\in N})$ be a non-cooperative cost game. A strategy profile $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n) \in \Pi_{j\in N} X_j$ is called a *Nash equilibrium* (Nash (1951)) of G if for all $x_i \in X_i$ and all $i \in N$,

$$f_i(\hat{x}) \leq f_i(x_i, \hat{x}_{-i}),$$

where x_{-i} is a shorthand notation for $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. A game can have no, one or multiple Nash equilibria and the set of Nash equilibria of G is denoted by $E(G)$.

An alternative characterization of a Nash equilibrium can be provided by using best reply correspondences. Let $G = ((X_j)_{j\in N}, (f_j)_{j\in N})$ be a non-cooperative cost game. For $i \in N$, the *best reply correspondence* $b_i(x_{-i})$ explains how player i could optimally react to strategy profile $x_{-i} \in \Pi_{j\in N\backslash\{i\}} X_j$, *i.e.*,

$$b_i(x_{-i}) = \arg\min\{f_i(x_i, x_{-i}) | x_i \in X_i\}.$$

Note that there can be no, one or multiple best replies. If, for a strategy profile $\hat{x} \in \Pi_{j\in N} X_j$, for all $i \in N$, $\hat{x}_i \in b_i(\hat{x}_{-i})$ then $\hat{x} \in E(G)$, and conversely.

A strategy $x_i^* \in X_i$ is called a *dominant strategy* for player i if for all $x_i \in X_i$ and all strategy profiles $x_{-i} \in \Pi_{j\in N\backslash\{i\}} X_j$,

$$f_i(x_i^*, x_{-i}) \leq f_i(x_i, x_{-i})$$

or alternatively, if for all $x_{-i} \in \Pi_{j\in N\backslash\{i\}} X_j$, $x_i^* \in b_i(x_{-i})$.

If player $i \in N$ has dominant strategy x_i^*, in finding a Nash equilibrium of G we can limit ourselves to the reduced strategy space $\{x_i^*\} \times \Pi_{j\in N\backslash\{i\}} X_j$. If there exists a sequence of players, such that in every reduced strategy space, there exists a player with a dominant strategy, we have constructed a *recursive dominant Nash equilibrium*.

If the strategy spaces of all players are *finite*, *i.e.*, if for all $i \in N$, $X_i = \{x_i^1, \ldots, x_i^{m_i}\}$ with $m_i \in \mathbb{N}_+$, the game is called finite and one can allow for mixed strategies. For a finite game $G = ((X_j)_{j\in N}, (f_j)_{j\in N})$ the corresponding *mixed extension* $\Delta(G) = (\Delta(X_j)_{j\in N}, (\bar{f}_j)_{j\in N})$ is defined in the following way. In a mixed strategy, a player $i \in N$ can play each of his actions $x_i \in X_i$

with a probability $\delta^i(x_i) \geq 0$ such that $\sum_{x_i \in X_i} \delta^i(x_i) = 1$. The set of mixed strategies of player $i \in N$ is given by

$$\Delta(X_i) = \left\{ \delta^i : X_i \to [0,1] | \sum_{j=1}^{m_i} \delta^i(x_i^j) = 1 \right\}.$$

The costs, $\bar{f}_i(\delta)$ for $i \in N$, corresponding to a profile of mixed strategies $(\delta^1, \ldots, \delta^n)$ are in fact expected costs and can be computed using the Von Neumann Morgenstern expected cost (payoff) function (Von Neumann and Morgenstern (1944)). The mixed extension of a finite game always has at least one Nash equilibrium (Nash (1951)).
On the other hand, the existence of Nash equilibria for arbitrary games is not guaranteed. Sufficient conditions have been developed by, *e.g.*, Rosen (1965). Especially if the cost functions of the players are not continuous, the existence of Nash equilibria is hard to verify.

5.4 Infinite ordering games

To define an ordering game corresponding to a CRSP-situation we need to define the strategy space of the purchasers, which depends on possible limits on their order sizes, and we need to define the cost function for a profile of strategies, which depends on the fulfillment policies of the suppliers. In this section we will define an *infinite ordering game* corresponding to a CRSP-situation. First, we will illustrate the underlying idea behind the strategy space by means of an example.

Example 5.4.1 Let $N = \{1,2,3\}$ and consider the CRSP-situation of Example 5.2.2 with $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^N$ given by $q = (8,9,15)$, $Q_A = 16$, $Q_B = 20$ and unit price functions

$$\begin{cases} p_A(t) = 18\frac{1}{4} - \frac{1}{3}t, \\ p_B(t) = 20\frac{1}{4} - \frac{1}{2}t. \end{cases}$$

Assuming that for each player $i \in N$, an order has to sum up to q_i and does not exceed the capacity restrictions of A and B, the possible ordering strategies of player 1 are given by

$$X_1 = \left\{ (x_1^A, x_1^B) | x_1^A + x_1^B = 8, x_1^A \in [0,16], x_1^B \in [0,20] \right\}.$$

Note that, due to the feasibility restrictions on the orders, a strategy (x_1^A, x_1^B) of player 1 can be represented by a scalar: $y_1 \in [0,8]$ corresponds to the order x_1^A at A and $x_1^B = 8 - y_1$ at B. We set $Y_1 = [0,8]$. Similarly we set $Y_2 = [0,9]$ and $Y_3 = [0,15]$. ◁

As illustrated in Example 5.4.1, assuming that each player i in total orders q_i, a strategy $y_i \in Y_i$ of player $i \in N$ can be represented by a scalar. Here, y_i denotes the amount player $i \in N$ would like to purchase at A. Automatically he would like to purchase the complementary amount $q_i - y_i$ at firm B. Since, $0 \leq y_i \leq Q_A$ and at the same time $0 \leq q_i - y_i \leq Q_B$, we formally define the strategy space of player $i \in N$, with $\underline{y}_i = (q_i - Q_B)^+$ and $\overline{y}_i = \min\{q_i, Q_A\}$, by

$$Y_i = [\underline{y}_i, \overline{y}_i].$$

Next, we define the cost functions $\{g_i\}_{i \in N}$. Given a profile of orders $y \in \Pi_{j \in N} Y_j$, the suppliers announce the levels of fulfillment, *i.e.*, $\pi^A(y)$ and $\pi^B(q - y)$. In case for some player $i \in N$, $\pi_i^A(y) + \pi_i^B(q - y) < q_i$ he is allowed to increase his order at the supplier that has enough supply capacity left, such that he will obtain exactly q_i. The reason to allow for order adjustment is that, although player i might know that he will only obtain $\pi_i^A(y) < y_i$ from A, initially he is not allowed to order $q_i - \pi_i^A(y)$ at B.
Thus given $y \in \Pi_{j \in N} Y_j$ and $i \in N$, if $\sum_{j \in N} y_j > Q_A$, the ordering costs for player i equal

$$c_A(\pi_i^A(y)) + c_B(q_i - \pi_i^A(y)).$$

If $\sum_{j \in N}(q_j - y_j) > Q_B$, the ordering costs equal

$$c_A(q_i - \pi_i^B(q - y)) + c_B(\pi_i^B(q - y))$$

and if both $\sum_{j \in N} y_j \leq Q_A$ and $\sum_{j \in N}(q_j - y_j) \leq Q_B$ the ordering costs equal

$$c_A(y_i) + c_B(q_i - y_i).$$

The cost function $g_i : \Pi_{j \in N} Y_j \to \mathbb{R}_+$ for player $i \in N$ is given by

$$g_i(y) = \begin{cases} c_A(\pi_i^A(y)) + c_B(q_i - \pi_i^A(y)) & \text{if } \sum_{j \in N} y_j \geq Q_A, \\ c_A(q_i - \pi_i^B(q - y)) + c_B(\pi_i^B(q - y)) & \text{otherwise.} \end{cases} \quad (5.6)$$

In the next subsections we vary the fulfillment policies π^A and π^B of the suppliers, based on the ideas we formulated in Example 5.2.2.

