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Flexibility in Technology Choice: A Real Options Approach

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Verena Hagspiel
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CONTENTS

CHAPTER 1

INTRODUCTION

The three chapters comprising the main body of this dissertation all evaluate investment decisions by applying the theory of real options. Chapter 2 and Chapter 3 both analyze optimal investment strategies in flexible technology. Section 1.1.1 gives motivation for these chapters.

Chapter 4 studies the optimal timing decision of technology adoption. The second part of the motivation is devoted to Chapter 4. Section 1.2 summarizes the contents of the chapters.

1.1 Motivation

1.1.1 Investment in Flexible Technology

There can be no doubt concerning the increasing importance of flexible technologies in many sectors of industry. The automotive industry is an excellent example of a sector where the importance of flexibility is at an all-time high (Chappell (2005)). Historically, automotive manufacturers relied on high-volume and inflexible plants with two, or even three assembly lines making the same vehicle. This situation changed with the entry of Japanese manufacturers and continuing product proliferation. There are very few car models now for which demand is large enough to justify dedicating an entire plant to their production (Goyal et al. (2006)). Because of the use of flexible manufacturing systems, Japan became a serious competitor in the automotive industry. Flexible assembly lines allow a manufacturer to assemble multiple types of cars with varying interior equipment, color, engine etc. on the same assembly line. When the sale of one type of car falls, the manufacturer can easily decide to shift a bigger part of the production to another type of car, which can also be assembled on the same production line. This demonstrates a big advantage of flexible manufacturing systems, especially for car models that face highly volatile demand. Opposed to the flexible manufacturing system is the dedicated manufacturing system. In such a system there is no flexibility in the sense that each product needs its own assembly line.

With the term "value of flexibility" I indicate throughout this thesis how valuable a flexible manufacturing system is opposed to a dedicated manufacturing system. In the situation of uncertain demand, firms would like to be able to shift some production around within their capacity (Goyal and Netessine (2007)). This means that the value of flexibility is higher in an uncertain market, a result that was confirmed by many researchers in the literature.

While the Japanese automotive industry lead the way in manufacturing with flexible systems, the North American firms where lagging behind, installing fewer flexible systems. This was partly due to the high costs of acquisition of flexible systems and to the lack of appropriate evaluation methods that measure the advantages of flexibility correctly (Li and Tirupati (1994)). This is widely seen as one of the main reasons that allowed the Japanese automotive industry to take a great share of the total market size. In August 2004, Toyota Motor Co. reported a quarterly profit of 2.6 billion dollars, which was higher than the combined profits of rivals General Motors and Ford Motor. Toyota credited simpler and more modular car designs, platform sharing, and flexible capacity for increasing its quarterly operating profit by $361 million (Van Mieghem (2008)). With a delay, also the North American automotive industry started investing heavily in product flexibility. Recently, Ford, for example, adopted flexibility in 75% of its 21 North American Assembly plants and also General Motors underwent crucial investments (Goyal et al. (2006)).

Empirical studies indicate that the available tools for evaluating the cost/ benefit tradeoffs of investment in flexible manufacturing systems are often contradicting to the intuition of managers, many of whom perceive significant benefits from acquiring flexible manufacturing systems (Fine and Freund (1990)). Van Mieghem (2008) states that the greatest obstacle to achieving flexibility is the fact that flexible assets cost more than dedicated or inflexible do, while it is difficult to measure, value and convey their benefits. The three main value sources or "drivers" of flexibility are (1) scale economies, (2) diversification and (3) allocation flexibility and information updating. The third value driver of flexibility stems from contingent decision making, which provides two real options: the option to wait for more information, and the option to switch capacity allocation or adapt capacity utilization.

One needs to understand the specific advantages of flexible technologies and take this knowledge into account when building investment decision models. In order to include all crucial aspects that should be considered when making investment decisions in general, and specifically when considering flexible technology, I explicitly take into account technological flexibility, uncertainty, investment timing and size in Chapter 2 and 3 of this thesis. The main focus is on the use of flexible capacity to hedge against uncertainty in future demand. Continuous time models are designed in order to apply the real options methodology to these problems. In particular, this allows to establish the effect of uncertainty on the value of flexibility within a dynamic framework. The use of the real options approach implies that the optimal timing of investment can be determined. In reality firms have the opportunity to wait with their investment until the market is big enough to enter with their product. They might enter with a higher capacity in the market at this optimal time.

Flexible Technology

Advances in manufacturing technologies and changing market conditions have led to a shift in production from dedicated to flexible production systems. Recent developments in manufacturing technologies permit the production of a wide variety of products, allow to adapt product mix as well as production volume with small changeover costs and to react rapidly in the case of changes, whether predicted or unpredicted. Global competition, short product cycles as well as highly volatile demand have made it necessary for firms to introduce advanced technologies such as flexible manufacturing systems. But these modern technologies are typically capital intensive and capacity additions require substantial investments.

The subject of the economics of flexibility has been of interest to economists for a long time (for many references see Jones and Ostroy (1984)). In the operations management literature the investment in flexible technologies has received significant attention only during the last two decades, following the increasing viability of flexible, computer controlled manufacturing systems (Fine and Freund (1990)). The issue of flexibility as a tool to better deal with demand uncertainty, has been discussed in the current operations management literature. Firms have to determine the optimal investment type (flexible/ dedicated), optimal (lumpy/incremental) capacity to invest in, and the occupation rate of the capacity. Examples are Bish and Wang (2004), who discuss the value of flexibility for a monopolist and find that its investment decision follows a threshold policy. Chod and Rudi (2005) discuss two different values of flexibility, namely, resource flexibility and responsive pricing. Van Mieghem (1998) studies optimal investment in flexible manufacturing capacity as a function of product prices (margins), investment costs and multivariate demand uncertainty. Goyal and Netessine (2007) study the impact of competition on a firm's choice of technology (product-flexible or product-dedicated) and capacity investment decision in an economic environment characterized by price-dependent and uncertain demand. However, most contributions have one big limitation: the use of static models. And therefore, they do not include the dynamic aspect of flexible capacity.

Timing of Investment

Most investment decisions posses three important characteristics. First, the investment is partially or completely irreversible and involves some sunk costs. Second, there is uncertainty over the future rewards from the investment. Third, there is some flexibility about the timing of investment. One can postpone the investment to get more information about the future. To evaluate such investment decisions the theory of real options is used.

The idea behind real options theory is that an investment opportunity shows an analogy with a call option in financial markets: while a call option gives its owner the right but not the obligation, to buy a stock somewhere in the future, a firm having an investment opportunity has the right, but not the obligation, to buy an asset somewhere in the future. In order to stress this analogy, this investment opportunity is therefore called a "real option". At the moment that the firm actually makes the irreversible investment expenditure, the firm exercises, or loses, the real option. This reflects that at this time the firm gives up the option to delay the investment. Delaying the investment can be beneficial due to the new information that may arrive over time. Dixit and Pindyck (1994) provide an introduction to the real options approach and a review of early contributions.

Investment Size

When investing, it is not only the timing that is important but also the scale of investment. By investing at a large scale the firm takes a risk in case of uncertain demand. In particular, revenue may be too low to defray the investment cost if ex-post demand turns out to be disappointingly low. On the other hand, large scale investment gives a high revenue in case of a high demand realization. In the automotive industry, for example, manufacturers' decisions on investing in production capacity are very critical. On the one hand, expanding already installed capacity is very expensive (Andreou (1990)). Therefore, the installed capacity must be sufficient for the whole life cycle of the product and easily adaptable to new product lines. On the other hand, the profitability of the products are threatened by low utilization of capacity as well as under-capacity.

However, most real option models only determine the optimal timing of an investment project of given size. The fact that this theory has focused more on timing of the investment than on the size of the investment has been already brought up, amongst others, by Hubbard (1994) in a review of Dixit and Pindyck (1994).

Manne (1961) was one of the first to determine an optimal capacity level of a monopolistic firm within a new facility, incorporating the timing issue using a random walk. Manne (1961) finds that when uncertainty increases, the firm will invest in a larger capacity level. Anupindi and Jiang (2008) implement a limited version of the timing issue in a competitive market, by employing a three-stage decision model. They model a market where firms optimize capacity, production, and price. Bar-Ilan and Strange (1999) observe the value of flexibility under the optimization of both timing and the intensity of the investment. Analogous to Dangl (1999), they find that uncertainty delays investment and increases the size of investment.

1.1.2 Technology Adoption Timing

The technological progress has speeded up enormously during the last century. While it took a century for the telephone to expand from a simple gadget of privileged citizens to an 'unimaginable to do without' commodity of daily use, the cell phone underwent this process within just one decade[1]. New mod-

[1]See http://visualizingeconomics.com/2008/02/18/adoption-of-new-technology-since-1900/ for a historical graph showing the adoption of new technologies in the United States since 1900.

els and technological improvements arrive every day, getting faster, smaller and cover a wider range of needs. Firms as well as individual consumers have not just the flexibility to decide when to invest in new technology but also the choice of which new technology innovation to adopt. Therefore, it is important to consider models that take into account both, the timing of technology investment as well as the fact that several new technologies appear, when analyzing technology investment decisions. The theoretical models of adoption timing can be classified in four groups, according whether the particular model deals with uncertainty regarding the arrival and value of a new technology and/or strategic interaction in the product market (see Hoppe (2002) and Huisman (2001) for good overviews). The model in Chapter 4 contributes to the stream of theoretical models of adoption timing in which the profitability of a new technology and/or the rate of technological progress is uncertain. A firm will find it optimal to adopt if and only if the estimate of the profitability of adoption exceeds a certain value and if it is not more profitable to wait for new information on the arrival of better technology (Hoppe (2002)). Uncertainty about the value of a new technology reduces or increases a firm's adoption incentive at any date, while the possibility of resolving uncertainty over time by receiving more information about the arrival and value of new technology depicts an incentive to delay adoption.

Specifically, Chapter 4 follows up on work of Huisman (2001) that studies a decision theoretic model of technology adoption of a monopolist. Huisman applies real options methods to the problem of technology adoption of a firm that faces uncertainty about both the value and arrival of new technology. He extends the traditional decision theoretic models on technology adoption with a model in which technologies arrive according to a Poisson process. One limitation of Huisman (2001) is that the arrival rate of new technologies is constant. This is however a rather strong assumption for many applications in practice. Chapter 4 relaxes this assumption taking into account that the arrival rate of new technologies can change over time.

1.2 Overview of Chapters

In Chapter 2 we analyze the investment and production decisions of a monopolist in a stochastic dynamic environment. Uncertainty is present in the sense that the future demand level is unknown and dynamics is taken into account by adopting a continuous time framework. Before the firm is able to produce goods it has to install capacity. The firm has to choose the optimal time as well as investment size. Once capacity is installed, the firm can decide the optimal production level. We allow for production flexibility where at each moment in time production can fluctuate between zero and the capacity level without additional adjustment costs when adapting production to demand. This chapter is based on (Hagspiel et al., 2011, a).

We find that the initial occupation rate of the flexible firm can be quite low, especially when investment costs are concave and demand uncertainty is high. In order to show the implications of production flexibility, we study the case of the inflexible firm, where the firm is restricted to a fixed output rate. Comparing the optimal investment decision of both firms, we find that the flexible firm invests in higher capacity than the inflexible firm. The capacity difference increases with uncertainty. Regarding the timing of investment there are two contrary effects. On the one hand the flexible firm has an incentive to invest earlier, because production flexibility increase the value of the project. While on the other hand, the flexible firm has an incentive to invest later, because the higher capacity level makes investment more costly. The later effect dominates when uncertainty is high.

The contribution of Chapter 3 is three-fold. First, I derive and analyze the optimal investment strategy in product-flexible contrary to product-dedicated manufacturing systems of a firm operating in an economic environment characterized by uncertain demand. The firm is a price setting monopolist that is selling two products that differ in substitutability and profitability. Similar to Chapter 2 a continuous time framework is applied with the future demand level assumed to be uncertain and following a stochastic process. I derive the optimal investment strategy for the case of flexible and dedicated capacity separately in the first run. The following three decisions of the firm are optimized. First, when is it optimal to invest, i.e. the investment timing. Second, how much capacity to install, i.e. the size of investment. And third, once capacity has been installed, how to optimally use this capacity. Here the use of product-flexible capacity distinguishes from dedicated capacity. Flexible capacity allows the firm to switch costlessly between products and handle changes in relative volumes among products in a given product mix. Dedicated capacity restricts to manufacture one specific product. I find that in the flexible case, under high demand the firm just produces the most profitable product. If demand is low the firm produces both products to make total market demand bigger. In the dedicated case the firm invests in both capacities only if the substitutability rate is low and profitability of both products high enough. Otherwise, it restricts investment to one dedicated capacity for the more profitable product.

Second, I compare the two investment strategies and specify the value of flexibility. Here I find that flexibility especially pays off when uncertainty is high, substitutability low, and profit levels between the two products are substantially different. In a third step, I change the model setup in Chapter 3 and consider a firm's decision to change from producing with dedicated to producing with flexible capacity. Analyzing the optimal timing of the switch, I find that the specific product combination has a remarkably high impact on the firm's decision. Chapter 3 is based on Hagspiel (2011).

Chapter 4 contributes to the literature of technology adoption. Our model builds on the model presented in (Huisman, 2001, Chapter 2). We investigate the role of a firm that decides about technology adoption with an investment to change from old to new technology facing uncertain improvement size and timing of future technology improvements. The firm's adoption decision is described as the solution of an infinite horizon dynamic programming problem in a continuous time setting. The technological process is assumed to advance exogenously to the firm. Once the firm decides to adopt new technology it faces large fixed cost. Unlike prevalent in the technology adoption literature, we assume that the arrival rate of new technology is not constant but changing over time. Chapter 4 is based on (Hagspiel et al., 2011, b).

The improved modeling of technological innovation arrival allows us to explain the fact that firms often adopt new technology a time lag after its arrival, while models with constant arrival rate neglect this phenomenon. Furthermore, it adds to the significance of our results by one additional degree of freedom and allows to study the effect of variance of time between two consecutive technology arrivals. Depending on whether the arrival rate is assumed to change or be constant over time, the optimal technology adoption timing changes significantly. Our analysis shows that the probability of a time lag between innovation and adoption is substantially high. We present two possible applications of the model and analyze numerical examples suited to those.

CHAPTER 2

Production Flexibility and Capacity Investment under Demand Uncertainty[1]

This chapter takes a real option approach to consider optimal capacity investment decisions under uncertainty. Besides the timing of the investment, the firm also has to decide on the capacity level. Concerning the production decision, we study a flexible and an inflexible scenario. The flexible firm can costlessly adjust production over time with the capacity level as the upper bound, while the inflexible firm fixes production at capacity level from the moment of investment onwards.

We find that the flexible firm invests in higher capacity than the inflexible firm, where the capacity difference increases with uncertainty. For the flexible firm the initial occupation rate can be quite low, especially when investment costs are concave and the economic environment is uncertain. As to the timing of the investment there are two contrary effects. First, the flexible firm has an incentive to invest earlier, because flexibility raises the project value. Second, the flexible firm has an incentive to invest later, because costs are larger due to the higher capacity level. The latter effect dominates in highly uncertain economic environments. Investment in flexible capacity leads to a significantly larger expected project value than for inflexible capacity, especially in case of highly volatile demand.

[1] This chapter is based on (Hagspiel et al., 2011, a).

2.1 Introduction

Nowadays firms often face high demand volatility. This uncertainty in demand influences the desirability to invest in production capacity, the choice of the capacity level, and it raises the value of being able to adapt the production decision. Bengtsson and Olhager (2002) argue that, in order to cope with unpredictable changes in demand, the firm needs to possess some degrees of flexibility in order to stay competitive and profitable. We analyze the investment and production decisions of the monopolist in a stochastic dynamic environment. Uncertainty is present in the sense that the future demand level is unknown and dynamics is taken into account by adopting a continuous time framework. Consumers' demand is driven by a demand intercept following a geometric Brownian motion. Before the firm is able to produce goods, it has to install capacity. In deciding about capacity investment the firm has to choose the timing as well as the capacity level. Once the capacity is installed, the firm can decide about the production level. We allow for production flexibility where at each moment in time production can fluctuate between zero and the capacity level without facing additional adjustment costs. A main input for the production decision is the current consumer demand level.

The current demand level is also the main input for the investment decision. In particular, we show in this study that three demand level regions can be distinguished. In the first region demand is so low that the firm will not invest. In the second region demand is at an intermediate level, implying that investment is optimal but initially the firm does not produce up to capacity. The corresponding occupation rate can be quite low, especially if there is a lot of demand uncertainty and when the investment cost function is concave. The intuition is based on the known result in the literature that, once uncertainty is large, then it is optimal for the firm to invest late in a large capacity level (Bar-Ilan and Strange (1999) and Dangl (1999)). Assuming an concave investment cost function an additional unit of investment is relatively cheap when the investment magnitude is large so that, once the firm decides to install a large capacity, it is relatively cheap to make this capacity "very large" implying that the ratio between current production and capacity will be small. Indeed, the occupation rate is significantly higher when investment costs are convex. We also show that the gap between the moment of investing in capacity and the moment where the firm uses full capacity for the first time is very large and increases more than proportionally with uncertainty. In the third region demand is so high that the firm immediately starts producing at full capacity after the investment.

In order to show the implications of production flexibility, we also study the case of the inflexible firm, where the firm is restricted to a fixed production rate. The capacity choice at the moment of the investment fixes the inflexible firm's quantity at which it will produce forever. We show that the flexible firm invests in larger capacity. This effect is reinforced by uncertainty. As to the timing of the investment there are two contrary effects. On the one hand the flexible firm has an incentive to invest earlier, because production flexibility increases the value of the project. On the other hand, the flexible firm has an incentive to invest later, because the higher capacity level makes investment more costly. The latter effect dominates as uncertainty goes up.

In today's economy production flexibility is an important means to adjust to fluctuations in demand. During the credit crunch recession that started in 2008 the demand in the car industry dropped significantly. Companies reacted by downscaling production, which resulted in low occupation rates[2]. Another example is the LCD industry. During its initial stage (2003-04) production flexibility was not crucial because firms were producing at full capacity. The reason was that capacity was lagging behind demand, since it took about one to two years to build these advanced and expensive production facilities, while demand for the new products was high. Later on competition on the supply side led to overcapacity, implying that production flexibility became a far more important issue[3].

Production flexibility is to some extent determined by flexibility of labor. One might criticize that a firm's potential to flexible adapt output is strongly curtailed by legal constraints. However, as recently shown during the financial crisis (2008-2011), even in European countries with traditionally tight labor laws, governments were reacting fast to the industry's call for possibilities

[2]In January 2009, for example, Honda forced British workers to start an enforced four-month layoff against the backdrop of a further dire warning over the trading outlook from the Japanese car giant. See article "Honda suspends UK workers for 16 weeks" published by 'The Times' on January 30, 2009. Toyota even closed down several of its factories in Japan for a few days as a consequence to the financial crisis. See, for example, the article "Toyota in 11-day factory shutdown" published by Guardian on January 6, 2009.

[3]On the site http://www.purchasing.com/article/340083-LCD_prices_will_fall_in_fourth_quarter.php it is stated that "LCD prices increased 30–40% this year because of production cutbacks by LCD manufacturers. Suppliers had a disastrous fourth quarter 2008 and cut production. Factory utilization rates fell to 30% at some suppliers with the average being about 50%."

in cutting working hours in order to down-scale production[4].

Our work adds to two streams of literature. The first stream considers the issue of production flexibility. The value of production flexibility has been brought up in the literature mainly considering two- or three-stage decision models. In order to consider the dynamic character of production flexible capacity, i.e. to adapt production flexible to demand changes over time by scaling down or up, we adopt a continuous time setting. This also allows us to consider the investment timing decision.

An important difference between our setup and the multi-stage models is that we adopt a continuous time framework. This enables us to analyze the timing of the investment, where we establish that uncertainty delays investment and that a flexible firm both has incentives to invest earlier or later than an inflexible firm. Anupindi and Jiang (2008) consider a model where firms make decisions on capacity, production, and price under demand uncertainty in a three-stage decision making framework. In their model the firm always decides about capacity before and price after the demand realization, while there is a difference in the timing of the production decision. A flexible firm can postpone production decisions until the actual demand curve is observed, while the inflexible cannot. Our results that capacity increases with uncertainty and flexibility coincide with Anupindi and Jiang's finding that, when the market is more volatile, flexibility allows a firm to increase investment in capacity and earn a higher profit. Chod and Rudi (2005) confirm this result as well. They study two types of flexibility - resource flexibility and responsive pricing. The firm is selling two products facing linear demand curves for these products. They consider a situation in which a single flexible resource can be used to satisfy two distinct demand classes, where they characterize the effects of demand variability and demand correlation. As Anupindi and Jiang (2008) they do not consider the timing issue of this investment problem but apply a two-stage decision problem.

Van Mieghem and Dada (1999) consider a two stage model where demand is linear with a stochastic intercept. The firm has to decide about capacity investment, production (inventory) quantity and price. They analyze several strategies which differ in the timing of the operational decisions (i.e. capacity, output and price) relative to the realization of uncertainty and show how the

[4] In consequence of the financial crisis many European countries introduced or extended the possibilities for "Kurzarbeit". This model, first established in Germany, is a short-term, recession-related program in which companies have entered into an agreement to avoid laying-off any of their employees by instead reducing the working hours of all or most of their employees, with the government making up some of the employees' lost income.

different strategies influence the strategic investment decision of the firm and its value. Similar characteristics to our flexible production shows their formalization of "production postponement strategies". Though, they restrict their work to a static environment.

The second stream of literature deals with the theory of real options, which mainly considers problems where a firm must find the optimal time to invest in a certain project (McDonald and Siegel (1985) and McDonald and Siegel (1986)). In general, this literature acknowledges partial irreversibility of investment and predicts that uncertainty delays investment. The real options theory is elaborately comprised in Dixit and Pindyck (1994). The fact that this theory has focused more on the timing of the investment than on the size of the investment has been brought up already in a review of this book by Hubbard (1994). He argues that"the new view models...do not offer specific predictions about the level of investment". Hubbard (1994) claims that in order to take this extra step"it requires the specification of structural links between the marginal profitability of capital and the desired capital stock (the usual research focus in the traditional, neoclassical literature)".

However, there are still a few real options papers that, besides the timing, also consider the size of the investment. Dixit (1993) picks up the capacity choice issue by evaluating a model with irreversible choice among mutually exclusive projects under uncertainty. He considers a project with output price uncertainty, sunk capital cost but no operating cost. Decamps et al. (2006) renew this model, reducing it to a choice among two alternative investment projects of different scales. They provide parameter restrictions under which the optimal investment strategy is not a trigger strategy and the optimal investment region is dichotomous. Lee and Shin (2000) determine the relationship between investment and uncertainty exploring the role of a variable input, e.g. labor, in striking a balance between a positive effect, due to the convexity of the profit function, and a negative effect, due to the usual option value of waiting.

Bar-Ilan and Strange (1999) consider both the timing and intensity of investment. Furthermore, they examine the evaluation of capital stock under incremental and lumpy investment. In the 'lumpy' investment model of Bar-Ilan and Strange the investment project is such that by paying a fixed investment cost the firm receives a production technology that allows it to produce forever at a certain production rate. This also holds for our inflexible firm model. In contrast to Bar-Ilan and Strange (1999) we also study the optimal decisions of a flexible firm, where we allow for flexibility in the production level at any instant after the investment time.

The paper most closely related to our work is Dangl (1999). The setup of our model is similar to Dangl (1999) in the sense that the firm has to determine the investment timing and the investment size. However, Dangl concentrates on the effect of demand uncertainty on these investment decisions. We elaborate on his paper by analyzing the specific implications of production flexibility on investment timing and size. Therefore we derive also the optimal investment strategy in inflexible capacity. This allows us to analyze the difference of optimal investment strategies in production flexible and inflexible capacity. respectively. While Dangl does not address the possibility of investing in the third region, i.e. demand is so large that directly after the investment the firm produces at full capacity, we show that for specific situations this is the optimal strategy for the firm. We analyze these scenarios and present examples and illustrations for this case.

Recent work of Chronopoulos et al. (2011) also takes into account both timing and size of investment. They analyze the impact of risk aversion as well as operational flexibility in the form of suspension and resumption options on these decision. Similar to us they do not take into account suspension costs. Production flexibility is not considered in their model. When the firm is in an operating mode, it always produces up to full capacity. Among other results, they find that increasing risk aversion facilitates investment and reduces the optimal capacity of the project. Operational flexibility increases the value of the investment opportunity and therefore, the incentive to invest, resulting in the decrease of the optimal capacity size. The option to suspend operations when demand is to low and resume operations again when demand increases is already considered in early real options literature, like the pioneering work of Brennan and Schwartz (1985) and McDonald and Siegel (1985). Adkins and Paxson (011b) recently presented a two factor multiple switching option model, with switching from an operating state with an option to suspend operations, or from suspended state to an operating state, when both output price and input cost are stochastic and switching is costly. They provide the value of such facilities and the optimal switching input and output triggers and present an illustration for an heavy crude oil field production which requires natural gas as an input, with shut-down and start-up switching costs.

The paper is structured as follows. Beyond this introduction we present the model for the flexible and the inflexible firm in Section 2.2, where we also develop results regarding the timing and the size of the investment. In this framework the investment cost function is concave while demand is linear.

Section 2.3 contains results about the occupation rate and the effects of production flexibility. Section 2.4 looks into some robustness issues, where we analyze the impact of different investment cost and demand functions. In particular we study the cases of convex investment cost and iso-elastic demand. Section 2.5 concludes. The appendix contains additional mathematical results and proofs.

2.2 Model, Size and Timing of Investment

2.2.1 Flexible Case

Consider a firm that has to decide about capacity investment. This involves two decisions, namely when to invest and determining the size of the capacity. After the investment is made the firm is able to produce goods. Production is flexible so that it can be adapted to demand changes, while it is bounded by the capacity size.

The investment costs are sunk and, following Dangl (1999), assumed to be equal to $I(K) = \delta K^{\lambda}$, in which K stands for the capacity level while λ is a constant being less than one. This means that the marginal investment costs are decreasing with increasing installed capacity. Later on, in the robustness section we study the implications of a convex investment cost function.

Denote production quantity at time t by q_t. Since production cannot exceed capacity, it holds that

$$0 \leq q_t \leq K. \tag{2.1}$$

The firm is uncertain about future demand. Adopting a linear demand structure we have

$$p(q_t, t) = \theta_t - \gamma q_t, \tag{2.2}$$

where $p(q_t, t)$ is price, γ is a positive constant, and demand uncertainty is modeled by $\{\theta_t\}$ following the geometric Brownian motion

$$d\theta_t = \alpha\theta_t dt + \sigma\theta_t dW_t. \tag{2.3}$$

In this expression α is the trend parameter, σ is the volatility parameter, and dW_t is the increment of a Wiener process. From now on we drop the time subscript whenever there can be no misunderstanding.

The firm's production costs are fixed and denoted by c. It follows that the profit flow is

$$\pi(q) = p(q)q - cq. \tag{2.4}$$

Given the demand is equal to θ, the optimal output rate, q^*, is determined by maximizing the profit flow subject to $0 \leq q \leq K$. This gives

$$q^*(\theta, K) = \begin{cases} 0 & \text{for } 0 \leq \theta < c, \\ \frac{\theta}{2\gamma} - \frac{c}{2\gamma} & \text{for } c \leq \theta < 2\gamma K + c, \\ K & \text{for } \theta \geq 2\gamma K + c. \end{cases} \tag{2.5}$$

Production will be temporarily suspended[5] when θ falls below c, and resumed later if θ (again) rises above c. Expression (2.5) implies that the profit flow is given by

$$\pi(\theta, K) = \begin{cases} 0 & \text{for } 0 \leq \theta < c, \\ \frac{(\theta - c)^2}{4\gamma} & \text{for } c \leq \theta < 2\gamma K + c, \\ (\theta - \gamma K - c)K & \text{for } \theta \geq 2\gamma K + c. \end{cases} \tag{2.6}$$

In order to find the expected discounted value of this investment project ($V(\theta, K)$), we apply the dynamic programming approach. Then this value function must satisfy the Bellman equation

$$V(\theta, K) = \pi(\theta, K)dt + E\left[V(\theta + d\theta, K)e^{-rdt}\right], \tag{2.7}$$

where r is the (constant) discount rate. Applying Ito's Lemma, substituting and rewriting leads to the differential equation (see, e.g., Dixit and Pindyck (1994))

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \alpha\theta\frac{\partial V}{\partial\theta} - rV + \pi = 0. \tag{2.8}$$

Solving this equation for $V(\theta, K)$, considering that we have three different regions, and ruling out bubble solutions, we get the following value of the project:

$$V_{flex}(\theta, K) = \begin{cases} L_1(K)\,\theta^{\beta_1} & \text{for } 0 \leq \theta < c, \\ M_1(K)\,\theta^{\beta_1} + M_2\theta^{\beta_2} & \\ +\frac{1}{4\gamma}\left[\frac{\theta^2}{r - 2\alpha - \sigma^2} - \frac{2c\theta}{r - \alpha} + \frac{c^2}{r}\right] & \text{for } c \leq \theta < 2\gamma K + c, \\ N_2(K)\,\theta^{\beta_2} + \frac{K}{r-\alpha}\theta - \frac{K(K\gamma + c)}{r} & \text{for } \theta \geq 2\gamma K + c, \end{cases} \tag{2.9}$$

in which β_1 (β_2) is the positive (negative) root of the quadratic polynomial

$$\frac{1}{2}\sigma^2\beta^2 + \left(\alpha - \frac{1}{2}\sigma^2\right)\beta - r = 0. \tag{2.10}$$

[5] As an example notice that in Japan Toyota suspended production for 11 days during the credit crunch recession (see footnote 1 on page 3).

The lengthy expressions for $N_2(K)$, $M_1(K)$, M_2 and $L_1(K)$ are relegated to Appendix A.

Consider first the value of the investment project in the region $0 \leq \theta < c$. Here demand is that low that production is temporarily suspended. The term $L_1(K)\,\theta^{\beta_1}$, being increasing in θ, stands for the value of the option to start producing in the future, which happens once θ rises beyond c. This option value is larger the closer θ is to c. The fact that $L_1(K)$ is positive for the considered parameter ranges is shown in the appendix (see Corollary 1).

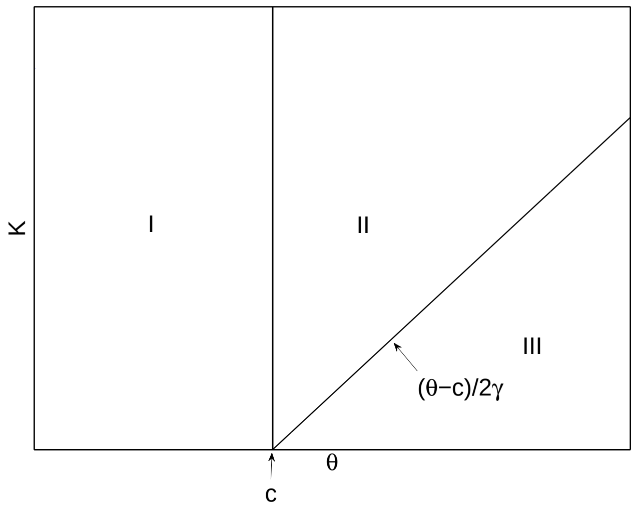
The value of the investment project in the region $c \leq \theta < 2\gamma K + c$ consists of three terms where the third term is the cash flow generated by the sales. The first term, which is negative, corrects for the fact that production is constrained by the capacity level, where the constraint becomes binding once θ reaches the level $2\gamma K + c$. The absolute value of this term increases with θ. The second term, $M_2\theta^{\beta_2}$, corrects for the fact that the quadratic profit function is positive even when θ falls below the unit production cost c. $M_1(K)$ and M_2 are negative for the considered parameter ranges (for the proof see Corollary 1 in the Appendix). We assume that switching between different levels of production is costless. One of the main goals of the paper is to analyze the differences of investment in flexible compared to investment in inflexible capacity. Therefore, we want to look at the two extreme cases, i.e. capacity that is fully flexible in adapting production to demand without any adjustment costs and production capacity that has to be used up to full extent. In practice firms might decide to install an in-between degree of flexibility.

In the region $\theta \geq 2\gamma K + c$ demand is that large that the firm produces at full capacity, which generates a discounted cash flow stream that is reflected in the second and third term of the value of the investment project associated with this region. The first term, $N_2(K)\,\theta^{\beta_2}$, describes the value of the option that in case demand decreases, in fact θ falling below $2\gamma K + c$, the firm is able to scale down production below capacity and is not forced to keep producing with full capacity. $N_2(K)$ is positive for the considered parameter range (see Appendix A).

Knowing the value of the project, $V(\theta, K)$, we are able to derive the optimal investment strategy. In general the procedure is as follows. First, we determine the optimal capacity choice $K^*(\theta)$ for a given level of θ. Second, we derive the optimal investment threshold θ^*. For this demand level θ^* it holds that the firm is indifferent between investment and waiting with investment. Investment (waiting) is optimal for a θ being larger (lower) than θ^*.

It is easy to understand that investment will not take place when demand

Figure 2.1: Three Investment Regions



is that low that the firm suspends production, i.e. the investment threshold θ^* will never be such that it falls below c. This is because the firm will not produce as long as $\theta < c$, so it does not lose anything when it saves on discounted investment expenses by waiting until θ becomes bigger than c. In the other two $\theta-$regions investment can take place. Investing while $c \leq \theta < 2\gamma K + c$ means that the firm leaves some capacity idle right after the investment has been undertaken, while investing for $\theta \geq 2\gamma K + c$ implies that the capacity level is fully used right after the moment of investment. Figure 2.1 visualizes the three regions.

The following proposition provides equations that implicitly determine the threshold θ^* and the corresponding capacity level $K^*(\theta^*)$ in each of the two cases. The optimal investment decision corresponds to the case that provides the largest value of the investment project.

