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BANKRUPTY AND THE PER CAPITA NUCLEOLUS

By

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6 November 2013

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Bankruptcy and the per capita nucleolus

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Abstract

This article characterizes the per capita nucleolus for bankruptcy games as a bankruptcy rule. This rule, called the cligths rule, is based on the well-known constrained equal awards principle. The essential feature of the rule however is that, for each bankruptcy problem, it takes into account a vector of clights. These clights only depend on the vector of claims while the height of the estate determines whether the clights should be interpreted as modified claims or as rights. Finally, it is seen that both the clights rule and the Aumann-Maschler rule can be captured within the family of so-called claim and right rules.

Keywords: Bankruptcy, (per capita) nucleolus.
JEL classification number: C71, G33.

1   Introduction

In a bankruptcy problem an insufficient monetary estate has to be divided over a number of claimants, each having a justified claim on this estate. Bankruptcy rules propose general principles and procedures to solve an arbitrary bankruptcy problem. From the wide variety of bankruptcy rules we just want to mention the constrained equal award rule, the constrained equal loss rule and the Aumann-Maschler rule (cf. Aumann and Maschler (1985)). An overview of bankruptcy rules and their properties can be found in Thomson (2003).
O'Neill (1982) associates a cooperative bankruptcy game with transferable utility to each bankruptcy problem. As a result, game theoretic solution concepts such as the Shapley value (cf. Shapley (1953)) and the nucleolus (cf. Schmeidler (1969)) can be viewed as bankruptcy rules, too. Interestingly, it turns out that the Aumann-Maschler rule coincides with the nucleolus in this context (cf. Aumann and Maschler (1985)).

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In this paper, we focus on the per capita nucleolus as first introduced by Grotte (1970). For each transferable utility game, this solution concept is based on the maximal dissatisfaction per player for each coalition. Hence, the per capita nucleolus is closely related to the nucleolus, which is based on the maximal dissatisfaction of each coalition.

This paper characterizes the per capita nucleolus for bankruptcy games as a bankruptcy rule, called the clights rule. The essential feature of this rule is that, for each bankruptcy problem, it takes into account a vector of clights. The clight vector only depends on the claim vector and does not depend on the estate. The clights rule allocates to each claimant at most his clight when the estate is less than the sum of the clights. In this case the clights can be viewed as a modified claim vector. However, each claimant will receive at least his clight when the estate is more than the sum of the clights. Hence, in the latter case the clights can be viewed as the rights of the claimant. When the clights represent modified claims, the clights rule divides the estate over the claimants using the constrained equal award rule with the clights as new claims. Whenever the clights represent rights, the clights rule first assigns to every claimant its right. Then, the remaining estate is divided using the constrained equal loss rule with the original claims minus the clights as the new claim vector. The proof that the per capita nucleolus coincides with the clights rule uses a new Kohlberg (1971) type of characterization of the per capita nucleolus which extends the idea of the characterization of the nucleolus presented by Groote Schaarsberg et al. (2012).

As a final result, we show that both the clights rule and the Aumann-Maschler rule can be captured within the general class of so-called claim and right bankruptcy rules. This class turns out to coincide with the class of increasing-constant-increasing bankruptcy rules as introduced by Thomson (2008).

The remainder of this paper is structured as follows. In Section 2, some basic definitions concerning cooperative transferable utility games and bankruptcy problems are presented. Also, new Kohlberg-like characterizations of the nucleolus and the per capita nucleolus are presented in this section. Section 3 formally introduces the clights rule and shows that this rule corresponds to the per capita nucleolus. In Section 4, claim and right bankruptcy rules are introduced.

2 Preliminaries

This section first recalls the definitions of the nucleolus and the per capita nucleolus. Secondly, it provides Kohlberg-like characterizations of both solution concepts. Finally, it surveys bankruptcy problems and bankruptcy rules, focussing on the concepts used in this paper.

2.1 The nucleolus and per capita nucleolus of transferable utility games

A *transferable utility* TU-game is defined by the pair (N, v), where $N = \{1, ..., n\}$ is the finite set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function*. The set of all TU-games with player set N is denoted by TU^N and a TU-game with player set N is abbreviated by v. For every coalition $S \in 2^N$, $v(S)$ is called the *worth* of the coalition with $v(\varnothing) = 0$ by convention.

The cardinality of a coalition $S \in 2^N$ is denoted by $|S|$. By $\mathbb{R}^N$ we denote the set of all real-valued vectors with $|N|$ elements in which each coordinate corresponds to a player $i \in N$. For $S \in 2^N$ we denote by $e^S \in \mathbb{R}^N$ the vector for which $e_i^S = 1$ for all $i \in S$ and $e_i^S = 0$ for all $i \in N \backslash S$.

The *imputation set*, $I(v)$, is defined by

$$I(v) = \{x \in \mathbb{R}^N \mid x_i \geq v(\{i\}) \text{ for all } i \in N, \sum_{i \in N} x_i = v(N)\}.$$

The core, $Core(v)$, (Gillies (1953)) consists of all imputations for which no coalition would be better off if it would separate itself and get its worth. Formally, the core is defined by

$$Core(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N, \sum_{i \in N} x_i = v(N)\}.$$

For $x, y \in \mathbb{R}^t$ we have $x \leq_L y$, *i.e.*, x is *lexicographically smaller* than (or equal to) y, if $x = y$ or if there exists an $\ell \in \{1, ..., t\}$ such that $x_k = y_k$ for all $k \in \{1, ..., \ell - 1\}$ and $x_\ell < y_\ell$.

Let $v \in TU^N$. Then the *excess* $exc(S, x)$ of coalition $S \in 2^N$ for an imputation $x \in I(v)$ is defined by

$$exc(S, x) = v(S) - x(S).$$

For a game $v \in TU^N$ and imputation $x \in I(v)$ the *excess vector* $\theta(x) \in \mathbb{R}^{2^{|N|}}$ has as its coordinates the excesses of all $2^{|N|}$ coalitions arranged in a weakly decreasing order, *i.e.*, $\theta_k(x) \geq \theta_{k+1}(x)$ for all $k \in \{1, ..., 2^{|N|} - 1\}$. The nucleolus is defined as follows.

Definition 2.1. *(cf. Schmeidler (1969))*
Let $v \in TU^N$ be such that $I(v) \neq \varnothing$. The nucleolus, $n(v)$, *is the unique imputation such that $\theta(n(v)) \leq_L \theta(y)$ for all $y \in I(v)$.*

For a game $v \in TU^N$ and an imputation $x \in I(v)$ we define the *per capita excess* of any non-empty coalition $S \in 2^N \backslash \{\varnothing\}$ by

$$exc^P(S, x) = \frac{v(S) - x(S)}{|S|}.$$

The *per capita excess vector* $\theta^P(x) \in \mathbb{R}^{2^{|N|}-1}$ has as its coordinates the per capita excesses of all non-empty coalitions arranged in a weakly decreasing order, *i.e.*, $\theta_k^P(x) \geq \theta_{k+1}^P(x)$ for all $k \in \{1, ..., 2^{|N|} - 2\}$.

Definition 2.2. *(cf. Grotte (1970))*
Let $v \in TU^N$ be such that $I(v) \neq \varnothing$. Then, the per capita nucleolus, *$pcn(v)$, is
the unique imputation such that $\theta^P(pcn(v)) \leq_L \theta^P(y)$ for all $y \in I(v)$.*

2.2 Characterizations of the (per capita) nucleolus using balanced collections

Besides the original definition of the nucleolus, there exist multiple characterizations that use balanced collections. One of the advantages of these characterizations is that they provide ways to quickly determine whether an imputation is the nucleolus or not. The first of this type of characterizations was provided by Kohlberg (1971). It uses the following definitions: A map $\rho : 2^N \backslash \{\varnothing\} \to [0, \infty)$ is called *balanced* if

$$\sum_{S \in 2^N \backslash \{\varnothing\}} \rho(S) e^S = e^N.$$

Furthermore, a collection $\mathcal{B} \subset 2^N \backslash \{\varnothing\}$ of coalitions is called *balanced* if there exists a balanced map ρ on N such that

$$\mathcal{B} = \{S \in 2^N \backslash \{\varnothing\} \mid \rho(S) > 0\}.$$

We call the grand coalition N and the empty coalition $\varnothing$ trivial. Let $x \in I(v)$ and define $\mathcal{B}_1(x)$ to be the set of the non-trivial coalitions for which the dissatisfaction with imputation x is the highest. Formally,

$$\mathcal{B}_1(x) = \big\{S \in 2^N \backslash \{\varnothing, N\} \mid exc(S, x) \geq exc(T, x) \text{ for all } T \in 2^N \backslash \{\varnothing, N\}\big\}.$$

Recursively, for $k = 2, 3, ...$ the sets $\mathcal{B}_k(x)$ are defined by,

$$\mathcal{B}_k(x) = \big\{S \in 2^N \backslash \{\{\varnothing, N\} \cup \bigcup_{\ell=1}^{k-1} \mathcal{B}_\ell(x)\} \mid exc(S, x) \geq exc(T, x)$$

$$\text{for all } T \in 2^N \backslash \{\varnothing, N\} \text{ with } T \notin \bigcup_{\ell=1}^{k-1} \mathcal{B}_\ell(x)\big\}.$$

It is clear that there exists a unique $t(x) \in \mathbb{N}$, such that

$$\begin{cases} \mathcal{B}_k(x) \neq \varnothing \text{ for all } k \in \{1, ..., t(x)\} \\ \mathcal{B}_k(x) = \varnothing \text{ for all } k \in \{t(x) + 1, ...\} \end{cases}$$

Theorem 2.3. *(cf. Kohlberg (1971))*
Let $v \in TU^N$ be such that $Core(v) \neq \varnothing$ and let $x \in I(v)$. Then, $x = n(v)$ if and only if $\bigcup_{\ell=1}^s \mathcal{B}_k(x)$ is balanced for all $s \in \{1, ..., t(x)\}$.

An alternative characterization is provided by Groote Schaarsberg et al. (2012). Let $\mathcal{D} \subseteq 2^N$ and let $H(\mathcal{D})$ be as follows:

$$H(\mathcal{D}) = \{S \in 2^N \mid e^S \in span(e^N, \{e^T\}_{T \in \mathcal{D}})\},$$

where *span* denotes the linear hull[1]. Note that $H(\{\varnothing\}) = \{\varnothing, N\}$.

Theorem 2.4. *(cf. Groote Schaarsberg et al. (2012))*
Let $v \in TU^N$ be such that $Core(v) \neq \varnothing$ and let $x \in I(v)$. Then, $x = n(v)$ if and only if there exists a sequence $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_\tau$ of non-empty subcollections of $2^N \backslash \{\varnothing, N\}$ with the following properties:

(i) *for all $r \in \{1, ..., \tau\}$ the collection $\bar{\mathcal{D}}_r = \bigcup_{k=1}^r \mathcal{D}_k$ is balanced.*

(ii) *there exists a sequence of real numbers $\gamma_1, \gamma_2, ..., \gamma_\tau$ such that $exc(T, x) = \gamma_r$ for every $T \in \mathcal{D}_r$ and all $r \in \{1, ..., \tau\}$ and that $\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_\tau$.*

(iii) *for all $T \in 2^N \backslash \{\{\varnothing, N\} \cup \bar{\mathcal{D}}_\tau\}$ it holds that $T \in H\big(\{S \in \bar{\mathcal{D}}_\tau : exc(S, x) \geq exc(T, x)\}\big)$.*

Both Theorems 2.3 and 2.4 require that (a part of) the coalitions are put into a sequence of collections, and that all coalitions in a collection have the same excess. Moreover, both sequences of collections have to satisfy the same balancedness requirement. However, there are two important differences between the theorems. First, in Theorem 2.4 it is allowed that several collections have the same excess as opposed to Theorem 2.3. In other words, in Theorem 2.4 it is allowed to split a large Kohlberg collection into multiple smaller collections. Second, Theorem 2.4 states that a non-trivial coalition either belongs to a collection or it is in the span of the collections with higher excesses, while Theorem 2.3 states that each non-trivial coalition belongs to collection. Hence, it is possible to use a subset of the Kohlberg collections to determine whether an imputation is the nucleolus or not, *i.e.*, some of the Kohlberg collections are irrelevant. Finally, note that the sequence of the original Kohlberg collections satisfies the three properties of Theorem 2.4.

