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Abstract

Proper equilibrium plays a prominent role in the literature on non-cooperative games. The underlying thought experiment is, however, unsatisfying, as it gives no justification for its fundamental idea that severe mistakes are made with a significantly smaller probability than innocuous ones. In this paper we provide a justification for this idea based on strategic choices of the players. In this way we provide a strategic foundation for proper equilibrium.

Keywords: proper equilibrium, fall back proper equilibrium

JEL Classification Number: C72

1 Introduction: proper equilibrium and its thought experiment

In this paper we reconsider the concept of *proper equilibrium* (Myerson (1978)) in mixed extensions of a finite strategic games, from now on just abbreviated to games. In order to adequately state our purposes and ideas, we first recall the underlying framework and basic notation and definitions. A game is given by $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$, with $N = \{1, \ldots, n\}$ the player set, Δ_{M^i} the mixed strategy space of player $i \in N$, with $M^i = \{1, \ldots, m^i\}$ the set of pure strategies, and $\pi^i :$

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$\prod_{j \in N} \Delta_{M^j} \to \mathbb{R}$ the von Neumann Morgenstern expected payoff function of player i. A pure strategy $k \in M^i$ of player i is alternatively denoted by e_k^i, a typical element of Δ_{M^i} by x^i. We denote the probability which x^i assigns to pure strategy k by x_k^i. The set of all strategy profiles is given by $\Delta = \prod_{i \in N} \Delta_{M^i}$, a typical element of Δ by x.

The most fundamental concept in games is that of Nash equilibrium (Nash (1951)). A strategy profile $\hat{x}$ is a *Nash equilibrium* of G, denoted by $\hat{x} \in NE(G)$, if $\pi^i(\hat{x}) \geq \pi^i(x^i, \hat{x}^{-i})$ for all $x^i \in \Delta_{M^i}$ and all $i \in N$. Here $(x^i, \hat{x}^{-i})$ is the frequently used shorthand notation for the strategy profile $(\hat{x}^1, \ldots, \hat{x}^{i-1}, x^i, \hat{x}^{i+1}, \ldots, \hat{x}^n)$.

The carrier of a strategy x^i is given by $C(x^i) = \{k \in M^i \mid x_k^i > 0\}$, the pure best reply correspondence of player i by $PB^i(x^{-i}) = \{k \in M^i \mid \pi^i(e_k^i, x^{-i}) \geq \pi^i(e_\ell^i, x^{-i})$ for all $\ell \in M^i\}$. Clearly, $\hat{x} \in NE(G)$ if and only if $C(\hat{x}^i) \subseteq PB^i(\hat{x}^{-i})$ for all $i \in N$.

The set of Nash equilibria may be very large and can contain counterintuitive outcomes. Selten (1965) introduced the concept of *perfect equilibrium* as a refinement of the set of Nash equilibria. The essential idea in the thought experiment underlying perfect equilibrium is that no pure strategy should ever be given zero probability, since there is always a small chance that any pure strategy might be chosen, if only by mistake. To further refine the set of (perfect) equilibria Myerson (1978) introduced the concept of *proper equilibrium*.

Definition [Myerson (1978)] Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player game. A strategy profile $x \in \Delta$ is a *proper equilibrium* of G if there exists a sequence $\{\varepsilon_t\}_{t \in \mathbb{N}}$ of positive real numbers converging to zero, and a sequence $\{x_t\}_{t \in \mathbb{N}}$ of completely mixed strategy profiles converging to x such that x_t is ε_t-proper for all $t \in \mathbb{N}$, *i.e.*,

$$\pi^i(e_\ell^i, x_t^{-i}) < \pi^i(e_k^i, x_t^{-i}) \quad \Rightarrow \quad x_{t,\ell}^i \leq \varepsilon_t x_{t,k}^i$$

for all $k, \ell \in M^i$ and all $i \in N$.

The properness concept plays an important role in the game theoretic literature and is widely studied in various directions, see, *e.g.*, Van Damme (1984), García-Jurado and Sánchez (1990), Blume et al. (1991), Yamamoto (1993). In the equilibrium refinement literature it is featured most prominently in the work on stable sets (Kohlberg and Mertens (1986), Mertens (1989), Hillas (1990) and Mertens (1991)), as each stable set contains a proper equilibrium. The attractiveness of the properness concept is mainly based on the fact that this concept selects the intuitively appealing strategy combinations in many (well-known) games (see, *e.g.*, Myerson (1978) and Van Damme (1991)). In that sense we recognize the selective power of proper equilibrium. In our opinion, however, the definition and underlying thought experiment of proper equilibrium are somewhat unsatisfying.

