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An Impulse Control Approach to Dike Height Optimization

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Abstract

This paper determines the optimal timing of dike heightenings as well as the corresponding optimal dike heightenings to protect against floods. To derive the optimal policy we design an algorithm based on the Impulse Control Maximum Principle. In this way the paper presents one of the first real life applications of the Impulse Control Maximum Principle developed by Blaquière. We show that the proposed Impulse Control approach performs better than Dynamic Programming with respect to computational time. This is caused by the fact that Impulse Control does not need discretization in time.

Key words: Impulse Control Maximum Principle, Optimal Control, flood prevention, dikes, cost-benefit analysis

JEL-codes: C61, D61, H54, Q54

1 Introduction

In February 1953 the south-western part of the Netherlands was struck by a flood disaster. The flood occurred in the night and resulted into the death of 1,835 people. Almost 200,000 hectares of land were flooded, 3,000 homes and 300 farms destroyed, and 47,000 herd of cattle drowned. In total there were 67 dike breaks. It was the biggest flood in the Netherlands for 300 years. Soon after this flood the Dutch government installed the Delta Committee with the main objective to prevent the occurrence of such events in the future, taking into account that 40% of the Netherlands is below sea level. The Delta Committee asked Van Dantzig (1956) to solve the economic cost-benefit decision model concerning the dike height problem. Because of sea-level rise and economic growth at some specific moments in time the height of the dike must be raised.

In 1995 again a critical situation occurred, where the water level of the major rivers Rhine and Meuse increased so much that 200,000 people where forced to evacuate. Fortunately, there was no serious flood and people could return to their homes. Protection against flooding is becoming an important issue all over the world. There are many deltas that need protection

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against floods. In Adikari and Yoshitania (2009) it is shown that the total number of natural disasters are continuously increasing in most regions of the world. They state that: *"Among all natural [...] disasters, water-related disasters are undoubtedly the most recurrent and pose major impediments to the achievement and sustainable socio-economic development."*

In Table 1 an overview of all recorded natural water-related disasters between 1900-2006 is presented.

1900-2006	Number of Disasters		Number killed ($\times 10^3$)		Total affected ($\times 10^6$)		Real damage US$ ($\times 10^9$)	
Flood	3,050	(42,34%)	6,899	(37,35%)	3,028	(50,18%)	343	(36,07%)
Windstorm	2,758	(38,28%)	10,009	(54,19%)	753	(12,48%)	536	(56,36%)
Drought	836	(11,60%)	1,209	(6,55%)	2,240	(37,12%)	61	(6,41%)
Slides	508	(7,05%)	56	(0,30%)	10	(0,17%)	3	(0,32%)
Wave/Surge	52	(0,72%)	296	(1,60%)	3	(0.05%)	8	(0,84%)
Total	7,204	(100%)	18,469	(100%)	6,034	(100%)	951	(100%)

Table 1: Statistics of recorded natural water-related disasters globally between 1900 and 2006.[1]

Between 1900 en 2006 floods accounted for more than 29.8% of the total number of natural disasters (including non-water related, like epidemics, earthquakes etc; see Adikari and Yoshitania (2009)). Of all casualties caused by natural disasters, 18,5% was due to flooding. Moreover more than 48% of the total number of people affected by natural disasters was flood related. In Table 2 the number of fatalities due to floods for different parts of the world between 1986 and 2006 are presented. These statistics show that not only the Netherlands, but many regions in the world have to deal with water-related disasters, such as floods.

1986-2006	Number of fatalities	
Asia	117,325	(64.4%)
Africa	14,573	(8.1%)
America	47,782	(26.2%)
Europe	2,120	(1.2%)
Oceania	218	(0.1%)
Total	182,118	(100%)

Table 2: The reported number of fatalities due to floods between 1986 and 2006 per continent.[1]

In 2007 the Delta Committee 2 was installed in the Netherlands. The objective of this committee was to advise the Dutch government concerning the consequences of the water level rise for the Dutch coast and the large river deltas. The Delta Committee 2 warned that the sea level could increase more than what was expected in the past. In particular, we should take into account a rise in sea water level between 0.65 m and 1.30 m around 2100 and a rise between 2 m and 4 m around 2200. In 2009 the Dutch government commissioned a project to develop a cost-benefit analysis and design a method to solve the resulting optimization model in order to set new safety standards. Results of this project can be found in Den Hertog and Roos (2009) and Eijgenraam et al. (2011).

[1]EM-DAT, The International Disaster Database of the Centre for Research on the Epidemiology of Disasters (CRED)

This paper presents an Impulse Control approach as an alternative method to the Dynamic Programming approach used in Eijgenraam et al. (2011) to solve the dike height problem. Brekelmans et al. (2012) develop a Mixed Integer Non-Linear Program (MINLP), but for homogeneous[2] dikes the best approach turns out to be Dynamic Programming. Therefore we choose to compare it with our Impulse Control approach. To develop the optimal policy we design an algorithm based on the Impulse Control Maximum Principle. We show that the proposed Impulse Control approach performs better than Dynamic Programming in computation time. This is caused by the fact that Impulse Control does not need discretization in time. Furthermore, this paper presents one of the first real life applications of the Impulse Control Maximum Principle. In the literature there are not many problems solved using the Impulse Control Maximum Principle. Luhmer (1986) and Kort (1989) design an algorithm to apply the Impulse Control Maximum Principle to theoretically solve (economic) problems. We consider a framework where the number of jumps is not restricted. This distinguishes our approach from, e.g., Liu et al. (1998), Augustin (2002, pp. 71-81) and Wu and Teo (2006), where the number of jumps is fixed (i.e. is taken as given).

The economic cost-benefit problem raised by the flood prevention is formulated by Van Dantzig (1956) as: *"Taking into account, the cost of dikebuilding, the material losses when a dike-break occurs, and the frequency distribution of different sea levels, determine the optimal height of the dikes"*. He assumes that both the economic value protected by the dikes and the probability of a dike breakthrough are constant over time. In his analysis he determines how much to invest in the heightening of a dike, but did not answer the question "when" to heighten this dike. Eijgenraam et al. (2011) adjusted Van Dantzig's analysis with respect to economic growth. Van Dantzig (1956) found that the height of a dike after every heightening should be such that the resulting flood probabilities are the same. Economic growth implies increasing potential damage, so it is optimal to have lower flood probabilities after every dike height increase. This can be achieved by raising the dike height to higher levels. In this paper all model assumptions are similar to Eijgenraam et al. (2011).

Impulse Control theory is a variant of optimal control theory where discontinuities (i.e. jumps) in the state variable are allowed. In Impulse Control the moments of these jumps as well as the sizes of the jumps are taken as (new) decision variables. In Blaquière (1985) an example is given that deals with optimal maintenance and life time of machines. Here one has to decide when to replace a machine by a new one (impulse control variable), and has to determine the rate of maintenance expenses (ordinary control variable), so that the profit is maximized over the planning period. In Kort (1989) a dynamic model of the firm is designed in which capital stock jumps upward at discrete points in time at which the firm invests. Blaquière (1977a; 1977b; 1979; 1985) extends the standard theory on optimal control by deriving a Maximum Principle, the so-called Impulse Control Maximum Principle, that gives necessary and sufficient optimality conditions for solving such problems.

Blaquière's Impulse Control analysis is based on the present value Hamiltonian form. In this paper we apply the Impulse Control theorem in the current value Hamiltonian framework as derived in Chahim et al. (2012).

This paper is organized as follows. In Section 2 we first build up the Impulse Control model and derive the necessary optimality conditions. In Section 3 we describe the algorithm used

[2] a homogeneous dike or dike ring consist of one segment

to solve the model and obtain an upper bound for the final dike height using the necessary optimality conditions. In in Section 4 we compare the Impulse Control model to the Dynamic Programming approach used in Eijgenraam et al. (2011) and present numerical results. Finally, in Section 5 we conclude.

2 Impulse Control Model

A dike or dike ring is an uninterrupted ring of water defences. There are 53 dike ring areas in the Netherlands with a higher safety standard (i.e. lower flood probability) than 1/1,000 per year. Each dike ring protects a certain area against flooding, see Figure 1. The model described in this section can be used for each dike ring separately.

