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Publication date:
2012

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May 24, 2012

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Abstract

For a graph Γ with adjacency matrix A, we consider a switching operation that takes Γ into a graph Γ' with adjacency matrix A', defined by $A' = Q^\top A Q$, where Q is a regular orthogonal matrix of level 2 (that is, $Q^\top Q = I$, $Q\mathbf{1} = \mathbf{1}$, $2Q$ is integral, and Q is not a permutation matrix). If such an operation exists, and Γ is nonisomorphic with Γ', then we say that Γ' is semi-isomorphic with Γ. Semi-isomorphic graphs are $\mathbb{R}$-cospectral, which means that they are cospectral and so are their complements. Wang and Xu ['On the asymptotic behavior of graphs determined by their generalized spectra', *Discrete Math.* **310** (2010)] expect that almost all pairs of $\mathbb{R}$-cospectral graphs are semi-isomorphic.

Regular orthogonal matrices of level 2 have been classified. By use of this classification we work out the requirements for this switching operation to work in case Q has one nontrivial indecomposable block of size 4, 6, 7 or 8. Size 4 corresponds to Godsil-McKay switching. The other cases provide new methods for constructions of $\mathbb{R}$-cospectral graphs. For graphs with eight vertices all these constructions are carried out. As a result we find that, out of the 1166 graphs on eight vertices which are $\mathbb{R}$-cospectral to another graph, only 44 are not semi-isomorphic to another graph.

Keywords: cospectral graphs, orthogonal matrices, switching.
Mathematics Subject Classifications: 05B20. JEL-code: C0.

1   Introduction

An orthogonal matrix Q is *regular* if it has constant row sum, that is, $Q\mathbf{1} = r\mathbf{1}$ (where $\mathbf{1}$ is the all-one vector). From $Q^\top Q = QQ^\top = I$, it follows that also $Q^\top \mathbf{1} = r\mathbf{1}$, and that $r = \pm 1$. Without loss of generality we will assume $r = 1$. A regular orthogonal matrix has *level ℓ*

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if ℓ is the smallest positive integer such that ℓQ is an integral matrix. We define $\ell = \infty$ if Q has irrational entries. Clearly $\ell = 1$ if and only if Q is a permutation matrix.

Consider two graphs Γ and Γ' with adjacency matrices A and A', respectively. The graphs are called *cospectral* if A and A' have the same spectrum. If $A + yJ$ and $A' + yJ$ (where J is the all-one matrix) have the same spectrum for every $y \in \mathbb{R}$, then Γ and Γ' are called $\mathbb{R}$-*cospectral*. Since A and A' are symmetric, Γ and Γ' are cospectral precisely when A and A' are similar, that is, there exist an orthogonal matrix Q such that $Q^\top A Q = A'$. If Q is a permutation matrix (i.e. Q is regular of level 1) then Γ and Γ' are isomorphic. If Γ and Γ' are nonisomorphic, and there exist a regular orthogonal matrix Q of level 2 such that $Q^\top A Q = A'$, we call Γ and Γ' *semi-isomorphic*. It easily follows that Γ and Γ' are $\mathbb{R}$-cospectral if Q is regular. (Indeed, $Q^\top \mathbf{1} = \mathbf{1}$ implies $Q^\top (A + yJ)Q = A' + yJ$). In particular, semi-isomorphic graphs are $\mathbb{R}$-cospectral. By taking $y = -1$ we see that $\mathbb{R}$-cospectral graphs have cospectral complements. The following result, due to Johnson and Newman [3] (see also [1, 4]) states that the converse of some of these observations is also true.

Theorem 1. *If Γ and Γ' are graphs with adjacency matrices A and A', respectively, then the following are equivalent.*
i. The graphs Γ and Γ' are cospectral, and so are their complements.
ii. The graphs Γ and Γ' are $\mathbb{R}$-cospectral.
iii. There exists a regular orthogonal matrix Q, such that $Q^\top A Q = A'$.

Any matrix of the form $xA + yJ + zI$ with $x, y, z \in \mathbb{R}$, $x \neq 0$ is called a *generalized adjacency matrix* of Γ. Clearly Γ and Γ' are $\mathbb{R}$-cospectral if and only if for every x, y, z the corresponding generalized adjacency matrices have the same spectrum.

A graph Γ is said to be *determined by its spectrum* if every graph cospectral with Γ is isomorphic with Γ. A graph Γ is *determined by its generalized spectrum* if every graph which is $\mathbb{R}$-cospectral with Γ is isomorphic with Γ. It has been conjectured by the second author that almost every graph is determined by its spectrum. A weaker version states that almost every graph is determined by its generalized spectrum. Both conjectures are still open, but Wang and Xu [8] have a number of results supporting these conjectures. They prove that for almost no graph there exists a graph semi-isomorphic with it, and in addition they provide experimental evidence showing that a positive fraction of all pairs of nonisomorphic $\mathbb{R}$-cospectral graphs, are in fact semi-isomorphic. This makes it interesting to investigate semi-isomorphism.