We formulate another variant of the characterization of Kohlberg. In this variant, the idea of Groote Schaarsberg that not all Kohlberg collections are relevant is further exploited. For collections $\mathcal{D} \subseteq 2^N$, denote by $\mathcal{F}(\mathcal{D})$ the set of the *free* coalitions. *i.e.*, coalitions which are not in the span of $\mathcal{D}$. Formally, the set of free coalitions is given by

$$\mathcal{F}(\mathcal{D}) = 2^N \backslash H(\mathcal{D}).$$

Theorem 2.5. *Let $v \in TU^N$ be such that $Core(v) \neq \varnothing$ and let $x \in I(v)$. Then, $x = n(v)$ if and only if there exists a sequence $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_\tau$ of non-empty collections such that[2] $\mathcal{D}_r \subseteq \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ for all $r \in \{1, ..., \tau\}$, that satisfies the following properties:*

(A) *for all $r \in \{1, ..., \tau\}$ the collection $\bar{\mathcal{D}}_r = \bigcup_{k=1}^r \mathcal{D}_k$ is balanced and $\mathcal{F}(\bar{\mathcal{D}}_\tau) = \varnothing$.*

---

[1] $span(e^N, \{e^T\}_{T \in \mathcal{D}}) = \{ \sum_{S \in \mathcal{D} \cup \{N\}} \gamma_S e^S \mid \gamma_S \in \mathbb{R} \text{ for all } S \in \mathcal{D} \cup \{N\}\}$

[2] Note that $\bar{\mathcal{D}}_r = \bigcup_{k=1}^r \mathcal{D}_k$ and that $\bar{\mathcal{D}}_0 = \varnothing$

(B) *for all $r \in \{1, ..., \tau\}$ and all $T \in \mathcal{D}_r$ it holds that*
$$exc(T, x) = \max_{S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})} exc(S, x).$$

Proof. "only if part". We show how to define a sequence $\mathcal{D}_1, ..., \mathcal{D}_\tau$ of relevant collections from the Kohlberg collections. In this proof, we abbreviate $\mathcal{B}_k(n(v))$ by $\mathcal{B}_k$.

Let $\mathcal{B}_1, ..., \mathcal{B}_{t(n(v))}$ be the sequence of Kohlberg collections of the nucleolus. Then, determine the sequence of relevant collections with the following algorithm:

1: $r = 1$ and $\mathcal{D}_1 = \mathcal{B}_1$
2: **while** $\mathcal{F}(\bar{\mathcal{D}}_r) \neq \varnothing$ **do**
3:      $r = r + 1$
4:      $k_r = \underset{\ell \in \{1, ..., t(n(v))\}}{\operatorname{argmin}} \{\mathcal{B}_\ell \cap \mathcal{F}(\bar{\mathcal{D}}_{r-1}) \neq \varnothing\}$
5:      $\mathcal{D}_r = \mathcal{B}_{k_r} \cap \mathcal{F}(\bar{\mathcal{D}}_{r-1})$
6: **end while**
7: $\tau = r$

By construction we have that $\mathcal{D}_r \subseteq \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ for all $r \in \{1, ..., \tau\}$ and that $\mathcal{F}(\mathcal{D}_\tau) = \varnothing$. Left to show is that the sequence satisfies (B) and the remaining part of (A).

For each $r \leq \tau$ we have $\mathcal{D}_r = \mathcal{B}_{k_r} \cap \mathcal{F}(\bar{\mathcal{D}}_{r-1})$, which implies that the coalitions in collection $\mathcal{D}_r$ have maximum excess with respect to the nucleolus over the set $\mathcal{F}(\bar{\mathcal{D}}_{r-1})$. This gives (B).

Left to prove is the balancedness of the collections $\bar{\mathcal{D}}_r$ for all $r \in \{1, ...\tau\}$, which is shown by induction.

Basis: $\bar{\mathcal{D}}_1$ is balanced, being equal to $\mathcal{B}_1$. Let $r \in \{2, ..., \tau\}$ and assume that $\bar{\mathcal{D}}_{r-1}$ is balanced. Define $\bar{\mathcal{B}}_{k_r} = \bigcup_{\ell=1}^{k_r} \mathcal{B}_\ell$ and denote $\mathcal{G} = \bar{\mathcal{B}}_{k_r} \cap H(\bar{\mathcal{D}}_{r-1})$. Then, $\bar{\mathcal{B}}_{k_r}$ is the disjoint union of $\mathcal{G}$ and $\mathcal{D}_r$, *i.e.*, $\mathcal{G} \cap \mathcal{D}_r = \varnothing$ and $\bar{\mathcal{B}}_{k_r} = \mathcal{G} \cup \mathcal{D}_r$. Because $\bar{\mathcal{B}}_{k_r}$ and $\bar{\mathcal{D}}_{r-1}$ are balanced, there exist for both collections a balanced map, *i.e.*, there exists a ρ, where $\rho(T) > 0$ for all $T \in \bar{\mathcal{B}}_{k_r}$, and an α, where $\alpha(T) > 0$ for all $T \in \bar{\mathcal{D}}_{r-1}$ with

$$e^N = \sum_{T \in \bar{\mathcal{B}}_{k_r}} \rho(T)e^T = \sum_{T \in \bar{\mathcal{D}}_{r-1}} \alpha(T)e^T.$$

Furthermore, since $\mathcal{G} \subseteq H(\bar{\mathcal{D}}_{r-1})$ we have that for every $S \in \mathcal{G}$ there exists a vector $\gamma_T^S \in \mathbb{R}^{\bar{\mathcal{D}}_{r-1} \cup \{N\}}$ such that

$$e^S = \sum_{T \in \bar{\mathcal{D}}_{r-1} \cup \{N\}} \gamma_T^S e^T.$$

Denote $\beta_T = \sum_{S \in \mathcal{G}} \rho(S)\gamma_T^S$ for all $T \in \bar{\mathcal{D}}_{r-1} \cup \{N\}$. By substituting the equation

6

above in the balancing equation of $\bar{\mathcal{B}}_{k_r}$, we find

$$
\begin{aligned}
e^N &= \sum_{T \in \bar{\mathcal{B}}_{k_r}} \rho(T) e^T \\
&= \sum_{S \in \mathcal{G}} \rho(S) e^S + \sum_{T \in \mathcal{D}_r} \rho(T) e^T \\
&= \sum_{S \in \mathcal{G}} \rho(S) \sum_{T \in \bar{\mathcal{D}}_{r-1} \cup \{N\}} \gamma_T^S e^T + \sum_{T \in \mathcal{D}_r} \rho(T) e^T \\
&= \sum_{T \in \bar{\mathcal{D}}_{r-1} \cup \{N\}} \beta_T e^T + \sum_{T \in \mathcal{D}_r} \rho(T) e^T.
\end{aligned}
$$

Let $\varepsilon \in (0, 1)$ and take a convex combination of the equation above and the balancing equation of $\bar{\mathcal{D}}_{r-1}$:

$$
e^N = \varepsilon \Big(\sum_{T \in \bar{\mathcal{D}}_{r-1} \cup \{N\}} \beta_T e^T + \sum_{T \in \mathcal{D}_r} \rho(T) e^T \Big) + (1 - \varepsilon) \Big(\sum_{T \in \bar{\mathcal{D}}_{r-1}} \alpha(T) e^T \Big),
$$

which rewrites to

$$
e^N = \frac{1}{1 - \varepsilon \beta_N} \Big(\sum_{T \in \bar{\mathcal{D}}_{r-1}} \Big(\varepsilon \beta_T + (1 - \varepsilon) \alpha(T) \Big) e^T + \sum_{T \in \mathcal{D}_r} \rho(T) e^T \Big).
$$

For ε sufficiently close to 0, this provides a balancing equation for $\bar{\mathcal{D}}_r$. Hence, $\bar{\mathcal{D}}_r$ is balanced, which proves (A).

"if part". Consider a sequence $\mathcal{D}_1, ..., \mathcal{D}_\tau$ of non-empty coalitions such that $\mathcal{D}_r \subseteq \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ for all $r \in \{1, ..., \tau\}$ and that satisfies (A) and (B) of Theorem 2.5. We show that this sequence satisfies conditions $(i), (ii)$ and (iii) of Theorem 2.4.

Condition (A) implies (i). Furthermore, (B) implies (ii) since $\mathcal{F}(\bar{\mathcal{D}}_r) \subsetneq \mathcal{F}(\bar{\mathcal{D}}_{r-1})$. Property (iii) is inferred as follows. Let $T \in 2^N \setminus \{\{\varnothing, N\} \cup \bar{\mathcal{D}}_\tau\}$. By (A) we have that $\mathcal{F}(\bar{\mathcal{D}}_\tau) = \varnothing$, hence there exists a unique $r \in \{1, ..., \tau\}$ with $T \in \mathcal{F}(\bar{\mathcal{D}}_{r-1}) \setminus \mathcal{F}(\bar{\mathcal{D}}_r)$. Furthermore, $T \in \mathcal{F}(\bar{\mathcal{D}}_{r-1}) \setminus \mathcal{F}(\bar{\mathcal{D}}_r)$ implies $exc(T, x) \leq \gamma_r$ and $exc(S, x) \geq \gamma_r$ for all $S \in \bar{\mathcal{D}}_r$. Therefore, we have

$$
\begin{aligned}
T \in \mathcal{F}(\bar{\mathcal{D}}_{r-1}) \setminus \mathcal{F}(\bar{\mathcal{D}}_r) &= H(\bar{\mathcal{D}}_r) \setminus H(\bar{\mathcal{D}}_{r-1}) \\
&\subseteq H(\bar{\mathcal{D}}_r) \\
&\subseteq H\big(\{S \in \bar{\mathcal{D}}_\tau : exc(S, x) \geq \gamma_r\}\big) \\
&\subseteq H\big(\{S \in \bar{\mathcal{D}}_\tau : exc(S, x) \geq exc(T, x)\}\big).
\end{aligned}
$$

This proves (iii) of Theorem 2.4. Hence, Theorem 2.5 implies Theorem 2.4. ∎

Wallmeier (1983) showed that Theorem 2.3 can be reformulated to provide a characterization of the per capita nucleolus. Similarly, the other two characterizations of the nucleolus above can be reformulated. In this paper, the following per capita variant of Theorem 2.5 is used.

Theorem 2.6. *Let $v \in TU^N$ be such that $Core(v) \neq \varnothing$ and let $x \in I(v)$. Then, $x = pcn(v)$ if and only if there exists a sequence $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_\tau$ of non-empty collections such that $\mathcal{D}_r \subseteq \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ for all $r \in \{1, ..., \tau\}$ that satisfy the following properties:*

(A) *for all $r \in \{1, ..., \tau\}$ the collection $\bar{\mathcal{D}}_r = \bigcup_{k=1}^r \mathcal{D}_k$ is balanced and $\mathcal{F}(\bar{\mathcal{D}}_\tau) = \varnothing$.*

(B) *for all $r \in \{1, ..., \tau\}$ and all $T \in \mathcal{D}_r$ it holds that*
$$exc^P(T, x) = \max_{S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})} exc^P(S, x).$$

2.3 Bankruptcy problems, bankruptcy rules and bankruptcy games

A *bankruptcy problem* is denoted by (N, E, c), where $N = \{1, ..., n\}$ is the set of *claimants*, which will be called *players*, $E \in \mathbb{R}_+$ is the monetary estate that has to be divided over the players, and $c \in \mathbb{R}_+^N$ is the vector of *claims*. By the nature of a bankruptcy problem, the sum of claims exceeds the estate, *i.e.*, $E \leq \sum_{i \in N} c_i$. The class of bankruptcy problems on N is denoted by BR^N.