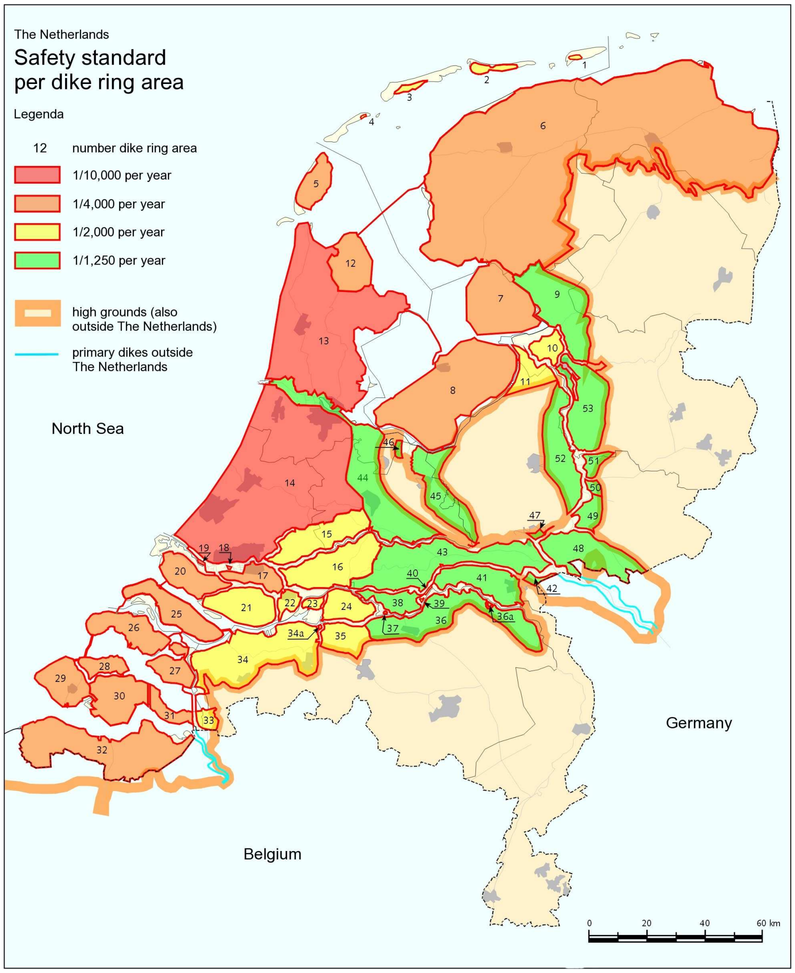


Figure 1: Dike ring areas and safety standards in the Netherlands.

In the first section we build up the mathematical model and show that this problem can be described as an Impulse Control problem. In the second section we derive necessary optimality conditions.

2.1 The Model

The economic cost-benefit decision problem defined in Eijgenraam (2006) contains two types of cost that we deal with in this problem, namely investment cost and cost due to damage (caused by failure of protection by the dikes). Clearly, there is a trade off between incurring cost due to investing or choosing not to invest and accept the probability that a dike is less protective leading to higher expected damage cost. The model minimizes the sum of the total expected damage cost and total investment cost. For a thorough discussion of the validity of the underlying model assumptions and parameter values we refer to Eijgenraam et al. (2011).

Let τ (with $\tau_1, \dots, \tau_K \in \mathbb{R}_+$) stand for the time of the dike heightening (years) and $H(t)$ denotes the dike height at time t (years) relative to the initial situation, i.e. $H(0) = 0$ (cm). The investment cost will be denoted as $I(u, H(\tau^-))$, with $H(\tau^-)$ the height of the dike (in cm) just before the dike heightening at time τ (i.e. $H(\tau^-) = \lim_{t \uparrow \tau} H(t)$) and u the amount of the dike heightening. Concerning the investment cost functions, we consider two different specifications. The exponential investment cost function is given by

$$I(u, H(\tau^-)) = \begin{cases} (c_0 + b_0 u)\, e^{a_0(H(\tau^-)+u)} & \text{for } u > 0 \\ 0 & \text{for } u = 0, \end{cases} \tag{1}$$

where a_0, b_0 and c_0 are positive constants. The quadratic investment cost functions is given by

$$I(u, H(\tau^-)) = \begin{cases} a_1(H(\tau^-) + u)^2 + b_1 u + c_1 & \text{for } u > 0 \\ 0 & \text{for } u = 0, \end{cases} \tag{2}$$

for suitably chosen constants a_1, b_1 and c_1. Observe that both the exponential and the quadratic investment cost functions depend on the height of the dike at the moment of heightening. This is contrary to Van Dantzig (1956), who uses a linear cost function that does not depend on the current height of the dike. Our investment cost specifications are in line with the engineering experience that making a dike higher also requires making it wider, implying that an additional dike height increase costs more if the current height is higher (see e.g. Sprong (2008)). Total (discounted) investment cost is then given by

$$\sum_{i=1}^{K} I(u_i, H(\tau_i^-))e^{-r\tau_i},$$

where r is the discount rate, u_i (cm) denotes the size of the i-th dike heightening, and τ_i is the time of the i-th dike heightening. Following Eijgenraam et al. (2011), we define the flood probability $P(t)$ (1/year) at time t as

$$P(t) = P_0 e^{\alpha\eta t} e^{-\alpha H(t)}, \tag{3}$$

where α (1/cm) stands for the parameter in the exponential distribution regarding the flood probability and η (cm/year) is the parameter that represents the increase of the water level per year. The flood probability at time $t = 0$ (i.e. the current flood probability) is denoted by P_0 (1/year), note that $P(0^-) = P_0$. We next describe the value of the damage by a flood, $V(t)$ (million euros):

$$V(t) = V_0 e^{\gamma t} e^{\zeta H(t)}, \tag{4}$$

in which γ (per year) is the parameter representing economic growth , and ζ (1/cm) stands for the damage increase per cm dike height. The loss by flooding at time $t = 0$ is denoted by V_0 (million euros). Note that $V(0^-) = V_0$. If $\zeta > 0$ (1/cm), the damage of a flood increases with the height of the dike. The intuition behind this is that when there is a flood, it holds that the higher the dike the higher the water level can get on the flooded land. This causes higher damage cost. Multiplying the flood probability with the value of the damage by a flood leads to the expected loss due to a flood. From (3) and (4) it follows that the expected damage at time t equals

$$S(t) = P(t)\, V(t) = S_0 e^{\beta t} e^{-\theta H(t)}, \tag{5}$$

with $S_0 = P_0 V_0$, $\beta = \alpha\eta + \gamma$, and $\theta = \alpha - \zeta$.

We consider a finite time horizon $[0, T]$. The total expected damage cost on the time interval $[0, T]$ equals

$$\int_0^T S(t) e^{-rt} dt = \int_0^T S_0 e^{\beta t} e^{-\theta H(t)} e^{-rt} dt,$$

and the expected damage cost after T, the so-called salvage value, is given by

$$S(T) \int_T^\infty e^{-rt} dt = \frac{S(T) e^{-rT}}{r}.$$

Hence, total (discounted) damage cost is given by

$$S_0 \int_0^T e^{\beta t} e^{-\theta H(t)} e^{-rt} dt + \frac{S(T) e^{-rT}}{r}.$$

The aim is to minimize the sum of the investment and expected damage cost:

$$\min \left(\int_0^T S_0 e^{\beta t} e^{-\theta H(t)} e^{-rt} dt + \sum_{i=1}^{K} I\left(u_i, H\left(\tau_i^-\right)\right) e^{-r\tau_i} + e^{-rT} \frac{S(T)}{r} \right),$$

where K is the endogenous number of dike heightenings in $[0, T]$.

The height of the dike, $H(t)$, between two dike heightenings does not change over time[3]:

$$\dot{H}(t) = 0 \text{ for } t \notin \{\tau_1, \ldots, \tau_K\}.$$

Dike heightenings occur at times $\tau_1, \ldots, \tau_K$. Then we have that

$$H\left(\tau_i^+\right) - H\left(\tau_i^-\right) = u_i > 0 \text{ for } i \in \{1, \ldots, K\},$$

where $H(\tau^+)$ denotes the height of the dike (in cm) just after the dike heightening at time τ. The dike heightening problem then becomes

$$\min_{\{u_i, \tau_i\}_{i=1}^K} \left(\int_0^T S_0 e^{\beta t} e^{-\theta H(t)} e^{-rt} dt + \sum_{i=1}^{K} I\left(u_i, H(\tau_i^-)\right) e^{-r\tau_i} + e^{-rT} \frac{S_0 e^{\beta T} e^{-\theta H(T)}}{r} \right) \tag{6}$$

s.t.

$$\begin{aligned} H(0^-) &= 0 \\ \dot{H}(t) &= 0 && \text{for } t \notin \{\tau_1, \ldots, \tau_K\} \\ H\left(\tau_i^+\right) - H\left(\tau_i^-\right) &= u_i > 0 && \text{for } i \in \{1, \ldots, K\}. \end{aligned}$$

This is an Impulse Control problem as described in Blaquière (1977a; 1977b; 1979; 1985). Note that this dike heightening model only contains an impulse control variable and not an ordinary control variable. In Blaquière (1979) an example is given of a linear model that contains both an ordinary and an impulse control variable. The example of Blaquière deals with machine maintenance, where the firm has to choose between preventive maintenance (ordinary control) and repair (or upgrade) of the machine (impulse control).