Proposition 1 *Concerning the firm's investment policy there are two possibilities:*

1. *Given that the firm does not produce up to full capacity right after the investment moment, the optimal capacity level $K^*(\theta)$ is implicitly determined by*

$$M_1'(K^*)\,\theta - \delta\lambda K^{*\lambda-1} = 0. \tag{2.11}$$

The expression for $M_1'(K)$ is stated by equation (2.72) in Appendix B.1. In case the obtained K^* is such that from the resulting production quantity (2.5) it follows that q^* is not an interior solution, i.e. in case $K^* \leq \frac{\theta-c}{2\gamma}$, then the *optimal capacity is replaced by the boundary solution* $\frac{\theta-c}{2\gamma}$. *Thus,*

$$K^*(\theta) = \max\left(K^*, \frac{\theta - c}{2\gamma}\right). \tag{2.12}$$

The investment threshold[6] θ^* *is implicitly determined by*

$$M_2\theta^{*\beta_2}\left(\frac{\beta_1-\beta_2}{\beta_1}\right)+\frac{1}{4\gamma}\left[\frac{\theta^{*2}}{r-2\alpha-\sigma^2}\left(\frac{\beta_1-2}{\beta_1}\right)\right.$$
$$\left.-\frac{2c\theta^*}{r-\alpha}\left(\frac{\beta_1-1}{\beta_1}\right)+\frac{c^2}{r}\right]-\delta\left(K^*\left(\theta^*\right)\right)^\lambda=0. \tag{2.13}$$

2. *Given that the firm produces up to full capacity right after the investment moment, the optimal capacity level* $K^*(\theta)$ *is implicitly determined by*

$$N_2'\left(K^*\right)\theta^{\beta_2}+\frac{\theta}{r-\alpha}-\frac{2K^*\gamma+c}{r}-\delta\lambda K^{*\lambda-1}=0. \tag{2.14}$$

The expression of $N_2'(K)$ *is given by equation (2.74) in the Appendix. In case the obtained* K^* *does not constitute an interior solution, i.e. in case* $K^* > \frac{\theta-c}{2\gamma}$*, the optimal capacity is replaced by the boundary solution* $\frac{\theta-c}{2\gamma}$. *Thus,*

$$K^*(\theta) = \min\left(K^*, \frac{\theta - c}{2\gamma}\right). \tag{2.15}$$

The investment threshold θ^* *is implicitly determined by*

$$N_2(K^*\left(\theta^*\right))\theta^{*\beta_2}\left(\frac{\beta_1-\beta_2}{\beta_1}\right)+\frac{K^*\left(\theta^*\right)}{r-\alpha}\theta^*\left(\frac{\beta_1-1}{\beta_1}\right)$$
$$-\frac{K^*\left(\theta^*\right)\left(K^*\left(\theta^*\right)\gamma+c\right)}{r}-\delta\left(K^*\left(\theta^*\right)\right)^\lambda=0. \tag{2.16}$$

Out of these two possibilities the firm chooses the one that gives the highest expected value of the project $V_{flex}\left(\theta^*, K^*(\theta^*)\right)$.

A numerical investigation based on this proposition will be provided in Section 2.3.

[6] One can show that given that there exists a threshold θ^*, this threshold is unique. For a detailed analysis of the two conditions that have to be satisfied to prove the uniqueness of the threshold see Dixit and Pindyck (1994) (Appendix 4.B).

2.2.2 Inflexible Case

The firm has to decide about when to undertake the capacity investment, and it has to determine the capacity size. The difference with the previous section is that the capacity size fixes the production level, i.e. at each point of time after the investment the firm produces up to capacity whenever it is an active producer. Hence, contrary to the flexible firm in the previous section, for this inflexible firm it is not possible to produce another positive quantity than the capacity level. However, we assume that the inflexible firm still has the suspension option in that it will not produce as soon as demand is such that price will fall below unit production cost, which implies that

$$p(K) = \theta - \gamma K < c \Rightarrow q = 0. \tag{2.17}$$

In all other cases it holds that

$$q = K. \tag{2.18}$$

The implication for the profit flow is that

$$\pi(\theta, K) = \begin{cases} 0 & \text{for } 0 \leq \theta < \gamma K + c, \\ (\theta - \gamma K - c)K & \text{for } \theta \geq \gamma K + c. \end{cases} \tag{2.19}$$

Considering the two different regions, familiar steps lead to the following value of the investment project:

$$V_{inflex}(\theta, K) = \begin{cases} Q_1(K)\,\theta^{\beta_1} & \text{for } 0 \leq \theta < \gamma K + c, \\ P_2(K)\,\theta^{\beta_2} + \frac{K}{r-\alpha}\theta - \frac{K(K\gamma + c)}{r} & \text{for } \theta \geq \gamma K + c, \end{cases} \tag{2.20}$$

in which β_1 (β_2) is the positive (negative) root of the quadratic polynomial (3.11). The lengthy expressions for $P_2(K)$ and $Q_1(K)$ are relegated to Appendix A. Both constants, P_2 and Q_1, are positive. The proof can be found in the Appendix (see Corollary 2).

In case $0 \leq \theta < \gamma K + c$, the firm does not produce, but it will start doing so as soon as θ exceeds $\gamma K + c$. The term in $Q_1(K)$ captures the expected profit from the option to resume operations in the future. If $\theta \geq \gamma K + c$ the firm produces at rate K, which generates a discounted cash flow stream represented by the two last terms of $V(\theta, K)$. The value of future suspension options, exercised at the moment that θ falls below $\gamma K + c$, is captured in $P_2(K)\,\theta^{\beta_2}$.

Let us turn to the investment decision that maximizes the project value. Like with the flexible firm case, also here it is never optimal to invest while demand is such that production will be suspended right after the investment decision. Hence, we only need to consider the case where $\theta \geq \gamma K + c$. Again we start out determining the optimal capacity level for every relevant value of θ, which we denote by $K^*(\theta)$. Then we proceed by determining the investment threshold θ^*. The following proposition presents the implicit equations that result from this procedure.

Proposition 2. *The optimal capacity level $K^*(\theta)$ satisfies*

$$P_2'(K^*)\,\theta^{\beta_2} + \frac{\theta}{r-\alpha} - \frac{2\gamma K^* + c}{r} - \delta\lambda K^{*\lambda-1} = 0. \tag{2.21}$$

The investment threshold θ^* is implicitly determined by

$$P_2\left(K^*\left(\theta^*\right)\right)\theta^{*\beta_2}\left(1-\frac{\beta_2}{\beta_1}\right) + \frac{K^*\left(\theta^*\right)}{r-\alpha}\theta^*\left(1-\frac{1}{\beta_1}\right) - \frac{(\gamma K^*\left(\theta^*\right)+c)K^*\left(\theta^*\right)}{r} - \delta\left(K^*\left(\theta^*\right)\right)^{\lambda} = 0. \tag{2.22}$$

The next section contains a numerical analysis based on this proposition.

2.3 Results

2.3.1 Occupation Rate

As we have seen in section 2.1, the flexible firm can either invest in the second $\theta-$region, i.e. $\theta \in [c, 2\gamma K + c)$, where the firm sets an upper bound for output at the moment of investment but does not produce up to full capacity yet, or invest in the third $\theta-$region, i.e. $\theta \in [2\gamma K + c, \infty)$, which means that the firm invests in a capacity level that is fully used right at the moment of investment. Later on it will adapt the production rate to the demand while this maximum output boundary stays fixed. In the following, we will present two examples which should illustrate that it might be optimal for the firm to invest in region 2 as well as region 3, depending on the economic environment it faces. In case of low uncertainty and a small drift rate of demand the firm prefers to invest in region 3, while it has a higher incentive to invest in region 2 when facing highly uncertain demand. Figure 2.2 shows an example where $\sigma = 0.1$, $\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$, and $\lambda = 0.7$. Solving equations (2.11) and (2.14) the optimal capacity choice for the two regions is derived.

After comparing the expected values of the investment project, we conclude that it is optimal to invest in the second region at the investment trigger $\theta^* = 440.13$, provided that the initial $\theta-$value lies below this θ^*. In particular, the firm invests immediately if the current value of θ exceeds θ^*, while

Figure 2.2: Investment Strategy of an example with the optimal investment moment laying **in region II**.
2.A: The optimal capacity $K^*(\theta)$ as well as production quantity $q^*(\theta)$ as a function of θ. Indicating the optimal investment threshold θ^* and capacity invested in $K^*(\theta^*)$ as well as the point θ^q where demand is high enough to use full capacity for production.
2.B: The bold dashed line shows the optimal production output as a function of θ after investment.
[Note: Parameter values are $\sigma = 0.1$, $\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 0.7$]

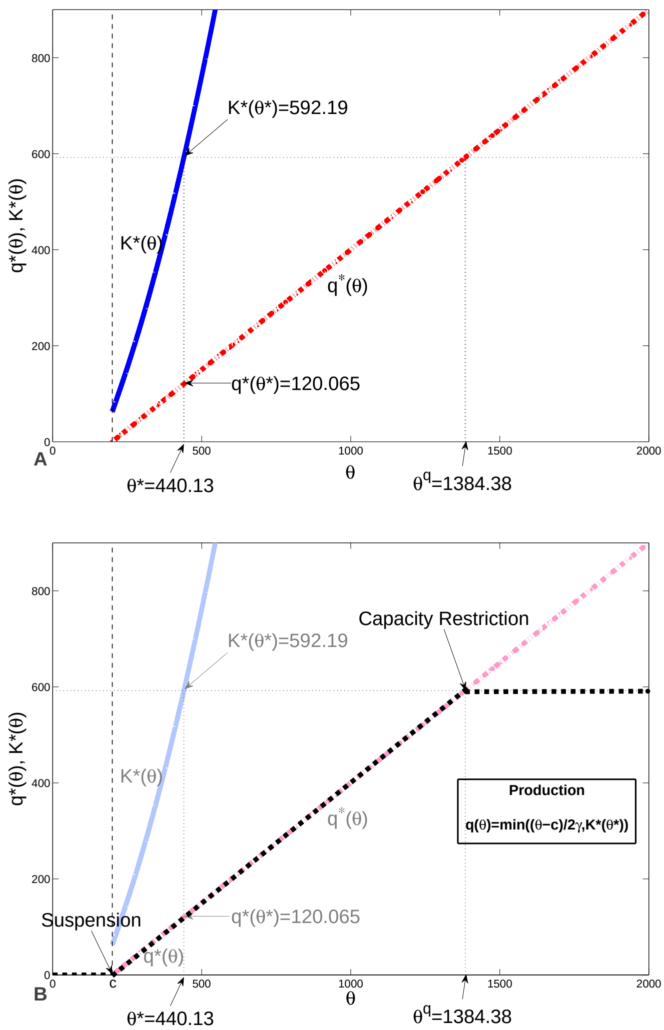


Table 2.1: Investment Strategy with the Occupation Rate $ocr = q^*(\theta^*)/K^*(\theta^*)$.
[Note: Parameter values are $\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\lambda = 0.7$ and $\delta = 1000$.]

σ	θ^*	$K^*(\theta^*)$	$q^*(\theta^*)$	ocr
0.1	440.13	592.19	120.07	20.3%
0.15	648.17	2819.94	224.08	7.9%
0.2	1769.72	93195.79	784.85	0.8%

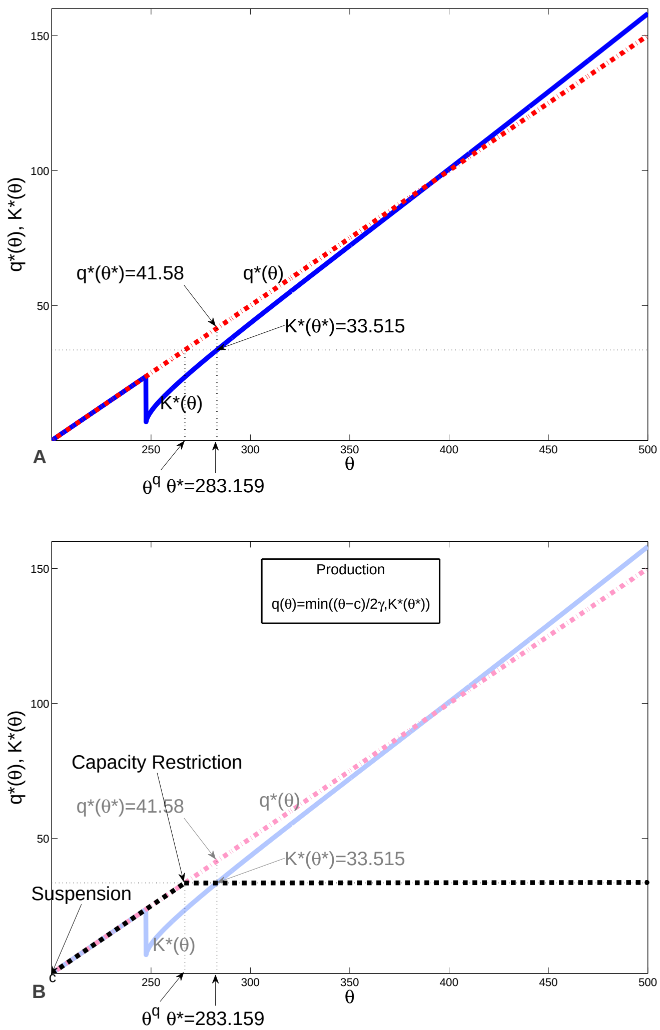
otherwise it delays investment until θ becomes equal to θ^*. At this demand realization the optimal capacity choice $K^*(\theta^*)$ is 592.19 while production is significantly less, $q^*(\theta^*) = 120.065$. This results in an occupation rate of 20.3%, implying that at the moment of investment the firm leaves almost 80% of the installed capacity idle. Demand would have to rise significantly up to the threshold $\theta^q = 1384.38$ in order to reach the point where the firm will produce up to capacity. After the investment is undertaken, the firm is producing the optimal quantity $q^*(\theta)$. At the moment that demand reaches θ^q the firm faces the capacity restriction $K^*(\theta^*)$. For $\theta > \theta^q$ the firm produces at full capacity.

Choosing a relatively low level of uncertainty ($\sigma = 0.03$) and a small drift rate ($\alpha = 0.002$), the firm will optimally produce at full capacity already at the moment of investment. Figure 2.3 illustrates this result. If the initial value for θ is sufficiently low, the firm invests when θ reaches the threshold value $\theta^* = 283.159$ for the first time. Then the corresponding capacity choice ($K^*(\theta^*) = 33.515$) is lower than the optimal quantity and therefore, the firm produces at full capacity right after the investment, as long as θ exceeds θ^q. Otherwise production equals $q^*(\theta)$.

In Dangl (1999), from which we adopted the parameter values of Figure 2.2, the main result is that the project size is exploding with increasing uncertainty, while on the other hand the project is unlikely to be installed because the incentive to wait with investment is increasing at the same time. He claims that the probability that the firm will invest in the near future vanishes with increasing uncertainty. This is illustrated in Table 2.1, in which both the capacity and the investment threshold increase more than proportionally with the volatility of demand.

In addition, Table 1 focuses on the effect of uncertainty on the occupation rate. The table shows that, while for $\sigma = 0.1$ the firm uses 20 percent of the

Figure 2.3: Investment Strategy of an example with the optimal investment moment laying **in region III**.
3.A: The optimal capacity $K^*(\theta)$ as well as production quantity $q^*(\theta)$ as a function of θ. Indicating the optimal investment threshold θ^* and capacity invested in $K^*(\theta^*)$ as well as the point θ^q where demand is high enough to use full capacity for production.
3.B: The bold dashed line shows the optimal production output as a function of θ after investment.
[Note: Parameter values are $\sigma = 0.03$, $\alpha = 0.002$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 0.7$.]



installed capacity at the moment of the investment, the occupation rate decreases enormously with uncertainty. For an uncertainty level of $\sigma = 0.15$, the rate drops to less than 10 percent (7.95%) and for $\sigma = 0.2$ the firms uses just 0.8 percent of the capacity installed at the moment of the investment. The reason why the firm decides to invest in such a high capacity level relative to the production level can be explained by, first, noting that investment costs are concave so that installing an additional unit of capacity is cheap when capacity is already large. Second, the low occupation rate is caused by the fact that the firm can invest only once. This implies that at one point in time it has to decide about how much capacity it will have at its disposal forever. Therefore, it is understandable that the firm will install a large capacity especially in case of highly uncertain demand and a positive demand trend.

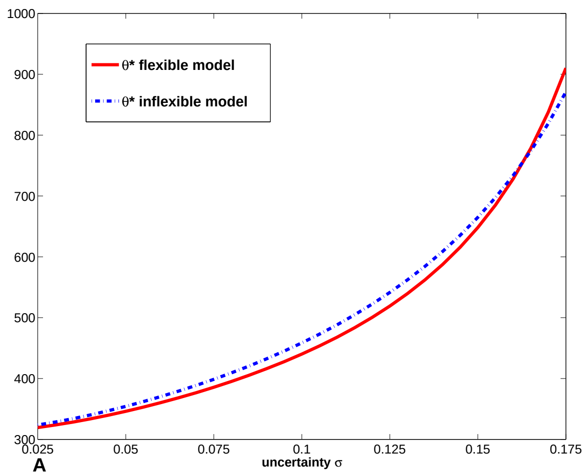
One might find it more realistic to include an upper boundary for the maximum capacity that can be installed, since. However, from results of Dangl (1999) we know that such an additional boundary would not change the results qualitatively. Dangl finds, that defining a limit for the capacity, damps the size of investment for highly volatile demand regions as well as the delay of investment. When volatility in demand exceeds a certain value, the firm will install the highest possible amount of capacity available. The effect of uncertainty on investment timing, i.e. increasing uncertainty increases the investment threshold, remains unchanged.

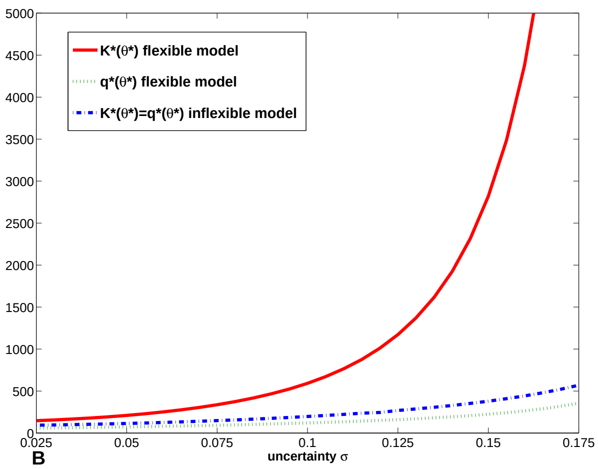
2.3.2 Impact of Flexibility

Figure 2.4 shows the impact of flexibility and uncertainty on the investment strategy. The left panel compares the investment thresholds for the flexible and inflexible model, and the right panel shows the difference in the optimal capacity choice of the two firms as a function of uncertainty. The capacity choice of the flexible firm is exploding for high uncertainty while the capacity choice of the inflexible firm is lower and also increasing with uncertainty albeit at a lower rate. The inflexible firm cannot adapt the production level, so instead it has to produce always up to capacity after the moment of investment whenever it is an active producer. Therefore, the inflexible firm chooses for less capacity. Furthermore, we see that the optimal production output at the moment of the investment of the flexible firm falls below the capacity level of the inflexible firm.

The left panel of Figure 2.4 shows that both for the flexible and for the inflexible firm the investment threshold increases in uncertainty. This implies

Figure 2.4: Impact of Flexibility and Uncertainty on the Investment Strategy. 4.A: Optimal Investment Thresholds (θ^*) for the Flexible and Inflexible Firm as a Function of σ. 4.B: Optimal Capacity Level ($K^*(\theta^*)$) and Production Level ($q^*(\theta^*)$) at the Moment of Investment as a Function of σ for the Flexible and Inflexible Firm.
[Note: Parameter values are $\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 0.7$.]





that when uncertainty goes up, a higher demand level is needed before it is optimal to invest. This is partly caused by the fact that in both cases capacity increases with uncertainty (right panel), and it is partly due to the standard real options result that in a more uncertain economic environment the firm has a higher incentive to wait for more information before undertaking the investment (see, e.g., Dixit and Pindyck (1994)).

Comparing the investment thresholds for the flexible and the inflexible firm, we note that there are two contrary effects. On the one hand the value of the project is higher for the flexible firm, which raises the incentive for this firm to invest earlier. The higher project value is caused by the ability to adapt quantity so that it can avoid overproduction in case demand falls. But on the other hand the inflexible firm invests in a lower capacity level, as shown in the right panel of Figure 2.4. This implies that the demand can be at a lower level for the investment of the inflexible firm to take place. This gives the inflexible firm an incentive to invest earlier than the flexible firm. Since the right panel of Figure 2.4 shows that the capacity choice of the flexible firm is exploding, the second effect dominates for high uncertainty and the inflexible firm invests earlier in this case. For lower levels of uncertainty the higher project value due to flexibility makes that the flexible firm invests earlier.

2.4 Robustness

2.4.1 Convex Investment Cost

So far, our analysis is based on the investment cost function, $I(K) = \delta K^{\lambda}$, where the constant λ has a value less than one. There are important economic reasons pleading for such a concavely shaped investment cost function, such as indivisibilities, use of information, fixed costs of ordering, and quantity discounts. However, a convexly shaped investment cost function, thus with $\lambda > 1$ so that marginal costs are increasing with investment expenditures, can also be motivated. Consider, for instance, a monopsonistic market of capital goods. In a monopsony there is only one firm which demands some factor of production. Facing an upward sloping supply curve and furthermore aiming to increase its rate of growth, the firm will be confronted with increasing prices resulting from increased demand of capital goods.

Table 2.2 shows the difference in investment triggers and occupation rate for a concave cost function ($\lambda = 0.7 < 1$) and a convex one ($\lambda = 1.1 > 1$). Clearly, the firm decides to invest later in less capacity in the convex case. This

Table 2.2: Comparing Investment Strategies with Convex ($\lambda < 1$) and Concave ($\lambda > 1$) Investment Cost Structure. The upper three lines present the results of a convex cost structure with $\lambda = 0.7$. The lower lines show the results assuming a concave cost structure with $\lambda = 1.1$.
[Note: Parameter values are $\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$ and $\delta = 1000$.]

σ	θ^*	$K^*(\theta^*)$	$q^*(\theta^*)$	ocr
0.1	440.13	592.19	120.07	20.3%
0.15	648.17	2819.94	224.08	7.9%
0.2	1769.72	93195.79	784.85	0.8%
0.1	767.90	311.89	283.95	91%
0.15	1390.96	1033.76	595.48	58%
0.2	6240.78	17738.30	3020.39	17%

is an expected result since the investment cost the firm is facing for the convex case is significantly higher for larger investments ($\lambda = 1.1$), and therefore installing a large amount of capacity is more expensive.

The effect of the change in the cost structure on the occupation rate is that, first, the influence of increasing uncertainty on the occupation rate remains to be of decreasing behavior, i.e. that higher uncertainty results in a lower relation between installed and used capacity. Second, for a given uncertainty level the occupation rate is significantly higher for convex investment costs. This is because installing an additional unit of capacity when capacity is already large, is significantly more expensive in the convex case.

Table 2.3 shows what happens when keeping the assumption of convex investment cost but increasing the parameter λ. For a low level of uncertainty, i.e. $\sigma = 0.05$, we get the expected result that the firm invests later in less capacity when λ is higher, because the investment costs are higher. However, at the uncertainty level of $\sigma = 0.1$ the firm facing investment costs with parameter $\lambda = 1.3$ is still investing later than the one facing investment costs with parameter $\lambda = 1.1$, but it invests in more capacity even though its investment costs are higher. This is because the larger threshold level θ^* implies that demand is higher at the moment of investment. This effect, namely larger capacity investment while investment costs are higher, is reinforced when uncertainty

Table 2.3: Investment Strategy of the Flexible Firm comparing two Levels of Convex Investment Cost Structure ($\lambda = 1.1$ is chosen for the upper four lines and $\lambda = 1.3$ for the lower lines.) [Note: Parameter values are $\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$ and $\delta = 1000$.]

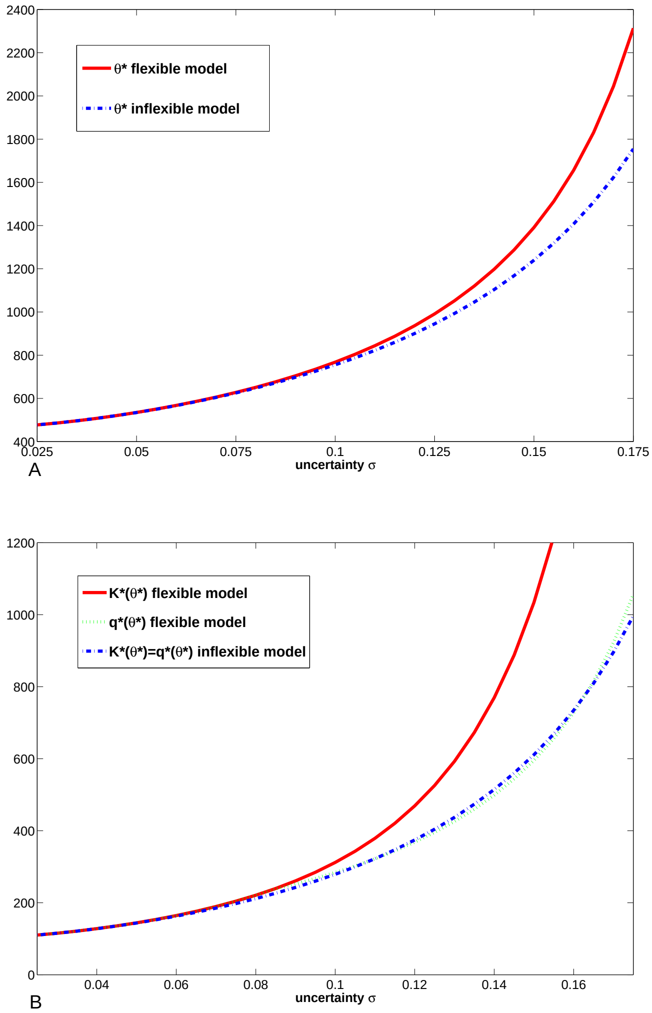
σ	θ^*	$K^*(\theta^*)$	$q^*(\theta^*)$	ocr
0.05	534.73	144.17	167.37	100%
0.1	767.90	311.89	283.95	91%
0.15	1390.96	1033.76	595.48	58%
0.2	6240.78	17738.30	3020.39	17%
0.05	736.63	100.914	268.31	100%
0.1	1260.23	321.833	530.11	100%
0.15	2678.08	1129.39	1239.04	100%
0.2	16674.00	18614.6	8237.02	44.25%

goes up from $\sigma = 0.1$ onwards.

Figure 2.5 illustrates the influence of flexibility in production volumes on the optimal investment decision while the investment cost function is convex. The right panel shows that the flexible firm still invests in a larger capacity level, but that the gap between the optimal capacity choice of the flexible firm and the inflexible firm is not so large as for the concave investment cost case. This is because the convex cost structure makes large investments a lot more expensive. The implication is that the upperbound on the production volume for the flexible firm does not differ a lot from the inflexible firm production volume. Consequently, under convex investment costs the project value is more equal for the flexible and inflexible firm. This means that the two contrary effects on investment timing, i.e. the flexible firm invests later because of larger capacity and earlier because of a higher project value, are not so big. Overall, in this particular example the “capacity effect” dominates, i.e. the flexible firm invests later, as shown in the left panel of Figure 2.5.

In case of low uncertainty, the difference between optimal investment in flexible and inflexible capacity is very small. This is not the case for the results of concave investment costs (see Figure 2.4). In the scale of Figure 2.5 the thresholds for both cases might appear to be very close to each other. However, numerically the investment threshold for flexible investment is slightly

Figure 2.5: Impact of Flexibility and Uncertainty on the Investment Strategy for Convex Investment Cost. 5.A: Optimal Investment Thresholds (θ^*) for the Flexible and Inflexible Firm as a Function of σ. 5.B: Optimal Capacity Level ($K^*(\theta^*)$) and Production Level ($q^*(\theta^*)$) at the Moment of Investment as a Function of σ for the Flexible and Inflexible Firm.



[Note: Parameter values are $\alpha = 0.02$, $r = 0.1$, $\gamma = 1$, $c = 200$, $\delta = 1000$ and $\lambda = 1.1$.]

larger than the threshold for inflexible investment. Since uncertainty is low and investment costs are higher than for the concave case, the firm invests in region III, i.e. it installs capacity and uses it up to the full extent at the moment of investment. Investment costs are too high and expectations of demand too low for the firm to have an incentive to install additional capacity that could potentially be used in case demand increases in the future.

2.4.2 Iso-elastic Inverse Demand Function

This section studies to what extent the assumption of linear demand was decisive for the results we obtained. To do so, here we adopt an iso-elastic demand function, i.e.

$$p(q_t, t) = \theta_t q_t^{-\gamma} \quad with \quad 0 < \gamma < 1, \tag{2.23}$$

where $\{\theta_t\}$ behaves according to expression (3.3).

Flexible Model

As in Section 2.2.1 we determine the optimal q^* by maximizing the profit flow, which implies that

$$q^* = \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}. \tag{2.24}$$

A remarkable feature is that this results in a constant price $p(q^*) = \frac{c}{1-\gamma}$, which is thus independent from the stochastic process $\{\theta_t\}$. Note that this price always exceeds unit production cost c. This makes that, while for the linear case the optimal quantity can fall to zero making it necessary to take into account the possibility of temporary suspension, for the iso-elastic demand function the optimal quantity produced is always positive. This implies that after the investment is undertaken, the firm will produce forever while the production rate will be adapted to the realization of the demand process at every instant.

Analogous to the linear demand case, we can derive the project value

$$V(\theta, K) = \begin{cases} M_1(K)\,\theta^{\beta_1} + L\frac{1}{r-\epsilon\alpha-\frac{1}{2}\epsilon(\epsilon-1)\sigma^2}\theta^{\epsilon} & \text{for } 0 \le \left[\frac{\theta(1-\gamma)}{c}\right]^{\epsilon} < K, \\ N_2(K)\,\theta^{\beta_2} + \frac{K^{1-\gamma}}{r-\alpha}\theta - \frac{cK}{r} & \text{for } \left[\frac{\theta(1-\gamma)}{c}\right]^{\epsilon} \ge K, \end{cases} \tag{2.25}$$

where $\epsilon = \frac{1}{\gamma}$ and the (lengthy) expressions for M_1, L and N_2 are presented in Appendix A.

The first line is the project value when the firm is currently not producing at full capacity. where the second term is the expected discounted cash flow stream when the firm produces a quantity $q^* = \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}$ forever. The (negative) first term corrects this cash flow stream for the presence of the capacity constraint.

The second line is the project value at the moment that the firm is producing at full capacity, where the last two terms stand for the expected discounted cash flow stream when the firm's production level is equal to the capacity size K. The first term is the value of the option to produce below capacity level, which takes place when demand falls with such an amount that it is not possible anymore for the firm to sell a quantity K against a price of $\frac{c}{1-\gamma}$.

The following proposition presents the optimal investment decision.

Proposition 3 *Concerning the firm's investment policy there are two possibilities:*

1. *Given that the firm does not produce up to full capacity right after the investment moment, the optimal capacity level $K^*(\theta)$ is implicitly determined by*

$$M_1'(K^*)\theta^{\beta_1} - \delta\lambda K^{*\lambda-1} = 0. \tag{2.26}$$

In case the obtained K^* is such that from the resulting production quantity (2.24) it follows that it is not an interior solution, i.e. in case $K^* \leq \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}$, it is replaced by the boundary solution $\left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}$. Thus,

$$K^*(\theta) = \max\left(K^*, \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}\right). \tag{2.27}$$

The investment threshold θ^* satisfies

$$\theta^{*\epsilon} = \delta K^{\lambda}\frac{r - \epsilon\alpha - \frac{1}{2}\sigma^2\epsilon(\epsilon-1)}{L}\frac{\beta_1}{\beta_1-\epsilon}. \tag{2.28}$$

2. *Given that the firm produces up to full capacity right after the investment moment, the optimal capacity level $K^*(\theta)$ is implicitly determined by*

$$N_2'(K^*)\theta^{\beta_2} + \frac{1-\gamma}{r-\alpha}K^{*-\gamma}\theta - \frac{c}{r} - \delta\lambda K^{*\lambda-1} = 0. \tag{2.29}$$

In case the obtained $K^*(\theta)$ does not constitute an interior solution, i.e. in case $K^* > \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}$, the optimal capacity is replaced by the boundary solution

$\left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}$. *Thus,*

$$K^*(\theta) = \min\left(K^*, \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}}\right). \tag{2.30}$$

The investment threshold θ^* is implicitly determined by

$$\begin{aligned} N_2(K^*(\theta^*))\theta^{*\beta_2}(\beta_1-\beta_2) &+ \frac{(K^*(\theta^*))^{(1-\gamma)}}{r-\alpha}\theta^*(\beta_1-1) - \beta_1\frac{c}{r}K^*(\theta^*) \\ &- \beta_1\delta\left(K^*(\theta^*)\right)^{\lambda} = 0. \end{aligned} \tag{2.31}$$

Out of these two possibilities the firm chooses the one that gives the highest expected value of the project $V(\theta^*, K^*(\theta^*))$.

Inflexible Model

The inflexible firm produces up to full capacity K forever. Consequently (cf. the last two terms of the second line of (2.25)), the value of the project is

$$V(\theta, K) = \frac{K^{1-\gamma}}{r-\alpha}\theta - \frac{cK}{r}. \tag{2.32}$$

The following proposition presents the optimal investment decision of the inflexible firm.

Proposition 4. *The optimal capacity level $K^*(\theta)$ satisfies*

$$(1-\gamma)\frac{K^{*-\gamma}}{r-\alpha}\theta - \frac{c}{r} - \delta\lambda K^{*\lambda-1} = 0. \tag{2.33}$$

The investment threshold θ^* is implicitly determined by

$$\left(\frac{\beta_1-1}{\beta_1}\right)\frac{(K^*(\theta^*))^{(1-\gamma)}}{r-\alpha}\theta^* - \frac{cK^*(\theta^*)}{r} - \delta\left(K^*(\theta^*)\right)^{\lambda} = 0.$$

Results

The optimal investment decision of the flexible firm is presented in Table 2.4. The table shows that the occupation rate is decreasing with increasing uncertainty. For lower uncertainty levels ($\sigma = 0 - 0.3$) the firm produces up to capacity at the moment of investment. For higher uncertainty levels it decides to install a higher capacity level than it uses at the moment of investment, where the occupation rate decreases with uncertainty. These results are qualitatively similar to what we obtained for our basic framework.