A *bankruptcy rule* $f : BR^N \to \mathbb{R}_+^N$ is a function that assigns to each bankruptcy problem $(N, E, c) \in BR^N$ a vector $f(N, E, c) \in \mathbb{R}_+^N$ such that $\sum_{i \in N} f_i(N, E, c) = E$ and $0 \leq f(N, E, c) \leq c$. The reader is referred to Thomson (2003) for a detailed overview on bankruptcy rules.

Definition 2.7. *The* constrained equal award rule (CEA) *is defined by*

$$CEA_i(N, E, c) = \min\{\alpha, c_i\}$$

for all bankruptcy problems $(N, E, c) \in BR^N$ and all $i \in N$, where α is such that $\sum_{i \in N} \min\{\alpha, c_i\} = E.$

The constrained equal award rule divides the estate as equal as possible among the players, given that no one can receive more than his claim.

Definition 2.8. *The* constrained equal loss rule (CEL) *is defined by*

$$CEL_i(N, E, c) = \max\{0, c_i - \beta\}$$

for all bankruptcy problems $(N, E, c) \in BR^N$ and all $i \in N$, where β is such that $\sum_{i \in N} \max\{0, c_i - \beta\} = E.$

The constrained equal loss rule divides the loss, which is the claim minus the amount received, as equal as possible among the players, given that no one can receive a negative amount. The constrained equal award rule and constrained equal loss rule are closely related, which is shown in the following well-known proposition.

Proposition 2.9. *The constrained equal award rule is the dual of the constrained equal loss rule and vice versa, i.e.,*

$$CEA(N, E, c) = c - CEL(N, \sum_{i \in N} c_i - E, c)$$

for all bankruptcy problems $(N, E, c) \in BR^N$.

A rule that combines the constrained equal award and constrained equal loss rule is the Aumann-Maschler rule.

Definition 2.10. *(cf. Aumann and Maschler (1985))*
The Aumann-Maschler rule (AM) *is defined by*

$$AM(N, E, c) = \begin{cases} CEA(N, E, \frac{1}{2}c) & \text{if } \sum_{i \in N} \frac{1}{2}c_i \geq E, \\ \frac{1}{2}c + CEL(N, E - \sum_{i \in N} \frac{1}{2}c_i, \frac{1}{2}c) & \text{if } \sum_{i \in N} \frac{1}{2}c_i < E, \end{cases}$$

for all bankruptcy problems $(N, E, c) \in BR^N$.

We refer to Aumann and Maschler (1985) for a motivation based on the concede and divide principle and consistency.

O'Neill (1982) associates with every bankruptcy problem $(N, E, c) \in BR^N$ a corresponding bankruptcy game $v_{E,c} \in TU^N$. In each bankruptcy game, the worth of coalition $S \in 2^N$ is the part of the estate that is left after the players outside the coalition, *i.e.*, $N \backslash S$, receive their claim. Formally,

$$v_{E,c}(S) = \max\{0, E - \sum_{i \in N \backslash S} c_i\} \text{ for all } S \in 2^N.$$

It is readily checked that $Core(v_{E,c}) \neq \varnothing$ for all $(N, E, c) \in BR^N$, so one can use the theorems in Section 2.2 in order to characterize the (per capita) nucleolus. The nucleolus for bankruptcy games corresponds to the Aumann-Maschler rule.

Theorem 2.11. *(cf. Aumann and Maschler (1985))*
Let $(N, E, c) \in BR^N$ *and let* $v_{E,c}$ *be the corresponding bankruptcy game. Then*

$$AM(N, E, c) = n(v_{E,c}).$$

3 A characterization of the per capita nucleolus as a bankruptcy rule

This section introduces a new bankruptcy rule σ that is based on so called clights. These clights can be interpreted as either the claims of the players when the estate is relatively small or as the rights of the players when the estate is relatively large. Moreover, it is proven that this new bankruptcy rule coincides with the per capita nucleolus of the corresponding bankruptcy game. Throughout the remainder of this paper we assume for notational ease and without loss of generality that claim vectors are weakly increasing.[3]

Definition 3.1. *The* clights bankruptcy rule σ *is defined by*

$$\sigma(N, E, c) := \begin{cases} CEA(N, E, \delta(c)) & \text{if } \sum_{i \in N} \delta_i(c) \geq E, \\ \delta(c) + CEL(N, E - \sum_{i \in N} \delta_i(c), c - \delta(c)) & \text{if } \sum_{i \in N} \delta_i(c) < E, \end{cases} \quad (1)$$

for all bankruptcy problems $(N, E, c) \in BR^N$, *where the* clight vector $\delta(c) \in \mathbb{R}^N$ *is recursively defined for all* $i \in N$ *by*

$$\delta_i(c) := \max_{j \in \{1, \dots, i\}} \left\{ \frac{1}{n+j-1} \left(jc_i - (n-1) \sum_{\ell=1}^{j-1} \delta_\ell(c) \right) \right\}. \quad (2)$$

First of all we observe that the clights are monotonic.

Lemma 3.2. *Let* $(N, E, c) \in BR^N$ *be a bankruptcy problem and let* $\delta(c)$ *be the corresponding clight vector. Then, for all* $i \in \{2, \dots, n\}$,

$$\delta_i(c) \geq \delta_{i-1}(c).$$

Proof. Let $i \in \{2, \dots, n\}$. Then

$$\delta_i(c) = \max_{j \in \{1, \dots, i\}} \left\{ \frac{1}{n+j-1} \left(jc_i - (n-1) \sum_{\ell=1}^{j-1} \delta_\ell(c) \right) \right\}$$

$$\geq \max_{j \in \{1, \dots, i-1\}} \left\{ \frac{1}{n+j-1} \left(jc_i - (n-1) \sum_{\ell=1}^{j-1} \delta_\ell(c) \right) \right\}$$

$$\geq \max_{j \in \{1, \dots, i-1\}} \left\{ \frac{1}{n+j-1} \left(jc_{i-1} - (n-1) \sum_{\ell=1}^{j-1} \delta_\ell(c) \right) \right\}$$

$$= \delta_{i-1}(c). \qquad \blacksquare$$

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[3]With $N = \{1, \dots n\}$ we assume that $c_1 \leq c_2 \leq \dots \leq c_n$.

Moreover, note that $\delta(c)$ is independent of E. Also, each clight is non-negative and less than the claim of the corresponding player, which follows from the following lemma.

Lemma 3.3. *Let $(N, E, c) \in BR^N$ be a bankruptcy problem and let $\delta(c) \in \mathbb{R}^N$ be the corresponding clight vector. Then, for all $i \in N$,*

$$\frac{1}{n}c_i \leq \delta_i(c) \leq \frac{i}{n+i-1}c_i.$$

Proof. Let $i \in N$. Then

$$\delta_i(c) = \max_{j \in \{1,\dots,i\}} \left\{ \frac{1}{n+j-1}\left(jc_i - (n-1)\sum_{\ell=1}^{j-1}\delta_\ell(c)\right)\right\} \geq \frac{1}{n}c_i,$$

since the right hand side corresponds to the case $j = 1$. Consequently, $\delta_j(c) \geq 0$ for all $j \in N$ and

$$\delta_i(c) = \max_{j \in \{1,\dots,i\}} \left\{ \frac{1}{n+j-1}\left(jc_i - (n-1)\sum_{\ell=1}^{j-1}\delta_\ell(c)\right)\right\}$$

$$\leq \max_{j \in \{1,\dots,i\}} \left\{ \frac{1}{n+j-1}jc_i\right\}$$

$$= \frac{i}{n+i-1}c_i.^4 \qquad\qquad \blacksquare$$

Example 3.1. *Part 1 (calculation of the clight vector):*
Consider a bankruptcy problem with player set $N = \{1, 2, 3, 4\}$ and vector of claims $c = (4, 9, 10, 19)$. Then

$$\delta_1(c) = \frac{1}{n+1-1}\left(1c_1 - (n-1)0\right) = \frac{1}{4}4 = 1$$

$$\delta_2(c) = \max_{j \in \{1,2\}}\left\{\frac{1}{n+j-1}\left(jc_2 - (n-1)\sum_{\ell=1}^{j-1}\delta_\ell(c)\right)\right\}$$

$$= \max\left\{\frac{1}{4}\cdot 9 - \frac{3}{4}\cdot 0, \ \frac{2}{5}\cdot 9 - \frac{3}{5}\cdot 1\right\}$$

$$= \max\left\{2\frac{1}{4}, \ 3\right\}$$

$$= 3,$$

$$\delta_3(c) = \max\left\{2\frac{1}{2}, \ 3\frac{2}{5}, \ 2\frac{3}{8}\right\} = 3\frac{2}{5},$$

$$\delta_4(c) = \max\left\{4\frac{3}{4}, \ 7, \ 7\frac{1}{2}, 7\frac{24}{35}\right\} = 7\frac{24}{35}.$$

---

[4] The final equation is due to the fact that $\frac{i}{n+1-1} < \frac{i+1}{n+(i+1)-1}$ for all $i \in \{1, ..., n-1\}$.

Part 2 (small estate):
Consider $N = \{1, 2, 3, 4\}$ and $c = (4, 9, 10, 19)$ as above and take $E = 10.5$. Consequently, $\delta(c) = (1, 3, 3\frac{2}{5}, 7\frac{24}{35})$ and $\sum_{i \in N} \delta_i(c) = 15\frac{3}{35} > 10.5 = E$. In this case, the clight vector $\delta(c)$ is interpreted as the appropriate vector of claims and $\sigma(N, E, c) = CEA(N, E, \delta(c)) = (1, 3, 3\frac{1}{4}, 3\frac{1}{4})$.

Part 3 (large estate):
Consider $N = \{1, 2, 3, 4\}$ and $c = (4, 9, 10, 19)$ as above but now take $E = 20.5$. Consequently, $\delta(c) = (1, 3, 3\frac{2}{5}, 7\frac{24}{35})$. Now $\sum_{i \in N} \delta_i(c) = 15\frac{3}{35} < 20.5 = E$. In this case, the clight vector $\delta(c)$ is interpreted as the vector of rights and $\sigma(N, E, c) = \delta(c) + CEL(N, E - \sum_{i \in N} \delta_i(c), c - \delta(c)) = (1, 3, 3\frac{3}{4}, 12\frac{3}{4})$.

Part 4 (A hydraulic interpretation):
Suppose that the estate symbolizes an amount of water and that the claims symbolize the amount of water claimed. Then, each claim can be represented by a bucket which has the volume of that claim. In the clights rule σ, each bucket is split into two smaller buckets, namely the clights buckets of volume $\delta(c)$ and the remainder bucket of volume $c - \delta(c)$. This is visualized in Figure 1, where the water will be poured into the buckets at the arrow and any overspill will flood from the buckets of volume $\delta(c)$ into the buckets of volume $c - \delta(c)$.
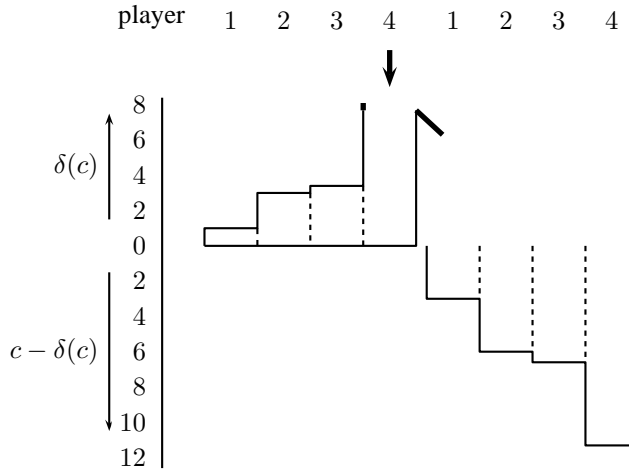

Figure 1: The buckets of the σ rule.