In this paper we show how semi-isomorphic graphs can be made by a switching procedure, that generalizes the switching method due to Godsil and McKay [5] (see also [1, 6]), called GM-switching. We start with the classification of indecomposable regular orthogonal matrices of level 2, and then consider the generalized switching for the case that Q has one nontrivial indecomposable block of order 4, 6, 7 or 8. In terms of the graph Γ it means that Γ must have a subgraph Δ of one of the mentioned orders that satisfies a number of properties. The four vertex case corresponds to GM-switching and the required properties are easily described; see Section 2. If Δ has six or seven vertices the required properties are worked out in detail. For eight vertices we restrict to the case $\Delta = \Gamma$.

As an application we determine all new switchings for graphs with eight vertices. We find 68 graphs for which GM-switching does not work, but the new switching does. It turns out that there exist only 22 pairs of $\mathbb{R}$-cospectral graphs on eight vertices which are not semi-isomorphic with each other or with another graph.

2 Switching

The following lemma describes the switching method of Godsil and McKay [5].

Lemma 2. *Let Γ be a graph and let $\{V_1, \ldots, V_m, W\}$ be a partition of the vertex set of Γ. Suppose that for every vertex $x \in W$ and every $i \in \{1, \ldots, m\}$, x has either 0, $\frac{1}{2}|V_i|$ or $|V_i|$ neighbors in V_i. Moreover, suppose that for every $i, j \in \{1, \ldots, m\}$ the number of neighbors in V_j of a vertex $x \in V_i$ only depends on i and j (in other words, $\{V_1, \ldots, V_m\}$ is an equitable partition of $\Gamma \setminus W$). Make a new graph Γ' as follows. For each $x \in W$ and $i \in \{1, \ldots, m\}$ such that x has $\frac{1}{2}|V_i|$ neighbours in V_i delete the corresponding $\frac{1}{2}|V_i|$ edges and join x instead to the $\frac{1}{2}|V_i|$ other vertices in V_i. Then Γ and Γ' are $\mathbb{R}$-cospectral.*

Proof. Let A and A' be the adjacency matrices of Γ and Γ', respectively (the vertex ordering is assumed to be in accordance with the partition). Let n be the number of vertices of Γ and Γ'. For $i = 1, \ldots, m$ define the $|V_i| \times |V_i|$ matrix $R_i = \frac{2}{|V_i|}J - I$, and the $n \times n$ block diagonal matrix $Q = \mathrm{diag}(R_1, \ldots, R_m, I)$. Then Q is orthogonal and regular, and it follows straightforwardly that $Q^\top A Q = A'$, and more generally, that $Q^\top(A + yJ)Q = A' + yJ$ for every $y \in \mathbb{R}$. $\qquad\square$

Note that the orthogonal matrix Q used in the above proof is regular of level $\mathrm{lcm}(|V_1|, \ldots, |V_m|)/2$. If $|V_i| = 2$ for some $i \in \{1, \ldots, m\}$, then GM-switching just interchanges the two vertices of V_i, and therefore the two vertices may be considered part of W. Thus we can assume that $|V_i| \geq 4$. If $|V_i| = 4$ for all $i \in \{1, \ldots, m\}$, then Q has level 2, and the graphs Γ and Γ' are semi-isomorphic, provided they are not isomorphic. The conditions for GM-switching are most easy to fulfill if $m = 1$ and $|V_1| = 4$. In this case the orthogonal matrix Q is regular of level 2 and has just one nontrivial indecomposable block $R_1 = \frac{1}{2}J - I$. For this switching to work, V_1 must induce a regular graph on four vertices, and each vertex outside V_1 should be adjacent to 0, 2, or 4 vertices of V_1. For example, the adjacency matrix A given below satisfies these conditions, and A' is obtained by GM-switching: $A' = Q^\top A Q$. Therefore the two graphs are $\mathbb{R}$-cospectral. The graphs are not isomorphic (because of different degree sequences), and therefore they are semi-isomorphic.

$$A = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right], \ A' = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

This is the situation we will generalize. If R is an indecomposable regular orthogonal $r \times r$ matrix of level 2, and Γ is a graph with $n \geq r$ vertices and adjacency matrix A. We define the $n \times n$ matrix

$$Q = \begin{bmatrix} R & O \\ O & I \end{bmatrix}$$

and investigate the required structure for A needed to ensure that $A' = Q^\top A Q$ is again the adjacency matrix of a graph. Note that it is sufficient to require that A' is a $(0,1)$ matrix, because A' is symmetric and trace $A' = \text{trace } A = 0$.

3 Regular orthogonal matrices of level 2

Let Q be a regular orthogonal matrix of level 2. Then after suitable reordering of rows and columns, Q takes the block diagonal form $\text{diag}(R_1, \ldots, R_m)$, or $\text{diag}(R_1, \ldots, R_m, I)$, where R_i is an indecomposable regular orthogonal matrix of level 2 for $i = 1, \ldots, m$. It follows easily that if R is an indecomposable regular orthogonal matrix of level 2, then all entries of $2R$ are equal to 0, 1 or -1, and each row and column of R has exactly three 1's and one -1. Using these observations and the orthogonality of R, Wang and Xu [8, 7] determined all indecomposable regular orthogonal matrices of level 2.