Table 2.4: Investment Strategy of the Flexible Firm.
[Note: Parameter values are $\alpha = 0.02$, $r = 0.1$, $\gamma = 0.7$, $c = 5$, $\lambda = 0.7$ and $\delta = 100$.]

σ	θ^*	$K^*(\theta^*)$	$q^*(\theta^*)$	ocr
0.1	18.85	1.192	1.192	100%
0.2	36.83	3.105	3.105	100%
0.3	69.65	7.713	7.713	100%
0.39	127.94	19.223	18.387	95.65%
0.4	289.28	98.093	58.974	60.12%
0.41	828.40	810.996	265.106	32.69%
0.42	3361.74	13660.900	1960.870	14.35%

Table 2.5 depicts the investment decision of the inflexible firm. The very low capacity choice is striking. Since the inflexible firm has to produce up to capacity without the possibility to reduce production, it has to keep capacity low compared to the flexible firm, which especially holds when uncertainty is high.

As to the timing, previously we detected that on the one hand the inflexible firm has an incentive to invest later, because inflexibility in the production decision leads to a lower project value. On the other hand the inflexible firm has an incentive to invest earlier because the lower capacity level does not require a high current demand level for the investment to take place. Since capacity is so much lower here for the inflexible firm, it makes sense that the latter effect dominates so that the inflexible firm will invest earlier.

2.5 Conclusions

This paper considers the timing and capacity choice of a firm facing stochastic demand. We distinguish between a flexible scenario, where the firm can continuously adjust production over time, and an inflexible scenario, where the firm always has to produce up to capacity. In both cases the firm has the option to suspend production, which could occur when demand is too low. Hence, the firm makes three decisions: choice of investment time, choice of capacity, and choice of production quantity. With respect to the last decision it holds that, where the flexible firm can choose any production quantity from

Table 2.5: Investment Strategy of the Inflexible Firm.
[Note: Parameter values are $\alpha = 0.02, r = 0.1, \gamma = 0.7, c = 5$ and $\delta = 100$.]

λ	σ	θ^*	$K^*(\theta^*)$
0.7	0.1	18.31	0.3372
0.7	0.2	20.77	0.4328
0.7	0.3	23.85	0.5676
0.7	0.4	27.76	0.7617
0.7	0.45	30.17	0.8937

zero to the capacity level, the inflexible firm can only choose between producing at full capacity or refraining from producing. Concerning the timing and capacity decision we develop implicit solutions, which we investigate numerically. We show that the firm invests later in more capacity if demand uncertainty rises. Furthermore, we find that being able to vary production over time implies that the flexible firm wants to install a larger capacity than the inflexible firm. This capacity difference goes up with uncertainty. This has a side-effect in that for the flexible firm right after the investment the occupation rate, i.e. the part of the capacity that is used for production, falls with uncertainty. Comparing the optimal investment timing of the flexible and the inflexible firm, we find that two contrary effects prevail. First, the flexible firm has an incentive to invest earlier because being able to vary the production quantity raises the value of the investment project. On the other hand, the fact that the flexible firm wants to install a larger capacity demands a larger current demand level for an investment to be optimal, which in turn implies that this gives the flexible firm an incentive to invest later. The latter effect dominates in a highly uncertain economic environment. Hence, when demand is highly uncertain, increased flexibility in production further delays investment.

Our results lead to several hypothesis that could be tested empirically. Namely, high uncertainty in combination with flexibility should lead to later and larger investments, and low initial occupation rates. Low uncertainty in combination with flexibility should lead to early investments.

Our results come with several limitations. In our work firms can invest only once, as is mainly done in the real options literature. Further, with regard to flexibility we only focus on volume flexibility, and not on, for example, product flexibility. Also, we disregard competition. In particular, extending our monopoly to a duopoly model in a continuous time setting promises interesting insights. Furthermore, we assume that holding capacity is costless. Introducing fixed cost for the maintenance of installed capacity in the model constitutes an interesting step for future research.

2.A Appendix

Additional Model Details and Results

Flexible Model

Following the methodology of McDonald (2005), the value of the project has to satisfy

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \alpha\theta\frac{\partial V}{\partial\theta} - rV(\theta) + \pi(\theta) = 0. \tag{2.34}$$

We solve this differential equation for V, where the non-homogeneous part is defined differently for the three regions. This results in the following value of the project:

$$V(\theta, K) = \begin{cases} L_1\theta^{\beta_1} + L_2\theta^{\beta_2} & \text{for } 0 \leq \theta < c, \\ M_1\theta^{\beta_1} + M_2\theta^{\beta_2} + \frac{1}{4\gamma}\left[\frac{\theta^2}{r-2\alpha-\sigma^2} - \frac{2c\theta}{r-\alpha} + \frac{c^2}{r}\right] & \text{for } c \leq \theta < 2\gamma K + c, \\ N_1\theta^{\beta_1} + N_2\theta^{\beta_2} + \frac{K}{r-\alpha}\theta - \frac{K(K\gamma+c)}{r} & \text{for } \theta \geq 2\gamma K + c. \end{cases} \tag{2.35}$$

Next, we derive L_1, L_2, M_1, M_2, N_1 and N_2 in (2.35). Following Dixit and Pindyck (1994) (see Section 6.2), we solve the differential equation separately for the different regions and stitch together the two solutions at the points $(\theta = \theta_1)$ and $(\theta = \theta_2)$. From the boundary conditions

$$V(0, K) = 0, \tag{2.36}$$

$$\lim_{\theta\to\infty} V_{\theta\geq\theta_2} = \frac{K}{r-\alpha}\theta - \frac{K(K\gamma + c)}{r}, \tag{2.37}$$

we derive that the parameters L_2 and N_1 in (2.35) are equal to zero. The other expressions in equation (2.35) are given by

$$\begin{aligned} M_1(K) = {} & \theta_2^{-\beta_1}\frac{1}{\beta_1 - \beta_2}\left[\frac{1}{4\gamma}\left[\frac{\theta_2^2}{r-2\alpha-\sigma^2}(\beta_2 - 2) - \frac{2c\theta_2}{r-\alpha}(\beta_2 - 1)\right.\right. \\ & \left.\left. + \beta_2\frac{c^2}{r}\right] + \beta_2\frac{K(\gamma K + c)}{r} - \frac{K\theta_2}{r-\alpha}(\beta_2 - 1)\right], \end{aligned} \tag{2.38}$$

$$\begin{aligned} M_2 = {} & \theta_1^{-\beta_2}\frac{1}{\beta_1 - \beta_2}\frac{1}{4\gamma}\left[\frac{\theta_1^2}{r-2\alpha-\sigma^2}(2 - \beta_1)\right. \\ & \left. - \frac{2c\theta_1}{r-\alpha}(1 - \beta_1) - \beta_1\frac{c^2}{r}\right], \end{aligned} \tag{2.39}$$

$$
\begin{aligned}
L_1(K) &= \theta_1^{-\beta_1}\Big[M_1(K)\theta_1^{\beta_1} + M_2\theta_1^{\beta_2} \\
&\quad + \frac{1}{4\gamma}\left[\frac{\theta_1^2}{r-2\alpha-\sigma^2} - \frac{2c\theta_1}{r-\alpha} + \frac{c^2}{r}\right]\Big], & (2.40)\\
&= M_1(K) + \theta_1^{-\beta_1}\frac{c^2}{4\gamma}\frac{1}{\beta_1-\beta_2}\left[\frac{(2-\beta_2)}{r-2\alpha-\sigma^2} - \frac{2(1-\beta_2)}{r-\alpha} - \frac{\beta_2}{r}\right] & (2.41)\\
N_2(K) &= \theta_2^{-\beta_2}\Bigg[M_1(K)\theta_2^{\beta_1} + M_2\theta_2^{\beta_2} + \frac{1}{4\gamma}\left[\frac{\theta_2^2}{r-2\alpha-\sigma^2} - \frac{2c\theta_2}{r-\alpha} + \frac{c^2}{r}\right] \\
&\quad - \frac{K\theta_2}{r-\alpha} + \frac{K(\gamma K + c)}{r}\Bigg], & (2.42)\\
&= M_2 + \theta_2^{-\beta_2}\frac{1}{\beta_1-\beta_2}\Bigg[\frac{1}{4\gamma}\Bigg(\frac{\theta_2^2}{r-2\alpha-\sigma^2}(\beta_1-2) - \frac{2c\theta_2}{r-\alpha}(\beta_1-1) \\
&\quad + \frac{c^2}{r}\beta_1\Bigg) + \frac{K(\gamma K+c)}{r}\beta_1 - \frac{K\theta_2}{r-\alpha}(\beta_1-1)\Bigg], & (2.43)
\end{aligned}
$$

with $\theta_1 = c$ and $\theta_2 = 2\gamma K + c$. We define

$$
\begin{aligned}
G(\beta) &= \Bigg[\frac{1}{4\gamma}\left[\frac{\theta_2^2}{r-2\alpha-\sigma^2}(\beta-2) - \frac{2c\theta_2}{r-\alpha}(\beta-1) + \beta\frac{c^2}{r}\right] \\
&\quad + \beta\frac{K(\gamma K+c)}{r} - \frac{K\theta_2}{r-\alpha}(\beta-1)\Bigg], & (2.44)\\
&= \frac{1}{4\gamma}\theta_2^2\left[\frac{\beta-2}{r-2\alpha-\sigma^2} - \frac{2(\beta-1)}{r-\alpha} + \frac{\beta}{r}\right], & (2.45)
\end{aligned}
$$

and

$$
F(\beta) = \frac{2-\beta}{2(r-2\alpha-\sigma^2)} - \frac{(1-\beta)}{r-\alpha} - \frac{\beta}{2r}. \quad (2.46)
$$

It holds that $G(\beta) = \frac{1}{4\gamma}\theta_2^2\,[-2F(\beta)]$. We can simplify the expression of the four parameters

$$
\begin{aligned}
M_1 &= \theta_2^{2-\beta_1}\frac{1}{\beta_1-\beta_2}\frac{1}{4\gamma}\left[(-2)F(\beta_2)\right], & (2.47)\\
M_2 &= c^{2-\beta_2}\frac{1}{\beta_1-\beta_2}\frac{1}{4\gamma}\left[2F(\beta_1)\right], & (2.48)\\
L_1 &= \frac{1}{\beta_1-\beta_2}\frac{1}{2\gamma}F(\beta_2)\left[\theta_1^{(2-\beta_1)} - \theta_2^{(2-\beta_1)}\right], & (2.49)\\
N_2 &= \frac{1}{\beta_1-\beta_2}\frac{1}{2\gamma}F(\beta_1)\left[\theta_1^{(2-\beta_2)} - \theta_2^{(2-\beta_2)}\right]. & (2.50)
\end{aligned}
$$

Corollary 1 *The following holds for the parameters L_1, M_1, M_2 and L_2 within the considered parameter ranges:*

$$
M_1 < 0, \quad (2.51)
$$

$$M_2 < 0, \tag{2.52}$$

$$L_1 > 0, \tag{2.53}$$

$$N_2 > 0. \tag{2.54}$$

Proof of Corollary 1 Considering the parameters M_1 and M_2 we know that the left part in the expression (2.47) as well as in expression (2.48) is positive since $\beta_1 > 0$ and $\beta_2 < 0$ hold. Therefore, it remains to show that the following holds: $F(\beta_1) < 0$ and $F(\beta_2) > 0$. Rewriting the function $F(.)$ one can easily see that it is linear with a negative slope, crossing the horizontal line in $\beta_0 = \frac{2r(\alpha+\sigma^2)}{2\alpha^2+\sigma^2(r+\alpha)} > 0$. Since $\beta_2 < 0$ we can conclude that $F(\beta_2)$ is strictly positive. And if $\beta_1 > \beta_0$ holds then $F(\beta_1) < 0$. In order to verify that $\beta_1 > \beta_0$ holds, we consider the quadratic equation (3.11) and derive the value of the function $Q(.)$ at β_0. This is given by $Q(\beta_0) = -\frac{(r-\mu)r\sigma^4(2\mu-r+\sigma^2)}{(2\mu^2+(r+\mu)\sigma^2)^2} < 0$. Since the quadratic function $Q(\beta)$ is an upward pointing parabola that goes to ∞ as β goes to $\pm\infty$ we can infer from $Q(\beta_0) < 0$ that the point β_0 has to be less than β_1. Considering the parameter L_1 it holds that

$$c^{(2-\beta_1)} - (2\gamma K + c)^{(2-\beta_1)} > 0, \tag{2.55}$$

since $\beta_1 > 2$. This and the fact that $F(\beta_2) > 0$ is sufficient to proof that $L_1 > 0$. Considering that $F(\beta_1) < 0$ and that the very left part in the expression of N_2 is negative one can conclude that N_2 is positive.

Inflexible Model

The expressions of $P_2(K)$ and $Q_1(K)$ in the project value function (2.20) are given by

$$P_2(K) = \theta_3^{-\beta_2} \frac{1}{\beta_1 - \beta_2} \left[\frac{K\theta_3}{r-\alpha}(1-\beta_1) + \frac{K(\gamma K + c)}{r}\beta_1 \right], \tag{2.56}$$

$$= \theta_3^{(1-\beta_2)} K \frac{1}{\beta_1 - \beta_2} \left[\frac{(1-\beta_1)}{r-\alpha} + \frac{\beta_1}{r} \right], \tag{2.57}$$

$$Q_1(K) = \theta_3^{-\beta_1} \frac{1}{\beta_1 - \beta_2} \left[\frac{K\theta_3}{r-\alpha}(1-\beta_2) + \frac{K(\gamma K + c)}{r}\beta_2 \right], \tag{2.58}$$

$$= \theta_3^{(1-\beta_1)} K \frac{1}{\beta_1 - \beta_2} \left[\frac{(1-\beta_2)}{r-\alpha} + \frac{\beta_2}{r} \right], \tag{2.59}$$

with $\theta_3 = \gamma K + c$.

Corollary 2 *Both parameters P_2 and Q_1 are positive within the considered parameter ranges.*

Proof of Corollary 2 In order to show that both parameters, P_2 and Q_1, are positive, we need

$$r > \alpha\beta_1 \text{ and } r > \alpha\beta_2. \tag{2.60}$$

To verify these, we evaluate the quadratic polynomial $Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta - r$, at $\beta = \frac{r}{\alpha}$. We get

$$\frac{1}{2}\sigma^2\frac{r}{\alpha}\left(\frac{r}{\alpha} - 1\right) > 0. \tag{2.61}$$

$Q(\beta)$ is an upward pointing parabola that goes to ∞ as β goes to $\pm\infty$. (For more details about the quadratic equation see Dixit and Pindyck (1994).) Therefore, $\frac{r}{\alpha}$ must lie to the right of the larger root β_1. It obviously does not lie to the left of the smaller root β_2 because it is strictly positive according to the assumption $r > \alpha$.

Expected project value

The expected present value of the project is

$$E\left[e^{-rT}\right][V(\theta, K^*(\theta)) - I(K)] = \left(\frac{\theta_0}{\theta}\right)^{\beta_1}[V(\theta, K^*(\theta)) - I(K)], \tag{2.62}$$

where T is the (random) first time when the stochastic process θ_t reaches θ starting at θ_0. The calculation of $E\left[e^{-rT}\right]$ can be found in Dixit and Pindyck (1994).

Iso-elastic Inverse Demand Function - Flexible Model

For the model with an iso-elastic inverse demand function we derive the profit flow

$$\pi(\theta) = \begin{cases} (\theta K^{-\gamma} - c)K & \text{for } \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}} \geq K, \\ \left[\theta\left(\frac{\theta(1-\gamma)}{c}\right)^{-1} - c\right]\left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}} & \text{for } 0 \leq \left[\frac{\theta(1-\gamma)}{c}\right]^{\frac{1}{\gamma}} < K. \end{cases} \tag{2.63}$$

Familiar steps lead to the differential equation

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \alpha\theta\frac{\partial V}{\partial\theta} - rV(\theta) + \pi(\theta) = 0. \tag{2.64}$$

Solving equation (2.64) gives the value of the project (2.25). The parameters in equation 2.25 are given by:

$$
\begin{aligned}
L &= \gamma \left[\frac{1-\gamma}{c}\right]^{\frac{1-\gamma}{\gamma}}, \qquad (2.65)\\
M_1(K) &= \hat{\theta}^{-\beta_1}\frac{1}{\beta_1-\beta_2}\frac{cK}{1-\gamma}\left[\frac{(1-\beta_2)}{r-\alpha}\right.\\
&\quad \left. + \frac{\beta_2(1-\gamma)}{r} + \frac{\gamma(\beta_2-\epsilon)}{r-\epsilon\alpha-\frac{1}{2}\epsilon(\epsilon-1)\sigma^2}\right], \qquad (2.66)\\
N_2(K) &= \hat{\theta}^{-\beta_2}\frac{1}{\beta_1-\beta_2}\frac{cK}{1-\gamma}\left[\frac{(1-\beta_1)}{r-\alpha} + \frac{\beta_1(1-\gamma)}{r}\right.\\
&\quad \left. + \frac{\gamma(\beta_1-\epsilon)}{r-\epsilon\alpha-\frac{1}{2}\epsilon(\epsilon-1)\sigma^2}\right], \qquad (2.67)
\end{aligned}
$$

where $\hat{\theta} = \frac{cK^\gamma}{1-\gamma}$.

Proof of Proposition 1

The procedure of this proof is split into two steps. First, the optimal investment capacity is determined for a given level of θ. This optimal capacity level K^* is determined by maximizing the objective value of the project. We have to calculate the optimal capacity level for region 2 and 3, respectively. We denote

$$V(\theta, K) = V_i(\theta, K) \text{ for region i,} \qquad (2.68)$$

where region 2 denotes $\theta_1 \leq \theta < \theta_2$ and region 3 is defined by $\theta \geq \theta_2$.

Setting the marginal value of the project equal to the marginal investment costs yields the equations (2.11) and (2.14), which the optimal capacity choice has to satisfy. Setting

$$\frac{\partial(V_i(\theta,K) - I(K))}{\partial K} = 0, \qquad (2.69)$$

leads to

$$
\begin{aligned}
\frac{\partial(V_2(\theta,K) - I(K))}{\partial K} &= M_1'(K)\theta^{\beta_1} - \delta\lambda K^{\lambda-1} = 0, \qquad (2.70)\\
\frac{\partial(V_3(\theta,K) - I(K))}{\partial K} &= N_2'(K)\theta^{\beta_2} + \frac{\theta}{r-\alpha} - \frac{2K\gamma + c}{r} - \delta\lambda K^{\lambda-1} = 0, \qquad (2.71)
\end{aligned}
$$

with

$$
\begin{aligned}
M_1'(K) &= \left(\frac{\beta_1 - 2}{\beta_1-\beta_2}\right)\theta_2^{(1-\beta_1)}F(\beta_2), \qquad (2.72)\\
M_2'(K) &= 0, \qquad (2.73)
\end{aligned}
$$

$$N_2'(K) \quad = \quad (-1)\left(\frac{2-\beta_2}{\beta_1-\beta_2}\right)\theta_2^{(1-\beta_2)}F(\beta_1). \tag{2.74}$$

In order to verify that the derived $K^*(\theta)$s are maxima for their respective region, the second order derivatives are calculated and proved to be less than zero at the derived optimal capacity.

For certain choices of the parameters, it may happen that equations (2.11) or (2.14) do not have an admissible solution, meaning that the function $V_i(\theta, K) - I(K))$ is monotonic in K in region 2 or 3. If this is the case, then we may have an increasing or a decreasing behavior in K. On the other side, if equations (2.11) or (2.14) have an admissible solution, then it means that the function $[V_i(\theta, K) - I(K))]$ is concave in that region, and therefore has a local maximum. Thus, one has to take into account the local behavior of $[V_i(\theta, K) - I(K))]$, in the case that it is monotonic. If we are in region 2, we can rule out the decreasing case, because it means that the optimal capacity level K is zero, and thus there is no production at all. Therefore, if $[V_2(\theta, K) - I(K))]$ is increasing in K, one should take the maximum allowed capacity level for this region, i.e. $\frac{\theta-c}{2\gamma}$. Similarly, if we are in region 3, one should rule out the increasing case because it would mean that optimally an infinite capacity level should be chosen. In the decreasing case, one should choose the minimum allowed capacity for this region, which is $\frac{\theta-c}{2\gamma}$.

We are able to find solutions of equations (2.11) and (2.14) numerically for different values of θ.

Knowing the optimal investment region and capacity level, the optimal timing strategy can be derived. With the following value of the option

$$F(\theta) = A_1\theta^{\beta_1}, \tag{2.75}$$

the value matching and smooth-pasting conditions are (cf. Dixit and Pindyck (1994))

$$A_1\theta^{*\beta_1} \quad = \quad V(\theta^*, K^*(\theta^*)) - I(K^*(\theta^*))|_{\theta=\theta^*}, \tag{2.76}$$

$$\beta_1 A_1 \theta^{*(\beta_1-1)} \quad = \quad \left.\frac{d(V(\theta, K^*(\theta)) - I(K^*(\theta)))}{d\theta}\right|_{\theta=\theta^*}. \tag{2.77}$$

Substituting (2.77) into equation (2.76) gives

$$V(\theta^*, K^*(\theta^*)) - I(K^*(\theta^*)) = \frac{1}{\beta_1}\theta^*\left(\frac{d(V(\theta, K^*(\theta)) - I(K^*(\theta)))}{d\theta}_{\theta=\theta^*}\right), \tag{2.78}$$

with

$$\frac{d(V(\theta, K^*(\theta)) - I(K^*(\theta)))}{d\theta} \quad = \quad \left.\frac{\partial(V(\theta, K(\theta)) - I(K(\theta)))}{\partial K}\right|_{K=K^*} \frac{\partial K^*(\theta)}{\partial\theta}$$

$$+\frac{\partial(V(\theta,K^*(\theta))-I(K^*(\theta)))}{\partial\theta}, \qquad (2.79)$$

where the first part of the right side is equal to zero. Therefore, it follows that

$$\frac{d(V(\theta,K^*(\theta))-I(K^*(\theta)))}{d\theta}=\frac{\partial(V(\theta,K^*(\theta))-I(K^*(\theta)))}{\partial\theta},$$

where

$$\begin{aligned}\frac{d(V_2(\theta,K^*(\theta))-I(K^*(\theta)))}{d\theta} &= \beta_1 M_1(K^*(\theta))\theta^{(\beta_1-1)}+\beta_2 M_2\theta^{(\beta_2-1)}\\ &\quad+\frac{1}{4\gamma}\left[\frac{2\theta}{r-2\alpha-\sigma^2}-\frac{2c}{r-\alpha}\right], \qquad (2.80)\end{aligned}$$

for region 2 and

$$\frac{d(V_3(\theta,K^*(\theta))-I(K^*(\theta)))}{d\theta}=\beta_2 N_2(K^*(\theta))\theta^{(\beta_2-1)}+\frac{K^*(\theta)}{r-\alpha}, \qquad (2.81)$$

for region 3, respectively. Substituting these results into (2.78) leads to the threshold equations (2.13) and (2.16).

Proof of Proposition 2

The proof is analogous to the proof of Proposition 1. Here we just have to derive the optimal capacity choice and investment trigger in the $\theta-$region, where $\theta \geq \theta_3$. The value of $P_2'(K)$ in equation (2.21) is equal to

$$\begin{aligned}P_2'(K)=\theta_3^{-\beta_2}\ \frac{1}{\beta_1-\beta_2}\ &\left[\left(\frac{\theta_3}{r-\alpha}+\frac{\gamma K}{r-\alpha}(1-\beta_2)\right)(1-\beta_1)\right.\\ -\ &\left.\frac{\gamma K}{r}\beta_1\beta_2+\frac{2\gamma K+c}{r}\beta_1\right]. \qquad (2.82)\end{aligned}$$

Proof of Proposition 3

The concept of the proof is the same as the one for the proof of Proposition 1 considering the two $\theta-$regions, $0 \leq \theta < \hat{\theta}$ and $\theta \geq \hat{\theta}$, with $\hat{\theta} = \frac{cK^\gamma}{(1-\gamma)}$. The values of $N_2'(K)$ and $M_1'(K)$ in equations (2.26) and (2.29) are equal to

$$\begin{aligned}M_1'(K) &= \hat{\theta}^{-\beta_1}\left(\frac{1-\beta_1\gamma}{\beta_1-\beta_2}\right)\frac{c}{1-\gamma}\left[\frac{\beta_2-\epsilon}{r-\epsilon\alpha-\frac{1}{2}\sigma^2\epsilon(\epsilon-1)}\gamma\right.\\ &\quad\left.+)\frac{\beta_2}{r}\right], \qquad (2.83)\end{aligned}$$

$$
\begin{aligned}
N_2'(K) \quad = \quad & \hat{\theta}^{-\beta_2}\left(\frac{1-\beta_2\gamma}{\beta_1-\beta_2}\right)\frac{c}{1-\gamma}\left[\frac{\beta_1-\epsilon}{r-\epsilon\alpha-\frac{1}{2}\sigma^2\epsilon(\epsilon-1)}\gamma+\frac{(1-\beta_1)}{r-\alpha}\right. \\
& \left.+(1-\gamma)\frac{\beta_1}{r}\right]. \qquad (2.84)
\end{aligned}
$$

Proof of Proposition 4

The concept is analogous to the proof of Proposition 1.

CHAPTER 3

Optimal Investment Strategies for Product-Flexible and Dedicated Production Systems under Demand Uncertainty[1]

This paper studies the optimal investment strategy of a firm having the managerial freedom to acquire either flexible or dedicated production capacity. Flexible capacity is more expensive but allows the firm to switch costlessly between products and handle changes in relative volumes among products in a given product mix. Dedicated capacities restrict to manufacture one specific product but for lower acquisition costs. Specifically, I model the investment decision of a monopolist selling two products in a market characterized by price-dependent and uncertain demand, in a continuous time setting.

I find that flexibility especially pays off when uncertainty is high, substitutability low, and profit levels of the two products are substantially different. In the flexible case, the firm just produces the most profitable product under high demand, while if demand is low the firm produces both products to make total market demand bigger. In the dedicated case the firm invests in both capacities only if the substitutability rate is low and profitability of both products high enough. Otherwise, it restricts investment to one dedicated capacity for the more profitable product.

Considering a firm's decision to change from dedicated to flexible capacity, it is shown that despite perfectly positively correlated demand the firm will

[1]This chapter is based on Hagspiel (2011).

undertake this switch even for very low demand cases if the profitability of the products is substantially different. The option to increase total capacity accelerates investment in flexible capacity when the profit levels of both products are high enough.

3.1 Introduction

Flexibility in manufacturing operations is becoming increasingly more important to industrial firms. Increasing market demand volatility, internationalization of markets and competition, as well as shorter product life cycles pose new challenges for companies. Since the investment cost in flexible capacity mostly exceeds the investment cost of dedicated capacity, firms need to know how much this flexibility is worth for them. This work focuses on the effect of demand volatility as well as product combination on the firm's investment decisions and the value of product flexibility, which is considered one of the most (if not the most) strategically important flexibility types (see for example Goyal and Netessine (2007), Jordan and Graves (1995)).

The automotive industry is a good example of an industry where manufacturers' decisions on investing in production capacity and on the optimal level of flexibility are critical. On the one hand, expanding already installed capacity is very expensive (Andreou (1990)) and, therefore, the installed capacity must be sufficient for the whole life cycle of the product and easily adaptable to new product lines. On the other hand, the profitability of the products are threatened by low utilization of capacity as well as under-capacity. Japanese carmakers very early implemented the concept of flexibility, which gave them a significant advantage to their US and European competitors that traditionally built plants that were dedicated to producing a single car model. Later, also the European and American car industry started to activate in terms of flexibility. The BMW group for example recently advertised that a new production factory in Leipzig added new production capacity with a high level of flexibility[2]. Despite this recent approach of European and US car manufacturers to strive for more manufacturing flexibility, Japanese companies still lead

[2]The BMW groups states that the production program includes not just the full range of cars of the BMW 1 Series (three-door, Coupé and Convertible) but also the BMW X1. At the Volkswagen plant in Zwickau (Germany) Passat and Golf run off the same assembly line. Additional to the flexibility of the assembly line also the supply to the production line is fully flexible. This allows to change between the models and different interior equipment without any adaption costs neither time lags. See http://www.bmwgroup.com/d/nav/index.html?

in this aspect, which is according to Goyal et al. (2006) "an advantage that is at least partially responsible for the increasing market share of the Japanese carmakers".

The investment in and management of flexible capacity has received significant attention in the operations management literature. While early research in this field has focused on scenarios with exogenously given prices and static time models, recent papers extend this approach including responsive pricing and multi-stage decision problems. These multi-stage decision models are built up in the structure of: first invest in capacity, then receive additional information and finally exploit capacity optimally according to revealed information. While this structure shows a characteristic of real option models, the models are restricted to one-period models not taking into account timing. This work applies a continuous time setting as done in the real options literature, which allows to gain insight in the optimal timing of the firm's investment decision. The theory of real options explored flexibility mainly as it is related to the timing structure of capacity acquisition and did not deal with technologies that exhibit flexibility per se. This work presents a model which takes into account both aspects: flexibility in timing and investment in flexible capacity.

Specifically, this paper studies the investment decision of a monopolist having the managerial freedom to acquire either flexible or dedicated manufacturing capacity, in a continuous time setting. Flexible capacity allows the firm to manufacture all of its products with the same production facility while dedicated capacity restricts to one product. Since product flexibility has not been clearly defined in the literature, for the purpose of this study product flexibility is defined as a system's ability to switch costlessly between products, and handle changes in relative volumes among products. Products differ in substitutability and profitability in the market. The firm wants to protect efficiently against uncertainty in demand for all of its products. It can choose the timing as well as the quantity of the investment and is free to invest in flexible or dedicated production capacity, choosing the one that leads to the highest expected profit. In a further step I change the assumptions about the initial conditions of the firm assuming that it has already entered the market in an earlier stage and is currently producing with dedicated capacity. The firm has the possibility to undertake an investment in order to switch to flexible capacity.

I analyze the results with a specific focus on four cases of product combinations. The specific product combination will turn out to be a key factor for the firm's investment decision. The four cases differ regarding profitability and substitutability rate between the products: (1) The first case considers a product combination of two almost similarly profitable products with a low substitutability rate. The car models Passat and Golf of Volkswagen are an example for such a product combination. In the Volkswagen plant in Mosel these two car models are produced at an assembly line that is fully flexible in switching between the two models without adaption time lags or costs. (2) National-brand manufacturers that, additionally to their brand product, also produce private label products, are an example for firms that produce two highly substitutable products for the same market where one product (their own manufacturer brand) is more profitable for them than the second product that is sold to and brought on the market by a private label retailer. Danone, the famous French food-products corporation for example, produces a variation of their popular cream cheese dessert "Fruchtzwerge" also for the private label product "Desira" of Germany's biggest discounter Aldi[3]. (3) The third case depicts the scenario of a firm producing two products for the same market that are almost equally profitable and highly substitutable. For brand manufacturers with less successful national brands, so called B and C brands, the production of private labels can be almost as profitable as producing their own brand product. Concorp group, a Dutch confectionery company for example states publicly that they do produce for dual branding[4]. (4) The fourth case considers a firm selling two products with a low substitutability rate and a substantial difference in profitability, in the same market. As an example think of a technology company producing a newly established touch screen mobile and a less successful obsolescent model in the same production facility.

This work focuses on the effect of demand variability - a key driver of flexibility - together with substitutability and product profitability effects. I show that in the flexible case, under high demand the firm just produces the most profitable product, if demand is low the firm produces both products to make total demand bigger. Comparing the optimal flexible investment strategies for the previously mentioned cases, the firm selling two products with a low

[3]See on p. 39 in 'Aldi - Welche Marke Steckt Dahinter? 100 Aldi-Top-Artikel und Ihre Prominenten Hersteller.' Muenchen: Suedwest Verlag by Schneider, Martina.

[4]Concorp group states on its website that they "...build brands and deliver private label concepts with added value in all segments of selected national and international confectionery markets." Concorp group produces candy foam and boiled sweets for three different brand names in its production site in Waddinxveen. See http //www.concorp.nl/international/pdf/concorp_international_overview.pdf

substitutability rate and high profitability difference invests in significantly higher capacity. Capacity size is growing more than proportionally with the uncertainty level. This confirms the intuition that a firm producing two almost equally profitable products with a low substitutability rate profits the most by the down size potential to increase the market size by producing both products. In the dedicated case the firm invests in both capacities if substitutability rate is low and profitability of both products high enough. In all other cases the firm decides to ignore demand for one product in the market and installs one dedicated capacity for the more profitable product. In this case the firm can just gain from the downside potential when demand levels are very low and therefore the negative effect of restriction to produce up to full capacity once it has installed capacity for both products, is dominating. For both dedicated and flexible capacity investment, I show that the firm invests later in higher capacity if demand uncertainty increases. This result is also obtained for production flexible capacity investment by Hagspiel et al. (2011) and Dangl (1999).

In order to study the value of flexibility, I consider as a benchmark the situation in which a firm relies on maximally two dedicated capacities rather than on one flexible capacity. Flexibility especially pays off when uncertainty is high, substitutability low, and profit levels between the two products are substantially different. In this case the flexible firm has the possibility to increase its total market demand if demand falls low by including the production of the less profitable second product. The dedicated firm on the other hand relinquishes production of one product completely by acquiring just one dedicated capacity for the more profitable product.

