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Publication date:
2012

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Citation for published version (APA):

Kleppe, J., Borm, P. E. M., & Hendrickx, R. L. P. (2012). *Fall Back Equilibrium for 2 x n Bimatrix Games*. (CentER Discussion Paper; Vol. 2012-044). Operations research.

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No. 2012-044

FALL BACK EQUILIBRIUM FOR 2 X n BIMATRIX GAMES

By

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May 30, 2012

ISSN 0924-7815

Fall back equilibrium for $2 \times n$ bimatrix games

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May 29, 2012

Abstract

In this paper we provide a characterisation of the set of fall back equilibria for $2 \times n$ bimatrix games. Furthermore, for this type of games we discuss the relation between the set of fall back equilibria and the sets of perfect, proper and strictly perfect equilibria. In order to do this we reformulate the existing characterisations for these three equilibrium concepts by the use of refinement-specific subgames.

Keywords: game theory, fall back equilibrium, $2 \times n$ bimatrix game, equilibrium refinement

JEL Classification Number: C72

1 Introduction

In this paper we provide a characterisation of the set of fall back equilibria (Kleppe et al. (2012)) for $2 \times n$ bimatrix games. Furthermore, for this type of games we discuss the relation between the set of fall back equilibria and the sets of perfect (Selten (1975)), proper (Myerson (1978)) and strictly perfect equilibria (Okada (1984)). In order to do this we reformulate the characterisations, as provided by Borm (1992), for these three equilibrium concepts.

In general it is hard to find *all* Nash equilibria of a non-cooperative strategic game (Papadimitriou (2001)). For matrix games linear programming solutions (Tucker (1960)) exist. Furthermore, for general two- and n -player strategic games several methods exist to find *a* Nash equilibrium, *e.g.*, the Lemke-Howson algorithm (Lemke and Howson (1964)) and the method by Porter, Nudelman and Shoham (Porter et al. (2004)).

For $2 \times n$ bimatrix games Borm et al. (1988) provide a geometric labelling method to find *all* Nash equilibria. In a subsequent paper (Borm (1992)) this method is used

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to characterise the sets of perfect, proper and strictly perfect equilibria for such games. Each of these characterisations consists of a list of necessary and sufficient conditions for a strategy pair to be an equilibrium of a certain type.

We start this paper by revisiting the papers by Borm et al. (1988) and Borm (1992) with the main objective to reformulate the characterisations of the three Nash equilibrium refinements in a more manageable way. The basis of these new formulations is formed by explicitly considering refinement-specific subgames obtained by eliminating pure strategies. We illustrate that determining the sets of equilibria by these new formulations is straightforward, especially when one uses the corresponding graphical representation of a $2 \times n$ bimatrix game.

In the main part of this paper we use the geometric labelling method in combination with the set-up with subgames to also characterise the set of fall back equilibria for $2 \times n$ bimatrix games.

The concept of fall back equilibrium is introduced by Kleppe et al. (2012). In the thought experiment underlying this Nash equilibrium refinement each player strategically chooses a back-up action, which he plays in case the strategy of his first choice is blocked, which happens with a small (but positive) probability. This probability is assumed to be independent of the chosen action(s), but may vary between players. Hence, in the thought experiment players act by choosing both a primary and a back-up strategy. These strategies in the original game together define a strategy in the fall back game. The payoffs in the fall back game are the expected payoffs in the original game given the blocking probabilities. Furthermore, in the fall back game players are also allowed to use mixed strategies. The limit point of a sequence of Nash equilibria of fall back games when the blocking probabilities converge to zero then gives rise to a fall back equilibrium in the original game.

The concept of fall back equilibrium shares the idea of “errors” converging to zero with the concepts of perfect, proper and strictly perfect equilibrium. The fundamental difference between the thought experiments regarding these “errors” is that in the fall back game players *choose* their back-ups strategically, while in the other thought experiments these “back-ups” are exogenously given. However, by the use of the new formulations we easily derive two relations between the ensuing sets of equilibria for $2 \times n$ bimatrix games. First of all, we obtain that each proper equilibrium is a fall back equilibrium. The second result is that whenever none of the two pure strategies of player 1 is dominant, each fall back equilibrium is a perfect equilibrium.

The outline of this paper is as follows. In Section 2 we provide the preliminaries with

respect to the geometric labelling method introduced by Borm et al. (1988) and recall the characterisation of the set of Nash equilibria for $2 \times n$ bimatrix games. In Section 3 we provide the reformulated characterisations of the sets of perfect, proper and strictly perfect equilibria. In Section 4 we characterise the set of fall back equilibria, and discuss the relation to the other Nash equilibrium refinements.

2 Nash equilibrium in $2 \times n$ bimatrix games

A $2 \times n$ bimatrix game is the mixed extension of a finite two-player strategic game. It is denoted by a pair (A, B) of real-valued matrices of size $2 \times n$. The players are named player 1 and player 2, where player 1 chooses a row and player 2 a column. The corresponding index sets (or rows and columns) are denoted by $M = \{1, 2\}$ and $N = \{1, \dots, n\}$, respectively. The spaces of mixed strategies for rows and columns are denoted by Δ_M and Δ_N , and together they define the space of strategy pairs $\Delta = \Delta_M \times \Delta_N$.

A pure strategy of player 1, which corresponds to a unit vector in Δ_M , is denoted by e_i , with $i \in M$. Similarly, a pure strategy of player 2 corresponds to a unit vector in Δ_N and is denoted by f_j , $j \in N$. Moreover, a typical element of Δ_M is given by p , a typical element of Δ_N by q . By $\dot{\Delta}_M$ we denote $\Delta_M \setminus \{e_1, e_2\}$.

Both players have a pure best reply correspondence, $PB^1(q) = \{e_i \mid e_i A q \geq e_k A q \text{ for all } k \in M\}$ and $PB^2(p) = \{f_j \mid p B f_j \geq p B f_\ell \text{ for all } \ell \in N\}$. Furthermore, the carrier of player 1's strategy p is given by $C(p) = \{e_i \mid p_i > 0\}$. Analogously, $C(q) = \{f_j \mid q_j > 0\}$. A Nash equilibrium (Nash (1951)) is a pair of mixed strategies $(p, q) \in \Delta$ such that $C(p) \subseteq PB^1(q)$ and $C(q) \subseteq PB^2(p)$. The non-empty set of all Nash equilibria of a bimatrix game (A, B) is denoted by $NE(A, B)$.

Let (A, B) be a $2 \times n$ bimatrix game. A pure strategy f_j of player 2 is provided with label [1] if $PB^1(f_j) = \{e_1\}$, with label [2] if $PB^1(f_j) = \{e_2\}$, and with label [12] if $PB^1(f_j) = \{e_1, e_2\}$. Let $J([1]) = \{f_j \mid PB^1(f_j) = \{e_1\}\}$ represent the set of player 2's pure strategies with label [1]. The sets $J([2])$ and $J([12])$ are defined analogously.

