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GEOMETRIC ASPECTS OF 2-WALK-REGULAR GRAPHS

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ABSTRACT. A t-walk-regular graph is a graph for which the number of walks of given length between two vertices depends only on the distance between these two vertices, as long as this distance is at most t. Such graphs generalize distance-regular graphs and t-arc-transitive graphs. In this paper, we will focus on 1- and in particular 2-walk-regular graphs, and study analogues of certain results that are important for distance-regular graphs. We will generalize Delsarte's clique bound to 1-walk-regular graphs, Godsil's multiplicity bound and Terwilliger's analysis of the local structure to 2-walk-regular graphs. We will show that 2-walk-regular graphs have a much richer combinatorial structure than 1-walk-regular graphs, for example by proving that there are finitely many non-geometric 2-walk-regular graphs with given smallest eigenvalue and given diameter (a geometric graph is the point graph of a special partial linear space); a result that is analogous to a result on distance-regular graphs. Such a result does not hold for 1-walk-regular graphs, as our construction methods will show.

1. INTRODUCTION

Walk-regular graphs were introduced by Godsil and McKay [26] in their study of cospectral graphs. They showed that the property that the vertex-deleted subgraphs of a graph Γ are all cospectral is equivalent to the property that the number of closed walks of a given length ℓ in Γ is independent of the starting vertex, for every ℓ. They also observed that walk-regular graphs generalize both vertex-transitive graphs and distance-regular graphs. Distance-regular graphs [5, 16] play a crucial role in the area of algebraic combinatorics, and it was shown by Rowlinson [36] that such graphs can be characterized in terms of the numbers of walks between two vertices; in particular that this number only depends on their length and the distance between the two vertices. Motivated by this characterization, Dalfó, Fiol, and Garriga [22, 10] introduced t-walk-regular graphs; such graphs have the property of Rowlinson's characterization at least for those vertices that are at distance at most t. These t-walk-regular graphs were further studied by Dalfó, Fiol, and coauthors [11, 12, 13, 14]. Dalfó, Van Dam, and Fiol [13] characterized t-walk-regular graphs

in terms of the cospectrality of certain perturbations, thus going back to the roots of walk-regular graphs. Dalfó, Van Dam, Fiol, Garriga, and Gorissen [14] among others raised the question of when t-walk-regularity implies distance-regularity.

Our motivation for studying t-walk-regular graphs lies in the generalization of distance-regular graphs. In order to better understand the latter, we would like to know which results for these graphs can be generalized to t-walk-regular graphs. In this way, we aim to have a better understanding of which properties of distance-regular graphs are most relevant.

Here we will focus on 1- and in particular 2-walk-regular graphs. We will for example generalize Delsarte's clique bound [17] to 1-walk-regular graphs. It seems however that 1-walk-regularity is still far away from distance-regularity, but going to 2-walk-regularity is an important step (or jump) forward. Indeed, we will see that several important results on distance-regular graphs have interesting generalizations to 2-walk-regular graphs (but not to 1-walk-regular graphs), such as Godsil's multiplicity bound [23] and Terwilliger's analysis of the local structure [39]. On the other hand, there are very basic construction methods for 1-walk-regular graphs that cannot be generalized to 2-walk-regular graphs; indeed, most known examples of the latter come from groups as graphs that are obtained in an elementary way (such as the line graph and halved graph) from s-arc-transitive graphs. We will indeed show that 2-walk-regular graphs have a much richer combinatorial structure than 1-walk-regular graphs. We will show that there are finitely many non-geometric 2-walk-regular graphs with given smallest eigenvalue and given diameter (a geometric graph is the point graph of a special partial linear space); a result that is analogous to a result on distance-regular graphs. In fact, this result shows that the class of 2-walk-regular graphs is quite limited. Again, such a result does not hold for 1-walk-regular graphs, as our construction methods (Proposition 3.5, in particular) will show.

This paper is organized as follows: in the next section, we give some technical background. In Section 3, we give elementary construction methods for t-walk-regular graphs that we will use in the remaining sections. In Section 4, Godsil's multiplicity bound for distance-regular graphs is generalized to 2-walk-regular graphs. Similarly we generalize in Section 5 Terwilliger's analysis of local graphs. In Section 6, we study t-walk-regular graphs with an eigenvalue with small multiplicity. Finally, in Section 7, we generalize Delsarte's clique bound and study geometric 2-walk-regular graphs.

2. Preliminaries

Let Γ be a connected graph with vertex set $V = V(\Gamma)$ and denote $x \sim y$ if the vertices $x, y \in V$ are adjacent. The *distance* $\mathrm{dist}_\Gamma(x, y)$ between two vertices $x, y \in V$ is the length of a shortest path connecting x and y (we omit the index Γ when this is clear from the context). The maximum distance between two vertices in Γ is the *diameter* $D = D(\Gamma)$. We use $\Gamma_i(x)$ for the set of vertices at distance i from x and write, for the sake of simplicity, $\Gamma(x) := \Gamma_1(x)$. The *degree* of x is the number $|\Gamma(x)|$ of vertices adjacent to it. A graph is *regular* with *valency* k if the degree of each of its vertices is k.

A graph Γ is called *bipartite* if it has no odd cycle. For a connected graph Γ, the *bipartite double* $\widetilde{\Gamma}$ of Γ is the graph whose vertices are the symbols $x^+, x^-(x \in V)$ and where x^+ is adjacent to y^- if and only of x is adjacent to y in Γ.

Let $\overline{\Gamma}$ be a graph with vertex set $V(\overline{\Gamma})$. Let Γ be a graph whose vertices are partitioned in $|V(\overline{\Gamma})|$ classes of the same size. We say that Γ is a *cover* of $\overline{\Gamma}$ if the following three properties hold: The vertices of each class induce an empty graph in Γ; the classes give an equitable partition in Γ (that is, for every two classes, every vertex in one of these classes has the same number of neighbors in the other class); and the quotient graph provided by the classes (that is, the graph on the classes, where two classes are adjacent if there are edges (of Γ) between them) is isomorphic to $\overline{\Gamma}$. This quotient graph is also called the folded graph of Γ.

Given a graph Γ and $x \in V$, the *local graph* $\Delta(x)$ *at vertex* x is the subgraph of Γ induced on the vertices that are adjacent to x. When all the local graphs are isomorphic, we simply write Δ, and say that Γ is locally Δ.

For a connected graph Γ with diameter D, the *distance-i graph* Γ_i of Γ ($1 \leq i \leq D$) is the graph whose vertices are those of Γ and whose edges are the pairs of vertices at mutual distance i in Γ. In particular, $\Gamma_1 = \Gamma$. The *distance-i matrix* $A_i = A_i(\Gamma)$ is the matrix whose rows and columns are indexed by the vertices of Γ and the (x,y)-entry is 1 whenever $\mathrm{dist}(x,y) = i$ and 0 otherwise. The *adjacency matrix* A of Γ equals A_1.

The *eigenvalues* of the graph Γ are the eigenvalues of A. We use $\{\theta_0 > \cdots > \theta_d\}$ for the set of distinct eigenvalues of Γ. The *multiplicity* of an eigenvalue θ is denoted by $m(\theta)$. Note that if Γ is connected and regular with valency k, then $\theta_0 = k$ and $m(\theta_0) = 1$. Let $\{v_1, \ldots, v_{m(\theta)}\}$ be an orthonormal basis of eigenvectors with eigenvalue θ, and let U be a matrix whose columns are these vectors. Then the matrix $E_\theta = UU^\top$ is called a *minimal idempotent* associated to θ. We abbreviate E_{θ_i} by E_i ($i = 0, \ldots, d$).

Fiol and Garriga [22] introduced t-walk-regular graphs as a generalization of both distance-regular and walk-regular graphs. A graph is t-*walk-regular* if the number of walks of every given length ℓ between two vertices $x, y \in V$ only depends on the distance between them, provided that $\mathrm{dist}(x,y) \leq t$ (where it is implicitly assumed that the diameter of the graph is at least t). The 'Spectral Decomposition Theorem' leads immediately to

$$A^\ell = \sum_{i=0}^{d} \theta_i^\ell E_i.$$

From that, we obtain that a graph is t-walk-regular if and only if for every minimal idempotent the (x,y)-entry only depends on $\mathrm{dist}(x,y)$, provided that the latter is at most t (see Dalfó, Fiol, and Garriga [10]). In other words, for a fixed eigenvalue θ with minimal idempotent E, there exist constants $\alpha_j := \alpha_j(\theta)$ ($0 \leq j \leq t$), such that $A_j \circ E = \alpha_j A_j$, where $\circ$ is the entrywise product.

If a graph comes from one of the relations in an association scheme (see Brouwer, Cohen, and Neumaier [5]), then the minimal idempotents of the graph as described above do not have to be the same as the minimal idempotents in the scheme. Mainly for this purpose, we will develop our theory somewhat more general than seems necessary at first. In particular, we define a t-*walk-regular idempotent* as a nonzero idempotent E such that $A_j \circ E = \alpha_j A_j$ for certain constants α_j ($0 \leq j \leq t$). We call E a t-*walk-regular idempotent for eigenvalue* θ if moreover $AE = \theta E$ holds.

If E is a t-walk-regular idempotent with rank m, then clearly its diagonal elements α_0 are positive, and we can write $E = UU^\top$, where the m columns of U form an orthonormal basis of the eigenspace of E for its eigenvalue 1 (note that we do not explicitly require U to contain eigenvectors of A, as we did above). For every

vertex $x \in V$ we now denote by $\hat{x}$ the x-th row of U. The map $x \mapsto \hat{x}$ is called a *representation* of Γ. Note that the vectors $\hat{x}$ ($x \in V$) all have the same length (the square of which is $E_{xx} = \alpha_0$); in this case we call the representation *spherical*. Given two vertices $x, y \in V$, we will often refer to $u_{xy} := E_{xy}/\alpha_0$ as the *xy-cosine*, as it can be interpreted as the cosine of the angle between the vectors $\hat{x}$ and $\hat{y}$. We remark that if Γ is t-walk-regular and $\text{dist}(x,y) = s \le t$, then $u_{xy} = \alpha_s/\alpha_0$ does not depend on x and y, but only on s. In this case, we write $u_s := \alpha_s/\alpha_0$.