Since firms are often not facing the direct choice between dedicated and flexible capacity, but need to evaluate whether or not to invest in flexible capacity while currently producing with dedicated once, I additionally analyze this scenario. This study shows that despite the assumption of perfectly positively correlated product demand, the firm undertakes investment to switch to flexible capacity also for low demand if the profitability of the products is substantially different. I conclude that the specific product combination has a crucial effect on the investment decision. Therefore, firms should take into account, besides demand volatility and correlation, the specific product combination when deciding between dedicated and flexible capacity. This contradicts early literature claiming that flexibility has no value in case of perfectly correlated demand and confirms more recent literature that claims that flexibility has value also in case of perfectly positively correlated demand. I extend this approach by analyzing the effect of specific product combinations on the value of flexibility.

As already mentioned earlier, there are two streams of literature that are most relevant to this study: the first considers the issue of product flexibility from an operations management perspective, while the second studies capital budgeting decision applying real options theory. Both will now be discussed more extensively.

The issue of resource. flexibility has become a significant interest in the operations management community in the beginning of the nineties, following the increasing viability of flexible, computer-controlled manufacturing systems. From the operations management literature this work is closely related to a stream of papers about resource flexibility initiated by work of Fine and Freund (1990). Fine and Freud derive necessary and sufficient conditions for the acquisition of flexible capacity that are based on a two-stage convex quadratic program. In the first stage a technology investment decision is made. After observing demand realization, an optimal production decision is made at the second stage. Inspired by Fine and Freud, Van Mieghem and Dada (1999) present closely related work that disproves Fine and Freud's claim that flexible capacity would not provide additional value when product demands are perfectly positively correlated. They show that in addition to its adaptability to demand mix changes, product-flexible technology provides another opportunity for revenue improvement through its ability to exploit differentials in price (margin) mix. They argue that product flexibility generates an option to produce and sell more of highly profitable products at the expense of less profitable products and show that this option can remain valuable even with perfectly positively correlated product demand. I extend this claim of Van Mieghem and Dada (1999) by analyzing specific differences in product combinations and analyze its individual effects on the value of flexibility.

Most closely related to my work are two recently published papers of Chod and Rudi (2005) and Bish and Wang (2004), who study the resource investment decision of a two-product, price setting firm that operates in a monopolistic setting. Chod and Rudi look at the effect of demand variability and demand correlation on the optimal flexible resource investment decision and show that expected profit is increasing in variability and decreasing in the correlation of normally distributed demand. Bish and Wang's model is more general than the one of Chod and Rudi by allowing the firm to invest in flexible and dedicated resources at the same time but they do not include cross-price effects.

Both previously mentioned papers present two-stage models that allow them to gain insight in the optimal resource size and allocation but deprives the timing aspect of investment decisions. Unlike these papers, I focus on an economic environment where uncertainty in demand of two products arises from one single market. Optimizing investment decisions facing uncertainty about the general economic situation became even more appealing for industry that just recently witnessed one of the biggest economic crises in history. The credit crunch affected the whole global market and companies faced large drops in demand for their products.

Applying a continuous time setting allows me to gain insight in the optimal timing of the firm's investment strategy. For evaluating investment decisions that have the following three characteristics: (1) the investment considered is irreversible, (2) there is uncertainty about future rewards and (3) a leeway about timing of investment, the theory of real options is used to evaluate such investment decisions. But real options theory explores flexibility mainly as it is related to the timing structure of capacity or information acquisition or commitment of resources: that means that the firm loses flexibility when it makes an irreversible commitment. Most papers do not deal with technologies that exhibit flexibility per se. Now that more and more firms undergo investment in flexible capacity because it appears for them to be a necessary tool to hedge against highly volatile demand, it is important to develop these models further with a special attention to include the ability of flexible capacity. This paper aims to take a crucial step in this direction. I explicitly consider the use of a (product-) flexible technology.

Real option papers that deal with investments in flexible capacity, mainly take into account capacity that allows to switch between different inputs or different outputs. With regard to product flexibility this would mean that a system is able to switch from producing product 1 to producing product 2 and backwards if the production of product 1 is becoming more profitable again. In contrast to these models, I take into account a system that is flexible in adapting the relative volume between two products, i.e. it can use total capacity to produce x units of one product and y units of the other and continuously adapt this split over time. Early approaches in evaluating such switching options have been presented by McDonald and Siegel (1986), Kulatilaka (1988) and Triantis and Hodder (1990). Conceptually, the switch between two volatile assets or commodities can be modelled as an exchange option. Margrabe (1978) and McDonald and Siegel (1986) model European finite and American perpetual exchange options, respectively, which are linear homogeneous in the underlying stochastic variables.

Triantis and Hodder evaluate product(-mix) flexibility based on option principles. Kulatilaka applies option pricing principles to the same problem using a stochastic dynamic programming formulation that includes costly switching between modes of operation. Andreou (1990) published a more applied study associated with the General Motors Research Laboratories that focuses on the economic evaluation of product flexibility. He presents a financial model for calculating the dollar value of flexible plant capacity for two products under conditions of uncertain market demand. In recent work Dockendor and Paxson (2011) present a two-factor model with continuous switching opportunities between two commodity outputs, taking into account operating, switching costs and the possibility of suspension of operation. Adkins and Paxson (011a) evaluate input switching options for single as well as multiple switching. Both papers present quasi-analytical solutions for two-factor models. Other current papers that deal with options to switch between inputs, outputs or between inputs and output are, for example, Adkins and Paxson (011b), Sigbjorn et al. (2008) and Bastian-Pinto et al. (2009). These papers do evaluate investment in flexible technology based on option theory, but do not include the timing decision nor capacity choice.

This work was inspired by the increasing interest in the development of the real options theory regarding technological flexibility shown by the operations and production management sector. Bengtsson (2001) presented a work that relates the real options literature to manufacturing flexibility from an industrial engineering/production management perspective. He refers to product flexibility as one of the flexibility types that have not been treated as real options yet. While his work addresses a wide range of manufacturing flexibility, Bengtsson and Olhager (2002) use real options theory to evaluate one specific type, i.e. product-mix flexibility, in a real case analysis. Their main focus is on solving for the value of a production system with multiple products which is applied to real case data, while the timing or capacity size decisions are not considered. Furthermore, there is an increasing number of real data cases and empirical analysis in this area. Two recent papers are for example Goyal et al. (2006) and Fleischmann et al. (2006). Both focus on the automotive industry.

The paper is organized as follows. The next section presents the general model and solves the optimization problems for the flexible capacity and dedicated capacity case. The optimal investment triggers for size and time of investment are derived. The first part of Section 3.3 analyzes the capacity and timing decision for flexible capacity investment and shows how investment timing and size are affected by demand uncertainty. The second part concentrates on analyzing investment in dedicated production capacity. Section 3.4 studies the optimal investment strategy of a firm having the option to choose between flexible and dedicated capacity investment and quantifies the value of flexibility. In Section 3.5 the optimal investment in flexible capacity is studied assuming that the firm is currently using dedicated capacities. Section 3.6 concludes.

3.2 Model

Consider a firm that produces two products, indicated by product A and B. The firm has to decide about the optimal capacity investment. This involves three decisions: when to invest, the size of the capacity and in which type of capacity to invest. The firm can invest in maximal two dedicated capacities, each of which can produce only one product, or in a more expensive, flexible production capacity, which can produce both products.

The firm is uncertain about future demand where the inverse demand function are assumed to be linear. The inverse demand functions for the two products are given by

$$p_A(\theta_t, q_A, q_B) = \theta_t - q_A - \gamma q_B, \tag{3.1}$$

$$p_B(\theta_t, q_B, q_A) = \alpha\theta_t - q_B - \gamma q_A, \tag{3.2}$$

where the demand intercept θ follows the geometric Brownian motion

$$d\theta_t = \mu\theta_t dt + \sigma\theta_t dW_t. \tag{3.3}$$

In this expression μ is a constant representing the trend, σ is the uncertainty parameter and dW_t is the increment of a Wiener process implying that it is independently and normally distributed with mean 0 and variance dt. I often refer to the uncertainty in demand intercepts simply as "demand uncertainty" in this paper. $\gamma \in (-1, 1)$ is the product substitutability parameter, and $\gamma > 0$ ($\gamma < 0$) signifies that the products are substitutes (complements). Since products made by the same flexible resource tend not to be complements, most applications are characterized by a nonnegative γ. Therefore, this work will focuses on the case of the products being substitutes. The two products are assumed to be sold in the same market. Product A is the more profitable product in this market, i.e. $\alpha < 1$. $\alpha \in (0, 1)$ is referred to as the profitability parameter of product B. Denote production quantity of product A (B) at time time by $q_{t,A}$

($q_{t,B}$). From now on I drop the time subscript whenever there can be no misunderstanding. For simplicity variable production costs are not considered yet. It follows that the profit flow is defined by

$$\Pi(\theta) = \max_{q_A, q_B}[p_A q_A + p_B q_B]. \tag{3.4}$$

Total production output, i.e. $q = q_A + q_B$, is restricted to be up to full capacity. This means that after investment the firm always produces up to full capacity, a constraint also referred to as capacity clearance. Two aspects motivate the introduction of this assumption here. First, it allows to concentrate on the effect of one specific type of technological flexibility, i.e. product-flexibility, in the analysis. Releasing this assumption and therefore allow the firm to produce under capacity, would mean to include the effect of production flexibility to the problem. See Anupindi and Jiang (2008) and Hagspiel et al. (2011) for examples of papers that consider investment in production flexible capacity. Second, producing below capacity is often linked with large fixed costs associated with, for example, labor and production ramp-up. Therefore, in practice firms often reduce prices to keep production lines running. See Goyal and Netessine (2007) and Chod and Rudi (2005) for examples of similar assumptions.

The flexible capacity is denoted by K_F and the dedicated capacities by K_{D_A} and K_{D_B}, respectively. The investment cost are sunk and assumed to be linear (for the same assumption see for example, Fine and Freund (1990), Van Mieghem and Dada (1999) and Chod and Rudi (2005)). Let c_i denote the unit cost of investing in resource K_i, $i = F, D_A, D_B$.

3.2.1 Flexible Capacity

Consider a firm that has to decide about investment in flexible capacity. Flexible capacity allows it to produce both products, A and B, on the same production line. It decides about the optimal time to invest and the optimal capacity size invested in, considering that it has to produce always up to full capacity after the moment of investment. The optimal output rate for the two products q_A^* and q_B^*, respectively, is determined by maximizing the profit flow considering the capacity clearance constraint $(q_A + q_B = K_F)$ and the upper and lower boundaries for each of the two output rates, $0 \leq q_B, q_A \leq K_F$. This gives

$$q_A^* = \begin{cases} \frac{\theta(1-\alpha)}{4(1-\gamma)} + \frac{K_F}{2} & \text{for } \theta < \hat{\theta}, \\ K_F & \text{for } \theta \geq \hat{\theta}, \end{cases} \tag{3.5}$$

$$q_B^* = \begin{cases} -\frac{\theta(1-\alpha)}{4(1-\gamma)} + \frac{K_F}{2} & \text{for } \theta < \hat{\theta}, \\ 0 & \text{for } \theta \geq \hat{\theta}, \end{cases} \tag{3.6}$$

where $\hat{\theta}$ denotes the boundary $\frac{2(1-\gamma)}{(1-\alpha)}K_F$. For low demand, $\theta \in [0, \hat{\theta})$, the firm will produce both products. If demand increases, i.e. $\theta \in [\hat{\theta}, \infty)$, the firm will switch to use full capacity K_F for production of the more profitable product A and suspend production of product B. Expressions (3.5) and (3.6) imply that the profit flow is given by

$$\Pi(\theta) = \begin{cases} \frac{(1-\alpha)^2}{8(1-\gamma)}\theta^2 + \frac{(1+\alpha)}{2}\theta K_F - \frac{(1+\gamma)}{2}K_F^2 & \text{for } \theta < \hat{\theta}, \\ (\theta - K_F)\,K_F & \text{for } \theta \geq \hat{\theta}. \end{cases} \tag{3.7}$$

In order to find the value of this investment project, the dynamic programming approach is applied. The value function, denoted here by $V(\theta, K_F)$, must satisfy the Bellman equation

$$V(\theta, K) = \pi(\theta, K)dt + E\left[V(\theta + d\theta, K)e^{-rdt}\right], \tag{3.8}$$

where r is the (constant) discount rate. Applying Ito's Lemma, substituting and rewriting leads to the differential equation (see, e.g., Dixit and Pindyck (1994))

$$\frac{1}{2}\sigma^2\theta^2\frac{\partial^2 V}{\partial\theta^2} + \mu\theta\frac{\partial V}{\partial\theta} - rV + \Pi(\theta) = 0. \tag{3.9}$$

Solving this equation for $V(\theta, K)$, considering that we have two different regions, and ruling out bubble solutions, we get the following value of the project:

$$V(\theta, K_F) = \begin{cases} A_1(K_F)\theta^{\beta_1} + a_1\theta^2 + a_2\theta K_F + a_3K_F^2 & \text{for } \theta < \hat{\theta}, \\ B_2(K_F)\theta^{\beta_2} + \frac{\theta K_F}{r-\mu} - \frac{K_F^2}{r} & \text{for } \theta \geq \hat{\theta}, \end{cases} \tag{3.10}$$

with $a_1 = \frac{(1-\alpha)^2}{8(1-\gamma)(r-2\mu-\sigma^2)}$, $a_2 = \frac{(1+\alpha)}{2(r-\mu)}$ and $a_3 = -\frac{(1+\gamma)}{2r}$. β_1 (β_2) is the positive (negative) root of the quadratic polynomial

$$\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0. \tag{3.11}$$

$V(\theta)$ must be continuously differentiable across the boundary $\hat{\theta} = \frac{2(1-\gamma)}{(1-\alpha)}K_F$.

Using the fact that $V(\theta, K_F)$ must be continuously differentiable across the boundary $\hat{\theta}$ one can derive the constants A_1 and B_2:

$$\begin{aligned} A_1(K_F) &= K_F^{2-\beta_1} \frac{1}{\beta_2 - \beta_1} \left[\frac{2(1-\gamma)}{1-\alpha} \right]^{-\beta_1} (1-\gamma) \left[\frac{(2-\beta_2)}{2(r-2\mu-\sigma^2)} \right. \\ & \qquad \left. - \frac{(1-\beta_2)}{(r-\mu)} - \frac{\beta_2}{2r} \right], \qquad (3.12) \\ B_2(K_F) &= K_F^{2-\beta_2} \frac{1}{\beta_2 - \beta_1} \left[\frac{2(1-\gamma)}{1-\alpha} \right]^{-\beta_2} (1-\gamma) \left[\frac{(2-\beta_1)}{2(r-2\mu-\sigma^2)} \right. \\ & \qquad \left. - \frac{(1-\beta_1)}{r-\mu} - \frac{\beta_1}{2r} \right]. \qquad (3.13) \end{aligned}$$

Corollary 1 in Appendix A shows that A_1 is negative for all parameter values and B_2 positive.

The value of the investment project in the region $\theta < \hat{\theta}$ consists of four terms where the last three terms constitute the cash flow generated by the sales. The first term $A_1\,(K_F)\,\theta^{\beta_1}$, which is negative, corrects for the fact that in a mathematically optimal case the production quantity of product B would turn negative for $\theta > \hat{\theta}$. Economically this does not make sense and therefore the output quantity is constrained by $q_B^* \geq 0$. The absolute value of this term decreases with θ.

In the region $\theta \geq \hat{\theta}$, demand is that large that the firm uses all of its installed capacity to produce the more profitable product A. This generates a discounted cash flow stream that is reflected in the second and third term of the value of the investment project associated with this region. The first term, $B_2\,(K_F)\,\theta^{\beta_2}$, describes the option value that accounts for the additional possibility that in case demand decreases the company can switch its production back to two products and therefore gains revenue. This option value is decreasing for large θ.

Knowing the value of the project, $V(\theta, K_F)$, one is able to derive the optimal investment strategy. In general the procedure is as follows. First, the optimal capacity choice $K_F^*(\theta)$ is determined for a given level of θ by setting the marginal value of the project equal to the marginal investment costs c_F. Second, one derives the optimal investment threshold θ^*. For this demand level θ^* it holds that the firm is indifferent between investment and waiting with investment. Investment (waiting) is optimal for a θ being larger (lower) than θ^*.

Investment can take place either in region I, which refers to the region $\theta < \hat{\theta}$, or in region II, i.e. the region $\theta \geq \hat{\theta}$. Investing while $\theta < \hat{\theta}$ means that the firm uses the capacity invested in, to produce both products right after the investment has been undertaken, while investing in region $\theta \geq \hat{\theta}$ implies that the full capacity level is used to produce only product A.

The following proposition provides equations that implicitly determine the threshold θ_F^* and the corresponding capacity level $K_F^*(\theta^*)$ in each of the two cases. The optimal investment decision corresponds to the case that provides the largest expected value of the investment project.

Proposition 1 *Concerning the firm's investment policy there are two possibilities:*

1. *Given that the firm produces a positive amount of both products right after the investment moment, the optimal capacity level $K_F^*(\theta)$ is implicitly determined by*

$$\frac{\partial A_1}{\partial K_F}\theta^{\beta_1} + a_2\theta + 2a_3K_F^* - c_F = 0. \tag{3.14}$$

If the obtained K^* is not an interior solution of the considered region, i.e. if $K^* > \frac{(1-\alpha)\theta}{2(1-\gamma)}$, the optimal capacity is replaced by the boundary solution, i.e. $\theta\frac{(1-\alpha)}{2(1-\gamma)}$. Thus, the optimal capacity choice for the demand realization θ is given by

$$K^*(\theta) = \max\left[K^*, \theta\frac{(1-\alpha)}{2(1-\gamma)}\right]. \tag{3.15}$$

The investment threshold θ^* is implicitly determined by

$$\begin{aligned} a_1(\beta_1-2)\theta^{*2} &+ a_2(\beta_1-1)\theta^*K_F^*(\theta^*) + \beta_1a_3K_F^*(\theta^*)^2 \\ &- \beta_1c_FK_F^*(\theta^*) = 0. \end{aligned} \tag{3.16}$$

2. *Given that the firm uses full capacity to produce the more profitable products A right after the investment moment, the optimal capacity level $K_F^*(\theta)$ is implicitly determined by*

$$\frac{\partial B_2}{\partial K_F}\theta^{\beta_2} + \frac{\theta}{r-\mu} - \frac{2}{r}K_F^* - c_F = 0. \tag{3.17}$$

If the obtained K^* does not constitute an interior solution of the region where $K^* \leq \frac{(1-\alpha)\theta}{2(1-\gamma)}$, the optimal capacity is given by the boundary $\theta\frac{(1-\alpha)}{2(1-\gamma)}$. Thus,

$$K^*(\theta) = \min\left[K^*, \theta\frac{(1-\alpha)}{2(1-\gamma)}\right]. \tag{3.18}$$

The investment threshold θ^* is implicitly determined by

$$\begin{aligned} B_2\theta^{*\beta_2}(\beta_1-\beta_2) &+ \frac{\theta^*K_F^*(\theta^*)}{r-\mu}(\beta_1-1) \\ &- \beta_1\frac{K_F^*(\theta^*)^2}{r} - \beta_1c_FK_F^*(\theta^*) = 0. \end{aligned} \tag{3.19}$$

Out of these two possibilities the firm chooses the one that gives the highest expected value of the project discounted back to an initial demand intercept level θ_0*, which is given by* $\left(\frac{\theta_0}{\theta_F^*}\right)^{\beta_1} V\left(\theta_F^*, K_F^*(\theta_F^*)\right)$.

3.2.2 Dedicated Capacity

The difference with the previous section is that the firm has to decide about optimal investment in dedicated capacities. Dedicated capacity can satisfy only one product. It is assumed that the firm invests at the same time in both capacity levels, provided that the firm wants to produce both products. The firm has to decide when to invest and in how much capacity. The firm has the following two investment options: it can invest in two dedicated production capacities, each of which is restricted to produce one of the two products A and B, respectively. The second option applies if the production of product B is not profitable enough in this market. In that case it is optimal for the firm to ignore demand for product B in the market and invest in just one dedicated production facility for product A. The unit costs of capacity are assumed to be equal for product A and product B, i.e.

$$c_{D_A} = c_{D_B} =: c_D. \tag{3.20}$$

Higher unit cost of dedicated capacity for one of the products would result in a proportionally lower capacity investment for this product. With assumption (3.20) I exclude a possible impact of cost differences of the two products in order to concentrate on the effect of product profitability and substitutability - one of the main aspects of this paper. The firm will choose the option with the highest expected project value. After investment the firm has to produce up to full capacity forever.

For the latter case the profit of the firm is given by

$$\Pi = (\theta - q_A)q_A. \tag{3.21}$$

Considering the capacity clearance constraint, $q_A = K_{D,A}$, the profit flow can be rewritten as a function of dedicated capacity $K_{D,A}$:

$$\Pi = (\theta - K_{D,A})K_{D,A}. \tag{3.22}$$

Familiar steps lead to the following value of the investment project:

$$V(\theta, K_{D,A}) = \frac{\theta K_{D,A}}{r - \mu} - \frac{K_{D,A}^2}{r}. \tag{3.23}$$

In case that the firm invests at the same time in two dedicated production facilities, for product A and product B, respectively, the profit of the firm $\Pi = p_A q_A + p_B q_B$ is maximize w.r.t the output rates q_A and q_B. Considering the capacity clearance constraints for each product respectively, $q_A = K_{D,A}$ and $q_B = K_{D,B}$, implies that the profit flow is given by

$$\Pi(\theta, K_{D,A}, K_{D,B}) = (\theta - K_{D,A})K_{D,A} + (\alpha\theta - K_{D,B})K_{D,B} - 2\gamma K_{D,A}K_{D,B}. \quad (3.24)$$

Familiar steps lead to the following project value

$$\begin{aligned} V(\theta, K_{D,A}, K_{D,B}) &= \frac{\theta}{r-\mu}\left[K_{D,A} + \alpha K_{D,B}\right] \\ &\quad - \frac{K_{D,A}^2 + 2\gamma K_{D,A}K_{D,B} + K_{D,B}^2}{r}. \end{aligned} \quad (3.25)$$

The optimal capacity level for every relevant value of θ is derived by maximizing the project value minus investment cost $c_D(K_{D,A} + K_{D,B})$ for a given demand intercept level θ. In case the obtained $K_{D,i}$ $(i = A, B)$ is negative, the optimal capacity is replaced by the boundary solution $K_{D,i}^*(\theta) = 0$. Knowing the optimal capacity level for all relevant demand levels, the optimal investment threshold θ^* is derived. The following proposition provides the expressions for threshold θ^* and the corresponding capacity level $K_D^*(\theta^*)$ in each of the two cases. The optimal investment decision corresponds to the investment strategy that provides the largest expected project value for the firm.

Proposition 2 *Concerning the firm's investment policy for dedicated capacity there are two possibilities:*

1. *Given that it is optimal for the firm to invest in dedicated production capacity for both products the optimal capacity levels for product A and B, respectively are given by*

$$K_{D,A}^*(\theta) = \begin{cases} 0 & \text{for } \theta < \hat{\theta}_A, \\ \frac{\theta r}{2(r-\mu)}\frac{(1-\alpha\gamma)}{1-\gamma^2} - \frac{c_D r}{2(1+\gamma)} & \text{for } \theta > \hat{\theta}_A, \end{cases} \quad (3.26)$$

$$K_{D,B}^*(\theta) = \begin{cases} 0 & \text{for } \theta < \hat{\theta}_B, \\ \frac{\theta r}{2(r-\mu)}\frac{(\alpha-\gamma)}{1-\gamma^2} - \frac{c_D r}{2(1+\gamma)} & \text{for } \theta > \hat{\theta}_B, \end{cases} \quad (3.27)$$

where $\hat{\theta}_A = \frac{c_D(r-\mu)(1-\gamma)}{(1-\alpha\gamma)}$ *and* $\hat{\theta}_B = \frac{c_D(r-\mu)(1-\gamma)}{(\alpha-\gamma)}$. *Total optimal dedicated capacity is given by*

$$K_D^*(\theta) = \begin{cases} 0 & \text{for } \theta < \hat{\theta}_A, \\ \frac{\theta r}{2(r-\mu)}\frac{(1-\alpha\gamma)}{1-\gamma^2} - \frac{c_D r}{2(1+\gamma)} & \text{for } \hat{\theta}_A < \theta < \hat{\theta}_B, \\ \theta\frac{r}{2(r-\mu)}\frac{(1+\alpha)}{(1+\gamma)} - c_D\frac{r}{(1+\gamma)} & \text{for } \hat{\theta}_B < \theta. \end{cases} \quad (3.28)$$

(3.29)

The investment thresholds are given by

$$\theta^*_{D,I} = \frac{\beta_1 c_D (r-\mu)(1+\gamma-2\gamma^2)}{\beta_1(1+\alpha\gamma) - 2(1+(\beta_1-1)\gamma^2)}, \tag{3.30}$$

for region $\hat{\theta}_A < \theta < \hat{\theta}_B$ and

$$\begin{aligned}\theta^*_{D,II} &= \frac{c_D(r-\mu)(1+\alpha)(1-\gamma)}{(1-2\alpha\gamma+\alpha^2)}\left(\frac{\beta_1-1}{\beta_1-2}\right) + \frac{2(r-\mu)^2(1-\gamma^2)}{r(1-2\alpha\gamma+\alpha^2)(\beta_1-2)} \\ &\sqrt{\frac{r^2c_D^2(1+\alpha)^2(\beta_1-1)^2}{4(r-\mu)^2(1+\gamma)^2} - \frac{r^2c_D^2(1-2\alpha\gamma+\alpha^2)\beta_1(\beta_1-2)}{2(r-\mu)^2(1-\gamma^2)(1+\gamma)}},\end{aligned} \tag{3.31}$$

for region $\theta > \hat{\theta}_B$.

2. *Given that it is more profitable for the firm to invest in just one production capacity for product A ignoring demand for product B, the optimal capacity level for product A is given by*

$$K^*_{D,A}(\theta) = \frac{r}{2(r-\mu)}\theta - \frac{r}{2}c_D. \tag{3.32}$$

The investment threshold is determined by

$$\theta^*_D = \left(\frac{\beta_1}{\beta_1-2}\right)(r-\mu)c_D. \tag{3.33}$$

Out of these choices the firm chooses the one that gives the highest expected value of the project discounted back to an initial time with demand intercept level θ_0*, which is given by* $\left(\frac{\theta_0}{\theta^*_D}\right)^{\beta_1} V\left(\theta^*_D, K^*_D(\theta^*_D)\right)$.

3.3 Results

This section presents results for investment in flexible and dedicated capacity independently. The optimal investment strategy of a firm considering both flexible and dedicated capacity, will be analyzed in Section 3.4.

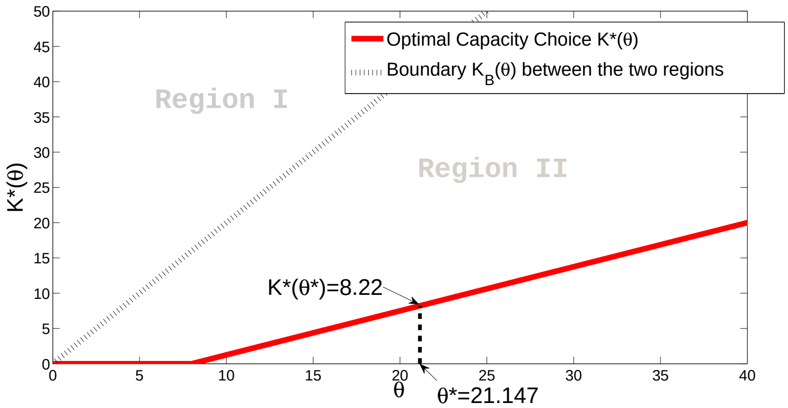
3.3.1 Flexible Capacity Investment

As shown in section 3.3, the flexible firm can either invest in the θ-region I, i.e. $\theta \in [0, \frac{2(1-\gamma)}{1-\alpha}K_F)$, where the firm sets an upper bound for output at the moment of investment and uses this capacity to produce both products right after the moment of investment, or invest in the second θ-region, i.e. $\theta \in [\frac{2(1-\gamma)}{1-\alpha}K_F, \infty)$. Investing in region II means that the firm invests in flexible capacity that is used up to full extent for production of the more profitable product A. Once investment has been made the firm is flexible to adapt the relative production volumes among products to the changing demand level. Facing low demand it will make use of the downside potential to produce both products in order to increase total market size. For high demand levels the firm will use the full available capacity to produce the more profitable product A.

Figure 3.1 shows an example of two highly substitutable products with substitutability parameter $\gamma = 0.8$, assuming a substantial difference in the profitability of the two products. Product B is much less profitable than product A with a profitability parameter of value $\alpha = 0.2$. The other parameter values assumed are $\mu = 0.02$, $\sigma = 0.1$, $r = 0.1$ and $c_F = 100$. Solving equations (3.14) and (3.17) the optimal capacity choice for the two regions is derived. Comparing the expected values of the investment project for the two regions it can be concluded that it is optimal to invest in region II at the investment trigger $\theta^* = 21.147$, provided that the initial θ-value lies below this θ^*. In particular, the firm invests immediately if the current value of θ exceeds θ^*, while otherwise it waits with investment until θ becomes equal to θ^*. The optimal size of acquired capacity is $K^*(\theta^*) = 8.22$. After the investment the firm can adapt the relative production volume among products according to changing demand to receive the highest possible profit. For this specific numerical example the firm will continue using full capacity to produce just product A unless demand drops drastically, i.e. below a θ-bound of $\hat{\theta} = 4.11$. Since the two products are good substitutes in the market but producing product B results in significantly less profit for the firm, it is optimal for a wide range of demand realization to keep producing just one product, i.e. the more profitable product A, with flexible capacity. Just for very low demand realizations the firm can gain profit from the downside potential to avoid overcapacity by increasing total market size including demand for product B.

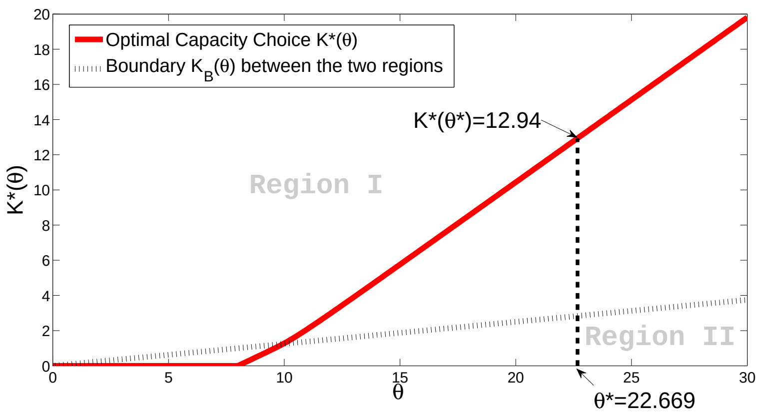
Choosing a relatively low substitutability parameter ($\gamma = 0.2$) but a high profitability for product B ($\alpha = 0.8$) increases the value of this downside potential for the firm significantly. In fact, it is optimal for the firm to invest in

Figure 3.1: Case: $\alpha = 0.2$, $\gamma = 0.8$ **—** Optimal Investment Capacity as a function of demand intercept θ. Region I constitutes the region where the capacity level for a specific demand realization is used to produce both products. Region II describes the area where it is optimal for the firm to use full capacity for the production of the more profitable product A. (Parameter values: $r = 0.1$, $\mu = 0.02$, $\sigma = 0.1$ and $c_F = 100$)



capacity at investment threshold $\theta_F^* = 22.669$. For this parameter choice the investment moment lies in region II which means that the firm uses purchased capacity to produce both products at the moment of investment. Figure 3.2 (which illustrates this example) shows the optimal capacity choice as a function demand intercept θ. The figure shows that unlike the previous example, the capacity function $K^*(\theta)$ switches at $\hat{\theta_S} = 9.902$ from optimal investment in region II to optimal investment in region I. Compared to the previous example flexible capacity is very valuable for a firm selling two almost equally profitable products with a low substitutability rate. The firm purchases significantly higher capacity $K^*(\theta^*) = 12.94$. Figure 3.3 illustrates the advantage of flexible production capacity for a firm selling two almost equally profitable products with a low substitutability rate by means of the following numerical example. The upper plot of Figure 3.3 shows a simulation of the demand intercept θ with a drift rate of $\mu = 0.02$ and volatility $\sigma = 0.1$ for a time period of 10 years, i.e. $t \in [0, 10]$. The firm will invest as soon as the demand intercept hits the value $\theta_F^* = 22.669$ for the first time, which is (for this specific simulation) after 1.6 years. The second plot (Figure 3.3) shows the optimal production decision from the moment of investment on. Flexible capacity allows the

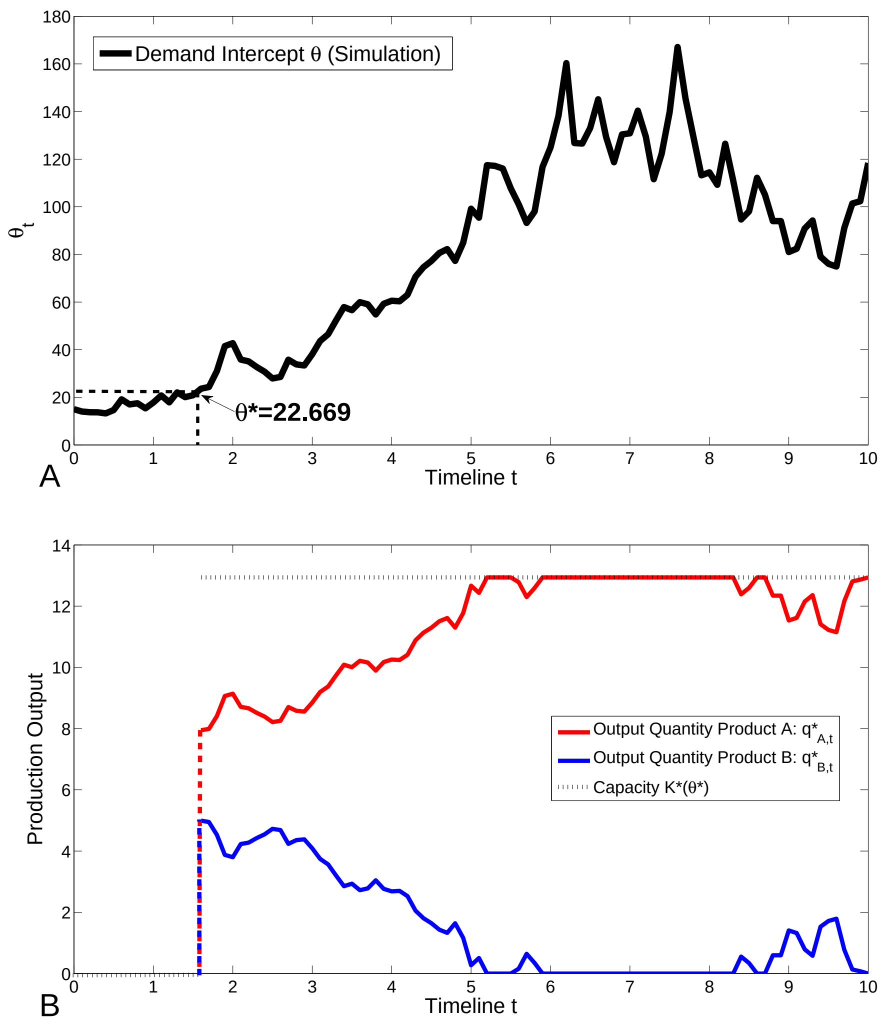
Figure 3.2: Case: $\alpha = 0.8$, $\gamma = 0.2$ **—** Optimal Investment Capacity as a function of demand intercept θ (Parameter values: $r = 0.1$, $\mu = 0.02$, $\sigma = 0.1$ and $c_F = 100$)



firm to adapt the relative production volume of the two products, relatively, in order to obtain the highest possible profit facing its capacity constraint of $K_F^*(\theta^*) = 12.94$. For low demand the firm uses full capacity to produce both products. If demand rises above a certain threshold, i.e. when demand intercept reaches the level $\hat{\theta} = 103.52$, it is most profitable to suspend production of product B and use full capacity (i.e. $K_F^*(\theta^*) = 12.94$) to produce just product A. Being able to adapt relative production volume optimally across the two products allows the firm to avoid over- as well as under capacity for a wide range of possible demand intercepts.