5.4.1 Order fulfillment based on fixed and identical preferences

In this section we analyze infinite ordering games that correspond to CRSP-situations in which suppliers' fulfillment policies π^A and π^B are based on specific preset preferences of the suppliers with respect to the players in N.

Let $W \in \mathcal{W}^N$. Supplier A has a fixed and strict preference order $\sigma^A \in \Pi(N)$ on the players in N, and $\sigma^B \in \Pi(N)$ represents the fixed and strict preferences of supplier B. These preferences imply that, independent of the ordering strategies, supplier A fulfills the order of player $\sigma^A(i)$ before he fulfills the order of player $\sigma^A(i+1)$. Fixed preference orders are called identical if $\sigma^A = \sigma^B$.
Given $y \in \Pi_{j \in N} Y_j$, supplier A's and supplier B's order fulfillments π^A and π^B are obtained as follows for $i \in \{1, \ldots, n\}$:

$$\pi^A_{\sigma^A(i)}(y) = \min \left\{ y_{\sigma^A(i)}, \left(Q_A - \sum_{j=1}^{i-1} y_{\sigma^A(j)} \right)^+ \right\}$$

and

$$\pi^B_{\sigma^B(i)}(q-y) = \min \left\{ q_{\sigma^B(i)} - y_{\sigma^B(i)}, \left(Q_B - \sum_{j=1}^{i-1} (q_{\sigma^B(j)} - y_{\sigma^B(j)}) \right)^+ \right\}.$$

By $\mathcal{W}^{N,FID}$ we denote the set of CRSP-situations in which π^A and π^B are based on *fixed and identical preferences (FID)*.

Example 5.4.2 Let $N = \{1, 2, 3\}$. Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^{N,FID}$ be the CRSP-situation from Example 5.2.2 given by $q = (8, 9, 15)$, $Q_A = 16$, $Q_B = 20$ and unit price functions

$$\begin{cases} p_A(t) & = 18\frac{1}{4} - \frac{1}{3}t, \\ p_B(t) & = 20\frac{1}{4} - \frac{1}{2}t. \end{cases}$$

Here, the fulfillment policies π^A and π^B are based on the fixed and identical preferences of A and B given by $\sigma^A = (2, 1, 3) = \sigma^B$. Let $G^W = ((Y_j)_{j \in N},$

$(g_j)_{j\in N})$ be the corresponding infinite ordering game.
As we have seen before, $Y_1 = [0,8]$, $Y_2 = [0,9]$ and $Y_3 = [0,15]$ are the strategy spaces of the players. For the strategy profile of Example 5.2.2, $y = (8,9,0)$, we find that $\pi^A(y) = (7,9,0)$ and $\pi^B(q-y) = (0,0,15)$. Hence,

$$g_1(y) = c_A(7) + c_B(1) = 131\frac{1}{6},$$
$$g_2(y) = c_A(9) = 137\frac{1}{4},$$
$$g_3(y) = c_B(15) = 191\frac{1}{4}.$$

This profile of strategies is not a Nash equilibrium of G^W. Player 1 can obtain lower ordering costs by changing his ordering strategy to $\tilde{y}_1 = 0$. Then, with $\tilde{y} = (0,9,0)$, fulfillments $\pi^A(\tilde{y}) = (0,9,0)$, and $\pi^B(q-\tilde{y}) = (8,0,12)$, the ordering costs of player 1 decrease to

$$g_1(\tilde{y}) = c_B(8) = 130\frac{1}{6} < 131\frac{1}{6}.$$

Next, we will argue that $y^* = (0,9,3)$ is a Nash equilibrium of G^W. Since player 2 is strictly preferred over the other players by both suppliers, he can simply solve the minimization problem

$$\min\{c_A(y_2) + c_B(9-y_2) | y_2 \in [0,Q_A], (9-y_2) \in [0,Q_B]\},$$

which is equivalent with

$$\min\{c_A(y_2) + c_B(9-y_2) | y_2 \in [0,9]\}.$$

Using Proposition 5.2.1, we find that $y_2^* = 9$ is a dominant strategy for player 2. In the reduced strategy space $Y_1 \times \{y_2^*\} \times Y_3$, players 1 and 3 can limit themselves to ordering in between 0 and 7 at A. Since player 1 is strictly preferred over player 3 by both suppliers, he can find his optimal strategy by solving

$$\min\{c_A(y_1) + c_B(8-y_1) | y_1 \in [0,Q_A-9], (8-y_1) \in [0,Q_B]\},$$

which is equivalent with

$$\min\{c_A(y_1) + c_B(8-y_1) | y_1 \in [0,7]\}.$$

Using Proposition 5.2.1, we find that $y_1^* = 0$ is a dominant strategy for player 1 in the reduced strategy space. For player 3, in the reduced strategy space $\{y_1^*\} \times \{y_2^*\} \times Y_3$, he can limit himself to ordering up between 3 and 7 at A (there is 12 left at B). Solving

$$\min\{c_A(y_3) + c_B(15 - y_3) | y_3 \in [0, 7], (15 - y_3) \in [0, 12]\}$$

we find that in this reduced strategy space $y_3^* = 3$ is a dominant strategy for player 3.

Clearly, no player can reduce his ordering costs, thus $y^* \in E(G^W)$. Furthermore, looking at the procedure we used in establishing y^* we can conclude that it is a recursive dominant equilibrium. $\triangleleft$

If both suppliers fulfill orders based on fixed and identical preferences the infinite ordering game has a Nash equilibrium.

Theorem 5.4.3 *Let $W \in \mathcal{W}^{N,FID}$ and let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding infinite ordering game. Then, there exists a Nash equilibrium $y^* \in E(G^W)$.*

Proof: Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q^B, \pi^B])$ and let $\sigma \in \Pi(N)$ be the fixed and identical preferences of the suppliers.

Define y^* by letting the players choose their optimal strategies *as if* they may decide according to order σ.

Thus, choose

$$y^*_{\sigma(1)} \in \arg\min\{g_{\sigma(1)}(y) | y_{\sigma(1)} \in Y_{\sigma(1)}\}. \tag{5.7}$$

Recursively, for $i = 2, \ldots, n$, with $\underline{y}^r_{\sigma(i)} = q_{\sigma(i)} - \min\{q_{\sigma(i)}, (Q_B - \sum_{k<i}(q_{\sigma(k)} - y^*_{\sigma(k)}))^+\} = (q_{\sigma(i)} - (Q_B - \sum_{k<i}(q_{\sigma(k)} - y^*_{\sigma(k)}))^+)^+$ and $\overline{y}^r_{\sigma(i)} = \min\{q_{\sigma(i)}, (Q_A - \sum_{k<i} y^*_{\sigma(k)})^+\}$, set

$$Y^r_{\sigma(i)} = [\underline{y}^r_{\sigma(i)}, \overline{y}^r_{\sigma(i)}]$$

as the remaining feasible orders for player $\sigma(i)$.
With

$$y_{-\sigma(i)} \in \big\{(y^*_{\sigma(1)}, \dots, y^*_{\sigma(i-1)}, y_{\sigma(i+1)}, \dots, y_{\sigma(n)}) | y_{\sigma(k)} \in Y_{\sigma(k)} \text{ for } k = i{+}1, \dots, n\big\},$$

recursively choose

$$y^*_{\sigma(i)} \in \arg\min\{g_{\sigma(i)}(y_{\sigma(i)}, y_{-\sigma(i)}) | y_{\sigma(i)} \in Y^r_{\sigma(i)}\}. \tag{5.8}$$

If we can show that the minima of (5.7) and (5.8) exist, then we have shown that

(i) $y^*_{\sigma(1)}$ is a dominant strategy for $\sigma(1)$, and that

(ii) $y^*_{\sigma(i)}$, $i \in \{2, \dots, n\}$, is a dominant strategy for $\sigma(i)$ within the reduced strategy space

$$\big\{(y^*_{\sigma(1)}, \dots, y^*_{\sigma(i-1)}, y_{\sigma(i)}, \dots, y_{\sigma(n)}) | y_{\sigma(k)} \in Y_{\sigma(k)} \text{ for } k = i, \dots, n\big\},$$

which shows that y^* is a Nash equilibrium of G^W.