In Figure 2, the small estate case, i.e., $E = 10.5$, is visualized. The water is poured into the buckets at the arrow and the result is visualized by the dashed area.
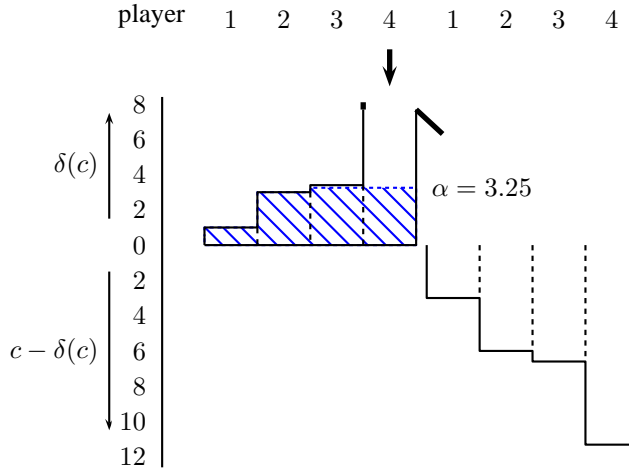

Figure 2: $\sigma(N, 10.5, c)$ visualized.

In Figure 3, the case with the large estate $E = 20.5$ is visualized. Again the water is poured into the buckets at the arrow, but now there is overflow of size $E - \sum_{i \in N} \delta(c) = 20.5 - 15\frac{3}{35} = 5\frac{29}{70}$. Again, the result is visualized by the dashed area.
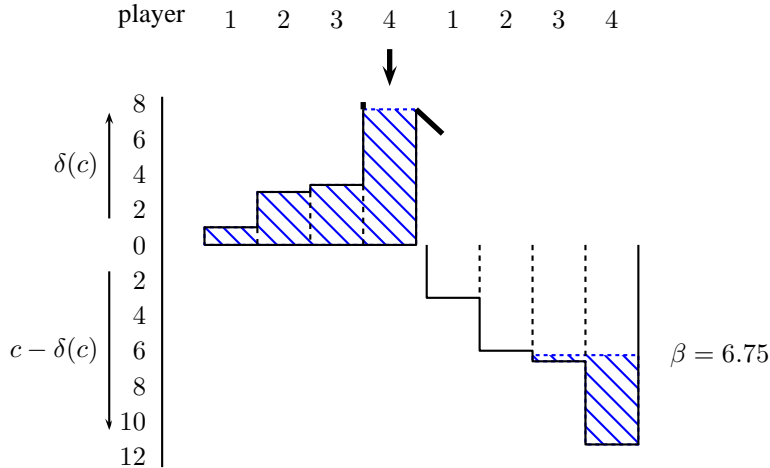
Figure 3: $\sigma(N, 20.5, c)$ visualized.

$\triangle$

The player $j \in \{1, ..., i\}$ with highest index for whom the maximum in (2) is attained for $\delta_i(c)$ is of importance later on. This player is called the clight-argument of claimant i and is formally defined below.

Definition 3.4. *Let $(N, E, c) \in BR^N$ be a bankruptcy problem, let $\delta(c)$ be the corresponding clight vector and let $i \in N$. Then, the $\delta(c)$-clight argument*

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$a(i) \in N$ *of player* $i \in N$ *is defined by*

$$a(i) := \max_{j \in \{1,...,i\}} \left\{ j \mid \delta_i(c) = \frac{1}{n+j-1} \left(jc_i - (n-1) \sum_{\ell=1}^{j-1} \delta_\ell(c) \right) \right\} \quad (3)$$

The next lemma states monotonicity in $\delta(c)$-clight arguments.

Lemma 3.5. *Let* $(N, E, c) \in BR^N$ *be a bankruptcy problem. Then, for all* $i \in \{2, ..., n\}$,

$$a(i) \geq a(i-1).$$

Proof. Let $i \in \{2, ..., n\}$ and let $k \in \{1, ..., a(i-1)\}$. We show that $a(i) \geq k$, which completes the proof. By the definition of $a(i-1)$ we have that

$$\frac{1}{n+a(i-1)-1} \left(a(i-1)c_{i-1} - (n-1) \sum_{\ell=1}^{a(i-1)-1} \delta_\ell(c) \right) \geq \frac{1}{n+k-1} \left(kc_{i-1} - (n-1) \sum_{\ell=1}^{k-1} \delta_\ell(c) \right)$$

which rewrites to

$$\left(\frac{a(i-1)}{n+a(i-1)-1} - \frac{k}{n+k-1} \right) c_{i-1} \geq \frac{n-1}{n+a(i-1)-1} \sum_{\ell=1}^{a(i-1)-1} \delta_\ell(c) - \frac{n-1}{n+k-1} \sum_{\ell=1}^{k-1} \delta_\ell(c).$$

Since $c_i \geq c_{i-1}$ and $\frac{a(i-1)}{n+a(i-1)-1} \geq \frac{k}{n+k-1}$ we have

$$\left(\frac{a(i-1)}{n+a(i-1)-1} - \frac{k}{n+k-1} \right) c_i \geq \left(\frac{a(i-1)}{n+a(i-1)-1} - \frac{k}{n+k-1} \right) c_{i-1}.$$

Hence,

$$\left(\frac{a(i-1)}{n+a(i-1)-1} - \frac{k}{n+k-1} \right) c_i \geq \frac{n-1}{n+a(i-1)-1} \sum_{\ell=1}^{a(i-1)-1} \delta_\ell(c) - \frac{n-1}{n+k-1} \sum_{\ell=1}^{k-1} \delta_\ell(c).$$

which rewrites back to

$$\frac{1}{n+a(i-1)-1} \left(a(i-1)c_i - (n-1) \sum_{\ell=1}^{a(i-1)-1} \delta_\ell(c) \right) \geq \frac{1}{n+k-1} \left(kc_i - (n-1) \sum_{\ell=1}^{k-1} \delta_\ell(c) \right).$$

Hence, $a(i) \geq a(i-1)$. ∎

In the following lemma it is shown that the clight-argument of all players except player 1 can not be player 1.

Lemma 3.6. *Let* $(N, E, c) \in BR^N$. *Then, for all* $i \in \{2, ..., n\}$,

$$a(i) \geq 2.$$

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Proof. Since $a(i) \geq a(i-1)$ (Lemma 3.5), we only have to prove that $a(2) \geq 2$. By definition we have that $\delta_1(c) = \frac{1}{n}c_1$. Note that

$$\delta_2(c) = \max\{\frac{1}{n}c_2, \frac{1}{n+2-1}\left(2c_2 - (n-1)\frac{1}{n}c_1\right)\}.$$

Since

$$\frac{1}{n+2-1}\left(2c_2 - (n-1)\frac{1}{n}c_1\right) \geq \frac{1}{n+1}\left(2c_2 - \frac{n-1}{n}c_2\right) = \frac{1}{n}c_2,$$

we obtain $a(2) \geq 2$. ∎

The following corollary provides an explicit expression for the clight vector of two- and three-player bankruptcy problems.

Corollary 3.7. *Let $c \in \mathbb{R}_+^N$ be a vector of claims. Then,*

$$\delta(c) = \begin{cases} \left(\frac{1}{2}c_1, \frac{2}{3}c_2 - \frac{1}{3}c_1\right) & \text{if } N = \{1,2\}, \\ \left(\frac{1}{3}c_1, \frac{1}{2}c_2 - \frac{1}{6}c_1, \max\{\frac{1}{2}c_3 - \frac{1}{6}c_1, \frac{3}{5}c_3 - \frac{1}{5}c_2 - \frac{1}{15}c_1\}\right) & \text{if } N = \{1,2,3\}. \end{cases}$$

The next lemma shows that the subsequent difference in the clights is less than the subsequent difference in the claims.

Lemma 3.8. *Let $(N, E, c) \in BR^N$ be a bankruptcy problem and let $\delta(c)$ be the corresponding clight vector. Then, for all $i \in \{2, ..., n\}$,*

$$\delta_i(c) - \delta_{i-1}(c) \leq \frac{i}{n+i-1}(c_i - c_{i-1}).$$

Proof. Let $i \in \{2, ..., n\}$. The proof is split into two cases, depending on the clight-argument.
Case (1): Assume $a(i) \leq i - 1$. Then

$$\begin{aligned} \delta_i(c) - \delta_{i-1}(c) &= \frac{1}{n+a(i)-1}\left(a(i)c_i - (n-1)\sum_{\ell=1}^{a(i)-1}\delta_\ell(c)\right) - \delta_{i-1}(c) \\ &\leq \frac{1}{n+a(i)-1}\left(a(i)c_i - (n-1)\sum_{\ell=1}^{a(i)-1}\delta_\ell(c)\right) \\ &\quad - \frac{1}{n+a(i)-1}\left(a(i)c_{i-1} - (n-1)\sum_{\ell=1}^{a(i)-1}\delta_\ell(c)\right) \\ &= \frac{a(i)}{n+a(i)-1}(c_i - c_{i-1}) \\ &\leq \frac{i}{n+i-1}(c_i - c_{i-1}). \end{aligned}$$

Case (2): Assume that $a(i) = i$. Then,

$$\delta_i(c) - \delta_{i-1}(c) = \frac{1}{n+i-1}\Big(ic_i - (n-1)\sum_{\ell=1}^{i-1}\delta_\ell(c)\Big) - \delta_{i-1}(c)$$

$$= \frac{1}{n+i-1}\Big(ic_i - (n-1)\sum_{\ell=1}^{i-2}\delta_\ell(c)\Big) - \frac{n-1}{n+i-1}\delta_{i-1}(c) - \delta_{i-1}(c)$$

$$= \frac{1}{n+i-1}\Big(ic_i - (n-1)\sum_{\ell=1}^{i-2}\delta_\ell(c)\Big) - \frac{n}{n+i-1}\delta_{i-1}(c) - \frac{n+i-2}{n+i-1}\delta_{i-1}(c)$$

$$\leq \frac{1}{n+i-1}\Big(ic_i - (n-1)\sum_{\ell=1}^{i-2}\delta_\ell(c)\Big) - \frac{1}{n+i-1}c_{i-1} - \frac{n+i-2}{n+i-1}\delta_{i-1}(c)$$

$$\leq \frac{1}{n+i-1}\Big(ic_i - (n-1)\sum_{\ell=1}^{i-2}\delta_\ell(c)\Big) - \frac{1}{n+i-1}c_{i-1}$$

$$\qquad - \frac{n+i-2}{n+i-1}\Big(\frac{1}{n+(i-1)-1}\big((i-1)c_{i-1} - (n-1)\sum_{\ell=1}^{i-2}\delta_\ell(c)\big)\Big)$$

$$= \frac{1}{n+i-1}\Big(ic_i - (n-1)\sum_{\ell=1}^{i-2}\delta_\ell(c)\Big) - \frac{1}{n+i-1}c_{i-1}$$

$$\qquad - \Big(\frac{1}{n+i-1}\big((i-1)c_{i-1} - (n-1)\sum_{\ell=1}^{i-2}\delta_\ell(c)\big)\Big)$$

$$= \frac{i}{n+i-1}(c_i - c_{i-1})$$

where the first inequality follows from the fact that $\delta_{i-1} \geq \frac{1}{n}c_{i-1}$ (Lemma 3.3). ∎

The next example acts as a stepping stone for the proof that the clights rule coincides with the per capita nucleolus of a bankruptcy game.

Example 3.2. *(Example 3.1 continued)*
Part 1 (small estate):
Consider the bankruptcy problem (N, E, c) with $N = \{1, 2, 3, 4\}$, $E = 10.5$ and $c = (4, 9, 10, 19)$. As we have seen, $\delta(c) = (1, 3, 3\frac{2}{5}, 7\frac{24}{35})$, $\sigma(N, E, c) = (1, 3, 3\frac{1}{4}, 3\frac{1}{4})$, $a(1) = 1$, $a(2) = a(3) = 2$ and $a(4) = 4$. The corresponding bankruptcy game and the per capita excesses of $\sigma(N, E, c)$ are as follows:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$
$v(S)$	0	0	0	0	0	0	0	0
$exc^P(S,\sigma)$	-1	-3	$-3\frac{1}{4}$	$-3\frac{1}{4}$	-2	$-2\frac{1}{8}$	$-2\frac{1}{8}$	$-3\frac{1}{8}$

S		$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
$v(S)$		0	0	0	0.5	1.5	6.5	10.5
$exc^P(S,\sigma)$		$-3\frac{1}{8}$	$-3\frac{1}{4}$	$-2\frac{5}{12}$	$-2\frac{1}{4}$	-2	-1	0

Set $\mathcal{D}_1 = \big\{\{1\},\{2,3,4\}\big\}$, $\mathcal{D}_2 = \big\{\{1,2\},\{1,3,4\}\big\}$ and $\mathcal{D}_3 = \big\{\{1,3\},\{1,4\}\big\}$. Then, $\mathcal{D}_1 \subseteq 2^N\backslash\{\varnothing,N\} = \mathcal{F}(\{\varnothing\})$ and clearly $\bar{\mathcal{D}}_1 = \big\{\{1\},\{2,3,4\}\big\}$ is balanced. Furthermore, the coalitions in $\mathcal{D}_1$ have the highest per capita excess of all (free) coalitions. Similarly, $\mathcal{D}_2 \subseteq 2^N\backslash\{\varnothing,\{1\},\{2,3,4\},N\} = \mathcal{F}(\bar{\mathcal{D}}_1)$ and $\bar{\mathcal{D}}_2 = \big\{\{1\},\{2,3,4\},\{1,2\},\{1,3,4\}\big\}$ is balanced since $\rho = (\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3})$ is a corresponding balanced map. The coalitions in $\mathcal{D}_2$ have the highest per capita excess of all current free coalitions.