Since player 1 has only two pure strategies, his strategy space Δ_M corresponds to the interval $[0, 1]$, indicating the probability with which he plays the first row e_1 . For $j \in \{1, \dots, n\}$, the line $p \mapsto p B f_j$ represents all possible payoffs to player 2 corresponding to the pure strategy f_j . In order to determine the set of Nash equilibria, the interesting part of these n lines is the piecewise linear maximum function, or rather all line segments constituting this upper envelope, as it fully describes the best reply correspondence of player 2.

Note that the piecewise linear maximum function has t line segments for some $t \in \{1, \dots, n\}$ corresponding to a subdivision of $[0, 1]$ into t intervals. Denote by c_0, c_1, \dots, c_t

the extreme points of these intervals, with $0 = c_0 < c_1 < \dots < c_{t-1}, c_t = 1$, such that for each $k \in \{0, 1, \dots, t\}$, $p_k = c_k e_1 + (1 - c_k) e_2$ is the corresponding strategy of player 1. Further, for each $k \in \{1, \dots, t\}$, define $I_k = \{\alpha e_1 + (1 - \alpha) e_2 \mid c_{k-1} < \alpha < c_k\}$ as the interval of strategies corresponding to the k^{th} line segment of the maximum function.

For $p \in \Delta_M$ let $PB^2(p, [1])$, $PB^2(p, [2])$ and $PB^2(p, [12])$ denote the sets of pure best replies to p with the corresponding label. Since $PB^2(p') = PB^2(p'')$ for all $p', p'' \in I_k$, $k \in \{1, \dots, t\}$, we unambiguously define $PB^2(I_k)$, as well as $PB^2(I_k, [1])$, $PB^2(I_k, [2])$ and $PB^2(I_k, [12])$.

For $p \in \Delta_M$, let $S(p)$ be the set of solutions to p , *i.e.*, the set of all strategies $q \in \Delta_N$ of player 2 such that $(p, q) \in NE(A, B)$. Since

$$\begin{aligned} S(p) &= \{q \in \Delta_N \mid (p, q) \in NE(A, B)\} \\ &= \Delta_N \cap \bigcap_{j=1}^n \{q \in \mathbb{R}^N \mid pBq \geq pBf_j\} \cap \bigcap_{i=1}^2 \{q \in \mathbb{R}^N \mid pAq \geq e_i Aq\}, \end{aligned}$$

the set $S(p)$ is bounded and determined by a finite system of linear inequalities. Hence, for each p , $S(p)$ is a (possibly empty) polytope.

The extreme points of $S(p)$ are provided by the set of *pure solutions* $PS(p)$, given by $PS(p) = \{q \in S(p) \mid |C(q)| = 1\}$, and the set of *coordination solutions* $CS(p)$, given by $CS(p) = \{q \in S(p) \mid |C(q)| = 2\}$, such that

$$S(p) = \text{conv}(PS(p) \cup CS(p)),$$

where $\text{conv}(A)$ denotes the convex hull of the set A . The set of pure solutions is given by

$$PS(p) = \begin{cases} PB^2(p, [12]) & \text{if } p \in \dot{\Delta}_M, \\ PB^2(p, [12]) \cup PB^2(p, [2]) & \text{if } p = e_2, \\ PB^2(p, [12]) \cup PB^2(p, [1]) & \text{if } p = e_1. \end{cases}$$

Further, with respect to the coordination solutions it holds that $CS(p)$ consists of all strategies $q(j, \ell) \in \Delta_N$ with $f_j \in PB^2(p, [1])$ and $f_\ell \in PB^2(p, [2])$ such that $C(q(j, \ell)) = \{f_j, f_\ell\}$ and $e_1 A q(j, \ell) = e_2 A q(j, \ell)$. Note that for any such $j, \ell \in N$, $q(j, \ell)$ is unique.

Note that $PS(p') = PS(p'')$ and $CS(p') = CS(p'')$ for all $p', p'' \in I_k$, with $k \in \{1, \dots, t\}$. This implies that we can unambiguously define $PS(I_k)$, $CS(I_k)$ and $S(I_k) = \text{conv}(PS(I_k) \cup CS(I_k))$. We illustrate the concepts and notation in the following example.

Example 2.1 Consider the following 2×4 bimatrix game (A, B) .

$$\begin{array}{c} e_1 \\ e_2 \end{array} \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ 1, 5 & 1, -1 & 1, 5 & 0, -16 \\ 0, 0 & 0, 7 & 1, 5 & 1, 8 \end{bmatrix}$$

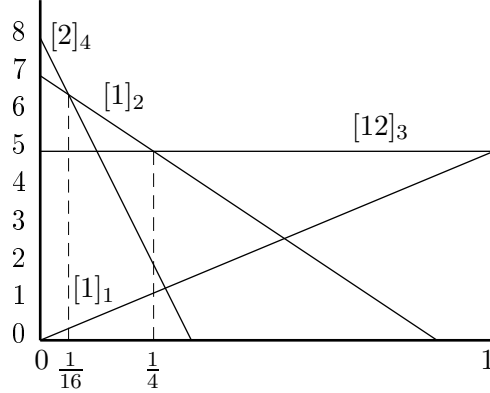


Figure 2.1: The graphical representation of the game (A, B)

The introduced notation allows us to represent this game graphically, see Figure 2.1. On the horizontal axis the strategy space of player 1 is displayed, on the vertical axis the payoff to player 2. Then the line with index 4 represents all possible payoffs to player 2 if he plays f_4 : 8 if player 1 plays e_2 ($p = 0$) and -16 if e_1 ($p = 1$) is played. The label $[2]$ indicates that e_2 is player 1's best reply against f_4 .

Furthermore, the piecewise linear maximum function consists of three intervals, with $p_0 = e_2$, $p_1 = \frac{1}{16}e_1 + \frac{15}{16}e_2$, $p_2 = \frac{1}{4}e_1 + \frac{3}{4}e_2$ and $p_3 = e_1$. We first consider the extreme points of the intervals, starting with $p_0 = e_2$. Since $|PB^2(e_2)| = 1$, it holds that $CS(e_2) = \emptyset$. Also $PB^2(e_2, [12]) = \emptyset$, but $PB^2(e_2, [2]) = \{f_4\}$, which implies that $PS(e_2) = \{f_4\}$ and $S(e_2) = \{f_4\}$. Secondly, we consider $p_1 = \frac{1}{16}e_1 + \frac{15}{16}e_2$. $PB^2(p_1, [12]) = \emptyset$ and therefore $PS(p_1) = \emptyset$, but as $PB^2(p, [1]) = \{f_2\}$ and $PB^2(p, [2]) = \{f_4\}$ we obtain $CS(p_1) = \{q(2, 4)\}$. Considering the second and fourth column of A , $q(2, 4)$ solves $e_1 A q(2, 4) = e_2 A q(2, 4)$, which gives $q_2 = q_4$ and hence, $CS(p_1) = \{\frac{1}{2}f_2 + \frac{1}{2}f_4\}$. The analysis of the other two extreme points can be done in a similar way.