In the thought experiment underlying properness the idea is that, just as in the thought experiment underlying perfectness, players make mistakes. Contrary to the concept of perfectness, however, these mistakes are not made randomly; the trembles are somehow sensible, meaning that innocuous mistakes are made with a significantly higher probability than mistakes that have a substantial negative impact on the payoff of the players. However, in the thought experiment players have a *passive role* in the sense that they do not consciously decide on (an ordering of the) alternatives to their preferred strategies. More precisely, in the thought experiment underlying properness the alternatives are exogenously ordered based upon the corresponding payoffs (given the opponent's strategies). Hence, what is missing is an appropriate justification for obtaining this specific ordering. This problem is also addressed in Van Damme (1991) who shows that the use of control costs does not provide such a justification. We provide a justification for the fundamental idea underlying properness by starting out from a different thought experiment.

In this alternative approach each player in the thought experiment is conscious of the fact that both his intended strategy and the intended strategies of his opponents might not be executed. In this approach we then explicitly model how each player *actively anticipates* on the occurrence of such events. More specifically, in this thought experiment all the actions of each player are blocked with a small but positive probability. Since each player wants to play a best reply, each player has to strategically decide beforehand on a back-up action in case his first choice is blocked. However, since this back-up action might be blocked as well, he also has to decide on a second back-up action in case the first back-up action turns out to be unavailable, and so forth and so on. Hence, each player must decide on a complete ordering of his actions beforehand. The probability with which a player is unable to play a certain action is assumed to be independent of the particular choice he makes. This probability may, however, vary between players.

The described thought experiment results in the concept of *fall back proper equilibrium*, which alternatively can be seen as a hierarchical extension to the concept of fall back equilibrium, introduced by Kleppe et al. (2012a) and further discussed in Kleppe et al. (2012b).

To formalize the concept of fall back proper equilibrium we introduce some additional notation. The action set in the fall back proper game for player $i \in N$ within the thought experiment described above equals the set of all orderings of the action set M^i, and is denoted by Ω^i. Hence, the total number of actions in the fall back proper game for player i equals $\tilde{m}^i = m^i!$. A typical element of Ω^i is denoted by σ, where the action on position s of σ is given by $\sigma(s) \in M^i$. A pure strategy $\sigma \in \Omega^i$ will alternatively be denoted by e^i_σ. By $\Omega^i_k \subseteq \Omega^i$, $k \in M^i$, we denote the set of orderings of M^i for which $\sigma(1) = k$, hence $\Omega^i_k = \{\sigma \in \Omega^i \,|\, \sigma(1) = k\}$. The mixed strategy space of player i is given by Δ_{Ω^i}.

We assume that each action of player i is blocked with the same probability,

denoted by ε^i, but we allow for different probabilities among the players. Hence, let $\varepsilon = (\varepsilon^1, \ldots, \varepsilon^n)$ be an n-tuple of (small) non-negative probabilities. If player i plays action $\sigma \in \Omega^i$ in the fall back proper game he plays with probability $(1 - \varepsilon^i)(\varepsilon^i)^{s-1}$ action $\sigma(s)$ of the game G for $s \in \{1, \ldots, |m^i|\}$. With probability $(\varepsilon^i)^{m^i}$ all actions of player i are blocked, the game is not played and the payoff to *all* players is defined to be zero.