Subsequently I analyze the effect of demand variability on the flexible investment decision. Four specific "extreme" cases that arise from different combinations of product profitability and substitutability are compared: The first case considers a product combination of two almost similar profitable products with low substitutability rate, indicated as 'Case: H - L'. For the numerical example the parameter values $\alpha = 0.9$ and $\gamma = 0.1$ are chosen. 'Case H - H' represents a product combination of two highly substitutable and almost equally profitable products. The numerical parameter values are $\alpha = 0.9$ and $\gamma = 0.9$. The third case, indicated with 'Case: L - H', represents the setting of a firm producing two products that are highly substitutable but one product is significantly less profitable than the other (parameter values $\alpha = 0.1$ and $\gamma = 0.9$). And last but not least the case of two products with low substitutability rate ($\alpha = 0.1$) and significant difference in profitability ($\gamma = 0.1$) between the products is considered. This case is referred to as 'Case: L - L'.

Figure 3.3: Production Time Line for a simulation result of demand intercept θ. Panel A: Simulated demand intercept process θ_t plotted against time line. Panel B: Optimal production output of product A and B, respectively over time period $t \in [0, 10]$. (Parameter values: $r = 0.1$, $\mu = 0.02$, $\sigma = 0.1$ and $c_F = 100$)



The magnitude of the impact of demand variability on the optimal capacity size and investment threshold are illustrated in Figure 3.4. This figure is based on the values $r = 0.1$, $\mu = 0.02$ and $c_F = 100$ which forms the base case for numerical illustrations throughout the rest of the paper. It confirms the widely accepted result that higher uncertainty increases capacity size but delays investment. When uncertainty goes up, a higher demand level is needed before it is optimal to invest. This effect is partly caused by the fact that capacity increases with uncertainty, and partly due to the real options result that in a more uncertain economic environment the firm has a higher incentive to wait for more information before undertaking the investment (see Dixit and Pindyck (1994)).

Figure 3.4 shows that the capacity size is higher for the firm selling two products with a low substitutability rate and high profitability difference. The difference in capacity size between 'Case: L- H' and the other cases gets more significant for higher uncertainty. First the difference in capacity of the 4 cases is not so large while the profit is already 50% higher (see results in Table 3.3). While the difference in profit remains approximately constant, the difference in capacity is growing more than proportionally. This confirms the intuition that the high capacity size result is not just driven by the general argument of 'investing later in more capacity'. It is strengthened by the high value of product flexible capacity for a firm that produces two almost equally profitable products with a low substitutability rate, which increases its willingness to invest in a high capacity level. See Panel A of Figure 3.3 that shows the high demand range in which the firm will benefit from product-flexible capacity by producing both products for a wide demand range ($\theta \in [0, \hat{\theta})$) and suspend production of product B only facing extremely high demand.

Table 3.1 shows the effect of profitability on the optimal investment strategy keeping the substitutability rate constant and low ($\gamma = 0.1$). Observe that the optimal capacity size and the expected profit of the project are increasing in profitability of product B. The non monotonic effect of α on the investment threshold is striking. For a graphical illustration of the non-monotonic behavior of investment timing see Figure 3.5. This result is driven by two contrary effects: on the one hand the firm invests later in more capacity while on the other hand higher value of the project would lead the firm to invest earlier. The capacity effect is stronger for low profitability parameters while the effect of higher project value dominates for cases of two almost equally profitable products.

Figure 3.5 shows the optimal investment thresholds for the situation when

Figure 3.4: Optimal Investment Strategy comparing the following Cases: ′**Case H - L**′: $\alpha = 0.9$ and $\gamma = 0.1$. ′**Case H - H**′: $\alpha = 0.9$ and $\gamma = 0.9$. ′**Case L - L**′: $\alpha = 0.1$ and $\gamma = 0.1$. ′**Case L - H**′:$\alpha = 0.1$ and $\gamma = 0.9$. ***Panel A:*** Optimal Investment Threshold θ^* as a function of demand volatility σ; ***Panel B:*** Optimal Capacities invested in $K^*(\theta^*)$, as a function of volatility σ. (Parameter values: $r = 0.1$, $\mu = 0.02$ and $c_F = 100$)

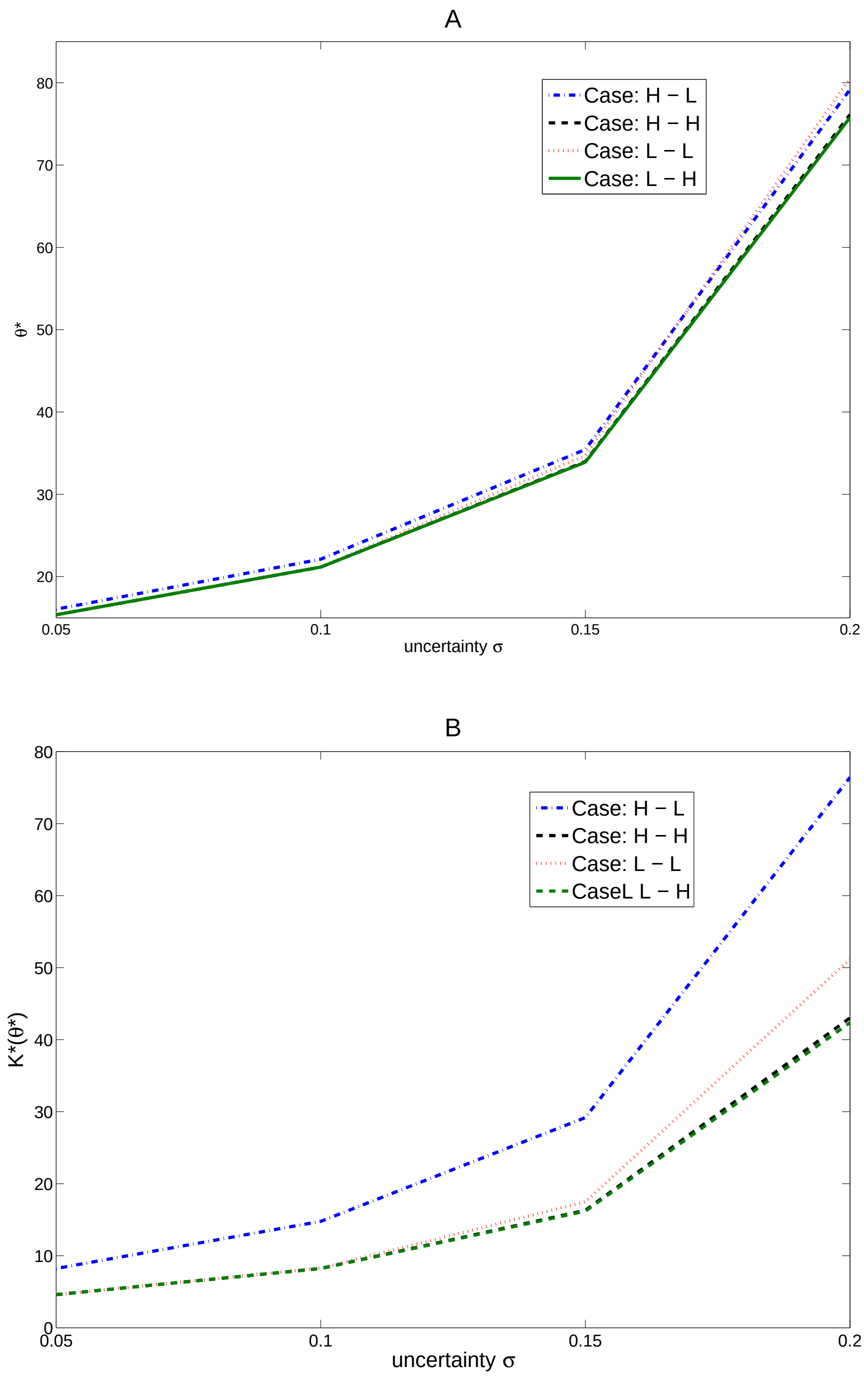
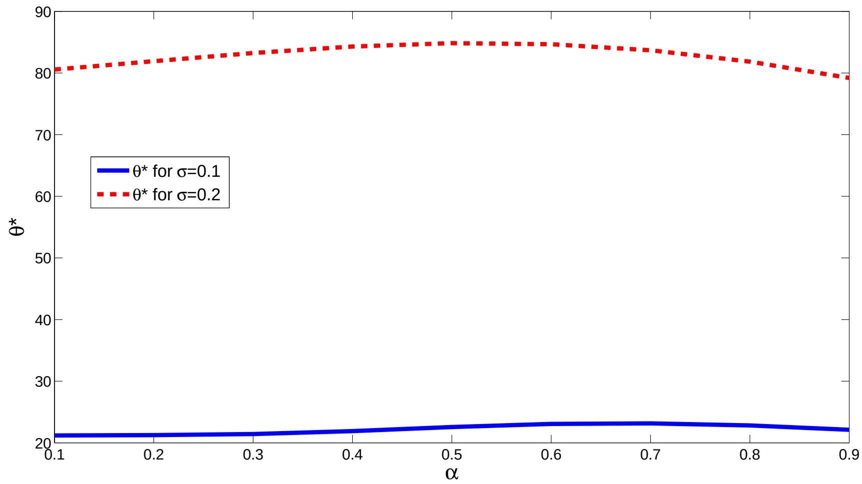


Table 3.1: Shows the optimal investment strategy for changing profitability parameter α and fixed substitutability parameter $\gamma = 0.1$. The expected profit is discounted back to an initial demand value $\theta_0 = 10$ for reasons of comparison. The left Panel shows the results for the case of uncertainty $\sigma = 0.1$, the right one for $\sigma = 0.2$. (Parameter Values: $r = 0.1$, $\mu = 0.02$ and $c_F = 100$)

α	θ_F	K_F^*	region	Π_F	θ_F	K_F^*	region	Π_F
0.1	21.19	8.28	II	60.75	80.55	51.17	I	204.14
0.2	21.25	8.36	II	60.83	81.91	54.13	I	208.05
0.3	21.42	8.61	I	61.04	83.23	57.64	I	213.72
0.4	21.90	9.35	I	61.75	84.28	61.48	I	221.67
0.5	22.56	10.61	I	63.64	84.83	65.41	I	232.46
0.6	23.06	12.06	I	67.42	84.67	69.12	I	246.71
0.7	23.15	13.33	I	73.58	83.68	72.34	I	265.04
0.8	22.81	14.25	I	82.50	81.84	74.83	I	288.15
0.9	22.11	14.78	I	94.60	79.18	76.43	I	316.85

Figure 3.5: Optimal Investment Threshold as a function of profitability parameter α (Parameter values: $\gamma = 0.1$, $r = 0.1$, $\mu = 0.02$ and $c_F = 100$)



the substitutability rate of the two products is low and the profitability of product B changes from low (0.1) to high (0.9). The optimal investment threshold is increasing in α for low value of α and decreasing for high values of α. This effect is stronger for environment with high demand volatility. Capacity size and expected profit are both increasing in α.

3.3.2 Dedicated Capacity Investment

Deriving the optimal investment thresholds for dedicated capacity, it is surprising that for most cases the firm will decide to purchase just one dedicated capacity for the more profitable product and fully ignores demand for product B. Table 3.2 shows the optimal investment thresholds for a specific parameter choice, comparing the previously introduced four cases. For the case of almost equally profitable products with a low substitutability rate the firm purchases significantly more dedicated capacity for product A than for the (slightly) less profitable product B. For all other cases the firm would commit itself at the moment of investment to have just one capacity for product A at its disposal forever. To build intuition for this result, note that in the three latter cases demand would have to fall very low so that the firm can actually gain from the possibility to make the total market size bigger with producing the second (less profitable) product B. For most demand intercept realizations it can gain

Table 3.2: Investment Strategies of Dedicated Capacity investment for the previously introduced cases (description see figure 3.4). (Parameter values: $L = 0.1$, $H = 0.9$, $r = 0.1$, $\mu = 0.02$ and $c_D = 100$)

CASE: $\alpha = H$, $\gamma = L$

σ	θ^*	$K^*_{D,A}(\theta^*)$	$K^*_{D,B}(\theta^*)$	$K^*_D(\theta^*)$
0.05	16.0587	4.6802	3.56501	8.24522
0.1	22.1392	8.17342	6.63597	14.8094
0.15	35.5252	15.8636	13.3966	29.2602
0.2	79.4644	41.1065	35.5881	76.6946

CASE: $\alpha = H$, $\gamma = H$; **Case:** $\alpha = L$, $\gamma = L$; **and CASE:** $\alpha = L$, $\gamma = H$

σ	θ^*	$K^*_{D,A}(\theta^*)$
0.05	15.3644	4.60274
0.1	21.1472	8.21699
0.15	33.8987	16.1867
0.2	75.7771	42.3607

highest profit satisfying just the demand for product A. The threat of possible overcapacity of product B dominates the value of the downside potential to increase total market demand by producing both products for low demand realizations.

Only the firm that faces a product combination of two similarly profitable products with a low substitutability rate can profit from this downside potential at a wide range of demand.

3.4 Value of Flexibility

One of the main objectives of this paper is to quantify the value of flexible capacity. In order to derive the flexibility value, the situation in which a firm relies on maximally two dedicated capacities rather than on one flexible capacity is considered as a benchmark. The two optimal investment strategies for flexible and dedicated capacity investment are compared assuming that the unit investment cost of flexible and dedicated capacity are equally high, i.e. $c_D = c_F =: c$. The value of flexibility is therefore given by the difference in expected profit of the flexible and dedicated capacity investment strategies.

In order to compare two investment strategies that have different optimal moments of investment, one needs to compare the discounted expected project values. Assuming the optimal investment thresholds derived in Section 3.2, the expression of the value of flexibility is equal to:

$$V_f = \left(\frac{\theta_0}{\theta_F^*}\right)^{\beta_1} V\left(\theta_F^*, K_F^*(\theta_F^*)\right) - \left(\frac{\theta_0}{\theta_D^*}\right)^{\beta_1} V\left(\theta_D^*, K_D^*(\theta_D^*)\right), \tag{3.34}$$

where the expected project values are discounted back to the beginning of the considered time period with the initial demand intercept θ_0.

Table 3.3 shows the results of the value of flexibility (V_f) for the numerical example presented in the previous section. The discounted expected project values, denoted by Π_i for $i = D, F$ are discounted back to the beginning of the considered time period, where the demand level is given by $\theta_0 = 10$. Furthermore, the relation of expected profit of dedicated investment to flexible investment, i.e. $\frac{\Pi_D}{\Pi_F}$ is given.

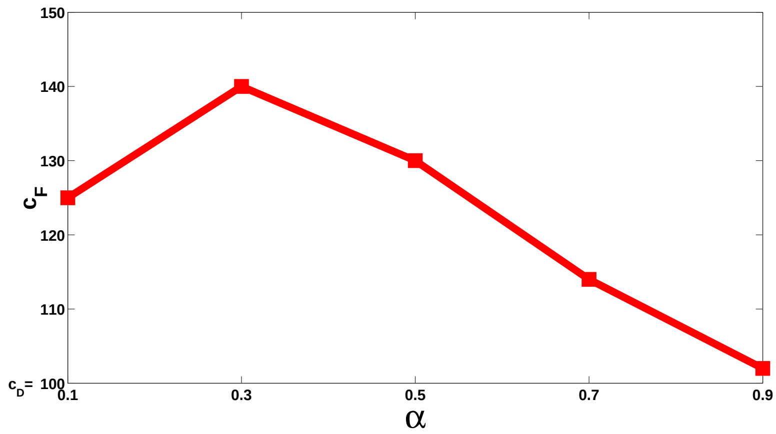
Table 3.3 shows that the value of flexibility is most significant if uncertainty is high, substitutability low, and profit levels between the two products are substantially different (see Case $\alpha = L$, $\gamma = L$). Assuming demand uncertainty of $\sigma = 0.2$, substitutability parameter γ equal to 0.1 and low profitability of product B compared to product A, the value of flexibility is substantially higher than for the other cases. Flexibility especially pays off in this case because it allows the firm to avoid over-capacity by increasing the market size including the less profitable product in production for cases of low demand, while a dedicated firm restricts itself in the optimal case to just one capacity for the more profitable product. As the optimal investment capacity for the flexible firm is significantly higher ($K_F^* = 51.17$) than for the dedicated firm ($K_D^* = 42.36$), the flexible firm is additionally less threatened by under-capacity in high demand periods.

Table 3.4 shows a wider range of profitability and substitutability parameter combinations. Considering the previous result, that flexibility is most valuable in case of high demand uncertainty, σ is chosen to be equal to $\sigma = 0.2$. Three cases, of low ($\gamma = 0.1$), medium ($\gamma = 0.5$) and high ($\gamma = 0.9$) product substitutability, are shown. The profitability parameter is ranging from a value close to zero ($\alpha = 0.1$) to a value close to one ($\gamma = 0.9$). As expected the value of flexibility is highest in case of low substitutability and low levels of the profitability parameter, specifically $\alpha = 0.1, 0.3$ and 0.5. Increasing α from 0.1 to 0.5, the difference in adopted capacity between the flexible and the dedicated case ($K_F^* - K_D^*$) is high and decreasing in α. This indicates, that the

Table 3.3: Optimal Profits of flexible and dedicated investment, respectively, discounted back to the initial demand level $\theta_0 = 10$ comparing four cases (description of these cases can be found in caption of Figure 3.4).
Π_F (Π_D)...discounted expected profit of the flexible (dedicated) capacity. Expression V_f denotes the **Value of Flexibility**. (Parameter values: $L = 0.1$, $H = 0.9$, $r = 0.1$, $\mu = 0.02$ and $c = 100$)

σ	θ_F^*	K_F^*	Π_F	θ_D^*	$K_{D,A}^*$	$K_{D,B}^*$	K_D^*	Π_D	V_f	$\frac{\Pi_D}{\Pi_F}\%$
CASE: $\alpha = H, \gamma = L$										
0.05	16.05	8.23	52.67	16.06	4.68	3.57	8.25	52.59	0.09	99.83
0.1	22.11	14.78	94.60	22.14	8.17	6.64	14.81	94.38	0.	99.77
0.15	35.44	29.18	172.06	35.53	15.86	13.40	29.26	171.49	0.57	99.67
0.2	79.18	76.43	316.85	79.46	41.11	35.59	76.69	315.41	1.44	99.55
CASE: $\alpha = H, \gamma = H$										
0.05	15.36	4.60	35.30	15.36	4.60	0	4.60	35.30	0	100
0.1	21.15	8.22	60.69	21.15	8.22	0	8.22	60.69	0.01	99.99
0.15	33.97	16.30	107.42	33.90	16.19	0	16.19	107.25	0.17	99.84
0.2	76.14	43.02	194.67	75.78	42.36	0	42.36	193.74	0.93	99.52
CASE: $\alpha = L, \gamma = L$										
0.05	15.36	4.60	35.30	15.36	4.60	0	4.60	35.30	0	100
0.1	21.19	8.28	60.75	21.15	8.22	0	8.22	60.69	0.07	99.89
0.15	34.71	17.49	108.98	33.90	16.19	0	16.19	107.25	1.73	98.41
0.2	80.55	51.17	204.14	75.78	42.36	0	42.36	193.74	**10.40**	**94.90**
CASE: $\alpha = L, \gamma = H$										
0.05	15.36	4.60	35.30	15.36	4.60	0	4.60	35.30	0	100
0.1	21.15	8.22	60.69	21.15	8.22	0	8.22	60.69	0	100
0.15	33.90	16.19	107.25	33.90	16.19	0	16.19	107.25	0	100
0.2	75.78	42.37	193.75	75.78	42.36	0	42.36	193.74	0.01	99.99

Figure 3.6: c_F as the unit cost of flexible capacity that the firm is willing to pay so that the expected project values for flexible and dedicated capacity investment are equal, i.e. $\Pi_F = \Pi_D$, assuming that $c_D = 100$. (Parameter values: $\gamma = 0.1$, $r = 0.1$, $\mu = 0.02$, $\sigma = 0.2$ and $\theta_0 = 10$.)



advantage of flexible capacity in high demand regions decreases with α. On the other hand the demand range for which the firm will produce both products, i.e. $\hat{\theta}$, increases with α, which allows the firm to profit more from the downside potential of flexible capacity. The value of flexibility is highest for $\alpha = 0.3$.

These results are for matter of comparison derived assuming that the unit cost of dedicated and of flexible capacity are equally high. However, flexible capacity is in most cases more expensive than dedicated capacity. Increasing the unit cost of flexible capacity to a level at which the expected value of flexible investment (Π_F) is equally high to the expected value of dedicated investment (Π_D), shows how much more a firm is willing to pay for flexible capacity. The results are striking. Table 3.6 shows the level of unit cost of flexible capacity for which $\Pi_F = \Pi_D$, as a function of the profitability parameter in case substitutability is low $\gamma = 0.1$ and demand uncertainty high $\sigma = 0.2$. For a high difference in profitability levels of the two products, i.e. $alpha = 0.1$, the unit cost of flexible capacity can increase by 25% compared to dedicated unit cost. In case $\alpha = 0.3$ the firm would even pay 40% more for flexible capacity.

The relatively low value of flexibility in the other product combination cases is partly driven by the fact that the firm can freely decide the optimal amount of capacity invested in for the flexible and the dedicated case, respec-

Table 3.4: Optimal Profits of flexible and dedicated investment, respectively, discounted back to the initial demand level $\theta_0 = 10$. Π_F (Π_D)...discounted expected profit of the flexible (dedicated) capacity. Expression V_f denotes the **Value of Flexibility**. $\hat{\theta}$ is the boundary as which it is optimal for the firm to change from producing both products to use full capacity just for product A. (Parameter values: $\sigma = 0.2$, $r = 0.1$, $\mu = 0.02$ and $c = 100$)

α	θ_F^*	K_F^*	$\hat{\theta}$	Π_F	θ_D^*	$K_{D,A}^*$	$K_{D,B}^*$	K_D^*	Π_D	V_f	$\frac{\Pi_D}{\Pi_F}$
CASE: $\gamma = L = 0.1$											
0.1	80.6	51.2	102	204.1	75.8	42.4	0	42.4	193.7	10.4	94.9
0.3	83.2	57.6	148.1	213.7	82.3	45.8	5.8	51.7	196.6	17.1	92.0
0.5	84.8	65.4	235.5	232.5	87.5	47.9	17.6	65.5	217.1	15.4	93.4
0.7	83.7	72.3	433.8	265.0	85.49	45.7	27.8	73.5	256.6	8.4	96.8
0.9	79.2	76.4	1375	316.9	79.5	41.1	35.6	76.7	315.4	1.5	99.5
CASE: $\gamma = M = 0.5$											
0.1	76.3	43.3	48.1	195.0	75.8	42.4	0	42.4	193.7	1.3	99.3
0.3	76.3	43.76	62.5	196.0	75.8	42.4	0	42.4	193.7	2.3	98.9
0.5	77.9	46.3	92.6	198.9	75.8	42.4	0	42.4	193.7	5.2	97.4
0.7	79.7	51.0	170	208.6	80.6	40.3	10.1	50.4	200.8	7.8	96.3
0.9	78.6	55.6	556.0	234.7	79.0	32.9	23.0	55.9	232.7	2.0	99.1
CASE: $\gamma = H = 0.9$											
0.1	75.8	42.4	9.4	193.7	75.8	42.4	0	42.4	193.7	0	100
0.3	75.8	42.4	12.1	193.8	75.8	42.4	0	42.4	193.7	0.1	99.9
0.5	75.8	42.4	17.0	193.8	75.8	42.4	0	42.4	193.7	0.1	99.9
0.7	75.8	42.4	28.3	193.8	75.8	42.4	0	42.4	193.7	0.1	99.9
0.9	76.1	43.0	86.0	194.7	75.8	42.4	0	42.4	193.7	1.0	99.5

tively, and choose to ignore the demand for the less profitable product B by just purchasing one dedicated capacity. Often, though, firms are faced with the decision whether or not to invest in flexible capacity when they already installed dedicated capacity at an earlier point of time and are currently producing with this dedicated capacity. Therefore, I evaluate this problem in the following section.

3.5 Incentive to Change from Dedicated to Flexible Capacity

Assume that a firm currently producing with two installed dedicated capacities for each of its products, A and B, can switch to flexible capacity by paying an investment cost of $c_F K_F$. In the following I will analyze for which specific product mix and the parameter ranges a firm has more or less incentive to switch from dedicated to flexible capacity. The dedicated capacities installed are given by $K_{D,A}$ and $K_{D,B}$. Let $V_D(\theta, K_{D,A}, K_{D,B})$ be the expected value of the firm producing with the installed dedicated capacities forever starting with a demand intercept θ. Similarly the value of the firm producing with flexible capacity forever is given by $V_F(\theta, K_F)$, assuming an amount of flexible capacity of K_F. The expressions for V_F and V_D are equal to the project values derived in Section 3.2 and 2.2, respectively. The threshold at which it is optimal to switch from dedicated capacity to flexible capacity, must satisfy the standard pair of conditions for optimal exercise, which are the value matching

$$\left[V_D(\theta, K_{D,A}, K_{D,B}) + D_1\theta^{\beta_1} + D_2\theta^{\beta_2}\right]\Big|_{\theta=\theta^*} = \left[V_F(\theta, K_F) - c_F K_F\right]\big|_{\theta=\theta^*}, \quad (3.35)$$

and the smooth pasting condition

$$\left[\frac{\partial}{\partial\theta}V_D(\theta, K_{D,A}, K_{D,B}) + \beta_1 D_1\theta^{\beta_1-1} + \beta_2 D_2\theta^{\beta_2-1}\right]\Bigg|_{\theta=\theta^*} = \left[\frac{\partial}{\partial\theta}V_F(\theta, K_F)\right]\Bigg|_{\theta=\theta^*}. \quad (3.36)$$

$D_1\theta^{\beta_1}$ and $D_2\theta^{\beta_2}$ are option values. Specifically, $D_1\theta^{\beta_1}$ is the value of the option of the firm to purchase flexible capacity and increase the firm's profit in case θ increases. $D_2\theta^{\beta_2}$ is the value of the option to gain profit at a later point in time by installing flexible capacity in case demand decreases. As shown in the previous section, flexibility is especially preferable in very low and very high demand regions. Therefore, we know that the function $[V_F - V_D](\theta)$ is convex in the demand intercept θ. There are two effects that may increase the firm's profit when adopting flexible technology. On the one hand it can optimally adapt the product mix to demand. This means that it can use full capacity to produce the more profitable product if demand is high. If demand is low it can increase total market demand by producing both products optimally adapting the output rates of the two products. On the other hand, the firm can also gain by investing in flexible capacity because it can update the optimal total capacity size having more actual information about the economic environment than at the moment when dedicated capacity was installed. If demand is high the firm can gain by increasing total capacity. If demand is low it can profit by downscaling total capacity. Recall that downscaling can be valuable for the firm because the firm is restricted to use full capacity for production, i.e. the capacity clearance assumption. $D_2\theta^{\beta_2}$ can therefore partly be seen as the value of a disinvestment option. However, I want to concentrate in the analysis on the additional value gained by flexibility in comparison to dedicated capacity in high demand regions. Therefore, I assume that $K_F \geq K_{D,A} + K_{D,B}$, which excludes the possibility of disinvestment from the problem. Furthermore, I rule out scenarios where it might be favorable to invest in flexible capacity with regard to its downside potential. Therefore, I set a lower boundary for the investment cost, so that the value of flexibility won't exceed the cost of investment in low demand regions. Assuming that $c_F > \hat{c}_F = \left[\frac{1-\gamma}{2r}\frac{(K_{D,A}-K_{D,B})^2}{(K_{D,A}+K_{D,B})}\right]$, it follows that $D_2 = 0$. For the derivation of this boundary I refer to Appendix D. Therefore, the value matching and smooth pasting conditions are given by

$$\left[V_D(\theta, K_{D,A}, K_{D,B}) + D_1\theta^{\beta_1}\right]\Big|_{\theta=\theta^*} = \left[V_F(\theta, K_F) - c_F K_F\right]\big|_{\theta=\theta^*}, \tag{3.37}$$

$$\left[\frac{\partial}{\partial\theta}V_D(\theta, K_{D,A}, K_{D,B}) + \beta_1 D_1\theta^{\beta_1-1}\right]\Bigg|_{\theta=\theta^*} = \left[\frac{\partial}{\partial\theta}V_F(\theta, K_F)\right]\Bigg|_{\theta=\theta^*}. \tag{3.38}$$

The following proposition states the implicit equations for the investment trigger for both regions.

Proposition 3 *Consider the firm's policy to switch to flexible capacity K_F, while currently producing with two dedicated capacities $K_{D,A}$ and $K_{D,B}$. Assuming that $K_F \geq K_{D,A} + K_{D,B}$, the optimal investment trigger is implicitly determined by the following equations.*

$$\begin{aligned}\left(\frac{\beta_1-2}{\beta_1}\right)a_1\theta^{*2} \quad &+ \quad \left(\frac{\beta_1-1}{\beta_1}\right)a_2\theta^* K_F + a_3 K_F^2 - \left(\frac{\beta_1-1}{\beta_1}\right)\frac{\theta^*}{r-\mu}\left[K_{D,A} + \alpha K_{D,B}\right] \\ &+ \quad \frac{K_{D,A}^2 + 2\gamma K_{D,A}K_{D,B} + K_{D,B}^2}{r} = c_F K_F, \end{aligned} \tag{3.39}$$

for region I (i.e. $\theta^* < \frac{2(1-\gamma)K_F}{1-\alpha}$*) and*

$$\left(\frac{\beta_1-\beta_2}{\beta_1}\right) B_2\theta^{*\beta_2} + \left(\frac{\beta_1-1}{\beta_1}\right)\frac{\theta^* K_F}{r-\mu} - \frac{K_F^2}{r} - \frac{\theta^*}{r-\mu}\left[K_{D,A}+\alpha K_{D,B}\right] + \frac{K_{D,A}^2+2\gamma K_{D,A}K_{D,B}+K_{D,B}^2}{r} = c_F K_F, \quad (3.40)$$

for region II (i.e. $\theta^* \geq \frac{2(1-\gamma)K_F}{1-\alpha}$*).*

Table 3.5 presents the optimal investment thresholds for different product combinations. The results in Table 3.5 are derived assuming parameter values $r = 0.1$, $\mu = 0.02$, unit cost of flexible capacity $c_F = 10$ and two dedicated capacities for each product of size $K_{D,A} = 8$ and $K_{D,B} = 6$, respectively. Demand volatility ranges from $\sigma = 0.05$, 0.1, 0.15 to 0.2. As in the previous sections $\alpha = L$ means that the difference in profitability of the two products is high, i.e. product A is much more profitable than product B. If $\alpha = H$ the two products are almost equally profitable. For the particular numerical example presented in Table 3.5, $L = 0.1$ and $H = 0.9$. The same numerical values for L and H are chosen for the substitutability parameter γ.

Regarding the optimal timing to change from dedicated to flexible capacity, I find that the investment threshold differs substantially for different product combinations. This crucial impact is surprising, given the fact that the demand of the products is assumed to be perfectly positively correlated. The firm invests earlier in flexible capacity if the profitability of the two products is substantially different (see case $\gamma = L$), while for a product mix of two almost equally profitable products, a higher demand threshold is required in order to undergo investment in flexible capacity. If demand uncertainty is equal to $\sigma = 0.1$ for example, the optimal investment threshold is equal to $\theta^* = 156.27$ for low substitutability and equal to $\theta^* = 45.63$ for high substitutability if the profit level of both products is high. If the products differ substantially in profitability, the optimal investment threshold is substantially lower and equal to $\theta^* = 17.36$ in case of low substitutability and equal to $\theta^* = 5.07$ in case of high substitutability. Since product A is much more profitable in case α is low, the firm could increase its profit substantially by rising the output rate for product A. Therefore, the value of waiting is dominated by the "cost" of lost profit the firm would face if it continues producing too much of product B in relation to the more profitable product A.