We also consider player 1's strategy intervals. First of all, $I_1 = \{\alpha e_1 + (1 - \alpha)e_2 \mid 0 < \alpha < \frac{1}{16}\}$. Since $|PB^2(I_1)| = 1$ and $PB^2(I_1, [12]) = \emptyset$, both $PS(I_1) = \emptyset$ and $CS(I_1) = \emptyset$. A similar result holds for I_2 . Finally, we consider $I_3 = \{\alpha e_1 + (1 - \alpha)e_2 \mid \frac{1}{4} < \alpha < 1\}$, where $CS(I_3) = \emptyset$, but $PB^2(I_3, [12]) = \{f_3\}$ and hence, $PS(I_3) = \{f_3\}$. \triangleleft

The set $NE(A, B)$ of Nash equilibria can be determined in $2t + 1$ steps by the use of the following theorem.

Theorem 2.2 (Borm et al. (1988)) The set of Nash equilibria is given by

$$NE(A, B) = \bigcup_{k=0}^t \{p_k\} \times S(p_k) \cup \bigcup_{k=1}^t I_k \times S(I_k).$$

Note that for all $k \in \{1, \dots, t\}$, $S(I_k) \subseteq S(p_{k-1})$ and $S(I_k) \subseteq S(p_k)$. Consequently, as a general rule one can first determine all possible equilibria with respect to the intervals I_k , $k \in \{1, \dots, t\}$, and include the corresponding extreme points in the description of the set of equilibria. Then one only has to consider an extreme point p_k , $k \in \{0, \dots, t\}$, separately if at such an extreme point an additional pure or coordination solution arises with respect to the interval(s) with equilibria the extreme point belongs to.

It follows that $NE(A, B)$ is the union of maximally $2t + 1$ polytopes. We illustrate in the next example how to find the set of Nash equilibria for a $2 \times n$ bimatrix game.

Example 2.3 Consider the 2×4 bimatrix game (A, B) of Example 2.1. With respect to the intervals we only have to consider I_3 , as $PB^2(I_3, [12]) \neq \emptyset$. As a consequence, $\text{conv}(\{\frac{1}{4}e_1 + \frac{3}{4}e_2, e_1\}) \times \{f_3\} \subseteq NE(A, B)$. Further, there are three extreme points at which additional equilibria arise. First of all, $p_0 = e_2$. Since $PB^2(e_2, [2]) = \{f_4\}$, it holds that $(e_2, f_4) \in NE(A, B)$. Secondly, $p_1 = \frac{1}{16}e_1 + \frac{15}{16}e_2$. Since $CS(p_1) = \{\frac{1}{2}f_2 + \frac{1}{2}f_4\}$ we obtain that $(\frac{1}{16}e_1 + \frac{15}{16}e_2, \frac{1}{2}f_2 + \frac{1}{2}f_4) \in NE(A, B)$. Finally, $p_3 = e_1$. Since $PS(e_1) = \{f_1, f_3\}$, it holds that $\{e_1\} \times \text{conv}(\{f_1, f_3\}) \subseteq NE(A, B)$. Altogether this results in $NE(A, B) = T_1 \cup T_2 \cup T_3 \cup T_4$, with $T_1 = \{(e_2, f_4)\}$, $T_2 = \{(\frac{1}{16}e_1 + \frac{15}{16}e_2, \frac{1}{2}f_2 + \frac{1}{2}f_4)\}$, $T_3 = \text{conv}(\{\frac{1}{4}e_1 + \frac{3}{4}e_2, e_1\}) \times \{f_3\}$ and $T_4 = \{e_1\} \times \text{conv}(\{f_1, f_3\})$. \triangleleft

3 Perfect, proper and strictly perfect equilibrium

In Borm (1992) the sets of perfect (Selten (1975)), proper (Myerson (1978)) and strictly perfect equilibria (Okada (1984)) are characterised for $2 \times n$ bimatrix games. Each of those characterisations consists of a list of necessary and sufficient conditions for a strategy pair to be an element of the set of equilibria under consideration. In this section we revisit those results and reformulate them in a more manageable and insightful way by the use of refinement-specific subgames, which are obtained by the elimination of pure strategies of player 2.

Let $(A, B)^{UND}$ be defined as the subgame of (A, B) in which all of player 2's pure strategies f_j are deleted for which $f_j \notin \bigcup_{p \in \Delta_M} PB^2(p)$. Hence, to get from (A, B) to $(A, B)^{UND}$ we leave out all player 2's pure dominated strategies¹.

¹A strategy $q \in \Delta_N$ is dominated if there exists a strategy $\bar{q} \in \Delta_N$ such that $pB\bar{q} \geq pBq$ for all $p \in \Delta_M$ and $pB\bar{q} > pBq$ for some $p \in \Delta_M$.

Let (A, B) be a $2 \times n$ bimatrix game. The sets of perfect, proper and strictly perfect equilibria are denoted by $PE(A, B)$, $PR(A, B)$ and $SPE(A, B)$, respectively. All results in this section are a direct consequence of the characterisations in Borm (1992).

Theorem 3.1 Let (A, B) be a $2 \times n$ bimatrix game with $J([1]) = \emptyset$ or $J([2]) = \emptyset$. Then $PE(A, B) = PR(A, B) = SPE(A, B)$, and

$$PE(A, B) = \begin{cases} NE((A, B)^{UND}) & \text{if both } J([1]) = \emptyset \text{ and } J([2]) = \emptyset, \\ NE((A, B)^{UND}) \cap \{e_1\} \times \Delta_N & \text{if } J([1]) \neq \emptyset \text{ and } J([2]) = \emptyset, \\ NE((A, B)^{UND}) \cap \{e_2\} \times \Delta_N & \text{if } J([2]) \neq \emptyset \text{ and } J([1]) = \emptyset. \end{cases}$$

If $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$ the sets of perfect, proper and strictly perfect equilibria do not coincide in general.

Theorem 3.2 Let (A, B) be a $2 \times n$ bimatrix game with $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$. Then $PE(A, B) = NE((A, B)^{UND})$.

We define the game $(A, B)_{-[12]}$ as the subgame of (A, B) in which all of player 2's pure strategies $f_j \in J([12])$ are deleted. Then we define the game $(A, B)^{PR}$ as $((A, B)_{-[12]})^{UND}$. Hence, to obtain the game $(A, B)^{PR}$ we first delete from (A, B) all of player 2's strategies with a label [12] and from that game we delete player 2's pure dominated strategies, which may differ from his dominated strategies in the original game. Let Δ_N^{PR} be the strategy space of player 2 in the subgame $(A, B)^{PR}$. The strategies in this subgame can be interpreted as strategies in the original game (A, B) , with zero probability for the deleted pure strategies. By slight abuse of notation we therefore use Δ_N instead of Δ_N^{PR} for expositional purposes.

Theorem 3.3 Let (A, B) be a $2 \times n$ bimatrix game with $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$. Then $PR(A, B) = \{(p, q) \in PE(A, B) \mid \exists \bar{q} \in \Delta_N : (p, \bar{q}) \in NE((A, B)^{PR})\}$.