The fall back proper game $\tilde{G}(\varepsilon) = (N, \{\Delta_{\Omega^i}\}_{i \in N}, \{\pi_\varepsilon^i\}_{i \in N})$ is the mixed extension of the corresponding finite game with $\tilde{m}^i$ pure strategies for each player $i \in N$. The payoff functions $\{\pi_\varepsilon^i\}_{i \in N}$ on mixed strategy combinations in $\Pi_{i \in N} \Delta_{\Omega^i}$ are derived in the standard way using expected payoffs from the payoff functions on pure strategy combinations in $\Pi_{i \in N} \Omega^i$, as described by

$$\pi_\varepsilon^i((e_\sigma^j)_{j \in N}) = \sum_{(k^1, \ldots, k^n) \in \prod_{r \in N} M^r} (\prod_{j \in N} (1 - \varepsilon^j)(\varepsilon^j)^{\sigma^{-1}(k^j)-1}) \pi^i((e_{k^j}^j)_{j \in N})$$

for all $i \in N$. The residual probability in which at least one player is unable to play any of his actions is implicitly incorporated in this payoff function, as in that case the payoff to every player is zero. Note that the zero payoff is arbitrary and will not influence the equilibria of the game, because it does not depend on the players' strategy choices.

A typical element of Δ_{Ω^i} is denoted by ρ^i, the probability which ρ^i assigns to pure strategy σ is given by ρ_σ^i. The set of all strategy profiles is given by $\tilde{\Delta} = \prod_{i \in N} \Delta_{\Omega^i}$, an element of $\tilde{\Delta}$ by ρ.

Definition Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player game. A strategy profile $x \in \Delta$ is a *fall back proper equilibrium* of G if there exists a sequence $\{\varepsilon_t\}_{t \in \mathbb{N}}$ of n-tuples of positive real numbers converging to zero, and a sequence $\{\rho_t\}_{t \in \mathbb{N}}$ such that $\rho_t \in NE(\tilde{G}(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $\rho \in \tilde{\Delta}$, with $x_k^i = \sum_{\sigma \in \Omega_k^i} \rho_\sigma^i$ for all $k \in M^i$ and all $i \in N$. The set of fall back proper equilibria of a game G is denoted by $FBPR(G)$.

In the thought experiment underlying fall back proper equilibrium all the actions

of each player are blocked with a small but positive probability. Therefore, players decide beforehand on a complete ordering of their actions. This is modeled by letting players play the fall back proper game in which each action consists of a full ordering of the actions of the original game such that the first action is played with a probability close to one and each following action with a smaller probability of a fixed factor. A fall back proper equilibrium of the original game is then deduced from the limit point of a sequence of Nash equilibria of the corresponding fall back proper games when the blocking probabilities converge to zero.

Since fall back proper equilibrium can be seen as a hierarchical extension of fall back equilibrium (Kleppe et al. (2012a)), one might think that the set of fall back proper equilibria refines the set of fall back equilibria. We refer to Kleppe (2010) for an example which shows that this is not the case.

The outline of the remainder of the paper is as follows. In Section 2 we provide an alternative characterization of fall back proper equilibrium based only on limitations of the strategy spaces. Using that characterization we show in Section 3 that the set of fall back proper equilibria is a (possibly strict) non-empty and closed subset of the set of proper equilibria, and in Section 4 that for two-player games the sets of proper and fall back proper equilibria coincide.

2   A characterization of fall back proper equilibrium

In this section we provide an alternative characterization of fall back proper equilibrium in which the perturbations of the thought experiment are fully captured by limitations of the strategy spaces. This allows for a perturbed game of the same dimensions as the original one. For a (sufficiently small) *blocking vector* $\delta \in \mathbb{R}_+^N$, the blocking game $G(\delta) = (N, \{\Delta_{M^i}(\delta^i)\}_{i \in N}, \{\pi^i\}_{i \in N})$ is defined to be the game which only differs from $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ in the sense that the strategy spaces

are restricted to

$$\Delta_{M^i}(\delta^i) = \{x^i \in \Delta_{M^i} \mid \sum_{k \in T^i} x_k^i \le \frac{1 - (\delta^i)^{|T^i|}}{1 - (\delta^i)^{m^i}} \text{ for all } T^i \subseteq M^i\}$$

for all $i \in N$, with the domains of the payoff functions restricted accordingly. We define the set of all strategy profiles of the blocking game by $\Delta(\delta) = \Pi_{j \in N} \Delta_{M^j}(\delta^j)$.

Note that this blocking game gives the maximum probability by which each number of actions can be played, *e.g.*, if player i puts the maximum allowed probability on the actions in a set T^i, then any other strategy $k \notin T^i$ can be played with a probability of at most $(1 - \delta^i)(\delta^i)^{|T^i|}$.