Theorem 3. *Let R be an indecomposable regular orthogonal matrix with level 2 and row sum 1. Then after suitable reordering of rows and columns R is one of the following:*

$$(i) \ \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \quad (ii) \ \frac{1}{2} \begin{bmatrix} J & O & \cdots & \cdots & O & Y \\ Y & J & O & \cdots & \cdots & O \\ O & Y & J & O & \cdots & O \\ & \ddots & \ddots & \ddots & \ddots & \\ O & \cdots & O & Y & J & O \\ O & \cdots & \cdots & O & Y & J \end{bmatrix},$$

$$(iii) \ \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}, \quad (iv) \ \frac{1}{2} \begin{bmatrix} -I & I & I & I \\ I & -Z & I & Z \\ I & Z & -Z & I \\ I & I & Z & -Z \end{bmatrix},$$

where I, J, O, $Y = 2I - J$ and $Z = J - I$, are square matrices of order 2.

We observed that $W = 2R$ is a matrix with entries 0, 1 and -1, satisfying $WW^\top = 4I$, and $W\mathbf{1} = W^\top \mathbf{1} = 2 \cdot \mathbf{1}$. Such a matrix W is known as a regular weighing matrix of weight 4. Two weighing matrices are called equivalent if one can be obtained by the other

by row and column permutations and/or multiplication of a number of rows and columns by -1. The inequivalent weighing matrices of weight 4 have been classified in 1986 by Chan, Rodger and Seberry [2], and from their result the classification of the regular ones is straightforward. Therefore, Theorem 3 should be attributed to the authors of [2].

Case (ii) of the above theorem gives an infinite family of matrices of even order starting with order 6. So for the order 8 there exist two different indecomposable regular orthogonal matrices of level 2. If

$$Q = \begin{bmatrix} R & O \\ O & I \end{bmatrix}$$

and R is as in case (i), then the transformation $A' = Q^\top A Q$ corresponds to GM-switching. In the next sections we will investigate the required structure for A for the other three cases.

The product of two regular orthogonal matrices of level 2 is again a regular orthogonal matrix, but the level need not be 2, but can also be 1 or 4. Therefore we may not conclude that the relation: 'being isomorphic or semi-isomorphic' is an equivalence relation. In fact, this is false. This is illustrated by the following example.

Example 4. Consider the three nonisomorphic graphs Γ, Γ_1, Γ_2 with adjacency matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad A_1 = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right], \quad A_2 = \left[\begin{array}{ccc|cccc|c} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

The graphs Γ_1 and Γ_2 can both be obtained from Γ by GM-switching. Therefore Γ_1 and Γ_2 are both semi-isomorphic with Γ. The regular orthogonal matrices that represent the switching are (with R as in Case (i) of Theorem 3):

$$Q_1 = Q_1^\top = \begin{bmatrix} R & O \\ O & I_4 \end{bmatrix}, \text{ and } Q_2 = Q_2^\top = \begin{bmatrix} I_3 & O & \mathbf{0} \\ O & R & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{0}^\top & 1 \end{bmatrix}.$$

Clearly, $Q = Q_1 Q_2$ is orthogonal and regular and satisfies $Q^\top A_1 Q = A_2$. But Q has level 4. Moreover, it has been checked (by computer) that there exists no other orthogonal regular Q of smaller level for which $Q^\top A_1 Q = A_2$. Therefore, Γ_1 and Γ_2 are not semi-isomorphic.

In some cases the product of two regular orthogonal matrices Q_1 and Q_2 of level 2 has level 2 again. This is obviously the case, if the rows of the nontrivial indecomposable blocks of Q_1 are all different from the rows of the nontrivial indecomposable blocks of Q_2. A nontrivial example is given by:

$$Q_1 = Q_1^\top = \begin{bmatrix} R_2 & O \\ O & I_2 \end{bmatrix}, \text{ and } Q_2 = Q_2^\top = \begin{bmatrix} I & O \\ O & R_1 \end{bmatrix},$$

with R_1 as in Case (i), and R_2 as in Case (i) or (ii) of Theorem 3. Then Q_1Q_2 has again level 2 and belongs to Case (ii) of Theorem 3. In case both R_1 and R_2 belong to Case (i), then Q_1Q_2 correspond to a six vertex switching of Case (ii). This shows that the six vertex switching, can sometimes be obtained by applying GM-switching twice.

4   Six vertex switching

Here we consider switching with a regular orthogonal matrix Q of order n, having just one nontrivial indecomposable block of order 6. Thus with a suitable ordering of rows and columns we have:

$$Q = \begin{bmatrix} R & O \\ O & I \end{bmatrix}, \text{ where } R = \tfrac{1}{2} \begin{bmatrix} J & O & Y \\ Y & J & O \\ O & Y & J \end{bmatrix}, \text{ and } Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Let Γ be a graph with n vertices and adjacency matrix

$$A = \begin{bmatrix} B & V \\ V^\top & C \end{bmatrix},$$

where B is the adjacency matrix of a graph Δ of order 6. For the six vertex switching with respect to Δ to work we need that the switched matrix

$$A' = Q^\top A Q = \begin{bmatrix} R^\top B R & R^\top V \\ V^\top R & C \end{bmatrix}$$

is a $(0,1)$ matrix again. First we determine the possible columns of V. This means that we have to find the vectors $\mathbf{v} \in \{0,1\}^6$ for which $R^\top \mathbf{v}$ is again a $(0,1)$ vector.

