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Contents

Chapter 1

A Swift Tour of Auction Theory and its Applications

1.1 Introduction

In the past few decades, auction theory has become one of the most active research areas in economic sciences. The focus on auctions is not surprising, as auctions have been widely used over thousands of years to sell a remarkable range of commodities. One of the earliest reports of an auction is by the old Greek historian Herodotus of Halicarnassus, who writes about men in Babylonia around 500 B.C bidding for women to become their wives.[1] Perhaps the most astonishing auction in history took place in 193 A.D. when the Praetorian Guard put the entire Roman Empire up for auction. Didius Julianus was the highest bidder. However, he fell prey to what, today, is known as the winner's curse: he was beheaded two months later when Septimus Severus conquered Rome.[2]

Nowadays, the use of auctions is widespread. There are auctions for perishable goods such as cattle, fish, and flowers; for durables including art, real estate, and wine; and for abstract objects like treasury bills, licenses for "third generation" (3G) mobile telecommunication (or UMTS), and electricity distribution contracts.[3] In some of these auctions, the amount of money raised is almost beyond imagination. In the 1990s, the US government collected tens of billions of dollars from auctions for licenses for second generation mobile telecommunications,[4] and in 2000, the British and German governments, together, raised almost 100 billion euros

[1] Some have called Herodotus the Father of History, while others have called him the Father of Lies (Pipes, 1998-1999). There may be some doubt, therefore, about whether auctions for women really took place.

[2] These, and other examples of remarkable auctions, can be found in Cassady (1967) and Shubik (1983).

[3] Empirical investigations on these auctions include Zulehner (2009) (cattle), Pezanis-Christou (2000) (fish), van den Berg et al. (2001) (flowers), Ashenfelter and Graddy (2002) (art), Lusht (1994) (real estate), Ashenfelter (1989) (wine), Binmore and Swierzbinski (2000) (treasury bills), van Damme (2002) (UMTS licenses), and Littlechild (2002) (electricity distribution contracts).

[4] Cramton (1998).

in UMTS auctions.[5]

The Dutch government is also becoming accustomed to auctions as allocation mechanisms. A beauty contest was used, as recently as 1996, to assign a GSM[6] license to Libertel. That year, however, seems to have been the turning point. A proposal to change the Telecommunication Law to allow auctions reached the Dutch parliament in 1996; the new law was implemented in 1997.[7] In 1998, GSM licenses were sold through an auction,[8] in 2000, the UMTS auction took place (although this auction was not as successful as the English and German UMTS auctions in terms of money raised),[9] and in 2010, frequencies in the 2.6 GHz-band were auctioned.[10] Moreover, since 2002, the Dutch government has auctioned licenses for petrol stations on a yearly basis.

In this paper, we present an overview of the theoretical literature on auctions.[11] Auction theory is an important theory for two very different reasons. First, as mentioned, many commodities are being sold at auctions. Therefore, it is important to understand how auctions work, and which auctions perform best, for instance, in terms of generating revenues or in terms of efficient allocation. Second, auction theory is a fundamental tool in economic theory. It provides a price formation model, whereas the widely used Arrow-Debreu model, from general equilibrium theory, is not explicit about how prices form.[12] In addition, the insights generated by auction theory can be useful when studying several other phenomena which have structures that resemble auctions like: lobbying contests, queues, wars of attrition, and monopolists' market behavior.[13] For instance, the theory of monopoly pricing is mathematically the same as the theory of revenue maximizing auctions.[14] Reflecting its importance, auction theory has become a substantial field in economic theory.

Historically, the field of auction theory roughly developed along the following lines. William Vickrey's 1961 paper is usually recognized as the seminal work in auction theory. Vickrey studies auctions of a single indivisible object. In the symmetric independent private values (SIPV) model, Vickrey derives equilibrium bidding for the first-price and the second-price auction.[15] He finds that the outcome of both auctions is efficient in the sense that it is always the bidder that attaches the highest value to the object who wins. Moreover, he comes to the surprising conclusion that the two auctions yield the same expected revenue for the seller.

[5] See, e.g., Binmore and Klemperer (2002), van Damme (2002), and Maasland and Moldovanu (2004).

[6] Global System for Mobile communications: a second generation mobile telecommunications standard.

[7] Verberne (2000).

[8] van Damme (1999).

[9] van Damme (2002).

[10] Maasland (2010).

[11] This paper is a slightly revised version of Maasland and Onderstal (2006).

[12] Arrow and Debreu (1954).

[13] Klemperer (2003).

[14] Bulow and Roberts (1989).

[15] In the SIPV model, risk neutral bidders with unlimited budgets bid competitively for an object whose value each bidder independently draws from the same distribution function. If a bidder does not win the object, she is indifferent about who wins, and how much the winner pays. More details about this model can be found in the next section.

Vickrey's paper largely contribute to his 1996 Nobel prize in economics, which he shares with Sir James Mirrlees.[16]

It takes until the end of the 1970s before Vickrey's work is further developed. Roger Myerson, John Riley, and William Samuelson derive results with respect to auctions that maximize the seller's expected revenue. They show that Vickrey's 'revenue equivalence result' extends far beyond the revenue equivalence of the first-price auction and the second-price auction. In addition, they discover that the seller can increase his revenue by inserting a reserve price. In fact, in the SIPV model, both the first-price and the second-price auction maximize the seller's expected revenue if the seller implements the right reserve price.

From the early eighties onwards, the attention shifts to the effects of relaxing the assumptions underlying the SIPV model. Particularly, under which circumstances does the revenue equivalence between the first-price auction and the second-price auction ceases to hold? Important contributions, in this respect, are the affiliated signals[17] model of Paul Milgrom and Robert Weber, and the risk aversion model of Eric Maskin and John Riley. Under affiliated signals, the second-price auction turns out to dominate the first-price auction in terms of expected revenue, while with risk aversion, the opposite result holds true.

In the mid-eighties, Jean-Jacques Laffont, Jean Tirole, Preston McAfee, and John McMillan further develop auction theory by focusing on the auctioning of incentive contracts. In contrast to Vickrey's framework, the principal does not wish to establish a high revenue or an efficient allocation of an object, but aims at inducing effort from the winner after the auction. An example is the procurement for the construction of a road. The procurer hopes that the winner of the procurement will build the road at the lowest possible cost. The question that arises is then: What is the optimal procurement mechanism? One of the main results is that it is not sufficient to simply sell the project to the lowest bidder and make her the residual claimant of the social welfare that she generates. This is because the winner would put too much effort in the project relative to the optimal mechanism.[18]

The most recent burst of auction theory follows in the mid-nineties and the first years of the new millennium, as a response to the FCC[19] auctions in the US, and the UMTS auctions in Europe. The main focus shifts from single-object auctions to auctions in which the seller offers several objects simultaneously. Rather simple efficient auctions can be constructed if all objects are the same, or if each bidder only demands a single object. In the general case, where multi-object demand and heterogeneous objects are concerned, the Vickrey-Clarke-Groves mechanism is efficient. Unfortunately, however, the mechanism has many practical

[16] We will see that the techniques developed by Mirrlees to construct optimal taxation schemes (and other incentive schemes), turned out to be useful for auction theory as well.

[17] Affiliation roughly means that the signals of the bidders are strongly correlated.

[18] The methods used to derive the results are essentially all those of Mirrlees who developed them within the framework of optimal taxation. We thank an anomymous referee for pointing this out to us.

[19] FCC stands for Federal Communications Commission, the agency that organized the auctions for licenses for second generation mobile telecommunications.

drawbacks. Larry Ausubel, Peter Cramton, and Paul Milgrom have recently proposed the 'clock-proxy auction' to deal with these. However, more research is needed to determine the circumstances under which this auction generates desirable outcomes.

The aim of this paper is to give an easily accessible overview of the most important insights of auction theory. The paper adds the following to earlier surveys like Klemperer (1999) and Krishna (2002).[20] First, when discussing the results for single-object auctions, we try to find a compromise between the mainly non-technical treatment of Klemperer and the advanced treatment of Krishna by giving easily accessible proofs to the most elementary propositions. Second, we elaborate more on what happens if the assumptions of the SIPV model are relaxed. Third, we cover auctions of incentive contracts, which have been almost entirely ignored in earlier surveys, although the problem of auctioning incentive contracts is interesting from both a theoretical and practical point of view. Fourth, our treatment of multi-object auctions captures the progress of auction theory since Klemperer's and Krishna's work was published. The fact that most of the cited articles are recently dated shows that the previous surveys are a little outdated with respect to multi-object auctions.