Lemma 2.1 *Let* $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ *be an n-player game. Let* $\delta \in \mathbb{R}_+^N$ *be a blocking vector, and let* $\tilde{G}(\delta) = (N, \{\Delta_{\Omega^i}\}_{i \in N}, \{\pi_\delta^i\}_{i \in N})$ *and* $G(\delta) = (N, \{\Delta_{M^i}(\delta^i)\}_{i \in N}, \{\pi^i\}_{i \in N})$ *be the corresponding fall back proper and blocking game, respectively. Then there exists an onto map* $f_\delta : \tilde{\Delta} \to \Delta(\delta)$ *such that* $\pi_\delta^i(\rho) = \pi^i(f_\delta(\rho)) \cdot \Pi_{j \in N}(1 - (\delta^j)^{m^j})$ *for all* $\rho \in \tilde{\Delta}$ *and all* $i \in N$.

Proof: We explicitly construct a map f_δ satisfying the conditions of the lemma. Let $\rho \in \tilde{\Delta}$. We define $f_\delta(\rho) = x$, with

$$x_k^i = \frac{\sum_{\sigma \in \Omega^i}(1 - \delta^i)(\delta^i)^{\sigma^{-1}(k)-1}\rho_\sigma^i}{1 - (\delta^i)^{m^i}}$$

for all $k \in M^i$ and all $i \in N$. By considering the most extreme case in which ρ_σ^i is a pure strategy in the fall back proper game, it is readily checked that $\sum_{k \in T^i} x_k^i \le \frac{1 - (\delta^i)^{|T^i|}}{1 - (\delta^i)^{m^i}}$ for all $T^i \subseteq M^i$ such that $x \in \Delta(\delta)$. Furthermore, the probabilities put by strategy profile x on all the action profiles in the game G are equal to the probabilities put by ρ on these action profiles multiplied by $\frac{1}{\Pi_{j \in N}(1 - (\delta^j)^{m^j})}$. Hence, $\pi_\delta^i(\rho) = \pi^i(x) \cdot \Pi_{j \in N}(1 - (\delta^j)^{m^j}) = \pi^i(f_\delta(\rho)) \cdot \Pi_{j \in N}(1 - (\delta^j)^{m^j})$ for all $i \in N$. Finally, it is readily checked that f_δ is onto. $\square$

As a consequence of Lemma 2.1, a fall back proper equilibrium can also be defined in terms of a sequence of Nash equilibria of blocking games.

Theorem 2.2 Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player game. Then, a strategy profile $x \in \Delta$ is a fall back proper equilibrium of G if and only if there exists a sequence $\{\delta_t\}_{t \in \mathbb{N}}$ of blocking vectors of positive real numbers converging to zero and a sequence $\{x_t\}_{t \in \mathbb{N}}$ converging to x such that $x_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$.

Proof: We just prove the "only if" part, the reverse statement can be shown analogously. Assume $\hat{x} \in FBPR(G)$. Then by definition there exists a sequence $\{\delta_t\}_{t \in \mathbb{N}}$ of n-tuples of positive real numbers converging to zero, and a sequence $\{\hat{\rho}_t\}_{t \in \mathbb{N}}$ converging to $\hat{\rho} \in \tilde{\Delta}$, with $\hat{x}_k^i = \sum_{\sigma \in \Omega_k^i} \hat{\rho}_\sigma^i$ for all $k \in M^i$ and all $i \in N$, such that $\hat{\rho}_t \in NE(\tilde{G}(\delta_t))$ for all $t \in \mathbb{N}$. By Lemma 2.1 there exists a sequence $\{\hat{x}_t\}_{t \in \mathbb{N}}$ converging to $\hat{x} \in \Delta$, with $\hat{x}_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$, such that $\pi^i(\hat{x}_t) = \dfrac{\pi_{\delta_t}^i(\hat{\rho}_t)}{\Pi_{j \in N}(1 - (\delta^j)^{m^j})}$ for all $i \in N$ and all $t \in \mathbb{N}$.