Given a vertex x in a graph Γ and a vertex y at distance j from x, we consider the numbers $a_j(x,y) = |\Gamma(y) \cap \Gamma_j(x)|$, $b_j(x,y) = |\Gamma(y) \cap \Gamma_{j+1}(x)|$, and $c_j(x,y) = |\Gamma(y) \cap \Gamma_{j-1}(x)|$. A graph Γ with diameter D is *distance-regular* if these parameters do not depend on x and y, but only on j, for $0 \le j \le D$. If this is the case then these numbers are denoted simply by a_j, b_j, and c_j, for $0 \le j \le D$, and they are called the *intersection numbers* of Γ. Also, if a graph Γ is t-walk-regular, then the intersection numbers are well-defined for $0 \le j \le t$, as they do not depend on x nor on the chosen $y \in \Gamma_j(x)$ (see Dalfó et al. [14, Prop. 3.15]). More generally, let x and y be two vertices at distance h in a t-walk-regular graph. Then the numbers $p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$ exist (i.e., they only depend on h, i and j) for nonnegative integers $h, i, j \le t$. This follows from working out the product $A_i A_j \circ A_h$, for example; see also Dalfó, Fiol, and Garriga [12, Prop. 1]. Moreover, if $k_h = |\Gamma_h(x)|$, then relations such as $k_h p_{ij}^h = k_i p_{hj}^i$ hold. From the above it is clear that a D-walk-regular graph is distance-regular. For such a graph (and a minimal idempotent for θ), the sequence $(u_0, \ldots, u_D)$ is known as the *standard sequence* of Γ with respect to θ.

Now let $E = UU^\top$ be a t-walk-regular idempotent for eigenvalue θ, with $t \ge 1$. From $AE = \theta E$ and $U^\top U = I$, it follows that $AU = \theta U$, so in this case the columns of U are clearly also eigenvectors of A (even though we did not require this in the definition; note also that U does not necessarily contain a basis of the eigenspace of A for eigenvalue θ). For the corresponding representation, this implies that

$$(1) \qquad \theta \hat{x} = \sum_{y \sim x} \hat{y}.$$

By looking at an (x,y)-entry with $\text{dist}(x,y) = j$ in the equation $AE = \theta E$ we obtain the following relations:

$$(2) \qquad k\alpha_1 = \theta\alpha_0$$

$$(3) \qquad c_j\alpha_{j-1} + a_j\alpha_j + b_j\alpha_{j+1} = \theta\alpha_j \qquad (1 \le j \le t-1).$$

In particular, θ follows from the first cosine (given k; and the other way around).

3. Construction methods

Highly symmetric examples of t-walk-regular graphs exist for $t \le 7$ in the form of t-arc-transitive graphs. For example, the infinite family of 3-arc-transitive graphs constructed by Devillers, Giudici, Li, and Praeger [19] is also an infinite family of 3-walk-regular graphs. Indeed, every t-arc-transitive graph with diameter at least t is t-walk-regular. By a covering construction due to Conway (see [2, Ch. 19]) and independently Djoković [21], infinite families of 5-arc-transitive graphs with valency 3 and 7-arc-transitive graphs with valency 4 were constructed. Conder and Walker [8] also constructed infinitely many 7-arc-transitive graphs with valency 4. In turn, these give rise to infinite families of cubic 5-walk-regular graphs and 7-walk-regular

graphs with valency 4. The validity of the Bannai-Ito conjecture [1] (in particular
the fact that there are finitely many distance-regular graphs with valency four [6])
for example implies that there are infinitely many 7-walk-regular graphs that are
not distance-regular.

It is worth mentioning that less-known (and less restrictive) concepts such as t-geodesic-transitivity and t-distance-transitivity have been introduced by Devillers,
Jin, Li, and Praeger [20], and both concepts are stronger than t-walk-regularity.

It is rather straightforward to show that the bipartite double of a t-arc-transitive
graph is again t-arc-transitive. This could for example be applied to the infinite
family of non-bipartite 2-arc-transitive graphs constructed by Nochefranca [35], to
obtain also an infinite family of bipartite such graphs. For t-walk-regular graphs, a
similar result holds, but we have to take into account the odd-girth (note that for
t-arc-transitive graphs with diameter at least t, the odd-girth is at least $2t + 1$).

Proposition 3.1. *Let Γ be a t-walk-regular graph with odd-girth $2s + 1$. Then the
bipartite double of Γ is $\min\{s, t\}$-walk-regular.*

Proof. Let $r = \min\{s, t\}$, and let N be the adjacency matrix of Γ, with minimal
idempotents E_i ($i = 0, \ldots, d$). Then the bipartite double $\widetilde{\Gamma}$ of Γ has adjacency
matrix
$$A = \begin{bmatrix} O & N \\ N & O \end{bmatrix}$$
with (not necessarily minimal) idempotents
$$\frac{1}{2} \begin{bmatrix} E_i & \pm E_i \\ \pm E_i & E_i \end{bmatrix} \quad (i = 0, \ldots, d).$$
Combinatorially it means that for every vertex x of Γ, there are two vertices, x^+, x^-,
of $\widetilde{\Gamma}$ (one in each of the stable sets) and x^+ is adjacent to y^- in $\widetilde{\Gamma}$ if and only if x
is adjacent to y in Γ.

We will now show that each distance matrix of $\widetilde{\Gamma}$ can be expressed in terms of
the corresponding distance matrix of Γ, at least up to distance r. Indeed, first of
all, if ℓ is even and $\text{dist}_\Gamma(x, y) = \ell$, then also $\text{dist}_{\widetilde{\Gamma}}(x^\epsilon, y^\epsilon) = \ell$, where the symbol ϵ
can be $+$ or $-$. If moreover $\ell \leq r$, then $\text{dist}_{\widetilde{\Gamma}}(x^\epsilon, y^{-\epsilon}) \geq 2s + 1 - \ell \geq r + 1$ (the first
inequality follows because otherwise there would be a walk between x and y in Γ
of odd length less than $2s + 1 - \ell$, which together with the walk of even length ℓ
would give a closed walk of odd length less than $2s + 1$, a contradiction). Similarly,
it follows that if ℓ is odd and $\text{dist}_\Gamma(x, y) = \ell$, then also $\text{dist}_{\widetilde{\Gamma}}(x^\epsilon, y^{-\epsilon}) = \ell$, and if
moreover $\ell \leq r$, then $\text{dist}_{\widetilde{\Gamma}}(x^\epsilon, y^\epsilon) \geq r + 1$. Altogether this shows that for $\ell \leq r$ we
have that
$$A_\ell = \begin{bmatrix} O & N_\ell \\ N_\ell & O \end{bmatrix} \quad (\ell \text{ odd}) \quad \text{and} \quad A_\ell = \begin{bmatrix} N_\ell & O \\ O & N_\ell \end{bmatrix} \quad (\ell \text{ even}),$$
where N_ℓ is the distance-ℓ matrix of Γ. By taking entrywise products of these
matrices with the idempotents, the result now follows. $\square$

The graph on the flags of the 11-point biplane as described by Dalfó, Van Dam,
Fiol, Garriga, and Gorissen [14] and characterized by Blokhuis and Brouwer [3] (see
also Figure 1) is 3-walk-regular with odd-girth 5, so its bipartite double is 2-walk-regular (and it is not 3-walk-regular). Proposition 3.1 also shows that the bipartite
double of the dodecahedral graph is 2-walk-regular because the dodecahedral graph
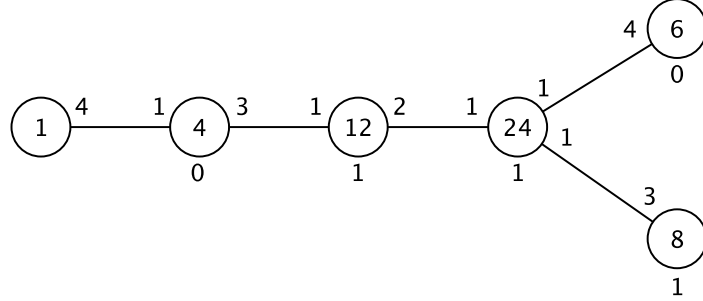
FIGURE 1. Distance distribution diagram of a graph on the flags of the eleven point biplane

is 5-walk-regular with odd-girth 5. This bipartite double is even 3-walk-regular, however.

The following result is in some sense going in the opposite direction.

Proposition 3.2. *Let $t \geq 2$, and let Γ be a t-walk-regular graph with valency k and odd-girth $2s + 1$. If Γ is not complete multipartite, then the distance-2 graph Γ_2 of Γ is $\min\{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor\}$-walk-regular.*

Proof. Let $r = \min\{\lfloor s/2 \rfloor, \lfloor t/2 \rfloor\}$, and let A be the adjacency matrix of Γ. Because Γ is 2-walk-regular, the adjacency matrix $B = A_2$ of Γ_2 satisfies

$$B = \frac{1}{c_2}(A^2 - a_1 A - kI)$$

and hence it has the minimal idempotents of A as (not necessarily minimal) idempotents. This, together with the claim that $B_i = A_{2i}$ for $i = 0, \ldots, r$ proves the desired result, as long as Γ_2 is connected.

In order to prove this claim, consider two arbitrary vertices x and y of Γ. If $\mathrm{dist}_\Gamma(x, y) = 2j \leq 2r$, then $\mathrm{dist}_{\Gamma_2}(x, y) = j$. If $\mathrm{dist}_\Gamma(x, y) = 2j + 1 < 2r + 1$, then $\mathrm{dist}_{\Gamma_2}(x, y) \geq 2r - j > r$ (where the first inequality follows because otherwise this would give a walk in Γ between x and y of even length less than $2(2r - j)$, which together with the odd walk of length $2j+1$ gives a closed walk of odd length less than $4r + 1$, a contradiction). Finally, also if $\mathrm{dist}_\Gamma(x, y) \geq 2r + 1$, then $\mathrm{dist}_{\Gamma_2}(x, y) > r$, which shows the claim.