The setup of this paper follows the historical development of auction theory as above. We study single-object auctions in Section 1.2. We start this section by studying equilibrium bidding in the SIPV model for standard auctions such as the English auction, and auctions that are important for modelling other economic phenomena such as the all-pay auction. Then we discuss the revenue equivalence theorem, and construct auctions that maximize the seller's expected revenue. We conclude this section by relaxing the assumptions of the SIPV model and discussing what happens to the revenue ranking of standard auctions. In Section 1.3, we solve the problem of auctioning incentive contracts. Section 1.4 moves our attention to multi-object auctions. Finally, Section 1.5 concludes with a short summary and outlines the remainder of the thesis. The proofs of all propositions and lemmas are relegated to the appendix.

1.2 Single-Object Auctions

In this section, we study auctions of a single object. In Section 1.2.1, we introduce the symmetric independent private values (SIPV) model. In Section 1.2.2, we analyze equilibrium bidding for several auction types. Section 1.2.3 contains a treatment of the revenue equivalence theorem and optimal auctions. In Section 1.2.4, we relax the assumptions of the SIPV model, and discuss the effects on the revenue ranking of standard auctions. Section 1.2.5 contains a summary of the main findings.

[20] An overview of field studies on auctions can be found in Laffont (1997). Kagel (1995) presents a survey of laboratory experiments on auctions, while the books by Klemperer (2004) and Milgrom (2004) discuss the use of auction theory in the design of real-life auctions.

1.2.1 The SIPV Model

The SIPV model was introduced by Vickrey (1961). He models an auction game as a non-cooperative game with incomplete information. The SIPV model applies to any auction in which a seller offers one indivisible object to $n \geq 2$ bidders, and is built around the following set of assumptions.[21]

(A1) *Risk neutrality*: All bidders are risk neutral.

(A2) *Private values:* Bidder i, $i = 1, ..., n$, has value v_i for the object. This number is private information to bidder i, and not known to the other bidders and the seller.

(A3) *Value independence*: The values v_i are independently drawn.

(A4) *No collusion among bidders*: Bidders do not make agreements among themselves in order to achieve the object cheaply. More generally, bidders play according to a Bayesian Nash equilibrium, i.e., each bidder employs a bidding strategy that tells her what to bid contingent on her value, and given the conditional bids of the other bidders, she has no incentive to deviate from this strategy.

(A5) *Symmetry*: The values v_i are drawn from the same smooth distribution function F on the interval $[0, \bar{v}]$ with density function $f \equiv F'$.

(A6) *No budget constraints*: Each bidder is able to fulfill the financial requirements that are induced by her bid.

(A7) *No allocative externalities*: Losers do not receive positive or negative externalities when the object is transferred to the winner of the auction.

(A8) *No financial externalities*: The utility of losing bidders is not affected by how much the winner pays.

1.2.2 Equilibrium Bidding in the SIPV Model

In this section we analyze equilibrium bidding in commonly studied auctions under the assumptions of the SIPV model. We start with the four 'standard' auctions that are used to allocate a single object: the first-price sealed-bid auction, the Dutch auction, the Vickrey auction, and the English auction. In addition, we examine two other auctions that are rarely used as allocation mechanisms, but that are useful in modeling other economic phenomena: the all-pay auction and the war of attrition. We focus on three types of questions. First, how much do bidders bid in equilibrium? Second, is the equilibrium outcome efficient?[22] And third, which of these auctions yields the highest expected revenue?

[21] For a more detailed discussion on this model, see for instance McAfee and McMillan (1987a).

[22] By efficiency we mean that the auction outcome is always such that the bidder who wins the object is the one who attaches the highest value to it.

First-Price Sealed-Bid Auction

In the first-price sealed-bid auction (sealed high-bid auction), bidders independently submit sealed bids. The object is sold to the highest bidder at her own bid.[23] In the US, mineral rights are sold using this auction. In the appendix, we consider two methods for deriving symmetric equilibrium bidding strategies, the 'direct' and the 'indirect' method. These methods turn out to be useful for determining equilibrium bidding, not only for the first-price sealed-bid auction, but for other auctions as well. The seller's expected revenue R_{FPSB} is the expectation of the bid of the highest bidder, which is equal to $E\{Y_2^n\}$, where Y_2^n is the second-order statistic of n draws from F. In other words, the expected revenue from the first-price sealed-bid auction is the expectation of the second highest value.

Proposition 1.1 *The n-tuple of strategies* $(B_{FPSB}, ..., B_{FPSB})$*, where*

$$B_{FPSB}(v) = v - \frac{\int_0^v F(x)^{n-1} dx}{F(v)^{n-1}},$$

constitutes a Bayesian-Nash equilibrium of the first-price sealed-bid auction.[24] *The equilibrium outcome is efficient. In equilibrium, the expected revenue is equal to*

$$R_{FPSB} = E\{Y_2^n\}.$$

Observe that all bidders bid less than their value for the object, i.e., they shade their bids with an amount equal to

$$\frac{\int_0^v F(x)^{n-1} dx}{F(v)^{n-1}}.$$

This amount decreases when the number of bidders increases. In other words, more competition decreases a bidder's profit given that she wins.[25]

Dutch Auction

In the Dutch auction (descending-bid auction), the auctioneer begins with a very high price, and successively lowers it, until one bidder bids, i.e., announces that she is willing to accept the current price. This bidder wins the object at that price, unless the price is below the reserve price. Flowers are sold this way in the Netherlands. The Dutch auction is strategically

[23] Sometimes, a reserve price is used, below which the object will not be sold. Throughout the paper, when we do not explicitly specify a reserve price, we assume it to be zero.

[24] Milgrom and Weber (1982) show that this equilibrium is the unique symmetric equilibrium. Maskin and Riley (2003) show that there can be no asymmetric equilibrium under the assumptions of the SIPV model.

[25] This result does not hold generally, though. In models with common values, increased competition may lead to lower bids. See, e.g., Goeree and Offerman (2003).

equivalent to the first-price sealed-bid auction because an n-tuple of bids $(b_1, ..., b_n)$ in both auctions yields the same outcome, i.e., the same bidder wins and she has to pay the same price.[26,27] This implies that the Bayesian-Nash equilibria of these two auctions must coincide, and that both are equally efficient and yield the same expected revenue.

Proposition 1.2 *The n-tuple of strategies* $(B_{Dutch}, ..., B_{Dutch})$*, where*

$$B_{Dutch}(v) = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}},$$

constitutes a Bayesian-Nash equilibrium of the Dutch auction. The equilibrium outcome is efficient. In equilibrium, the expected revenue is equal to

$$R_{Dutch} = E\{Y_2^n\}.$$

Vickrey Auction

In the Vickrey auction (second-price sealed-bid auction), bidders independently submit sealed bids. The object is sold to the highest bidder (given that her bid exceeds the reserve price). However, in contrast to the first-price sealed-bid auction, the price the winner pays is not her own bid, but the second highest bid (or the reserve price if it is higher than the second highest bid).

The Vickrey auction has an equilibrium in weakly dominant strategies[28] in which each bidder bids her value. To see this, imagine that bidder i wishes to bid $b < v_i$. Let $\bar{b}$ be the highest bid of the other bidders. Bidding b instead of v_i only results in a different outcome if $b < \bar{b} < v_i$. If $\bar{b} > v_i$, bidder i does not win in either case. If $\bar{b} < b$, bidder i wins and pays $\bar{b}$ in both cases. However, in the case that $b < \bar{b} < v_i$, bidder i receives zero utility by bidding b, while she obtains $v_i - \bar{b} > 0$ when bidding v_i. Bidding $b > v_i$ only results in a different outcome if $v_i < \bar{b} < b$. A bid of v_i results in zero utility, whereas bidding b yields her a utility of $v_i - \bar{b} < 0$. Therefore, bidder i is always (weakly) better off by submitting a bid equal to her value. As all bidders bid their value and the winner pays the second highest value, the revenue from the Vickrey auction can be straightforwardly expressed as

$$R_{Vickrey} = E\{Y_2^n\}.$$

[26] Strictly speaking, in the Dutch auction, only one bidder submits a bid, namely the winner. However, each bidder has a price in her mind at which she wishes to announce that she is willing to buy the object. We consider this price as her bid.