With respect to (5.7). For all $y \in \Pi_{j \in N} Y_j$,

$$\begin{cases} \pi^A_{\sigma(1)}(y) & = y_{\sigma(1)} \\ \pi^B_{\sigma(1)}(q - y) & = q_{\sigma(1)} - y_{\sigma(1)}. \end{cases}$$

Hence,

$$g_{\sigma(1)}(y_{\sigma(1)}, y_{-\sigma(1)}) = c_A(y_{\sigma(1)}) + c_B(q_{\sigma(1)} - y_{\sigma(1)}).$$

and thus

$$\min\{g_{\sigma(1)}(y) | y_{\sigma(1)} \in Y_{\sigma(1)}\} = \min\{c_A(y_{\sigma(1)}) + c_B(q_{\sigma(1)} - y_{\sigma(1)}) | y_{\sigma(1)} \in Y_{\sigma(1)}\},$$

and by Proposition 5.2.1, the minimum exists.

With respect to (5.8), take $i \in \{2, \dots, n\}$ and assume that for all $\ell \in \{1, \dots, i-1\}$, $y^*_{\sigma(\ell)}$ is a dominant strategy in the reduced strategy space

$$\big\{(y^*_{\sigma(1)}, \dots, y^*_{\sigma(\ell-1)}, y_{\sigma(\ell)}, \dots, y_{\sigma(n)}) | y_{\sigma(k)} \in Y_{\sigma(k)} \text{ for } k = \ell, \dots, n\big\},$$

We have that $\sigma(i)$ is, after $\sigma(1)$ up to $\sigma(i-1)$, first in line with respect to order fulfillment. Thus for all $y \in \big\{(y^*_{\sigma(1)}, \ldots, y^*_{\sigma(i-1)}, y_{\sigma(i)}, \ldots, y_{\sigma(n)})|y_{\sigma(k)} \in Y_{\sigma(k)}$ for $k = i, \ldots, n\big\}$, satisfying $y_{\sigma(i)} \in Y^r_{\sigma(i)}$,

$$\begin{cases} \pi^A_{\sigma(i)}(y) & = y_{\sigma(i)} \\ \pi^B_{\sigma(i)}(q-y) & = q_{\sigma(i)} - y_{\sigma(i)}. \end{cases}$$

Hence, for all $y_{\sigma(i)} \in Y^r_{\sigma(i)}$,

$$g_{\sigma(i)}(y_{\sigma(i)}, y_{-\sigma(i)}) = c_A(y_{\sigma(i)}) + c_B(q_{\sigma(i)} - y_{\sigma(i)}).$$

Again, by Proposition 5.2.1 there exists a minimum in (5.8). □

Note that the Nash equilibrium we constructed in the proof of Theorem 5.4.3 is a dominant recursive equilibrium. Moreover, in the equilibrium we constructed, there is no need for the players to adjust their orders at one of the suppliers after announcement of the order fulfillment levels.

If suppliers' preferences are fixed but not identical, it is not clear whether there exists an equilibrium in the infinite ordering game. This remains an open problem.

5.4.2 Order fulfillment based on order size

Suppliers' preferences can also be based on the orders that are placed. If supplier A faces a set of orders y that cannot be fulfilled by Q_A, he can maximize his turnover by splitting Q_A over as much orders as possible. The smaller the order quantity, the higher the unit price. This preference order myopically maximizes the turnover of A. It is myopic in the sense that suppliers do not take into account possible reactions of the purchasers towards the fulfillment policy.

More formally, the set $\mathcal{W}^{N,SBL}$ describes the set of CRSP-problems where suppliers' fulfillment policies are based on *small before large preferences (SBL)*. Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q-B, \pi^B]) \in \mathcal{W}^{N,SBL}$ and let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding infinite ordering game. Let $y \in \Pi_{j\in N} Y_j$ and take $i \in N$. Then, $S_i^A(y)$ denotes the amount of smaller orders that are fulfilled by A before player i's order is fulfilled. Hence, with $T_1 = \{j \in N | y_j < y_i\}$ and with $T_2 = \{j \in N | y_j = y_i \text{ and } j < i\}$,

$$S_i^A(y) = \sum_{j \in T_1 \cup T_2} y_j.$$

Thus small orders are fulfilled before larger orders, and in case of a tie the supplier prefers the player with smallest index.
Due to this tie breaking rule, for each profile of strategies $y \in \Pi_{j \in N} Y_j$ it is determined which orders are (partially) fulfilled at A.
Similar for supplier B, $S_i^B(q-y)$ denotes the amount of smaller orders that are fulfilled by B before player i's order is fulfilled. With $T_1 = \{j \in N | q_j - y_j < q_i - y_i\}$ and with $T_2 = \{j \in N | q_j - y_j = q_i - y_i \text{ and } j < i\}$,

$$S_i^B(q-y) = \sum_{j \in T_1 \cup T_2} (q_j - y_j).$$

Thus for each $i \in N$, the SBL-fulfillment policies of A and B are given by

$$\pi_i^A(y) = \min\left\{ y_i, \left(Q_A - S_i^A(y) \right)^+ \right\}$$

and

$$\pi_i^B(q-y) = \min\left\{ q_i - y_i, \left(Q_B - S_i^B(q-y) \right)^+ \right\}.$$

The next example shows that when both suppliers fulfill orders based on small before large (SBL), there need not be a Nash equilibrium in the infinite ordering game.

Example 5.4.4 Let $N = \{1, 2\}$ and let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^{N,SBL}$ be a CRSP-situation given by $q = (18, 19)$, $Q_A = 20$, $Q_B = 25$ and unit price functions

$$\begin{cases} p_A(t) & = 20\frac{1}{4} - \frac{1}{2}t, \\ p_B(t) & = 18\frac{1}{4} - \frac{1}{3}t. \end{cases}$$

The fulfillment policies π^A and π^B are based on SBL.

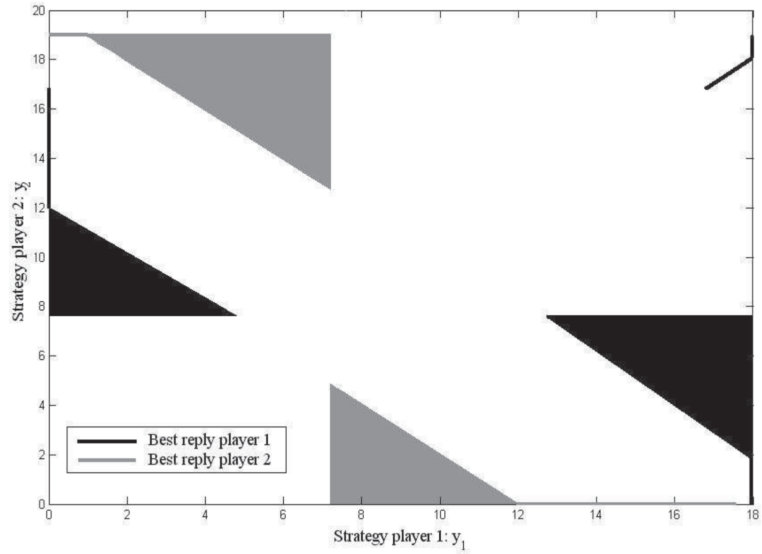

Figure 5.4.1: Best reply correspondences in the infinite ordering game of Example 5.4.4

Let $G^W = ((Y_j)_{j \in N}, (g_j)_{j \in N})$ be the corresponding infinite ordering game. To show that G^W does not have a Nash equilibrium, we will construct the best reply correspondences for both players.

