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Abstract In this note we generalize a numerical algorithm presented in [9] to calculate all solutions of the scalar algebraic Riccati equations that play an important role in finding feedback Nash equilibria of the scalar N-player linear affine-quadratic differential game. The algorithm is based on calculating the positive roots of a polynomial matrix.

Keywords: linear-quadratic differential games, linear feedback Nash equilibrium, affine systems, numerical solution, Riccati equations.

JEL-codes: C02,C61,C63,C72

1 Introduction

In the last decades, there is an increased interest in studying diverse problems in economics and optimal control theory using dynamic games. In particular, in environmental economics and macroeconomic policy coordination, dynamic games are a natural framework to model policy coordination problems (see e.g. the books and references in [6], [18], [29] and [14]). In engineering the theory is used to model problems in, e.g., finance, robust optimal control and pursuit-evasion problems. Particularly in the area of robust optimal control the theory of linear quadratic differential games has been extensively developed (see, e.g., [1], [21] and [3]). In engineering applications, using this framework, are reported from diverse areas (robot control formation [15], interconnection of electric power systems [26], multipath routing in communication networks [23], solving mixed H_2/H_∞ control problems [24]).

In the theory of linear quadratic differential games the environment is modeled by a set of linear differential equations and the objectives are modeled as functions containing just affine quadratic terms. Assuming that players don't cooperate and look for linear feedback strategies which lead to a worse performance if they unilaterally deviate from it, leads to the study of so-called linear feedback Nash equilibria (FNE). These strategies have the important property that they are strong time consistent. A property which, e.g., does not hold under an open-loop information structure (see, e.g., [2, Chapter 6.5]). This problem has been considered by many authors and dates back to the seminal work of Starr and Ho [30]. For the fixed finite planning horizon there exists at most one FNE (see e.g. [25], [27]). Whether a solution exists depends on the solvability of a related set of coupled Riccati-type differential equations. Global existence and convergence properties of solutions of these differential equations have, e.g., been studied in [28], [13] and [32]. Further, the problem of calculating the solutions of these differential equations was considered in, e.g., [5] and [17]. In both [2, Section 6.5] and [10, Chapter 8] one can find additional references and generalizations of these results.

For an infinite planning horizon, the affine-quadratic differential game is solved in [12]. To find the FNE in this game involves solving a set of coupled algebraic Riccati equations. Only a few existence results are known for some special cases of these equations (see, e.g., [11] for an overview). Moreover, for the multivariable case there are no computational algorithms available which provide all equilibrium points. Some iterative schemes have been proposed in literature to find an equilibrium for some special cases (see e.g. [20], [31], [27] and [22]). However, all of them just provide one equilibrium (if convergence occurs). Since the number of equilibria can vary between zero and infinity it is clear that, particularly when there is no additional information that a certain type of equilibrium point is preferred or the number of equilibria is unknown, one would like to have an overview of all possible equilibria.

Papavassilopoulos et al. considered in [28] a geometric approach for calculating the stabilizing solutions of a set of feedback Nash algebraic Riccati equations. In that approach subspaces have to be calculated which satisfy simultaneously some invariance properties. However, up to now, it is unknown how to find these subspaces.

Also for the scalar case these set of equations can have multiple solutions. For instance, for the most simple two-player scalar case where the performance criterion is a strict positive quadratic function of both states and controls, the game can have one up to three different solutions (see [7] or [10][Chapter 8.4], or [8] for the corresponding N-player game result). In [9] (or [10][Chapter 8.5]) a numerical algorithm, based on the calculation of invariant subspaces for a certain matrix, was given to calculate all equilibria for this special scalar game.

In this note we generalize the approach taken in [9] to find all FNE for the affine-quadratic N-player scalar case. The algorithm we will present is based on determining the real positive roots and corresponding nullspaces of a polynomial matrix that can be derived from the with this game associated set of coupled algebraic Riccati equations. The approach generalizes the approach taken in [9].

This note is organized as follows. Section 2 introduces the problem and some notation. The main result of the paper is stated in Section 3. Section 4 contains some concluding remarks. In the Appendix we provide details on the derivation of the polynomial matrix which is essential in solving the problem.

2 Preliminaries

In this section we introduce the set of algebraic Riccati equations that play an essential role in finding the FNE for the linear affine-quadratic differential game. Since the same equations occur in solving a simplified version of this game we will consider this simplified game here.
Consider the next N-person linear quadratic differential game, with dynamics

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t),\ x(0) = x_0 \in I\!R^n, \tag{1}$$

and where player i wishes to choose his control $u_i \in I\!R^{m_i}$ to minimize: $\lim_{t_f \to \infty} J_i(0, t_f, x_0, u_1, \cdots, u_N)$ where J_i equals

$$\int_0^{t_f} [x^T(t),\ u_1^T(t), \cdots,\ u_N^T(t)] M_i [x^T(t),\ u_1^T(t), \cdots,\ u_N^T(t)]^T dt. \tag{2}$$

Here $M_i = \begin{bmatrix} Q_i & V_{i11}^T & \cdots & \cdots & V_{i1N}^T \\ V_{i11} & R_{i1} & V_{i22}^T & \cdots & V_{i2N}^T \\ & & \ddots & & \\ V_{i1N} & V_{i2N} & \cdots & \cdots & R_{iN} \end{bmatrix} =: \begin{bmatrix} Q_i & V_i^T \\ V_i & R_i \end{bmatrix}$, with $M_i = M_i^T$ and $R_{ii} > 0,\ i \in \bar{N}$[1].
Notice that we make no definiteness assumptions w.r.t. matrix Q_i and that the minimization problem has no solution if R_{ii} is an indefinite matrix, i.e. if R_{ii} has one or more negative eigenvalues.