The assumption that Γ is not complete multipartite nor bipartite (because it has finite odd-girth) ensures that Γ_2 is connected. Indeed, if Γ_2 is not connected, then by using that the largest eigenvalue of $B = \frac{1}{c_2}(A^2 - a_1 A - kI)$ occurs with multiplicity at least two, we find that $a_1 - k$ must be an eigenvalue of Γ. Because Γ is not bipartite, this implies that $a_1 \neq 0$. We may also assume that $a_2 = 0$, because otherwise $p_{22}^1 \neq 0$, which implies that Γ_2 is connected. Thus, if the diameter of Γ is two, then Γ is complete multipartite. But if the diameter is at least three, then consider vertices x and y_i ($i = 1, 2, 3$) such that $y_i \in \Gamma_i(x)$ and $y_1 \sim y_2 \sim y_3$. Then $0 < a_1 = |\Gamma(y_2) \cap \Gamma(y_3)| \leq |\Gamma_2(x) \cap \Gamma(y_2)| + |\Gamma_2(y_1) \cap \Gamma(y_3)| = 2a_2 = 0$ (cf. [5, Prop. 5.5.1]), a contradiction.   □

For example, the distance-2 graph of the dodecahedral graph is 1-walk-regular but not 2-walk-regular. Another example comes from the Biggs-Smith graph, whose distance-2 graph is 2-walk-regular.

We remark that the halved graphs of a bipartite graph are degenerate cases of the distance-2 graph. We thus obtain the following.

Corollary 3.3. *Let $t \geq 2$, and let Γ be a t-walk-regular bipartite graph. Then the halved graphs of Γ are $\lfloor t/2 \rfloor$-walk-regular.*

Using that the minimal idempotents of the line graph of a regular graph are easily deduced from the minimal idempotents of the graph, we obtain the following.

Proposition 3.4. *Let $t \geq 0$. Let Γ be a $(t+1)$-walk-regular graph with valency k and girth larger than $2t+1$. Then the line graph of Γ is t-walk-regular.*

Proof. Let $k = \theta_0 > \cdots > \theta_d$ be the distinct eigenvalues of Γ. Consider an eigenvalue $\theta_i \neq -k$ and let E_i be the minimal idempotent associated to θ_i. Let N be the vertex-edge incidence matrix of Γ, then $A = NN^\top - kI$ is the adjacency matrix of Γ, and $B = N^\top N - 2I$ is the adjacency matrix of the line graph $L(\Gamma)$ of Γ. It is well-known and easy to check that if u is an eigenvector of A with eigenvalue $\theta_i \neq -k$, then $N^\top u$ is an eigenvector of B with eigenvalue $\sigma_i = \theta_i + k - 2$. It follows that $F_{\sigma_i} = \frac{1}{k+\theta_i} N^\top E_i N$ is a minimal idempotent for $L(\Gamma)$ with corresponding eigenvalue σ_i. Moreover, if -2 is an eigenvalue of $L(\Gamma)$, then

$$(4) \qquad F_{-2} = I - \sum_{\sigma_i \neq -2} F_{\sigma_i}$$

completes the set of all minimal idempotents of $L(\Gamma)$.

Let $e_1 = uv$ and $e_2 = xy$ be two edges in Γ, that is, two vertices in $L(\Gamma)$. Assume that $\operatorname{dist}_{L(\Gamma)}(e_1, e_2) = s \leq t$, then, as the girth is larger than $2s+1$, we can assume without loss of generality that $\operatorname{dist}_\Gamma(u, y) = s+1$, $\operatorname{dist}_\Gamma(u, x) = \operatorname{dist}_\Gamma(v, y) = s$, and $\operatorname{dist}_\Gamma(v, x) = s - 1$ (except for the case $s = 0$; then $\operatorname{dist}_\Gamma(v, x) = 1$). Let $\sigma_i \neq -2$, $F = F_{\sigma_i}$, and $E = E_i$, then the (e_1, e_2)-entry of F will be $F_{e_1 e_2} = \frac{1}{k+\theta_i}(E_{ux} + E_{uy} + E_{vx} + E_{vy})$, which does not depend on the chosen e_1 and e_2, but only on the distance s between them in $L(\Gamma)$. As this holds for every $\sigma_i \neq -2$, by (4) also the entries of F_{-2} depend only on distance. So we conclude that $L(G)$ is t-walk-regular. $\qquad\square$

An example is the already mentioned graph on the flags of the 11-point biplane whose distance distribution diagram is in Figure 1. Since this graph has girth 5 and it is 3-walk-regular (and therefore 2-walk-regular), its line graph is 1-walk-regular (and not 2-walk-regular). This shows that the condition on the girth is necessary. Also the line graphs of s-arc-transitive graphs (with large girth) provide new examples of t-walk-regular graphs. Note by the way that the line graph of a $(t+1)$-arc-transitive graph with valency at least 3 is not t-arc-transitive (for $t \geq 2$), since it has triangles.

We will proceed with a straightforward construction method for 1-walk-regular graphs. Let us first recall the coclique extension of a graph Γ, that is, the graph with adjacency matrix $A \otimes J$, where A is the adjacency matrix of Γ, J is a square all-ones matrix and $\otimes$ stands for the Kronecker product. It is fairly easy to see (combinatorially) that if Γ is a 1-walk-regular graph, then also every coclique extension of Γ is 1-walk-regular. A variation on the coclique extension is the Kronecker product

$\Gamma \otimes \Gamma'$ of two graphs Γ and Γ', that is, the graph with adjacency matrix $A \otimes B$, where A and B are the adjacency matrices of Γ and Γ'.

Proposition 3.5. *Let Γ and Γ' be two 1-walk-regular graphs. Then the Kronecker product $\Gamma \otimes \Gamma'$ is 1-walk-regular.*

Proof. Let A and B be the adjacency matrices of Γ and Γ', and let E_i and F_j be the respective minimal idempotents, say with $AE_i = \theta_i E_i$, $A \circ E_i = \gamma_i A$ $(i = 0, \ldots, d)$, $BF_j = \theta'_j F_j$, and $B \circ F_j = \beta_j B$ $(j = 0, \ldots, d')$. It follows from standard multiplication rules of the Kronecker product that $E_i \otimes F_j$ $(i = 0, \ldots, d; j = 0, \ldots, d')$ are idempotents of $\Gamma \otimes \Gamma'$, with eigenvalues $\theta_i \theta'_j$ (see [27]). We remark that these are not necessarily the minimal idempotents, because some eigenvalues $\theta_i \theta'_j$ may coincide. If the latter is the case, then the corresponding minimal idempotent is a sum of idempotents of the form $E_i \otimes F_j$.

First of all, $E_i \otimes F_j$ has constant diagonal because both Γ and Γ' are walk-regular, and so every minimal idempotent has constant diagonal. So $\Gamma \otimes \Gamma'$ is also walk-regular.

Secondly, it follows that $(A \otimes B) \circ (E_i \otimes F_j) = (A \circ E_i) \otimes (B \circ F_j) = \gamma_i \beta_j A \otimes B$, which shows that $\Gamma \otimes \Gamma'$ is 1-walk-regular. $\qquad \square$

Note that it was already observed by Godsil and McKay [26, Thm. 4.5] that several kinds of products, such as the Kronecker product and the Cartesian product of walk-regular graphs are again walk-regular. Still, the Cartesian product (or sum [9]) $\Gamma \oplus \Gamma'$ of two 1-walk-regular graphs Γ and Γ', that is, the graph with adjacency matrix $A \otimes I + I \otimes B$, is in general not 1-walk-regular. However, the particular case $\Gamma \oplus \Gamma$ is again 1-walk-regular, as one can easily show (the idempotents are $E_i \otimes E_j + E_j \otimes E_i$ $(i \neq j)$ and $E_i \otimes E_i$).

As announced in the Introduction of this paper, we will observe a structural gap between 1- and 2-walk-regular graphs. Clearly, we cannot speak of such a gap for graphs with valency 2, because all such (connected) graphs are distance-regular. The final result of this section is that also valency 3 is special, in the sense that cubic 1-walk-regular graphs are also 2-walk-regular. In particular, every cubic 1-arc-transitive graph is 2-walk-regular.

Proposition 3.6. *Let Γ be a cubic 1-walk regular graph with diameter at least two. Then Γ is 2-walk regular.*

Proof. Let θ be an eigenvalue of Γ and consider the cosines with respect to this eigenvalue. Let u_1 be the cosine for two vertices at distance 1. Let x and y be two vertices at distance 2. Consider a common neighbor z of x and y, and let w be the third neighbor of z. By considering the (z, x)-entry in the equation $AE_\theta = \theta E_\theta$, we obtain that $1 + u_{yx} + u_{wx} = \theta u_1$. Similarly, we find that $1 + u_{xy} + u_{wy} = \theta u_1$ and $1 + u_{xw} + u_{yw} = \theta u_1$. These three equations imply that $u_{xy} = (\theta u_1 - 1)/2$. So the cosine for two vertices at distance 2 is constant (for every eigenvalue), hence Γ is 2-walk-regular. $\qquad \square$

4. Godsil's multiplicity bound

Let $m \geq 2$ and let Γ be a connected regular graph with an eigenvalue $\theta \neq \pm k$ with multiplicity m. Godsil [23] proved that if such a graph is distance-regular and not complete multipartite, then both its diameter and its valency are bounded by a

function of m. In particular, this assures that there are finitely many such distance-regular graphs. In this section we extend some of Godsil's results and reasonings to the class of 2-walk-regular graphs. The main difference with distance-regular graphs is that we are not able to bound the diameter.

We start by pointing out that, as it happens with distance-regular graphs, the images of two vertices at distance at most 2 under a representation associated to $\theta \neq \pm k$ cannot be collinear. The following lemma can indeed be read between the lines in a proof by Godsil [23].

Lemma 4.1. *Let Γ be a 2-walk-regular graph different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq \pm k$. Let x and y be vertices of Γ and consider a representation associated to a 2-walk-regular idempotent for θ. If $\hat{x} = \pm\hat{y}$, then $\mathrm{dist}(x, y) > 2$.*

Proof. Assume that $x, y \in V$ are such that $\hat{x} = \pm\hat{y}$. Then $u_{xy} = \langle \hat{x}, \hat{y} \rangle / \alpha_0 = \pm 1$. If x and y are adjacent, then this implies that $u_1 = u_{xy} = \pm 1$. From (2) we find that $u_1 = \theta/k$, so $\theta = \pm k$, a contradiction. Suppose now that $\mathrm{dist}(x, y) = 2$. Then $u_2 = \pm 1$, so for every pair of vertices v and w at distance 2, we have that $\hat{v} = \pm\hat{w}$.