[3] The dike height can decrease slightly due to damage and wear, however these changes are so small that we can neglect them in our model.

2.2 Necessary Optimality Conditions

In this section we state necessary optimality conditions to solve the Impulse Control dike heightening model given by (6). Here we employ the current value Hamiltonian form derived in Chahim et al. (2012). This is done, because the model described in this paper involves discounting. Other references stating the necessary optimality conditions for impulse control problems are Blaquière (1977a; 1977b; 1979; 1985), Seierstad (1981) and Seierstad and Sydsæter (1987).

To apply the Impulse Control Maximum Principle the functions $S(t)$ and $I(u, H(\tau^-))$ should be continuously differentiable in H and u_i on $\mathbb{R}_+$. Moreover $S(T)/r$ should be continuously differentiable in $H(T)$ on $\mathbb{R}_+$, and finally that $I(u_i, H(\tau^-))$ is continuous in τ.

The current value Hamiltonian is

$$Ham(t, H(t)) = -S_0 e^{\beta t} e^{-\theta H(t)},$$

and the current value Impulse Hamiltonian is given by

$$IHam(t, H(t), u_i, \lambda(t)) = -I\left(u_i, H(t)\right) + \lambda(t) u_i,$$

in which $\lambda(t)$ represents the costate variable.

Applying the necessary optimality conditions from Chahim et al. (2012) to our problem yields:

$$\left\{\begin{array}{lr}
\dot{\lambda}(t) = r\lambda(t) - Ham_H = r\lambda(t) - \theta S_0 e^{\beta t} e^{-\theta H(t)} & (7)\\
\lambda(T) = \dfrac{\theta S_0 e^{\beta T} e^{-\theta H(T)}}{r} & (8)\\
\lambda\left(\tau_i^+\right) - \lambda\left(\tau_i^-\right) = I_H\left(u_i, H(\tau_i^-)\right) \quad \text{for } i = 1, \ldots, K & (9)\\
-I_u\left(u_i, H\left(\tau_i^-\right)\right) + \lambda\left(\tau_i^+\right) = 0 \quad \text{for } i = 1, \ldots, K & (10)\\
S_0 e^{\beta \tau_i}\left(e^{-\theta H(\tau_i^-)} - e^{-\theta H(\tau_i^+)}\right) - rI\left(u_i, H(\tau_i^-)\right) \left\{\begin{array}{l} > 0 \text{ if } \tau_i = 0 \\ = 0 \text{ if } \tau_i \in (0, T) \\ < 0 \text{ if } \tau_i = T \end{array}\right\} \text{for } i = 1, \ldots, K & (11)\\
\dfrac{\partial IHam(t, H(t), 0, \lambda(t))}{\partial u_i} u_i \leq 0 \quad \text{for } u_i \geq 0,\ t \neq \tau_i\ (i = 1, \ldots, K), & (12)
\end{array}\right.$$

where $\dot{\lambda}(t)$ denotes the time derivative of the costate variable $\lambda(t)$, Ham_H the derivative of the Hamiltonian with respect to the state variable $H(t)$, and I_H and I_u denote the derivatives of the investment cost function with respect to the state variable $H(t)$ and u, respectively. The state variable $H(t)$ as well as the costate variable $\lambda(t)$ are piecewise-continuous functions in $\mathbb{R}_+$. The domain of the impulse control u is $\mathbb{R}_+$.

When there is no jump (i.e. $t \neq \tau_i$ $(i = 1, \ldots, K)$) equation (7) denotes the change of the costate variable and (8) gives the transversality condition. Both (9) and (10) state that at a jump point the marginal cost is equal to the corresponding marginal gains. In equation (9) the jump in the costate variable is equal to I_H, which can be interpreted as the marginal investment cost of increasing the dike height just before a dike height jump of size u_i occurs. Equation (10) states that the costate variable $\lambda(t)$, which can be interpreted as the reduction in expected damage of an additional centimeter dike increase, equals the investment cost of an additional centimeter of a dike increase, i.e. I_u. When dividing equation (11) by the discount rate r, the first term can be interpreted as the decrease of the discounted value of expected damage on the interval $[\tau_i, \infty]$ due to the increase of the dike at τ_i, while the last term is the investment cost of the dike heightening. So, at the jump point τ_i it must also hold that the total gain of increasing the dike should be equal to the cost of increasing the dike. It follows that optimal behavior requires that the Net Present Value (NPV) of the investment to increase the dike height equals zero. The NPV equals the difference between of discounted future gains and current investment cost.

Since $I(u_i, H(\tau^-))$ is not continuous differentiable in u (i.e. the derivative at $u = 0$ does not exists, due to the fixed cost) one of the conditions for applying the Impulse Control Maximum Principle is violated and we have a problem applying condition (12). Chahim et al. (2012) deals with this problem and provides a transformation for the impulse cost function $I(u_i, H(\tau^-))$, which ensures that the application of the Impulse Control Maximum Principle still provides the optimal solution even in the case of a fixed cost. This transformation is based on a continuously differentiable approximation of the impulse cost function (see Section 2.3 of Chahim et al. (2012)). Combining equation (12) with the correct approximation implies that $\lim_{\epsilon \downarrow 0} \frac{\partial IHam_\epsilon}{\partial u}(t, H(t), 0, \lambda(t))u = -\infty u \leq 0$ for every $u \in [0, \infty)$, where $IHam_\epsilon$ is the continuously differentiable approximation of $IHam$. Hence, (12) is satisfied, since it holds for all $t \neq \tau_i$ $(i = 1, \ldots, K)$.

3 Impulse Control Algorithm for a Dike Ring

In this section we present an algorithm that can be used to solve the problem described in the previous section and explain how we apply the necessary optimality conditions to find all dike heightenings that are candidates for occurrence in our optimal solution. In the algorithm $H(T)$ (i.e. the height of the dike at $t = T$) is a search variable. We show how to obtain an upper bound for the optimal $H(T)$ using the necessary optimality conditions. Finally, we explain how to find the optimal $H(T)$.

3.1 Algorithm

In Chahim et al. (2012) it is shown that the Impulse Control sufficient conditions do not hold in all relevant economic problems found in the literature. For our dike height problem the sufficient conditions do not hold due to the fixed cost in the investment cost function, which breaks down the concavity of the Impulse Hamiltonian. Therefore, solutions satisfying the necessary optimality conditions presented in the previous subsection are just candidate optimal solutions. Based on the necessary optimality conditions, we design an algorithm that finds all *candidate* solutions. The candidate that minimizes (6) is the optimal solution. This algorithm can lead to multiple candidate solutions already described in Luhmer (1986). Contrary to Luhmer, who designs a forward algorithm, we implement a backward algorithm, as described by Kort (1989). This algorithm starts at the horizon date T instead of starting at $t = 0$. We do this since the forward algorithm uses the costate variable $\lambda(0)$ as a search parameter to start the algorithm. In other words, the forward algorithm needs $\lambda(0)$ as input to initialize the algorithm. Contrary to the forward algorithm, the backward algorithm uses the dike height at the end of the planning period, $H(T)$, as the search parameter. Since $\lambda(t)$ is only an auxiliary variable, $\lambda(0)$ is harder to guess than $H(T)$. Moreover, Section 3.3 shows that an upper bound for $H(T)$ can be easily derived using the model characteristics. Figure 2 shows a flowchart of the algorithm. The next paragraph explains the algorithm in broad terms. In Appendix A the algorithm is presented in more detail.