Let $i \in N$. We show that $\pi^i(\hat{x}_t) \geq \pi^i(x_t^i, \hat{x}_t^{-i})$ for all $x_t^i \in \Delta_{M^i}(\delta_t^i)$ and all $t \in \mathbb{N}$, which proves that $\hat{x}_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$ and therefore completes the proof. Let $t \in \mathbb{N}$ and let $(x_t^i, \hat{x}_t^{-i}) \in \Delta(\delta_t)$. Then by Lemma 2.1 we can take a strategy $(\rho_t^i, \hat{\rho}_t^{-i}) \in \tilde{\Delta}$ such that $\pi_{\delta_t}^i(\rho_t^i, \hat{\rho}_t^{-i}) = \pi^i(x_t^i, \hat{x}_t^{-i}) \cdot \Pi_{j \in N}(1 - (\delta^j)^{m^j})$.

Since $\hat{\rho}_t \in NE(\tilde{G}(\delta_t))$, we obtain

$$
\begin{aligned}
\pi^i(x_t^i, \hat{x}_t^{-i}) &= \frac{\pi_{\delta_t}^i(\rho_t^i, \hat{\rho}_t^{-i})}{\Pi_{j \in N}(1 - (\delta^j)^{m^j})} \\
&\leq \frac{\pi_{\delta_t}^i(\hat{\rho}_t)}{\Pi_{j \in N}(1 - (\delta^j)^{m^j})} \\
&= \pi^i(\hat{x}_t).
\end{aligned}
$$

Consequently, $\pi^i(\hat{x}_t) \geq \pi^i(x_t^i, \hat{x}_t^{-i})$ for all $x_t^i \in \Delta_{M^i}(\delta_t^i)$ and all $t \in \mathbb{N}$. $\qquad \square$

Note that it immediately follows from Theorem 2.2 that each completely mixed Nash equilibrium is a fall back proper equilibrium.

3   General results

In this section we show that the set of fall back proper equilibria is a (possibly strict) non-empty and closed subset of the set of proper equilibria.

Theorem 3.1 Let G be an n-player game. Then each fall back proper equilibrium of G is a proper equilibrium of G.

Proof: Let $G = (N, \{\Delta_{M^i}\}_{i \in N}, \{\pi^i\}_{i \in N})$ be an n-player game and let $x \in FBPR(G)$. Then by Theorem 2.2 there exists a sequence $\{\delta_t\}_{t \in \mathbb{N}}$ of blocking vectors converging to zero, and a sequence $\{x_t\}_{t \in \mathbb{N}}$ such that $x_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$, converging to $x \in \Delta$.

Let the sequence $\{\varepsilon_t\}_{t \in \mathbb{N}}$ be given by $\varepsilon_t = \max_{i \in N} \delta_t^i$ for all $t \in \mathbb{N}$. Let $i \in N$ and let $\pi^i(e_\ell^i, x_{\hat{t}}^{-i}) < \pi^i(e_k^i, x_{\hat{t}}^{-i})$ for some $k, \ell \in M^i$ and some $\hat{t} \in \mathbb{N}$. Since $x_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$, it holds that $x_{\hat{t}, \ell}^i \leq \delta_{\hat{t}}^i x_{\hat{t}, k}^i$. Hence, $x_{\hat{t}, \ell}^i \leq \varepsilon_{\hat{t}} x_{\hat{t}, k}^i$.

Consequently, $\{\varepsilon_t\}_{t \in \mathbb{N}}$ is a sequence of positive real numbers converging to zero and $\{x_t\}_{t \in \mathbb{N}}$ is a sequence of completely mixed strategy profiles converging to x such that for all $t \in \mathbb{N}$

$$\pi^i(e_\ell^i, x_t^{-i}) < \pi^i(e_k^i, x_t^{-i}) \quad \Rightarrow \quad x_{t, \ell}^i \leq \varepsilon_t x_{t, k}^i$$

for all $k, \ell \in M^i$ and all $i \in N$. Hence, x is a proper equilibrium. $\square$

Hence, the set of fall back proper equilibria is a subset of the set of proper equilibria. The following theorem states that this subset is non-empty and closed.

Theorem 3.2 Let G be an n-player game. Then the set of fall back proper equilibria of G is non-empty and closed.

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Proof: We first show non-emptiness. Let $\{\delta_t\}_{t \in \mathbb{N}}$ be a sequence of blocking vectors converging to zero. Take a sequence $\{x_t\}_{t \in \mathbb{N}}$ such that $x_t \in NE(G(\delta_t))$ for all $t \in \mathbb{N}$. Since the strategy spaces are compact, there exists a subsequence of $\{x_t\}_{t \in \mathbb{N}}$ converging to, say, $x \in \Delta$. By Theorem 2.2, $x \in FBPR(G)$.