Lemma 5. *Let $\mathbf{v} \in \{0,1\}^6$. With R as above, $R^\top \mathbf{v} \in \{0,1\}^6$ if and only if the number of ones in each class of the partition is even, or the number of ones in each class of the partition is odd. In the first case $R^\top \mathbf{v} = \mathbf{v}$. In the second case, multiplication by $R^\top$ gives a permutation of the eight involved $(0,1)$ vectors represented by the following two cycles ($[101010]^\top$ and $[010101]^\top$ are fixed):*

$$\left([101001]^\top, [100110]^\top, [011010]^\top \right) \left([100101]^\top, [010110]^\top, [011001]^\top \right).$$

Proof. With $\mathbf{v} = [\, v_1 \ldots v_6 \,]^\top$ we have

$$\mathbf{0} = 2R^\top \mathbf{v} = \begin{bmatrix} v_1 + v_2 + v_3 - v_4 \\ v_1 + v_2 - v_3 + v_4 \\ v_3 + v_4 + v_5 - v_6 \\ v_3 + v_4 - v_5 + v_6 \\ v_1 - v_2 + v_5 + v_6 \\ -v_1 + v_2 + v_5 + v_6 \end{bmatrix} = \begin{bmatrix} v_{1,2} + v_{3,4} \\ v_{1,2} + v_{3,4} \\ v_{3,4} + v_{5,6} \\ v_{3,4} + v_{5,6} \\ v_{1,2} + v_{5,6} \\ v_{1,2} + v_{5,6} \end{bmatrix} \pmod 2,$$

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where $v_{i,i+1} = v_i + v_{i+1}$ for $i = 1, 3, 5$. It follows that $R^\top \mathbf{v}$ is a $(0, 1)$ vector if and only if $v_{1,2} = v_{3,4} = v_{5,6} \pmod 2$. The second part of the lemma follows by straightforward verification. $\qquad\square$

Next we determine the set $\mathcal{B}$ of adjacency matrices B of order 6, that have the property that $B' = R^\top B R$ is a $(0, 1)$ matrix again. To do so, the following observations are useful. The matrix R is invariant under certain reorderings of row and columns, more precisely: $R = P^\top R P$, when P is any permutation matrix generated by

$$P_1 = \begin{bmatrix} O & I & O \\ O & O & I \\ I & O & O \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} Z & O & O \\ O & Z & O \\ O & O & Z \end{bmatrix}, \text{ where } Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, $B' = R^\top B R$ implies $P^\top B' P = R^\top (P^\top B P) R$, so $\mathcal{B}$ is invariant under the mentioned permutations and $(P^\top B P)' = P^\top B' P$. Moreover, $B' = R^\top B R$ implies $J - B' - I = R^\top (J - B - I) R$, so $\mathcal{B}$ is also invariant under taking complements and $(J - B - I)' = J - B' - I$. But there is more. The permutation matrix P_2 commutes with R, and therefore $P_2 + B' = R^\top (P_2 + B) R$, so if $B \in \mathcal{B}$, and the three diagonal blocks of B are O, then $B + P_2 \in \mathcal{B}$ and $(P_2 + B)' = P_2 + B'$.

Lemma 6. *Let B be an adjacency matrix of of order six. With R as above, the matrix $B' = R^\top B R$ is again an adjacency matrix if and only if B can be obtained from one of the following $B_0 \ldots B_7$ by the above mentioned operations.*

$$B_0 = O, \ B_1 = \begin{bmatrix} O & J & O \\ J & O & O \\ O & O & O \end{bmatrix}, \ B_2 = \begin{bmatrix} O & I & I \\ I & O & I \\ I & I & O \end{bmatrix}, \ B_3 = \begin{bmatrix} O & I & I \\ I & O & Z \\ I & Z & O \end{bmatrix},$$

$$B_4 = \begin{bmatrix} O & N & N^\top \\ N^\top & O & N \\ N & N^\top & O \end{bmatrix}, \ B_5 = \begin{bmatrix} O & M & N^\top \\ M^\top & O & N \\ N & N^\top & O \end{bmatrix}, \ B_6 = \begin{bmatrix} O & O & I \\ O & O & M \\ I & M^\top & O \end{bmatrix}, \ B_7 = \begin{bmatrix} O & O & Z \\ O & O & N \\ Z & N^\top & O \end{bmatrix},$$

where $N = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $M = J - N$ and $Z = J - I$. The switched matrices $B_i' = R^\top B_i R$ are:

$$B_0' = O, \ B_1' = B_1, \ B_2' = B_2, \ B_3' = \begin{bmatrix} O & I & Z \\ I & O & I \\ Z & I & O \end{bmatrix},$$

$$B_4' = \begin{bmatrix} O & N^\top & N \\ N & O & N^\top \\ N^\top & N & O \end{bmatrix}, \ B_5' = \begin{bmatrix} O & N^\top & N \\ N & O & M^\top \\ N^\top & M & O \end{bmatrix}, \ B_6' = \begin{bmatrix} O & O & M \\ O & O & I \\ M^\top & I & O \end{bmatrix}, \ B_7' = \begin{bmatrix} O & O & N \\ O & O & Z \\ N^\top & Z & O \end{bmatrix}.$$