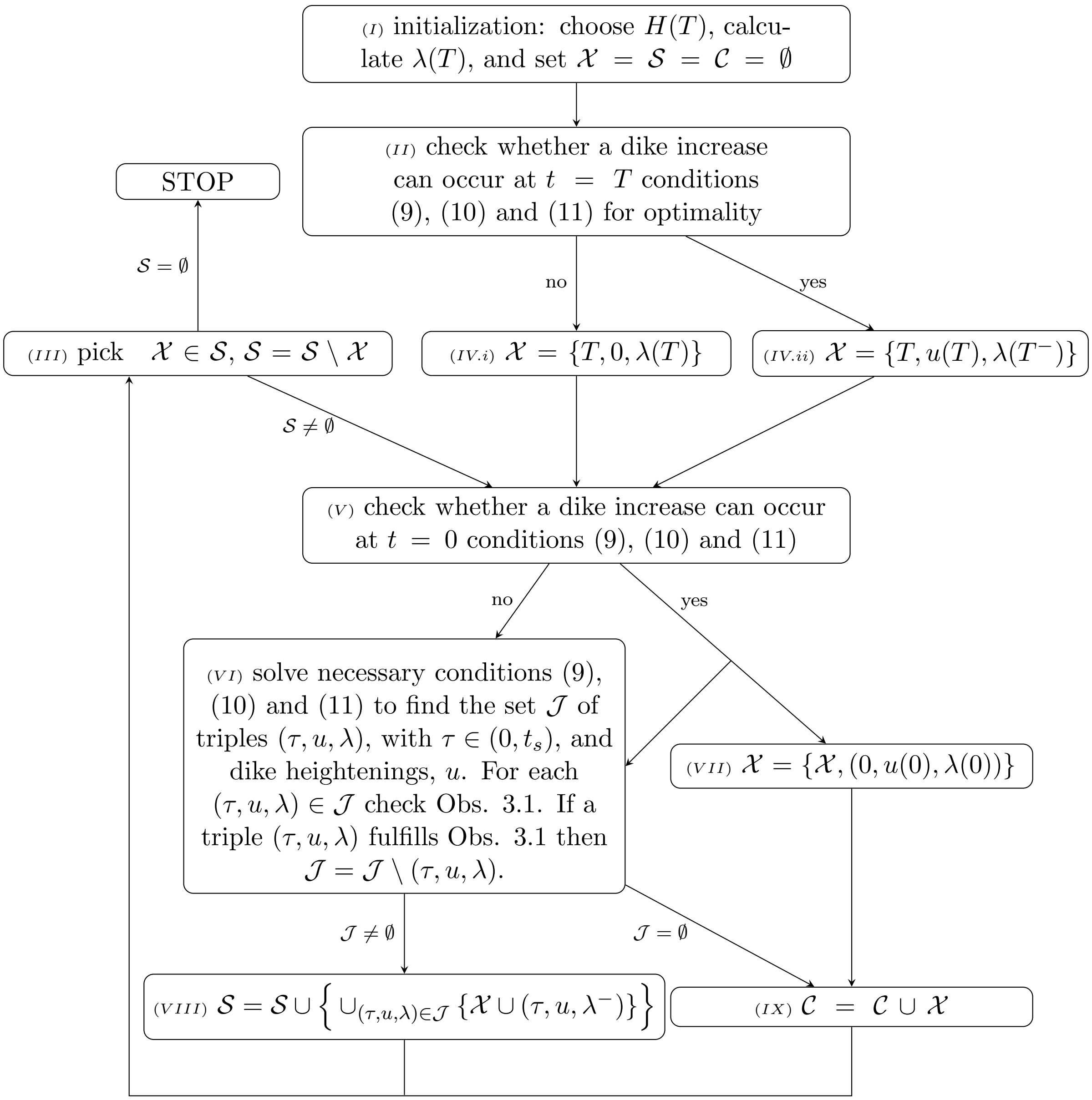


Figure 2: Flowchart of Impulse Control Algorithm for a given $H(T)$.

First, we define $\mathcal{X}$ as a set of triples (τ, u, λ) that represents (part of) a solution based on the necessary optimality conditions, $\mathcal{S}$ as the stack (set) of unfinished (partial) solutions, and $\mathcal{C}$ the set of candidate solutions represented by a set of triples. Let t_s denote the time of the earliest update in $\mathcal{X}$ or T if $\mathcal{X}$ is empty. We refer to the flowchart depicted in Figure 2 using roman capital numbers. To initialize the algorithm (I) we choose a final dike height $H(T)$ and calculate $\lambda(T)$ via equation (8). Then we check whether a dike increase can occur at the horizon date T, and whether it satisfies the necessary optimality conditions (II). If it does not satisfy these conditions, we go via $(IV.i)$, where we set $\mathcal{X} = (T, 0, \lambda(T))$, to (V). If the necessary optimality conditions are satisfied, we go via $(IV.ii)$, where we set $\mathcal{X} = (T, u(T), \lambda(T^-))$, to (V). In (V) we check whether a dike increase can occur at $t = 0$. If a dike heightening at $t = 0$ can occur and satisfies the necessary optimality conditions we save this candidate solution. More precise, in (VII) we add this triple to $\mathcal{X}$, i.e. $\mathcal{X} = \{\mathcal{X}, (0, u(0), \lambda(0))\}$ and save this sequence of triples as a candidate solution in (IX), i.e. $\mathcal{C} = \mathcal{C} \cup \mathcal{X}$. Parallel to this we go to (VI) to find all other candidate solutions (i.e. in (VI) we check whether other candidate solutions can be found, neglecting the jump at $t = 0$). If a dike heightening at $t = 0$ can not occur or does not satisfy the necessary optimality conditions, we go to (VI). In (VI) we solve the necessary optimality conditions to find the set $\mathcal{J}$ of all triples, with $\tau \in (0, t_s)$ and dike heightenings u. If no such triple is found we go to (IX) and save the current sequence $\mathcal{X}$ of triples as a candidate solution. If at least one tripple is found, then in $(VIII)$ we add each triple $(\tau, u, \lambda) \in \mathcal{J}$ to the current sequence $\mathcal{X}$, and add the results to the set of unfinished sequences. From $(VIII)$ and (IX) we go to (III) where we pick a sequence $\mathcal{X}$ from the set of open solutions and continue the procedure as shown in Figure 2. Finally, if the stack (set) of unfinished (partial) solutions is empty, we stop.

We neglect solutions that are associated with a negative dike heightening, since these are infeasible. Such solutions are discarded and not investigated any further. We also neglect sequences of triples for which the sum of the investment cost for the dike heightening u_j and its predecessor u_{j-1} is larger than the investment cost for increasing the dike with $u_j + u_{j-1}$ at time t_j. If this is the case, this solution can never be part of the optimal solution since updating with $u_j + u_{j-1}$ at t_j has lower discounted investment cost and induces more safety (note that $t_j < t_{j-1}$). This results in the following observation.

Observation 3.1. *If:*

(i) $$u_j \leq 0,$$
or
(ii) $$e^{-r\tau_j} I(u_j, H(\tau_j^-)) + e^{-r\tau_{j-1}} I(u_{j-1}, H(\tau_{j-1}^-)) \geq e^{-r\tau_j} I(u_j + u_{j-1}, H(\tau_j^-)),$$

then the corresponding solution can never be optimal.

This approach yields a set of candidates and we select the candidate with the lowest expected cost. Furthermore, we have to check whether $H(0) = 0$. If this is not satisfied, then the initial $H(T)$ is not optimal we restart the algorithm with a new initial $H(T)$, more on this in Section 3.4.

3.2 Solving the Necessary Optimality Conditions

In Figure 2 it is noted in box (VI) that the necessary optimality conditions are used to find all candidate solutions, i.e. all candidate dike heightenings. Equation (10) is of the following form

$$y_1 e^{\alpha_1 t} + y_2 e^{\alpha_2 t} + y_3 e^{\alpha_3 t} - I_u = 0, \tag{13}$$

where y_1, y_2, y_3, α_1, α_2 and α_3 are constants. Expression (11) is of the following form:

$$z_1 e^{\beta_1 t} + z_2 e^{\beta_2 t + \beta_3 u} - rI \begin{cases} > 0 \text{ for } t = 0 \\ = 0 \text{ for } t \in (0, T) \\ < 0 \text{ for } t = T, \end{cases} \tag{14}$$

where z_1, z_2, β_1, β_2 and β_3 are constants. If (13) depends on u and t this can be rewritten into a function $u(t)$ which can be substituted into (14). The resulting non-linear equation has only one unknown t. Solving this leads to all possible jumps points τ, and $u(\tau)$ gives us the corresponding jump size. It can also be the case that (13) depends only on t. Then (13) can be solved to find all τ. Using (14) we find all corresponding jump sizes u. Finally, equation (9) gives us the value of the costate variable before the dike update. This results in a set $\mathcal{J}$ of triples (τ, u, λ).