The linear feedback information structure of the game means that all players know at any point in time the current state of the system and the cost function of their opponents. Furthermore, the set of admissible control actions, $\mathcal{U}_s$, are linear functions of the current state of the system[2], i.e.:

$$\Big\{ u = [u_1^T\ \cdots u_N^T]^T \mid u_i(t) = F_i x(t),\ \text{where } \sigma(A + BF) \subset\ \mathbb{C}^- \Big\}.$$

Here B is the block-row matrix $B := [B_1, \cdots, B_N]$ and F is the block-column matrix $F := [F_1^T, \cdots, F_N^T]^T$. To assure that $\mathcal{U}_s$ is nonempty we assume (A, B) to be stabilizable. Notice that the assumption that the players use simultaneously stabilizing controls introduces the cooperative meta-objective of all players to stabilize the system (see, e.g., the introduction of Sections 7.2 and 7.4 of [10] for a discussion).

[1] $\bar{N} := \{1, \cdots, N\}$ and $A > 0$ means that matrix A is positive definite.
[2] $\sigma(H)$ denotes the spectrum of matrix H; $\mathbb{C}^- = \{\lambda \in\ \mathbb{C} \mid \text{Re}(\lambda) < 0\}$; $\mathbb{C}^+ = \{\lambda \in\ \mathbb{C} \mid \text{Re}(\lambda) > 0\}$.

Then, $u^* \in \mathcal{U}_s$ is called a feedback Nash equilibrium if the usual inequalities apply, i.e., no player can improve his performance by a unilateral deviation from this set of equilibrium actions. Introducing the notation $u^*_{-i}(\alpha) := u^*$ where u^*_i has been replaced by the arbitrary input function α the formal definition reads as follows

Definition 2.1 $(F_1^*, \cdots, F_N^*)$, or $u^* \in \mathcal{U}_s$, is called a *feedback Nash equilibrium* if for $i \in \bar{N}$, for all $t_0 \geq 0$ and x_0, $\lim_{t_f \to \infty} J_i(t_0, t_f, x_0, u^*) \leq \lim_{t_f \to \infty} J_i(t_0, t_f, x_0, u^*_{-i}(\alpha))$, where $x(.)$ satisfies (1) with $x(t_0) = x_0$, and the input α is such that $u^*_{-i}(\alpha) \in \mathcal{U}_s$. □

To facilitate a brief statement of the next Theorem 2.3 we use some shorthand notation.

Notation 2.2 $\mathrm{diag}(D_i)$ *is the block-diagonal matrix where the* i^{th} *diagonal block-entry equals* D_i. $\widetilde{B}$ *is the block-diagonal matrix* $\widetilde{B}^T := \mathrm{diag}(B_1^T, B_2^T, \cdots, B_N^T)$; K *is the block-column matrix* $K := [K_1^T, \cdots, K_N^T]^T$; I_n *is the* $n \times n$ *identity matrix;* E_{i+1} *is obtained from the block-column matrix containing* $N+1$ *zero blocks, where block* $i+1$ *is replaced by the identity matrix, i.e.*
$E_{i+1}^T = [0_{m_i \times n}\ 0_{m_i \times m_1} \cdots 0_{m_i \times m_{i-1}}\ I_{m_i}\ 0_{m_i \times m_{i+1}} \cdots 0_{m_i \times m_N}]$, $i \in \bar{N}$; *Block-row* i *of matrix* G *equals block-row* $i{+}1$ *of* M_i, *excluding its first block-entry,* $i \in \bar{N}$, *i.e.* $G := [M_1 E_2\ \cdots\ M_N E_{N+1}]^T [0_{\bar{m} \times n}\ I_{\bar{m}}]^T$, where $\bar{m} = \sum_{1=1}^{N} m_i$. *Block-entry* i *of block-column matrix* Z *is the* $(i+1)^{th}$ *block-entry of the first block-column of matrix* M_i, *i.e.* $Z := [M_1 E_2\ \cdots\ M_N E_{N+1}]^T [I_n\ 0\ \cdots 0]^T = [V_{111}^T\ \cdots\ V_{N1N}^T]^T$. □

Theorem 2.3 *Assume* G *is invertible. The linear quadratic differential game (1,2) has a feedback Nash equilibrium* $(F_1, \cdots, F_N)$ *for every initial state if and only if*

$$F = -G^{-1}(Z + \tilde{B}^T K). \tag{3}$$

Here K_i, $i \in \bar{N}$, *are symmetric solutions of the coupled algebraic Riccati-type equations*

$$A_{cl}^T K_i + K_i A_{cl} + [I_n\ F^T] M_i [I_n\ F^T]^T = 0,\ i \in \bar{N}, \tag{4}$$

satisfying $\sigma(A_{cl}) \subset \mathbb{C}^-$, *where* $A_{cl} := A + BF$. *Further,* $J_i = x_0^T K_i x_0$. □

3 The algorithm

From Theorem 2.3 it follows that the solvability of (4) plays a crucial role in finding all FNE. For special cases of (4) it has been shown that these equations may have no, one, or more than one set of stabilizing solutions (see e.g. [10, Chapter 8]).