The stand alone strategies for both players are to order at A, since $p_A(18) < p_B(18)$ and $p_A(19) < p_B(19)$.

Let us look at the best replies of player 2. Recall that it is always optimal to buy as much as possible at one of the suppliers. If y_1 is sufficiently small, player 2 can best respond by buying as much as possible at A. The first turnover point is $y_1 = 7.2$, since $c_A(20-7.2)+c_B(19-(20-7.2)) = 277.62 = c_B(25-(18-7.2)) + c_A(19-(25-(18-7.2)))$. The second turnover point is $y_1 = 17.6$. If y_1 lies in between 7.2 and 17.6, player 2 better purchases as much as possible at supplier B. If $y_1 > 17.6$, then since the suppliers treat smaller orders first, player 2 should order slightly less than y_1, hence for $y_1 > 17.6$ there is no best reply. Thus,

$$b_2(y_1) = \begin{cases} 19 & \text{if } y_1 \in [0,1], \\ [20-y_1, 19] & \text{if } y_1 \in (1,7.2), \\ [0,4.8] \cup [12.8,19] & \text{if } y_1 = 7.2, \\ [0, 12-y_1] & \text{if } y_1 \in (7.2,12), \\ 0 & \text{if } y_1 \in [12,17.6], \\ \emptyset & \text{if } y_1 \in (17.6,18]. \end{cases}$$

Similarly, we can find the best reply correspondence for player 1,

$$b_1(y_2) = \begin{cases} 18 & \text{if } y_2 \in [0,1], \\ [20-y_2, 18] & \text{if } y_2 \in (1,7.6), \\ [0,4.4] \cup [12.4,18] & \text{if } y_2 = 7.6, \\ [0, 12-y_2] & \text{if } y_2 \in (7.6,12), \\ 0 & \text{if } y_2 \in [12,16.8), \\ \{0, 16.8\} & \text{if } y_2 = 16.8, \\ y_2 & \text{if } y_2 \in (16.8,18), \\ 18 & \text{if } y_2 \in [18,19]. \end{cases}$$

In Figure 5.4.1 one can find a sketch of the best reply correspondences of the players. We can see that they do not intersect, so this ordering game does not have a Nash equilibrium. ◁

The opposite of SBL constitutes another fulfillment policy for A and B. In this case a supplier invests in a relation with a large purchaser by preferring him over a smaller purchaser. The possible fixed ordering costs are not necessarily an argument for a supplier to choose a large before small fulfillment, as opposed to the myopic SBL. These economies of scale are already translated in the decreasing unit price functions of the suppliers.

The set $\mathcal{W}^{N,LBS}$ contains all CRSP-situations where the fulfillment policies are based on large before small. Take $W \in \mathcal{W}^{N,LBS}$ and let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding infinite ordering game. Then the fulfillment policies of A and B are based on *large before small preferences (LBS)* if orders are fulfilled in the following way. Let $y \in \Pi_{j\in N} Y_j$ and take $i \in N$. Then $L_i^A(y)$ denotes the amount of larger orders that are fulfilled by

A before player i's order is fulfilled. Hence, with $T_1 = \{j \in N | y_j > y_i\}$ and with $T_2 = \{j \in N | y_j = y_i \text{ and } j > i\}$,

$$L_i^A(y) = \sum_{j \in T_1 \cup T_2} y_j.$$

Similarly for supplier B, $L_i^B(q - y)$ denotes the amount of larger orders that are fulfilled by B before player i's order is fulfilled. With $T_1 = \{j \in N | q_j - y_j > q_i - y_i\}$ and with $T_2 = \{j \in N | q_j - y_j = q_i - y_i \text{ and } j > i\}$,

$$L_i^B(q - y) = \sum_{j \in T_1 \cup T_2} (q_j - y_j).$$

Thus for each $i \in N$, the fulfillment policies of A and B, based on LBS, are given by

$$\pi_i^A(y) = \min \left\{ y_i, \left(Q_A - L_i^A(y) \right)^+ \right\}$$

and

$$\pi_i^B(q - y) = \min \left\{ q_i - y_i, \left(Q_B - L_i^B(q - y) \right)^+ \right\}.$$

Thus a supplier fulfills a large order before a smaller order, and in case of a tie the order of the player with largest index goes first.
Due to this tie breaking rule for each profile of strategies $y \in \Pi_{j \in N} Y_j$ it is determined which orders are (partially) fulfilled.[6]
Whereas for SBL fulfillment policies, the ordering game does not necessarily have an equilibrium, for CRSP-situations with LBS fulfillment policies, there exists a Nash equilibrium in the corresponding infinite ordering game.

Theorem 5.4.5 *Let $W \in \mathcal{W}^{N,LBS}$ and let $G^W = ((Y_j)_{j \in N}, (g_j)_{j \in N})$ be the corresponding infinite ordering game. Then, there exists a Nash equilibrium of G^W.*

Proof: Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B])$. Define y^* in the following backward recursive way.

[6] If we replace this tie breaking rule with another one, Theorem 5.4.5 remains valid.

Set

$$y_n^* = \begin{cases} \overline{y}_n & \text{if } c_A(\overline{y}_n) + c_B(q_n - \overline{y}_n) \leq c_B(q_n - \underline{y}_n) + c_A(\underline{y}_n), \\ \underline{y}_n & \text{otherwise,} \end{cases} \tag{5.9}$$

and set $t_n = y_n^*$.

For $i \in \{n-1, \ldots, 1\}$, with $\underline{y}_i^r = (q_i - (Q_B - \sum_{j>i}(q_j - t_j)))^+$ and $\overline{y}_i^r = \min\{q_i, Q_A - \sum_{j>i} t_j\}$, recursively set

$$y_i^* = \begin{cases} \overline{y}_i & \text{if } c_A(\overline{y}_i^r) + c_B(q_i - \overline{y}_i^r) \leq c_B(q_i - \underline{y}_i^r) + c_A(\underline{y}_i^r), \\ \underline{y}_i & \text{otherwise,} \end{cases} \tag{5.10}$$

and set

$$t_i = \begin{cases} \overline{y}_i^r & \text{if } y_i^* = \overline{y}_i, \\ \underline{y}_i^r & \text{if } y_i^* = \underline{y}_i. \end{cases}$$

We will show that

(i) y_n^* is a dominant strategy for n,

(ii) y_i^* is a dominant strategy for $i \in \{n-1, \ldots, 1\}$ within the reduced strategy space

$$\{(y_1, \ldots, y_i, y_{i+1}^*, \ldots, y_n^*) | y_k \in Y_k \text{ for } k = 1, \ldots, i\},$$

which shows that y^* is a Nash equilibrium of G^W.