[27] The strategic equivalence between the Dutch auction and the first-price sealed-bid auction is generally valid, i.e., not restricted to the SIPV model.

[28] For a definition of this equilibrium concept, see, e.g., Krishna (2002), p. 280.

Proposition 1.3 *The n-tuple of strategies* $(B_{Vickrey}, ..., B_{Vickrey})$, *where*

$$B_{Vickrey}(v) = v,$$

constitutes a Bayesian-Nash equilibrium of the Vickrey auction. The equilibrium is in weakly dominant strategies and the equilibrium outcome is efficient. In equilibrium, the expected revenue is equal to

$$R_{Vickrey} = E\{Y_2^n\}.$$

Despite its useful theoretical properties, the Vickrey auction is seldom used in practice.[29] There may be several reasons why this is the case. First, bidding in the auction is not as straightforward as the theory suggests. At least in laboratory experiments, a substantial number of subjects deviates from the weakly dominant strategy, in contrast to the English auction.[30] Second, the Vickrey auction may cause political inconveniences. For instance, in a spectrum auction in New Zealand, the winner, who submitted a bid of NZ$ 7 million, paid only NZ$ 5,000, the bid of the runner-up.[31] Third, a reason why the auction may not be as efficient as the theory predicts is that bidders are reluctant to reveal their true value for the object, as the seller may use this information in later interactions. In the English auction, as we will see next, the highest bidder does not have to reveal how much she values the object, as the auction stops after the runner-up has left the auction.

Still, the Vickrey auction is closely related to the so-called proxy auction, which is frequently used in reality. For instance, Internet auction sites such as eBay.com, Amazon.com and ricardo.nl, use this auction format, and in the Netherlands, special telephone numbers, such as 0900-flowers, are also allocated via this auction.[32] In a proxy auction, a bidder indicates until which amount of money the auctioneer (commonly a computer) is allowed to increase her bid (in case she is outbid by another bidder). The proxy auction is strategically equivalent to the Vickrey auction if bidders are only allowed to submit a single bid, and no information about the bids of the other bidders is revealed.

English Auction

In the English auction (also known as English open outcry, oral, open, or ascending-bid auction), the price starts at the reserve price, and is raised successively until one bidder remains. This bidder wins the object at the final price. The price can be raised by the auctioneer, or by having bidders call the bids themselves. We study here a version of the English auction called the Japanese auction, in which the price is raised continuously, and bidders announce to quit the auction at a certain price (e.g., by pressing or releasing a button). The English auction

[29] Rothkopf et al. (1990).
[30] Kagel et al. (1987), Kagel and Levin (1993), Harstad (2000), and Englmaier et al. (2009).
[31] McMillan (1994).
[32] Staatscourant (2004).

is the most famous and most commonly used auction type. Art and wine are sold using this type of auction.

In the SIPV model, the English auction is equivalent to the Vickrey auction in the following sense. In both auctions, bidders have a weakly dominant strategy to bid their own valuation.[33] In the English auction, no bidder has a reason to step out at a price that is below or above her value. Therefore, the equilibrium outcome in terms of revenue and efficiency is the same for both auctions. However, unlike the first-price sealed-bid auction and the Dutch auction, these two auctions are not strategically equivalent. In the English auction bidders can respond to rivals leaving the auction, which is not possible in the Vickrey auction. Therefore, the equilibrium outcomes are the same as long as the bidders' valuations are not affected by observing rivals' bidding behavior.

Proposition 1.4 *The n-tuple of strategies* $(B_{English}, ..., B_{English})$*, where*

$$B_{English}(v) = v,$$

constitutes a Bayesian-Nash equilibrium of the English auction. The equilibrium is in weakly dominant strategies and the equilibrium outcome is efficient. In equilibrium, the expected revenue is equal to

$$R_{English} = E\{Y_2^n\}.$$

Of course, the English auction has several equilibria. For example, the strategy combination where bidder 1 bids very aggressively and bidders 2 to n hold back, is also an equilibrium. However, the equilibrium in Proposition 1.4 is the only one that is not weakly dominated.

All-Pay Auction

Now we turn to the all-pay auction and the war of attrition, mechanisms that are rarely used to allocate objects, but turn out to be useful in modeling other economic phenomena. The all-pay auction has the same rules as the first-price sealed-bid auction, with the difference that all bidders must pay their bid, even those who do not win the object. Although the all-pay auction is rarely used as a selling mechanism, there are at least three reasons why economists are interested in it. First, all-pay auctions are used to model several interesting economic phenomena, such as political lobbying, political campaigns, research tournaments, and sport tournaments.[34] Efforts of the agents in these models are viewed as their bids. Second, this auction has useful theoretical properties, as it maximizes the expected revenue for the auctioneer if bidders are risk averse or budget constrained.[35] Third, all-pay auctions are far better able to raise money for a public good than winner-pay auctions (such as the four

[33] See, e.g., Milgrom and Weber (1982).
[34] See, e.g., Che and Gale (1998a) and Moldovanu and Sela (2001).
[35] See Matthews (1983) and Laffont and Robert (1996), respectively.

auctions described above).[36] The reason is that in winner-pay auctions, in contrast to all-pay auctions, bidders forgo a positive externality if they top another's high bid. The optimal fund-raising mechanism is an all-pay auction augmented with an entry fee and reserve price.

Most of the early literature on the all-pay auction and its applications focuses on the complete information setting.[37] This is somewhat surprising, as it seems to be more natural to assume incomplete information, i.e., the 'bidders' (e.g. interest groups) do not know each other's value for the 'object' (e.g. obtaining a favorable decision by a policy maker). In addition, in some situations, there is not less than a continuum of equilibria for the all-pay auction with complete information.[38] In contrast, there is a unique symmetric equilibrium for the all-pay auction with incompletely informed bidders, at least in the SIPV model.[39] The following proposition gives the equilibrium properties of the all-pay auction in the SIPV model.

Proposition 1.5 *The n-tuple of strategies* $(B_{All-Pay}, ..., B_{All-Pay})$*, where*

$$B_{All-Pay}(v) = (n-1)\int_0^v xF(x)^{n-2}dF(x),$$

constitutes a Bayesian-Nash equilibrium of the all-pay auction. The equilibrium outcome is efficient. In equilibrium, the expected revenue is equal to

$$R_{All-Pay} = E\{Y_2^n\}.$$

Note that, in contrast to the earlier models of the all-pay auction with complete information, there is no 'full rent dissipation': the total payments are below the value of the object to the winner. This finding suggests that Posner (1975) overestimates the welfare losses of rent-seeking when he assumes that firms' rent-seeking costs to obtain a monopoly position are equal to the monopoly profits.

War of Attrition

The war of attrition game was defined by biologist Maynard Smith (1974) in the context of animal conflicts.[40] For economists, this game has turned out to be useful to model certain (economic) interactions between humans. An example is a battle between firms to control new technologies, for instance in mobile telecom the battle between the CDMA (code division multiple access), the TDMA (time division multiple access), and the GSM techniques to become the single surviving standard worldwide.[41]

[36] See Goeree et al. (2005) [**Chapter 3 of this thesis**].
[37] See, e.g., Tullock (1967, 1980), and Baye et al. (1993).
[38] Baye et al. (1996).
[39] Moldovanu and Sela (2001).
[40] Maynard Smith speaks about 'contests' and 'displays' instead of 'wars of attrition'.
[41] Bulow and Klemperer (1999).

Although at first sight, the war of attrition is not an auction, its rules could be used in the auction of a single object. In such an auction, the price is raised successively until one bidder remains. This bidder wins the object at the final price. Bidders who do not win the object pay the price at which they leave the auction. Observe that there are two differences between the war of attrition and the all-pay auction. First, the all-pay auction is a sealed-bid auction, whereas the war of attrition is an ascending auction. Second, in the war of attrition, the highest bidder only pays an amount equal to the second highest bid, and in the all-pay auction, the highest bidder pays her own bid.

For $n > 2$, it is not straightforward to construct a symmetric Bayesian Nash equilibrium of the war of attrition. Bulow and Klemperer (1999) show that in any efficient equilibrium all but the bidders with the highest two values should step out immediately. The remaining two bidders then submit bids according to a strictly increasing bid function. Strictly speaking, this cannot be an equilibrium, as there is no information available about whom of the bidders should step out immediately. Therefore, we restrict ourselves to the two-player case in the following proposition.