We define the game $(A, B)^{SPE}$ as $((A, B)^{UND})_{-[12]}$. Hence, opposed to the game $(A, B)^{PR}$ we now first delete all the pure dominated strategies of player 2 and then the strategies with a label [12]. Hence, the game $(A, B)^{SPE}$ is a subgame of $(A, B)^{PR}$. Note that $(A, B)^{SPE}$ may be a vacuous game, in which case $SPE(A, B) = \emptyset$.

Let Δ_N^{SPE} be the strategy space of player 2 in the subgame $(A, B)^{SPE}$, then similarly as for proper equilibria we use Δ_N instead of Δ_N^{SPE} .

Theorem 3.4 Let (A, B) be a $2 \times n$ bimatrix game with $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$. Then $SPE(A, B) = \{(p, q) \in PE(A, B) \mid \exists \bar{q} \in \Delta_N : (p, \bar{q}) \in NE(A, B) \cap NE((A, B)^{SPE})\}$.

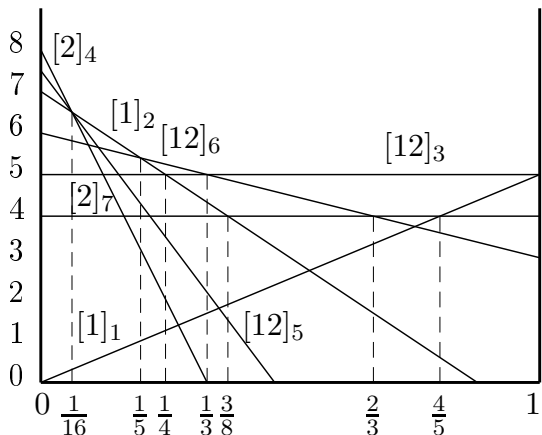


Figure 3.1: The game (A, B)

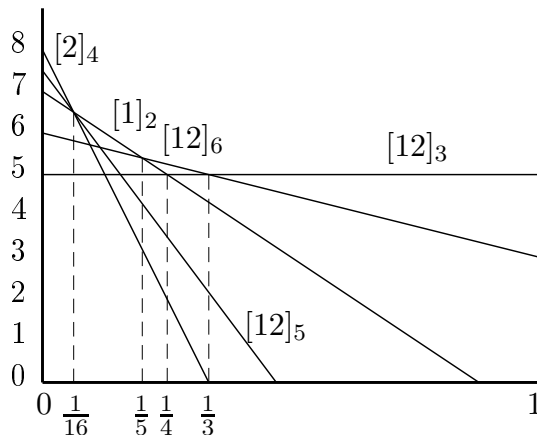


Figure 3.2: The game $(A, B)^{UND}$

By the above two characterisations it easily follows that for $2 \times n$ bimatrix games each strictly perfect equilibrium is a proper equilibrium.

Corollary 3.5 *Let (A, B) be a $2 \times n$ bimatrix game. Then $SPE(A, B) \subseteq PR(A, B)$.*

Note that the inclusion $SPE(A, B) \subseteq PR(A, B)$ does not hold for bimatrix games in general, as is shown by Vermeulen and Jansen (1996).

In the following example we illustrate the way to compute the sets of perfect, proper and strictly perfect equilibrium by the use of the above theorems and the geometric labelling method.

Example 3.6 Consider the following 2×7 bimatrix game (A, B) and its graphical representation in Figure 3.1.

$$\begin{array}{c} e_1 \\ e_2 \end{array} \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 \\ 1, 5 & 1, -1 & 1, 5 & 0, -16 & 1, -8\frac{1}{2} & 1, 3 & 0, 4 \\ 0, 0 & 0, 7 & 1, 5 & 1, 8 & 1, 7\frac{1}{2} & 1, 6 & 1, 4 \end{bmatrix}$$

The set of Nash equilibria is given by $NE(A, B) = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6$, with

- $T_1 = \{(e_2, f_4)\}$,
- $T_2 = \{\frac{1}{16}e_1 + \frac{15}{16}e_2\} \times \text{conv}(\{f_5, \frac{1}{2}f_2 + \frac{1}{2}f_4\})$,
- $T_3 = \text{conv}(\{\frac{1}{5}e_1 + \frac{4}{5}e_2, \frac{1}{3}e_1 + \frac{2}{3}e_2\}) \times \{f_6\}$,

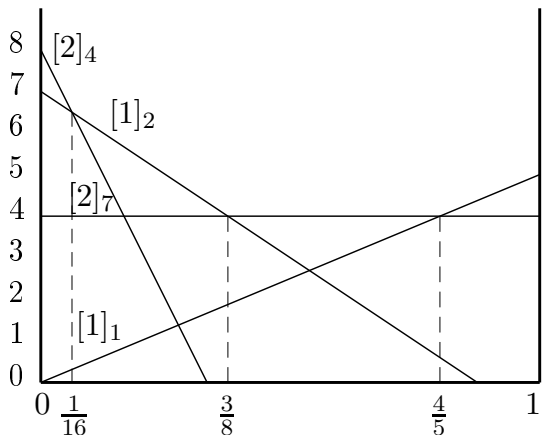


Figure 3.3: The game $(A, B)^{PR}$

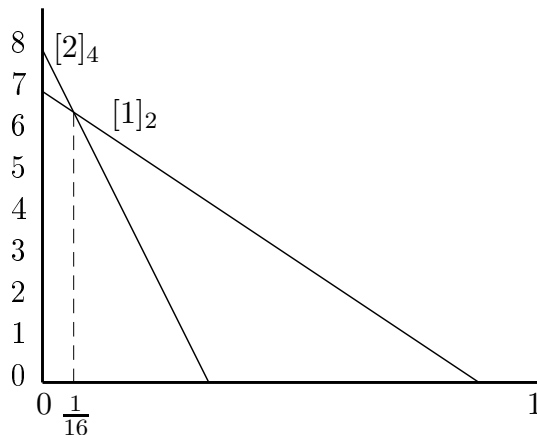


Figure 3.4: The game $(A, B)^{SPE}$

- $T_4 = \{\frac{1}{3}e_1 + \frac{2}{3}e_2\} \times \text{conv}(\{f_3, f_6\})$,
- $T_5 = \text{conv}(\{\frac{1}{3}e_1 + \frac{2}{3}e_2, e_1\}) \times \{f_3\}$,
- $T_6 = \{e_1\} \times \text{conv}(\{f_1, f_3\})$.

To obtain the game $(A, B)^{UND}$ we delete the pure dominated strategies f_1 and f_7 , see Figure 3.2. Then $PE(A, B) = NE((A, B)^{UND}) = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5$.