Assume first that Γ is triangle free. Let z be a common neighbor of x and y. Then, as $k \geq 3$, we must have $\hat{x} = \hat{y}$ (otherwise $u_2 = -1$, so if w is another neighbor of z, then $\hat{x} = -\hat{y} = \hat{w} = -\hat{x}$, a contradiction). Then (1) gives $\theta\hat{z} = \sum_{w \sim z} \hat{w} = k\hat{x}$. Since $\|\hat{x}\| = \|\hat{z}\|$, this implies that $\theta = \pm k$, again a contradiction.

Suppose now that Γ is not triangle free. As Γ is 2-walk-regular, every edge is in a triangle. Let v and w be two vertices at distance 2. Let z_1 be a common neighbor of v and w and let z_2 be a common neighbor of v and z_1. Then $\hat{v} = \pm\hat{w}$. If w is not adjacent to z_2, then $\hat{v} = \pm\hat{w} = \pm\hat{z_2}$ and $u_1 = \pm 1$, a contradiction. So w is adjacent to z_2. By the same argument, every other neighbor z_3 of v is adjacent to z_1 or z_2, and hence to w. So every two vertices at distance 2 have k common neighbors, so Γ is complete multipartite, a final contradiction. $\square$

An immediate corollary is the following.

Corollary 4.2. *Let Γ be a 2-walk-regular graph different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq k$, and consider the representation associated to a 2-walk-regular idempotent for θ. If $u_2 = \pm 1$, then $\theta = -k$ and Γ is bipartite.*

Let $\theta \neq \pm k$ be an eigenvalue of a 2-walk-regular graph Γ (with valency k) and consider the (spherical) representation associated to a 2-walk-regular idempotent with rank m for θ. Let x be a vertex of Γ and consider the set of vectors $\{\hat{y} \mid y \in \Gamma(x)\}$. These vectors lie in the hyperplane of all vectors having inner product α_1 with $\hat{x}$, so they lie in an $(m-1)$-dimensional sphere (in $\mathbb{R}^m$). Lemma 4.1 ensures that the cardinality of the set is k. Also, the inner product between two of its elements is either α_1 or α_2, so it is a (spherical) 2-distance set. As pointed out by Godsil [23, Lemma 4.1], Delsarte, Goethals, and Seidel [18, Ex. 4.10] provide a bound for the size of such a set, and we have the following (cf. [23, Thm. 1.1]):

Theorem 4.3. *Let Γ be a 2-walk-regular graph, not complete multipartite, with valency $k \geq 3$. Assume that Γ has a 2-walk-regular idempotent with rank m for an eigenvalue $\theta \neq \pm k$. Then $k \leq \frac{(m+2)(m-1)}{2}$.*

The assumption in this result is of course satisfied if Γ has an eigenvalue with multiplicity $m > 1$. The obtained bound will be key in Section 7, as well as

for the study of 2-walk-regular graphs with an eigenvalue with multiplicity 3 in Section 6.3. In both cases we will also use properties of the local graph of 2-walk-regular graphs; we will study these in the next section. Note that also some of the results in Terwilliger's 'tree bound' paper [38] on t-arc-transitive graphs and in Hiraki and Koolen's paper [28] with improvements of Godsil's bound can be generalized to t-walk-regular graphs with large enough girth.

5. The local structure of 2-walk-regular graphs

In [39] Terwilliger gave bounds for the eigenvalues of the local graphs of a distance-regular graph (see also [5, Thm. 4.4.3] and [24, Cor. 4.3, Ch. 13]). We start this section showing that these bounds also hold for 2-walk-regular graphs. We follow the proof as given by Godsil [24, Ch. 13].

Proposition 5.1. *Let Γ be a 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \cdots > \theta_d$. Let x be a vertex of Γ and let Δ be the subgraph of Γ induced on the neighbors of x. Let $a_1 = \eta_0 \geq \cdots \geq \eta_{k-1}$ be the eigenvalues of Δ. Then*

$$\eta_{k-1} \geq -1 - \frac{b_1}{\theta_1 + 1},$$

$$\eta_1 \leq -1 - \frac{b_1}{\theta_d + 1}.$$

Proof. Let $\theta \neq k$ be an eigenvalue of Γ and let $E := E_\theta$ be the minimal idempotent corresponding to θ. Since Γ is 2-walk-regular, the intersection numbers a_j, b_j and c_j $(j = 0, 1, 2)$ are well-defined and there exist constants α_i for $i \in \{0, 1, 2\}$ such that $E \circ A_i = \alpha_i A_i$, where $\circ$ is the entrywise product. Also, (2) and (3) lead to $\alpha_1 = \alpha_0 \frac{\theta}{k}$ and $\alpha_2 = \frac{1}{b_1}((\theta - a_1)\alpha_1 - \alpha_0) = \alpha_0(\frac{\theta^2}{b_1 k} - \frac{a_1 \theta}{b_1 k} - \frac{1}{b_1})$.

Let E_Δ be the principal submatrix of E on the vertices of Δ. Clearly, E_Δ is positive semidefinite and has the form $E_\Delta = \alpha_0 I + \alpha_1 A(\Delta) + \alpha_2(J - I - A(\Delta))$, where J is the all-ones matrix and $A(\Delta)$ is the adjacency matrix of Δ.

Let w be an eigenvector of $A(\Delta)$ with corresponding eigenvalue η that is orthogonal to the all-ones vector. Then $E_\Delta w = (\alpha_0 + \alpha_1 \eta + \alpha_2(-1-\eta))w$, which implies that $\alpha_0 + \alpha_1 \eta + \alpha_2(-1-\eta) \geq 0$ as E_Δ is positive-semidefinite. Since $\alpha_0 > 0$, $\alpha_1 = \alpha_0 \frac{\theta}{k}$, and $\alpha_2 = \alpha_0(\frac{\theta^2}{b_1 k} - \frac{a_1 \theta}{b_1 k} - \frac{1}{b_1})$, we find that $(\theta - k)((1+\eta)\theta - (a_1 - k - \eta)) \leq 0$ and this shows that $(1+\eta)\theta - (a_1 - k - \eta) \geq 0$ as $\theta < k$. So we have $\eta(\theta + 1) \geq -(\theta + 1) - b_1$. This completes the proof. $\square$

We remark that the 2-coclique extensions of the lattice graphs $L_2(n)$ provide examples of 1-walk-regular graphs for which the upper bound for the eigenvalues of the local graphs in the above proposition is not valid. In this case $\eta_1 = a_1 = 2n - 4$ (the local graph consists of 2 cocktailparty graphs), $b_1 = 2n - 1$, and $\theta_d = -4$.

In what follows the symbol $\delta_{x,y}$ stands for the Kronecker delta, that is, $\delta_{x,y} = 1$ if $x = y$ and 0 otherwise.

Proposition 5.2. *Let Γ be a 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \cdots > \theta_d$. Let x be a vertex of Γ and let Δ be the subgraph of Γ induced on the neighbors of x. Let E be a 2-walk-regular idempotent with rank m for an eigenvalue $\theta \neq \pm k$. If $m < k$, then $\theta \in \{\theta_1, \theta_d\}$ and $b := -1 - \frac{b_1}{\theta+1}$ is an eigenvalue of Δ with multiplicity at least $k - m + \delta_{b,a_1}$.*

Proof. Let E_Δ be the principal submatrix of E indexed by the vertices of Δ. Clearly, $E_\Delta = \alpha_0 I + \alpha_1 A(\Delta) + \alpha_2 (J - I - A(\Delta))$.

Now it follows first of all that $\theta \neq -1$, because if $\theta = -1$, then (2) and (3) imply that $k\alpha_1 = -\alpha_0$ and $\alpha_0 + \alpha_1 a_1 + \alpha_2(k-1-a_1) = -\alpha_1$, and from this it follows that $\alpha_2 = \alpha_1 = -\alpha_0/k$. This implies that $m = \mathrm{rk}(E) \geq \mathrm{rk}(E_\Delta) = \mathrm{rk}(\alpha_0 I + \alpha_1(J-I)) = k$, which is indeed a contradiction.

Next, we obtain that 0 is an eigenvalue of E_Δ with multiplicity at least $k - m$ as $\mathrm{rk}(E_\Delta) \leq \mathrm{rk}(E) = m < k$ and E_Δ is a $k \times k$ matrix. Let us the consider the possible eigenvectors for this eigenvalue.

If w is an eigenvector of $A(\Delta)$ orthogonal to the all-ones vector, with corresponding eigenvalue η, then w is an eigenvector of E_Δ with eigenvalue $\alpha_0 + \alpha_1 \eta + \alpha_2(-1-\eta)$. If the latter eigenvalue equals 0, then by a derivation similar to that in the proof of Proposition 5.1 and using that $\theta \neq -1$, it follows that $\eta = -1 - \frac{b_1}{\theta+1}$. By Proposition 5.1, we have that $\eta \leq -1 - \frac{b_1}{\theta_d+1}$, so if $\theta < -1$, then $\theta = \theta_d$. Similarly, it follows that if $\theta > -1$, then $\theta = \theta_1$.

What remains is to check the all-ones eigenvector. The corresponding eigenvalue of E_Δ is equal to $\alpha_0 + \alpha_1 a_1 + \alpha_2(k - 1 - a_1) = \theta\alpha_1 = (\theta^2/k)\alpha_0$, where the two equalities follow as before. Because $\alpha_0 \neq 0$, it follows that if this eigenvalue of E_Δ is 0 then $\theta = 0$. Because θ_1 and θ_d are both nonzero, the above shows that in this case $\mathrm{rk}(E_\Delta) \geq k - 1$. Now consider the subgraph $\overline{\Delta}$ of Γ induced on x and its neighbors. Because $\alpha_1 = \alpha_0 \frac{\theta}{k} = 0$, the corresponding submatrix $E_{\overline{\Delta}}$ of E has rank $\mathrm{rk}(E_\Delta) + 1$, which is at least k, and this contradicts $m < k$.