Proof. With the vertex ordering used for R, we write

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{bmatrix}, \text{ and } B' = R^\top B R = \begin{bmatrix} B'_{1,1} & B'_{1,2} & B'_{1,3} \\ B'_{2,1} & B'_{2,2} & B'_{2,3} \\ B'_{3,1} & B'_{3,2} & B'_{3,3} \end{bmatrix}.$$

This leads to

$$4B'_{i,i} = JB_{i,i}J + JB_{i,i+1}Y + YB_{i+1,i}J + YB_{i+1,i+1}Y, \tag{1}$$

for $i = 1, 2, 3$ (addition mod 3), where $B_{i,j} = B_{j,i}^\top$. Without loss of generality we take $B_{1,1} = O$. Taking traces in Equation 1 yields trace$(YB_{2,2}Y) = 0$, and therefore $B_{2,2} = O$. Thus $B_{i,i} = O$ for $i = 1, 2, 3$. Equation 1 becomes $4B'_{i,i} = JB_{i,i+1}Y + (JB_{i,i+1}Y)^\top$. For every 2×2 matrix X, $JXY = \alpha(M - N)$ for some scalar α. Since $B'_{i,i}$ has no negative entries it follows that $\alpha = 0$ when $X = B_{i,i+1}$. Therefore $JB_{i,i+1}Y = O$, which reflects that $B_{i,i+1}$ has constant column sums for $i = 1, 2, 3$. Equivalently, $B_{i,i+2} = B_{i+2,i}^\top = B_{i+2,i+3}^\top$ has constant row sums for $i = 1, 2, 3$. Now it is straightforward to find all admissible matrices B and the corresponding B'. $\qquad\square$

For example the following matrix A has the desired form (indeed, $B = B_4 + P_2$ and V has columns $[001100]^\top$ and $[101001]^\top$). With the above Lemmas we conclude that the switched matrix A' is cospectral with A.

$$A = \left[\begin{array}{cccccc|cc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \quad A' = \left[\begin{array}{cccccc|cc} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

The two graphs are not isomorphic, because the degree sequences differ, but they are semi-isomorphic. In addition, the graphs are not related by GM-switching.

Out of the eight adjacency matrices presented in Lemma 6, the graphs with matrices B_4 and B_5 are isomorphic, and the same is true for B_6 and B_7. In addition, the complement of B_4 (and B_5) is isomorphic with $B_4 + P_2$, and the complement of B_2 is isomorphic with $B_3 + P_2$. Therefore, the total number of nonisomorphic graphs Δ for which the six vertex switching works is 18. The total number of matrices B for which $R^\top B R$ is a $(0, 1)$ matrix equals 96.

We note that in Lemma 6 in all cases the graph Δ' with matrix B' is isomorphic to Δ with matrix B. This implies that with a suitable reordering of the rows and columns of R we can establish that $B' = B$. However, this would require a reordering of the entries of the vectors in Lemma 5 depending on the choice of B. So it would not have made the presentation easier. Besides that, the phenomenon is not general, as we shall see in the next section.

5   Seven vertex switching

Here we consider switching with a regular orthogonal matrix Q of order n, having just one nontrivial indecomposable block R of order 7. Theorem 3 gives

$$R = \tfrac{1}{2} \begin{bmatrix} -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

Let Γ be a graph with n vertices and adjacency matrix

$$A = \begin{bmatrix} B & V \\ V^\top & C \end{bmatrix},$$

where now B is the adjacency matrix of a graph Δ with seven vertices. For the seven vertex switching with respect to Δ to work we need that the switched matrix

$$A' = Q^\top A Q = \begin{bmatrix} R^\top B R & R^\top V \\ V^\top R & C \end{bmatrix}$$

is a $(0,1)$ matrix again. Note that the matrix R is invariant under a cyclic shift, that is, $P_1 R P_1^\top = R$ for the cyclic permutation matrix $P_1 = \mathrm{cycle}(0,1,0,0,0,0,0)$. Moreover, also the following permutation leaves R invariant:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Thus the permutation group G generated by P_1 and P_2 is an automorphism group of R.

Remark. The group G is known as the Frobenius group $F_{7,3}$, which can be described as the additive group of the field $\mathbb{F}_7$ extended with the multiplications by a nonzero square. It is the automorphism group of R, but also an automorphism group of the Fano plane. Indeed, $2R + I$, and also $J - 2R - 2I$ are incidence matrices of the Fano plane.

First we determine the possible columns of V. This means that we have to find the vectors $\mathbf{v} \in \{0,1\}^7$ for which $R^\top \mathbf{v}$ is again a $(0,1)$ vector.