Secondly, we show that $FBPR(G)$ is closed. Take a converging sequence $\{x_t\}_{t \in \mathbb{N}}$ with $x_t \in FBPR(G)$ for all $t \in \mathbb{N}$, with limit x. For all $t \in \mathbb{N}$ there exists a sequence $\{\delta_{tr}\}_{r \in \mathbb{N}}$ of blocking vectors converging to zero and a sequence $\{x_{tr}\}_{r \in \mathbb{N}}$ converging to x_t such that

$$x_{tr} \in NE(G(\delta_{tr}))$$

for all $r \in \mathbb{N}$. Considering the sequences $\{\delta_{tt}\}_{t \in \mathbb{N}}$ and $\{x_{tt}\}_{t \in \mathbb{N}}$ one readily establishes that $x \in FBPR(G)$. $\qquad\square$

The following example shows that the set of fall back proper equilibria can be a strict subset of the set of proper equilibria.

Example 3.3 Consider the following three-player game in which the third player chooses the left (e_1^3) or the right (e_2^3) matrix.

$$
\begin{array}{c}
 \\
e_1^1 \\
e_2^1 \\
e_3^1
\end{array}
\begin{array}{c}
\begin{array}{ccc}
e_1^2 & e_2^2 & e_3^2 \\
\end{array} \\
\left[
\begin{array}{ccc}
10,10,10 & 0,10,0 & 0,0,1 \\
10,1,0 & 2,0,0 & 0,0,0 \\
0,0,0 & 0,0,0 & 0,0,0
\end{array}
\right]
\end{array}
\begin{array}{c}
\begin{array}{ccc}
e_1^2 & e_2^2 & e_3^2 \\
\end{array} \\
\left[
\begin{array}{ccc}
1,0,10 & 0,1,0 & 0,0,0 \\
0,0,0 & 0,0,0 & 0,0,0 \\
0,0,5 & 0,0,0 & 0,0,0
\end{array}
\right]
\end{array}
$$

In this example it is possible to coordinate the probabilities on the lower-level actions in such a way that $x = (e_1^1, e_1^2, e_1^3)$ is a proper equilibrium. This type of coordination is, however, not possible in the thought experiment underlying fall back proper equilibrium, as players are not free to make these lower-level mistakes that just happen to make things work, as their assumed active role requires them to play a (hierarchical) best reply.

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Consider the sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$, with $\varepsilon_t = \frac{1}{t}$ for all $t \in \mathbb{N}$, converging to zero and the sequence $\{\bar{x}_t\}_{t\in\mathbb{N}}$ converging to $x \in \Delta$, with $\bar{x}_t$ for all $t \in \mathbb{N}$ given by $\bar{x}_t^1 = (1 - \frac{1}{25t} - \frac{1}{1000t^2})e_1^1 + \frac{1}{25t}e_2^1 + \frac{1}{1000t^2}e_3^1$, $\bar{x}_t^2 = (1 - \frac{1}{100t} - \frac{1}{100t^2})e_1^2 + \frac{1}{100t}e_2^2 + \frac{1}{100t^2}e_3^2$ and $\bar{x}_t^3 = (1 - \frac{3}{100t})e_1^3 + \frac{3}{100t}e_2^3$. Then $\bar{x}_t$ is ε_t-proper for all $t \in \mathbb{N}$ and hence, x is a proper equilibrium.