Proposition 1.6 *Let $n = 2$. The strategies (B_{WoA}, B_{WoA}), where*

$$B_{WoA}(v) = \int_0^v \frac{xf(x)}{1 - F(x)} dx,$$

constitutes a Bayesian-Nash equilibrium of the war-of-attrition. The equilibrium outcome is efficient. In equilibrium, the expected revenue is equal to

$$R_{WoA} = E\{Y_2^n\}.$$

Nalebuff and Riley (1985) show that there is a continuum of asymmetric equilibria where one bidder bids “aggressively” and the other “passively”. The greater the degree of aggression, the larger is the equilibrium expected gain of the aggressive bidder.

1.2.3 Revenue Equivalence and Optimal Auctions

Observe that in the SIPV model, all of the above auctions yield the same expected revenue if the bidders bid according to the symmetric Bayesian Nash equilibrium. Is this result more general? Are there auctions that generate more revenue? And which auction yields the highest expected revenue? In his remarkable paper, published in 1981, Myerson answers these questions in a model that includes the SIPV model as a special case. In order to find the answers, Myerson derives two fundamental results, the *revelation principle*, and the *revenue-equivalence theorem*. In this section, we will discuss Myerson’s results in the context of the

SIPV model.[42] For simplicity, we assume that the seller does not attach any value to the object.[43]

A special class of auctions is the class of *direct revelation games.* In a direct revelation game, each bidder is asked to announce her value, and depending on the announcements, the object is allocated to one of the bidders, and one bidder, or several bidders, pay a certain amount to the seller. More specifically, let (p, x) denote a direct revelation game, where $p_i(\mathbf{v})$ is the probability that bidder i wins, and $x_i(\mathbf{v})$ is the expected payments by i to the seller when $\mathbf{v} \equiv (v_1, ..., v_n)$ is announced. There are two types of constraints that must be imposed on (p, x), an *individual rationality constraint* and an *incentive compatibility constraint.* The individual rationality constraint follows from the assumption that each bidder expects nonnegative utility. The incentive compatibility constraint is imposed as we demand that each bidder has an incentive to announce her value truthfully.

Lemma 1.1 (Revelation Principle) *For any auction there is an incentive compatible and individually rational direct revelation game that gives the seller the same expected equilibrium revenue as the auction.*

Lemma 1.1 implies that when solving the seller's problem, there is no loss of generality in only considering direct revelation games that are individually rational and incentive compatible. Now, consider the following definition of bidder i's marginal revenue:

$$MR(v_i) \equiv v_i - \frac{1 - F(v_i)}{f(v_i)}, \forall v_i, i. \tag{1.1}$$

We call the seller's problem *regular* if MR is an increasing function.

Lemma 1.2 *Let (p, x) be a feasible direct revelation mechanism. The seller's expected revenue from (p, x) is given by*

$$U_0(p, x) = E_{\mathbf{v}}\{\sum_{i=1}^{n} MR(v_i)p_i(\mathbf{v})\} - \sum_{i=1}^{n} U_i(p, x, \underline{v}_i), \tag{1.2}$$

where $U_i(p, x, \underline{v}_i)$ is the expected utility of the bidder with the lowest possible value.

Several remarkable results follow from Lemma 1.2. We start with the revenue equivalence theorem.

Proposition 1.7 (Revenue Equivalence Theorem) *The seller's expected revenue from an auction is completely determined by the allocation rule p related to its equivalent direct revelation game (p, x), and the expected utility of the bidder with the lowest possible value.*

[42] Independently, Riley and Samuelson (1981) derived similar results.

[43] Myerson (1981) assumes that the seller attaches some value to the object, which is commonly known among all bidders.

From this proposition, it immediately follows that in the SIPV model, all standard auctions yield the same expected utility for the seller and the bidders, provided that all bidders play the efficient Bayesian Nash equilibrium. Efficiency implies that the allocation rule is such that it is always the bidder with the highest value who wins the object, so that the allocation rule is the same for all standard auctions. In addition, in the efficient equilibrium of all standard auctions, the expected utility of the bidder with the lowest possible value is zero.

Now, we use Lemma 1.2 to construct the revenue maximizing auction. Observe that in (1.2), apart from a constant, the seller's expected revenue is equal to the sum of each bidder's marginal revenue multiplied by her winning probability. If the sellers' problem is regular, then marginal revenues are increasing in v_i, so that the following result follows.[44]

Proposition 1.8 *Suppose that the seller's problem is regular, and that there is an auction that in equilibrium, (1) assigns the object to the bidder with the highest marginal revenue, provided that the marginal revenue is nonnegative, (2) leaves the object in the hands of the seller if the highest marginal revenue is negative, and (3) gives the lowest types zero expected utility. Then this auction is optimal.*

This proposition has an interesting interpretation for the standard auctions:

Proposition 1.9 *When the seller's problem is regular, all standard auctions are optimal when the seller imposes a reserve price r with $MR(r) = 0$.*[45]

We only sketch the proof of this proposition. In the equilibrium of a standard auction with reserve price, bidders with a value below the reserve price abstain from bidding, and bidders with a value above the reserve price bid according to the same strictly increasing bid function. If the reserve price is chosen such that the marginal revenue at the reserve price is equal to zero, then all standard auction are optimal as (1) if the object is sold, it is always assigned to the bidder with the highest value and hence the highest nonnegative marginal revenue, (2) the object remains in the hands of the seller in the case that the highest marginal revenue is negative, and (3) the expected utility of the bidder with the lowest value is zero. Note that the reserve price does not depend on the number of bidders. In fact, it is the same as the optimal take-it-or-leave price when the seller faces a single potential buyer. The following example illustrates these findings in a simple setting with the uniform distribution.

Example 1.1 *Suppose all bidders draw their value from the uniform distribution on the interval $[0, 1]$. Then*

$$MR(v_i) = 2v_i - 1.$$

[44] See Myerson (1981) for further discussion on the consequences of relaxing this restriction.

[45] Myerson (1981) does not mention this result explicitly, but it follows from his study. Riley and Samuelson (1981) formally derive the result in an independent private values model.

As MR is strictly increasing in v_i, the seller's problem is regular. Moreover, $MR(r) = 0$ implies $r = \frac{1}{2}$. So, from Proposition 1.9 it follows that all standard auctions are optimal when the seller choose reserve price $r = \frac{1}{2}$. Observe that the seller keeps the object with probability $\left(\frac{1}{2}\right)^n$. ▲

1.2.4 Relaxing the SIPV Model Assumptions

In the previous section, we have observed, that in the SIPV model, all efficient auctions yield the same revenue to the seller as long as the bidder with the lowest possible value obtains zero expected utility. In this section, we relax the Assumptions (A1)-(A8) underlying the SIPV model and study the effect on the revenue ranking of the most commonly studied auctions, first-price auctions (like the first-price sealed-bid auction and the Dutch auction), and second-price auctions (like the Vickrey auction and the English auction).[46] In order to obtain a clear view of the effect of each single assumption, we relax the assumptions one by one, while keeping the others satisfied. Table 1.1 gives an overview.[47]

Assumption	**Alternative Model**
(A1) Risk neutrality	Risk aversion
(A2) Private values	Almost common values
(A3) Value independence	Affiliation
(A4) No collusion	Collusion
(A5) Symmetric bidders	Asymmetry
(A6) No budget constraints	Budget constraints
(A7) No allocative externalities	Allocative externalities
(A8) No financial externalities	Financial externalities

Table 1.1: Models that relax assumptions (A1)-(A8).

Risk Aversion

The first assumption in the SIPV model is risk neutrality. Under risk aversion, the expected revenue in the first-price auction is higher than in the second-price auction. A model of risk aversion is the following. The winning bidder receives utility $u(v-p)$ if her value of the object

[46] From this section on, when writing 'standard auctions', we only refer to the first four auction types dealt with in Section 1.2.2: the first-price sealed-bid auction, the Dutch auction, the Vickrey auction, and the English auction.