For the scalar case where all entries in the cost functions, except Q_i and R_{ii}, are zero, an eigenvalue based approach was used in [9] to find all solutions of (4). In this section we generalize this eigenvalue based approach to calculate all stabilizing solutions of (4).

To stress we consider the scalar case, we will use lower-case notation. Assume $\bar{s}_i := BG^{-1}\tilde{B}^T e_i \neq 0$, $i \in \bar{N}$, where $e_i \in I\!R^N$ is the i^{th} standard basis vector. Following [9] (see also [10, Chapter 8.5.3]) let $\lambda := -a_{cl}$. Then, with $y_i := \bar{s}_i k_i$, $y^{[1]} := y := [y_1, \cdots, y_N]^T$ and $\bar{S} := \mathrm{diag}(\bar{s}_i)$, we have $\bar{S}k = y$.

Therefore, with $F^T = -(Z + \tilde{B}^T \bar{S}^{-1} y)^T G^{-T}$, (4) can be rewritten as a set of quadratic polynomial equations in y_i:

$$-2\lambda y_i + \bar{s}_i [1 \; F^T] M_i [1 \; F^T]^T = 0, \; i \in \bar{N}. \tag{5}$$

The above equations (5) constitute a set of N polynomial equations f_i in N unknowns $y_1, \cdots, y_N$, i.e.,

$$f_i(y_1, \cdots, y_N) = 0, \; i \in \bar{N}. \tag{6}$$

For all $i \in \bar{N}$ the degree[3], d_i, of f_i is two. Consequently, if the solution set of (5) is a zero-dimensional set, an upper bound is given by the Bézout number $m := \Pi_{i=1}^N d_i = 2^N$.

Below we give an algorithm from which in most cases one can determine all stabilizing solutions of (5). Let $v := [1 \; y^T \; y^{[2]^T} \; \cdots \; y^{[N]^T}]^T$, where $y^{[i]}$ is the vector[4] which elements contains all $\begin{pmatrix} N \\ i \end{pmatrix}$ monomials consisting of the product of i variables from $y_1, \cdots, y_N$. By convention, let $y^{[1]} = y$ and $y^{[0]} = 1$. Furthermore, let $U(\lambda^k)$ denote a matrix polynomial in λ where the greatest degree of the polynomials appearing as entries of U is k. The algorithm is based on the next theorem which is proved in the Appendix.

Theorem 3.1 *Let y be a solution of (5), v be as introduced above and $\lambda := -a_{cl}$. Then, provided some invertibility conditions are met (see Appendix for details), there exists a polynomial matrix $U(\lambda^N) \in I\!R^{2^N \times 2^N}$ such that λ is an eigenvalue of $U(\lambda^N)$ and v is a corresponding eigenvector.* □

This results then in the next algorithm to calculate all solutions of (5).

Algorithm 3.2
Step 1: Calculate the polynomial matrix $U(\lambda^N)$ such that $\lambda v = U(\lambda^N) v$ (see Appendix).
Step 2: Calculate $\Lambda := \{\lambda > 0 \mid \det(\lambda I - U(\lambda^N)) = 0\}$.
Step 3: For all $\lambda_i \in \Lambda$ repeat the following steps.
3.1: Calculate the eigenspace of $U(\lambda_i^N)$.
3.2: If the dimension of the eigenspace is one proceed with Step 3.3, else terminate the algorithm.
3.3: Calculate an eigenvector v of $U(\lambda_i^N)$. Let $v =: [v_0, \cdots, v_N, v_{N+1}, \cdots]^T$. Then $k_i := \frac{v_i}{v_0 \bar{s}_i}$, $i \in \bar{N}$, is a candidate solution of (4). Verify by substituting k_i into (4), whether it solves (4).
Step 4: end of algorithm. □

[3]The degree of a monomial $y_1^{\alpha_1} . y_2^{\alpha_2} \cdots y_N^{\alpha_N}$ is $\alpha_1 + \cdots + \alpha_N$. The (total) degree of a polynomial f in $y_1, \cdots, y_N$ is the maximum degree of all (nonzero) monomials in f

[4]For convenience we will use the lexicographic ordering here. That is, if $\alpha = (\alpha_1, \cdots, \alpha_N)$ and $\beta = (\beta_1, \cdots, \beta_N)$ are the exponents associated with two monomials $y_1^{\alpha_1} \cdots y_N^{\alpha_N}$ and $y_1^{\beta_1} \cdots y_N^{\beta_N}$, respectively, $\alpha > \beta$ if in the vector difference $\alpha - \beta$ the left-most nonzero entry is positive. The entries of $y^{[i]}$ are arranged in decreasing order (so the first entry contains the monomial with highest order etc.). This ordering of the entries of v is also known as the graded lexicographic ordering (see e.g. [4]).