It thus follows that $\theta \in \{\theta_1, \theta_d\}$ and that $b = -1 - \frac{b_1}{1+\theta}$ is an eigenvalue of Δ with multiplicity at least $k - m$. Moreover, if $b = a_1$, then the all-ones vector is also an eigenvector for eigenvalue b, so that the multiplicity is at least $k - m + \delta_{b,a_1}$. $\quad\square$

By taking for the matrix E the minimal idempotent corresponding to an eigenvalue of Γ we obtain (cf. [5, Thm. 4.4.4] and [24, Thm. 4.2, Ch. 13]):

Corollary 5.3. *Let Γ be a 2-walk-regular graph with distinct eigenvalues $k = \theta_0 > \cdots > \theta_d$ and local graph Δ. Let $\theta \neq k$ be an eigenvalue of Γ with multiplicity m. If $m < k$, then $\theta \in \{\theta_1, \theta_d\}$ and $b := -1 - \frac{b_1}{\theta+1}$ is an eigenvalue of Δ with multiplicity at least $k - m + \delta_{b,a_1}$.*

We remark that instead of taking the local graph Δ, we may take any regular induced subgraph Σ with the property that every two distinct non-adjacent vertices in Σ have distance 2 in Γ. See also Koolen [31].

In the next part we are going to derive the 'fundamental bound' for 2-walk-regular graphs. This bound was obtained for distance-regular graphs by Jurišić, Koolen, and Terwilliger [30]. We follow the proof of this bound as given by Jurišić and Koolen [29] and start with the following lemma. For the convenience of the reader we provide a proof of it.

Lemma 5.4. [29, Thm. 2.1] *Let Δ be a regular graph with valency k and n vertices. Let $k = \eta_0 \geq \cdots \geq \eta_{n-1}$ be the eigenvalues of Δ. Let σ and τ be numbers such that $\sigma \geq \eta_1 \geq \eta_{n-1} \geq \tau$. Then $n(k + \sigma\tau) \leq (k - \sigma)(k - \tau)$, with equality if and only if $\eta_i \in \{\sigma, \tau\}$ $(1 \leq i \leq n - 1)$. In particular, if equality holds then Δ is empty, complete, or strongly regular.*

Proof. Note that by assumption of the lemma we have

$$(5) \qquad \sum_{i=1}^{n-1} (\eta_i - \sigma)(\eta_i - \tau) \leq 0.$$

As $\sum_{i=0}^{n-1} \eta_i = 0$, $\sum_{i=0}^{n-1} \eta_i^2 = nk$ and $\eta_0 = k$, the inequality in the lemma immediately follows.

In case of equality we obtain equality in (5) which in turn implies that $\eta_i \in \{\sigma, \tau\}$ $(1 \leq i \leq n-1)$. $\qquad \square$

As a consequence of Proposition 5.1 and Lemma 5.4 we obtain the following 'fundamental bound' (cf. [30, Thm. 6.2] and [29, Thm. 2.1]).

Theorem 5.5. *Let* Γ *be a 2-walk-regular graph with distinct eigenvalues* $k = \theta_0 > \cdots > \theta_d$. *Then*

$$(\theta_1 + \frac{k}{a_1 + 1})(\theta_d + \frac{k}{a_1 + 1}) \geq -\frac{ka_1b_1}{(a_1 + 1)^2}.$$

If $a_1 \neq 0$, *then equality holds if and only if every local graph* Δ *is strongly regular with eigenvalues* a_1, $-1 - \frac{b_1}{\theta_d + 1}$, *and* $-1 - \frac{b_1}{\theta_1 + 1}$. *If* $a_1 = 0$, *then equality holds if and only if* Γ *is bipartite.*

Proof. Let x be a vertex of Γ and let $\Delta := \Delta(x)$ be the subgraph of Γ induced on the neighbors of x. Let $a_1 = \eta_0 \geq \cdots \geq \eta_{k-1}$ be the eigenvalues of Δ. Let $\sigma = -1 - \frac{b_1}{\theta_d + 1}$ and $\tau = -1 - \frac{b_1}{\theta_1 + 1}$. Then we have $\sigma \geq \eta_1 \geq \eta_{k-1} \geq \tau$ by Proposition 5.1. As Δ is a regular graph with valency a_1 and k vertices, we obtain the fundamental bound by reformulating the inequality in Lemma 5.4; we omit the technical details. If $a_1 = 0$ (and the local graph is empty), then equality holds if and only if $\theta_d = -k$. $\qquad \square$

6. SMALL MULTIPLICITY

This section is devoted to study t-walk-regular graphs having eigenvalues with small multiplicity. We start by answering the following question: How small can the multiplicity of an eigenvalue be of a t-walk-regular graph that is not distance-regular? Afterwards, in Sections 6.2 and 6.3, we will use this answer and the results in the previous sections to describe 1- and 2-walk-regular graphs having an eigenvalue (with absolute value smaller than the spectral radius) with small multiplicity.

6.1. **Distance-regularity from a small multiplicity.** Dalfó, Van Dam, Fiol, Garriga and Gorissen [14] posed the following problem: What is the smallest t such that every t-walk-regular graph is distance-regular? More precisely, they considered t as a function of either the diameter D of Γ or the number $d + 1$ of distinct eigenvalues. We will give an answer to this question, but in terms of the minimum multiplicity of an eigenvalue $\theta \neq \pm k$ of Γ (where k is the valency), or actually a bit stronger, in terms of the minimum rank of a t-walk-regular idempotent for θ. Notice that the minimum multiplicity is related to d and the number of vertices. The following result follows from revisiting the proof of a result by Godsil [23, Thm. 3.2].

Proposition 6.1. *Let $t \geq 2$ and let Γ be a t-walk-regular graph with valency $k \geq 3$ and diameter $D > t$. If Γ has a t-walk-regular idempotent for an eigenvalue $\theta \neq \pm k$ with rank at most t, then $b_t = 1$.*

Proof. Consider the representation associated to the t-walk-regular idempotent E for θ. Let $x = x_0$ and $y = x_t$ be two vertices at distance t in Γ, and let $P = x_0 \ldots x_t$ be a (shortest) path joining them. Let $Q = x_0 \ldots x_q$ be the longest subpath of P starting at x such that $\{\hat{x}_i\}_{0 \leq i \leq q}$ are linearly independent (clearly $q + 1 \leq t$, since the maximum number of linearly independent $\hat{x}_i$ is at most the rank of E). Therefore, $\hat{x}_{q+1} = p_0\hat{x}_0 + p_1\hat{x}_1 + \cdots + p_q\hat{x}_q$, for certain coefficients p_i. If z is a vertex adjacent to x_{q+1} that is at distance $q + 2$ from x, then we claim that $\hat{z} = p_0\hat{x}_1 + p_1\hat{x}_2 + \cdots + p_q\hat{x}_{q+1}$. Indeed,

$$0 = \|x_{q+1} - (p_0\hat{x}_0 + p_1\hat{x}_1 + \cdots + p_q\hat{x}_q)\| = \|z - (p_0\hat{x}_1 + p_1\hat{x}_2 + \cdots + p_q\hat{x}_{q+1})\|,$$

which holds because $\langle \hat{z}, \hat{x}_{q+2-i} \rangle = \langle \hat{x}_{j+i}, \hat{x}_j \rangle = \alpha_i$ for $i = 1, \ldots, q + 1$ and $j = 0, \ldots, q + 1 - i$.

Recall that the intersection numbers a_i, b_i, and c_i are well-defined for $i = 0, \ldots, t$. Suppose now that $b_{q+1} > 1$, and let z_1 and z_2 be two vertices adjacent to x_{q+1} and at distance $q + 2$ from x. Then by the above, we have that $\hat{z}_1 = p_0\hat{x}_1 + p_1\hat{x}_2 + \cdots + p_q\hat{x}_{q+1} = \hat{z}_2$. By Lemma 4.1, this implies that $\theta = \pm k$, a contradiction, so $b_{q+1} = 1$. Now observe that in the same way as for distance-regular graphs (see [5, Prop. 4.1.6]), we have that $b_i \leq b_j$ if $j \leq i \leq t$. Therefore $b_t = 1$. $\square$

Proposition 6.2. *Let Γ be a t-walk-regular graph. If $b_t = 1$, then Γ is distance-regular.*

Proof. We will show that if $t < D$ and $b_t = 1$, then Γ is also $(t + 1)$-walk-regular. Since $b_{t+1} \leq b_t$ (cf. [5, Prop. 4.1.6]), the statement then follows by induction.

Let x and z be vertices at distance $t + 1$, and let y be a neighbor of z at distance t from x. Because $b_t = 1$, the only neighbor of y at distance $t + 1$ from x is z. Let E be the minimal idempotent of an eigenvalue θ of Γ. By considering the (x, y)-entry of $AE = \theta E$, we thus obtain that $c_t\alpha_{t-1} + a_t\alpha_t + E_{xz} = \theta\alpha_t$. This shows that E_{xz} does not depend on x and z, but only on their distance $t + 1$. Therefore Γ is $(t + 1)$-walk-regular. $\square$

We remark that the second part of the proof generalizes, in the sense that it actually proves that if E is a t-walk-regular idempotent for an eigenvalue in a t-walk-regular graph with $b_t = 1$, then E is a $(t + 1)$-walk-regular idempotent. The following result now follows immediately.

Theorem 6.3. *Let Γ be a t-walk-regular graph with a t-walk-regular idempotent for an eigenvalue $\theta \neq \pm k$ with rank at most t. If $t \geq 2$, then Γ is distance-regular.*

Let us stress once more that if Γ has an eigenvalue $\theta \neq \pm k$ with multiplicity at most t, then it has a corresponding t-walk-regular idempotent for θ with rank at most t, so we obtain the following result.

Corollary 6.4. *Let Γ be a t-walk-regular graph with an eigenvalue $\theta \neq \pm k$ with multiplicity at most t. If $t \geq 2$, then Γ is distance-regular.*

Note that we can extend this result with $t = 1$, as we will show next that 1-walk-regular graphs with an eigenvalue $\theta \neq \pm k$ with multiplicity 1 do not exist.

6.2. **1-Walk-regular graphs with a small multiplicity.** Let Γ be a 1-walk-regular graph, and suppose that it has an eigenvalue θ with multiplicity 1. Let x and y be two adjacent vertices. Since the minimal idempotent E_θ has rank 1, by considering the determinant of the 2×2 principal submatrix of E_θ on x and y, it follows that $\alpha_1 = \pm\alpha_0$, and hence by (2) we obtain that $\theta = \pm k$. In other words, a 1-walk-regular graph has no eigenvalues different from $\pm k$ with multiplicity 1. In the following proposition we consider 1-walk-regular graphs with an eigenvalue with multiplicity 2.