A firm that produces two products with a low substitutability rate and similar profitability, will change from two dedicated to the flexible capacity, keeping the total amount of capacity the same, only in case of high demand. The

Table 3.5: The optimal investment threshold ($\theta^*(K_F)$) to adopt flexible technology currently using dedicated capacities of amount $K_{D,A} = 8$ and $K_{D,B} = 6$ for different values of K_F. (Parameter values: $L = 0.1$, $H = 0.9$, $r = 0.1$, $\mu = 0.02$ and $c_F = 10$)

σ	$\theta^*(14)$	$q_A(\theta^*)$	$q_B(\theta^*)$	$\theta^*(15)$	$q_A(\theta^*)$	$q_B(\theta^*)$	$\theta^*(16)$	$q_A(\theta^*)$	$q_B(\theta^*)$
CASE: $\alpha = H, \gamma = L$									
0.05	143.3	10.98	3.02	34.26	8.45	6.55	27.05	8.75	7.25
0.1	156.3	11.34	2.66	37.76	8.55	6.45	29.84	8.83	7.17
0.15	170.3	11.73	2.27	42.04	8.67	6.33	33.25	8.92	7.08
0.2	184.5	12.12	1.88	46.89	8.80	6.20	37.14	9.03	6.97
CASE: $\alpha = L, \gamma = L$									
0.05	15.92	10.98	3.02	18.63	12.16	2.84	20.61	13.15	2.85
0.1	17.36	11.34	2.66	20.36	12.59	2.41	22.56	13.64	2.36
0.15	18.93	11.73	2.27	22.29	13.07	1.93	24.75	14.19	1.81
0.2	20.50	12.12	1.88	24.29	13.57	1.43	27.05	14.76	1.24
CASE: $\alpha = H, \gamma = H$									
0.05	41.38	14	0	35.25	15	0	34.63	16	0
0.1	45.63	14	0	38.83	15	0	38.15	16	0
0.15	50.68	14	0	43.14	15	0	42.41	16	0
0.2	56.20	14	0	47.96	15	0	47.21	16	0
CASE: $\alpha = L, \gamma = H$									
0.05	4.6	14	0	8.81	15	0	12.17	16	0
0.1	5.07	14	0	9.72	15	0	13.43	16	0
0.15	5.63	14	0	10.84	15	0	14.97	16	0
0.2	6.24	14	0	12.1	15	0	16.73	16	0

firm does not have a strong incentive to change to flexible capacity because dedicated capacities of size $K_{D,A} = 8$ and $K_{D,B} = 6$ are a good choice for this specific product combination and therefore, the firm cannot increase its profit a lot by changing to flexible capacity. However, if the firm would be able to increase its total capacity with investment in flexible capacity, this would accelerate the investment substantially to a threshold $\theta^* = 37.76$ for the case of $K_F = 15$ and even more to an investment trigger of $\theta^* = 29.84$ for the case of $K_F = 16$. The favorable product combination results in a large option value to increase total capacity and therefore the firm invests earlier if total capacity can be increased.

In case $\alpha = L$, i.e. product B is much less profitable than product A, the opposite holds. The possibility to increase total capacity delays investment.

There are two contrary effects that either accelerate or delay investment if the firm can increase total capacity. On the one hand, the firm can increase its profit due to larger production capacity and therefore wants to invest earlier, while on the other hand higher capacity investment increases total investment costs. The firm waits longer before investing, because a higher demand level is necessary to profit from this investment. In case $(\alpha = L, \gamma = H)$, the cost effect dominates.

The following two cases presented in Table 3.5 refer to the investment decision of a firm producing two highly substitutable goods. Due to the high substitutability of the two products, the firm tends to concentrate on the production of the more profitable product A. It strives to avoid competing with itself in the market. Therefore, it is optimal for the firm to switch to flexible capacity early in order to change the unfavorable output rate mix, to which it is restricted by the currently installed dedicated capacities. I distinguish again two cases: If one of the products is significantly more profitable than the other, the firm is eager to adapt the reinforced output rate mix and changes to flexible capacity soon. Since the market is not so large the firm prefers in the first run to optimally adapt the output rates and not to extend capacity. Therefore, the investment threshold increases with the amount of capacity K_F invested in. The investment cost effect dominates here. However, if the two products are almost equally profitable the firm faces a better market situation and therefore prefers to increase total capacity in order to increase the output rate for the more profitable product A. That leads to the fact that the investment threshold decreases with higher capacity purchase.

Furthermore, the optimal output levels at the moment of investment do not change with the profitability parameter α. Uncertainty determines the optimal output split at the moment of investment, while α determines the timing. I analyze the investment timing considering the optimal investment threshold θ^*. However, the expected time of investment is equal to the expected time the demand intercept process hits the investment threshold θ^*. Since the expected time of investment[5] is monotonic in the investment threshold for the considered parameter ranges, one can analyze the qualitative results regarding the investment timing taking into account the investment threshold θ^*.

3.6 Conclusions

This paper considers the timing and capacity choice of a firm facing stochastic demand. Two types of capacity investment are distinguished. The flexible capacity investment allows the firm to produce both products with the same production facility, while investment in dedicated capacity investment restricts the firm to produce just one product by purchased dedicated production facility. The firm makes three decisions: choice of investment time, choice of capacity, and type of capacity investment, i.e. flexible or dedicated. Concerning the timing and capacity decision I develop implicit solutions, which are investigated numerically. I show that for both flexible and dedicated capacity investment, the firm invests later in higher capacity if demand uncertainty increases. Flexibility especially pays off when uncertainty is high, substitutability low, and profit levels between the two products are substantially different. In the flexible case, under high demand the firm just produces the most profitable product, if demand is low the firm produces both products to increase total demand. In the dedicated case the firm invests in both capacities if the substitutability rate is low and profitability of both products high enough. Otherwise the firm will ignore demand for the less profitable product in the market and install just one dedicated capacity.

Considering a firm's decision to change from dedicated to flexible capacity, currently producing with dedicated one, it is shown that apart from demand volatility this decision is substantially affected by the specific product combination of the firm. The firm undertakes the change to flexible investment even in low demand scenarios in case the profitability of the two products is

[5]The expected time to hit the investment threshold θ^* starting from a level θ_0, denoted by T^*, equals

$$E\left[T^*\right] = \begin{cases} -\frac{1}{\mu - \frac{1}{2}\sigma^2} ln\left(\frac{\theta_0}{\theta^*}\right) & \text{for } \sigma^2 < 2\mu, \\ \infty & \text{for } \sigma^2 \geq 2\mu. \end{cases}$$

substantially different. The option to increase total capacity accelerates investment if demand for both products is substantially high.

Numerous extensions of this model deserve further analysis. This includes analyzing different cost structures, asymmetric demand curves, and different demand functions. Another interesting extension would be to allow multiple investments. Proceeding stepwise would add additional flexibility as the firm can respond to resolving uncertainty by choosing the investment timing individually for each step. Extending this model to a two-factor model with each products' demand intercept following a separate stochastic process, would allow to include the effect of demand correlation into the problem. This promises a challenging and interesting idea for future research.

The framework I analyzed was a single firm model. However, as applications for this model I mentioned several examples in car or food industry, which are not monopolies. Therefore, a natural next step is certainly to add strategic interactions to the problem.

3.A Appendix

Flexible Capacity

The expected project value in case of flexible capacity is given by

$$V(\theta, K_F) = \begin{cases} A_1\theta^{\beta_1} + a_1\theta^2 + a_2\theta K_F + a_3 K_F^2 & \text{for } \theta < \frac{2(1-\gamma)}{(1-\alpha)}K_F, \\ B_2\theta^{\beta_2} + \frac{\theta K_F}{r-\mu} - \frac{K_F^2}{r} & \text{for } \theta > \frac{2(1-\gamma)}{(1-\alpha)}K_F. \end{cases} \tag{3.41}$$

In order to check the second order condition for the optimal capacities I derive the second order derivative of the value function w.r.t. to capacity K_F:

$$\frac{\partial V^2(\theta, K_F)}{\partial K_F^2} = \begin{cases} \frac{\partial^2 A_1}{\partial K_F^2}\theta^{\beta_1} + 2a_3 & \text{for } \theta < \frac{2(1-\gamma)}{(1-\alpha)}K_F, \\ \frac{\partial^2 B_2}{\partial K_F^2}\theta^{\beta_2} - \frac{2}{r} & \text{for } \theta > \frac{2(1-\gamma)}{(1-\alpha)}K_F, \end{cases} \tag{3.42}$$

with the first order derivative of A_1 and B_2 w.r.t. K_F,

$$\frac{\partial A_1}{\partial K_F} = K_F^{1-\beta_1}\frac{(2-\beta_1)}{\beta_2-\beta_1}\left[\frac{2(1-\gamma)}{1-\alpha}\right]^{-\beta_1}(1-\gamma)F(\beta_2), \tag{3.43}$$

$$\frac{\partial B_2}{\partial K_F} = K_F^{1-\beta_2}\frac{(2-\beta_2)}{\beta_2-\beta_1}\left[\frac{2(1-\gamma)}{1-\alpha}\right]^{-\beta_2}(1-\gamma)F(\beta_1), \tag{3.44}$$

and the second order derivative of A_1 and B_2 w.r.t. K_F,

$$\frac{\partial^2 A_1}{\partial K_F^2} = K_F^{-\beta_1}\frac{(1-\beta_1)(2-\beta_1)}{\beta_2-\beta_1}\left[\frac{2(1-\gamma)}{1-\alpha}\right]^{-\beta_1}(1-\gamma)F(\beta_2), \tag{3.45}$$

$$\frac{\partial^2 B_2}{\partial K_F^2} = K_F^{-\beta_2}\frac{(1-\beta_2)(2-\beta_2)}{\beta_2-\beta_1}\left[\frac{2(1-\gamma)}{1-\alpha}\right]^{-\beta_2}(1-\gamma)F(\beta_1), \tag{3.46}$$

where $F(\beta) := \left[\frac{(2-\beta)}{2(r-2\mu-\sigma^2)} - \frac{(1-\beta)}{(r-\mu)} - \frac{\beta}{2r}\right]$.

Corollary 1 *The variable A_1 is negative and the variable B_2 is positive for the considered parameter ranges.*

Proof of Corollary 1 The two variables A_1 and B_2 are given by

$$\begin{aligned} A_1 &= K_F^{2-\beta_1}\frac{(1-\gamma)}{(\beta_2-\beta_1)}\left[\frac{2(1-\gamma)}{1-\alpha}\right]^{-\beta_1} \\ &\quad \left[\frac{(2-\beta_2)}{2(r-2\mu-\sigma^2)} - \frac{(1-\beta_2)}{(r-\mu)} - \frac{\beta_2}{2r}\right], \\ B_2 &= K_F^{2-\beta_2}\frac{(1-\gamma)}{(\beta_2-\beta_1)}\left[\frac{2(1-\gamma)}{1-\alpha}\right]^{-\beta_2} \end{aligned} \tag{3.47}$$

$$\left[\frac{(2-\beta_1)}{2(r-2\mu-\sigma^2)} - \frac{(1-\beta_1)}{r-\mu} - \frac{\beta_1}{2r}\right]. \tag{3.48}$$

Since it holds that $\beta_2 < 0$ and $\beta_1 > 1$, one can easily see that the left part of the expression of A_1, i.e. $\left[K_F^{2-\beta_1}\frac{1}{\beta_2-\beta_1}\left[\frac{2(1-\gamma)}{1-\alpha}\right]^{-\beta_1}(1-\gamma)\right]$, is negative, as well as the left part of the expression of B_2, i.e. $\left[K_F^{2-\beta_2}\frac{1}{\beta_2-\beta_1}\left[\frac{2(1-\gamma)}{1-\alpha}\right]^{-\beta_2}(1-\gamma)\right]$. Denote the following function $F(\beta) := \left[\frac{(2-\beta)}{2(r-2\mu-\sigma^2)} - \frac{(1-\beta)}{(r-\mu)} - \frac{\beta}{2r}\right]$. The right part of A_1 is given by $F(\beta_2)$ and the right part of B_2 is given by $F(\beta_1)$. If it holds that $F(.)$ is positive at the point β_2 it is proved that the variable A_1 is negative for the considered parameter ranges. A similar argument holds for variable B_2. If it holds that $F(.)$ is negative at the point β_1 it is proved that the variable B_2 is positive for the considered parameter ranges. Rewriting the function $F(\beta)$ by

$$F(\beta) = \left[\frac{\mu+\sigma^2}{(r-\mu)(r-2\mu-\sigma^2)}\right] - \beta\left[\frac{2\mu^2+(r+\mu)\sigma^2}{2r(r-\mu)(r-2\mu-\sigma^2)}\right], \tag{3.49}$$

one can easily see that the function is linear with a negative slope, crossing the horizontal axes at the point $\beta_0 = \frac{2r(\mu+\sigma^2)}{2\mu^2+(\mu+r)\sigma^2} > 0$. We know that $\beta_2 < 0$ and therefore $F(\beta_2)$ must be positive. It remains to show that $\beta_0 < \beta_1$ in order to verify that $F(\beta_1) < 0$. To show that $\beta_0 < \beta_1$, we evaluate the quadratic equation $Q(\beta)$ (see equation (3.11)) at the point β_0. The quadratic expression at β_0 is given by

$$Q(\beta_0) = -\frac{(r-\mu)r\sigma^4(2\mu-r+\sigma^2)}{(2\mu^2+(r+\mu)\sigma^2)^2} < 0. \tag{3.50}$$

The graph $Q(\beta)$ is an upward-pointing parabola that goes to ∞ as β goes to $\pm\infty$ (see Dixit and Pindyck (1994) about the "Fundamental Quadratic") and crosses the horizontal axis at β_1 and β_2. Since $Q(\beta_0)$ is negative we know that the relation $\beta_0 < \beta_1$ must hold. And therefore, it is shown that $F(\beta_1) < 0$. □

Dedicated Capacity

Derivations for Proposition 2:

1. **Two Capacities Case:** The optimal capacity choice is derived by maximizing the project value minus investment cost. Therefore, one should derive

$$\frac{\partial(V(\theta, K_{D,A}) - c_D K_{D,A})}{\partial K_{D,A}} = 0. \tag{3.51}$$

Solving the value-matching and smooth pasting conditions assuming that the capacity is given as function of the demand intercept θ that solves equation (3.51), leads to the optimal investment thresholds. For region $\hat{\theta}_A < \theta < \hat{\theta}_B$ this results in the following two thresholds:

$$\theta^*_{1,1} = \frac{c_D(r-\mu)(1-\gamma)}{(1-\alpha\gamma)}, \tag{3.52}$$

$$\theta^*_{1,2} = \frac{\beta_1 c_D(r-\mu)(1+\gamma-2\gamma^2)}{\beta_1(1+\alpha\gamma)-2(1+\gamma^2(\beta_1-1))}. \tag{3.53}$$

Since $\theta^*_{1,1} = \hat{\theta_A}$, $\theta^*_{1,2}$ is the unique optimal investment threshold for this region.

The threshold equation for region $\theta > \hat{\theta}_B$ is given by

$$\left(\frac{\beta_1-2}{\beta_1}\right)\frac{\theta^{*2}r(1+\alpha^2-2\alpha\gamma)}{4(r-\mu)^2(1-\gamma^2)} - \left(\frac{\beta_1-1}{\beta_1}\right)\frac{\theta^* c_D r(1+\alpha)}{2(r-\mu)(1+\gamma)} + \frac{rc_D^2}{2(1+\gamma)} = 0. \tag{3.54}$$

One can derive the following two solutions of equation (3.54)

$$\theta^*_1 = \frac{c_D(r-\mu)(1+\alpha)(1-\gamma)}{(1-2\alpha\gamma+\alpha^2)}\left(\frac{\beta_1-1}{\beta_1-2}\right) + \frac{2(r-\mu)^2(1-\gamma^2)}{r(1-2\alpha\gamma+\alpha^2)(\beta_1-2)} \sqrt{\frac{r^2c_D^2(1+\alpha)^2(\beta_1-1)^2}{4(r-\mu)^2(1+\gamma)^2} - \frac{r^2c_D^2(1-2\alpha\gamma+\alpha^2)\beta_1(\beta_1-2)}{2(r-\mu)^2(1-\gamma^2)(1+\gamma)}}, \tag{3.55}$$

$$\theta^*_2 = \frac{c_D(r-\mu)(1+\alpha)(1-\gamma)}{(1-2\alpha\gamma+\alpha^2)}\left(\frac{\beta_1-1}{\beta_1-2}\right) - \frac{2(r-\mu)^2(1-\gamma^2)}{r(1-2\alpha\gamma+\alpha^2)(\beta_1-2)} \sqrt{\frac{r^2c_D^2(1+\alpha)^2(\beta_1-1)^2}{4(r-\mu)^2(1+\gamma)^2} - \frac{r^2c_D^2(1-2\alpha\gamma+\alpha^2)\beta_1(\beta_1-2)}{2(r-\mu)^2(1-\gamma^2)(1+\gamma)}}. \tag{3.56}$$

Since $\hat{\theta}_B > \theta^*_2$, θ^*_1 is the investment threshold of this region.

2. **One Capacity Case:** The optimal capacity choice is derived by maximizing the project value minus investment cost. Therefore, one should derive

$$\frac{\partial(V(\theta, K_{D,A}) - c_D K_{D,A})}{\partial K_{D,A}} = 0, \tag{3.57}$$

resulting in an optimal capacity choice K^*_F for given demand intercept θ of

$$K^*_{D,A}(\theta) = \frac{r}{2(r-\mu)}\left[\theta - (r-\mu)c_D\right]. \tag{3.58}$$

The project value assuming that the capacity is chosen optimally is given by

$$V(\theta, K^*_{D,A}) - c_D K^*_{D,A} = \frac{r}{4(r-\mu)^2}\theta^2 - \frac{r}{2(r-\mu)}c_D\theta + \frac{r}{4}c_D^2. \tag{3.59}$$

The value of the option is given by

$$F(\theta) = A_1\theta^{\beta_1}. \tag{3.60}$$

Value matching and smooth pasting the project value with the value of the option results in the following investment threshold

$$\theta^* = \left(\frac{\beta_1}{\beta_1 - 1}\right)(r-\mu)\left[\frac{K_D}{r} + c_D\right]. \tag{3.61}$$

For further explanation for the derivation of the investment threshold by value-matching and smooth pasting see Dixit and Pindyck (1994). Combining equation (3.58) and (3.61) gives the optimal investment threshold θ^*_D shown in equation (3.33). The optimal capacity choice at the moment of investment is given by

$$K^*_D(\theta^*) = \frac{1}{(\beta_1 - 2)} r c_D. \tag{3.62}$$

Expected Present Value

The formula for $E[e^{-rT}]$, when θ follows the geometric Brownian motion presented in equation (3.3), and T is the random first time the process reaches a fixed level $\bar{\theta}$ starting from the general initial position θ_0, is given by

$$E[e^{-rT}] = \left(\frac{\theta_0}{\bar{\theta}}\right)^{\beta_1}. \tag{3.63}$$

where T is the (random) first time when process θ reaches $\bar{\theta}$. See e.g. Dixit and Pindyck (1994) for further explanation.

Incentive to Change from Dedicated to Flexible Capacity

In order to rule out the scenarios where the downside potential of flexibility exceeds the effect of investment cost, one needs to impose a downside boundary on the unit investment cost c_F. The function $[V_F - V_D](\theta)$ is convex in the demand intercept θ. Therefore, one can conclude that there is no downside potential of flexibility present, if the following holds

$$[V_F - V_D]\,(\theta = 0) - c_F K_F < 0, \tag{3.64}$$

$$a_3 K_F^2 + \frac{K_{D,A}^2 + 2\gamma K_{D,A} K_{D,B} + K_{D,B}^2}{r} - c_F K_F < 0. \tag{3.65}$$

Since I assume that $K_F \geq K_{D,A} + K_{D,B}$, the condition (3.64) has to be fulfilled at $K_F = K_{D,A} + K_{D,B}$. This holds for

$$c_F > \frac{(1-\gamma)}{2r} \frac{[K_{D,A} - K_{D,B}]^2}{[K_{D,A} + K_{D,B}]}. \tag{3.66}$$

The boundary is denoted by $\hat{c}_F := \frac{(1-\gamma)}{2r} \frac{[K_{D,A} - K_{D,B}]^2}{[K_{D,A} + K_{D,B}]}$.

Proofs

Proof of Proposition 1

The procedure of this proof is analogous to the steps in the proof of Proposition 1 presented in the Appendix of Chapter 2. For certain choices of the parameters, it may happen that equations (3.14) or (3.17) do not have an admissible solution, meaning that the function $V(\theta, K_F) - c_F K_F)$ is monotonic in K_F in region I or II. If this is the case, then $V(\theta, K_F) - c_F K_F)$ shows an increasing or a decreasing behavior in K_F. However, if equations (3.14) or (3.17) have an admissible solution, then it means that the function $[V(\theta, K_F) - c_F K_F)]$ is concave in that region, and therefore has a local maximum. One has to take into account the local behavior of $[V_i(\theta, K_F) - c_F K_F)]$ if it is monotonic K_F. For region I, one can rule out the decreasing case, because it means that the optimal capacity level K_F would be zero, and thus the output rate would be zero. Therefore, if $[V(\theta, K_F) - c_F K_F)]$ is increasing in K_F, one should take the maximum allowed capacity level for this region, i.e. $\theta \frac{(1-\alpha)}{2(1-\gamma)}$. Similarly, for region II, one should rule out the increasing case, because it would mean that optimally an infinite capacity level should be chosen. In the decreasing case, one should choose the minimum allowed capacity for this region, which is $\theta \frac{(1-\alpha)}{2(1-\gamma)}$.

Proof of Proposition 3

The equations that implicitly determine the investment threshold θ^* are derived combining the value matching and smooth pasting conditions shown in equations (3.37) and (3.38).

CHAPTER 4

Optimal Technology Adoption when the Arrival Rate of New Technologies Changes[1]

This chapter contributes to the literature of technology adoption. In most of these models it is assumed that after the arrival of a new technology the probability of the next arrival is constant. This chapter extends this approach by assuming that after the last technology jump the probability of a new arrival can change. Right after the arrival of a new technology the intensity equals a specific value that switches if no new technology arrival has taken place within a certain period after the last technology arrival. We look at different scenarios, dependent on whether the firm is threatened by a drop in the arrival rate after a certain time period or expects the rate of new arrivals to rise.

We analyze the effect of variance of time between two consecutive arrivals on the optimal investment timing and show that larger variance accelerates investment in a new technology in case the arrival rate increases if no new arrival takes place within a specific time period after the last arrival. For the case that the arrival rate is supposed to decrease, increasing variance has a non-monotonic effect on investment timing. We find that firms often adopt a new technology a time lag after its introduction, which is a phenomenon frequently observed in practice. Regarding a firm's technology releasing strategy we explain why clear signals set by regular and steady release of new product generations stimulates customers buying behavior. Depending on whether the arrival rate is assumed to change or be constant over time, the optimal

[1]This chapter is based on (Hagspiel et al., 2011, b).

technology adoption timing changes significantly. In a further step we add an additional source of uncertainty to the problem. We assume that the length of the time period, after which the arrival intensity changes, is not known to the firm in advance. Here, we find that increasing uncertainty accelerates investment, a result that is opposite to the standard real options theory.

4.1 Introduction

The trend in innovation arrivals regarding capabilities of digital electronic devices is strongly linked to the famous statement of Gordon E. Moore about computing hardware. He described that 'the number of transistors that can be placed inexpensively on an integrated circuit doubles approximately every two years'. Observed over several periods of decades we see that several measures of digital technology are improving at exponential rates related to Moore's law, including the size, cost, density and speed of components. Processing speed, memory capacity and sensors, all of these are improving at (roughly) exponential rates as well. His law is now used in the semiconductor industry to guide long-term planning and to set targets for research and development. On the other hand, managers have to consider that this technological innovation progress has natural boundaries. At some point the physical possibilities of improvement are exhausted. This means that the rate of improvements approaches zero at some point. Ignoring this fact, and instead assuming that technological improvements evolve at exponential rates forever, would lead to crucial mistakes in a firm's technology adoption decisions. However, the literature of technology adoption widely assumes that arrival rates for technological innovations are constant. This assumption has been made among others in McCardle (1985), Farzin et al. (1998) and Huisman (2001). In this paper we want to relax this assumption and assume that the arrival rate of technological innovation changes.

Another example of a technology adoption decision, where arrival rates should not be considered as constant, is typical for the consumer electronics industry. Big companies like Apple, release new improved versions of their most popular electronic devices on a regular basis. Apple released, for example, a first version of the iPod Mini in February 2004 and announced a second generation of the iPod Mini in February of the following year. In September 2005 Apple officially discontinued the iPond Mini line and replaced it by the iPod Nano. The first generation of iPod Nano was replaced by the second generation in September 2006. From then on Apple updated the iPod Nano on a regular basis every year[2]. From the point of view of the consumer this means that right after the release of a new iPod Nano generation, one does not expect a new release soon. However, after one year without any release, one would expect Apple to announce a new series of the product line. In this paper we raise the question of how such a clear product release policy would influence the consumer's consumption behavior.

In this paper, we investigate the role of a firm that decides about technology adoption with an investment to change from old to new technology facing uncertain improvement size and timing of future technology improvements. The technological process advances exogenously to the firm. A large fixed cost occurs upon the adoption, which becomes a sunk cost because technology choice is irreversible. We assume that the arrival rate is changing over time, unlike the assumption of constant arrival prevalent in the technology adoption literature. Specifically, we assume a specific arrival rate to be present right after the last technology arrival. This arrival rate changes (increase/decrease) to another value if no arrival should take place for a certain time period (of Δ time units) since the last technology arrival.

We do not impose a specific ordering of these two values of the arrival rate, which allows to look at two different scenarios that constitute for different technology adoption environments faced by firms or customers. The first scenario gives attention to technology adoption decisions when the arrival of a new innovation would increase in case a certain time period has passed. Second, we analyze the optimal investment behavior of firms that are aware of the fact that the rate of a new arrival could decrease.

Specifically, we study the technology adoption decision of a single firm in the absence of strategic considerations. We describe the firm's adoption decision as the solution of an infinite horizon dynamic programming problem in a continuous time setting. The firm decides about the optimal moment to adopt a new technology, while it currently uses a less efficient technology. The improvement in the value of the available technology follows a compound Poisson process and evolves exogenously to the firm. Technologies become more valuable over time. At each moment in time the firm learns whether an innovation occurs or not.

We focus our analysis on the optimal adoption strategies of a firm facing changing arrival rate, also in comparison to the results of a constant arrival rate model. We introduce possible applications and analyze numerical exam-

[2]See $http://en.wikipedia.org/wiki/Ipod\#Timeline\_of\_iPod_models$ for a graphical illustration of the time line of iPod models and the releases of improved versions from 2001 on.

ples suited to those in order to gain more insight about the optimal timing strategy for technology adoption in specific scenarios. Generally, we show that a firm expecting a decrease in the arrival rate after a certain period without any arrival, should apply a different adoption timing strategy than a firm assuming constant arrival rate forever. In case a firm is confronted with a low arrival rate right after the last technology arrival while it expects an increase in the arrival rate after a certain period of time without any arrival, it should invest later than a firm that in expectation assumes the same but constant arrival intensity.

Introducing a model with changing arrival rate allows us to explain the fact that firms often adopt new technology a time lag after its arrival, while models with constant arrival rate neglect this phenomenon. As criticized by Cho and McCardle (2009) and Doraszelski (2004), "due to the memoryless and stationary probability distribution (resulting from the fact that the technology process advances according to a compound Poisson process), Huisman (2001)'s model does not explain a time lag between the occurrence of an innovation and its adoption." Extending Huisman's model to changing arrival rate we can explain in which cases the firm optimally adopts a new innovation a time lag after the innovation took place. In fact our analysis shows that the probability of a time lag between innovation and adoption is substantially high.

Furthermore, we analyze the effect of the variance of time between two consecutive technology innovations on the firm's adoption decision keeping the expected time between two technology arrivals fixed. The introduction of a changing arrival rate to the basic problem introduced by Huisman (2001) adds to the significance of our results by one additional degree of freedom. Which effect increasing variance has on the technology adoption decision depends on the scenario considered: In case the arrival rate would increase if no new arrival takes place within a certain time period after the last technology arrival, we find that increasing variance decelerates technology adoption. Regarding a firm's product release strategy we show that it can accelerate the buying behavior of its customers by following a consistent release schedule. Considering the other case, (i.e. the arrival rate would decrease if no new arrival happens for a period of time while it is relatively high right after the last technology arrival,) increasing variance has a non-monotonic effect on technology adoption timing. Furthermore, we find that increasing variance decreases the probability that the firm will a adopt better technology a time lag after it has been introduced.

In a further step we generalize the problem so that the moment at which the arrival rate switches is not known to the firm beforehand but assumed to be stochastic. This generalization leads to the result that increasing uncertainty accelerates investment.

The remainder of this paper is organized as follows. In Section 4.2 we review the related literature in technology adoption models. A short summary of the standard model with constant arrival rate follows in Section 4.3 before we present our model with changing arrival rate and characterize the optimal investment policy. Section 4.4 continues with comparative statics of the optimal adoption policy and presents two numerical examples. In Section 4.5 we extend our analysis to the fact that the moment of change in arrival rate is not known by the firm but uncertain. Section 4.6 concludes.

4.2 Related Literature

Our work extends the literature on technology adoption, especially by improving the modeling of the technological progress. While in most models it is assumed that the probability of the next technology arrival is constant, we consider the technology adoption problem of a firm facing a changing arrival probability.

Therewith, we add to a strand of papers that started with early work of Baldwin (1982), Balcer and Lippman (1984) as well as McCardle (1985). For an extensive survey about decision theoretic models of technology adoption see Huisman (2001) and Hoppe (2002).

Balcer and Lippman (1984) study the optimal time to adopt the best currently available technology when multiple adoptions are allowed. In their model, timing and the value of future innovations is uncertain although the profitability of the currently available technology is known. They show that it is optimal to adopt the best currently available technology if the technological lag exceeds a certain threshold, which depends upon the elapsed time since the last innovation and the pace of the technological progress. McCardle (1985) concentrates on the fact that the profitability of a new technology is rarely known with certainty at its announcement date. The firm can sequentially gather information, updating its prior estimate of profitability in a Bayesian manner. He derives two thresholds where crossing the upper threshold the firm stops collecting information and adopts technology. The lower threshold suggests the rejection of the technology. McCardle shows that it is optimal for the firm to continue to collect information until its estimate of profitability crosses one of two thresholds. Crossing the upper threshold the firm will adopt technology while it will reject if the lower threshold is crossed. He finds that even firms that behave optimally will occasionally adopt unprofitable technologies and reject profitable ones.

Doraszelski (2004) distinguishes between generation of a new technology and its further improvement and assumes that the occurrence of the next improvement depends on the time elapsed since the previous innovation. Doraszelski concludes that the firm may have an incentive to delay adoption of a new technology until it is sufficiently advanced. Also Alvarez and Stenbacka (2001) introduce a model dealing with the possibility to improve the incumbent technology. They study the optimal time to adopt an incumbent technology, incorporating as an embedded option a technologically uncertain prospect of opportunities for updating the technology in response to the emergence of future superior versions of the technology. They derive the optimal adoption timing facing technological uncertainty in combination with uncertainty of future market conditions.

Empirical econometric work on technology adoption goes back to the 1950s with Griliches (1957). Our approach to extend the technology adoption literature is closest related to Farzin et al. (1998) and Huisman (2000). Huisman (2001) introduced an approach to technology adoption timing decisions in a real options context and extended the traditional decision theoretic models on technology adoption with a model in which the technologies arrive according to a Poisson process. We briefly explain Huisman's basic model, which forms the starting point for our work, in Section 3.1. Huisman studies the optimal time to irreversibly switch to a new technology when the value and the arrival date of future improvements are uncertain. Technology advances according to a compound Poisson process with constant arrival rate. As in Huisman we model the technological process with a Poisson process but assume that the arrival rate changes.

Two recent papers dealing with technology adoption are Kwon (2010) and Cho and McCardle (2009). Kwon (2010) focuses upon investment and exit decisions of a firm facing a declining profit stream. The firm can continue operation, exit or use the one time option to undertake an investment that boosts the project's profit rate. This leads to the results that in case of a sufficiently large profit boost upon investment, this investment threshold decreases in volatility.

Cho and McCardle (2009) investigate the role of economic and stochastic technological dependence on the adoption of multiple type of new technologies. They show that the dependence among different types of technologies has a material impact on a firm's adoption decisions. This impact is not unidirectional, it can either delay or expedite the adoption of an improved technology.

While we model the technological innovation progress with a stationary stochastic process, literature on learning in the financing of innovation shows a current modeling approach with non stationary models. An example for this approach is Bergemann et al. (2009) that consider the innovation process from the point of view of investors that provide the financing for a project. They analyze how investors make optimal dynamic investment decisions as a function of their information about failure risk and potential final value. While a larger investment flow into the project promises faster success, it is likely to reduce the efficiency of the investment. The investors can influence the speed which with a project is developed. There is uncertainty present about the true value of the project which is resolved over time. The uncertainty about the likelihood of the project's success is modeled by a non-stationary process.

4.3 Model

4.3.1 Constant Arrival Probability

In the following we briefly present the technology arrival model with constant arrival rate as presented by Huisman (2001, Chapter 2, single switch case). Huisman (2001) considers a risk-neutral firm, whose profit flow is determined by its own technology choice. $\Pi(\theta)$ denotes the profit function which is increasing in the technology level, i.e. $\frac{\partial \Pi}{\partial \theta} > 0$. The firm maximizes its value over an infinite planning horizon. The discount rate r is assumed to be constant. At the beginning of the planning horizon the firm produces with a technology whose efficiency equals $\xi_0 (\geq 0)$. ξ_t denotes the efficiency level of the technology that the firm uses at time t ($t \geq 0$). As time passes new and more efficient technologies are invented. θ_t denotes the efficiency level of the most efficient technology available at this time t. This means that $\theta_t - \xi_0$ denotes the highest possible improvement in technological efficiency available to the firm if it decides to invest at time t. It is assumed that $\{\theta_t\}$ follows the Poisson jump process:

$$d\theta_t = \begin{cases} u & \text{w.p. } \lambda dt, \\ 0 & \text{w.p. } (1 - \lambda dt), \end{cases} \tag{4.1}$$

with λ assumed to be constant. As in Huisman (2001) we will also assume in the following that the firm can not influence the innovation itself, i.e. it is assumed exogenous to the firm. Concerning the size of the jump we assume it to be constant for the moment.