To obtain the game $(A, B)_{-[12]}$ we first delete the strategies in the set $J([12])$: f_3, f_5 and f_6 . Since player 2 has no pure dominated strategies in $(A, B)_{-[12]}$ it holds that $(A, B)^{PR} = (A, B)_{-[12]}$. See Figure 3.3. Then, $NE((A, B)^{PR}) = \{(e_2, f_4)\} \cup \{(\frac{1}{16}e_1 + \frac{15}{16}e_2, \frac{1}{2}f_2 + \frac{1}{2}f_4)\} \cup \{(\frac{3}{8}e_1 + \frac{5}{8}e_2, \frac{1}{2}f_2 + \frac{1}{2}f_7)\} \cup \{(\frac{4}{5}e_1 + \frac{1}{5}e_2, \frac{1}{2}f_1 + \frac{1}{2}f_7)\} \cup \{(e_1, f_1)\}$. Consequently, $PR(A, B) = T_1 \cup T_2 \cup T'_5 \cup T''_5 \cup T'''_5$, with $T'_5 = \{(\frac{3}{8}e_1 + \frac{5}{8}e_2, f_3)\}$, $T''_5 = \{(\frac{4}{5}e_1 + \frac{1}{5}e_2, f_3)\}$ and $T'''_5 = \{(e_1, f_3)\}$.

To obtain the game $(A, B)^{SPE}$ we first delete player 2's pure dominated strategies f_1 and f_7 (see Figure 3.2), and then the strategies out of $J([12])$, which are f_3, f_5 and f_6 . See Figure 3.4. $NE((A, B)^{SPE}) = \{(e_2, f_4)\} \cup \{(\frac{1}{16}e_1 + \frac{15}{16}e_2, \frac{1}{2}f_2 + \frac{1}{2}f_4)\} \cup \{(e_1, f_2)\}$. This implies that $SPE(A, B) = T_1 \cup T_2$. \triangleleft

4 Fall back equilibrium in $2 \times n$ bimatrix games

In this section we show how to determine the set of fall back equilibria for $2 \times n$ bimatrix games. In the underlying thought experiment players play a fall back game in which each

player strategically chooses a back-up action for the possible event that his primary action is blocked. The action set for player 1 in the fall back game is given by $\tilde{M} = \{(i, k) \in M \times M \mid i \neq k\}$. Similarly, $\tilde{N} = \{(j, \ell) \in N \times N \mid j \neq \ell\}$ is the action set of player 2. Hence, the total number of actions in the fall back game for player 1 is $\tilde{m} = m(m-1) = 2$, and $\tilde{n} = n(n-1)$ for player 2. An action $(j, \ell) \in \tilde{N}$ consists of a *primary action* j and a *back-up action* ℓ . Let $\varepsilon = (\varepsilon^1, \varepsilon^2)$ be a pair of (small) non-negative probabilities. The interpretation of player 2's action (j, ℓ) in the fall back game is that in the original game he plays with probability $1 - \varepsilon^2$ primary action j and with probability ε^2 back-up action ℓ . The *fall back game* $(\tilde{A}(\varepsilon), \tilde{B}(\varepsilon))$ is given by

$$\begin{aligned}\tilde{A}_{ik,j\ell}(\varepsilon) &= (1 - \varepsilon^1)(1 - \varepsilon^2)A_{i,j} + \varepsilon^1(1 - \varepsilon^2)A_{k,j} + (1 - \varepsilon^1)\varepsilon^2A_{i,\ell} + \varepsilon^1\varepsilon^2A_{k,\ell}, \\ \tilde{B}_{ik,j\ell}(\varepsilon) &= (1 - \varepsilon^1)(1 - \varepsilon^2)B_{i,j} + \varepsilon^1(1 - \varepsilon^2)B_{k,j} + (1 - \varepsilon^1)\varepsilon^2B_{i,\ell} + \varepsilon^1\varepsilon^2B_{k,\ell},\end{aligned}$$

for all $i, k \in M$, $j, \ell \in N$, with $i \neq k$ and $j \neq \ell$. The pure strategy $(i, k) \in \tilde{M}$ is alternatively denoted by e_{ik} , the pure strategy $(j, \ell) \in \tilde{N}$ by $f_{j\ell}$. A typical element of $\Delta_{\tilde{M}}$ is denoted by ρ , a typical element of $\Delta_{\tilde{N}}$ by σ . As a consequence, the payoff in mixed strategies of player 1 can be given by $\rho\tilde{A}(\varepsilon)\sigma$, and for player 2 by $\rho\tilde{B}(\varepsilon)\sigma$.

Definition Let (A, B) be a bimatrix game. A strategy pair $(p, q) \in \Delta$ is a *fall back equilibrium* of (A, B) if there exists a sequence $\{\varepsilon_t\}_{t \in \mathbb{N}}$ of pairs of positive real numbers converging to zero, and a sequence $\{(\rho_t, \sigma_t)\}_{t \in \mathbb{N}}$ such that $(\rho_t, \sigma_t) \in NE(\tilde{A}(\varepsilon_t), \tilde{B}(\varepsilon_t))$ for all $t \in \mathbb{N}$, converging to $(\rho, \sigma) \in \tilde{\Delta}$, with $p_i = \sum_{k \in M \setminus \{i\}} \rho_{ik}$ for all $i \in M$ and $q_j = \sum_{\ell \in N \setminus \{j\}} \sigma_{j\ell}$ for all $j \in N$. The set of fall back equilibria of (A, B) is denoted by $FBE(A, B)$.

As a starting point for the analysis of fall back equilibrium in bimatrix games we use the characterisation by Kleppe et al. (2012), which is based on blocking probabilities. We first recall the set of secondary replies. Let (A, B) be a bimatrix game and let $(p, q) \in \Delta$. Then, we define the players' pure secondary reply correspondences² by

$$\begin{aligned}PSR^1(q) &= \left\{ e_i \mid \exists k \in M \setminus \{i\} : e_k A q \geq e_i A q, \right. \\ &\quad \left. e_i A q \geq e_r A q \text{ for all } r \in M \setminus \{k\} \right\},\end{aligned}$$

$$\begin{aligned}PSR^2(p) &= \left\{ f_j \mid \exists \ell \in N \setminus \{j\} : p B f_\ell \geq p B f_j, \right. \\ &\quad \left. p B f_j \geq p B f_s \text{ for all } s \in N \setminus \{\ell\} \right\}.\end{aligned}$$

²In Kleppe et al. (2012) the notation *PS* is used for the set of pure secondary replies. However, to clearly distinguish between the set of pure solutions and the set of pure secondary replies, we use the notation *PSR* for the latter in this paper.

Note that if $|PB^1(q)| > 1$, then $PSR^1(q) = PB^1(q)$. Similarly, if $|PB^2(p)| = 1$, then $PSR^2(p) = PB^2(p)$.

Theorem 4.1 (Kleppe et al. (2012)) Let (A, B) be a bimatrix game. Then a strategy pair $(p, q) \in \Delta$ is a fall back equilibrium if and only if one of the following four statements is satisfied.