Furthermore, $\mathcal{D}_3 \subseteq \big\{\{3\},\{4\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\}\big\}=\mathcal{F}(\bar{\mathcal{D}}_2)$ and $\bar{\mathcal{D}}_3 = \big\{\{1\},\{2,3,4\},\{1,2\},\{1,3,4\},\{1,3\},\{1,4\}\big\}$ is balanced since $\rho = (\frac{1}{6},\frac{4}{6},\frac{2}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$ is a corresponding balanced map. The coalitions in $\mathcal{D}_3$ have the highest per capita excess of all current free coalitions. Finally, note that $\mathcal{F}(\bar{\mathcal{D}}_3) = \varnothing$. Using Theorem 2.6 we conclude that $\sigma(N,E,c) = pcn(v_{E,c})$.

Part 2 (large estate):
Consider the bankruptcy problem (N,E,c) with $N = \{1,2,3,4\}$, $E = 20.5$ and $c = (4,9,10,19)$. As we have seen, $\delta(c) = (1,3,3\frac{2}{5},7\frac{24}{35})$, $\sigma(N,E,c) = (1,3,3\frac{3}{4},12\frac{1}{4})$, $a(1) = 1$, $a(2) = a(3) = 2$ and $a(4) = 4$. The corresponding bankruptcy game and the per capita excesses of $\sigma(N,E,c)$ are as follows:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$
$v(S)$	0	0	0	0	0	0	1.5	0
$exc^P(S,\sigma)$	-1	-3	$-3\frac{3}{4}$	$-12\frac{3}{4}$	-2	$-2\frac{3}{8}$	$-6\frac{1}{8}$	$-3\frac{3}{8}$

S		$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
$v(S)$		6.5	7.5	1.5	10.5	11.5	16.5	20.5
$exc^P(S,\sigma)$		$-4\frac{5}{8}$	$-4\frac{1}{2}$	$-2\frac{1}{12}$	$-2\frac{1}{12}$	-2	-1	0

Now take $\mathcal{D}_1=\big\{\{1\},\{2,3,4\}\big\}$, $\mathcal{D}_2=\big\{\{1,2\},\{1,3,4\}\big\}$ and $\mathcal{D}_3=\big\{\{1,2,4\},\{1,2,3\}\big\}$. Regarding $\mathcal{D}_1$ and $\mathcal{D}_2$ we refer to part (1). Moreover, $\mathcal{D}_3\subseteq\big\{\{3\},\{4\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\}\big\}=\mathcal{F}(\bar{\mathcal{D}}_2)$ and $\bar{\mathcal{D}}_3 = \big\{\{1\},\{2,3,4\},\{1,2\},\{1,3,4\},\{1,2,4\},\{1,2,3\}\big\}$ is balanced since $\rho = (\frac{1}{6},\frac{3}{6},\frac{1}{6},\frac{2}{6},\frac{1}{6},\frac{1}{6})$ is a corresponding balanced map. The coalitions in $\mathcal{D}_3$ have the highest per capita excess of all the current free coalitions. Finally, note that $\mathcal{F}(\bar{\mathcal{D}}_3) = \varnothing$. Again, using Theorem 2.6 we can conclude that $\sigma(N,E,c) = pcn(v_{E,c})$.

Part 3 (general remarks on the cases):
The relevant collections of these examples can be expressed by the clight-argument

vector. For both cases ($E = 10.5$ and $E = 20.5$) we have that

$$\mathcal{D}_1 = \{\{1\}, \{2, 3, 4\}\} = \{\{1, ..., a(1) - 1\} \cup \{1\}, N \backslash \{1\}\},$$
$$\mathcal{D}_2 = \{\{1, 2\}, \{1, 3, 4\}\} = \{\{1, ..., a(2) - 1\} \cup \{2\}, N \backslash \{2\}\}.$$

Furthermore, when $E = 10.5$, we have that

$$\mathcal{D}_3 = \{\{1, 3\}, \{1, 4\}\} = \{\{1, ..., a(3) - 1\} \cup \{3\}, \{1, ..., a(4) - 1\} \cup \{4\}\},$$

and when $E = 20.5$, we have that

$$\mathcal{D}_3 = \{\{1, 3, 4\}, \{1, 2, 3\}\} = \{N \backslash \{3\}, N \backslash \{4\}\}.$$

This structure of the relevant collections will form the basis of the proof of our main result. $\triangle$

Theorem 3.9. *Let $(N, E, c) \in BR^N$ and let $v_{E,c}$ be the corresponding bankruptcy game. Then,*

$$\sigma(N, E, c) = pcn(v_{E,c}).$$

Proof. In this proof $\sigma(N, E, c)$ is abbreviated to σ and $v_{E,c}$ is abbreviated to v. Furthermore, since σ and the per capita nucleolus both depend continuously on the estate E, we assume that $\sum_{\ell \in N} \delta_\ell(c) \neq E$.

In order to apply Theorem 2.6 we will do the following:
Part I: Define τ, and for all $r \in \{1, ..., \tau\}$, define appropriate relevant collections $\mathcal{D}_r$ and show that $\mathcal{D}_r \subset \mathcal{F}(\bar{\mathcal{D}}_{r-1})$.
Part II: Show that the sequence $\mathcal{D}_1, ..., \mathcal{D}_\tau$ satisfies condition (A) of Theorem 2.6, *i.e.*, $\bar{\mathcal{D}}_r = \bigcup_{\ell=1}^r \mathcal{D}_\ell$ is balanced for all $r \in \{1, ..., \tau\}$ and $\mathcal{F}(\bar{\mathcal{D}}_\tau) = \varnothing$.
Part III: Show that the sequence $\mathcal{D}_1, ..., \mathcal{D}_\tau$ satisfies condition (B) of Theorem 2.6, *i.e.*, for all $r \in \{1, ..., \tau\}$ and all $S \in \mathcal{D}_r$, it holds that
$exc^P(S, \sigma) = \max\limits_{T \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})} exc^P(T, \sigma)$.

Part I: Define

$$t := \begin{cases} \min\{ i \in N \mid \delta_i(c) \geq \alpha\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) > E, \\ \min\{ i \in N \mid c_i - \delta_i(c) \geq \beta\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) < E, \end{cases} \tag{4}$$

in which α and β are determined by $\sum_{\ell \in N} \min\{\alpha, \delta_\ell(c)\} = E$ and $E - \sum_{\ell \in N} \delta_\ell(c) = \sum_{\ell \in N} \max\{0, c_\ell - \delta_\ell(c) - \beta\}$, respectively. Note that the clights rule allocates the estate in the following way:

$$\sigma_i = \begin{cases} \delta_i(c) & \text{if } i < t, \\ \alpha & \text{if } i \geq t \text{ and } \sum_{\ell \in N} \delta_\ell(c) > E, \\ c_i - \beta & \text{if } i \geq t \text{ and } \sum_{\ell \in N} \delta_\ell(c) < E. \end{cases} \tag{5}$$

Additionally, define $S_r := \{1, ..., a(r) - 1\} \cup \{r\}$ for each $r \in \{1, ..., t - 1\}$ and define

$$\mathcal{D}_r = \big\{S_r, N\backslash\{r\}\big\} \text{ for all } r \in \{1, ..., t - 1\}.$$

Furthermore, if $t < n$ and $\sum_{\ell \in N} \delta_\ell(c) > E$, then, define $S_r := \{1, ..., m-1\} \cup \{r\}$ for each $r \in \{t, ..., n\}$, where m is defined by

$$m = \underset{s \in \{1, ..., t\} \,:\, v(\{1, ..., s-1\} \cup \{t\})=0}{\operatorname{argmax}} \frac{-\alpha - \sum_{\ell=1}^{s-1} \delta_\ell(c)}{s}. \tag{6}$$

Such an m exists because $v\big(\{1, ..., a(t) - 1\} \cup \{t\}\big) = 0$ (Lemma .4, where it is used that $t < n$).

Moreover, if $t < n$, define

$$\mathcal{D}_t = \begin{cases} \{S_t, ..., S_n\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) > E, \\ \{N\backslash\{t\}, ..., N\backslash\{n\}\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) < E. \end{cases}$$

For all $r < t$ we have

$$H(\bar{\mathcal{D}}_r) = \big\{S : S \subset \{1, ..., r\}\big\} \cup \big\{N\backslash S : S \subset \{1, ..., r\}\big\}.$$

Moreover, if $t < n$ we have
$$H(\bar{\mathcal{D}}_t) = 2^N$$

and if $t = n$ we have
$$H(\bar{\mathcal{D}}_{t-1}) = 2^N,$$

so we define $\tau := \min\{t, n - 1\}$.

This gives that $\mathcal{F}(\bar{\mathcal{D}}_{r-1}) = \{S : 1 \le |S \cap \{r, ..., n\}| \le n-r\}$ for all $r \le \tau$ which implies that $\mathcal{D}_r \subset \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ for all $r \in \{1, ..., \tau\}$.

Part II: By construction we have that $\mathcal{F}(\bar{\mathcal{D}}_\tau) = \varnothing$. It remains to prove that $\bar{\mathcal{D}}_r$ is balanced for all $r \in \{1, ..., \tau\}$. The balancedness proof is split into two parts depending on whether $t = 1$ or not.
Part II.a: If $t = 1$, then

$$\mathcal{D}_1 = \begin{cases} \{\{1\}, ..., \{n\}\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) > E, \\ \{N\backslash\{1\}, ..., N\backslash\{n\}\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) < E. \end{cases}$$

Choose $\rho(\{i\}) = 1$ for all $i \in N$ to define a balanced map if $\sum_{\ell \in N} \delta_\ell(c) > E$. Choose $\rho(N\backslash\{i\}) = \frac{1}{n-1}$ for all $i \in N$ to define a balanced map if $\sum_{\ell \in N} \delta_\ell(c) < E$.
Part II.b: Assume $t > 1$ and let $r \in \{1, ..., \tau\}$. Take $\varepsilon > 0$ sufficiently small. Again this part is split into two parts. First, we show that $\bar{\mathcal{D}}_r$ is balanced for all

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$r < t$. Then we show that $\bar{\mathcal{D}}_t$ is balanced.

Part II.b1: Let $t > 1$, assume $r < t$ and let

$$\rho(S_i) = \varepsilon \text{ for all } i \in \{2, ..., r\},$$

$$\rho(N\backslash\{i\}) = \varepsilon + \sum_{k=i+1}^{r} \mathbb{1}_{\{i \leq a(k)-1\}}\rho(S_k) \text{ for all } i \in \{2, ..., r\},$$

$$\rho(N\backslash\{1\}) = 1 - \sum_{k=2}^{r} \rho(N\backslash\{k\}),$$

$$\rho(\{1\}) = \rho(N\backslash\{1\}) - \sum_{k=2}^{r} \rho(S_k).$$

$$\rho(S) = 0 \text{ else.}$$

Since all ρ-values except $\rho(N\backslash\{1\})$ and $\rho(\{1\})$ are in the order of ε, both $\rho(N\backslash\{1\})$ and $\rho(\{1\})$ are strictly positive.