3.3 Finding an Upper Bound for the optimal $H(T)$

Let $H^*(T)$ denote the end height (i.e. the height at $t = T$) of the optimal solution to our problem (6). An upper bound can be obtained by using the necessary optimality conditions. Investing in a dike is only "profitable" if the marginal cost of the investment is at most equal to the marginal revenue. In the cases of exponential and quadratic cost function the following results can be established.

Proposition 3.2 (Upper bounds for $H^*(T)$)**.**
For exponential cost (see (1)):

Let $T > \frac{1}{\beta} \ln \frac{r(b_0 + a_0 c_0)}{\theta S_0}$*, and let* $\bar{H}_e$ *be defined by the solution of the following equation:*

$$\frac{\theta S_0 e^{\beta T} e^{-\theta \bar{H}_e}}{r} = b_0 e^{a_0 \bar{H}_e} + a_0 c_0 e^{a_0 \bar{H}_e}. \tag{15}$$

Furthermore, let

$$\hat{H}_e = \frac{1}{\theta + a_0} \ln \left(\frac{\theta S_0 e^{\beta T}}{r b_0} \right).$$

Then, it holds that $H^*(T) \leq \bar{H}_e \leq \hat{H}_e$*.*

For quadratic cost (see (2)):

Let $T > \frac{1}{\beta} \ln \frac{r b_1}{\theta S_0}$*, and let* $\bar{H}_q$ *be defined by the solution of the following equation:*

$$\frac{\theta S_0 e^{\beta T} e^{-\theta \bar{H}_q}}{r} = 2 a_1 \bar{H}_q + b_1.$$

Furthermore, let

$$\hat{H}_q = \frac{1}{\theta} \ln \left(\frac{\theta S_0 e^{\beta T}}{r b_1} \right).$$

Then, it holds that $H^*(T) \leq \bar{H}_q \leq \hat{H}_q$*.*

Proof: An upper bound for $H^*(T)$ is the end height for which the following equation (10) holds at the the time horizon T:

$$\lambda(T^+) \quad = \quad I_u\left(u_i, H\right),$$

with

$$\lambda\left(T^+\right) = \lambda\left(T\right) = \frac{\theta S_0 e^{\beta T} e^{-\theta H}}{r}.$$

For exponential investment cost this (with no dike heightening at $t = T$) boils down to solving the following equation:

$$\frac{\theta S_0 e^{\beta T} e^{-\theta \bar{H}_e}}{r} = b_0 e^{a_0 \bar{H}_e} + a_0(c_0 + b_0 u)e^{a_0 \bar{H}_e}, \tag{16}$$

where $\bar{H}_e$ denotes the upper bound for $H^*(T)$. The left-hand side of (16) gives the marginal gain of a dike heightening and is decreasing in $\bar{H}_e$. The right-hand side of (16) gives the marginal cost of such a heightening and is increasing in $\bar{H}_e$. We lower the right-hand side of (16) by omitting $a_0 b_0 u e^{a_0 \bar{H}_e}$, this shifts the graph of to the right and results in a lower marginal cost at $t = T$. Additionally, this gives us equation (15). Since $T > \frac{1}{\beta} \ln \frac{r(b_0 + a_0 c_0)}{\theta S_0}$, we have that the left-hand side of (15) is larger than the right-hand side of (15) at $\bar{H}_e = 0$. Combining the latter with the fact that left-hand side of (15) is decreasing in $\bar{H}_e$, that the right-hand side of (15) is increasing in $\bar{H}_e$, that

$$\lim_{\bar{H}_e \to \infty} b_0 e^{a_0 \bar{H}_e} + a_0 c_0 e^{a_0 \bar{H}_e} = \infty,$$

and that

$$\lim_{\bar{H}_e \to \infty} \frac{\theta S_0 e^{\beta T} e^{-\theta \bar{H}_e}}{r} = 0,$$

results in a unique solution $\bar{H}_e$ for equation (15). Furthermore, we lower the right-hand side of (15) by now omitting $a_0(c_0 + b_0 u)e^{a_0 \bar{H}_e}$, this again shifts the graph of the right-hand side to the right and results in a lower marginal cost at $t = T$. Hence, an upper bound for $\bar{H}_e$ results from solving the following equation

$$\frac{\theta S_0 e^{\beta T} e^{-\theta \hat{H}_e}}{r} = b_0 e^{a_0 \hat{H}_e}, \tag{17}$$

where $\hat{H}_e$ denotes the upper bound for $\bar{H}_e$ (i.e. $H^*(T) \leq \bar{H}_e \leq \hat{H}_e$). Solving (17) we get that $\hat{H}_e$ is given by:

$$\hat{H}_e = \frac{1}{\theta + a_0} \ln\left(\frac{\theta S_0 e^{\beta T}}{r b_0}\right).$$

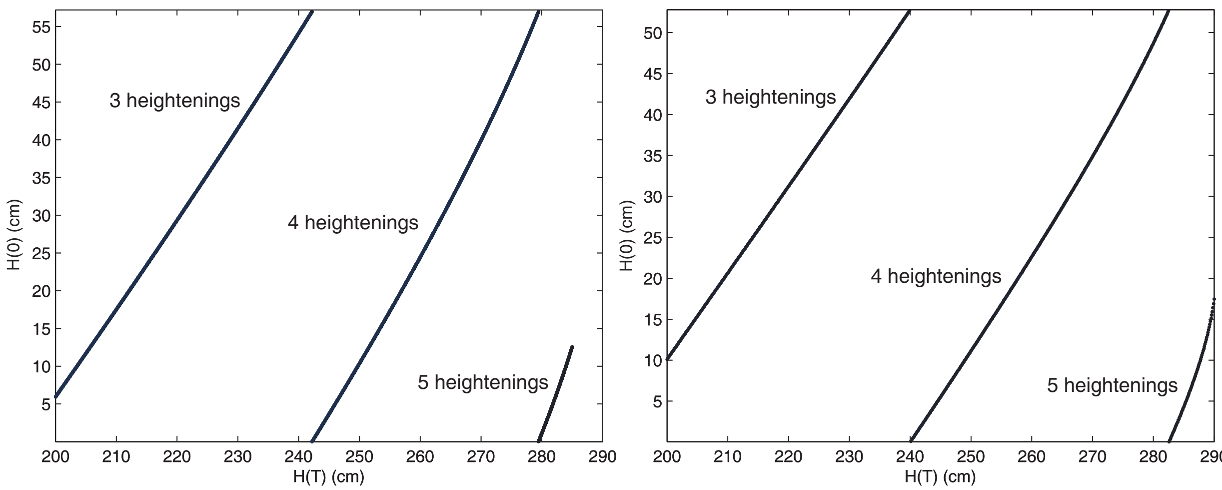
The proof for the quadratic cost function goes analogously. □

Note that these upper bounds for $H(T)$ can also be used for the Dynamic Programming approach in Eijgenraam et al. (2011) to decrease the number of states, see Section 4.2. Moreover, we have that $\theta S_0 > r(b_0 + a_0 c_0)$ and $\theta S_0 > r b_1$ for all dikes (in the Netherlands). Hence, we have that the condition on T for both cost function is always satisfied.

3.4 Finding the Optimal $H(T)$

Recall that an ending height $H(T)$ is required as an input to the algorithm in Section 3.1. For an arbitrary $H(T)$, the algorithm is not guaranteed to produce a feasible solution to problem (4), because the condition on the initial height $H(0) = 0$ might be violated. In that case we always have $H(0) > 0$—since negative heightenings are not allowed—and apparently there does not exist a feasible solution for the chosen $H(T)$ that satisfies all necessary optimality conditions. Thus, we need a procedure to find an ending height for which the algorithm returns a feasible solution.

If we find all ending heights for which the algorithm returns feasible solutions, then we know that the optimal solution must be among them, because all solutions, by construction, satisfy all necessary optimality conditions—and there are no other solutions with this property. The dependency on $H(T)$ of any solution produced by the algorithm is piecewise continuous, with discontinuities occurring when the total number of heightenings in $[0, T]$ changes. This is illustrated by Figure 3, which shows the residual height $H(0)$ corresponding to the candidate solution that results from the selected ending height $H(T)$. At each discontinuity point the total number of heightenings changes as indicated in the figure. Hence, a bisection method on $H(0)$ could be used to search for an ending height that produces a feasible solution, i.e., $H(0) = 0$. For now, we propose the simpler approach of discretization of $H(T)$ as is also necessary for the dynamic programming approach to the problem (see Eijgenraam et al. (2011))



(a) Exponential investment cost. (b) Quadratic investment cost.