The firm can adopt a new technology by paying a sunk cost $I(>0)$. The problem addressed concerns the timing of the firm's technology switch. The firm faces an optimal stopping problem, where stopping means that the firm invests and thus adopts a new technology and continuation resembles waiting with investing. As shown in Huisman (2001) there is a unique value of θ_t for which the firm is indifferent between investing and waiting. In order to derive this threshold θ^* one needs to derive the termination as well as the continuation payoff. The termination payoff is given by the firm's value at the moment that it undertakes the investment:

$$V(\xi) = \int_{t=0}^{\infty} \Pi(\xi) e^{-rt} dt - I = \frac{\Pi(\xi)}{r} - I. \tag{4.2}$$

The continuation region can (given that the jump size u is constant) be split into two parts. In the first part investing is not optimal even after the next jump, i.e. $\{\theta | \theta < \theta^* - u\}$, while in the second part investing is optimal after the next jump, i.e. $\{\theta | \theta^* - u \leq \theta < \theta^*\}$. Given ξ_0 and $\theta_t = \theta$ the value of the firm is denoted by $F(\theta, \xi_0)$. $F(\theta, \xi_0)$ is given by

$$F(\theta, \xi_0) = \begin{cases} \left(\frac{\lambda}{r+\lambda}\right)^{\frac{\theta^*-\theta}{u}} \left(V(\theta^*) - \frac{\Pi(\xi_0)}{r}\right) + \frac{\Pi(\xi_0)}{r} & \text{if } \theta < \theta^* - u, \\ \frac{\Pi(\xi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda} V(\theta + u) & \text{if } \theta^* - u \leq \theta < \theta^*, \\ V(\theta) & \text{if } \theta \geq \theta^*. \end{cases} \tag{4.3}$$

The critical investment level θ^* is found by solving the value matching condition at $\theta = \theta^*$:

$$V(\theta^*) = \frac{\Pi(\xi_0)}{r+\lambda} + \frac{\lambda}{r+\lambda} V(\theta^* + u). \tag{4.4}$$

At the threshold θ^* the firm is indifferent between investing right away (see left-hand side of equation (4.4)) and investing after the next technology arrival (see right-hand side of equation (4.4)). The value matching condition ensures the continuity of the value function at this point.

The following proposition shows that for the model with constant arrival rate the adoption threshold is increasing in the arrival rate λ for concave profit flow functions. This result is supported numerically in Huisman (2001). We provide the analytical proof for it (see Appendix B).

Proposition 1 *For concave profit function, the optimal adoption threshold is increasing in the arrival rate, i.e., if* $\theta^*(\lambda)$ *denotes the threshold when the arrival rate* *is* λ *then for* $\lambda_1 < \lambda_2$ *it follows that:*

$$\theta^*(\lambda_1) < \theta^*(\lambda_2). \tag{4.5}$$

4.3.2 Changing Arrival Rate

This work extends the constant arrival rate model assuming that the arrival rate of new technology changes, i.e. λ in equation (4.1) is not constant. We assume that right after a technology arrival has happened the arrival rate is equal to λ_1. If no arrival should happen for a time period of length Δ the arrival rate changes to λ_2. λ_2 is assumed to be strictly positive. We denote X as the time between consecutive technology arrivals:

$$X \sim Exp(\lambda_1)1_{\{X<\Delta\}} + (\Delta + Exp(\lambda_2))\,1_{\{X\geq\Delta\}}, \tag{4.6}$$

where $1_{\{A\}}$ is equal to one if A is true. Δ is set to be constant for the moment. In Section 4.5 we will extend the presented model assuming random Δ. This approach allows us to model a range of practical problems we had in mind, for example the characteristics in technological innovation process in the CPU transistor industry or the product release strategy of an electronics company, while maintaining still a certain degree of analytical tractability of the problem.

As X denotes the random variable corresponding to the time elapsed between consecutive arrivals, the expected value and variance of X are given by

$$\mathbb{E}[X] = \begin{cases} \Delta + \frac{1}{\lambda_2} & \text{for } \lambda_1 = 0, \\ \frac{1}{\lambda_1} + e^{-\Delta\lambda_1}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) & \text{for } \lambda_1 > 0, \end{cases} \tag{4.7}$$

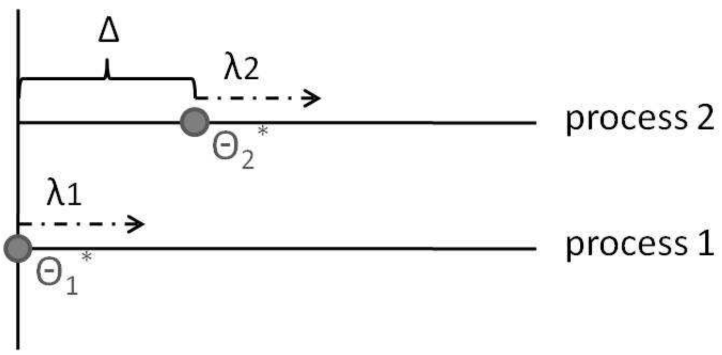
and

$$Var[X] = \begin{cases} \frac{1}{\lambda_2^2} & \text{for } \lambda_1 = 0, \\ \left(\frac{1}{\lambda_1^2\lambda_2^2}\right) e^{-2\Delta\lambda_1}\left(e^{2\Delta\lambda_1}\lambda_2^2 - (\lambda_1 - \lambda_2)^2\right. & \\ \left.+2e^{\Delta\lambda_1}\lambda_1(\lambda_1 - \lambda_2)(1 + \Delta\lambda_2)\right) & \text{for } \lambda_1 > 0. \end{cases} \tag{4.8}$$

For a more detailed explanation of the derivation of this expressions we refer to Appendix A.

In case of changing arrival rate we have to distinguish the two cases, $X < \Delta$ and $X \geq \Delta$. For matter of clearness we refer to these two cases as 'region 1' and 'region 2'. The fact that there are two regions formally means that we have a mixture of processes (one with rate λ_1 and another with rate λ_2), with

Figure 4.1: Illustration of the two processes



corresponding probability-weights $\int_0^\Delta e^{-rx}\lambda_1 e^{-\lambda_1 x}dx = 1 - e^{-(r+\lambda_1)\Delta}$ (the discounted probability that an arrival takes place before time Δ, meaning that we are in region 1) and $e^{-(r+\lambda_1)\Delta}$ (the discounted probability that there is no arrival in region 1). For the case of the process with arrival rate λ_1, the decision about the investment takes place at time $t = 0$; remark that this is exactly the constant arrival model as in Huisman (2001). Due to the constant investment cost, the exponential interarrival times, and due to discounting, the firm will adopt the technology only at the moment of arrival of new technology. For the process with rate λ_2 the reasoning is similar; but in this case the decisions can just be taken after a time period Δ, as what happens before Δ is not taken into account. Due to the lack of memory of the exponential distribution, the fact that only what happens after time Δ is taken into account, does not change the probabilistic structure of the process. And thus it can also be addressed as the constant arrival model, with an additional waiting period of length Δ before any arrival is possible.

Deriving the optimal investment moment, both processes have to be taken into account and therefore the optimal investment thresholds for each of them. In the following we will denote the process with arrival rate λ_1 as 'process 1' and therefore denote the corresponding investment threshold with θ_1^*. Similarly θ_2^* will denote the optimal investment threshold for process 2 (i.e. the process with arrival rate λ_2 and waiting period of Δ). Figure 1 illustrates the difference of the two processes and their corresponding investment thresholds.

The following proposition provides equations that implicitly determine the technology adoption thresholds θ_i^* for both processes ($i = 1, 2$).

Proposition 2 *For the derivation of the optimal technology adoption thresholds for process 1, the following holds:*

1. *If $\lambda_1 < \lambda_2$, the optimal adoption threshold for process 1, i.e. θ_1^*, is implicitly defined as the solution of the following equation*

$$\begin{aligned} V(\theta_1^*) &= \left(1 - e^{-(r+\lambda_1)\Delta}\right)\left[\frac{\Pi(\xi_0)}{r+\lambda_1} + \frac{\lambda_1}{r+\lambda_1} V(\theta_1^* + u)\right] \\ &\quad + e^{-(r+\lambda_1)\Delta} V(\theta_1^*). \end{aligned} \tag{4.9}$$

2. *If $\lambda_1 > \lambda_2$, the optimal adoption threshold θ_1^*, is implicitly defined as the solution of the following equation*

$$\begin{aligned} V(\theta_1^*) &= \left(1 - e^{-(r+\lambda_1)\Delta}\right)\left[\frac{\Pi(\xi_0)}{r+\lambda_1} + \frac{\lambda_1}{r+\lambda_1} V(\theta_1^* + u)\right] \\ &\quad + e^{-(r+\lambda_1)\Delta}\left[\frac{\Pi(\xi_0)}{r+\lambda_2} + \frac{\lambda_2}{r+\lambda_2} V(\theta_1^* + u)\right]. \end{aligned} \tag{4.10}$$

The optimal adoption threshold for process 2, i.e. θ_2^*, is implicitly determined by the following equation

$$V(\theta_2^*) = \frac{\Pi(\xi_0)}{r+\lambda_2} + \frac{\lambda_2}{r+\lambda_2} \begin{cases} V(\theta_2^* + u) & \text{for } \theta_1^* \leq \theta_2^* + u, \\ G(\theta_2^* + u) & \text{otherwise,} \end{cases} \tag{4.11}$$

where function $G(\theta)$ is given by

$$\begin{aligned} G(\theta) &= \left(1 - e^{-(r+\lambda_1)\Delta}\right)^{M(\theta)} \left(\frac{\lambda_1}{r+\lambda_1}\right)^{M(\theta)} \left(V(\theta + M(\theta)u)\right) \qquad (4.12) \\ &+ \sum_{n=0}^{M(\theta)-1} e^{-(r+\lambda_1)\Delta} \left(1 - e^{-(r+\lambda_1)\Delta}\right)^n \left(\frac{\lambda_1}{r+\lambda_1}\right)^n \left(V(\theta + nu)\right), \end{aligned}$$

and $M(\theta) = \lceil \frac{\theta_1^* - \theta}{u} \rceil$, with the convention that $\sum_0^{-1} ... = 0$.

Generally the optimal investment threshold θ^* for the model with constant arrival rate is given by the trigger at which the value of immediate investment is equal to the value of investment after the next technology arrival (for further explanations see Huisman (2001)). The same procedure can be applied to derive the optimal investment threshold in region 1. One has to find the optimal θ_1^* where the firm is indifferent between investing immediately and investing after the next arrival. The left hand side of equations (4.9) and (4.10) describes the value of immediate investment, i.e. the value of the firm when it produces with technology $\xi = \theta_1^*$ forever minus the investment cost (according to equation (4.2) or the constant arrival rate model). The right hand side consists of two terms. The first term corresponds to the profit gained by investing just after the arrival in region 1 (which occurs with probability $\left(1 - e^{-(r+\lambda_1)\Delta}\right)$). The second term describes the profit gained in case there is no arrival in region 1. Here we have to distinguish the following two cases: First, if there is no arrival in region 2 there is no further possibility to invest and therefore the firm has to continue producing with the current technology level. In case an arrival happens in region 2, which constitutes the second case, the firm will immediately invest in the new technology at the moment of arrival and therefore the discounted profit of producing with this technology forever is added to the firm's value.

Deriving the threshold for region 2, one has to look for the optimal trigger θ_2^* where the firm is indifferent between investing Δ time units after the previous technology arrival and investing after the next arrival. One has to differentiate here between the following two cases: In case $\theta_1^* \leq \theta_2^* + u$ the firm will invest right after the next arrival happened in region 2. The first line of equation (4.11) summarizes this case. Otherwise, the firm will invest as soon as region 2 is reached again (see the second line of equation (4.11)). In this case one has to take into account two scenarios:

1. Region 2 will not be reached because it is optimal for the firm to invest before in region 1. This happens if the following $M(\theta)$ arrivals take place in region 1. This happens with probability $\left[\left(\frac{\lambda_1}{r+\lambda_1}\right)^{M(\theta)}\left(1 - e^{-(r+\lambda_1)\Delta}\right)^{M(\theta)}\right]$.
2. Region 2 will be reached because there are at most $(M(\theta) - 1)$ arrivals happening in region 1 before region 2 is reached again. The probability of reaching region 2 is equal to $e^{-(r+\lambda_1)\Delta}$. The term $(1 - e^{-(r+\lambda_1)\Delta})^n$ describes the probability that there are n arrivals in region 1. The second term of function $G(\theta)$ includes all possible repetitions of arrivals in region 1 before region 2 is entered.

The following proposition describes the optimal investment strategy of the firm considering the two thresholds specified by Proposition 2:

Proposition 3 *For the optimal investment strategy the firm considers two different scenarios:*

1. *If $\lambda_1 < \lambda_2$, threshold θ_1^* is the solution of equation (4.9). The firm does not consider threshold θ_2^* because threshold θ_1^* will always be reached first.*

2. *If $\lambda_1 > \lambda_2$, threshold θ_1^* is the solution of equation (4.10). θ_2^* is given by the admissible solution of equation (4.11). Considering both thresholds, the firm will invest as soon as*

   (a) *it is in region 2 and has reached θ_2^* or*

   (b) *it is in region 1 having reached θ_1^*.*

The relation of the two arrival rate parameters λ_1 and λ_2 directly specifies the relation of the two thresholds in region 1 and region 2 as stated in the following proposition.

Proposition 4 *If $\lambda_1 > \lambda_2$ then the following relation between the two thresholds of region 1 and region 2, respectively, holds: $\theta_1^* > \theta_2^*$, with θ_1^* implicitly given by equation (4.10).*

The following corollary states that according to the optimal investment strategy it is possible that there occurs a time lag between a new technology innovation and its adoption.

Corollary 1 *If $\lambda_2 < \lambda_1$ (i.e. for the optimal investment strategy both thresholds need to be considered) it is optimal for the firm to invest Δ time units after the last technology innovation if the actual technology level has increased up to level θ_2^* and Δ time units have elapsed since the last innovation without having reached threshold θ_1^* yet.*

4.3.3 Expected Time of Technology Adoption

Knowing the optimal adoption triggers we will derive the expected value of the adoption time T^*. For reasons of comparison of the models with constant arrival rate and changing arrival rate we first state the expected time of adoption for the model with constant arrival rate as shown in Huisman (2001). Therefore, let

$$n(\theta) = \lceil \frac{\theta - \xi_0}{u} \rceil,$$

Then it follows that T^* is just the sum of $n(\theta)$ i.i.d. exponential random variables, with rate λ, and therefore

$$\mathbb{E}[T^\star] = \frac{n(\theta^*)}{\lambda}. \tag{4.13}$$

where $\theta^\star$ is the optimal investment level derived as solution of equation (4.4).

For the model with changing arrival rate, let M denote an integer value, and let $E[t^*](M)$ denote the expected time of adoption given that M jumps occurred before the adoption of the new technology. Remark that these M jumps can occur as a combination of jumps when the arrival rate is λ_1 and λ_2; in the first case the expected time between consecutive arrivals is $1/\lambda_1$, whereas in the second it is $\Delta + 1/\lambda_2$. Thus, given that M jumps occurred, we have

$$\begin{aligned}\mathbb{E}[t^*](M) &= \sum_{n=0}^{M} \binom{M}{n} \left(\frac{n}{\lambda_1} + \frac{M-n}{\lambda_2} + (M-n)\Delta \right) \\ &\quad [P(X < \Delta)]^n \, [P(X > \Delta)]^{M-n}, \end{aligned} \tag{4.14}$$

where X is the time elapsed between consecutive arrivals, so that

$$P(X < \Delta) = 1 - e^{-\lambda_1 \Delta}.$$

The following proposition gives expressions for the expected value of T^* assuming a changing arrival rate for innovation. The proof is given in Appendix B.

Proposition 5 *Let $n(\theta) = \lceil \frac{\theta - \theta_0}{u} \rceil$, that is the number of steps necessary to reach an adoption threshold θ starting at current technology level θ_0. Concerning the expected time of adoption we have the following results:*

- ***Case 1:*** *If $\lambda_1 \leq \lambda_2$ the firm will just consider adoption of a new technology immediately after a technology arrival happens, i.e. it just considers θ_1^*. Therefore, the expected time of adoption is given by the expected time of $n(\theta) = \lceil \frac{\theta - \theta_0}{u} \rceil$ technology arrivals to happen, which is given by*

$$\mathbb{E}[T^*] = E[t^*](n(\theta_1^*)). \tag{4.15}$$

- ***Case 2:*** *If $\lambda_1 > \lambda_2$ the firm will either invest as soon as investment threshold θ_1^* is reached or at the moment that threshold θ_2^* is reached and region 2 has been entered again, i.e. Δ has passed since the last technology arrival. For this case the expected time of technology adoption is given by expected time to reach threshold θ_2^* plus the expected time to either reach threshold θ_1^* or enter region 2 before. This is formally given by:*

$$\begin{aligned}\mathbb{E}[T^*] = E[t^*](n(\theta_2^*)) + \sum_{j=0}^{m^*-1} & \left(\frac{j}{\lambda_1} + \Delta \right) \ (1 - e^{-\Delta\lambda_1})^j e^{-\Delta\lambda_1} \\ + & \quad \frac{m^*}{\lambda_1}(1 - e^{-\Delta\lambda_1})^{n^*}, \end{aligned} \tag{4.16}$$

with $m^* = \lceil \frac{\theta_1^* - \theta_2^*}{u} \rceil$.

For the special case that $\lambda_1 = 0$ the expected value of the adoption time T^* is equal to

$$\mathbb{E}[T^*] = \frac{n(\theta_2^*)}{\Delta + \frac{1}{\lambda_2}}. \tag{4.17}$$

4.4 Results

In this section we analyze the optimal adoption strategy of a firm considering changing arrival rate and compare the results to the adoption strategy in case the arrival rate is constant. We will consider two specific applications of the model and analyze numerical examples suited to those. The applications relate to the cases of $\lambda_1 > \lambda_2$ and $\lambda_1 < \lambda_2$, respectively. In order to compare the model with changing arrival rate to the one with constant arrival rate in a meaningful way, we choose the parameter values in such a way that the expected time between two consecutive technology arrivals is equal for both models. This means that we set the parameters of the two models in the following relation:

$$\frac{1}{\lambda} = \begin{cases} \Delta + \frac{1}{\lambda_2} & \text{for } \lambda_1 = 0, \\ \frac{1}{\lambda_1} + e^{-\Delta\lambda_1}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) & \text{for } \lambda_1 > 0. \end{cases} \tag{4.18}$$

The derivation of the expected value of the time between two consecutive arrivals for both models is shown in Appendix A.1.

First, we consider the case of $\lambda_1 > \lambda_2$: This means that the arrival rate of new technology is assumed to decrease if a specific time period Δ has passed without any new arrival taking place. As shown in Proposition 4 we know that if $\lambda_1 > \lambda_2$ the threshold in region 1 is lower than the threshold of region 2. This means that the probability that the firm adopts a new technology a time lag after its release is strictly positive. That is consistent with the fact that firms in practice often adopt a new technology a time lag after its introduction.

4.4.1 Example 1

In the following we will present an application for the case of $\lambda_1 > \lambda_2$, and analyze the effect of several parameters on the derived investment strategy. The application we have in mind deals with the statement of Moore's law that describes a long-term trend in the history of computing hardware. It suggests that "the number of transistors that can be placed inexpensively on an integrated circuit doubles approximately every two years. This trend has continued for more than half a century and is expected to continue until 2015, 2020 or later"[3] when the possibilities of further improvement are exhausted.

We will show on the basis of the following numerical example that the probability that a firm will adopt a new innovation with a time lag between innovation and adoption can be substantially high in case the arrival rate in region 1 is higher than the arrival rate in region 2, i.e. $\lambda_1 > \lambda_2$. Inspired by the statement of Moore's law and the characteristics in technological innovation process in the CPU transistor sector we introduce the following example:

EXAMPLE 1. We assume that on average an innovation takes place every two years, i.e. $\mathbb{E}[X] = 2$. Δ is equal to 3. The arrival rate in region 1, λ_1, ranges from 1.0, 1.1 to 1.5 with λ_2 aligned so that $\mathbb{E}[X]$ is for all cases equal to 2. The investment costs for new technology are equal to $I = 500$. The step size with which the technology process increases by technology arrival is given with $u = 0.5$. In order to model the fact that a new innovation doubles the efficiency of technology (for example the number of transistors that can be placed on an integrated circuit considering the example of computing hardware) we choose a profit function that approximately doubles per step of improvement in technology efficiency. We choose the profit function $\Pi(\theta) = \varphi\theta^{b}$[4], with $\varphi = 1000$ and $b = 2$. The other parameters are $r = 0.1$, $\theta_0 = 1$ and an arrival rate of $\lambda = 0.5$ for the standard model. For a graphical illustration of transistor counts improvements from 1971-2011 see Figure 4.2.

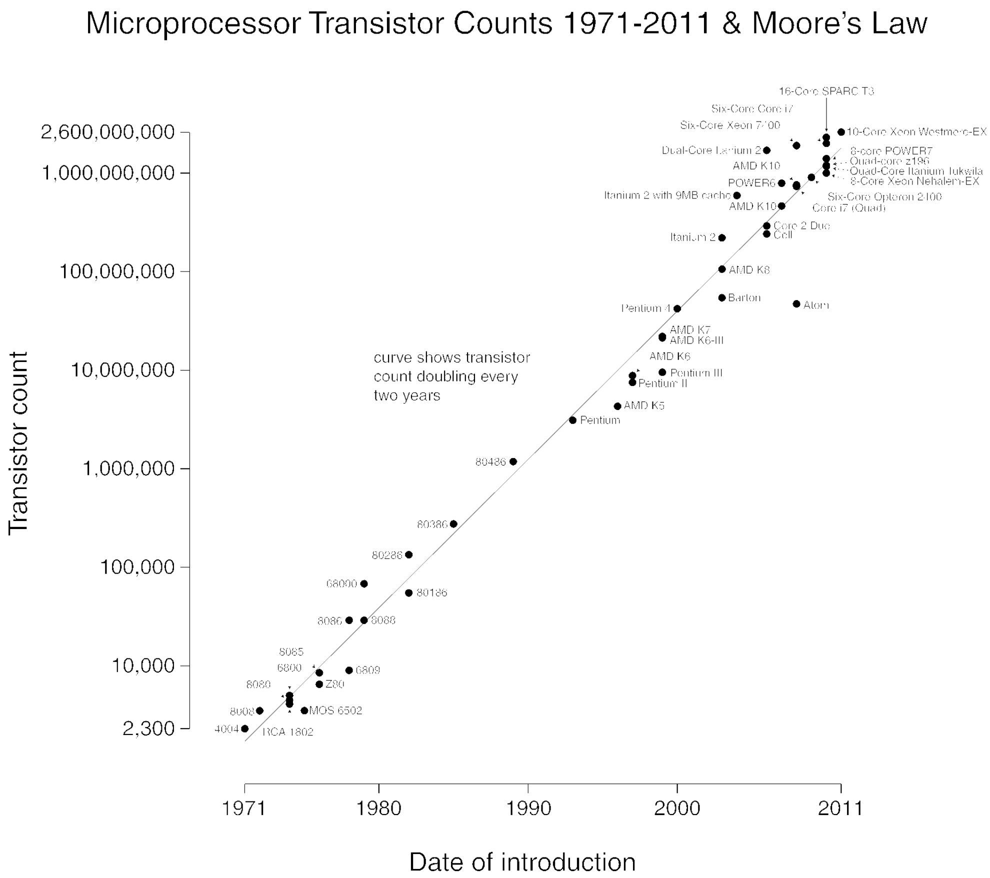
We solve for the optimal investment strategy using the changing arrival rate model as well as the constant arrival rate model. The result of Proposition 2 is based on the fact that the adoption threshold for the standard model with constant arrival rate, is increasing in the arrival rate. We proved this for concave profit functions (see Proposition 1). However, for Example 1 we choose a convex profit function of the form $\varphi\theta^2$. Therefore, it remains to show that the statement in Proposition 1 is also valid for this specific convex profit function. This is the content of the following proposition. The proof can be found in the Appendix. Therewith, the statement of Proposition 2 that explicitly specifies the two cases $\lambda_1 < \lambda_2$ and $\lambda_1 > \lambda_2$ is also valid for convex profit functions of the form $\Pi(\theta) = \varphi\theta^2$.

Proposition 6 *For a given profit function of the form $\Pi(\theta) = \varphi\theta^2$, the optimal*

[3] We refer to http://en.wikipedia.org/wiki/Moore's_law for this statement.

[4] Here we choose the same profit function as considered in Farzin et al. (1998). The firm's production function is given by $h(v, \xi) = \xi v^a$, where $v (\geq 0)$ is a variable input, $\xi (\geq 0)$ is the efficiency parameter, and $a \in (0, 1)$ is the constant output elasticity. The output and input price are constant and equal to p and w. The profit flow equals $\pi(\xi) = \max_v (p\xi v^a - wv)$. Solving the maximization leads to $\pi(\xi) = \varphi\xi^b$ with $\varphi = (1-a)\left(\frac{a}{w}\right)^{\frac{a}{1-a}} p^{\frac{1}{1-a}}$ and $b = \frac{1}{1-a}$.

Figure 4.2: Graphical Illustration of Microprocessor Transistor Counts from 1971-2011: Transistor counts for integrated circuits are plotted against their dates of introduction. The curve shows that the transistor counts double approximately every two years.
(Source: http://en.wikipedia.org/wiki/File:Transistor_Count_and_Moore%27s_Law_-_2008.svg)



adoption threshold is increasing in the arrival rate, i.e., if $\theta^*(\lambda)$ *denotes the threshold when the arrival rate is* λ*, then for* $\lambda_1 < \lambda_2$ *it follows that:*

$$\theta^*(\lambda_1) < \theta^*(\lambda_2). \tag{4.19}$$

The standard model assumes that the arrival rate of new technology is equal to λ forever. This assumption ignores that at some point all possibilities for further improvements are likely to be exhausted and further improvement is not possible. We account for this phenomenon and allow that the arrival rate can decrease to a value close to zero in case the last arrival dates back long indicating that the natural boundary of the innovation process is reached. For our specific example we assume that in case three time periods have passed without a new arrival the arrival rate drops from $\lambda_1 = (0.8, 1.0, 1.25, 1.5 \text{ or } 1.75)$ to $\lambda_2 = (0.1051, 0.0474, 0.0193, 0.0083 \text{ or } 0.0037)$. The numerical results of Example 1 are presented in Table 4.1. Recall that for the case of $\lambda_1 > \lambda_2$ there is a positive probability that the firm adopts new technology a time lag after its arrival. The expression for the probability that the firm will invest a time lag after the innovation at θ_2^*, is content for Proposition 7. Proposition 7 also states the expression of the expected number of jumps before technology adoption.

Proposition 7 *If* $\lambda_1 > \lambda_2$*, the probability that the firm will adopt new technology at a level of* θ *such that* $\theta \in [\theta_2^*, \theta_2^* + u)$*, assuming a current adopted technology value of* ξ_0*, is* $\in [1 - (1 - e^{-\lambda_1 \Delta})^{m^*-1}, 1 - (1 - e^{-\lambda_1 \Delta})^{m^*}]$*, where* $m^* = \lceil \frac{\theta_1^* - \theta_2^*}{u} \rceil$*. The expected number of jumps before technology adoption is given by*

$$\mathbb{E}[n^*] = n(\theta_1^*) \left[1 - P(\text{adopt at } \theta_2^*)\right] + n(\theta_2^*) P(\text{adopt at } \theta_2^*). \tag{4.20}$$

where $n(\theta)$ *denotes the number of jumps necessary to reach the technology level* θ *when the firm's currently used technology level is equal* θ_0*.* $P(\text{adopt at } \theta_2^*)$ *denotes the probability that the firm adopts technology a time lag after its innovation at the threshold* θ_2^**.*

A crucial advantage to the standard model is that we can explain a possible time lag between innovation and adoption. In case that $\lambda_1 > \lambda_2$ the probability of a time lag between innovation and adoption is strictly positive and it can be substantially high as the results of Example 1 illustrate. For the case of $\lambda_1 = 0.8$ and $\lambda_2 = 0.1051$, the firm will adopt new technology a time lag of $\Delta = 3$ after the technology innovation with a probability of more than 31%.

Another advantage of the extended model is that we gain an additional degree of freedom. It allows to show the effect of increasing variance or standard deviation, respectively, of time between two consecutive technology arrivals on the optimal adoption strategy while keeping the expected time fixed

Table 4.1: Effect of increasing variance between two consecutive arrivals on the optimal adoption strategy, keeping the expected time between consecutive arrivals fixed to $\mathbb{E}[X] = 2$ by increasing λ_1 and decreasing λ_2. (Parameter values: $\Delta = 3.0$, $r = 0.1$, $u = 0.5$, $\theta_0 = 1$, $I = 500$, $b = 2.0$ and $\varphi = 1000$. $P_{\theta_2^*}$:=Prob(investing at θ_2^*))

	$\lambda_1 = 0.8$	$\lambda_1 = 1.0$	$\lambda_1 = 1.25$	$\lambda_1 = 1.5$	$\lambda_1 = 1.75$
	$\lambda_2 = 0.105$	$\lambda_2 = 0.047$	$\lambda_2 = 0.019$	$\lambda_2 = 0.008$	$\lambda_2 = 0.004$
θ_1^*	6.72	8.25	10.78	13.69	16.67
θ_2^*	4.48	4.2	3.89	3.65	3.47
$\mathbb{E}[T^*]$	21.78	23.18	22.69	25.24	24.82
$std[X]$	4.45	6.94	11.44	18.13	28.04
$P_{\theta_2^*} \geq$	31.64%	33.54%	26.61%	20.02%	12.79%
$\mathbb{E}[n^*]$	10.42	12.32	16.28	22	28.55

(in our numerical example $\mathbb{E}[X] = 2$). Table 4.1 focuses on the effect of increasing standard deviation on the optimal adoption timing, presenting all relevant values of the optimal adoption strategy for increasing standard deviation ($std[X] = 4.45, 6.94, 11.44, 18.13$ and 28.04). We find that an increase in the standard deviation decreases the probability of a time lag between innovation and adoption to 12.79%. We increase the standard deviation keeping the expected value fixed by increasing the arrival rate in region 1 while decreasing the arrival rate in region 2. This leads to a lower probability of ending up in region 2 (i.e. no arrival in region 1 within Δ time periods) and therefore, the probability of a time lag between innovation and adoption decreases (see Table 4.1). This is, on the one hand due to the fact that an increasing λ_1 lowers the chance that region 2 will be reached. On the other hand, lower λ_2 makes an arrival in region 2 more unlikely. The amount of arrivals necessary to reach the adoption threshold, increases with standard deviation. While for a standard deviation of $std[X] = 6.94$ 11 technology arrivals are necessary to reach the adoption threshold, the amount of necessary technology arrivals increases to 29 if the standard deviation increases to $std[X] = 28.04$. The effect of increasing variance on the expected time of arrival is non-monotonic.

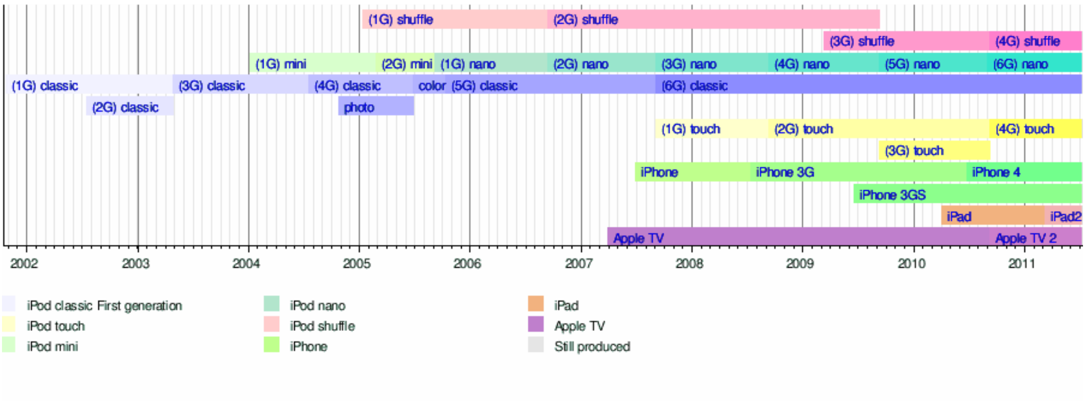
Table 4.2: Effect of increasing variance between two consecutive arrivals, keeping the expected arrival time between consecutive arrivals fixed to $\mathbb{E}[X] = 2$ by increasing Δ and decreasing λ_2. (Parameter values: $\lambda_1 = 1.0$, $r = 0.1$, $u = 0.5$, $\theta_0 = 1$, $I = 500$, $b = 2.0$ and $\varphi = 1000$). $P_{\theta_2^*}$:=Prob(investing at θ_2^*))

	$\Delta = 3$	$\Delta = 4$	$\Delta = 5$	$\Delta = 6$
	$\lambda_2 = 0.0474$	$\lambda_2 = 0.01799$	$\lambda_2 = 0.00669$	$\lambda_2 = 0.00247$
θ_1^*	8.25	9.33	9.94	10.2
θ_2^*	4.20	3.66	3.37	3.25
$\mathbb{E}[T^*]$	23.19	23.9	23.93	24.02
$\mathbb{E}[X]$	2	2	2	2
$Var[X]$	48.2	119.2	308.83	820.86
$std[X]$	6.94	10.92	17.57	28.65
$P_{\theta_2^*} \geq$	33.54%	18.4%	8.41%	3.18%
$\mathbb{E}[n^*]$	12.32	14.98	16.91	18.56

4.4.2 Example 2

The second application deals with technological innovations in the electronics sector and customers' buying behavior. We look at a technology adoption problem from the point of view of a single consumer that will decide when it is optimal to buy a new version of an electronic product. While consumers as a group can have influence on the technological progress, i.e. for example Apple's product release schedule, a single consumer's actions do not influence this progress. The technology progress therefore is exogenous to an individual consumer. However, he might have certain expectations about the arrival of new product series. Many companies in the electronics sector present new series of their product lines regularly. If Apple has just released a new line of its iPod or iPad series, consumers expect that the chance of a new release is very unlikely for a specific period of time, depending on Apple's previous release schedules. If Apple used to release a new iPod generation once per year, consumers think that a new update is very likely in case one year has passed after the last release already. For a graphical illustration of the steady iPod-product release schedule of Apple see Figure 4.3. Example 2 addresses the technology investment strategy of a consumer that decides about the optimal moment to purchase the next generation of an electronic product. The

Figure 4.3: Illustration of iPod release schedule of Apple



consumer assumes that a new product generation is released on average every two years. We raise the question of how the consumer's optimal adoption timing of a new product line depends on his expectation about a companies product release schedule.