With respect to (i), note that for all $y = (\overline{y}_n, y_{-n}) \in \Pi_{j \in N} Y_j$, $L_n^A(y) = 0$. Similarly, for all $y = (\underline{y}_n, y_{-n}) \in \Pi_{j \in N} Y_j$, $L_n^B(q - y) = 0$. Hence, for all $y_{-n} \in \Pi_{j \in N \setminus \{n\}} Y_j$,

$$\begin{cases} \pi_n^A(\overline{y}_n, y_{-n}) & = \overline{y}_n, \\ \pi_n^B(q_n - \underline{y}_n, q_{-n} - y_{-n}) & = q_n - \underline{y}_n. \end{cases}$$

Thus for all $y_{-n} \in \Pi_{j \in N \setminus \{n\}} Y_j$,

$$\begin{cases} g_n(\overline{y}_n, y_{-n}) & = c_A(\overline{y}_n) + c_B(q_n - \overline{y}_n), \\ g_n(\underline{y}_n, y_{-n}) & = c_B(q_n - \underline{y}_n) + c_A(\underline{y}_n). \end{cases}$$

By Proposition 5.2.1, at least one of the following two inequalities is true, for all $y_n \in Y_n$:

$$c_A(\overline{y}_n) + c_B(q_n - \overline{y}_n) \leq c_A(y_n) + c_B(q_n - y_n),$$
$$c_A(\underline{y}_n) + c_B(q_n - \underline{y}_n) \leq c_A(y_n) + c_B(q_n - y_n).$$

Hence, by (5.9), y_n^* is a dominant strategy for player n.
Note that for all $y_{-n} \in \Pi_{j \in N \setminus \{n\}} Y_j$,

$$\begin{cases} \pi_n^A(y_n^*, y_{-n}) = t_n, \\ \pi_n^B(q_n - y_n^*, q_{-n} - y_{-n}) = q_n - t_n. \end{cases}$$

With respect to (ii) take $i \in \{1, \ldots, n-1\}$ and assume that for all $\ell \in \{i+1, \ldots, n\}$, y_ℓ^* is a dominant strategy in the reduced strategy space

$$\big\{(y_1, \ldots, y_\ell, y_{\ell+1}^*, \ldots, y_n^*) | y_k \in Y_k \text{ for } k = 1, \ldots, \ell\big\},$$

and that for all $y \in \big\{(y_1, \ldots, y_\ell, y_{\ell+1}^*, \ldots, y_n^*) | y_k \in Y_k \text{ for } k = 1, \ldots, \ell\big\}$,

$$\begin{cases} \pi_\ell^A(y) = t_\ell & \text{if } y_\ell^* = \overline{y}_\ell, \\ \pi_\ell^B(q - y) = q_\ell - t_\ell & \text{if } y_\ell^* = \underline{y}_\ell. \end{cases}$$

Take $y = (\overline{y}_i, y_{-i}) \in \big\{(y_1, \ldots, y_i, y_{i+1}^*, \ldots, y_n^*) | y_k \in Y_k \text{ for } k = 1, \ldots, i\big\}$. Then, for all $k < i$, $\overline{y}_i \geq y_k$, thus

$$L_i^A(y) \leq \sum_{j>i} t_j.$$

Furthermore, if $L_i^A(y) < \sum_{j>i} t_j$, then there is a player $k > i$ with $y_k^* = \underline{y}_k$ and with $q_k > Q_B - (\sum_{j>k} q_j - t_j)$. By the global sufficient capacity of A and B, $q_i \leq Q_A - \sum_{j>i} t_j$. Thus

$$\pi_i^A(\overline{y}_i, y_{-i}) = \min\{q_i, Q_A - \sum_{j>i} t_j\} = \overline{y}_i^r.$$

Therefore,

$$g_i(\overline{y}_i, y_{-i}) = c_A(\overline{y}_i^r) + c_B(q_i - \overline{y}_i^r).$$

Let $y = (\underline{y}_i, y_{-i}) \in \{(y_1, \ldots, y_i, y^*_{i+1}, \ldots, y^*_n) | y_k \in Y_k \text{ for } k = 1, \ldots, i\}$. Using the same reasoning as above we find that

$$\pi_i^B(q_i - \underline{y}_i, q_{-i} - y_{-i}) = q_i - \underline{y}_i^r$$

and thus that

$$g_i(\underline{y}_i, y_{-i}) = c_A(\underline{y}_i^r) + c_B(q_i - \underline{y}_i^r).$$

For any $y \in \{(y_1, \ldots, y_i, y^*_{i+1}, \ldots, y^*_n) | y_k \in Y_k \text{ for } k = 1, \ldots, i\}$, we have that $\pi_i^A(y) \leq \overline{y}_i^r$ and $\pi_i^B(q - y) \leq q_i - \underline{y}_i^r$. By Proposition 5.2.1 at least one of the following two inequalities is true, for all $y_i \in [\underline{y}_i^r, \overline{y}_i^r]$:

$$c_A(\overline{y}_i^r) + c_B(q_i - \overline{y}_i^r) \leq c_A(y_i) + c_B(q_i - y_i)$$
$$c_A(\underline{y}_i^r) + c_B(q_i - \underline{y}_i^r) \leq c_A(y_i) + c_B(q_i - y_i).$$

Hence, by (5.10), y_i^* is a dominant strategy for player i in the reduced strategy space. Furthermore, for all $y \in \{(y_1, \ldots, y_i, y^*_{i+1}, \ldots, y^*_n) | y_k \in Y_k \text{ for } k = 1, \ldots, i\}$,

$$\begin{cases} \pi_i^A(y) = t_i & \text{if } y_i^* = \overline{y}_i, \\ \pi_i^B(q - y) = q_i - t_i & \text{if } y_i^* = \underline{y}_i. \end{cases}$$

□

In the Nash equilibrium we constructed in the proof above, players choose a (dominant) strategy that is either $\underline{y}_i$ or $\overline{y}_i$, just to be next in line when it comes to order fulfillment. Hence, there are equilibria in an infinite ordering game corresponding to a CRSP-situation with LBS fulfillment, that can be found at the boundaries of the combined strategy space. Also here, the equilibrium we constructed in the proof is a recursive dominant equilibrium.

Hence, we can conclude that a myopic fulfillment policy does not result in a Nash equilibrium, whereas a more farsighted fulfillment policy does. A possible explanation for this phenomenon is that LBS combined with the feasibility restrictions induces some sort of truth telling by the purchasers. In the equilibrium constructed in the proof of Theorem 5.4.5 each purchaser truthfully reveals the maximum amount he would like to receive from his preferred supplier, whereas in Example 5.4.4 with SBL fulfillment the purchasers undermine each other's ordering strategies by ordering slightly less than they actually would like to receive.

5.4.3 Proportional order fulfillment

The last fulfillment policy we study, in this section, is proportional order fulfillment. A reason for suppliers to fulfill orders proportionally is that they are indifferent with respect to the identity of the players or do not want to discriminate for another reason.

We denote by $\mathcal{W}^{N,PROP}$ the set of CRSP-situations in which both suppliers fulfill orders proportionally. Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^{N,PROP}$ be a CRSP-situation and let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding infinite ordering game. Given $y \in \Pi_{j\in N} Y_j$, the *proportional order fulfillment* is given by

$$\pi^A(y) = y \frac{Q_A}{\max\{\sum_{j\in N} y_j, Q_A\}}$$

and

$$\pi^B(q-y) = (q-y) \frac{Q_B}{\max\{\sum_{j\in N}(q_j - y_j), Q_B\}}.$$

Note that for $i \in N$, π_i^A and π_i^B are strictly monotonic in y_i. Let $y \in \Pi_{j\in N} Y_j$, take $i \in N$ and $\tilde{y}_i > y_i$. Then,

$$\pi_i^A(y) < \pi_i^A(\tilde{y}_i, y_{-i})$$

and

$$\pi_i^B(q-y) > \pi_i^B(q_i - \tilde{y}_i, q_{-i} - y_{-i}).$$

In Proposition 5.2.1 we have shown that for increasing and strictly concave c_A and c_B, it is optimal to order as much as possible at one of the two suppliers. If we combine this fact with the the strict monotonicity of π^A and π^B, we can show that against a strategy profile y_{-i}, the set of best replies consists of one or both of the two extreme strategies $\underline{y}_i = (q_i - Q_B)^+$ and $\overline{y}_i = \min\{q_i, Q_A\}$.