If x would be a fall back proper equilibrium, there should exist a sequence $\{\delta_t\}_{t\in\mathbb{N}}$ of blocking vectors converging to zero and a sequence $\{\hat{x}_t\}_{t\in\mathbb{N}}$ converging to x such that $\hat{x}_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$, with a $t \in \mathbb{N}$ such that $\pi^1(e_1^1, \hat{x}_t^{-1}) \geq \pi^1(e_2^1, \hat{x}_t^{-1})$, $\pi^2(e_1^2, \hat{x}_t^{-2}) \geq \pi^2(e_2^2, \hat{x}_t^{-2})$ and $\pi^3(e_1^3, \hat{x}_t^{-3}) \geq \pi^3(e_2^3, \hat{x}_t^{-3})$. However, note that $\pi^1(e_1^1, \hat{x}_t^{-1}) \geq \pi^1(e_2^1, \hat{x}_t^{-1})$ implies that $\delta_t^3 \geq 2\delta_t^2$, $\pi^2(e_1^2, \hat{x}_t^{-2}) \geq \pi^2(e_2^2, \hat{x}_t^{-2})$ implies that $\delta_t^1 \geq \delta_t^3$ and $\pi^3(e_1^3, \hat{x}_t^{-3}) \geq \pi^3(e_2^3, \hat{x}_t^{-3})$ implies that $\delta_t^2 \geq \sqrt{5}\delta_t^1$. Combining all this results in $\delta_t^1 \geq 2\sqrt{5}\delta_t^1$, which is not possible for $\delta_t^1 > 0$. Consequently, x is not a fall back proper equilibrium. ◁

4   Results for two-player games

In the previous section we showed that in general the set of fall back proper equilibria is a (possibly strict) subset of the set of proper equilibria. Interestingly, for two-player games the sets of proper and fall back proper equilibria coincide.

Theorem 4.1 Let G be a two-player game. Then the sets of proper and fall back proper equilibria of G coincide.

Proof: Let $G = (\{1,2\}, \{\Delta_{M^i}\}_{i\in\{1,2\}}, \{\pi^i\}_{i\in\{1,2\}})$ be a two-player game. Since $FBPR(G) \subseteq PR(G)$ for all n-player games (Theorem 3.1), we only have to show that $PR(G) \subseteq FBPR(G)$. Let $x \in PR(G)$. Then there exists a sequence $\{\varepsilon_t\}_{t\in\mathbb{N}}$ of positive real numbers converging to zero, and a sequence $\{x_t\}_{t\in\mathbb{N}}$ of completely mixed strategy profiles converging to x such that x_t is ε_t-proper for all $t \in \mathbb{N}$, *i.e.*,

$$\pi^i(e_\ell^i, x_t^{-i}) < \pi^i(e_k^i, x_t^{-i}) \;\; \Rightarrow \;\; x_{t,\ell}^i \leq \varepsilon_t x_{t,k}^i$$

for all $k, \ell \in M^i$ and all $i \in N$.

Let $i \in \{1, 2\}$ and $t \in \mathbb{N}$. We divide the actions of player i recursively in a finite number S_t^i of best reply sets such that $Q_t^i(s) = \{k \in M^i \setminus \cup_{r \in \{1,\ldots,s-1\}} Q_t^i(r) \mid \pi^i(e_k^i, x_t^j) \geq \pi^i(e_\ell^i, x_t^j)$ for all $\ell \in M^i \setminus \cup_{r \in \{1,\ldots,s-1\}} Q_t^i(r)\}$ for all $s \in \{1, \ldots, S_t^i\}$. Note that since x_t is ε_t-proper, $x_{t,\ell}^i \leq \varepsilon_t x_{t,k}^i$ for all $k \in Q_t^i(s)$ and $\ell \in Q_t^i(s')$ with $s < s'$.

For each set $Q_t^i(s)$, with $s = \{1, \ldots, S_t^i\}$, we construct a strategy $\bar{x}_t^i(s)$ such that

$$\bar{x}_{t,k}^i(s) = \begin{cases} \dfrac{x_{t,k}^i}{\sum_{k \in Q_t^i(s)} x_{t,k}^i} & \text{if } k \in Q_t^i(s), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\bar{x}_t^i(s)$ is a strategy in which actions outside $Q_t^i(s)$ are not played and the probabilities on the actions in $Q_t^i(s)$ are *relatively* the same as in x_t^i.

Let $\delta_t^i = \varepsilon_t$ for all $i \in N$ and all $t \in \mathbb{N}$. Then we construct for each $t \in \mathbb{N}$ the strategy $\hat{x}_t$ such that

$$\hat{x}_t^i = \frac{\sum_{s=1}^{S_t^i} ((1 - \delta_t^i) \sum_{a=b}^{b + |Q_t^i(s)|} (\delta_t^i)^a) \bar{x}_t^i(s)}{1 - (\delta_t^i)^{m^i}}$$

for all $i \in \{1, 2\}$, with $b = |\cup_{r<s} Q_t^i(r)|$.