Let $i > r$. Then

$$\sum_{S:S \ni i} \rho(S) = \sum_{k=1}^{r} \rho(N\backslash\{k\})$$

$$= \rho(N\backslash\{1\}) + \sum_{k=2}^{r} \rho(N\backslash\{k\})$$

$$= 1 - \sum_{k=2}^{r} \rho(N\backslash\{k\}) + \sum_{k=2}^{r} \rho(N\backslash\{k\})$$

$$= 1. \tag{7}$$

Let $i \leq r$. Then,

$$\sum_{S:S \ni i} \rho(S) = \rho(S_i) + \sum_{k=1, k \neq i}^{r} \rho(N\backslash\{k\}) + \sum_{k=i+1}^{r} \mathbb{1}_{\{i \leq a(k)-1\}}\rho(S_k)$$

$$= \sum_{k=1}^{r} \rho(N\backslash\{k\}) + \rho(S_i) - \rho(N\backslash\{i\}) + \sum_{k=i+1}^{r} \mathbb{1}_{\{i \leq a(k)-1\}}\rho(S_k)$$

$$= \sum_{k=1}^{r} \rho(N\backslash\{k\})$$

$$= 1, \tag{8}$$

in which equation (8) has been shown in (7). Hence, $\sum_{S \in \bar{\mathcal{D}}_r} \rho(S)e^S = e^N$ which completes the proof that this ρ forms a balanced map for the case $r < t$.

Part II.b2: Let $t > 1$, assume that $r = t$. Since $r \leq \tau = \min\{t, n-1\}$, we have $t = \tau$. Again, this part is split into two, depending on whether $\sum_{\ell \in N} \delta_\ell(c) > E$ or

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$\sum_{\ell \in N} \delta_\ell(c) < E$.

Part II.b2i: Let $t > 1$, $r = t = \tau$ and assume $\sum_{\ell \in N} \delta_\ell(c) > E$. Then,

$$\rho(S_i) = 2\varepsilon \text{ for all } i \in \{2, ..., t-1\},$$
$$\rho(S_i) = \varepsilon \text{ for all } i \in \{t, ..., n\},$$
$$\rho(N \backslash \{i\}) = \varepsilon + \sum_{k=i+1}^{t-1} \mathbb{1}_{\{i \leq a(k)-1\}} \rho(S_k) + \mathbb{1}_{\{i \leq m-1\}} \sum_{k=t}^{n} \rho(S_k)$$
$$\text{for all } i \in \{2, ..., t-1\},$$

$$\rho(N \backslash \{1\}) = 1 - \varepsilon - \sum_{k=2}^{t-1} \rho(N \backslash \{k\}),$$

$$\rho(\{1\}) = \rho(N \backslash \{1\}) + \varepsilon - \sum_{k=2}^{t-1} \mathbb{1}_{\{1 \leq a(k)-1\}} \rho(S_k) - \mathbb{1}_{\{1 \leq m-1\}} \sum_{k=t}^{n} \rho(S_k).$$

$$\rho(S) = 0 \text{ else.}$$

Since all ρ-values except $\rho(N \backslash \{1\})$ and $\rho(\{1\})$ are in the order of ε, both $\rho(N \backslash \{1\})$ and $\rho(\{1\})$ are strictly positive.

Let $i \geq t$. Then

$$\sum_{S : S \ni i} \rho(S) = \rho(S_i) + \sum_{k=1}^{t-1} \rho(N \backslash \{k\})$$

$$= \varepsilon + \rho(N \backslash \{1\}) + \sum_{k=2}^{t-1} \rho(N \backslash \{k\})$$

$$= \varepsilon + 1 - \varepsilon - \sum_{k=2}^{t-1} \rho(N \backslash \{k\}) + \sum_{k=2}^{t-1} \rho(N \backslash \{k\})$$

$$= 1.$$

Let $i < t$. Then

$$
\sum_{S:S\ni i} \rho(S) = \rho(S_i) + \sum_{k=1,k\neq i}^{t-1} \rho(N\setminus\{k\}) + \mathbb{1}_{\{i\leq m-1\}} \sum_{k=t}^{n} \rho(S_k)
$$

$$
+ \sum_{k=i+1}^{t-1} \mathbb{1}_{\{i\leq a(k)-1\}}\rho(S_k)
$$

$$
= \sum_{k=1}^{t-1} \rho(N\setminus\{k\}) + \rho(S_i) - \rho(N\setminus\{i\}) + \mathbb{1}_{\{i\leq m-1\}} \sum_{k=t}^{n} \rho(S_k)
$$

$$
+ \sum_{k=i+1}^{t-1} \mathbb{1}_{\{i\leq a(k)-1\}}\rho(S_k)
$$

$$
= \sum_{k=1}^{t-1} \rho(N\setminus\{k\}) + \varepsilon
$$

$$
= \sum_{k=2}^{t-1} \rho(N\setminus\{k\}) + \rho(N\setminus\{1\}) + \varepsilon
$$

$$
= (1 - \varepsilon) + \varepsilon
$$

$$
= 1,
$$

Hence, $\sum_{S\in\bar{\mathcal{D}}_t} \rho(S)e^S = e^N$, which completes Part II.b2i.

Part II.b2ii: Let $t > 1$, $r = t = \tau$ and assume $\sum_{\ell\in N} \delta_\ell(c) < E$. Then, $\bar{\mathcal{D}}_t = \bar{\mathcal{D}}_{t-1} \cup \mathcal{B}$, where $\mathcal{B} = \{N\setminus\{1\}, ..., N\setminus\{n\}\}$. Since $\mathcal{B}$ is balanced (see **Part II.a**) and $\bar{\mathcal{D}}_{t-1}$ is balanced (see **Part II.b1**) we have that $\bar{\mathcal{D}}_t$ is balanced.

Part II: For all cases it is shown that $\bar{\mathcal{D}}_r$ is balanced for all $r \in \{1, ..., \tau\}$.

Part III: The proof is split into two parts. In the first part (**Part IIIA**), we provide an upper bound for the per capita excesses of coalitions in $\mathcal{F}(\bar{\mathcal{D}}_{r-1})$. In the second part (**Part IIIB**), it is shown that the coalitions in $S \in \mathcal{D}_r$ are equal to this upper bound.

First, note that $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ implies that there exists at least one $j \geq r$ such that $j \in S$.

Part IIIA: Let $r \in \{1, ..., \tau\}$ and let $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$. The proof is split into two parts, depending on whether $v(S) = 0$ or $v(S) > 0$.

Part IIIA.a: Let $r \in \{1, ..., \tau\}$ and let $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$. Assume that $v(S) = 0$ and define $s = \min\{|S|, r\}$. Again there are two parts, depending on whether $\delta_r(c) \leq \sigma_r$ or $\delta_r(c) > \sigma_r$.

Part IIIA.a1: Let $r \in \{1, ..., \tau\}$, $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ and let $v(S) = 0$. Assume that

$\delta_r(c) \leq \sigma_r$. We have

$$exc^P(S, \sigma) = \frac{0 - \sum_{\ell \in S} \sigma_\ell}{|S|}$$

$$\leq \frac{0 - \sum_{\ell=1}^{s-1} \sigma_\ell - \sigma_r}{s} \tag{9}$$

$$\leq \frac{0 - \sum_{\ell=1}^{s-1} \delta_\ell(c) - \delta_r(c)}{s} \tag{10}$$

$$= \frac{-\sum_{\ell=1}^{s-1} \delta_\ell(c) - \frac{n+s-1}{n-1}\delta_r(c) + \frac{s}{n-1}\delta_r(c)}{s}$$

$$= \frac{-\sum_{\ell=1}^{s-1} \delta_\ell(c) - \frac{n+s-1}{n-1}\delta_r(c)}{s} + \frac{\delta_r(c)}{n-1}$$

$$\leq \frac{-\sum_{\ell=1}^{s-1} \delta_\ell(c) - \frac{1}{n-1}\left(sc_r - (n-1)\sum_{\ell=1}^{s-1} \delta_\ell(c)\right)}{s} + \frac{\delta_r(c)}{n-1} \tag{11}$$

$$= \frac{\delta_r(c) - c_r}{n-1}.$$

Let us clarify the steps in the elaboration above. At (9) the following is used: If $r > s$, then we first remove the $r - s$ players which receive the most. Then, we replace the remaining players with players $1, ..., s-1$ and r, which is possible since there is a player $j \in S$ such that $j \geq r$. At (10), it is used that $\delta_r(c) \leq \sigma_r$, which implies that $\delta_\ell(c) \leq \sigma_\ell$ for all $\ell \in \{1, ..., r\}$. Finally, (11) follows from (2).

To conclude this part: for all $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ with $v(S) = 0$ and $\delta_r(c) \leq \sigma_r$ we have that $exc^P(S, \sigma) \leq \frac{\delta_r(c) - c_r}{n-1}$.

Part IIIA.a2: Let $r \in \{1, ..., \tau\}$, let $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ and let $v(S) = 0$. Assume that $\delta_r(c) > \sigma_r$. This implies that $\sum_{\ell \in N} \delta_\ell > E$ and $r \geq t$. Then, since $r \in \{1, ..., \tau\}$, the definition of τ and $r \geq t$ we have that $r = t$. Furthermore, we have

$$exc^P(S, \sigma) \leq \frac{0 - \sum_{\ell=1}^{s-1} \sigma_\ell - \sigma_r}{s} \tag{12}$$

$$= \frac{0 - \sum_{\ell=1}^{s-1} \sigma_\ell - \sigma_t}{s}$$

$$= \frac{-\sum_{\ell=1}^{s-1} \delta_\ell(c) - \alpha}{s} \tag{13}$$

$$\leq \frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m}. \tag{14}$$

We clarify the elaboration. (12) follows from **PartIIIA.a1i** until (9). At (13) we use (5) and at (14) we use (6) together with the fact that $v(S) = 0$ implies that $v(\{1, ..., s-1\} \cup \{r\}) = 0$.

To conclude this part: if $\delta_t(c) > \sigma_t$ (and hence $\sum_{\ell \in N} \delta_\ell > E$) we have for all $S \in \mathcal{F}(\bar{\mathcal{D}}_{t-1})$ with $v(S) = 0$ that $exc^P(S, \sigma) \leq \frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m}$.

Part IIIA.a: From **Part IIIA.a1** and **Part IIIA.a2** we obtain that for all $r \in \{1, ..., \tau\}$ and all $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ with $v(S) = 0$,

$$exc^P(S, \sigma) \leq \begin{cases} \frac{\delta_r(c) - c_r}{n-1} & \text{if } r < t, \\ \frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m} & \text{if } r = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) > E, \\ \frac{\delta_t(c) - c_t}{n-1} & \text{if } r = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) < E. \end{cases}$$

Part IIIA.b: Let $r \in \{1, ..., \tau\}$, let $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$, assume that $v(S) > 0$ and let $k \in \{r, ..., n\} \setminus S$. We have

$$exc^P(S, \sigma) = \frac{v(S) - \sum_{\ell \in S} \sigma_\ell}{|S|}$$

$$\leq \frac{v(S) - \sum_{\ell \in S} \sigma_\ell + \sum_{\ell \in N \setminus (S \cup \{k\})} c_\ell - \sum_{\ell \in N \setminus (S \cup \{k\})} \sigma_\ell}{|S|} \quad (15)$$

$$= \frac{E - \sum_{\ell \in N \setminus S} c_\ell - \sum_{\ell \in S} \sigma_\ell + \sum_{\ell \in N \setminus (S \cup \{k\})} c_\ell - \sum_{\ell \in N \setminus (S \cup \{k\})} \sigma_\ell}{|S|}$$

$$= \frac{E - c_k - (E - \sigma_k)}{|S|}$$

$$\leq \frac{\sigma_k - c_k}{n - 1}$$

$$\leq \frac{\sigma_r - c_r}{n - 1}. \quad (16)$$

To clarify, (15) uses the fact that $\sigma_i \leq c_i$ for all $i \in N$. Inequality (16) follows from the fact that $r \leq k$ together with the fact that $c_{i-1} - \sigma_{i-1} \leq c_i - \sigma_i$ for all $i \in \{2, ..., n\}$ (Lemma 3.8).