1. $|C(p)| > 1$, $|C(q)| > 1$ and $(p, q) \in NE(A, B)$.
2. $|C(p)| > 1$, $|C(q)| = 1$ and there exists a strategy $\bar{q} \in \Delta_N$ such that $C(\bar{q}) \cap C(q) = \emptyset$ and a blocking probability $\bar{\delta}^2 > 0$, such that for all $\delta^2 \in (0, \bar{\delta}^2]$ the strategy pair (p, \hat{q}) , with $\hat{q} = (1 - \delta^2)q + \delta^2\bar{q}$, satisfies

$$\begin{aligned} C(p) &\subseteq PB^1(\hat{q}), \\ C(q) &\subseteq PB^2(p), \\ C(\bar{q}) &\subseteq PSR^2(p). \end{aligned}$$

3. $|C(p)| = 1$, $|C(q)| > 1$ and there exists a strategy $\bar{p} \in \Delta_M$ such that $C(\bar{p}) \cap C(p) = \emptyset$ and a blocking probability $\bar{\delta}^1 > 0$, such that for all $\delta^1 \in (0, \bar{\delta}^1]$ the strategy pair (\hat{p}, q) , with $\hat{p} = (1 - \delta^1)p + \delta^1\bar{p}$, satisfies

$$\begin{aligned} C(p) &\subseteq PB^1(q), \\ C(q) &\subseteq PB^2(\hat{p}), \\ C(\bar{p}) &\subseteq PSR^1(q). \end{aligned}$$

4. $|C(p)| = |C(q)| = 1$ and there exist strategies $\bar{p} \in \Delta_M$ and $\bar{q} \in \Delta_N$ such that $C(\bar{p}) \cap C(p) = \emptyset$ and $C(\bar{q}) \cap C(q) = \emptyset$ and there also exist blocking probabilities $\bar{\delta}^1 > 0$ and $\bar{\delta}^2 > 0$, such that for all $\delta^1 \in (0, \bar{\delta}^1]$ and for all $\delta^2 \in (0, \bar{\delta}^2]$, the strategy pair (\hat{p}, \hat{q}) , with $\hat{p} = (1 - \delta^1)p + \delta^1\bar{p}$ and $\hat{q} = (1 - \delta^2)q + \delta^2\bar{q}$, satisfies

$$\begin{aligned} C(p) &\subseteq PB^1(\hat{q}), \\ C(q) &\subseteq PB^2(\hat{p}), \\ C(\bar{p}) &\subseteq PSR^1(\hat{q}), \\ C(\bar{q}) &\subseteq PSR^2(\hat{p}). \end{aligned}$$

Recall that in the thought experiment underlying fall back equilibrium each player can decide on a back-up action, which is played when the action of his first choice turns out to be unavailable. Define for all $j \in N$ the game $(A, B)_{-j}$ as the subgame of (A, B) in which

player 2's pure strategy f_j is deleted. Then $FBS(A, B)_{-j} = \{p \in \Delta_M \mid \exists \tilde{q} \in \Delta_N : (p, \tilde{q}) \in NE(((A, B)_{-j})^{UND})\}$ is the projection of the set of Nash equilibria of $((A, B)_{-j})^{UND}$ onto the strategy space Δ_M of player 1. Moreover, $FBS(A, B) = \bigcap_{f_j \in J([12])} FBS(A, B)_{-j}$ is the intersection of these projections for all pure strategies $f_j \in J([12])$ of player 2. Hence, if $p \in FBS(A, B)$ and $f_j \in J([12])$, then $(p, q) \in NE(((A, B)_{-j})^{UND})$ for some $q \in \Delta_N$.

This leads us to the central theorem of our paper: a characterisation of the set of fall back equilibria for $2 \times n$ bimatrix games.

Theorem 4.2 Let (A, B) be a $2 \times n$ bimatrix game. Then $FBE(A, B) = \{(p, q) \in NE((A, B)^{UND}) \mid p \in FBS(A, B)\}$.

Proof: This proof consists of two parts. We first assume that $(p, q) \in NE((A, B)^{UND})$ and $p \in FBS(A, B)$, and we show that either one of the four statements of Theorem 4.1 is satisfied. Note that $(p, q) \in NE((A, B)^{UND})$ implies that $(p, q) \in NE(A, B)$.

1. We assume $|C(p)| > 1$ and $|C(q)| > 1$. Then we immediately obtain $(p, q) \in NE(A, B)$.

2. We assume $|C(p)| > 1$ and $|C(q)| = 1$. Since $(p, q) \in NE(A, B)$ we obtain that $q = f_j \in J([12])$ for some $j \in N$, and $C(q) \subseteq PB^2(p)$.

Since $p \in FBS(A, B)$ it holds that $p \in FBS(A, B)_{-j}$. Therefore, there exists a \bar{q} , $C(\bar{q}) \cap C(q) = \emptyset$ such that $(p, \bar{q}) \in NE(((A, B)_{-j})^{UND})$, which also implies $(p, \bar{q}) \in NE((A, B)_{-j})$. Hence, $C(\bar{q}) \subseteq PSR^2(p)$.

Furthermore, $PB^1(f_j) = \{e_1, e_2\}$. Since $(p, \bar{q}) \in NE((A, B)_{-j})$ we obtain $PB^1(\hat{q}) = \{e_1, e_2\}$, with $\hat{q} = (1 - \delta^2)f_j + \delta^2\bar{q}$, with $\delta^2 > 0$ sufficiently small. Hence, $C(p) \subseteq PB^1(\hat{q})$.

3. We assume $|C(p)| = 1$ and $|C(q)| > 1$. Without loss of generality we assume $p = e_2$. Since $(p, q) \in NE((A, B)^{UND})$ it holds that $q_j = 0$ for all f_j that are dominated. Therefore, $C(q) \subseteq PB^2(\hat{p})$, with $\hat{p} = (1 - \delta^1)e_2 + \delta^1e_1$, $\delta^1 > 0$ sufficiently small.

Since $(p, q) \in NE(A, B)$, $C(p) \subseteq PB^1(q)$.

It is immediate that $C(\bar{p}) = \{e_1\} \subseteq PSR^1(q)$.

4. We assume $|C(p)| = 1$ and $|C(q)| = 1$. Without loss of generality we assume $p = e_2$. Since $(p, q) \in NE((A, B)^{UND})$ it holds that $q = f_j \in J([12]) \cup J([2])$, $j \in N$. As $(p, q) \in NE((A, B)^{UND})$ we obtain $C(q) = \{f_j\} \subseteq PB^2(\hat{p})$, with $\hat{p} = (1 - \delta^1)e_2 + \delta^1e_1$, with $\delta^1 > 0$ sufficiently small.

If $f_j \in J([2])$, then let \bar{q} be such that $C(\bar{q}) \subseteq PSR^2(\hat{p})$.

In that case $C(\bar{p}) = \{e_1\} \subseteq PSR^1(\hat{q})$, with $\hat{q} = (1 - \delta^2)q + \delta^2\bar{q}$, with $\delta^2 > 0$ sufficiently small, is an immediate result.

And also $C(p) = \{e_2\} \subseteq PB^1(\hat{q})$.

If $f_j \notin J([2])$, then $f_j \in J([12])$, in which case $C(p) = \{e_2\} \subseteq PB^1(\hat{q})$.

And $C(\bar{p}) = \{e_1\} \subseteq PSR^1(\hat{q})$ is an immediate result.