In case Algorithm 3.2 is completed successfully one has found all solutions of (4). Clearly a more detailed study of the structure of matrix $U(\lambda^N)$ is needed to get more insight into the question how restrictive the invertibility assumptions we made are. Furthermore, the case that the dimension of the eigenspace in Step 3.2 is larger than one has to be elaborated. This is in particular important for providing an estimate of the number of solutions. This is, however, beyond the scope of this paper.

Example 3.3 Consider 2 players who like to track a predefined signal $w(.)$, for the system $\dot{s}(t) = s(t)+u_1(t)+u_2(t)$. The players base their actions on the gap $x(t) := s(t)-w(t)$ between the signal and system. Player i controls u_i and wants to achieve tracking at minimal cost J_i. So he has an incentive to shift the burden of tracking the system to the other player. Assume $J_1 = \int_0^\infty x^2(t)+u_1^2(t)+\frac{1}{2}u_2^2(t)dt$ and $J_2 = \int_0^\infty 2x^2(t) + \frac{1}{4}u_1^2(t) + u_2^2(t)dt$. Then $x(t)$ satisfies $\dot{x}(t) = x(t) + u_1(t) + u_2(t) + c(t)$, where $c(t) := w(t) - \dot{w}(t)$ and $x(0) = s(0) - w(0)$. Assume $c(t) \in L_2(0,\infty)$. Then, with $A = 1$, $B = [1\ 1]$, $M_1 = \mathrm{diag}(1, 1, \frac{1}{2})$, $M_2 = \mathrm{diag}(2, \frac{1}{4}, 1)$, $Z^T = [0\ 0]$ and $G = \tilde{B} = I_2$, Theorem 2.3 applies. (4) reduces to $2(-1 + k_1 + k_2)k_1 - k_1^2 - 1 - \frac{1}{2}k_2^2 = 0$ and $2(-1 + k_1 + k_2)k_2 - k_2^2 - 2 - \frac{1}{4}k_1^2 = 0$. Following Algorithm 3.2 we first calculate matrix $U(\lambda^2)$. Using the procedure outlined in the appendix, with $\bar{a} = -1$, $\bar{s}_i = 1$, $y_i = k_i$, $\lambda = -1 + y_1 + y_2$ and $v^T = [1\ y_1\ y_2\ y_1y_2]$, we obtain $\lambda I - U(\lambda^2) = \begin{bmatrix} 1+\lambda & -1 & -1 & 0 \\ 0 & 1-\frac{9}{7}\lambda & \frac{8}{7}\lambda & -1 \\ 2 & \frac{4}{7}\lambda & 1-\frac{9}{7}\lambda & -1 \\ -\frac{16}{7}\lambda & 2+\frac{32}{49}\lambda^2 & \frac{96}{49}\lambda^2 & 1-\frac{25}{7}\lambda \end{bmatrix}$. The determinant of $\lambda I - U(\lambda^2)$ is $p(\lambda) = -\frac{431}{49}\lambda^4 + 4\lambda^3 + \frac{1290}{49}\lambda^2 - \frac{172}{7}\lambda + 9$. $p(\lambda)$ has one positive root $\lambda = 1.5092$. A corresponding eigenvector of $U(\lambda^2)$ is $v = [.3899\ .4090\ .5693\ .5972]^T$. This yields the unique solution $K^T = [k_1, k_2] = [1.0491, 1.4601]$ of (4). So this affine-quadratic game has a unique FNE. The equilibrium strategies $u^T(t) = [u_1(t), u_2(t)]$, are given by (see [12]):

$$u(t) = -Kx(t) - \int_t^\infty e^{\begin{bmatrix} 1.5092 & -0.4110 \\ 0.4110 & 1.5092 \end{bmatrix}(t-s)} Kc(s)ds.$$

In case the signal $w(t) = e^{-\mu t}$ and $s(0) = 0$, we obtain from the above equation that, with $c(s) = (1+\mu)e^{-\mu t}$, $a = \lambda$ and $b = -0.4110$, the FNE are

$$\begin{aligned} u_1(t) &= -k_1x(t) + \frac{(1+\mu)(bk_2 - (\mu+a)k_1)}{(\mu+a)^2+b^2}e^{-\mu t} \text{ and} \\ u_2(t) &= -k_2x(t) + \frac{(1+\mu)(-bk_1 - (\mu+a)k_2)}{(\mu+a)^2+b^2}e^{-\mu t}, \end{aligned}$$

respectively. From the cost functions we infer that player 2 has the incentive to achieve accurate tracking with as much as possible control used by player 1. This is confirmed in Figure 1. Player 2 uses the first period most control in order to achieve good tracking. This point is illustrated in the left-hand panel. After $t = 0.2$ this role is shifted to player 1. The right-hand panel provides a complete picture. Furthermore, we see from this plot that some overreaction occurs. □