Proposition 5.4.6 *Let $W \in \mathcal{W}^{N,PROP}$ be a CRSP-situation. Let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding infinite ordering game. Let $y \in \Pi_{j\in N} Y_j$*

be such that for $i \in N$, $y_i \in (\underline{y}_i, \overline{y}_i)$. Then, at least one of the following two inequalities is true,

$$\text{(i) } g_i(\underline{y}_i, y_{-i}) < g_i(y), \tag{5.11}$$

$$\text{(ii) } g_i(\overline{y}_i, y_{-i}) < g_i(y). \tag{5.12}$$

Proof: Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B])$. We first show that $\underline{y}_i$ or $\overline{y}_i$ is a best reply for player i against y_{-i}.

With $y_i \in (\underline{y}_i, \overline{y}_i)$, we have that

$$\pi_i^A(\underline{y}_i, y_{-i}) < \pi_i^A(y) < \pi_i^A(\overline{y}_i, y_{-i}),$$
$$\pi_i^B(q_i - \overline{y}_i, q_{-i} - y_{-i}) < \pi_i^B(q - y) < \pi_i^B(q_i - \underline{y}_i, q_{-i} - y_{-i}).$$

Thus, given y_{-i}, player i can maximally realize $\pi_i^A(\overline{y}_i, y_{-i})$ at supplier A. Similarly at B, the maximal order fulfillment equals $\pi_i^B(q_i - \underline{y}_i, q_{-i} - y_{-i})$. Since in the end, player i will obtain q_i, player i will obtain at least $q_i - \pi_i^B(q_i - \underline{y}_i, q_{-i} - y_{-i})$ from A and at least $q_i - \pi_i^A(\overline{y}_i, y_{-i})$ from B.

Hence the possible obtained orders at A can be represented by the interval $[\underline{t}, \overline{t}]$, define by

$$[\underline{t}, \overline{t}] = [q_i - \pi_i^B(q_i - \underline{y}_i, q_{-i} - y_{-i}), \pi_i^A(\overline{y}_i, y_{-i})].$$

Note that for any $t \in [\underline{t}, \overline{t}]$, $q_i - t \in [q_i - \pi_i^A(\overline{y}_i, y_{-i}), \pi_i^B(q_i - \underline{y}_i, q_{-i} - y_{-i})]$, *i.e.*, the possible obtained orders at B. For determining a best reply against y_{-i}, player i has to decide how much is optimal to be obtained from supplier A (and the remainder from B) and choose y_i accordingly.
We need to solve the following minimization problem

$$\min \left\{ c_A(t) + c_B(q_i - t) | t \in [\underline{t}, \overline{t}] \right\}. \tag{5.13}$$

Define $h : [\underline{t}, \overline{t}] \to \mathbb{R}_+$ in the following way,

$$h(t) = c_A(t) + c_B(q_i - t).$$

Note that h is strictly concave on $[\underline{t}, \overline{t}]$. Thus, $\underline{t} = q_i - \pi_i^B(q_i - \underline{y}_i, q_{-i} - y_{-i})$ or $\overline{t} = \pi_i^A(\overline{y}_i, y_{-i})$ minimizes (5.13).

We finish the proof by stating that $g_i(\underline{y}_i, y_{-i}) = h(\underline{t})$, that $g_i(\overline{y}_i, y_{-i}) = h(\overline{t})$ and that $g_i(y) = h(t)$ for some $t \in (\underline{t}, \overline{t})$. □

Using Proposition 5.4.6 we know where to look for equilibria. A possible best reply for player i against strategy y_{-i} is either $\underline{y}_i = (q_i - Q_B)^+$ or $\overline{y}_i = \min\{Q_A, q_i\}$. Hence, a possible Nash equilibrium can be found in one of the extreme points of $\Pi_{j\in N} Y_j$.

Example 5.4.7 We consider the CRSP-situation from Example 5.4.4, but now the suppliers fulfill orders proportionally. Let $N = \{1, 2\}$ and let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^{N,PROP}$ be given by $q = (18, 19)$, $Q_A = 20$, $Q_B = 25$ and

$$p_A(t) = 20\frac{1}{4} - \frac{1}{2}t \text{ if } t \in [0, Q_A],$$
$$p_B(t) = 18\frac{1}{4} - \frac{1}{3}t \text{ if } t \in [0, Q_B].$$

Let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding infinite ordering game. We have $Y_1 = [0, 18]$ and $Y_2 = [0, 19]$.

Using Proposition 5.4.6 we know that if G^W has a Nash equilibrium, it can only be at the boundaries of the strategy space. Hence, the candidates are $y^1 = (0, 0)$, $y^2 = (18, 0)$, $y^3 = (18, 19)$ and $y^4 = (0, 19)$. We can eliminate y^1, since $g_1(y^1) \approx 273.83 > g_1(y^2) = 202.5$. And since $g_1(y^2) < g_1(y^3)$, $g_2(y^2) < g_2(y^3)$ and also $g_2(y^2) < g_1(y^1)$, y^2 is an equilibrium of G^W. Because $g_2(y^2) < g_2(y^3)$, y^3 is not an equilibrium. Similarly one can check that for y^4 there are no unilateral deviations that decrease the purchasing costs of the deviating player. Hence $E(G^W) = \{(18, 0), (0, 19)\}$. ◁

The observations of Example 5.4.7 can be generalized. One only needs to compare the extreme points of the combined strategy space to verify whether the infinite ordering game has a Nash equilibrium.

Theorem 5.4.8 *Let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the infinite ordering game corresponding to CRSP-situation $W \in \mathcal{W}^{N,PROP}$. If $\hat{y} \in E(G^W)$, then $\hat{y} \in \Pi_{j\in N}\{\underline{y}_j, \overline{y}^j\}$.*

It is an open problem whether in an infinite ordering game corresponding to a CRSP-situation in which both suppliers fulfill orders proportionally, there exists an equilibrium.

5.5 Matrified ordering games

In this section we use a different approach for analyzing equilibrium behavior in CRSP-situations. Instead of infinitely many options, the purchasers can only choose from a limited but representative set of ordering options. We *matrify* the infinite ordering game to a finite game.
Proposition 5.2.1 and the result of the previous section can be used as an argument to limit the ordering possibilities of the purchasers to two extreme actions: either order as much as possible at A or as much as possible at B.

Let $W \in \mathcal{W}^N$ be a CRSP-situation and let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding infinite ordering game. In this section we define a *matrified ordering game*

$$G^W_m = \Big((\Delta(\bar{Y}_j))_{j\in N}, (\bar{g}_j)_{j\in N}\Big)$$

corresponding to $W \in \mathcal{W}^N$.

For each player $i \in N$ the limited set of actions is given by

$$\bar{Y}_i = \{\underline{y}_i, \overline{y}_i\}.$$

The mixed strategy space of player $i \in N$ is given by

$$\Delta(\bar{Y}_i) = \{\delta^i : \bar{Y}_i \to [0,1] | \delta^i(\underline{y}_i) + \delta^i(\overline{y}_i) = 1\}.$$

The interpretation of $\delta^i \in \Delta(\bar{Y}_i)$ is that player i chooses strategy $\underline{y}_i$ with probability $\delta^i(\underline{y}_i)$ and $\overline{y}_i$ with probability $\delta^i(\overline{y}_i)$. The extreme points of $\Delta(\bar{Y}_i)$

are identified with $\underline{y}_i$ and $\overline{y}_i$ and are called the pure strategies of player i. Every mixed strategy $\delta^i \in \Delta(\bar{Y}_i)$ corresponds to an ordering strategy $y_i \in Y_i$, with $y_i = \delta^i(\underline{y}_i)\underline{y}_i + \delta^i(\overline{y}_i)\overline{y}_i$. A mixed strategy is a linear combination of $\underline{y}_i$ and $\overline{y}_i$. Since $Y_i = [\underline{y}_i, \overline{y}_i]$, every linear combination of $\underline{y}_i$ and $\overline{y}_i$ falls within this set.