[47] The alternative model is a model in which one particular assumption of the SIPV model (see first column) does not hold, but in which another specific assumption applies (see second column) besides the other seven SIPV model assumptions.

is v and she pays p, where u is a concave increasing function with $u(0) = 0$. Note that risk aversion may play a role as a bidder has uncertainty about the values, and hence the bids, of the other bidders. In a second-price auction, bidding one's value remains a dominant strategy. In a first-price auction, a risk-averse bidder will bid higher than a risk-neutral bidder as she prefers a smaller gain with a higher probability. By bidding higher she insures herself against ending up with zero. This result still holds true if the value of the object is ex ante unknown to the bidders.[48]

Almost Common Values

Assumption (A2) states that each bidder knows her own value for the object, which may be different than the values of the other bidders. In almost common value auctions, the actual value of the object being auctioned is almost the same to all bidders - but the actual value is not known to anyone. For instance, in the case of two bidders, bidder 1 attaches value $v_1 = t_1 + t_2$ to the object, and bidder 2 has value $v_2 = t_1 + t_2 + \epsilon$, where t_i is bidder i's signal, and ϵ is strictly positive but small. In the equilibrium of the second-price auction, bidder 1 bids 0 and bidder 2 a strictly positive amount, so that the revenue to the seller will be zero. The intuition is as follows. Suppose that bidder 1 intends to continue bidding until B. If the high-valuation bidder goes beyond B, the low-valuation bidder's profit is zero. If the high-valuation bidder stops bidding before B, she obviously is of the opinion that the object is worth less than B to her. But in that case, it is certain that it is worth less than B to the low-valuation bidder. For each positive B for the low-valuation bidder, there is an expected loss. Therefore, bidder 1 bids zero in equilibrium. In the first-price auction, in contrast, the auction proceeds are strictly positive. Bidder 1 bids more than zero as she knows that she has a chance of winning as bidder 2 does not know exactly how much she should shade her bid in order to still win the auction.[49]

Affiliation

The third assumption is value independence, i.e., the values are independently drawn. If these values are 'affiliated', the second-price auction yields more expected revenue than the first-price auction. Affiliation roughly means that there is a strong positive correlation between the signals of the bidders. In other words, if one bidder receives a high signal about the value of the good, she expects the other to receive a high signal as well. Let us consider a situation with pure common values, i.e., all bidders have the same value for the object, for instance the right to drill oil in a certain area. As the actual value of the oil field is not known to the bidders before the auction, they run the risk of bidding too high, and fall prey to what

[48] Maskin and Riley (1984).
[49] Klemperer (1998).

is called the winner's curse:[50] for a bidder, winning is bad news as she is the one who has the most optimistic estimate for the true value of the object. Taking the winner's curse into account, a bidder is inclined to shade her bid substantially. However, if bidders can base their final bid on other bidders' information, then they feel more confident about bidding - and will hence bid less conservatively. An auction generates more revenue if the payment of the winning bidder has greater linkage to the value estimates of other bidders. In a first-price sealed-bid auction, there is no such linkage (the winner pays her own bid). A second-price auction has more linkage since the winner pays the second highest bid - which is linked to the value estimate of the second highest bidder.[51]

Collusion

According to Assumption (A4), bidders do not collude, i.e., they play according to a Bayesian Nash equilibrium. Collusive agreements are easier to sustain in a second-price auction than in a first-price auction, so that the expected revenue is higher in the latter. Assuming no problems in coming to agreement among all the bidders, or in sharing the rewards between them, and abstracting from any concerns about detection, etc., the optimal agreement in a second-price auction is for the bidder with the highest value to bid her true value and for all other bidders to abstain from bidding. This agreement is stable as the bidders with the lower values cannot improve their situation by bidding differently. In a first-price auction, the optimal agreement for the highest value bidder is to bid a very small amount and for all other bidders to abstain from bidding. This agreement is much harder to sustain as the bidders with the lower values have a substantial incentive to cheat on the agreement by bidding just a little bit higher than the bid of the highest value bidder.[52]

Asymmetry

The fifth assumption is symmetry, which means that the values are drawn from the same distribution function. Asymmetry in bidders' value distributions has an ambiguous effect on the revenue ranking of the first-price and second-price auctions. In some situations, the expected revenue from a first-price auction is higher. Imagine, for instance, that the strong bidder's distribution is such that, with high probability, her valuation is a great deal higher than that of a weak bidder. In a first-price auction the strong bidder has an incentive to outbid the weak bidder (to enter a bid slightly higher than the maximum valuation in the weak bidder's support) in order to be sure that she will win. In a second-price auction the expected payment will only be the expected value of the weak bidder's valuation, as for both bidders it is a weakly dominant strategy to bid their own value. In other situations, however,

[50] Capen et al. (1971) claim that oil companies indeed fall prey to the winner's curse in early Outer Continental Shelf oil lease auctions.

[51] Milgrom and Weber (1982).

[52] Robinson (1985).

the expected revenue from the first-price auction may be lower. Suppose, for instance, that across bidders, distributions have different shapes but approximately the same support. A strong bidder, with most mass in the upper range of the distribution, has not much reason to bid high in the first-price auction as she has a substantial probability to beat the weak bidder by submitting a low bid. This incentive to 'low ball' is absent in a second-price auction, so that the expected revenue from the latter may be higher.[53]

Budget Constraints

Under Assumption (A6), bidders face no budget constraints. If this assumption is violated, the first-price auction yields more revenue than the second-price auction. This is trivially true when all bidders face a budget constraint $\bar{b}$ such that $B_{FPSB}(\bar{v}) < \bar{b} < \bar{v}$. Clearly, the expected revenue of the first-price auction is not affected, as no bidder wishes to submit a bid above $\bar{b}$ in equilibrium. In contrast, in the second-price auction, bidders with a value in the range $[\bar{b}, \bar{v}]$ cannot bid higher than $\bar{b}$, so that the expected revenue from the second-price auction decreases relative to the situation that there are no budget constraints. This finding turns out to hold more generally.[54]

Allocative Externalities

According to Assumption (A7), losers face no allocative externalities when the object is transferred to the winner. If allocative externalities are present, the second-price auction and the first-price auction are only revenue equivalent under specific circumstances. Allocative externalities arise when losing bidders receive positive or negative utility when the auctioned object is allocated to the winner. As an example, think about a monopolist suffering a negative externality when a competitor wins a license to operate in 'his' market.[55] Jehiel et al. (1999) show that the Vickrey auction (weakly) dominates other sealed-bid formats, such as the first-price sealed-bid auction. Das Varma (2002) derives circumstances under which first-price auctions and second-price auctions are revenue equivalent, namely when externalities are 'reciprocal', i.e., for each pair of bidders, the externality imposed on each other is the same. However, when externalities are nonreciprocal, the revenue ranking becomes ambiguous.

The following example shows why this is the case. Imagine that two bidders bid for a single object in an auction. We assume that all conditions of the SIPV model hold, except that the utility of bidder i when bidder j wins at a price of p is given by

$$U_i(j,p) = \begin{cases} v_i - p & \text{if } i = j \\ -a_i & \text{if } i \neq j, \end{cases}$$

[53] Maskin and Riley (2000).
[54] Che and Gale (1998b).
[55] Gilbert and Newbery (1982).

$i, j \in \{1, 2\}$, where a_i is the negative externality imposed on bidder i when the other bidder wins. Assume also that a_i is private information to bidder i. Note that in equilibrium, each bidder submits a bid as if her value for the object were $v_i + a_i$. Now, if a_i is drawn from different distribution functions, this model is isomorphic to a model with asymmetry in bidders' value distributions. Recall from above that in such a model, the revenue ranking between the two auctions is ambiguous.

Financial Externalities

Finally, we relax the assumption that the bidders face no financial externalities. The seller generates more revenue in the second-price auction than in the first-price auction in situations with financial externalities. A losing bidder enjoys financial externalities when she obtains a positive externality from the fact that the winning bidder has to pay some money from winning the object. In soccer, the Spanish team FC Barcelona may obtain positive utility when the Italian club AC Milan spends a lot of money when buying a new striker. Assuming that AC Milan faces a budget constraint, AC Milan becomes a weaker competitor to FC Barcelona in future battles for other soccer players.