Proposition 6.5. *Let Γ be a 1-walk-regular graph with a 1-walk-regular idempotent for an eigenvalue with rank 2. Then Γ is a cover of a cycle.*

Proof. Let E be a 1-walk-regular idempotent for an eigenvalue θ with rank 2, then θ has multiplicity at least 2, so $\theta \neq \pm k$. Consider the representation associated to θ, and notice that the images $\hat{x}$ under this representation are in $\mathbb{R}^2$, i.e., the representation is in the plane.

We may assume that Γ is not a complete graph, because it is straightforward to derive that the only complete graph with a 1-walk-regular idempotent for an eigenvalue with rank 2 is the 3-cycle. Consider two vertices x and z at distance 2 and let y be a common neighbor of them. Because the determinant of the principal submatrix of $\frac{1}{\alpha_0}E$ on x, y, z equals 0, and the fact that $u_{xy} = u_{yz} = u_1$, it follows that the cosine u_{xz} between two vertices at distance 2 equals 1 or $2u_1^2 - 1$ (note that the latter is the cosine of twice the angle with cosine u_1).

Consider the quotient graph $\overline{\Gamma}$ obtained by identifying vertices that have the same image under the representation (that is, x and x' are identified if and only if $\hat{x} = \hat{x}'$). Let $\overline{x}$ be a class of vertices, with $x \in \overline{x}$. Assume that the class of vertices $\overline{y}$, with $y \in \overline{y}$, is adjacent to $\overline{x}$ in $\overline{\Gamma}$. Then $\langle \hat{x}, \hat{y} \rangle = u_1$. In the plane, this is only possible for two different vectors $\hat{y}$, so $\overline{x}$ will be adjacent to at most two other classes.

Consider two adjacent vertices x and y in Γ. Let s denote the number of neighbors z of y that are at distance 2 from x with $\langle x, z \rangle = 2u_1^2 - 1$. Then by looking at the (y, x)-entry of $\frac{1}{\alpha_0}AE = \frac{1}{\alpha_0}\theta E$, we find that

$$1 + a_1 u_1 + s(2u_1^2 - 1) + (k - 1 - a_1 - s)1 = \theta u_1,$$

so $s = \frac{k}{2}\left(1 - a_1/(\theta + k)\right)$. If $a_1 = 0$, then $s = k/2$, and $\overline{y}$, and hence all the vertices in $\overline{\Gamma}$, have degree 2, so $\overline{\Gamma}$ is a cycle. Moreover, the partition given by the classes of vertices is equitable (with every vertex being adjacent to $k/2$ vertices in each neighboring class).

Assume finally that $a_1 > 0$. By considering the principal submatrix of $\frac{1}{\alpha_0}E$ on the three vertices of a triangle, we find that $u_1 = -1/2$ (because $u_1 \neq 1$), and hence that $2u_1^2 - 1 = -1/2$ and $\theta = -k/2$. Let $x \sim y$, then the common neighbors of x and y, and the vertices $z \in \Gamma_2(x) \cap \Gamma(y)$ such that $\langle \hat{x}, \hat{z} \rangle = 2u_1^2 - 1 = -1/2$ are in the same class in the quotient graph. Hence, the quotient graph $\overline{\Gamma}$ is a triangle, and again the partition is equitable. $\square$

Every coclique extension of a cycle is 1-walk-regular (see Section 3), and it has eigenvalues with multiplicity 2, except for coclique extensions of the 4-cycle (which are complete bipartite graphs). But this certainly does not cover all the possibilities.

Indeed, let Γ be any 1-walk-regular graph (for example, a strongly regular graph) and let Γ' be any cycle, except the 4-cycle. Then by applying Proposition 3.5 one

obtains a 1-walk-regular graph, which typically has an eigenvalue with multiplicity 2. Indeed, if k is the valency of Γ and $\theta \neq 0$ is an eigenvalue of Γ' with multiplicity 2, then the product $k\theta$ is a good candidate eigenvalue with multiplicity 2 of $\Gamma \otimes \Gamma'$; sometimes however this eigenvalue coincides with other (product) eigenvalues. The latter clearly happens when Γ' is the 4-cycle, because its only eigenvalue with multiplicity 2 is $\theta = 0$.

To show that not all 1-walk-regular graphs with an eigenvalue with multiplicity 2 come from the above product construction, we next present examples that have eigenvalue 0 with multiplicity 2, and the 4-cycle as a quotient.

Consider a connected regular graph Γ with n vertices and adjacency matrix A, minimal idempotents $E_0 = \frac{1}{n}J, E_1, \ldots, E_d$, and spectrum $k = \theta_0^1, \theta_1^{m_1}, \ldots, \theta_d^{m_d}$, where the superscripts stand for the multiplicities. Let $\overline{A}$ be the adjacency matrix of the complement of Γ. Then the graph Γ' with adjacency matrix

$$N = \begin{bmatrix} A & \overline{A} \\ \overline{A} & A \end{bmatrix}$$

has (not necessarily minimal) idempotents

$$\frac{1}{2n}J, \frac{1}{2}\begin{bmatrix} E_0 & -E_0 \\ -E_0 & E_0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} I - E_0 & I - E_0 \\ I - E_0 & I - E_0 \end{bmatrix}, F_i = \frac{1}{2}\begin{bmatrix} E_i & -E_i \\ -E_i & E_i \end{bmatrix} (i = 1, \ldots, d),$$

and (corresponding) spectrum $n - 1^1, 2k - n + 1^1, -1^{n-1}, 2\theta_1 + 1^{m_1}, \ldots, 2\theta_d + 1^{m_d}$. Clearly, Γ' is walk-regular if Γ is walk-regular. If Γ is strongly regular with parameters (n, k, λ, μ), then one can show that Γ' has an eigenvalue $(2k - n + 1)$ with multiplicity 1 if and only if $n \neq 4k - 2\mu - 2\lambda$. We remark that in the exceptional case where $n = 4k - 2\mu - 2\lambda$, the graph Γ is in the switching class of a regular two-graph, and Γ' is the corresponding distance-regular Taylor graph (cf. [5, Thm. 1.5.6]). Note that this shows that there are infinitely many walk-regular graphs with an eigenvalue (whose absolute value is smaller than the valency) with multiplicity 1. Several such infinite families (but with only four distinct eigenvalues) were already constructed by Van Dam [15], who also studied the structure of such graphs.

Now consider the bipartite double $\widetilde{\Gamma}'$ of Γ'. If Γ is a conference graph, then the above eigenvalue of Γ' with multiplicity 1 equals 0, so $\widetilde{\Gamma}'$ has eigenvalue 0 with multiplicity 2 (its entire spectrum consists of eigenvalues $\pm(n-1)^1, 0^2, \pm 1^{n-1}, \pm\sqrt{n}^{n-1}$). Moreover, $\widetilde{\Gamma}'$ is 1-walk-regular (whereas Γ' is not; cf. Proposition 3.1), which follows by considering the minimal idempotents of $\widetilde{\Gamma}'$ (which can be obtained by summing the appropriate idempotents given in the proof of Proposition 3.1); the crucial point is that $N \circ (F_1 - F_2)$ is a multiple of N, but we omit details.

Thus, for every $n \geq 3$, there are infinitely many examples of 1-walk-regular graphs with an eigenvalue with multiplicity 2 being covers of C_n.

We end this section by observing that the smallest multiplicity of an eigenvalue different from k in a 1-walk-regular graph provides a bound for its clique number.

Proposition 6.6. *Let Γ be a 1-walk-regular graph with valency k. Let E be a 1-walk-regular idempotent with rank m for an eigenvalue $\theta \neq k$. Then every clique of Γ has at most $m + 1$ vertices.*

Proof. Let C be a clique with size c. Then the principal submatrix of E indexed by the vertices of C equals $\alpha_0 I + \alpha_1 (J - I)$. Therefore, since $\alpha_0 \neq \alpha_1$ (see (2)), E has rank at least $c - 1$. $\square$

The coclique extensions of the triangle satisfy the bound with equality (with $m = 2$), for example.

6.3. **2-Walk-regular graphs with a small multiplicity.** Let $\theta \neq k$ be an eigenvalue of a 2-walk-regular graph Γ with valency k. Recall that θ, as proven in Section 6.2, cannot have multiplicity one. If θ has multiplicity 2, then by Corollary 6.4 we know that Γ is distance-regular, and the only distance-regular graphs with an eigenvalue with multiplicity 2 are the polygons and the regular complete tripartite graphs (see [5, Prop. 4.4.8]). In Theorem 6.8, we will discuss the case of multiplicity 3. For that we use the following lemma (cf. [38, Thm. 1]), which is interesting on its own.

Lemma 6.7. *Let Γ be a 2-walk-regular graph with valency k. If Γ has a 2-walk-regular idempotent for an eigenvalue $\theta \neq \pm k$ with rank less than k, then the intersection number a_1 is positive.*

Proof. Assume that $a_1 = 0$, and so $b_1 = k - 1$. Let x be a vertex of Γ and let Δ be the subgraph of Γ induced on the neighbors of x. Note that the local graph Δ has no edges and hence Δ has only 0 as an eigenvalue. As (the rank) $m < k$, by Proposition 5.2 we know that $-1 - \frac{b_1}{\theta+1}$ is an eigenvalue of Δ, which shows that $\theta = -k$. This is a contradiction, so a_1 should be positive. $\square$

Theorem 6.8. *Let Γ be a 2-walk-regular graph different from a complete multipartite graph, with valency $k \geq 3$ and eigenvalue $\theta \neq \pm k$ with multiplicity 3. Then Γ is a cubic graph with $a_1 = a_2 = 0$ or a distance-regular graph. Moreover, if Γ is distance-regular, then Γ is the cube, the dodecahedron, or the icosahedron.*

Proof. As Γ is 2-walk-regular and not complete multipartite, we have that $k \leq \frac{(m+2)(m-1)}{2} = 5$ by Theorem 4.3.