Finally, as $p \in FBS(A, B)_{-j}$ there exists a \bar{q} such that $(p, \bar{q}) \in NE((A, B)_{-j})^{UND}$, which implies $C(\bar{q}) \subseteq PSR^2(p)$ and therefore, $C(\bar{q}) \subseteq PSR^2(\hat{p})$.

In the second part of the proof we assume that $(p, q) \in FBE(A, B)$ and we show that $(p, q) \in NE((A, B)^{UND})$ and $p \in FBS(A, B)$, by distinguishing the same four cases as in Theorem 4.1. Note that it easily follows from each of the four cases that $(p, q) \in NE(A, B)$.

1. Since $|C(p)| > 1$, $p \in \dot{\Delta}_M$. Combining this with $(p, q) \in NE(A, B)$ results in $(p, q) \in NE((A, B)^{UND})$. Further, if $q_j = 0$ for all $f_j \in J([12])$, then $(p, q) \in NE(((A, B)_{-j})^{UND})$ for all $j \in J([12])$ and hence, $p \in FBS(A, B)$.

If $q_j > 0$ for some $f_j \in J([12])$, then by the fact that $|C(p)| > 1$ and $C(p) \subseteq PB^1(q)$ either one of the following two things must hold: (i) $q_\ell > 0$, $\ell \neq j$, for some $f_\ell \in J([12])$ in which case it easily follows that $p \in FBS(A, B)$, or (ii) $q_\ell > 0$, $\ell \neq j$, for some $f_\ell \in J([1])$ and $q_m > 0$, $m \neq j$, for some $f_m \in J([2])$, which also implies $p \in FBS(A, B)$.

2. Let $q = f_j$, $j \in N$. Since $(p, q) \in NE(A, B)$ and $|C(p)| > 1$ it holds that $f_j \in J([12])$. Hence, $p \in FBS(A, B)_{-\ell}$ for all $\ell \in N \setminus \{j\}$. Since $C(\bar{q}) \subseteq PSR^2(p)$ it holds that \bar{q} is a best reply of player 2 against p in the game $((A, B)_{-j})^{UND}$. Since $\{e_1, e_2\} = C(p) \subseteq PB^1(\hat{q})$ and $f_j \in J([12])$ it holds that $PB^1(\bar{q}) = \{e_1, e_2\}$. Therefore, p is a best reply of player 1 against \bar{q} in the game $((A, B)_{-j})^{UND}$. Hence, $(p, \bar{q}) \in NE(((A, B)_{-j})^{UND})$, which implies that $p \in FBS(A, B)$. And as $p \in \dot{\Delta}_M$, $(p, q) \in NE((A, B)^{UND})$.

3. Without loss of generality we assume $p = e_2$. Since $C(q) \subseteq PB^2(\hat{p})$ it holds that $q_j = 0$ for all $f_j \notin \bigcup_{p \in \dot{\Delta}_M} PB^2(p)$. Hence, $(p, q) \in NE((A, B)^{UND})$. If $q_j = 0$ for all $f_j \in J([12])$, then obviously, $p \in FBS(A, B)$.

Otherwise, let $q_j > 0$ for some $f_j \in J([12])$. Since $(p, q) \in NE((A, B)^{UND})$ it holds that there exists an $f_\ell \in J([12]) \cup J([2])$ with $f_\ell \in PB^2(\hat{p})$ such that $q_\ell > 0$, $\ell \neq j$. Hence, $p \in FBS(A, B)$.

4. Without loss of generality we assume $p = e_2$ and $q = f_j, j \in N$. As $(p, q) \in NE(A, B)$, $f_j \in J([12]) \cup J([2])$. Since $C(q) \in PB^2(\hat{p})$ it holds that $(p, q) \in NE((A, B)^{UND})$. If $f_j \in J([2])$, then $p \in FBS(A, B)$.

Otherwise, $f_j \in J([12])$. In that case $C(\bar{q}) \subseteq PSR^2(\hat{p})$ implies that \bar{q} is undominated in $(A, B)_{-j}$. Moreover, in the game $((A, B)_{-j})^{UND}$, strategy \bar{q} is a best reply for player 2, because $C(\bar{q}) \subseteq PSR^2(\hat{p})$, and strategy p is a best reply for player 1 because $\{e_2\} = C(p) \subseteq PB^1(\hat{q}) = PB^1(\bar{q})$, where the last equality sign follows from the fact that $f_j \in J([12])$. Hence, $p \in FBS(A, B)$. \square

Note that from Theorem 4.2 it easily follows that the set of fall back equilibria is the union of finitely many polytopes, a result generalised for all bimatrix games in Kleppe et al. (2012).

For the sake of completeness we also provide a characterisation of the set of fall back equilibria which is more in line with the terminology of the characterisations in Borm (1992). Let $PB^2(p, [1, 12])$ denote the set of pure best replies against p with a label [1] or [12]; the sets $PB^2(p, [2, 12])$, $PSR^2(p, [1, 12])$ and $PSR^2(p, [2, 12])$ are defined analogously.

Theorem 4.3 Let (A, B) be a $2 \times n$ bimatrix game. Then $(p, q) \in FBE(A, B)$ if one of the following three statements holds.

1. $p \in \hat{\Delta}_M$, $q \in S(p)$ and there either exist strategies $f_j \in PB^2(p, [12])$ and $f_\ell \in PSR^2(p, [12])$ or strategies $f_j \in PSR^2(p, [1])$ and $f_\ell \in PSR^2(p, [2])$.

2. $p = e_2$ and

$$\begin{cases} q \in \text{conv}\{CS(I_1) \cup PB^2(I_1, [2, 12])\} & \text{if there exists an } f_\ell \in PSR^2(e_2, [2, 12]), \\ q \in \text{conv}\{CS(I_1) \cup PB^2(I_1, [2])\} & \text{otherwise.} \end{cases}$$

3. $p = e_1$ and

$$\begin{cases} q \in \text{conv}\{CS(I_t) \cup PB^2(I_t, [1, 12])\} & \text{if there exists an } f_\ell \in PSR^2(e_1, [1, 12]), \\ q \in \text{conv}\{CS(I_t) \cup PB^2(I_t, [1])\} & \text{otherwise.} \end{cases}$$

We illustrate the method to compute the set of fall back equilibria of a $2 \times n$ bimatrix game in the following example.