Figure 3: Plot of the residual height (i.e. $H(0)$) vs. $H(T)$ for dike 10.

An upper bound for the discretization of $H(T)$ is readily provided by $\bar{H}$ (see Section 3.3) and a suitable lower bound is the current dike height plus the (future) sea-level rise. The set of solutions produced by the algorithm applied to a discretization of $H(T)$ in this range is unlikely to contain exact feasible solutions. To cope with the infeasibility of the solutions, we transform each solution to a feasible solution by adding the residual height $H(0)$ to the solution's first heightening. In that way, both the investment cost of the first heightening and the expected damage from $t = 0$ until the first heightening increase, and which makes that there is some difference between the objective values of the original and the transformed solution. Note that if the residual height $H(0)$ is small—and for any reasonably fine grid, solutions with $H(0)$ close to zero should be found—then this difference will be small as well. Of all transformed solutions obtained in this way, we pick the one with the smallest objective value.

4 Comparing Impulse Control to Dynamic Programming

This section consists of two parts. First, we compare the numerical results obtained using the Impulse Control approach to the results found in Eijgenraam et al. (2011) using Dynamic Programming. Second, we derive the computation order of both methods.

4.1 Numerical Results for 5 Dike Rings

In this section we apply the algorithm described in Section 3. The data used in this section are taken from Den Hertog and Roos (2009) and are presented in Table 3. The data are made available by Rijkswaterstaat/Deltares (i.e. a bureau concerned with practical execution of the public works and water management part of the Dutch Ministry of Infrastructure and the Environment) and were generated by water experts. It is clear that the choice of T will influence the solution. If we choose T too small then this can affect the solution in the beginning of the planning period. We choose T such that the solution in the beginning of the planning period remains stable when T increases. As in Eijgenraam et al. (2011) we set $T = 300$. Taking $T = 600$ gives similar results for the beginning of the planning period compared to $T = 300$. This is caused by the fact that the discount factor ($e^{-0.04*300} \approx 0.00000614$) is small for large values of t. Hence, the effect of the salvage value is very small when $T = 300$. In Tables 4 and 5 the solutions obtained by using the algorithm described in Section 3.1 for both exponential and quadratic investment cost can be found.

Dike No.	10	11	15	16	22
a_0	0.0014	0	0.0098	0.01	0.0066
b_0	0.6258	1.7068	1.1268	2.1304	0.9325
c_0	16.6939	42.62	125.6422	324.6287	154.4388
a_1	0.0004	0	0.027	0.102	0.0154
b_1	0.7637	1.7168	3.779	3.1956	2.199
c_1	12.603	42.3003	67.699	319.25	141.01
V_0	1564.9	1700.1	11810.4	22656.5	9641.1
r	0.04	0.04	0.04	0.04	0.04
P_0	1/2270	1/855	1/729	1/906	1/1802
H_0	0	0	0	0	0
α	0.033027	0.032	0.0502	0.0574	0.07
η	0.32	0.32	0.76	0.76	0.62
γ	0.02	0.02	0.02	0.02	0.02
ζ	0.003774	0.003469	0.003764	0.002032	0.002893

Table 3: Parameter values for dikes 10, 11, 15, 16 and 22.

Dike ring no.	10	11	15	16	22
Heightenings($\tau_i : u_i$)	272.8 : 52.18	275.9 : 54.56	259.2 : 57.33	271.6 : 47.89	261.6 : 50.97
	217.0 : 56.43	218.9 : 61.71	206.2 : 54.16	219.2 : 51.69	199.9 : 53.37
	160.1 : 56.90	160.2 : 62.35	154.3 : 53.47	165.3 : 52.41	137.6 : 53.65
	103.0 : 56.95	101.3 : 62.42	103.7 : 53.32	111.5 : 52.55	75.2 : 53.68
	45.9 : 56.96	42.4 : 62.42	51.2 : 53.29	57.5 : 52.57	12.7 : 53.71
			0 : 55.82	3.5 : 52.58	
$H(T)$	279.41	303.47	327.39	309.69	265.37
$\bar{H}_e$	290.93	311.48	347.14	320.48	278.75
$\hat{H}_e$	292.12	311.48	360.28	334.65	288.77
Investment cost	10.16	30.18	414.59	797.75	198.42
Damage cost	29.87	80.05	130.55	291.84	110.82
Total cost	40.03	110.23	545.14	1089.59	309.24

Table 4: Impulse Control solutions for dikes 10, 11, 15, 16 and 22, with exponential cost function.

Dike ring no.	10	11	15	16	22
Heightenings($\tau_i : u_i$)	275.9 : 57.15	274.6 : 55.09	282.0 : 62.62	245.3 : 76.90	262.1 : 56.36
	213.0 : 61.35	217.8 : 61.39	214.1 : 77.43	176.7 : 69.35	194.5 : 58.53
	153.4 : 57.30	159.4 : 61.97	149.7 : 69.92	113.8 : 61.03	130.5 : 54.13
	98.0 : 53.99	100.9 : 62.03	92.3 : 59.86	56.9 : 52.51	70.7 : 50.15
	45.2 : 52.78	42.4 : 62.05	42.6 : 49.39	3.2 : 48.25	12.7 : 49.74
			0 : 46.44		
$H(T)$	282.57	302.53	365.66	308.04	268.91
$\bar{H}_q$	290.22	311.28	370.28	331.79	283.82
$\hat{H}_q$	299.30	311.28	410.25	387.76	304.39
Investment cost	10.17	30.16	421.30	822.41	201.35
Damage cost	29.96	80.06	160.91	334.72	115.74
Total cost	40.13	110.23	582.21	1157.13	317.09

Table 5: Impulse Control solutions for dikes 10, 11, 15, 16, 22, with quadratic cost function.

Dike ring no.	10	11	15	16	22
Heightenings($\tau_i : u_i$)	274 : 51.84	272 :42.24	262 : 54.72	274 : 45.60	254 : 52.08
	219 : 55.68	218 :59.52	209 : 54.72	223 : 50.16	194 : 52.08
	162 : 57.60	160 :61.44	156 : 54.72	171 : 50.16	133 : 52.08
	104 : 57.60	101 :63.36	103 : 54.72	116 : 54.72	73 : 52.08
	46 : 57.60	43 :61.44	50 : 54.72	60 : 54.72	12 : 52.08
			0 : 54.72	4 : 54.72	
$H(T)$	280.32	288.00	328.32	310.08	260.4
Investment cost	10.16	29.33	413.39	796.31	202.09
Damage cost	29.87	80.90	131.95	294.13	107.33
Total cost	40.04	110.24	545.34	1090.44	309.41

Table 6: Dynamic Programming solutions for dikes 10, 11 15, 16, 22, with exponential cost function.

Dike ring no.	10	11	15	16	22
Heightenings($\tau_i : u_i$)	277 : 55.68	272 : 42.24	280 : 63.84	274 : 45.60	265 : 55.80
	214 : 61.44	218 : 59.52	212 : 77.52	223 : 50.16	197 : 59.52
	155 : 57.60	160 : 61.44	149 : 68.40	171 : 50.16	131 : 55.80
	99 : 53.76	101 : 63.36	92 : 59.28	116 : 54.72	69 : 52.08
	46 : 53.76	43 : 61.44	42 : 50.16	60 : 54.72	12 : 48.36
			0 : 45.60	4 : 54.72	
$H(T)$	282.24	288.00	364.80	310.08	271.56
Investment cost	9.97	29.33	418.94	840.70	208.15
Damage cost	30.17	80.90	163.35	317.51	112.09
Total cost	40.14	110.24	582.28	1158.21	317.24

Table 7: Dynamic Programming solutions for dikes 10, 11, 15, 16, 22, with quadratic cost function.

After comparing the results presented in Table 4 and 5 with the Dynamic Programming results taken from Eijgenraam et al. (2011) presented in Table 6 and 7, we can make the following observations:

- The (total) cost using the Impulse Control approach are always lower. The reason for this (minor) difference is due to the discretization of the problem in time and dike height in the Dynamic Programming approach.

- Comparing the results between the exponential and quadratic investment cost functions for the Impulse Control approach given in Table 4 and 5, respectively, no significant difference can be found. The first dike heightening for Impulse control using a quadratic cost function takes place slightly earlier comparing it with the exponential cost function. However, the corresponding amount of this first dike heightening is lower. This difference is also observed for the Dynamic Programming approach.