For $k = 3$, if at least one of the intersection numbers a_1 and a_2 is not zero, then $b_2 \leq 1$ and this shows that the graph Γ is distance-regular by Proposition 6.2. All cubic distance-regular graphs are known (see [5, Thm. 7.5.1]), and it follows that Γ is either the cube or the dodecahedron.

For the cases with $k > 3$, we first observe that $a_1 > 0$ by Lemma 6.7. Note also that $a_1 < k - 1$ because Γ is not complete.

Assume first that $k = 5$. Let x be a vertex of Γ and let Δ be the subgraph of Γ induced on the neighbors of x. Then the number of edges in Δ equals $5a_1/2$, which implies that a_1 should be even, and hence $a_1 = 2$. Now the local graph Δ is a pentagon, and hence Γ is the icosahedron (see [5, Prop. 1.1.4]), which is indeed distance-regular with an eigenvalue with multiplicity 3 (it has spectrum $\{3^1, \sqrt{5}^3, 1^5, 0^4, -2^4, -\sqrt{5}^3\}$).

Next, we assume that $k = 4$. In this case $a_1 \in \{1, 2\}$. If $a_1 = 2$, then the local graph Δ is a quadrangle and hence the graph Γ is the octahedron. The octahedron is distance-regular and has spectrum $\{4^1, 0^3, -2^2\}$, but it is a complete multipartite graph $K_{3\times 2}$.

To complete the proof, we may assume that $k = 4$ and $a_1 = 1$. As $a_1 = 1$, the local graph Δ is a disjoint union of two edges, i.e., the spectrum of Δ is $\{1^2, -1^2\}$. By Proposition 5.2 we know that $-1 - \frac{b_1}{\theta+1}$ is an eigenvalue of Δ, so $\theta = -2$ because $b_1 = 2$. Because every vertex is in precisely two triangles, and every edge is precisely one triangle, it will be useful to consider the triangle-vertex $(0, 1)$-incidence matrix N, where $N_{T,x} = 1$ if vertex x is in triangle T. If B is the adjacency matrix of

Γ, then $B = N^\top N - 2I$ (in fact, Γ is a line graph, cf. Proposition 3.4). If n is the number of vertices and c the number of triangles, then $3c = 2n$. Because the rank of N is at most c, it follows that Γ has eigenvalue -2 with multiplicity at least $n - c = n/3$, which implies that $n \leq 9$. Consider now the intersection number c_2 and the number of vertices k_2 at distance 2 from a fixed vertex in Γ. Because $c_2 k_2 = k b_1 = 8$ and Γ is not complete multipartite, it follows that $c_2 \in \{1, 2\}$. If $c_2 = 1$, then $n \geq 1 + k + k_2 = 13$, a contradiction. If $c_2 = 2$, then $n \geq 1 + k + k_2 = 9$, with equality if and only if Γ is strongly regular with parameters $(9, 4, 1, 2)$. This implies that Γ is the lattice graph $L_2(3)$, which however has no eigenvalue with multiplicity 3. So, there is no possible graph for $k = 4$. $\qquad\square$

Contrary to many of the other results, it seems difficult to generalize the condition on the eigenvalue multiplicity in Theorem 6.8 to a condition on the rank of a 2-walk-regular idempotent. The multiplicity condition gives a bound on n, and without this bound the case $k = 4, a_1 = 1, b_2 = 2$ causes difficulties.

Notice that the complete multipartite graph $K_{(m+1) \times \omega}$ has eigenvalue $-\omega$ with multiplicity m, and hence the complete multipartite graph $K_{4 \times \omega}$ has an eigenvalue with multiplicity 3. The only complete multipartite graph having eigenvalue 0 with multiplicity 3 is the earlier mentioned $K_{3 \times 2}$. Examples of other 2-walk-regular graphs (not being distance-regular) with an eigenvalue with multiplicity 3 can be found in the Foster census of symmetric cubic graphs [37], such as the graphs $F056A$, $F060A$, $F104A$, $F112C$, as well as the generalized Petersen graphs $G(8, 3)$, $G(12, 5)$, and $G(24, 5)$, which correspond to graphs $F016A$ (also known as the Möbius-Kantor graph; the double cover of the cube without quadrangles), $F024A$, and $F048A$, respectively. In a forthcoming paper, we will present an infinite family of 2-walk-regular graphs with a multiplicity 3.

7. The Delsarte bound and geometric graphs

In this section we start by observing that the Delsarte bound [17] for the size of a clique also holds for 1-walk-regular graphs. We will in fact prove a somewhat stronger statement and study the cases when equality is attained. After that, we will focus our attention on the highly related notion of geometric graphs. We will show that there are finitely many non-geometric 2-walk-regular graphs with bounded smallest eigenvalue and fixed diameter.

7.1. **The Delsarte bound.**

Proposition 7.1. *Let Γ be a connected k-regular graph with a 1-walk-regular idempotent E for an eigenvalue $\theta < 0$. If C is a clique in Γ with characteristic vector χ, then $|C| \leq 1 - \frac{k}{\theta}$, with equality if and only if $E\chi = 0$.*

Proof. Let C be a clique of Γ of size c. As in Proposition 6.6, the principal submatrix of E indexed by the vertices of C equals $\alpha_0 I + \alpha_1 (J - I)$. Since E is positive semidefinite, it follows that $0 \leq \chi^\top E \chi = c(\alpha_0 + \alpha_1(c-1))$. Now the bound on c follows by using (2): $\alpha_0/\alpha_1 = k/\theta$. If equality holds, then $0 = \chi^\top E \chi = \chi^\top E^2 \chi = \|E\chi\|^2$, so $E\chi = 0$ (and the other way around). $\qquad\square$

We call a clique with size attaining this bound a *Delsarte clique*. Note that if the multiplicity of θ equals $|C| - 1$, that is, the bound of Proposition 6.6 is tight, then C is a Delsarte clique. Clearly, Proposition 7.1 applies to 1-walk-regular graphs, so that we obtain the following Delsarte bound.

Theorem 7.2. *Let* Γ *be a 1-walk-regular graph with valency* k *and smallest eigenvalue* θ_d. *Then every clique of* Γ *has at most* $1 - \frac{k}{\theta_d}$ *vertices.*

We remark that if the graph is 1-walk-regular, then equality in Proposition 7.1 can only occur for $\theta = \theta_d$. Line graphs of regular graphs with valency at least 3 constitute a class of graphs for which the bound is satisfied with equality. However, the minimal idempotent corresponding to its smallest eigenvalue does not necessarily satisfy the conditions of Proposition 7.1. On the other hand, the Cartesian product $K_m \oplus K_n \oplus K_p$ of three complete graphs (a generalized Hamming graph) is 0-walk-regular with maximal cliques of size m, n, and p, while the Delsarte 'bound' equals $(m + n + p)/3$, so for particular values of m, n, and p, it has maximal cliques of size attaining the Delsarte bound, but also larger cliques. A final remark is that the same approach works for bounding the maximum number of vertices mutually at distance t in a t-walk-regular graph.

In a distance-regular graph with diameter D, a Delsarte clique C has covering radius (that is, the maximum distance of a vertex to the clique) equal to $D-1$ (note that in every connected graph with diameter D, the covering radius of a clique is either $D-1$ or D). Moreover, C is *completely regular* in the sense that every vertex at distance i from C is at distance i from the same number of vertices ϕ_i of C (and hence it is at distance j from the same number of vertices $\phi_{i,j}$ of C for every j), for $i = 0, \ldots, D-1$. We can generalize this as follows.

Proposition 7.3. *Let* Γ *be a 1-walk-regular graph with* $d + 1$ *distinct eigenvalues, and let* C *be a Delsarte clique. Then the covering radius of* C *is at most* $d - 1$. *Moreover, if* Γ *is* t-*walk-regular, then every vertex at distance* i *from* C *is at distance* i *from the same number of vertices* ϕ_i *of* C, *for* $i = 0, \ldots, t - 1$.

Proof. Let E be the minimal idempotent for the smallest eigenvalue θ_d. By Proposition 7.1, we have that $E\chi = 0$. Consider the adjacency algebra $\mathbb{A} = \langle I, A, A^2, \ldots, A^d \rangle$ of Γ. Because E is a nonzero matrix in $\mathbb{A}$, it follows that $\mathbb{A}\chi$ has dimension at most d. This implies that if $(A^d\chi)_x \neq 0$, then $(A^i\chi)_x \neq 0$ for some $i < d$, hence every vertex is at distance at most $d - 1$ from C.

Now let c be the size of C and assume that Γ is t-walk-regular. Let x be a vertex at distance i from C, for $i < t$. Let ϕ_i be the number of vertices of C that are at distance i from x (we intend to show that this number does not depend on x), then the number of vertices of C that are at distance $i+1$ from x equals $c - \phi_i$. Because $(E\chi)_x = 0$, it follows that $\phi_i \alpha_i + (c - \phi_i)\alpha_{i+1} = 0$. Indeed, this shows that ϕ_i does not depend on x, as long as $\alpha_i \neq \alpha_{i+1}$. Suppose however that $\alpha_i = \alpha_{i+1}$. Then it follows that $\alpha_{i+1} = 0$, and by repeatedly using (3), it follows that $\alpha_j = 0$ for all $j \leq i+1$, in particular $\alpha_0 = 0$, a contradiction. $\square$

In the above we used the original definition of completely regular codes by Delsarte [17]. Neumaier's [34] alternative definition (which is equivalent for codes in distance-regular graphs) is in terms of the distance partition with respect to the code being equitable, and also this property can be adjusted to Delsarte cliques in t-walk-regular graphs in a straightforward manner; cf. Neumaier [34, Thm. 4.1]. Note however that for arbitrary codes (not just cliques), the two concepts of 'partial complete regularity' are not equivalent. 'Partially Delsarte completely regular' seems to be stronger than 'partially Neumaier completely regular' (at least in

distance-regular graphs), in the same way as t-walk-regularity is stronger than t-partial distance-regularity. An example showing this is the code consisting of two vertices at distance $n - 1$ in the n-cube.

7.2. **Geometric graphs.** A graph is *geometric* if there exists a set of Delsarte cliques such that every edge lies on exactly one of them. The notion of geometric graph in this sense was introduced by Godsil [25] for distance-regular graphs. Examples of geometric graphs are bipartite graphs (trivially) and line graphs of a regular graphs with valency at least 3.