Example 4.4 Reconsider the game of Example 3.6 and the corresponding graphical representation in Figure 3.1. By the use of Theorem 4.2 we determine the set of fall

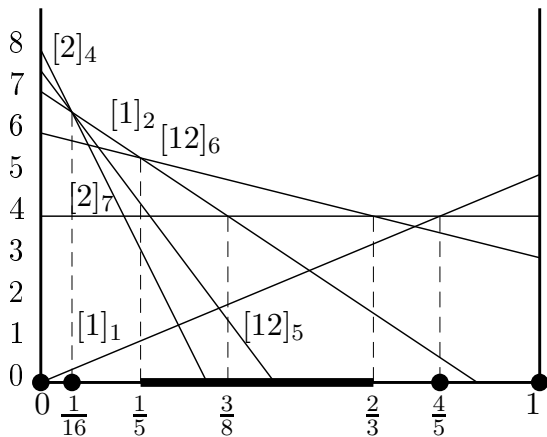


Figure 4.1: $((A, B)_{-3})^{UND}$

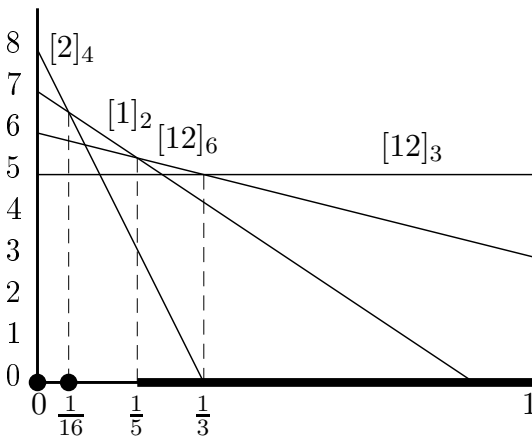


Figure 4.2: $((A, B)_{-5})^{UND}$

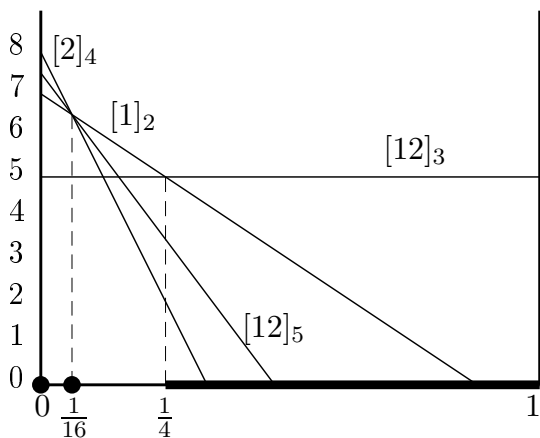


Figure 4.3: $((A, B)_{-6})^{UND}$

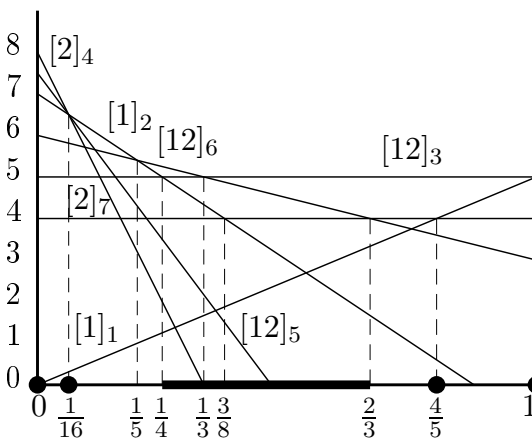


Figure 4.4: $FBS(A, B)$

back equilibria of the game (A, B) . Player 2 has three strategies with a label $[12]$. We delete them one at a time, starting with f_3 . Since $(A, B)_{-3}$ has no pure dominated strategies $(A, B)_{-3} = ((A, B)_{-3})^{UND}$, see Figure 4.1. Then $FBS(A, B)_{-3} = \{e_2\} \cup \{\frac{1}{16}e_1 + \frac{15}{16}e_2\} \cup \text{conv}(\{\frac{1}{5}e_1 + \frac{4}{5}e_2, \frac{2}{3}e_1 + \frac{1}{3}e_2\}) \cup \{\frac{4}{5}e_1 + \frac{1}{5}e_2\} \cup \{e_1\}$, which is illustrated by the bold parts on the horizontal axis (Δ_M) of Figure 4.1.

Secondly, we delete f_5 . To obtain $((A, B)_{-5})^{UND}$ we also delete f_1 and f_7 . Then $FBS(A, B)_{-5} = \{e_2\} \cup \{\frac{1}{16}e_1 + \frac{15}{16}e_2\} \cup \text{conv}(\{\frac{1}{5}e_1 + \frac{4}{5}e_1, e_1\})$. See Figure 4.2.

Finally, we delete f_6 . To obtain $((A, B)_{-6})^{UND}$ we also delete f_1 and f_7 . Then $FBS(A, B)_{-6} = \{e_2\} \cup \{\frac{1}{16}e_1 + \frac{15}{16}e_2\} \cup \text{conv}(\{\frac{1}{4}e_1 + \frac{3}{4}e_2, e_1\})$. See Figure 4.3.

This implies that $FBS(A, B) = \{e_2\} \cup \{\frac{1}{16}e_1 + \frac{15}{16}e_2\} \cup \text{conv}(\{\frac{1}{4}e_1 + \frac{3}{4}e_2, \frac{2}{3}e_1 + \frac{1}{3}e_2\}) \cup \{\frac{4}{5}e_1 + \frac{1}{5}e_2\} \cup \{e_1\}$, which is illustrated in Figure 4.4. Consequently, $FBE(A, B) = T_1 \cup T_2 \cup T_3' \cup T_4 \cup T_5'''' \cup T_5'' \cup T_5'''$, with $T_3' = \text{conv}(\{\frac{1}{4}e_1 + \frac{3}{4}e_2, \frac{1}{3}e_1 + \frac{2}{3}e_2\}) \times \{f_6\}$, $T_5'''' = \text{conv}(\{\frac{1}{3}e_1 + \frac{2}{3}e_2, \frac{2}{3}e_1 + \frac{1}{3}e_2\}) \times \{f_3\}$, $T_5'' = \{(\frac{4}{5}e_1 + \frac{1}{5}e_2, f_3)\}$ and $T_5''' = \{(e_1, f_3)\}$. \triangleleft

Note that by the characterisations of Theorems 3.1, 3.3 and 4.2 it is readily checked that the set of proper equilibria is a subset of the set of fall back equilibria for $2 \times n$ bimatrix games. This result is generalised for all bimatrix games in Kleppe et al. (2012).

Corollary 4.5 (Kleppe et al. (2012)) *Let (A, B) be a $2 \times n$ bimatrix game. Then $PR(A, B) \subseteq FBE(A, B)$.*

Further, since properness implies both perfectness and fall back for $2 \times n$ bimatrix games, it follows immediately that the intersection between the sets of fall back and perfect equilibria is non-empty. Note, however, that this intersection may include more than the set of proper equilibria, as in the game of Examples 3.6 and 4.4 the strategy pair $(\frac{1}{2}e_1 + \frac{1}{2}e_2, f_3)$ is not a proper equilibrium, but both a fall back and a perfect equilibrium.

Finally, by the characterisations of Theorems 3.2 and Theorem 4.2 we obtain the following corollary.

Corollary 4.6 *Let (A, B) be a $2 \times n$ bimatrix game with $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$. Then $FBE(A, B) \subseteq PE(A, B)$.*

Hence, whenever none of the two pure strategies of player 1 is dominant, each fall back equilibrium is a perfect equilibrium. It is easily verified that when $J([1]) = \emptyset$ or $J([2]) = \emptyset$ the set of fall back equilibria generally is not a subset of the set of perfect equilibria.

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