The cost function for a profile of mixed strategies is based on the cost function of the infinite ordering game. The costs are an approximation of the actual ordering costs for a mixed strategy. Let $\delta \in \Pi_{j\in N}\Delta(\bar{Y}_j)$, for each $i \in N$, the ordering costs are given by

$$\bar{g}_i(\delta) = \sum_{\bar{y}\in\Pi_{k\in N}\bar{Y}_k} \Pi_{j\in N}\delta^j(\bar{y}_j)g_i(\bar{y}).$$

If the mixed extension of a finite game G has an equilibrium in which each player plays a pure strategy, this is called a *pure Nash equilibrium*.

In the next subsections we show that for some of the possible order fulfillment policies we discussed in Section 5.4, there exist pure equilibria in the matrified ordering game. Moreover, we explain the relation between equilibria in the infinite ordering game and (pure) equilibria in the matrified ordering game.

5.5.1 Order fulfillment based on small before large

In this section we analyze matrified ordering games corresponding to CRSP-situations with SBL-fulfillment policies. We start by providing an example of a matrified ordering game.

Example 5.5.1 We consider the CRSP-situation from Example 5.2.2, but we assume that suppliers' fulfilment policies are based on SBL. Let $N = \{1, 2, 3\}$ and let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^{N,SBL}$ be given by $q = (8, 9, 15)$, $Q_A = 16$, $Q_B = 20$ and unit price functions

$$\begin{cases} p_A(t) = & 18\frac{1}{4} - \frac{1}{3}t \\ p_B(t) = & 20\frac{1}{4} - \frac{1}{2}t. \end{cases}$$

Let $G_m^W = \left((\Delta(\bar{Y}_j))_{j\in N}, (\bar{g}_j)_{j\in N}\right)$ be the corresponding matrified ordering game. Then, $\bar{Y}_1 = \{0, 8\}$, $\bar{Y}_2 = \{0, 9\}$ and $\bar{Y}_3 = \{0, 15\}$.

In Table 5.5.1, we present the costs for the pure strategies. The vectors in the cells represent the ordering costs of the three players, given the combinations of pure strategies.

Table 5.5.1: Matrified ordering game of Example 5.5.1

Player 1 $\overline{y}_1$		
	Player 3 $\overline{y}_3$	Player 3 $\underline{y}_3$
Player 2 $\overline{y}_2$	$(124\frac{2}{3}, 144\frac{5}{12}, 191\frac{1}{4})$	$(124\frac{2}{3}, 144\frac{5}{12}, 191\frac{1}{4})$
Player 2 $\underline{y}_2$	$(124\frac{2}{3}, 141\frac{3}{4}, 241\frac{11}{12})$	$\mathbf{(124\frac{2}{3}, 141\frac{3}{4}, 229\frac{11}{12})}$
Player 1 $\underline{y}_1$		
	Player 3 $\overline{y}_3$	Player 3 $\underline{y}_3$
Player 2 $\overline{y}_2$	$(132, 137\frac{1}{4}, 241\frac{5}{12})$	$(132, 137\frac{1}{4}, 222\frac{3}{4})$
Player 2 $\underline{y}_2$	$(132, 141\frac{3}{4}, 198\frac{3}{4})$	$(132, 141\frac{3}{4}, 227\frac{1}{4})$

For example, the profile of pure strategies $y = (\overline{y}_1, \overline{y}_2, \underline{y}_3)$ has corresponding fulfillments $\pi^A(y) = (8, 8, 0)$ and $\pi^B(q - y) = (0, 0, 15)$. The corresponding ordering costs for player 1 are given by $g_1(y) = c_A(8) = 124\frac{2}{3}$, for player 2 are given by $g_2(y) = c_A(8) + c_B(1) = 144\frac{5}{12}$ and for player 3 by $g_3(y) = c_B(15) = 191\frac{1}{4}$.

If all players play both of their pure strategies with probability $\frac{1}{2}$, *i.e.*, for all $i \in N$ and all $y_i \in \bar{Y}_i$, $\delta^i(y_i) = \frac{1}{2}$, then the corresponding costs in the matrified ordering game can be computed using the payoffs for the pure strategies, *e.g.*, for player 1,

$$\bar{g}_1(\delta) = \sum_{\tilde{y}\in\Pi_{j\in N}\bar{Y}_j} \left(\frac{1}{2}\right)^3 g_1(\tilde{y}) = 128\frac{1}{3}.$$

This mixed strategy δ corresponds to player 1 ordering $\frac{1}{2}\underline{y}_1 + \frac{1}{2}\overline{y}_1 = 4$ at A and thus 4 at B, player 2 ordering 4.5 at A, and player 3 ordering 7.5 at A. Thus δ corresponds with the strategy profile $y = (4, 4.5, 7.5)$ of the infinite ordering game. In fact, $\bar{g}_1(\delta)$ can be seen as an approximation of $g_1(y)$.

The combination of pure strategies $\hat{y} = (\overline{y}_1, \underline{y}_2, \underline{y}_3) \in \Pi_{j \in N} \bar{Y}_j$ results in a Nash equilibrium. This is a pure Nash equilibrium of the matrified ordering game. $\triangleleft$

In the previous section, Example 5.4.4 showed that in the infinite ordering game corresponding to a CRSP-situation with fulfillments based on SBL there need not be a Nash equilibrium. In the matrified game corresponding to such CRSP-situations, however, we can always find a pure Nash equilibrium.

Theorem 5.5.2 *Let $W \in \mathcal{W}^{N,SBL}$ be a CRSP-situation. Let $G_m^W = (\Delta(\bar{Y}_j)_{j \in N}, (\bar{g}_j)_{j \in N})$ be the corresponding matrified ordering game. Then, there exists a pure Nash equilibrium of G_m^W.*

Proof: Let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B])$. The proof follows the same recursive structure as the proof of Theorem 5.4.5. Define y^* in the following way. Set

$$y_1^* = \begin{cases} \overline{y}_1 & \text{if } c_A(\overline{y}_1) + c_B(q_1 - \overline{y}_1) \leq c_B(\underline{y}_1) + c_A(\underline{y}_1) \\ \underline{y}_1 & \text{otherwise} \end{cases} \tag{5.14}$$

and set $t_1 = y_1^*$.

For $i \in \{2, \ldots, n\}$, recursively set

$$y_i^* = \begin{cases} \overline{y}_i & \text{if } c_A(\overline{y}_i^r) + c_B(q_i - \overline{y}_i^r) \leq c_B(q_i - \underline{y}_i^r) + c_A(\underline{y}_i^r), \\ \underline{y}_i & \text{otherwise,} \end{cases} \tag{5.15}$$

where $\underline{y}_i^r = (q_i - (Q_B - \sum_{j<i}(q_j - t_j)))^+$ and $\overline{y}_i^r = \min\{(q_i, Q_A - \sum_{j<i} t_j)\}$, and with

$$t_i = \begin{cases} \overline{y}_i^r & \text{if } y_i^* = \overline{y}_i, \\ \underline{y}_i^r & \text{if } y_i^* = \underline{y}_i. \end{cases}$$

Using the same reasoning as in the proof of Theorem 5.4.5, one can show that

(i) the pure strategy y_1^* is a dominant strategy for player 1, *i.e.*, for all $y \in \Pi_{j\in N}\bar{Y}_j$,

$$g_1(y_1^*, y_{-1}) \leq g_1(y),$$

(ii) the pure strategy y_i^* is a dominant strategy for player $i \in \{2, \ldots, n\}$ within the reduced strategy space, *i.e.*, for all $y \in \{(y_1^*, \ldots, y_{i-1}^*, y_i, \ldots, y_n) | y_k \in \bar{Y}_k \text{ for } k = i, \ldots, n\}$

$$g_i(y_i^*, y_{-i}) \leq g_i(y),$$

which shows that y^* is a pure Nash equilibrium of G_m^W. □

5.5.2 Order fulfillment based on large before small

In Theorem 5.4.5 we have shown that an infinite ordering game corresponding to a CRSP-situation with LBS-fulfillment has a Nash equilibrium. In this section we analyze the correspondence between a Nash equilibrium in the infinite ordering game and a pure Nash equilibrium in the matrified game.