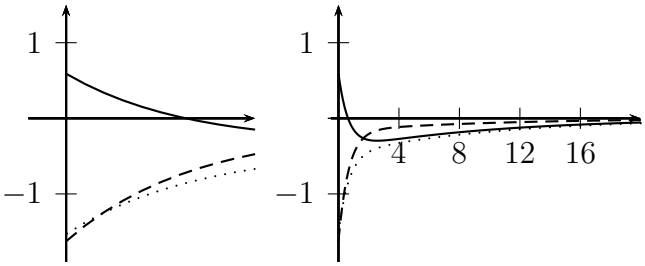

Figure 1: Graphs Example 3.3. u_1 : dotted line; u_2 : dashed line; x : full line.

4 Concluding Remarks

In this note we considered a numerical algorithm to calculate all feedback Nash equilibria of the regular indefinite infinite-planning horizon linear-quadratic differential game. To that end we had to trace all stabilizing solutions of a set of coupled algebraic Riccati equations. For the scalar case we indicated a numerical algorithm by which in most cases all solutions of the algebraic Riccati-type equations can be calculated. The algorithm is based on determining the real positive eigenvalues and corresponding eigenspace of a $2^N \times 2^N$ polynomial matrix that has degree N. To determine the points where the polynomial matrix becomes singular, we calculated the roots of its determinant. This determinant is an 2^N order polynomial. A more detailed study of this algorithm and a numerical stable implementation of it remains a topic of future research.

Appendix

Let Q be a square matrix with entries q_{ij}. Then $d(Q) := [q_{11} \cdots q_{nn}]$ denotes the row vector consisting of all main diagonal entries of Q. Further $t(Q) := [q_{12} \cdots q_{1n} q_{23} \cdots \ q_{n-1n}]$ is the row vector obtained by stacking all the upper triangular entries of Q row by row.

Lemma 4.1 *Let Q be a symmetric matrix, $d = [d_1 \cdots d_N]^T$ and $D = \mathrm{diag}(d^T) = \mathrm{diag}(d_i)$ be a diagonal matrix. Then*
1. $y^T Q y = d(Q)y^2 + 2t(Q)y^{[2]}$.
2. $y^T DQDy = d(Q)\mathrm{diag}(d^{2^T})y^2 + 2t(Q)\mathrm{diag}(d^{[2]^T})y^{[2]}$.

Proof Lemma 4.1.
1. Since $q_{ij} = q_{ji}$ this follows directly from $y^T Q y = \sum_{i=1,j=1} q_{ij} y_i y_j$.
2. Let $z := Dy$. Then, by item 1, $y^T DQDy = d(Q)z^2 + 2t(Q)z^{[2]}$. Next notice that $z^2 = \mathrm{diag}(d_i^2)y^2$ and the entry of $z^{[2]}$ containing $z_i z_j$ equals $d_i d_j y_i y_j$. So $z^{[2]} = \mathrm{diag}(d^{[2]^T})y^{[2]}$. □

Next, let $\tilde{M}_i := \mathrm{diag}(I_{m_i}, G^{-T}) M_i \mathrm{diag}(I_{m_i}, G^{-1}) =: \begin{bmatrix} Q_i & \tilde{V}_i^T \\ \tilde{V}_i & \tilde{R}_i \end{bmatrix}$; $\tilde{b} := [\frac{b_1}{\bar{s}_1} \cdots \frac{b_N}{\bar{s}_N}]^T$;

$$C_0 := \bar{S}\begin{bmatrix} \begin{bmatrix}1 & -Z^T\end{bmatrix}\tilde{M}_1\begin{bmatrix}1 & -Z^T\end{bmatrix}^T \\ \vdots \\ \begin{bmatrix}1 & -Z^T\end{bmatrix}\tilde{M}_N\begin{bmatrix}1 & -Z^T\end{bmatrix}^T \end{bmatrix};$$

$$C_1 := -2\left(\lambda I + \bar{S}\begin{bmatrix} \tilde{V}_1^T Z^T \tilde{R}_1 \\ \vdots \\ \tilde{V}_N^T - Z^T\tilde{R}_N \end{bmatrix}\mathrm{diag}(\tilde{b}^T)\right);$$

$$C_2 := \bar{S}\begin{bmatrix} d(\tilde{R}_1) \\ \vdots \\ d(\tilde{R}_N)\end{bmatrix}\mathrm{diag}(\tilde{b}^{2^T}); \text{ and}$$

$$C_3 := 2\bar{S}\begin{bmatrix} t(\tilde{R}_1) \\ \vdots \\ t(\tilde{R}_N)\end{bmatrix}\mathrm{diag}(\tilde{b}^{[2]^T}).$$

As a first step in the derivation of matrix U we have the next Lemma 4.2.