To conclude this part: For all $r \in \{1, ..., \tau\}$ and all $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ with $v(S) > 0$ we have due to (5) that

$$exc^P(S, \sigma) \leq \frac{\sigma_r - c_r}{n - 1} = \begin{cases} \frac{\delta_r(c) - c_r}{n-1} & \text{if } r < t, \\ \frac{\alpha - c_r}{n-1} & \text{if } r = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) > E, \\ \frac{-\beta}{n-1} & \text{if } r = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) < E. \end{cases}$$

Part IIIA: We have from **part IIIA.a** and **part IIIA.b** that for all $r \in \{1, ..., \tau\}$

and all $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ it holds that

$$
exc^P(S,\sigma) \leq
\begin{cases}
\dfrac{\delta_r(c) - c_r}{n-1} & \text{if } r < t, \\[2ex]
\dfrac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m} & \text{if } r = t, \sum_{\ell \in N} \delta_\ell(c) > E \text{ and } v(S) = 0, \\[2ex]
\dfrac{\alpha - c_t}{n-1} & \text{if } r = t, \sum_{\ell \in N} \delta_\ell(c) > E \text{ and } v(S) > 0, \\[2ex]
\dfrac{\delta_t - c_t}{n-1} & \text{if } r = t, \sum_{\ell \in N} \delta_\ell(c) < E \text{ and } v(S) = 0, \\[2ex]
\dfrac{-\beta}{n-1} & \text{if } r = t, \sum_{\ell \in N} \delta_\ell(c) < E \text{ and } v(S) > 0.
\end{cases}
$$

Now we will show for the case $r = t$ that $\frac{a - c_t}{n-1} \leq \frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m}$ and $\frac{\delta_t - c_t}{n-1} \leq \frac{-\beta}{n-1}$. If $r = t$ and $\sum_{\ell \in N} \delta_\ell(c) > E$, then

$$
\frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m} \geq \frac{-\sum_{\ell=1}^{a(t)-1} \delta_\ell(c) - \alpha}{a(t)} \tag{17}
$$

$$
= \frac{-\left(\frac{a(t)}{n-1} c_t - \frac{n+a(t)-1}{n-1} \delta_t(c)\right) - \alpha}{a(t)} \tag{18}
$$

$$
\geq \frac{-\frac{a(t)}{n-1} c_t + \frac{n+a(t)-1}{n-1} \alpha - \alpha}{a(t)} \tag{19}
$$

$$
= \frac{\alpha - c_t}{n-1}.
$$

To clarify, (17) uses (6), which also uses Lemma .4, and (18) uses (3). Finally, at (19) we use (4).

Furthermore, for the case $r = t$ and $\sum_{\ell \in N} \delta_\ell(c) < E$ we have by (4) that $-\beta \geq \delta_t(c) - c_t$ which implies that $\frac{\delta_t - c_t}{n-1} \leq \frac{-\beta}{n-1}$. Hence, we obtain for all $r \in \{1, ..., \tau\}$ and all $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ the following:

$$
exc^P(S,\sigma) \leq
\begin{cases}
\dfrac{\delta_r(c) - c_r}{n-1} & \text{if } r < t, \\[2ex]
\dfrac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m} & \text{if } r = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) > E, \\[2ex]
\dfrac{-\beta}{n-1} & \text{if } r = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) < E.
\end{cases}
\tag{20}
$$

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Part IIIB: Let $r \in \{1, ..., \tau\}$. The proof is split into two parts, depending on whether $r < t$ or not.

Part IIIB.a: Let $r \in \{1, ..., \tau\}$ and assume that $r < t$. Then, $\mathcal{D}_r = \{S_r, N\backslash\{r\}\}$ and we have for S_r that

$$exc^P(S_r, \sigma) = \frac{0 - \sum_{\ell=1}^{a(r)-1} \sigma_\ell - \sigma_r}{a(r)} \tag{21}$$

$$= \frac{0 - \sum_{\ell=1}^{a(r)-1} \delta_\ell - \delta_r}{a(r)} \tag{22}$$

$$= \frac{-\sum_{\ell=1}^{a(r)-1} \delta_\ell(c) - \frac{n + a(r) - 1}{n - 1}\delta_r(c) + \frac{a(r)}{n-1}\delta_r(c)}{a(r)}$$

$$= \frac{-\sum_{\ell=1}^{a(r)-1} \delta_\ell(c) - \frac{1}{n-1}\left(a(r)c_r - (n-1)\sum_{\ell=1}^{a(r)-1} \delta_\ell(c)\right)}{a(r)} + \frac{\delta_r(c)}{n-1} \tag{23}$$

$$= \frac{\delta_r(c) - c_r}{n - 1}.$$

We clarify the elaboration. At (21), (22) and (23), we use Lemma .4, (5), and (3), respectively.

And for $N\backslash\{r\}$ we have

$$exc^P(N\backslash\{r\}, \sigma) = \frac{v(N\backslash\{r\}) - \sum_{\ell \in N\backslash\{r\}} \sigma_\ell}{n - 1}$$

$$= \frac{E - c_r - (E - \sigma_r)}{n - 1} \tag{24}$$

$$= \frac{\delta_r(c) - c_r}{n - 1}. \tag{25}$$

We clarify the elaboration. At (24) we use Lemma .5 and at (25) we use (5).

To conclude this part: $exc^P(S_r, \sigma) = exc^P(N\backslash\{r\}, \sigma) = \frac{\delta_r(c)-c_r}{n-1}$.

Part IIIB.b: Let $r \in \{1, ..., \tau\}$ and assume that $r = t$. There are two cases, depending on whether $\sum_{\ell \in N} \delta_\ell(c) > E$ or $\sum_{\ell \in N} \delta_\ell(c) < E$.

Part IIIB.b1: Let $r \in \{1, ..., \tau\}$, let $r = t$ and assume that $\sum_{\ell \in N} \delta_\ell(c) > E$. Then, $\mathcal{D}_t = \{S_t, ..., S_n\}$ and by (6) and the construction of S_t we have that $v(S_t) = 0$ and

$$exc^P(S_t, \sigma) = \frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m}.$$

Furthermore, for S_j with $j > t$ we have that

$$\frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m} \geq exc^P(S_j, \sigma) \tag{26}$$

$$\geq \frac{0 - \sum_{\ell \in S_j} \sigma_\ell}{|S_j|} \tag{27}$$

$$= \frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m}. \tag{28}$$

To clarify: (26), (27) and (28), follow from (20), the fact that $v(S_j) \geq 0$ and the definition of S_j, respectively.

To conclude this part: $exc^P(T, \sigma) = \frac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m}$ for all $T \in \mathcal{D}_t$.

Part IIIB.b2: Let $r = t$ and assume $\sum_{\ell \in N} \delta_\ell(c) < E$. Then, $\mathcal{D}_t = \{N \setminus \{t\}, ..., N \setminus \{n\}\}$ and for all $j \geq t$ we have

$$exc^P(N \setminus \{j\}, \sigma) = \frac{v(N \setminus \{j\}) - \sum_{\ell \in N \setminus \{j\}} \sigma_\ell}{n - 1}$$

$$= \frac{E - c_j - (E - \sigma_j)}{n - 1} \tag{29}$$

$$= \frac{-\beta}{n - 1}. \tag{30}$$

We clarify the elaboration. At (29) we use Lemma .5 and at (30) we use $r = t$ together with (5).

To conclude this part: $exc^P(T, \sigma) = \frac{-\beta}{n-1}$ for all $T \in \mathcal{D}_t$.

Part IIIB: From **Part IIIB.a**, **Part IIIB.b1** and **Part IIIB.b2** we obtain for all $r \in \{1, ..., \tau\}$ and all $S \in \mathcal{D}_r$ that

$$exc^P(S, \sigma) = \begin{cases} \dfrac{\delta_r(c) - c_r}{n - 1} & \text{if } r < t, \\[2ex] \dfrac{-\sum_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m} & \text{if } r = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) > E, \\[2ex] \dfrac{-\beta}{n - 1} & \text{if } r = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) < E. \end{cases} \tag{31}$$

Part III: From **Part IIIA**, especially (20), and from **Part IIIB**, especially (31), we

obtain for all $r \in \{1, ..., \tau\}$, all $S \in \mathcal{F}(\bar{\mathcal{D}}_{r-1})$ and all $T \in \mathcal{D}_r$

$$exc^P(S, \sigma) \leq exc^P(T, \sigma) = \begin{cases} \dfrac{\delta_r(c) - c_r}{n - 1} & \text{if } r < t, \\[2ex] \dfrac{-\sum\limits_{\ell=1}^{m-1} \delta_\ell(c) - \alpha}{m} & \text{if } r = t \text{ and } \sum\limits_{\ell \in N} \delta_\ell(c) > E, \\[2ex] \dfrac{-\beta}{n - 1} & \text{if } r = t \text{ and } \sum\limits_{\ell \in N} \delta_\ell(c) < E. \end{cases} \tag{32}$$

Hence, by Theorem 2.6 we obtain $\sigma(N, E, c) = pcn(v_{E,c})$. ∎

In the proof of Theorem 3.9 it is seen that the relevant coalitions of the per capita nucleolus have a special structure for bankruptcy games. This is formalized in the following corollary.

Corollary 3.10. *Let* $(N, E, c) \in BR^N$ *be a bankruptcy problem and let* $x = pcn(v_{E,c})$. *Then there exists a player* $t \in \{1, ..., n\}$ *and a sequence* $\mathcal{D}_1, \ldots, \mathcal{D}_\tau \subset 2^N \setminus \{\varnothing, N\}$, *where* $\tau = \min\{t, n - 1\}$, *that satisfy (A) and (B) of Theorem 2.6, where*

$$\mathcal{D}_r = \big\{ \{1, ..., a(r) - 1\} \cup \{r\}, N \setminus \{r\} \big\},$$

if $r < \tau$. *Furthermore,*

$$\mathcal{D}_\tau = \begin{cases} \{\{1, ..., a(\tau) - 1\} \cup \{\tau\}, N \setminus \{\tau\}\} & \text{if } \tau < t, \\ \{\{1, ..., m - 1\} \cup \{\tau\}, ..., \{1, ..., m - 1\} \cup \{n\}\} & \text{if } \tau = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) > E, \\ \{N \setminus \{\tau\}, ..., N \setminus \{n\}\} & \text{if } \tau = t \text{ and } \sum_{\ell \in N} \delta_\ell(c) < E, \end{cases}$$

for some $m \in \{1, ..., \tau\}$.

4   The claim and right family of bankruptcy rules

This section shows that both the Aumann-Maschler rule and the clights rule belong to the same family of bankruptcy rules: the claim and right family.

Both the Aumann-Maschler rule and the clights rule have two different regimes depending on the size of the estate. For the Aumann-Maschler rule, the estate is considered to be small if the estate is less than half of the total amount claimed and large otherwise. Hence, half of the sum of the claims can be seen as a switch-point for the Aumann-Maschler rule. Moreover, each player receives at most half of his claim in the first regime. Therefore, half of his claim can be seen as his modified claim. On the other hand, in the second regime, half of the claim is considered to be his right, since each player will receive at least half of his claim.

The clights rule has a similar setup. Namely, the estate is considered to be small if the estate is less than the total amount of the clights and the estate is large otherwise. Hence, the switch-point for the clights rule is the sum of the clights. Similar to the Aumann-Maschler rule, the clights act as new claims in the first regime and rights in the second regime. Note that in both the Aumann-Maschler rule and the clights rule the constraint equal award rule is used in the first regime and the constraint equal loss rule in the second.

To show that both the Aumann-Maschler rule and the clight based rule are based on the same conceptual idea, we use the concept of claim and right functions, which are formalized below. Let $\mathcal{C} \subseteq \mathbb{R}_+^N$ be the set of weakly increasing (claim) vectors.