Lemma 7. *Let* $\mathbf{v} \in \{0,1\}^7$. *With R and P_1 as above, $R^\top \mathbf{v} \in \{0,1\}^7$ if and only if the vector $\mathbf{v}$ or the complement $\mathbf{1} - \mathbf{v}$ is equal to $\mathbf{0}$, or $P_1^i[1101000]^\top$ for some $i \in \{0, \ldots, 6\}$. If $\mathbf{v} = P_1^i[1101000]^\top$, or $P_1^i[0010111]^\top$, then $R^\top \mathbf{v} = P_1^i[0010110]^\top$, or $P_1^i[1101001]^\top$, respectively.*

9

Proof. This follows by straightforward verification. Using the above mentioned automorphisms of R, and the fact that $R^\top(\mathbf{1} - \mathbf{v}) = \mathbf{1} - R^\top\mathbf{v}$, there are just a few cases to be checked. $\square$

Next we determine the set $\mathcal{B}$ of adjacency matrices B of order 7, that have the property that $B' = R^\top BR$ is a $(0,1)$ matrix again. In the determination and description of $\mathcal{B}$ we use that $\mathcal{B}$ is invariant under the action of G, and under complementation. More precisely, if $B \in \mathcal{B}$, then so is $J - B - I$, and $P^\top BP$ for $P \in G$. Moreover, $(J - B - I)' = R^\top(J - B - I)R = J - B' - I$ and $(P^\top BP)' = R^\top P^\top BPR = P^\top B'P$.

Lemma 8. *Let B be an adjacency matrix of order seven. With R, P_1 and P_2 as above, the matrix $B' = R^\top BR$ is again an adjacency matrix if and only if B can be obtained from one of the following $B_0 \ldots B_{11}$ by complementation and/or a permutation of rows and columns generated by P_1 and P_2.*

$$B_0 = O, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_6 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad B_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The switched matrices $B'_i = R^\top B_i R$ satisfy $B'_0 = B_0$, $B'_1 = B_1$, $B'_i = Z_7 B_i Z_7$ for $i = 2, \ldots, 5$, $B'_6 = Z_7 B_9 Z_7$, $B'_9 = Z_7 B_6 Z_7$, $B'_{10} = Z_7 B_2 Z_7$, and

$$B'_7 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B'_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B'_{11} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix},$$

where Z_7 is the reverse identity matrix of order 7, that is, $(Z_7)_{i,j} = 1$ if $i + j = 7$, and 0 otherwise.

Again the proof goes by straightforward verification. Observe that B_0 to B_{11} are all nonisomorphic, and together with the complements this gives 24 nonisomorphic graphs for which the seven vertex switching works. Out of these graphs B_0 and its complement are the only ones invariant under the group G. Of the remaining cases B_1 and its complement are invariant under the cyclic permutation P_1, and $B_5, \ldots, B_9$ and their complements are invariant under P_2. So in total there are 288 adjacency matrices B of order 7 for which $B' = R^\top B R$ is again an adjacency matrix. For the six vertex switching we observed that B_i' is isomorphic with B_i in all cases. This is not true anymore for the seven vertex switching. Indeed, B_i' is nonisomorphic (and hence semi-isomorphic) to B_i for $i = 6, \ldots, 10$. It is not difficult to see that these semi-isomorphic pairs can also be made by GM-switching with respect to four vertices. However, the following example on eight vertices gives semi-isomorphic graphs that can be made by the seven vertex switching described above, but not by GM-switching.

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
A' = \begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.
$$

6 Eight vertex switching

In this section we consider the case that Q has one nontrivial indecomposable block R of order 8. Theorem 3 gives two nonequivalent possibilities for R, being:

$$
R_1 = \tfrac{1}{2} \begin{bmatrix}
J & O & O & Y \\
Y & J & O & O \\
O & Y & J & O \\
O & O & Y & J
\end{bmatrix}, \text{ and } R_2 = \tfrac{1}{2} \begin{bmatrix}
-I & I & I & I \\
I & -Z & I & Z \\
I & Z & -Z & I \\
I & I & Z & -Z
\end{bmatrix},
$$

with I, J, O, $Y = 2I - J$, and $Z = J - I$ of order 2. We had hoped to find a general description of matrices B for which $R^\top B R$ is a $(0,1)$ matrix again, when R has the form of Case (ii) in Theorem 3, but failed. Already for the above matrix R_1 of order 8, we found 3584 such matrices, and we were not able to discover a general structure. Also for R_2 we found a large number (1504) of such matrices B, so we decided not to give a complete description of the switching conditions as we did in the previous sections for six and seven vertex switching. However, in the next section we will investigate semi-isomorphism for graphs on eight vertices. Therefore we also have to consider eight vertex switching with no additional vertices, that is, $Q = R$. In this case we only have to consider adjacency matrices B for which $B' = R^\top B R$ is nonisomorphic with B. With the help of a computer we found the following:

00110110	00111011	00011010	00101001	00111010	00101011	00110110	00111011
00111010	00111000	00101010	00101010	00001010	00101000	00111010	00111000
11000011	11000001	01001100	11001100	10001111	11001110	11001111	11001101
11000000	11000010	10001100	00001100	10001100	00001110	11001100	11001110
01000000	11000000	11110000	11110000	11110000	11110000	01110000	11110000
10000000	00000000	00110000	00110000	00110000	00110000	10110000	00110000
11100000	10010000	11000000	01000000	11100000	10110000	11100000	10010000
00100000	10100000	00000000	10000000	00100000	10000000	00100000	10100000
00110100	00111100	00111011	00101111	00110111	00111111	00110011	00010011
00111000	00110000	00001011	00100011	00111011	00110011	00000011	00100011
11001110	11001001	10001110	11001110	11001001	11001001	10000110	01001010
11000010	11001010	10000010	00001010	11000010	11001010	10001010	10001010
01100000	10110000	11100000	10110000	01100000	10110000	00010000	00110000
10100000	10000000	00100000	10000000	10100000	10000000	00100000	00000000
00110000	00010000	11110000	11110000	11110000	11010000	11110000	11110000
00000000	00100000	11000000	11000000	11000000	11100000	11000000	11000000
00111111	00011111	00011010	00101001				
00001111	00101111	00101010	00101010				
10000110	01001010	01001111	11001111				
10001010	10001010	10001111	00001111				
11010000	11110000	11110000	11110000				
11100000	11000000	00110000	00110000				
11110000	11110000	11110000	01110000				
11000000	11000000	00110000	10110000				

Table 1: Nonisomorphic pairs $B_i, B'_i = R_1^\top B_i R_1$ $(i = 1, \ldots, 10)$ mentioned in Lemma 9.