Lemma 4.2 *Let $p_0 := -C_2^{-1}C_0$; $p_1(\lambda)$ the linear polynomial matrix $-C_2^{-1}C_1(\lambda)$ and $p_2 := -C_2^{-1}C_3$. Then, $y^2 := [y_1^2 \cdots y_N^2]^T$ satisfies:*

$$y^2 = p_0 + p_1(\lambda)y + p_2 y^{[2]}, \tag{7}$$

where $y^{[2]} := [y_1y_2 \cdots y_1y_N\ y_2y_3 \cdots y_2y_N \cdots y_{N-1}y_N]^T$ is the vector consisting of all monomials of the product of 2 variables from $y_1, \cdots, y_N$.

Proof Lemma 4.2. First note that (5) can be rewritten as

$$-2\lambda y_i + \bar{s}_i[1\ -(Z+\tilde{B}^T\bar{S}^{-1}y)^T]\tilde{M}_i[1\ -(Z+\tilde{B}^T\bar{S}^{-1}y)^T]^T = 0.$$

Or,

$$\begin{aligned}-2\lambda e_i^T y + \bar{s}_i[1\ -Z^T]\tilde{M}_i[1\ -Z^T]^T - 2\bar{s}_i(\tilde{V}_i^T - Z^T\tilde{R}_i)\tilde{B}^T\bar{S}^{-1}y \\ +\bar{s}_i y^T\bar{S}^{-1}\tilde{B}\tilde{R}_i\tilde{B}^T\bar{S}^{-1}y = 0,\ i\in\bar{N}.\end{aligned}$$

Since $\bar{S}^{-1}\tilde{B} = \mathrm{diag}(\frac{b_i}{\bar{s}_i})$, using Lemma 4.1.2, the above equation can be rewritten as

$$\begin{aligned}-2\lambda e_i^T y + \bar{s}_i[1\ -Z^T]\tilde{M}_i[1\ -Z^T]^T + \bar{s}_i d(\tilde{R}_i)\mathrm{diag}(\tilde{b}^{2^T})y^2 + \\ -2\bar{s}_i(\tilde{V}_i^T - Z^T\tilde{R}_i)\mathrm{diag}(\tilde{b}^T)y + 2\bar{s}_i t(\tilde{R}_i)\mathrm{diag}(\tilde{b}^{[2]^T})y^{[2]} = 0.\end{aligned}$$

Which yields: $C_0 + C_1(\lambda)y + C_2y^2 + C_3y^{[2]} = 0$. □

Next, let $y_{-j}^{[i]}$ denote the vector that contains the entries of $y^{[i]}$ except the monomials that contain y_j and, with $w_2 := y^2$, $w_{i+2} := [y_1^2 y_{-1}^{[i]^T} \cdots y_N^2 y_{-N}^{[i]^T}]^T$, $i = 0, \cdots, N-1$. Then, using induction, the next Lemma 4.4 shows that under some invertibility assumptions w_{i+2} is a linear combination of $y^{[j]}$, $j = 0, \cdots, i+2$. The parameters are in this case polynomial matrices in λ. To prove this result, we use the next intermediate result.

Lemma 4.3 *For every k and j there exist matrices C_1 and C_2 such that:*

$$y^{[j]}y_k = C_{1,j,k}y^{[j+1]} + C_{2,j,k}w_{j+1},\ j \in \overline{N-1}.$$

Proof Lemma 4.3: $y^{[j]}y_k = C_{1,j,k}y^{[j+1]} + y_k^2\tilde{C}_{2,j,k}y_{-k}^{[j-1]} = C_{1,j,k}y^{[j+1]} + C_{2,j,k}w_{j+1}$. □

Lemma 4.4 *Assume matrices $\bar{P}_{i+2},\ i \in \overline{N-1}$, (see proof below) are invertible. Then there exist matrices $P_{0,i+2}(\lambda^i)$ and $P_{j,i+2}(\lambda^{i+2-j})$, $i = 0, \cdots, N-1$, $j = 1, \cdots, i+2$, such that:*

$$w_{i+2} = P_{0,i+2}(\lambda^i) + \sum_{j=1}^{i+2} P_{j,i+2}(\lambda^{i+2-j})y^{[j]},\ i = 0, \cdots, N-2, \tag{8}$$

$$w_{N+1} = P_{0,N+1}(\lambda^{N-1}) + \sum_{j=1}^{N} P_{j,N+1}(\lambda^{N+1-j})y^{[j]}. \tag{9}$$

Proof Lemma 4.4. We prove this lemma by induction.
For $i = 0$ this follows from (7) with $P_{0,2} := p_0$, $P_{1,2} := p_1(\lambda)$ and $P_{2,2} := p_2$.
Next assume that (8) holds for $i = m,\ m < N-2$. We show that (8) also holds for $i = m+1$ and how matrices $P_{j,k+1}$ are related to $P_{r,s},\ r = 1, \cdots, N,\ s = 1, \cdots, k$. To that end first note that there exist matrices $D_{k,m+2}$ such that: $w_{m+3} = \sum_{k=1}^{N} D_{k,m+2}y_k w_{m+2}$. So, using the induction argument and Lemma 4.3, respectively, we have that w_{m+3} equals:

$$\sum_{k=1}^{N} D_{k,m+2}\left(P_{0,m+2}(\lambda^m)y_k + \sum_{j=1}^{m+2} P_{j,m+2}(\lambda^{m+2-j})y^{[j]}y_k\right) =$$
$$\sum_{k=1}^{N} D_{k,m+2}P_{0,m+2}(\lambda^m)y_k +$$
$$\sum_{k=1}^{N} D_{k,m+2}\left(\sum_{j=1}^{m+2} P_{j,m+2}(\lambda^{m+2-j})(C_{1,j,k}y^{[j+1]} + C_{2,j,k}w_{j+1})\right).$$