Example 5.5.3 We consider the CRSP-situation from Example 5.2.2, but we assume that suppliers' fulfilment policies are based on LBS. Let $N = \{1, 2, 3\}$ and let $W = (q, [p_A, Q_A, \pi^A], [p_B, Q_B, \pi^B]) \in \mathcal{W}^{N,LBS}$ be given by $q = (8, 9, 15)$, $Q_A = 16$, $Q_B = 20$ and unit price functions

$$\begin{cases} p_A(t) = & 18\frac{1}{4} - \frac{1}{3}t \\ p_B(t) = & 20\frac{1}{4} - \frac{1}{2}t. \end{cases}$$

Let $G^W = ((Y_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding infinite ordering game and let $G_m^W = \left((\Delta(\bar{Y}_j))_{j\in N}, (\bar{g}_j)_{j\in N}\right)$ be the corresponding matrified ordering game.

Then, for the infinite ordering game, following the proof of Theorem 5.4.5, we find that $y^* = (8, 9, 0)$ is an equilibrium of G^W, with $g_1(y^*) = c_A(7) + c_B(1)$, $g_2(y^*) = c_A(9)$ and $g_3(y^*) = c_B(15)$. Note that $y^* \in \Pi_{j\in N}\bar{Y}_j$ and thus that

y^* is a pure equilibrium of G_m^W with exactly the same ordering costs.

The strategy profile $\tilde{y} = (7, 9, 0)$, however, has exactly the same ordering costs as y^*. Hence, $\tilde{y}$ is also an equilibrium of G^W. But the strategy profile $\tilde{y}$ does not correspond to a pure strategy in the matrified ordering game, because it corresponds to $\tilde{\delta}^1(\overline{y}_1) = \frac{7}{8}$, $\tilde{\delta}^2(\overline{y}_2) = 1$ and $\tilde{\delta}^3(\underline{y}_3) = 1$. Here, $\bar{g}_1(\tilde{\delta}) = \frac{7}{8} g_1(y^*) + \frac{1}{8}(c_A(3) + c_B(5) < g_1(y^*)$. Thus $\tilde{\delta}$ is not an equilibrium of G_m^W. ◁

In the Nash equilibrium we have constructed in the proof of Theorem 5.4.5, each player chooses one of the two extreme strategies. This Nash equilibrium corresponds to a pure equilibrium in the matrified ordering game. Hence, for a CRSP-situation in which both suppliers have relationship maximizing preferences, the matrified ordering game has a pure equilibrium. Without proof, we can state the following.

Theorem 5.5.4 *Let $W \in \mathcal{W}^{N,LBS}$ be a CRSP-situation. Let $G^W = ((Y_j)_{j \in N}, (g_j)_{j \in N})$ be the corresponding infinite ordering game and let $G_m^W = (\Delta(\bar{Y}_j)_{j \in N}, (g_j)_{j \in N})$ be the corresponding matrified ordering game. Then, there exists a pure equilibrium $y^* \in E(G_m^W)$ such that $y^* \in E(G^W)$.*

5.5.3 Proportional order fulfillment

In the previous subsections we showed that for SBL and LBS fulfillment policies, there is not a one-to-one correspondence between equilibria of the infinite ordering game and pure equilibria of the matrified ordering game.
For proportional order fulfillment, however, one can easily verify that the existence of a pure equilibrium of the matrified ordering game corresponds to the existence of a Nash equilibrium of the infinite ordering game and vice versa.

Theorem 5.5.5 *Let $W \in \mathcal{W}^{N,PROP}$ be a CRSP-situation. Let $G^W = ((Y_j)_{j \in N}, (g_j)_{j \in N})$ be the corresponding infinite ordering game and let $G_m^W =$*

$(\Delta(\bar{Y}_j)_{j\in N}, (g_j)_{j\in N})$ be the corresponding matrified ordering game. Then, G_m^W has a pure equilibrium $\hat{y}$ if and only if $\hat{y} \in E(G^W)$.

Proof: For the "only if"-part. Let $W \in \mathcal{W}^{N,PROP}$ be such that the corresponding matrified ordering game has a pure equilibrium $\hat{y} \in E(G_m^W)$. Take $i \in N$. Then, for all $y_i \in \{\underline{y}_i, \overline{y}_i\}$

$$g_i(\hat{y}) \leq g_i(y_i, \hat{y}_{-i}). \tag{5.16}$$

By Proposition 5.4.6 and by (5.16), for all $y_i \in (\underline{y}_i, \overline{y}_i)$,

$$g_i(\hat{y}) \leq g_i(y_i, \hat{y}_{-i}).$$

Hence, for all $y_i \in Y_i = [\underline{y}_i, \overline{y}_i]$,

$$g_i(\hat{y}) \leq g_i(y_i, \hat{y}_{-i}).$$

Thus, $\hat{y} \in E(G^W)$.

For the "if"-part. Let $W \in \mathcal{W}^{N,PROP}$ be such that the corresponding infinite ordering game has an equilibrium $\hat{y} \in E(G^W)$. By Proposition 5.4.8 $\hat{y} \in \Pi_{j\in N}\{\underline{y}_j, \overline{y}_j\}$ and thus

$$\hat{y} \in \Pi_{j\in N}\bar{Y}_j.$$

Since $\hat{y} \in E(G^W)$, for all $y_i \in \{\underline{y}_i, \overline{y}_i\} = \bar{Y}_j$,

$$g_i(\hat{y}) \leq g_i(y_i, \hat{y}_{-i}).$$

Thus $\hat{y} \in E(G_m^W)$. □

5.6 Concluding remarks

This section briefly discusses the effects of the order adjustment assumption we made in this chapter. Furthermore, we summarize the remaining open questions.

We assumed that, since purchasers' orders are restricted by feasibility, these purchasers are allowed to adjust their orders after announcement of the fulfillment levels. One *could* argue that purchasers, that have obtained insufficient supplies, have to place an extra *separate* order at the supplier with remaining capacity, or even worse, have to place a rush order for which they do not receive quantity discounts. These alterations change the cost function of the ordering game and this can affect some of the obtained results, *e.g.*, Proposition 5.4.6. However, if purchasers are responsible for obtaining enough supplies, they can also demand that they are given the opportunity to obtain enough, which would imply dropping the feasibility restriction on the ordering strategies. This gives rise to a completely different class of CRSP-situations.
For the SBL and LBS fulfillment policies we have introduced a tie breaking rule that depends on the identity of the players. In our setting the tie breaking rule is identical for both suppliers. If the tie breaking rule is different per supplier, the results of Theorem 5.4.5, Theorem 5.5.2 and Theorem 5.5.4 still hold.

Which brings us to one of the open problems. If suppliers fulfillment policies are based on fixed but non-identical preferences, does there exist an equilibrium in the infinite ordering game (Section 5.4.1)? The last open problem is whether in an infinite ordering game where suppliers fulfill orders proportionally there is an equilibrium (Section 5.4.3). Using Theorem 5.5.5, the answer to this question also verifies whether the matrified game has a pure equilibrium (Section 5.5.3).

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