Lemma 9. *There exist exactly 20 nonisomorphic graphs $\Gamma_1, \ldots, \Gamma_{20}$, which have an adjacency matrix B_i for which $B'_i = R_1^\top B_i R_1$ is the adjacency matrix of a graph nonisomorphic with Γ_i for $i = 1, \ldots, 20$. The matrices $B_1, \ldots, B_{10}$ of $\Gamma_1 \ldots \Gamma_{10}$ are displayed in Table 1, and $\Gamma_{11}, \ldots, \Gamma_{20}$ are the complements of $\Gamma_1, \ldots, \Gamma_{10}$.*
There exist exactly 36 nonisomorphic graphs $\Gamma_{21}, \ldots, \Gamma_{56}$, which have an adjacency matrix B_i for which $B'_i = R_2^\top B_i R_2$ is the adjacency matrix of a graph nonisomorphic with Γ_i for $i = 21, \ldots, 56$. The matrices $B_{21}, \ldots, B_{38}$ of $\Gamma_{21} \ldots \Gamma_{38}$ are displayed in Table 2, and $\Gamma_{39}, \ldots, \Gamma_{56}$ are the complements of $\Gamma_{21}, \ldots, \Gamma_{36}$.

7   Semi-isomorphic graphs with eight vertices

With the results of the previous sections, we were able to generated by computer all graphs on eight vertices for which the six vertex switching (with two extra vertices), the seven vertex switching (with one extra vertex), or one of the eight vertex switching (with no extra vertex) applies and gives a nonisomorphic (and therefore semi-isomorphic) mate. In total we found 427 nonisomorphic graphs; 227 by six vertex switching, 144 by seven vertex switching, and 56 (see Lemma 9) by eight vertex switching.

00001010 00100110 01011010 00101100 10110000 01010000 11100000 00000000	00111100 00010000 10011001 11100000 10100010 10000001 00001000 00100100	01100011 10100110 11011010 00101100 00110000 01010000 11100000 10000000	00111000 00010000 10011101 11100000 10100110 00101011 00001100 00100100	01001100 10101010 01001001 00001010 11110000 10000000 01010000 00100000	00011100 00000000 00010011 10101010 10010001 10000010 00110100 00101000	01110101 10001100 10010011 10100110 01000000 11010000 00110000 10100000	00001000 00101100 01011101 00100011 11100100 01101010 00010100 00110000
00011111 00000100 00010010 10101100 10010110 11011000 10101000 10000000	00101000 00111010 11011001 01100001 11100010 00000000 01001001 00110010	00111001 00010010 10010010 11101100 10010110 00011000 01101000 10000000	00101001 00110010 11011000 01101001 10110110 00001000 01001000 10010000	01011111 10010010 00010010 11101100 10010110 10011000 11101000 10000000	00101000 00110010 11011001 01101001 10110111 00001000 01001001 00111010	00001010 00101110 01010010 00101100 11010110 01011000 11101000 00000000	00111101 00010000 10011001 11100001 10100010 10000000 00001001 10110010
00111111 00000100 10001011 10001100 10110110 11011000 10101000 10100000	00101010 00111010 11011011 01100001 11100010 00000010 11101100 00110000	00001111 00100100 01011000 00101001 10110110 11001000 10001000 10010000	00111000 00011010 10010011 11100100 11000010 00010000 01101001 00100010	00101001 00110010 11011000 01101001 10110110 00001000 01001000 10010000	00111001 00010010 10010010 11101100 10010110 00011000 01101000 10000000	01000011 10111110 01010100 01100101 01000110 01111000 11001000 10010000	00110001 00010100 10010101 11100101 00000111 01111000 00001001 10111010
00011001 00001101 00010100 10100101 11000110 01111000 00001000 11010000	00100001 00111100 11010000 01100111 01000010 01010000 00011001 10010010	01011001 10011011 00010100 11100101 11000110 00111000 01001000 11010000	00100001 00110100 11010000 01101111 00010111 01011000 00011001 10011010	00110111 00000100 10000011 10001100 00010011 11010000 10101000 10101000	00001010 00111010 01001101 01000001 11100010 00100010 11001100 00110000	00011110 00010010 00000011 11001100 10010011 10010000 11101000 00101000	00001110 00110010 01001001 01001001 10110011 10000000 11001000 00111000
00111101 00000001 10001001 10001001 10110011 10000000 00001000 11111000	00001010 00111010 01000010 01001110 11010010 00010010 11111100 00000000	00101000 00101011 11001001 00001001 11110011 00000000 01001000 01111000	00011111 00010000 00000010 11001110 10010010 10010010 10111100 10000000				

Table 2: Nonisomorphic pairs $B_i, B'_i = R_2^\top B_i R_2$ $(i = 21, \ldots, 38)$ mentioned in Lemma 9.