Next, introduce
$\tilde{P}_{0,m+2}(\lambda^m) := [D_{1,m+2}P_{0,m+2}(\lambda^m) \cdots D_{N,m+2}P_{0,m+2}(\lambda^m)]$ and
$E_{s,j}(\lambda^{m+2-j}) := \sum_{k=1}^{N} D_{k,m+2}P_{j,m+2}(\lambda^{m+2-j})C_{s,j,k},\ s = 1, 2$. Then, the above expression for w_{m+3} can be rewritten as

$$\tilde{P}_{0,m+2}(\lambda^m)y + \sum_{j=1}^{m+2}\left(E_{1,j}(\lambda^{m+2-j})y^{[j+1]} + E_{2,j}(\lambda^{m+2-j})w_{j+1}\right).$$

Therefore, in case matrix $\bar{P}_{m+2} := I - E_{2,m+2}(\lambda^0)$ is invertible, it follows from the above equation that w_{m+3} equals:

$$\bar{P}_{m+2}^{-1}\tilde{P}_{0,m+2}(\lambda^m)y + \sum_{j=1}^{m+2} \bar{P}_{m+2}^{-1}E_{1,j}(\lambda^{m+2-j})y^{[j+1]} +$$
$$\sum_{j=1}^{m+1} \bar{P}_{m+2}^{-1}E_{2,j}(\lambda^{m+2-j})w_{j+1} = \bar{P}_{m+2}^{-1}\tilde{P}_{0,m+2}(\lambda^m)y +$$
$$\sum_{j=1}^{m+2} \bar{P}_{m+2}^{-1}E_{1,j}(\lambda^{m+2-j})y^{[j+1]} + \sum_{j=1}^{m+1} \bar{P}_{m+2}^{-1}E_{2,j}(\lambda^{m+2-j}) *$$
$$\left(P_{0,j+1}(\lambda^{j-1}) + \sum_{s=1}^{j+1} P_{s,j+1}(\lambda^{j+1-s})y^{[s]}\right).$$

So, with

$$P_{0,m+3}(\lambda^{m+1}) := \sum_{j=1}^{m+1} \bar{P}_{m+2}^{-1}E_{2,j}(\lambda^{m+2-j})P_{0,j+1}(\lambda^{j-1}),$$
$$P_{1,m+3}(\lambda^{m+2}) := \sum_{j=1}^{m+1} \bar{P}_{m+2}^{-1}E_{2,j}(\lambda^{m+2-j})P_{1,j+1}(\lambda^{j}) +$$
$$\bar{P}_{m+2}^{-1}\tilde{P}_{0,m+2}(\lambda^m),$$
$$P_{k,m+3}(\lambda^{m+3-k}) := \sum_{j=k-1}^{m+1} \bar{P}_{m+2}^{-1}E_{2,j}(\lambda^{m+2-j})P_{k,j+1}(\lambda^{j+1-k})$$
$$+\bar{P}_{m+2}^{-1}E_{1,k-1}(\lambda^{m+3-k}),\ k = 2, \cdots, m+2,$$
$$P_{m+3,m+3}(\lambda^0) := \bar{P}_{m+2}^{-1}E_{1,m+2}(\lambda^0),$$

equation (8) results for $i = m + 1$.

Finally, assume (8) holds for $i = 0, \cdots, N-2$. Then, since $w_{N+1} = \sum_{k=1}^{N} D_{k,N}y_k w_N$, we can rewrite w_{N+1} as:

$$\sum_{k=1}^{N} D_{k,N}\left(P_{0,N}(\lambda^{N-2})y_k + \sum_{j=1}^{N-1} P_{j,N}(\lambda^{N-j})y^{[j]}y_k +\right.$$
$$\left.P_{N,N}(\lambda^0)y^{[N]}y_k\right) = \sum_{k=1}^{N} D_{k,N}P_{0,N}(\lambda^{N-2})y_k +$$
$$\sum_{k=1}^{N} D_{k,N}\left(\sum_{j=1}^{N-1} P_{j,N}(\lambda^{N-j})(C_{1,j,k}y^{[j+1]} + C_{2,j,k}w_{j+1})\right) +$$
$$[D_{1,N}P_{N,N}\cdots D_{N,N}P_{N,N}]\, w_{N+1}.$$

So, introducing
$\tilde{P}_{0,N}(\lambda^{N-2}) := [D_{1,N}P_{0,N}(\lambda^{N-2})\cdots D_{N,N}P_{0,N}(\lambda^{N-2})]$ and