It follows that the sequence $\{\hat{x}_t\}_{t \in \mathbb{N}}$ converges to x and that $\hat{x}_t \in \Delta(\delta_t)$ for all $t \in \mathbb{N}$. It remains to be shown that for all $i \in \{1, 2\}$ and all $t \in \mathbb{N}$, $\pi^i(\hat{x}_t) \geq \pi^i(\dot{x}_t^i, \hat{x}_t^{-i})$ for all $\dot{x}_t^i \in \Delta_{M^i}(\delta_t^i)$. Since each player has only one opponent, for all $i \in \{1, 2\}$ and all $\ell \in M^i$, $\{k \in M^i \mid \pi^i(e_k^i, x_t^{-i}) \geq \pi^i(e_\ell^i, x_t^{-i})\} = \{k \in M^i \mid \pi^i(e_k^i, \hat{x}_t^{-i}) \geq \pi^i(e_\ell^i, \hat{x}_t^{-i})\}$. Hence, let $i \in \{1, 2\}$ and $t \in \mathbb{N}$, and let $k \in Q_t^i(s)$ and $\ell \in Q_t^i(s')$, with $s < s'$. Then there is number $U \in \{1, \ldots, S_t^{-i}\}$ such that

$$\pi^i(e_k^i, \bar{x}_t^{-i}(u)) = \pi^i(e_\ell^i, \bar{x}_t^{-i}(u))$$

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for all $1 \le u < U$, and

$$\pi^i(e^i_k, \bar{x}^{-i}_t(U)) \quad > \quad \pi^i(e^i_\ell, \bar{x}^{-i}_t(U)).$$

This implies that in $\hat{x}_t$ player i recursively puts the maximum allowed probability on each following best reply level. Consequently, $\pi^i(\hat{x}_t) \ge \pi^i(\dot{x}^i_t, \hat{x}^{-i}_t)$ for all $\dot{x}^i_t \in \Delta_{M^i}(\delta^i_t)$. Therefore, $x \in FBPR(G)$. $\qquad\square$

References

Blume, L., A. Brandenburger, and E. Dekel (1991). Lexicographic probabilities and equilibrium refinements. *Econometrica*, **59**, 81–98.

Damme, E. van (1984). A relation between perfect equilibria in extensive form games and proper equilibria in normal form games. *International Journal of Game Theory*, **13**, 1–13.

Damme, E. van (1991). *Stability and Perfection of Nash Equilibria.* Berlin: Springer Verlag.

García-Jurado, I. and J. P. Sánchez (1990). A remark on Myerson's concept of proper equilibrium. *International Journal of Game Theory*, **19**, 11–16.

Hillas, J. (1990). On the definition of the strategic stability of equilibria. *Econometrica*, **59**, 1365–1390.

Kleppe, J. (2010). *Modelling Interactive Behaviour, and Solution Concepts.* Ph. D. thesis, Tilburg University, Tilburg.

Kleppe, J., P. Borm, and R. Hendrickx (2012a). Fall back equilibrium. *European Journal of Operational Research*, **223**, 372–379.

Kleppe, J., P. Borm, and R. Hendrickx (2012b). Fall back equilibrium for 2 x n bimatrix games. CentER Discussion Paper 044, Tilburg University, Tilburg.

Kohlberg, E. and J. Mertens (1986). On strategic stability of equilibria. *Econometrica*, **54**, 1003–1037.

Mertens, J. (1989). Stable equilibria - A reformulation. Part I. Definition and basic properties. *Mathematics of Operations Research*, **14**, 575–625.

Mertens, J. (1991). Stable equilibria - A reformulation. Part II. Discussion of the definition, and further results. *Mathematics of Operations Research*, **16**, 694–753.

Myerson, R. (1978). Refinements of the Nash equilibrium point concept. *International Journal of Game Theory*, **7**, 73–80.

Nash, J. (1951). Non-cooperative games. *Annals of Mathematics*, **54**, 286–295.

Selten, R. (1965). Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit. *Zeitschrift für die gesamte Staatswissenschaft*, **12**, 301–324.

Yamamoto, Y. (1993). A path-following procedure to find a proper equilibrium of finite games. *International Journal of Game Theory*, **22**, 249–259.