Example 2 The arrival rate in region 2 is assumed to be $\lambda_2 = 1$, while the arrival rate in region 1 is small $\lambda_1 = 0.05$. The expected time between two consecutive product releases is equal to $E[X] = 1.93$. The other parameters are: $b = 1$, $\theta_0 = 1$, $u = 0.5$, $r = 0.1$ and $I = 500$.

Tables 4.3 presents numerical results for Example 2. The optimal adoption thresholds of Example 2 are $\theta_1^* = 1.3$ for region 1 and $\theta_2^* = 15.05$ for region 2. Since $\theta_1^* < \theta_2^*$, the firm will always adopt new technology at the moment of innovation. This confirms what is observed in practice. Consumers do not buy a product of the old series when expecting a new release soon. Furthermore, our results show that there is no time lag effect. This finding explains the fact that companies are forced to reduce prices of old product generations when announcing a new one. Since there is no time lag effect, firms need to drop prices of the old product generation in order to increase their sales of the old product generation to empty their inventories. After Apple announced the release of iPad2 on March 2 2011 (release in U.S. on March 11 2011), they dropped the price of the old generation iPad 1 immediately by 20% in order to unload the remaining inventory of the table computer's first generation[5].

Table 4.3 shows the effect of increasing variance $Var[X]$ on the optimal adoption strategy. It shows that the investment threshold θ_1^* as well as ex-

[5]See for example article in CNN Tech ($http://articles.cnn.com/2011-03-03/tech/ipad.1.price\_1\_ipad-16gb-wi-fi-64gb?\_s=PM:TECH$) from March 3, 2011.

Table 4.3: Optimal adoption strategy and the effect of changing variance. (Parameter values: $\mathbb{E}[X] = 1.93$, $\Delta = 1$, $u = 0.5$, $r = 0.1$, $\theta_0 = 1$, $I = 500$, $b = 1.0$ and $\varphi = 1000$).

λ_1	λ_2	$Var[X]$	$std[X]$	θ_1^*	θ_2^*	$\mathbb{E}[T^*]$	$E[n^*]$
0.01	1.063	0.89796	0.95	1.1	15.37	2.91	1
0.05	1	1.06026	1.03	1.3	15.05	2.88	1
0.1	0.928	1.27512	1.13	1.55	14.69	5.66	2

pected time of adoption $\mathbb{E}[T^*]$ is increasing in variance. While for variance equal to $Var[X] = 0.898$ the expected time of adoption is equal to 2.91, increasing variance to 1.28, keeping the expected time equal to 1.93, causes the expected time of adoption to increase to 5.66. This is in line with the standard real options result that increasing uncertainty postpones investment. For the standard case an expected interarrival time of $\mathbb{E}[X] = 1.93$ results from a constant arrival rate equal to $\lambda = 0.52$. The related investment threshold is equal to $\theta^* = 3.64$ and the expected time of adoption $\mathbb{E}[T^*] = 6$. The threshold derived for the constant arrival model is higher than for the changing arrival rate model. This is driven by the fact that the variance of time between two consecutive arrivals is higher for the standard case model ($Var[x] = 3.73$) than for the case of changing arrival rate with low λ_1.

With its product release strategy, Apple sends clear signals to their customers by regular steady releases of new product generations which theoretically speaking decreases the variance of time between arrivals from their customers' point of view and therefore uncertainty. Evaluating this product release strategy, we can confirm that it is successful in the sense of accelerating customers' purchases of new product generations.

4.5 Model Extension

So far we have assumed that the time period after which the value of the arrival rate changes in case no arrival happened before is constant and known. In the following we will release that assumption including an additional degree of uncertainty to the problem. Specifically we assume that the moment at which the arrival rate changes is uncertain introducing stochastic Δ that is exponentially distributed.

We assume that Δ is exponentially distributed, i.e. $\Delta \sim Exp(\mu)$. This means that at time τ the firm knows in which regime it is but it does not know when the value of λ changes. And therefore, the expected time between two consecutive arrivals is given (in view of equation (4.7)) by

$$\mathbb{E}[X] = E\left[E[X|\Delta]\right] = \begin{cases} E[\Delta] + \frac{1}{\lambda_2} = \frac{1}{\mu} + \frac{1}{\lambda_2} & \text{for } \lambda_1 = 0, \\ \frac{1}{\lambda_1} + E[e^{-\Delta\lambda_1}]\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) & \\ \quad = \frac{1}{\lambda_1} + \frac{\mu}{\lambda_1+\mu}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) & \text{for } \lambda_1 > 0. \end{cases} \tag{4.21}$$

The variance of time between two consecutive arrivals assuming exponentially distributed Δ is given by

$$Var[X] = \begin{cases} \frac{(\lambda_2^2+\mu(2\lambda_1+\mu))}{\lambda_2^2(\lambda_1+\mu)^2} & \text{for } \lambda_1 > 0, \\ \frac{1}{\lambda_2^2} + \frac{1}{\mu^2} & \text{for } \lambda_1 = 0. \end{cases} \tag{4.22}$$

The derivation of this expression is shown in Appendix A. For the case of exponentially distributed Δ the following proposition states the equations that implicitly determine the thresholds θ_1^* and θ_2^*:

Proposition 8 *For the derivation of the optimal technology adoption thresholds for process 1, the following two cases have to be distinguished:*

1. *If $\lambda_1 < \lambda_2$, the optimal adoption threshold for process 1, i.e. θ_1^*, is implicitly defined as the solution of the following equation*

$$\begin{aligned} V(\theta_1^*) &= \left(\frac{r+\lambda_1}{r+\lambda_1+\mu}\right)\left[\frac{\Pi(\xi_0)}{r+\lambda_1} + \frac{\lambda_1}{r+\lambda_1}V(\theta_1^*+u)\right] \\ &\quad + \left(\frac{\mu}{r+\lambda_1+\mu}\right)V(\theta_1^*). \end{aligned} \tag{4.23}$$

2. *If $\lambda_1 > \lambda_2$, the optimal adoption threshold θ_1^*, is implicitly defined as the solution of the following equation*

$$\begin{aligned} V(\theta_1^*) &= \left(\frac{r+\lambda_1}{r+\lambda_1+\mu}\right)\left[\frac{\Pi(\xi_0)}{r+\lambda_1} + \frac{\lambda_1}{r+\lambda_1}V(\theta_1^*+u)\right] \\ &\quad + \left(\frac{\mu}{r+\lambda_1+\mu}\right)\left[\left(\frac{\Pi(\xi_0)}{r+\lambda_2} + \frac{\lambda_2}{r+\lambda_2}V(\theta_1^*+u)\right)\right]. \end{aligned} \tag{4.24}$$

The optimal adoption threshold for process 2, i.e. θ_2^*, is implicitly determined by

$$V(\theta_2^*) = \frac{\Pi(\xi_0)}{r+\lambda_2} + \frac{\lambda_2}{r+\lambda_2}\begin{cases} V(\theta_2^*+u) & \text{for } \theta_1^* \leq \theta_2^*+u, \\ G(\theta_2^*+u) & \text{otherwise,} \end{cases} \tag{4.25}$$

with

$$G(\theta) = \left(\frac{r+\lambda_1}{r+\lambda_1+\mu}\right)^{M(\theta)}\left(\frac{\lambda_1}{r+\lambda_1}\right)^{M(\theta)}\left(V(\theta+M(\theta)u)\right) + \sum_{n=0}^{M(\theta)-1}\left(\frac{\mu}{r+\lambda_1+\mu}\right)\left(\frac{r+\lambda_1}{r+\lambda_1+\mu}\right)^{n}\left(\frac{\lambda_1}{r+\lambda_1}\right)^{M(\theta)}\left(V(\theta+nu)\right), \tag{4.26}$$

and $M(\theta) = \lceil \frac{\theta_1^*-\theta}{u} \rceil$, *with the convention that* $\sum_0^{-1} ... = 0$.

We state the expression of the expected time of technology adoption as well as its derivation in Appendix A.

In the following we will analyze the effect of stochastic Δ on the optimal adoption timing for the case of $\lambda_1 < \lambda_2$. In case $\lambda_1 > \lambda_2$ the optimal adoption time only depends on the threshold of region 1, that is independent of Δ, and therefore it is not influenced by the extension to this additional uncertainty of the problem. We concentrate on the case where the arrival rate decreases after a certain period of time has passed with any new arrival, i.e. $\lambda_1 > \lambda_2$. We consider Example 1 introduced in the previous section assuming exponentially distributed Δ with parameter $\mu = \frac{1}{3}$, i.e. $\Delta \sim Exp(\frac{1}{3})$. That means that in expectation Δ is equal to 3 as in the non-stochastic case. The numerical results are presented in Table 4.4. It shows that additional uncertainty about the moment that the arrival rate decreases from λ_1 to λ_2 accelerates the firm's investment. The firm threatened that the arrival rate changes to a low value making a new arrival and therefore improvement in the recent technology very unlikely, is investing earlier than in the case where it is aware of the moment of change. The result that increasing uncertainty accelerates investment is opposite to the standard real options results and arises from a property of the exponential distribution. Assuming that Δ is exponentially distributed with an expected value of 3 implies that an arrival of Δ before the expected value $\mathbb{E}[\Delta] = 3$ is more likely than after and therefore, the firm is threatened by an early decrease in arrival rate which makes it invest earlier.

4.6 Conclusion

We extend the literature on technology adoption by assuming that the arrival rate of new technologies is not constant but changes. The arrival rate changes in case no new technology arrives within a certain time period after the last innovation. We explain the economic fact that firms sometimes adopt new technology a time lag after its innovation. Our analysis shows that the probability that a firm adopts a new technology a time lag after its innovation is

Table 4.4: Effect of increasing variance between two consecutive arrivals on the optimal adoption strategy, keeping the expected time between consecutive arrivals fixed to $\mathbb{E}[X] = 2$ by increasing λ_1 and decreasing λ_2. (Parameter values: $\Delta = 3.0$, $r = 0.1$, $u = 0.5$, $\theta_0 = 1$, $I = 500$, $b = 2.0$ and $\varphi = 1000$).

	$\lambda_1 = 0.8$	$\lambda_1 = 1.0$	$\lambda_1 = 1.25$	$\lambda_1 = 1.5$	$\lambda_1 = 1.75$
	$\lambda_2 = 0.105$	$\lambda_2 = 0.047$	$\lambda_2 = 0.019$	$\lambda_2 = 0.008$	$\lambda_2 = 0.004$
θ_1^*	4.17	3.88	3.92	4.21	4.63
θ_2^*	3.78	3.44	3.34	3.25	3.22
$\mathbb{E}[T^*]$	16.08	12.5	14.02	13.12	14.93
$Var[X]$	46	195	1012	4799	21858.
$std[X]$	14.3513	13.98	31.81	69.27	147.84
$E[X]$	6.8	6.02	11.54	22.45	44.08

strictly positive in case the arrival rate would drop after a certain time period without any new arrival. If the firm expects the arrival rate to rise, this time lag effect is not present.

We analyze the effect of variance of time between two consecutive arrivals on the adoption decision and find that increasing variance postpones investment in case the arrival rate would drop after a certain period without any arrival. In case the arrival rate is supposed to rise, increasing variance affects the adoption timing in a non-monotonic way.

In a further step, we assume that the moment at which the arrival rate switches is not known beforehand, but uncertain. We find the that increasing uncertainty accelerates investment, which is opposite to the standard real options result that a firm facing higher uncertainty waits longer with investment.

Allowing for multiple technology adoptions, changing step size or time dependent investment costs could be interesting extensions for further research. We assume that the arrival rate of new technology can take two different values. At a specific period after the last arrival took place, the arrival rate switches from one value to the other. An interesting approach for further research would be to assume that the arrival rate is a smooth increasing/decreasing function of time, which would mean that the longer the last technology arrival dates back the higher/lower is the probability of a new arrival. Considering the electronics industry it would be also very appealing to include the fact that once a company introduces a new product, other companies are under pressure to introduce new products as well. When Parm, Inc. introduced the first smartphone to be deployed in widespread use in the United States, in the early 2011, Microsoft announced the "Microsoft Windows Powered" smartphone the end that year. In 2002 Handspring released the "Palm OS Treo" smartphone and RIM followed with the release of the first "BlackBerry" devices. One could employ a different model based on mutually exciting jump processes, known as Hawkes processes, which capture that each event generated by the process in turn generates a sequence of offspring events according to a Poisson distribution.

4.A Appendix

Additional Model Details

In the Appendix we will refer to the model with constant arrival rate as model (i) and to the extended model with changing arrival rate as model (ii).

Time between Consecutive Arrivals

In the following we derive the expectation and variance of the time between consecutive technology arrivals for both models. X denotes the random variable corresponding to the time elapsed between consecutive arrivals for model (ii) and Z denotes the time elapsed between consecutive arrivals for model (i).

In both cases the technology level process, $\{\theta(t), t \geq 0\}$, is a linear transformation of a Poisson process

$$\theta(t) = \theta(0) + N(t)u, \tag{4.27}$$

where $N(t)$ denotes the number of new technology arrivals up to time t. Note that in model (i) $\{N(t), t \geq 0\}$ is a Poisson process with rate λ, whereas in model (ii) $\{N(t), t \geq 0\}$ is a non-homogeneous Poisson process. In both cases one can derive the expected value of time between consecutive arrivals as follows:

For model (i), as we are in presence of a homogeneous Poisson process with rate λ, the expected time between consecutive arrivals of information concerning new technology is simply $\mathbb{E}[Z] = \frac{1}{\lambda}$ and the variance is $Var[Z] = \frac{1}{\lambda^2}$.

For model (ii) we have to distinguish two cases. Let us first assume that λ_1 is strictly positive ($\lambda_1 > 0$). As

$$P(N(t) = k) = \begin{cases} P(Poi(\lambda_1 t) = k) & \text{for } t < \Delta, \\ P(Poi(\lambda_1 \Delta) + Poi(\lambda_2(t - \Delta)) = k) & \text{for } \Delta < t, \end{cases}$$

where $Poi(t\lambda)$ denotes that $N(t)$ is Poisson distributed with parameter λt, and assuming independence of these two Poisson distributions, it follows that

$$P(N(t) = k) = e^{-\delta(t)}(\delta(t))^k / k!,$$

where

$$\delta(t) = \begin{cases} \lambda_1 t & t < \Delta, \\ (\lambda_1 - \lambda_2)\Delta + \lambda_2 t & t > \Delta. \end{cases}$$

Thus

$$\mathbb{E}[X] = \int_0^\infty P(X > x)dx = \int_0^\infty P(N(x) = 0)dx, \tag{4.28}$$
$$= \int_0^\Delta e^{-\lambda_1 x}dx + \int_\Delta^\infty e^{-((\lambda_1-\lambda_2)\Delta+\lambda_2 x)}dx, \tag{4.29}$$
$$= \frac{1}{\lambda_1} + e^{-\Delta\lambda_1}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right). \tag{4.30}$$

In order to derive the variance we derive

$$\begin{aligned}\mathbb{E}[X^2] &= \int_0^\Delta s^2\lambda_1 e^{-\lambda_1 s}ds + \mathbb{E}[(\Delta+X)^2]e^{-\lambda_1\Delta}, &(4.31)\\ &= \frac{1}{\lambda_1^2}\left(2 - e^{-\lambda_1\Delta}(2+\lambda_1\Delta(2+\lambda_1\Delta))\right)\\ &\quad + \left(\Delta^2 + \frac{2}{\lambda_2^2} + \frac{2\Delta}{\lambda_2}\right)e^{-\lambda_1\Delta}. &(4.32)\end{aligned}$$

Substituting equations (4.30) and (4.32) in $Var[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X]$ leads the expression for the variance shown in equation (4.8).

Now assume that $\lambda_1 = 0$. Then $X = \Delta + Y$ with $Y \sim Exp(\lambda_2)$, where the previous equality is to be read as with probability one. Thus the expected value is given by

$$\mathbb{E}[X] = \Delta + \frac{1}{\lambda_2}, \tag{4.33}$$

and the variance by

$$Var[X] = \frac{1}{\lambda_2^2}. \tag{4.34}$$

Time between Consecutive Arrivals for extension of stochastic Δ

In order to derive the variance of time between two consecutive technology arrivals we use the fact that $V[X] = E[Var[X]|\Delta] + Var[E[X|\Delta]]$ deriving expression $E[Var[X]|\Delta]$ and $Var[E[X|\Delta]]$, separately. For the case $\lambda_1 > 0$, we first derive $E[Var[X|\Delta]$, using the properties of the exponential distribution:

$$\begin{aligned}E[Var[X|\Delta]] &= E[\left(\frac{1}{\lambda_1^2\lambda_2^2}\right)e^{-2\Delta\lambda_1}\left(e^{2\Delta\lambda_1}\lambda_2^2 - (\lambda_1-\lambda_2)^2\right.\\ &\qquad \left.+ 2e^{\Delta\lambda_1}\lambda_1(\lambda_1-\lambda_2)(1+\Delta\lambda_2)\right)], &(4.35)\\ &= E[\left(\frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\lambda_2^2 - e^{-2\Delta\lambda_1}(\lambda_1-\lambda_2)^2\right.\end{aligned}$$

$$+2e^{-\Delta\lambda_1}\lambda_1(\lambda_1-\lambda_2)(1+\Delta\lambda_2)\Big)], \tag{4.36}$$

$$= \left(\frac{1}{\lambda_1^2\lambda_2^2}\right)\Big(\lambda_2^2 - E[e^{-2\Delta\lambda_1}](\lambda_1-\lambda_2)^2$$

$$+2\lambda_1(\lambda_1-\lambda_2)E[e^{-\Delta\lambda_1}(1+\Delta\lambda_2)]\Big), \tag{4.37}$$

$$= \left(\frac{1}{\lambda_1^2\lambda_2^2}\right)\Big(\lambda_2^2 - \frac{\mu(\lambda_1-\lambda_2)^2}{2\lambda_1+\mu}$$

$$+2\lambda_1(\lambda_1-\lambda_2)\mu\frac{\lambda_1+\lambda_2+\mu}{(\lambda_1+\mu)^2}\Big), \tag{4.38}$$

as

$$E[e^{-2\Delta\lambda_1}] = \int_0^\infty \mu e^{-(2\lambda_1+\mu)x}dx = \frac{\mu}{2\lambda_1+\mu}, \tag{4.39}$$

while the other term is given by

$$\begin{aligned} Var[E[X|\Delta]] &= Var\left[\frac{1}{\lambda_1} + e^{-\Delta\lambda_1}\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\right], \\ &= \left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)^2 Var[e^{-\Delta\lambda_1}], \\ &= \left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)^2 \frac{\lambda_1^2\mu}{(\lambda_1+\mu)^2(2\lambda_1+\mu)}, \end{aligned}$$

as

$$\begin{aligned} Var[e^{-\Delta\lambda_1}] &= E[e^{-2\Delta\lambda_1}] - E^2[e^{-\Delta\lambda_1}], \\ &= \frac{\mu}{2\lambda_1+\mu} - \left(\frac{\mu}{\lambda_1+\mu}\right)^2. \end{aligned}$$

Adding the two expressions up and simplifying results in equation (4.22).

For the case $\lambda_1 = 0$ we conclude that:

$$Var[X] = \frac{1}{\lambda_2^2} + \frac{1}{\mu^2}. \tag{4.40}$$

Expected time of Adoption for extension of stochastic Δ

Now for the expected time of technology adoption: we use again the fact that $E[t^\star] = E\left[E[t^\star|\Delta]\right]$, and thus we may use the expressions for $E[t^\star]$ derived in the previous section. Therefore, we end up with the following result:

- ***Case 1:*** $\theta_1^* \le \theta_2^*$ *In this case the firm will just consider adoption of new technology immediately after a technology arrival happens, without entering region 2. Thus, similarly to the case of fixed and deterministic* Δ*,*

$$\mathbb{E}[T^*] = E[t^*](n(\theta_1^*)). \tag{4.41}$$

- ***Case 2:*** $\theta_1^* > \theta_2^*$ *In this case, in view of Equation (4.16), it follows that*

$$\mathbb{E}[T^*] = E[t^*](n(\theta_2^*)) + \sum_{j=0}^{m^*-1} \int_0^\infty \left(\frac{j}{\lambda_1} + x\right)(1-e^{-x\lambda_1})^j e^{-x\lambda_1} + \frac{m^*}{\lambda_1}(1-e^{-x\lambda_1})^{n^*} \mu e^{-\mu x} dx, \tag{4.42}$$

with $m^* = \lceil \frac{\theta_1^* - \theta_2^*}{u} \rceil$.

For the special case that $\lambda_1 = 0$ *the expected value of the adoption time* T^* *is equal to*

$$\mathbb{E}[T^*] = n(\theta_2^*) \int_0^\infty \frac{1}{x + \frac{1}{\lambda_2}} \mu e^{-\mu x} dx. \tag{4.43}$$

Proofs

Proof of Proposition 1

We show that for the constant arrival model $\lambda_1 < \lambda_2$ *implies* $\theta^*(\lambda_1) < \theta^*(\lambda_2)$. We denote

$$F(\theta, \lambda) = \frac{\Pi(\theta)}{r} - I - \frac{\Pi(\theta_0)}{r+\lambda} - \frac{\lambda}{r+\lambda}\left(\frac{\Pi(\theta+u)}{r} - I\right). \tag{4.44}$$

At $\theta = \theta^*$ the derivative of $F(.,.)$ w.r.t λ is

$$\frac{dF}{d\lambda}|_{\theta=\theta^*} = \left(\frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial \lambda} + \frac{\partial F}{\partial \lambda}\right)\Bigg|_{\theta=\theta^*} = 0. \tag{4.45}$$

We want to show that the investment threshold θ^* is increasing in λ, i.e. $\frac{\partial \theta^*}{\partial \lambda} \geq 0$. This is equivalent to proving that

$$\frac{\partial \theta}{\partial \lambda}|_{\theta=\theta^*} = -\frac{\frac{\partial F}{\partial \lambda}}{\frac{\partial F}{\partial \theta}}|_{\theta=\theta^*} \geq 0, \tag{4.46}$$

where

$$\frac{\partial F(\theta,\lambda)}{\partial \theta} = \frac{1}{r}\frac{\partial \Pi(\theta)}{\partial \theta} - \frac{\lambda}{r+\lambda}\frac{1}{r}\frac{\partial \Pi(\theta+u)}{\partial \theta}, \tag{4.47}$$

$$\frac{\partial F(\theta,\lambda)}{\partial \lambda} = \frac{1}{(r+\lambda)^2}\left(\Pi(\theta_0) - \Pi(\theta+u) + rI\right). \tag{4.48}$$

Since $\frac{\Pi(\theta)}{r} - I > \frac{\Pi(\theta_0)}{r} \Rightarrow \frac{\Pi(\theta+u)}{r} - I > \frac{\Pi(\theta_0)}{r}$ at $\theta = \theta^*$ it holds that $\frac{\partial F(\theta,\lambda)}{\partial \lambda} < 0$. For $\frac{\partial \theta}{\partial \lambda}$ to be positive we need to show that $\frac{\partial F(\theta,\lambda)}{\partial \theta} \geq 0$. For a concave profit function $\Pi(.)$ it holds that $\frac{\partial^2 \Pi(.)}{\partial \theta^2} \leq 0$. Thus, $\frac{\partial \Pi(\theta)}{\partial \theta} \geq \frac{\partial \Pi(\theta+u)}{\partial \theta}$. It follows trivially that

$$\frac{\partial F(\theta,\lambda)}{\partial \theta} = \frac{1}{r}\left(\frac{d\Pi(\theta)}{d\theta} - \frac{\lambda}{r+\lambda}\frac{\partial \Pi(\theta+u)}{\partial \theta}\right) \geq 0. \tag{4.49}$$

Proof of Proposition 2

We note that equations (4.9) and (4.10) can be re-written in the following way:

$$V(\theta_1^*) = \begin{cases} V_{\lambda_1}(\theta_1^*), \\ cV_{\lambda_1}(\theta_1^*) + (1-c)V_{\lambda_2}(\theta_1^*), \end{cases} \tag{4.50}$$

where $V_s(\cdot)$ denotes $\frac{\Pi(\xi_0)}{r+s} + \frac{s}{r+s}V(\cdot + u)$ (i.e. case of constant arrival rate with arrival rate s), and c computed accordingly. This means that θ_1^* is either equal to the optimal level of the standard case with constant arrival rate λ_1, or it is a solution of an equation involving a linear combination of values $V_{\lambda_1}(\cdot)$ and $V_{\lambda_2}(\cdot)$.

Now assume that $\lambda_1 < \lambda_2$. Then, in view of the monotonicity of the optimal level in terms of the intensity rate of the Poisson process (see Proposition 4), we know that $\theta^\star(\lambda_1) < \theta^\star(\lambda_2)$, with $\theta^*(\lambda)$ denoting the optimal level in the standard case with constant arrival rate λ. Thus it follows trivially, that $V_{\lambda_1}(\theta^*) < V_{\lambda_2}(\theta^*)$. As a linear combination is always between the smallest and the largest values of the combination, we have $cV_{\lambda_1}(\theta_1^*) + (1-c)V_{\lambda_2}(\theta_1^*) > V_{\lambda_1}(\theta_1^*)$. Thus the threshold ($\theta_1^*$) computed by means of the upper expression for equation (4.50) presents the smallest optimal adoption trigger for the case $\lambda_1 < \lambda_2$.

Using similar arguments, we conclude that θ_1^* should be computed by the lower expression of equation (4.50) if $\lambda_1 > \lambda_2$. Thus the following holds:

$$\begin{aligned} V(\theta_1^*) &= \left(1 - e^{-(r+\lambda_1)\Delta}\right)\left[\frac{\Pi(\xi_0)}{r+\lambda_1} + \frac{\lambda_1}{r+\lambda_1}V(\theta_1^* + u)\right] \\ &\quad + e^{-(r+\lambda_1)\Delta}\begin{cases} V(\theta_1^*) & \text{if } \lambda_1 < \lambda_2, \\ \left(\frac{\Pi(\xi_0)}{r+\lambda_2} + \frac{\lambda_2}{r+\lambda_2}V(\theta_1^* + u)\right) & \text{if } \lambda_1 > \lambda_2. \end{cases} \end{aligned} \tag{4.51}$$

Proof of Proposition 4

Assume that $\lambda_1 > \lambda_2$. In that case θ_1^* is the solution of equation (4.10). θ_2^* is the solution of the upper part of equation (4.11) if $\theta_1^* \leq \theta_2^* + u$, otherwise it is the solution of the lower part of equation (4.11). In case of the latter, i.e. $\theta_1^* > \theta_2^* + u$, it is already implicitly assumed that $\theta_1^* > \theta_2^*$. Therefore, it remains to prove that the solution for θ_2^* of the upper part of equation (4.11) is smaller than the threshold θ_1^* of region 1 assuming that $\theta_1^* \leq \theta_2^* + u$.

Let $\theta_{i;std}^\star$ denote the optimal adoption threshold for the standard case with constant arrival rate λ_i. The solution of the upper branch of equation (4.11) gives exactly $\theta_{2;std}^\star$, i.e. in this case it holds $\theta_2^\star = \theta_{2;std}^\star$. On the other hand, the right part of equation (4.10) is a linear convex combination of $V_{\lambda_1}(\theta_1^\star)$ and $V_{\lambda_2}(\theta_1^\star)$. (For the specification of the function $V_s(.)$ see Proof of Proposition 2.) Therefore, it follows that $\min(\theta^*_{1;std}, \theta^*_{2;std}) = \theta^*_{2;std} < \theta^*_1 < \max(\theta^*_{1;std}, \theta^*_{2;std}) = \theta^*_{1;std}$ and thus it holds that $\theta_2^* < \theta_1^*$.

Proof of Proposition 5

For the derivation of the expected time of adoption assuming $\lambda_1 > 0$ two cases have to be distinguished: For $\lambda_1 \leq \lambda_2$, just threshold θ_1^* is considered for adoption and therefore the derivation of the expected time of adoption follows straightforwardly assuming equation (4.14). For case $\lambda_1 > \lambda_2$, both thresholds have to be considered for the optimal adoption time. In this case the company will wait at least $\mathbb{E}[T^*](M(\theta_2^*))$ plus a time $\mathbb{E}[T_{plus}]$, where T_{plus} is the time it takes to reach first θ_2^* if the arrival rate is λ_2 (which occurs after time Δ), or reach θ_1^*, if the arrival rate is λ_1 (which occurs before time Δ). Therefore, if $n^* = \lceil \frac{\theta_1^* - \theta_2^*}{u} \rceil$, then n^* denotes the necessary number of extra jumps that we need in order to reach the investment level θ_1^*, whereas if we just need $j \leq n^* - 1$ jumps then it means that we are above θ_2^* level (but under θ_1^* level) and we are running with arrival rate λ_2. Therefore, T_{plus} can take the values:

$$T_{plus} = \left\{ \begin{array}{ll} \Delta & \text{with probability } P(X > \Delta) \\ X + \Delta & \text{with probability } P(X < \Delta)P(X > \Delta) \\ 2X + \Delta & \text{with probability } P(X < \Delta)^2 P(X > \Delta) \\ \ldots & \\ (n^* - 1)X + \Delta & \text{with probability } P(X < \Delta)^{n^*-1} P(X > \Delta) \\ n^* X & \text{with probability } P(X < \Delta)^{n^*} \end{array} \right\} \tag{4.52}$$

Therefore, the expression for $\mathbb{E}[T^*]$ stated in equation (4.16) follows.

Case $\lambda_1 = 0$: in this case the optimal investment policy is driven only by θ_1^*. Thus the numbers of jumps until the investment is just $n^* = \lceil \frac{\theta_1^* - \theta}{u} \rceil$ and the expected time until adoption is according to equation (4.17). □

Proof of Proposition 6

Denote by $x(\lambda)$ the solution of equation (4.4) and let $F(.)$ denote the following function:

$$F(x(\lambda)) = (r + \lambda)x(\lambda)^b - \lambda(x(\lambda) + u)^b.$$

If we prove that the solution of the above equation is increasing with λ, then so is the solution of equation (4.4). According to the theorem of the implicit function, the proof that the optimal adoption threshold is increasing in the arrival rate is equivalent to prove that:

$$x'(\lambda) = -\frac{\frac{\delta F}{\delta \lambda}}{\frac{\delta F}{\delta x}} > 0, \tag{4.53}$$

for all $\lambda > 0$. Thus, as for $b > 0$, $(x(\lambda) + u)^b > x(\lambda)^b$, one needs to prove that

$$b(r+\lambda)x(\lambda)^{b-1} - \lambda b(x(\lambda) + u)^{b-1} > 0.$$

If $b = 1$, then the above inequality follows trivially. If $b < 1$: as $r > 0$, then it follows that

$$b(r+\lambda)x(\lambda)^{b-1} - \lambda b(x(\lambda) + u)^{b-1} > \lambda b(x(\lambda)^{b-1} - (x(\lambda) + u)^{b-1}).$$

For $b < 1$, the function $x(\lambda)^{b-1}$ is decreasing in x and therefore it holds that $x(\lambda)^{b-1} > (x(\lambda) + u)^{b-1}$. Thus inequality (4.53) is proven. Note that it is also possible that for $b > 1$ the optimal adoption threshold can also be increasing in the arrival rate. For instance, for $b = 2$ we have that θ^* is the solution of the following equation:

$$(r+\lambda)\theta = a + \lambda(\theta + u)^2, \tag{4.54}$$

where $a = \frac{\xi_0^2}{r+\lambda}$ is a constant. The only admissible solution is $\frac{2\lambda u + \sqrt{4\lambda^2 u^2 + 4r(a + \lambda u^2)}}{2r}$, which is clearly increasing with λ. □

Proof of Proposition 7

Assuming case $\theta_1^* > \theta_2^*$ we want to derive the probability that having reached a θ-level $\theta \in [\theta_2^*, \theta_2^* + u)$, region 2 will be reached before the technology level has increased up to a level greater than θ_1^*. This probability is given by one minus the probability that threshold θ_1^* will be reached, i.e. if and only if the following $\hat{n}$ jumps arrive within a time period smaller than Δ. The probability that the following $\hat{n} = \lceil \frac{\theta_1^* - \theta_2^*}{u} \rceil$ jumps (having reached a θ-level $\in [\theta_2^*, \theta_2^* + u)$) occur within a time interval $(0, \Delta)$, lays in the interval $\in [P(X < \Delta)^{\hat{n}}, P(X < \Delta)^{\hat{n}-1}] = [(1 - e^{-\lambda_1 \Delta})^{\hat{n}}, (1 - e^{-\lambda_1 \Delta})^{\hat{n}-1}]$ depending on the specific θ-level. Taking the reciprocal of this leads to the result.

Proof of Proposition 8

The result follows in view of the equations (4.9) and (4.10) and in view of the fact that, if Δ is exponentially distributed then the probability of an arrival in region 1 is given by

$$\int_0^\infty \left(1 - e^{-(r+\lambda_1)s}\right) \mu e^{-\mu s} ds = \frac{r + \lambda_1}{r + \lambda_1 + \mu}. \tag{4.55}$$

□

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