- Dike 15 needs to be heightened immediately (i.e. at $\tau_1 = 0$). This result is found for both the exponential and the quadratic cost function, and for both approaches.

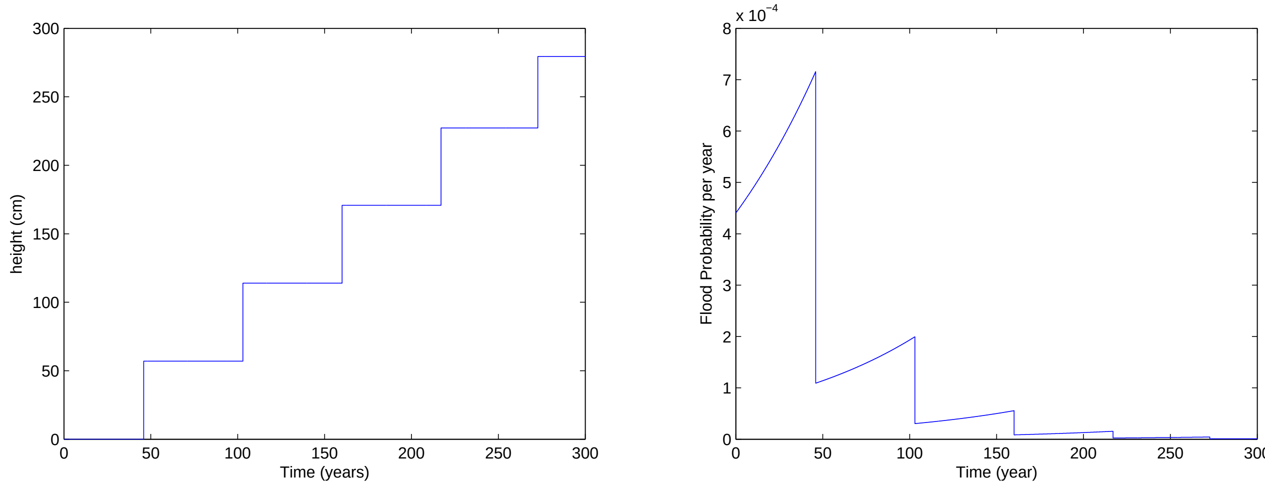
- The Impulse Control approach results in a significantly higher $H(T)$ for dike 11 compared to the Dynamic Programming approach. This is observed for both cost functions.

- For exponential investment cost the upper bound $\bar{H}_e$ is very close to the optimal $H(T)$ found for all 5 dikes. Comparing the upper bound for quadratic cost, $\bar{H}_q$, with $\bar{H}_e$ we observe that $\bar{H}_q$ is higher than $\bar{H}_e$ for dikes 15, 16 and 20. The values are comparable for dike rings 10 and 11.

- When the first dike heightening is far from time zero, $\bar{H}_e$ and $\hat{H}_e$ are closer to each other (same holds for $\bar{H}_q$ and $\hat{H}_q$). For dike ring 11 we have that $a_1 = a_0 = 0$ and hence $\bar{H}_e = \hat{H}_e$ and $\bar{H}_q = \hat{H}_q$.

In Figures 4 and 5 the optimal dike height and the corresponding flood probability of dike 10 is presented for both exponential and quadratic investment cost. It is striking to see that the upper bound(s) are very close to the optimal dike height at time T. Finally, in Figures 4 and 5 one can observe that at the time moments where a dike heightening occurs the flood probability drops instantaneously.

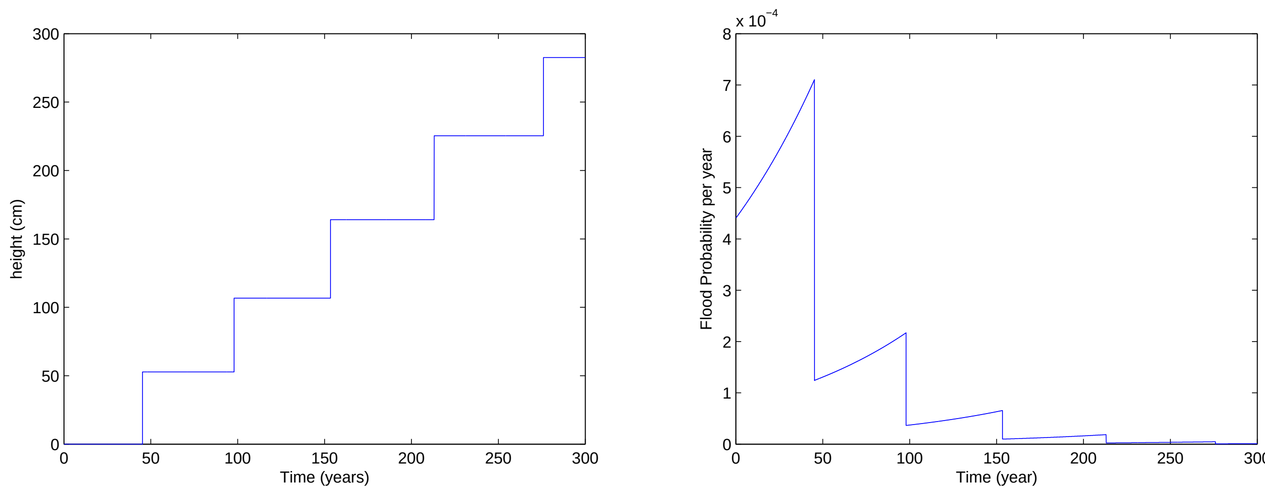
We also observe that after each dike heightening at most three candidate dike heightenings were found by the algorithm (stage VI). In case of three candidates we always found that two out of the three candidates could not be optimal, since one was always negative (Observation 3.1, (i))) and for the other one it holds that combining this heightening with its predecessor was an improvement (see Observation 3.1, (ii))).



(a) Dike height over time.

(b) Flood probability per year corresponding to the optimal dike height.

Figure 4: Optimal dike height of dike 10 and the corresponding flood probability using exponential investment cost



(a) Dike height over time.

(b) Flood probability per year corresponding to the optimal dike height.

Figure 5: Optimal dike height of dike 10 and the corresponding flood probability using quadratic investment cost

4.2 Computation Time

In Section 4 of Eijgenraam et al. (2011) a Dynamic Programming approach is described that corresponds to the above described Impulse Control model. A drawback of this approach is that the finite time horizon $[0, T]$ as well as the height of the dike $H(t)$ are discretized. This directly has an effect on the accuracy of the method. This can partly be resolved by taking a "finer" discretization. However, this will clearly affect the computation time of the problem. The discretization chosen in Eijgenraam et al. (2011) seemed to be fine enough for the dike heightening problem.

Dynamic Programming (DP)
The number of computations that have to be made in the DP approach depends on the number of stages and states (for each stage the value of each state should be calculated). The problem is discretized in both time and dike height. Let us call M the number of states per stage and T the number of stages. Cost are related to the transition from one state to another. The DP problem can be presented in a graph where the vertices in the graph are the states, and the arcs of the graph represent the transition. The aim of DP is to find the shortest path in the graph. In the DP approach used to solve the dike heightening problem in Eijgenraam et al. (2011), the stages defined as the years $t = -1, 0, 1, 2, ...T$, in which $t = -1$ is the time just before $t = 0$. The state at stage t is defined as $H(t)$. For the initial state at stage $t = -1$ it holds that $H(-1) = 0$. Also we know that a transition can only occur from state $H(t)$ in stage t to state $H(t+1)$ in stage $t+1$ such that $H(t+1) \geq H(t)$. In Eijgenraam et al. (2011) the discounted investment and damage cost in the period $[t, t+1]$, for $t = 0, 1, \ldots, T-1$ are given by

$$c_t(H(t), H(t+1)) = \int_t^{t+1} S(t)e^{-rt}dt + I(H(t+1) - H(t), H(t))e^{-r(t+1)},$$

and for $t = -1$, by

$$c_{-1} = I(H(-1), H(0)) = I(0, H(0)).$$

The recursive relation for the DP approach is

$$f_t(H(T)) = \min_{H(t)<H(t+1)\in\mathcal{H}_{t+1}} \{f_{t+1}(H(t+1) + c_t(H(t), H(t+1))\}, \quad t < T, \quad H(T) \in \mathcal{H}_t,$$

where $\mathcal{H}_t$ denotes the set of all feasible dike heights at time t. Starting in state $H(T)$, $f_t(H(T))$ denotes the minimal cost to cover the years $t, t+1, \ldots, T, T+1, \ldots, \infty$. The costs after $t = T$ are given by

$$f_T(H(T)) = \frac{S(T)e^{-rT}}{r}.$$

It is easy to see that this DP approach is of order $O(\alpha_{DP}M^2T)$, where α_{DP} denotes the basic operations needed to calculate the transitions cost from one state to another.