Formally, financial externalities can be described as follows. The utility of bidder i when bidder j wins at a price of p is given by

$$u_i(j, p) = \begin{cases} v_i - p & \text{if } j = i \\ \varphi p & \text{if } j \neq i, \end{cases}$$

where $\varphi > 0$ is the parameter indicating the financial externality. In this model, given that the other assumptions of the SIPV model hold, the expected revenue from a second-price auction is higher than from a first-price auction for reasonable values of φ. The intuition is that, in contrast to the first-price auction, a bidder in a second-price auction can directly influence the level of payments made by the winner by increasing her bid.[56]

1.2.5 Summary

In the SIPV model, a remarkable result arises with respect to the seller's expected revenue: it is the same for the four standard auctions! Vickrey (1961) was the first to show this result for the simple case of a uniform value distribution function on the interval $[0, 1]$. Also the all-pay auction and the two-player war of attrition turn out to yield the same revenue to the seller.[57] Observe that the seller does not always realize all gains from trade, although he has some

[56] Maasland and Onderstal (2007) [**Chapter 2 of this thesis**] and Goeree et al. (2005) [**Chapter 3 of this thesis**].

[57] In the next section, we will give an alternative proof of this 'revenue equivalence result', and argue that it holds more generally.

market power as he can determine the rules of the auction. In expectation, he obtains the expected value of the second highest value, whereas under complete information, his revenue could be equal to the highest value. The seller can exploit his market power a bit more by inserting a reserve price in any standard auction, which is indeed a way to implement an optimal auction.

Table 1.2 summarizes how the ranking of the standard auctions changes when one of the Assumptions (A1)-(A8) is relaxed while the other assumptions remain valid. In this table, we compare first-price auctions (F), like the first-price sealed-bid auction and the Dutch auction, and second-price auctions (S), like the Vickrey auction and the English auction. S $\prec$ F [S $\succ$ F] means that a second-price auction yields strictly lower [strictly higher] expected revenue than a first-price auction. S ? F implies that the revenue ranking is ambiguous, that is, in some circumstances S $\prec$ F holds, and in other S $\succ$ F.

Assumption	**Alternative Model**	**Ranking**
(A1) Risk neutrality	Risk aversion	S $\prec$ F
(A2) Private values	Almost common values	S $\prec$ F
(A3) Value independence	Affiliation	S $\succ$ F
(A4) No collusion	Collusion	S $\prec$ F
(A5) Symmetric bidders	Asymmetry	S ? F
(A6) No budget constraints	Budget constraints	S $\prec$ F
(A7) No allocative externalities	Allocative externalities	S ? F
(A8) No financial externalities	Financial externalities	S $\succ$ F

Table 1.2: Revenue ranking of standard auctions when the Assumptions (A1)-(A8) are relaxed.

1.3 Auctioning Incentive Contracts

In this section, we turn to the problem of auctioning incentive contracts. In a large range of countries, governments use procurements to select a firm to establish a certain project, e.g., constructing a road. These procurements give flesh and blood to Demsetz' (1968) idea of competition 'for' the market. McAfee and McMillan (1986, 1987b) and Laffont and Tirole (1987, 1993) study these types of situations, thus building a bridge between auction theory and incentive theory. Auction theory applies as the buyer wishes to select a bidder out of a set of several bidders, and incentive theory is relevant as the buyer may wish to stimulate the winning bidder to put effort in the project. The question that arises is then: what is the optimal procurement mechanism? Is it optimal, in the example of road construction, to simply select the cheapest bidder and make her the residual claimant of all cost savings, or are there more advanced mechanisms that increase the buyer's utility? McAfee and McMillan and Laffont and Tirole have answered these questions using the techniques that were first developed by Mirrlees (1971, 1976, 1999 (first draft 1975)).

The problem of auctioning incentive contracts is not only of theoretical interest. For instance, in several countries, the government procures welfare-to-work programs as a part of their active labor market policy.[58] In these procurements, the government allocates welfare-to-work projects to employment service providers. A welfare-to-work project typically consists of a number of unemployed people, and the winning provider is rewarded on the basis of the number of these people that find a job within a specified period of time. According to OECD (2001) procurements for welfare-to-work projects should be organized as follows. The government defines an incentive contract that guarantees an employment service provider a fixed reward for each person that finds a job. This reward is equal to the increase in social welfare if this person does find a job. The government sells the contract to the highest bidder in an auction, who has to pay her bid. Onderstal (2009) shows that the mechanism proposed by OECD indeed performs almost as well as the optimal mechanism.

In the next section, we present a simple model. In Section 1.3.2, we construct the optimal mechanism, and in Section 1.3.3, we summarize the main findings.

1.3.1 The Model

Let us describe a simple setting, in which a risk neutral buyer wishes to procure a project. We assume that n risk neutral firms participate in the procurement. Each firm i, $i = 1, ..., n$, when winning the project, is able to exert observable effort e_i at the cost

$$C_i(e_i, \alpha_i) = \frac{1}{2}e_i^2 + e_i - \alpha_i e_i.$$

In the road construction example, the effort level e_i may be interpreted as a decrease in the cost to build the road, while in procurements of welfare-to-work programs, effort is related to the number of people that find a job. We choose this specific cost function so that by construction, in the first-best optimum, i.e., the optimum under complete information, the winning firm's effort is equal to α_i. In addition, note that $C_i'''(e_i) = 1 > 0$. In other words, the marginal costs of effort is strictly increasing in effort. In road construction, this seems to make sense: the first euro in cost savings is easier to obtain than the second euro, and so forth. We assume that diseconomies of scale do not play a role, as otherwise the government would have a good reason to split up the project in smaller projects, and have several firms do the job.

The firms differ with respect to their efficiency level $\alpha_i \in [0, 1]$, which is only observable to firm i. Note that the costs per unit of effort are increasing in effort. The firms draw the

[58] Zwinkels et al. (2004) provide a comparison of welfare-to-work procurements in Australia, Denmark, the Netherlands, Sweden, the UK, and the US.

α_i's independently from the same distribution with a cumulative distribution function F on the interval $[0, 1]$ and a differentiable density function f. F is common knowledge. We assume that

$$\alpha_i - \frac{1 - F(\alpha_i)}{f(\alpha_i)} \text{ is strictly increasing in } \alpha_i, \tag{1.3}$$

which is the same as the regularity condition we imposed in the problem of revenue maximizing auctions.

Firm i has the utility function

$$U_i = t_i - C_i,$$

where t_i is the monetary transfer that it receives from the buyer. Let S denote the buyer's utility from the project. We assume that

$$\begin{aligned} S &= e_i - t_i \qquad (1.4)\\ &= e_i - U_i - C_i(e_i, \alpha_i), \end{aligned}$$

where i is the firm the buyer has selected for the project. In the road construction example, S can be viewed as the net cost savings for the government.[59] An optimal mechanism maximizes S under the restriction that the firms play a Bayesian Nash equilibrium, and that the mechanism satisfies a participation constraint (in equilibrium, each participating firm should at least receive zero expected utility).

The first-best optimum has the following properties. First, the buyer selects the most efficient firm, i.e., the firm with the highest type α_i, as this firm has the lowest C_i for a given effort level. Second, the buyer induces this firm to exert effort α_i. Finally, the buyer exactly covers the costs C_i. We will see that this first-best optimum cannot be reached in our setting with incomplete information: the buyer has to pay informational rents to the firm.

1.3.2 The Optimal Mechanism

What is the optimal mechanism, i.e., the mechanism that maximizes (1.4)? As in the problem of finding a revenue maximizing auction, we apply the revelation principle: without loss of generality we restrict our attention to incentive compatible and individually rational direct revelation games. Let $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, ..., \tilde{\alpha}_n)$ be the vector of announcements by firm $1, ..., n$ respectively. We consider mechanisms $(x_i, e_i, t_i)_{i=1,...,n}$ that induce a truthtelling Bayesian Nash equilibrium, where, given $\tilde{\boldsymbol{\alpha}}$, $x_i(\tilde{\boldsymbol{\alpha}})$ is the probability that firm i wins the contract, and, given that firm i wins the contract, $e_i(\tilde{\boldsymbol{\alpha}})$ is its effort and $t_i(\tilde{\boldsymbol{\alpha}})$ is the monetary transfer it receives from the buyer.

[59] When the government does not select one of the bidders in the procurement, a public firm builds the road. We assume that in that case $S = 0$.