Koolen and Bang [32] proved that there are only finitely many non-geometric distance-regular graphs with smallest eigenvalue at least $-\omega$ and diameter at least 3. It is also possible to state a similar result for 2-walk-regular graphs. More precisely, Koolen and Bang [32, Thm. 3.3] showed that there are finitely many distance-regular graphs with smallest eigenvalue $-\omega$, diameter $D \geq 3$, and small c_2 (compared with a_1). In order to prove this, they bound the valency k using Godsil's multiplicity bound (the analogue of Theorem 4.3), using the multiplicity of the second largest eigenvalue θ_1. In turn, a bound on $m(\theta_1)$ is derived from the analogue of Proposition 5.2, after showing that $m(\theta_1) < k$. One of the key points for the latter inequality is to give an upper bound for the number of vertices in Γ. Their argument, however, does not apply to 2-walk-regular graphs. The following lemma intents to solve this problem.

Lemma 7.4. *Let $\omega \geq 2$ be an integer. Let Γ be a 2-walk-regular graph with valency k, diameter D, and smallest eigenvalue at least $-\omega$. If ϵ is such that $0 < \epsilon < 1$ and $c_2 \geq a_1\epsilon$, then $|V| < \left(\frac{2\omega^2}{\epsilon}\right)^D Dk$.*

Proof. Let Δ be the subgraph of Γ induced on the neighbors of x. Observe first that the size of a coclique in Δ is at most ω^2, because otherwise by eigenvalue interlacing, Γ would have an eigenvalue smaller than $-\omega$. Because the number of vertices in Δ is k, it follows that $k \leq \omega^2(a_1 + 1)$.

Note that the assumptions on ϵ imply that $\frac{a_1+1}{c_2} < \frac{2}{\epsilon}$. Since $b_1 = k - 1 - a_1$, this implies that

$$\frac{b_1}{c_2} \leq \frac{(\omega^2 - 1)(a_1 + 1)}{c_2} < \frac{2\omega^2}{\epsilon}.$$

Fix $x \in V$ and let $k_i = |\Gamma_i(x)|$ ($i = 0, \ldots, D$). We claim that $k_{i+1} \leq \frac{b_1}{c_2}k_i$ for $i = 1, \ldots, D - 1$. In order to show this claim, first observe the following. For $y \in \Gamma_{i+1}(x)$, let $c_{i+1}(y) = |\Gamma(y) \cap \Gamma_i(x)|$. By taking $z \in \Gamma_{i-1}(x)$ with $\text{dist}(y, z) = 2$, and observing that $\Gamma(y) \cap \Gamma(z) \subset \Gamma(y) \cap \Gamma_i(x)$, it follows that $c_2 \leq c_{i+1}(y)$. Similarly, one can show that $b_i(y) \leq b_1$ for $y \in \Gamma_i(x)$, where $b_i(y) = |\Gamma(y) \cap \Gamma_{i+1}(x)|$. Now $k_{i+1}c_2 \leq \sum_{y \in \Gamma_{i+1}(x)} c_{i+1}(y) = \sum_{y \in \Gamma_i(x)} b_i(y) \leq k_ib_1$, which proves the claim. Thus,

$$|V| = \sum_{i=0}^{D} k_i < 1 + \sum_{i=1}^{D}\left(\frac{2\omega^2}{\epsilon}\right)^{i-1} k < \left(\frac{2\omega^2}{\epsilon}\right)^D Dk. \qquad \square$$

As a consequence of this lemma, the proof by Koolen and Bang [32] also applies to 2-walk-regular graphs, so we have the following result (cf. [32, Thm. 3.3]).

Theorem 7.5. *Let $0 < \epsilon < 1$, and let $\omega \geq 2$ and $D \geq 3$ be integers. Let Γ be a 2-walk-regular graph with valency k, diameter D, smallest eigenvalue at least $-\omega$,*

and with $c_2 \geq a_1 \epsilon$. Then $k < D^2 \left(\frac{2\omega^2}{\epsilon} \right)^{2D+4}$. In particular, there are finitely many such graphs.

Next is to show, as it happens with distance-regular graphs (see Koolen and Bang [32, Thm. 5.3]), that if a_1 is large enough (compared to c_2), then a 2-walk-regular graph with smallest eigenvalue at least $-\omega$ is geometric. The next result by Metsch [33] is a key point for that purpose.

Proposition 7.6. [33, Result 2.1] *Let $k \geq 2$, $\mu \geq 1$, $\lambda \geq 0$, and $s \geq 1$. Suppose that Γ is a regular graph with valency k such that every two non-adjacent vertices have at most μ common neighbors, and every two adjacent vertices have exactly λ common neighbors. Define a line as a maximal clique in Γ with at least $\lambda + 2 - (s-1)(\mu-1)$ vertices. If $\lambda > (2s-1)(\mu-1) - 1$ and $k < (s+1)(\lambda+1) - s(s+1)(\mu-1)/2$, then every vertex is in at most s lines, and each edge lies in a unique line.*

Proposition 7.7. *Let $\omega \geq 2$ be an integer and let Γ be a 2-walk-regular graph with valency k, diameter $D \geq 2$, and smallest eigenvalue in the interval $[-\omega, 1-\omega)$. If $a_1 > \omega^4 c_2$, then Γ is geometric.*

Proof. Let $s = \omega^2$, $\lambda = a_1$, $\mu = c_2$, and define a line as in Proposition 7.6. Using that $k \leq \omega^2(a_1+1)$, we can check that the conditions of Proposition 7.6 are fulfilled. So, every vertex is in at most ω^2 lines and every edge is in a unique line, where the size of a line is at least

$$a_1 + 2 - (\omega^2-1)(c_2-1) > \omega^4 c_2 - (\omega^2-1)(c_2-1) = \omega^2(\omega^2 c_2 - c_2 + 1) + c_2 - 1 \geq \omega^2 + 1.$$

From Theorem 7.2, we know that the size of a clique is less than $1 + \frac{k}{\omega-1}$. Let M_x be the number of lines through the vertex x and let $\omega_x + 1$ be the average size of these lines. Then $M_x = \frac{k}{\omega_x} > \frac{k}{k/(\omega-1)} = \omega - 1$, hence

$$(6) \qquad\qquad\qquad\qquad M_x \geq \omega.$$

Let $\mathcal{C}$ denote the set of lines, and let f be the number of flags (incident vertex-line pairs) (x, C). By counting these pairs in two ways, we obtain that $|V|\omega^2 \geq f \geq |\mathcal{C}|(\omega^2+1)$, and hence there are more vertices than lines.

Now let N be the vertex-line incidence matrix of Γ. Then $NN^\top = A + D$, where D is a diagonal matrix with $D_{xx} = M_x$; moreover, $NN^\top$ is singular. Because $z^\top A z \geq -\omega$ and $z^\top D z \geq \omega$ for every vector z of length 1, it follows that if z is an eigenvector of $NN^\top$ with eigenvalue 0, then equality holds in both inequalities. In particular, this implies that there is a vertex x with $M_x = \omega$. It follows that for this vertex x we have $\omega_x = \frac{k}{\omega}$ (and all lines through x are Delsarte cliques), hence $a_1 \geq \frac{k}{\omega} - 1$, or equivalently, $k \leq \omega(a_1+1)$.

Now that we have a better bound for k, we can apply again Proposition 7.6, but this time we can set $s = \omega$. Now every vertex is in at most ω lines, so we must have equality in (6). Thus, every vertex is in exactly ω lines, and hence every line is a Delsarte clique. So Γ is geometric. $\qquad\square$

As a consequence of Theorem 7.5 and Proposition 7.7, we have the following result.

Theorem 7.8. *Let $\omega \geq 2$ and $D \geq 3$. There are finitely many non-geometric 2-walk-regular graphs with diameter D and smallest eigenvalue at least $-\omega$.*

Let us remark that we need to fix both ω and D for the finiteness. Conder and Nedela [7, Prop. 2.5] constructed infinitely many 3-arc-transitive cubic graphs with girth 11. Because a geometric graph without triangles must be bipartite, this shows that there are infinitely many non-geometric 3-walk-regular graphs with smallest eigenvalue larger than -3. To show that we need to fix ω, we consider the symmetric bilinear forms graph. This graph has as vertices the symmetric $n \times n$ matrices over $\mathbb{F}_q$, where two vertices are adjacent if their difference has rank 1; see [5, Sec. 9.5.D]. For q even and $n \geq 4$, this graph is not distance-regular, but it is 2-walk-regular. For $n = 4$, these graphs have diameter 5, and one can show using the distance-distribution diagram (see [4, p. 22]) that the smallest eigenvalue equals $-1 - q^3$. Because the valency equals $q^4 - 1$, this graph cannot be geometric, even though there are 'lines' of size q, but these are not Delsarte cliques. Note that also for $n > 4$ (and q even), we can show that the symmetric bilinear forms graph is not geometric, but we omit the proof, because it is rather involved.

On the other hand, we need 2-walk-regularity, because the earlier mentioned 2-coclique extensions of the lattice graphs provide an infinite family of non-geometric 1-walk-regular graphs with diameter 2 and smallest eigenvalue -4. Theorem 7.8 thus illustrates once more the important structural gap between 1- and 2-walk-regular graphs.

Note finally that a geometric graph Γ is the point graph of the partial linear space of vertices and (some) Delsarte cliques, and that one can consider also the dual graph on the cliques, that is, the point graph of the dual of this partial linear space. In particular when Γ is locally a disjoint union of cliques (i.e., when $k = -\theta_d(a_1+1)$), this can be used to obtain new examples of t-walk-regular graphs, in the same spirit as in Proposition 3.4, although now one has to consider the so-called geometric girth instead of the usual girth. For example, the Hamming graphs have geometric girth 4 (as $c_2 > 1$), and the dual graphs of the Hamming graph (with diameter at least three) are only 1-walk-regular. The distance-regular near octagon coming from the Hall-Janko group (see [5, Sec. 13.6]) has geometric girth 6 and its dual is 2-walk-regular.

We finish by observing that besides distance-regular graphs and the above mentioned symmetric bilinear forms graphs, we do not know of many examples of 2-walk-regular graphs with $c_2 \geq 2$. We challenge the reader to construct more such examples.

ACKNOWLEDGMENTS

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