For $n \leq 11$, Table 1 of [6] gives exact numbers of nonisomorphic graphs on n vertices for which there exist a $\mathbb{R}$-cospectral mate (that is, the graph is not determined by the generalized spectrum); the column carries the name $A\&\overline{A}$. The table also presents the number of graphs for which a nonisomorphic cospectral mate can be obtained by GM-switching (the name of the column is GM). If $n \leq 8$, only GM-switching with respect to four vertices can give nonisomorphic mates. Therefore, nonisomorphic pairs related by GM-switching must be semi-isomorphic when $n \leq 8$. For $n \leq 6$, all graphs are determined by their generalized spectrum. On seven vertices, there exist 1044 graphs. Out of these, 40 graphs are not determined by the generalized spectrum, but for each of these graphs there exist a semi-isomorphic mate by GM-switching. Thus, every graph on seven vertices which is not determined by its generalized spectrum, is semi-isomorphic to some other graph. On eight vertices, there are 12346 nonisomorphic graphs. Out of these 1166 are not determined by their generalized spectrum, and for 1054 of these, an $\mathbb{R}$-cospectral mate can be obtained by GM-switching. Ted Spence (private communication) generated the remaining 112 graphs, and we compared these with the 427 graphs, for which six, seven or eight vertex switching applies. Only 44 of the 112 graphs in Spence's list did not occur in our list of 427. These 44 graphs consist of 22 pairs of $\mathbb{R}$-cospectral graphs, which are not isomorphic or semi-isomorphic. Thus we have:

Proposition 10. *On eight vertices, there exist 22 pairs of nonisomorphic $\mathbb{R}$-cospectral graphs for which no graph is semi-isomorphic with another graph. These are the twelve pairs of graphs displayed in Table 3, together with their complements (the last two pairs of the table are self-complementary).*

According to Theorem 1, each of the 22 pairs of matrices from Proposition 10 are similar by a regular orthogonal matrix Q. For example for the first pair in Table 3 we find

$$
Q = \frac{1}{3}
\begin{bmatrix}
1 & 1 & 1 & 2 & -1 & -1 & 0 & 0 \\
1 & 1 & 1 & -1 & 2 & -1 & 0 & 0 \\
1 & 1 & 1 & -1 & -1 & 2 & 0 & 0 \\
2 & -1 & -1 & 1 & 1 & 1 & 0 & 0 \\
-1 & 2 & -1 & 1 & 1 & 1 & 0 & 0 \\
-1 & -1 & 2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{bmatrix},
$$

which is a regular orthogonal matrix of level 3.

Warning. The 22 pairs of Proposition 10 are not the only $\mathbb{R}$-cospectral pairs that are not semi-isomorphic with each other. For example Γ_1 and Γ_2 from Example 4 have the same property, but the two graphs nor the complements do occur in Table 3. The reason is that both graphs have a nonisomorphic cospectral mate by GM-switching, therefore they are both semi-isomorphic with another graph, but not with each other.

01110010	01110011	01111000	01111100	01111000	01111100	01111000	01111100
10000011	10000010	10110000	10110000	10110100	10111000	10110100	10110010
10000010	10000010	11000000	11001000	11001000	11000010	11001010	11001000
10001101	10001100	11000100	11000011	11000000	11000000	11000001	11000001
00010000	00010001	10000111	10100000	10100100	11000000	10100001	10100001
00010000	00010000	00111010	10000011	01001011	10000011	01000001	10000010
11100000	11100000	00001100	00010100	00000100	00100100	00100000	01000100
01010000	10001000	00001000	00010100	00000100	00000100	00011100	00011000

01111100	01111100	01111100	01111100	01111100	01111100	01111100	01111100
10110010	10111000	10110010	10110010	10110010	10110010	10110010	10110010
11000010	11010011	11001000	11001001	11001000	11001000	11001001	11001010
11000010	11100000	11000001	11000000	11000001	11000001	11000000	11001001
10000111	11000000	10100100	10100000	10100101	10100000	10100100	10110000
10001000	10000011	10001011	10010001	10001000	10000000	10011010	10000000
01111000	00100100	01000100	01000001	01000000	01100001	01000100	01100001
00001000	00100100	00010100	00100110	00011000	00010010	00100000	00010010

01111100	01111100	01111100	01111110	01111100	01111100	01111110	01111110
10110010	10111010	10110011	10110001	10110010	10110010	10111001	10111110
11001010	11000110	11001010	11001000	11001000	11001010	11010001	11011000
11000011	11000100	11000000	11000100	11000011	11000011	11100000	11100000
10100001	11000001	10100100	10100010	10100101	10100001	11000001	11100000
10000000	10110001	10001001	10010000	10001010	10010000	10000000	11000001
01110001	01100000	01100000	10001001	01010100	01100001	10000000	11000000
00011010	00001100	01000100	01000010	00011000	00011010	01111000	00000100

Table 3: Pairs of $\mathbb{R}$-cospectral graphs mentioned in Proposition 10

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