$E_{s,j}(\lambda^{N-j}) := \sum_{k=1}^{N} D_{k,N} P_{j,N}(\lambda^{N-j}) C_{s,j,k},\ s = 1, 2$, the above equation for w_{N+1} can be rewritten as follows, provided matrix $\bar{P}_N := I - [D_{1,N} P_{N,N} \cdots D_{N,N} P_{N,N}]$ is invertible.

$$
\begin{aligned}
w_{N+1} = \bar{P}_N^{-1} \tilde{P}_{0,N}(\lambda^{N-2}) y + \sum_{j=1}^{N-1} \bar{P}_N^{-1} E_{1,j}(\lambda^{N-j}) y^{[j+1]} + \\
\sum_{j=1}^{N-1} \bar{P}_N^{-1} E_{2,j}(\lambda^{N-j}) w_{j+1} \\
= \bar{P}_N^{-1} \tilde{P}_{0,N}(\lambda^{N-2}) y + \sum_{j=1}^{N-1} \bar{P}_N^{-1} E_{1,j}(\lambda^{N-j}) y^{[j+1]} + \\
\sum_{j=1}^{N-1} \bar{P}_N^{-1} E_{2,j}(\lambda^{N-j}) \left(P_{0,j+1}(\lambda^{j-1}) + \sum_{s=1}^{j+1} P_{s,j+1}(\lambda^{j+1-s}) y^{[s]} \right).
\end{aligned}
$$

Then, with

$$
\begin{aligned}
P_{0,N+1}(\lambda^{N-1}) &:= \sum_{j=1}^{N-1} \bar{P}_N^{-1} E_{2,j}(\lambda^{N-j}) P_{0,j+1}(\lambda^{j-1}), \\
P_{1,N+1}(\lambda^{N}) &:= \sum_{j=1}^{N-1} \bar{P}_N^{-1} E_{2,j}(\lambda^{N-j}) P_{1,j+1}(\lambda^{j}) + \\
&\quad \bar{P}_N^{-1} \tilde{P}_{0,N}(\lambda^{N-2}), \\
P_{k,N+1}(\lambda^{N+1-k}) &:= \sum_{j=k-1}^{N-1} \bar{P}_N^{-1} E_{2,j}(\lambda^{N-j}) P_{k,j+1}(\lambda^{j+1-k}) \\
&\quad + \bar{P}_N^{-1} E_{1,k-1}(\lambda^{N+1-k}),\ k = 2, \cdots, N,
\end{aligned}
$$

equation (9) results. Which completes the proof. □

Remark 4.5 *Note that the calculation of the matrices $C_{i,j,k}$ and $D_{i,j}$ can be done off-line. Furthermore, following the lines of the proof of Lemma 4.4 one can recursively calculate the matrices $P_{i,j}$.* □

Construction matrix $U(\lambda^N)$.
To calculate matrix $U(\lambda^N)$ let $\bar{a} := a - BG^{-1}Z$ and U_i block-row i of U. Then,

1) $\lambda * 1 = -\bar{a} * 1 + 1_N^T y^{[1]} =: U_0 v.$

$$2)\ \lambda * y^{[i]} = -\bar{a} y^{[i]} + \sum_{j=1}^{N} y_j y^{[i]} =$$

$$= -\bar{a} y^{[i]} + \sum_{j=1}^{N} (C_{1,i,j} y^{[i+1]} + C_{2,i,j} w_{i+1})$$

$$= -\bar{a} y^{[i]} + \sum_{j=1}^{N} C_{1,i,j} y^{[i+1]} +$$

$$\sum_{j=1}^{N} C_{2,i,j} \left(P_{0,i+1}(\lambda^{i-1}) + \sum_{k=1}^{i+1} P_{k,i+1}(\lambda^{i+1-k}) y^{[k]} \right) =$$

$$\sum_{j=1}^{N} C_{2,i,j} P_{0,i+1}(\lambda^{i-1}) + \sum_{k=1}^{i-1} \left(\sum_{j=1}^{N} C_{2,i,j} P_{k,i+1}(\lambda^{i+1-k}) \right) y^{[k]}$$

$$+ \left(-\bar{a} I + \sum_{j=1}^{N} P_{i,i+1}(\lambda) \right) y^{[i]} +$$

$$\left(\sum_{j=1}^{N} C_{1,i,j} + \sum_{j=1}^{N} P_{i+1,i+1}(\lambda^0) \right) y^{[i+1]} =: U_i v,\ i \in \overline{N-1}.$$

$$3)\ \lambda * y^{[N]} = -\bar{a} y^{[N]} + \sum_{j=1}^{N} y_j y^{[N]} = -\bar{a} y^{[N]} + 1_N^T w^{[N+1]}$$

$$= 1_N^T P_{0,N+1}(\lambda^{N-1}) + \sum_{j=1}^{N-1} 1_N^T P_{j,N+1}(\lambda^{N+1-j}) y^{[j]} +$$

$$\left(1_N^T P_{N,N+1}(\lambda) - \bar{a} I \right) y^{[N]} =: U_N v.$$

□

Proof of Theorem 3.1. From the above construction it is clear that $U(\lambda^N)v = \lambda v$. □

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