Proposition 1.10 *The optimal mechanism* $(x_i^*, e_i^*, t_i^*)_{i=1,\ldots,n}$ *has the following properties:*

$$x_i^*(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i > \max_{j \neq i} \alpha_j \text{ and } \alpha_i \geq \underline{\alpha} \\ 0 & \text{otherwise,} \end{cases}$$

$$e_i^*(\boldsymbol{\alpha}) = \alpha_i - \frac{1 - F(\alpha_i)}{f(\alpha_i)}, \text{ and}$$

$$t_i^*(\boldsymbol{\alpha}) = C_i(e_i^*(\boldsymbol{\alpha}), \alpha_i) + \int_{\underline{\alpha}}^{\alpha_i} e_i^*(y) F(y)^n dy,$$

where $\underline{\alpha}$ *is the unique solution to* y *in* $y = \frac{1-F(y)}{f(y)}$.

The optimal mechanism has the property that the buyer optimally selects the most efficient firm, provided that its efficiency level exceeds $\underline{\alpha} > 0$. This firm exerts effort according to e_i^*, and t_i^* determines the payments it receives from the buyer. Observe that the desired effort level $e_i^*(\boldsymbol{\alpha})$ and $\underline{\alpha}$ do not depend on the number of bidding firms.

Finally, let us go back to the example of the road construction project. Is it optimal to simply auction the project to the lowest bidder and gives her a compensation equal to the cost savings e that she realizes? The answer turns out to be 'no'. This can be seen as follows. The winner i of the auction maximizes her utility, which is equal to

$$t_i + e_i - C(e_i, \alpha_i) = t_i - \frac{1}{2}e_i^2 + \alpha_i e_i, \tag{1.5}$$

where $t_i + e_i$ is the transfer the government makes to the winner. It is routine to derive that $\hat{e}_i = \alpha_i$ maximizes (1.5). In other words, the winner puts too much effort in the project relative to the optimal mechanism, as $\hat{e}_i > e_i^*(\boldsymbol{\alpha})$. It can be checked that the optimal mechanism can be implemented by selling a non-linear contract to the lowest bidder. For instance, if the efficiency levels are drawn from the uniform distribution on the interval $[0, 1]$, the government optimally pays the winner

$$b + \frac{1}{4}e^2 + \frac{1}{2}e$$

if her winning bid is b and she puts effort e in the project.

1.3.3 Summary

In this section, we have studied auctions of incentive contracts in a stylized model. Table 1.3 summarizes the main results of this model. Note that three types of inefficiency arise from the optimal mechanism under incomplete information relative to a situation with complete information. First, since $e_i^*(\boldsymbol{\alpha}) < \alpha_i$ for all $\alpha_i < 1$, the firm's effort is lower than in the full-information optimum. Second, the buyer will not contract with any firm whose efficiency level is below $\underline{\alpha}$, whereas in the full-information world, the buyer would contract with any provider. The latter is analogous to a reserve price in an optimal auction. Third, as $t_i^*(\boldsymbol{\alpha}) > C_i(e_i^*(\boldsymbol{\alpha}))$ for all $\alpha_i > 0$, the government covers more than the costs that are actually born by the winning provider. These types of inefficiency give the buyer the opportunity to reduce the informational rents that he has to pay to the winner because of incomplete information.

	First-Best Mechanism	Optimal Mechanism
Winner	$x_i^{**}(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i > \max_{j \neq i} \alpha_j \\ 0 & \text{otherwise} \end{cases}$	$x_i^*(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i > \max\{\max_{j \neq i} \alpha_j, \underline{\alpha}\} \\ 0 & \text{otherwise} \end{cases}$
Effort	$e_i^{**}(\boldsymbol{\alpha}) = \alpha_i$	$e_i^*(\boldsymbol{\alpha}) = \alpha_i - \frac{1-F(\alpha_i)}{f(\alpha_i)}$
Payment	$t_i^{**}(\boldsymbol{\alpha}) = C_i(e_i^{**}(\boldsymbol{\alpha}), \alpha_i)$	$t_i^*(\boldsymbol{\alpha}) = C_i(e_i^*(\boldsymbol{\alpha}), \alpha_i) + \int_{\underline{\alpha}}^{\alpha_i} e_i^*(y) F(y)^n dy$

Table 1.3: Properties of the first-best mechanism and the optimal mechanism under incomplete information.

1.4 Multi-Object Auctions

In the previous sections, we have observed that the seller faces a trade-off between efficiency and revenue. When selling a single object, the seller maximizes his revenue by imposing a reserve price. This causes inefficiency as the object remains unsold when none of the bidders turns out to be willing to pay the reserve price, while they may assign a positive value to it. Equivalently, in auctions of incentive contracts, the revenue maximizing buyer only assigns the incentive contract if a sufficiently efficient firm enters the auction.

In multi-object auctions, a new trade-off enters the picture: the trade-off between efficiency and complexity. We will see that if each bidder in the auction only demands one object, and if the seller offers homogeneous objects, the main results from the single-object case carry over: straightforward generalizations of the standard auctions are efficient (and revenue equivalent). However, as soon as objects are heterogeneous, or when bidders demand more than one object, an efficient outcome is no longer guaranteed. Luckily, rather simple efficient auctions can be constructed with multi-object demand if objects are homogeneous and with heterogeneous objects if there is single-object demand. In the general case, with multi-object demand and heterogeneous objects, the Vickrey-Clarke-Groves mechanism is efficient. However, this auction has several practical drawbacks, for instance that it is complex as bids are needed on a large range of packages. These disadvantages are only partially mitigated in innovative new designs that have been recently proposed in the literature, such as Ausubel, Cramton, and Milgrom's clock-proxy auction and Goeree and Holt's hierarchical package bidding auction.

From the above it is clear that the results are highly dependent on whether the objects are identical or not and whether the bidders are allowed to win several objects or only one. We have therefore decided to build up this section along these two crucial points. In the 2x2 matrix in Table 1.4 it is shown which part is covered in which section.

	Identical Objects	Non-Identical Objects
Single-Object Demand	Section 1.4.1	Section 1.4.2
Multi-Object Demand	Section 1.4.3	Section 1.4.4

Table 1.4: Set-up of Section 1.4.

In Section 1.4.1, we deal with auctions of multiple identical objects when bidders are allowed to win only one object/desire at most one object. A real-life example of such an auction is the Danish UMTS auction (by which licenses for third generation mobile telecommunication were sold).[60] Four identical licenses were put up for sale and firms were only allowed to win one license. Section 1.4.2 introduces auctions of multiple non-identical objects when bidders are allowed to win only one object/desire at most one object. A good example of such an auction is the Dutch UMTS-auction (and most of the other European UMTS-auctions).[61] In the Netherlands, five non-identical licenses (differing with respect to the amount of spectrum) were put up for sale and firms were only allowed to win one license. In Section 1.4.3, we analyze auctions of multiple identical objects when bidders are allowed to win multiple objects. Examples are treasury bond auctions, electricity auctions, and initial public offerings (IPOs) of companies shares (e.g. Google's IPO). In Section 1.4.4, we discuss auctions of multiple non-identical objects when bidders are allowed to win multiple objects. The Dutch GSM auction is an example of such an auction.[62] Section 1.4.5 contains a conclusion with the main findings of this section.

1.4.1 Auctions of Multiple Identical Objects with Single-Object Demand

In this section, we present the multi-unit extensions of the four standard auctions dealt with in Section 1.2 when bidders are allowed to win only one unit/each bidder wants at most one unit.[63] These four extensions are the pay-your-bid auction, the multi-unit Dutch auction, the uniform-price auction, and the multi-unit English auction. We assume that $2 \leq k < n$ units are put up for sale.

[60] http://en.itst.dk/spectrum-equipment/Auctions-and-calls-for-tenders/3g-hovedmappe/3g-auction-2001-1.
[61] van Damme (2002).
[62] van Damme (1999).
[63] Early articles on multiple identical object auctions with single-object demand are Vickrey (1962) and Weber (1983).

Pay-Your-Bid Auction

In the pay-your-bid auction, bidders independently submit sealed bids (each bidder submits one bid). The k units are sold to the k highest bidders at their own bid. This auction is sometimes called a discriminatory auction as it involves price discrimination (bidders pay different prices for an identical object), and can be seen as the generalization of the first-price sealed-bid auction. In the SIPV model, the equilibrium outcome is efficient.