Definition 4.1. *A function* $\lambda : \mathcal{C} \to \mathcal{C}$ *is called a* claim *and* right function *if* $c - \lambda(c) \in \mathcal{C}$ *for all* $c \in \mathcal{C}$. *The class of claim and right functions is denoted by* Λ.

Note that the functions $\lambda(c) = 0$, $\lambda(c) = c$, $\lambda(c) = \frac{1}{2}c$ and $\lambda(c) = \delta(c)$, for all $c \in C$, are all claim and right functions.

With each claim and right function one can define a bankruptcy rule that is based on two regimes. The first regime occurs when the estate is insufficient to cover $\lambda(c)$. In this case, $\lambda(c)$ is viewed as the claim vector rather than c itself. The second regime occurs when the estate is sufficient to cover $\lambda(c)$ and in this case $\lambda(c)$ is considered as a right vector and $c - \lambda(c)$ is considered as the vector of claims in the remaining bankruptcy problem. Subsequently, within the first regime, the constrained equal award rule is used and within the second regime, the constrained equal loss rule is used. The resulting family of rules is called the claim and right family and is formally defined as follows.

Definition 4.2. *Let* $\lambda \in \Lambda$ *be a claim and right function. The* claim and right bankruptcy rule CR^λ *is defined by*

$$
CR^\lambda(N, E, c) = \begin{cases} CEA(N, E, \lambda(c)) & \text{if } \sum_{i \in N} \lambda_i(c) \geq E, \\ \lambda(c) + CEL(N, E - \sum_{i \in N} \lambda_i(c), c - \lambda(c)) & \text{if } \sum_{i \in N} \lambda_i(c) < E, \end{cases}
$$

for all bankruptcy problems $(N, E, c) \in BR^N$.

Using the four examples of claim and right functions discussed above, we have the following.

Theorem 4.3. CEA, CEL, AM *and* σ *are claim and right bankruptcy rules.*

As a final remark, we want to state that is readily seen that the claim and right family of bankruptcy rules coincides with the increasing-constant-increasing family of bankruptcy rules introduced by Thomson (2008).

Appendix: The lemmas used in the proof of Theorem 3.9

Lemma .4. *Let $(N, E, c) \in BR^N$ be a bankruptcy problem and let $\delta(c)$, $a(i)$, and t defined as in (2), (3), and (4), respectively. Let $t < n$. Then, for all*

$$
i \in
\begin{cases}
\{1, \ldots, n-1\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) > E, \\
\{1, \ldots, t-1\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) < E,
\end{cases}
$$

it holds that

$$
v_{E,c}(\{1, \ldots, a(i) - 1\} \cup \{i\}) = 0.
$$

Proof. Let $k = n - 1$ if $\sum_{\ell \in N} \delta_\ell(c) > E$, and let $k = t - 1$ if $\sum_{\ell \in N} \delta_\ell(c) < E$. Proving that $v_{E,c}(\{1, ..., a(i) - 1\} \cup \{i\}) = 0$ for $i = k$ implies that it holds for all $i \in \{1, ..., k\}$. The worth of the coalition is given by

$$
v_{E,c}(\{1, ..., a(k) - 1\} \cup \{k\}) = \max\{0, E - \sum_{i=a(k)}^{k-1} c_i - \sum_{i=k+1}^{n} c_i\},
$$

hence, proving that $E - \sum_{i=a(k)}^{k-1} c_i - \sum_{i=k+1}^{n} c_i \leq 0$ is sufficient.
First we will prove that

$$
E \leq \sum_{i=1}^{k} \delta_i(c) + \sum_{i=k+1}^{n} c_i - (n-k)(c_k - \delta_k(c)). \tag{33}
$$

If $\sum_{\ell \in N} \delta_\ell(c) > E$, so $k = n - 1$, we have

$$
\begin{aligned}
E &\leq \sum_{i \in N} \delta_i(c) \\
&= \sum_{i=1}^{n-1} \delta_i(c) + \delta_n(c) \\
&\leq \sum_{i=1}^{n-1} \delta_i(c) + c_n - c_{n-1} + \delta_{n-1}(c), \tag{34}
\end{aligned}
$$

where (34) follows from Lemma 3.8.
If $\sum_{\ell \in N} \delta_\ell(c) < E$, so $k = t - 1$, we have

$$
\begin{aligned}
E &= \sum_{i=1}^{t-1} \delta_i(c) + \sum_{i=t}^{n} (c_i - \beta) \\
&= \sum_{i=1}^{t-1} \delta_i(c) + \sum_{i=t}^{n} c_i - (n - t + 1)\beta \\
&< \sum_{i=1}^{t-1} \delta_i(c) + \sum_{i=t}^{n} c_i - (n - t + 1)(c_{t-1} - \delta_{t-1}(c)), \tag{35}
\end{aligned}
$$

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where (35) follows from (4).

Now we will prove that $E - \sum_{i=a(k)}^{k-1} c_i - \sum_{i=k+1}^{n} c_i \leq 0$:

$$E - \sum_{i=a(k)}^{k-1} c_i - \sum_{i=k+1}^{n} c_i$$

$$\leq \sum_{i=1}^{k} \delta_i(c) + \sum_{i=k+1}^{n} c_i - (n-k)(c_k - \delta_k(c)) - \sum_{i=a(k)}^{k-1} c_i - \sum_{i=k+1}^{n} c_i \qquad (36)$$

$$= \sum_{i=1}^{a(k)-1} \delta_i(c) + \sum_{i=a(k)}^{k-1} \delta_i(c) + \delta_k(c) + (n-k)\delta_k(c) + (k-n)c_k - \sum_{i=a(k)}^{k-1} c_i$$

$$= \sum_{i=1}^{a(k)-1} \delta_i(c) + (n-k+1)\delta_k(c) + (k-n)c_k - \sum_{i=a(k)}^{k-1} (c_i - \delta_i(c))$$

$$\leq \sum_{i=1}^{a(k)-1} \delta_i(c) + (n-k+1)\delta_k(c) + (k-n)c_k$$

$$= \sum_{i=1}^{a(k)-1} \delta_i(c) + (n-k+1)\left(\frac{a(k)}{n+a(k)-1}c_k - \frac{n-1}{n+a(k)-1}\sum_{i=1}^{a(k)-1}\delta_i(c)\right) + (k-n)c_k \qquad (37)$$

$$= \left(\frac{(n-k+1)a(k)}{n+a(k)-1} + (k-n)\right)c_k + \left(\frac{(n-k+1)(1-n)}{n+a(k)-1} + 1\right)\sum_{i=1}^{a(k)-1}\delta_i(c)$$

$$= \left(\frac{a(k)+(k-n)(n-1)}{n+a(k)-1}\right)c_k + \left(\frac{a(k)+(k-n)(n-1)}{n+a(k)-1}\right)\sum_{i=1}^{a(k)-1}\delta_i(c)$$

$$\leq 0. \qquad (38)$$

We clarify the elaboration. At (36), (37), and (38) we have used (33), (3), and the fact that $a(k) \leq k < n$, respectively. ∎

Lemma .5. *Let $(N, E, c) \in BR^N$ be a bankruptcy problem with $\delta(c)$ and t defined as in (2) and (4) respectively. Then, for all*

$$i \in \begin{cases} \{1, \ldots, t-1\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) > E, \\ \{1, \ldots, n-1\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) < E \text{ and } t = n, \\ \{1, \ldots, n\} & \text{if } \sum_{\ell \in N} \delta_\ell(c) < E \text{ and } t < n, \end{cases}$$

it holds that

$$E - c_i \geq 0$$

Proof. Note that proving $E \geq c_i$ for $i = t - 1$ (or $i = n - 1$ or $i = n$) implies that it holds for all $i \in \{1, ..., t-1\}$ (or $i \in \{1, ..., n-1\}$ or $i \in N$). The proof

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is split into two cases depending on whether $\sum_{\ell \in N} \delta_\ell(c) > E$ or $\sum_{\ell \in N} \delta_\ell(c) < E$.

Case 1: Assume $\sum_{\ell \in N} \delta_\ell(c) > E$. Then by (4), we have

$$E > \sum_{\ell=1}^{t-1} \delta_\ell(c) + \sum_{\ell=t}^{n} \delta_{t-1}(c). \tag{39}$$

By (2) we have that

$$\delta_{t-1}(c) \geq \frac{t-1}{n+t-1-1} c_{t-1} - \frac{n-1}{n+t-1-1} \sum_{\ell=1}^{t-1-1} \delta_\ell(c),$$

which can be rewritten to

$$\sum_{\ell=1}^{t-2} \delta_\ell(c) \geq \frac{t-1}{n-1} c_{t-1} - \frac{n+t-2}{n-1} \delta_{t-1}(c). \tag{40}$$

Now we will prove that $E - c_{t-1} \geq 0$:

$$E - c_{t-1} > \sum_{\ell=1}^{t-2} \delta_\ell(c) + (n-t+2)\delta_{t-1}(c) - c_{t-1} \tag{41}$$

$$\geq \frac{t-1}{n-1} c_{t-1} - \frac{n+t-2}{n-1}\delta_{t-1}(c) + (n-t+2)\delta_{t-1}(c) + \frac{1-n}{n-1} c_{t-1} \tag{42}$$

$$= \frac{t-n}{n-1} c_{t-1} - \frac{n+t-2}{n-1}\delta_{t-1}(c) + \frac{n^2-nt+2n-n+t-2}{n-1}\delta_{t-1}(c)$$

$$= \frac{t-n}{n-1} c_{t-1} + \frac{n^2-nt}{n-1}\delta_{t-1}(c)$$

$$= \frac{n-t}{n-1}(n\delta_{t-1}(c) - c_{t-1})$$

$$\geq 0. \tag{43}$$

We clarify the elaboration. At (41), (42), and (43), we use (39), (40), and Lemma 3.3, respectively.

Case 2: Assume $\sum_{\ell \in N} \delta_\ell(c) < E$. Again, this case is split into two parts, depending whether $t = n$ or $t < n$.

Case 2a: We have $\sum_{\ell \in N} \delta_\ell(c) < E$ and assume that $t = n$. Then

$$E - c_{n-1} > \sum_{\ell \in N} \delta_\ell(c) - c_{n-1}$$

$$\geq \sum_{j=1}^{n-2} \delta_j(c) + 2\delta_{n-1}(c) - c_{n-1} \tag{44}$$

$$\geq \sum_{j=1}^{n-2} \delta_j(c) + 2\left(\frac{1}{2}c_{n-1} - \frac{1}{2}\sum_{j=1}^{n-2} \delta_j(c)\right) - c_{n-1} \tag{45}$$

$$= 0.$$

To clarify: at (44) we use Corollary 3.2 and at (45) we use (2).

Case 2b: We have $\sum_{\ell \in N} \delta_\ell(c) < E$ and assume that $t < n$. Then

$$
\begin{aligned}
E &= \sum_{\ell=1}^{t-1} \delta_\ell(c) + \sum_{\ell=t}^{n} (c_\ell - \beta) \\
&= \sum_{\ell=1}^{t-1} \delta_\ell(c) + c_n - \beta + \sum_{\ell=t}^{n-1} (c_\ell - \beta) \\
&\geq \sum_{\ell=1}^{t-1} \delta_\ell(c) + c_n - (c_{n-1} - \delta_{n-1}(c) + \sum_{\ell=t}^{n-1} (c_\ell - (c_\ell - \delta_\ell(c))) \quad (46) \\
&= \sum_{j=1}^{n-2} \delta_j(c) + 2\delta_{n-1}(c) + c_n - c_{n-1}.
\end{aligned}
$$

We clarify the elaboration. At (46) we use (4) together with the fact that $c_{i-1} - \sigma_{i-1} \leq c_i - \sigma_i$ for all $i \in \{2, ..., n\}$ (Lemma 3.8). Using this we have

$$
\begin{aligned}
E - c_n &\geq \sum_{j=1}^{n-2} \delta_j(c) + 2\delta_{n-1}(c) + c_n - c_{n-1} - c_n \\
&= \sum_{j=1}^{n-2} \delta_j(c) + 2\delta_{n-1}(c) - c_{n-1} \\
&\geq 0. \quad (47)
\end{aligned}
$$

We clarify the elaboration: At (47) we use Case 2a. ∎

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