Impulse Control (IC)
Let J be the number of dike heightenings found. To make an easy comparison with DP we run the algorithm for the same candidate final dike heights, i.e. we take the states used in the DP approach as input determining the optimal final dike height. In the dynamic programming approach for each stage a certain number of states are defined. Clearly, for the impulse control approach this is not necessary. Let us call the number of basic operations needed to solve the necessary optimality conditions (see Section 3.2) to find all candidate dike heightenings α_{IC}. Then it is easy to see that this problem is of order $O(\alpha_{IC}JM)$. In the previous section we have seen that the number of dike heightenings (5 or 6) in the dike heightening problem never exceeds the number of states ($M = 300$) used in the DP approach and α_{DP} and α_{IC} are comparable. Hence, we can conclude that IC needs less computation time DP.

5 Conclusions and Recommendations

In this paper we present the first real life application of the Impulse Control Maximum Principle. In doing so, we propose an alternative for the Dynamic Programming approach used in

Eijgenraam et al. (2011). We show that, compared to the Dynamic Programming approach, the Impulse Control approach has lower computation time. This can be explained since the Impulse Control approach does not need discretization in time and only discretization for the dike height at the end of the time horizon (final stage), unlike dynamic programming where discretization is needed for time and for the heights (states) for each stage. Comparing the total cost for the dike updating scheme for the five dikes presented in this paper with the total cost using the Dynamic Programming approach, we observe that the total cost for the Impulse Control approach is always lower. However, the differences are very small. Further, we identify upper bounds for the final dike height, by using the necessary optimality conditions at the end of the planning period. It is striking to see that both proposed upper bounds are very close to the optimal dike height at the end of the planning period. The way we derive these upper bounds can be used in general, so that these upper bounds can also be implemented in the Dynamic Programming approach. We show that the Impulse Control approach works well for both exponential and quadratic investment cost.

A possible extension of this paper would be adding some preventive dike maintenance. It would be interesting to analyze the interaction between preventive dike maintenance and the impulse dike heightening. This extension will quadratically increase the number of states for the Dynamic Programming approach and hence take more time to compute. Another possible extension is applying Impulse Control to nonhomogeneous dikes (i.e. dikes or dike rings that consist of multiple segments) for which the dynamic programming approach is not useful since it suffers from the well-known combinatorial explosion. Also other maintenance problems can be considered.

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Appendix A: Backward Algorithm for Impulse Control

In this section the algorithm described in Section 3.1 is presented in more detail. Before we start we define $\mathcal{X}$ as a sequence of triples (τ, u, λ), $\mathcal{S}$ the stack (set) of open solutions, and $\mathcal{C}$ the set of candidate solutions. We need one more variable t_s defined as

$$t_s = \begin{cases} T & \text{if} \quad \mathcal{X} = \emptyset, \\ \min_{(\tau,u,\lambda)\in\mathcal{X}} \tau & \text{if} \quad \mathcal{X} \neq \emptyset. \end{cases}$$

Step I: Initialization:

Choose $H(T)$
Determine the value of the co-state variable:

$$\lambda(T) = \frac{\theta S_0 e^{\beta T} e^{-\theta H(T)}}{r}.$$

Step II: Check whether a dike height increase can occur at time $t = T$ and whether it is optimal. Derive $H(T^-)$ and $u(T)$ from

$$H\left(T^+\right) - H\left(T^-\right) = u(T),$$

and

$$-I_u\left(u(T), H\left(T^-\right)\right) + \lambda\left(T^+\right) = 0.$$

The dike height increase is optimal at time T if

$$-S_0e^{\beta T}e^{-\theta H\left(T^+\right)}+S_0e^{\beta T}e^{-\theta H\left(T^-\right)}-rI\left(u\left(T\right),H\left(T^-\right)\right)<0.$$

If so, go to step $IV.ii$. Otherwise, define

$$\begin{aligned}H\left(T^+\right)&=H\left(T\right),\\ \lambda\left(T^+\right)&=\lambda\left(T\right),\end{aligned}$$

and go to step $IV.i$.

Step III: If $\mathcal{S}=\emptyset$ STOP. Else pick $\mathcal{X}\in\mathcal{S}$, set $\mathcal{S}=\mathcal{S}\setminus\mathcal{X}$ and go to step V.

Step $IV.i$: Set $\mathcal{X}=\{(T,0,\lambda(T))\}$.

Step $IV.ii$: Set $\mathcal{X}=\{(T,u(T),\lambda(T^-))\}$.

Step V: Check whether a dike height increase can occur at time 0 and whether it is optimal.
Solve (10) to find $u(0)$. The dike height increase is optimal if

$$-S_0e^{\beta T}e^{-\theta H\left(0^+\right)}+S_0e^{\beta T}e^{-\theta H\left(0^-\right)}-rI\left(u\left(0\right),H\left(0^-\right)\right)>0.$$

If so, go to VI and to VII. If not, go to step VI.

Step VI: Find all $t\in(0,\tau)$ such that

$$\lambda(t^+)=e^{-r(t-t_s)}\lambda(t_s)+\int_t^{t_s}e^{r(t-s)}\theta S_0e^{\beta s}e^{-\theta(H(t_s))}ds. \tag{18}$$

At the point in time where a dike increase can occur, equations (9), (10) and (11) hold. Combining equation (10) and (18) gives us a condition that holds at the jump point:

$$e^{-r(t-t_s)}\lambda(t_s)+\int_t^{t_s}e^{r(t-s)}\theta S_0e^{\beta s}e^{-\theta(H(t_s))}ds-I_u(u,H(t_s))=0. \tag{19}$$

Solving equation (19) results either in an explicit function $u(t)$ for the dike heightening or gives all τ for which (19) holds. When $u(t)$ can explicitly be identified, go to step VI.i, else go to step VI.ii.

Step $VI.i$: Substituting $u(t)$ in equation (11) yields

$$-S_0e^{\beta t}e^{-\theta H(t_s)}+S_0e^{\beta t}e^{-\theta(H(t_s)-u(t))}-rI(u(t),H(t_s))=0, \tag{20}$$

which is an equation that only depends on t and holds for each jump point $t\in(0,t_s)$.

If equation (20) is solvable, it gives us all potential jump points τ. Using this, we get all dike heightenings u using $u(t)$ (from equation (19)). This gives a set $\mathcal{J}$ of triples (τ,u,λ). For each triple $(\tau,u,\lambda)\in\mathcal{J}$ check Observation 3.1 conditions (i) and (ii). If a triple (τ,u,λ) satisfies condition (i) or (ii) of Observation 3.1 then $\mathcal{J}=\mathcal{J}\setminus(\tau,u,\lambda)$. If $\mathcal{J}\neq\emptyset$, go to $VIII$, else go to step IX.

Step $VI.ii$: For each τ found in step V solve

$$-S_0e^{\beta t}e^{-\theta H(t_s)} + S_0e^{\beta t}e^{-\theta(H(t_s)-u(t))} - rI(u(t),H(t_s)) = 0, \tag{21}$$

to find the corresponding u. This gives a set $\mathcal{J}$ of triples (τ, u, λ). For each triple $(\tau, u, \lambda) \in \mathcal{J}$ check Observation 3.1. If a triple (τ, u, λ) fulfills Observation 3.1 then $\mathcal{J} = \mathcal{J} \setminus (\tau, u, \lambda)$. If $\mathcal{J} \neq \emptyset$, go to $VIII$, else go to step IX.

Step VII: Save $\mathcal{X} = \{\mathcal{X}, (0, u(0), \lambda(0))\}$ and go to step IX.

Step $VIII$: Add each triple $(\tau, u, \lambda) \in \mathcal{J}$ to the current sequence $\mathcal{X}$ and add the result to the stack (set) of unfinished (partial) solutions, i.e.:

$$\mathcal{S} = \mathcal{S} \cup \left\{ \cup_{(\tau,u,\lambda)\in\mathcal{J}} \left\{\mathcal{X} \cup (\tau, u, \lambda^-)\right\}\right\}$$

and go to step III.

Step IX: Save set of sequences $\mathcal{X}$ as candidate solution, i.e.:

$$\mathcal{C} = \mathcal{C} \cup \mathcal{X},$$

and go to step III.

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