Multi-Unit Dutch Auction

In the multi-unit Dutch auction, the auctioneer begins with a very high price, and successively lowers it, until one bidder bids. This bidder wins the first unit at that price. The price then goes further down until a second bidder bids. This bidder wins the second unit for the price she bid. The auction goes on until all k units are sold (or until the auction has reached a zero price). Note that also this auction involves price discrimination. In contrast to the single-unit case in Section 1.2, the multi-unit Dutch auction is not strategically equivalent to the pay-your-bid auction, as bidders may update their bid when bidders leave the auction (after winning one of the k units). In the SIPV model, the Bayesian-Nash equilibria of the multi-unit Dutch auction and the pay-your-bid auction still coincide though, so that also the multi-unit Dutch auction is efficient.

Uniform-Price Auction

In the uniform-price auction with single unit demand, bidders independently submit sealed-bids (each bidder submits one bid). The k units are sold to the k highest bidders (given that these bids exceed the reserve price). The winners pay the $(k+1)$-th highest bid, i.e. the highest rejected bid. The uniform-price auction has an efficient equilibrium, as each bidder has a weakly dominant strategy to bid her value. The intuition is the same as in the Vickrey auction.

Multi-Unit English Auction

In the multi-unit English auction, the price starts at the reserve price, and is successively raised until k bidders remain. These bidders each win one unit at the final price. As in the SIPV model, the multi-unit English auction is equivalent to the uniform-price auction, the equilibrium outcome in terms of revenue and efficiency is the same for both auctions.

Results

When objects are identical and bidders desire at most one unit, several results from the single-unit case generalizes to the multi-unit case. In the SIPV model, all standard auctions remain efficient. Moreover, the revenue equivalence theorem continues to hold, which implies that the above four auction types yield the same revenue in expectation.[64] Another result is that all four auctions are revenue maximizing, provided that the seller imposes the optimal reserve price. What changes with respect to the single-unit case is that the multi-unit Dutch auction (open first-price format) is not strategically equivalent to the pay-your-bid auction (sealed-bid first-price format) anymore.

1.4.2 Auctions of Multiple Non-Identical Objects with Single-Object Demand

In this section, we keep the assumption that each bidder only desires one object, but now we assume that the seller auctions non-identical objects.

Simultaneous Ascending Auction

The best-understood auction format in this environment is the simultaneous ascending auction (SAA).[65] The rules of this auction are the following. Multiple objects are sold simultaneously and bidding occurs in a series of rounds. In each round, those bidders who are eligible to bid, make sealed bids for as many objects as they want. At the end of each round, the auctioneer announces the standing high bid for each object along with the minimum bids for the next round, which he computes by adding a pre-determined bid increment such as 5% or 10% to the standing high bids. A standing high bid remains valid until it is overbid or withdrawn. The auction concludes when no new bids are submitted. The standing high bids are then deemed to be winning bids, and the winners pay an amount equal to the standing high bid.

In the simple case in which each bidder can buy at most one object, the SAA is reasonably well understood in the SIPV context. An equilibrium is established when bidders bid 'straightforwardly', i.e., in each round, each bidder that currently does not have a standing high bid, bids for that object which currently offers the highest surplus (the highest difference between value and price), and they drop out once the highest available surplus becomes negative.[66] This equilibrium is efficient. Indeed, this is one important reason why auction experts have convincingly advocated the use of the SAA to sell license for mobile telecommunication, both in the US (second generation mobile telecommunication) and Europe (UMTS).

Although the SAA has nice theoretical properties, there are some practical disadvantages. For instance, the SAA may perform poorly with respect to revenue in uncompetitive situations, i.e. when the number of objects available exactly equals the number of 'advantaged' bidders. Weaker bidders are reluctant to participate in the auction, and those that are present bid especially cautiously because of the enhanced winner's curse they face. Klemperer (2002) suggests incorporating a first-price element to bolster competition in this case. Indeed, Goeree

[64] See, e.g., Harris and Raviv (1981), Maskin and Riley (1989).
[65] See, e.g., Milgrom (2004).
[66] Leonard (1983) and Demange et al. (1986).

et al. (2006) show in a laboratory experiment that the seller's revenues are the highest among a number of first-price formats when the licenses are sold sequentially, in decreasing order of quality.

Another practical drawback of the SAA is related to the time it takes for the auction to complete. For instance, in the UMTS auctions in Europe, it sometimes took several weeks for the auction to finish. The 'proxy auction' is much faster. This auction format implements straightforward bidding: a computer bids on behalf of the bidders, who indicate for each object which amount of money they are maximally willing to pay. The computer takes this maximal willingness to pay as a bidder's value, and then bids straightforwardly for all bidders until the auction ends. Suppose for simplicity that bidders have fixed valuations which are possibly different for different licenses, but which were not affected by information held by other companies, or by information released during the auction. Then it is a dominant strategy for each bidder to reveal her true willingness to pay for each object, so that the outcome of the proxy auction is efficient. When valuations are not private, a disadvantage of the proxy auction relative to the SAA is that the bidders cannot adjust bids when during the bidding process information is revealed which affects their estimation of the objects' values.

1.4.3 Auctions of Multiple Identical Objects with Multi-Object Demand

In this section, we return to the situation in which all objects are identical, now assuming that bidders are allowed to win more than one unit. We present multi-unit extensions of the four standard auctions dealt with in Section 1.2: the pay-your-bid auction, the uniform-price auction, the multi-unit Vickrey auction, and the Ausubel auction. We will see that several nice properties of single-object auctions which still remain valid under single-object demand, seize to hold under multi-object demand. Finally, we return to the SAA, and discuss some practical drawbacks of this auction under multi-object demand.

Pay-Your-Bid Auction

As said, the pay-your-bid auction can be viewed as the multi-unit extension of the first-price sealed-bid auction. Bidders now simultaneously submit several sealed bids. These bids should comprise weakly decreasing inverse demand curves $p_i(q)$, for $q \in \{1, ..., k\}$ and $i \in \{1, ..., n\}$. $p_i(q)$ represents the price offered by bidder i for the q-th unit. Each bidder wins the quantity demanded at the clearing price, and pays the amount that she bid for each unit won.

In a model where q can be any positive real number below a certain threshold value, Ausubel and Cramton (2002) show that under certain circumstances the pay-your-bid auction results in an efficient allocation of the object. To be more precise, the pay-your-bid auction is efficient if (1) bidders are symmetric, in the sense that the joint distribution governing the bidders' valuations is symmetric with respect to the bidders, (2) each bidder i has a constant marginal valuation for every quantity $q \in [0, \lambda_i]$, where λ_i is a capacity limitation on the quantity of units that bidder i can consume, and (3) the bidders are symmetric in their capacity limitations: $\lambda_i = \lambda$, for all bidders i. Otherwise, the auction may be inefficient. One reason for this is that bidders submit a higher bid on the first unit than on the second if they have the same value for each unit.[67]

However, Swinkels (1999) shows in a general setting that if the number of bidders gets arbitrarily large, the inefficiency in the pay-your-bid auction goes to zero. This result is interesting, as it shows that the pay-your-bid auction does rather well when the seller is able to attract a large number of bidders. This may partly explain why the pay-your-bid auction is popular in practice, for instance to sell treasury bills.[68] In addition, this result also formalizes the more general idea that competition in a market leads to an efficient allocation of resources when none of the agents has market power.

Uniform-Price Auction

In the uniform-price auction, bidders simultaneously submit sealed-bids comprising inverse demand curves. Each bidder wins the quantity demanded at the clearing price, and pays the clearing price for each unit she wins. This auction format raised quite some confusion among economists. For instance, Nobel Prize winners Milton Friedman and Merton Miller thought incorrectly that the uniform-price auction was the multi-unit extension of the single unit Vickrey auction. Both argued that, like in the Vickrey auction, bidders would truthfully reveal their values in the auction, so that the auction outcome is efficient.[69] This is actually only true under similarly strong symmetry conditions as for the pay-your-bid auction.[70] Usually, however, bidders have an incentive to shade their bidding on some units, so that the uniform-price auction is inefficient.

The reason is simple: if a bidder can demand more than one unit, she can influence the market price and she will profit from a lower market price on all the units that she gets. Suppose that a bidder is bidding for two units to which she attaches the same value. Imagine that she bids her true value on both units. Then the other bidders may happen to bid in such a way that her bid on the second unit is the highest rejected bid. In that case, she wins one unit for which she pays a price equal to her own bid on the second unit. Therefore, she strictly prefers to bid lower on the second unit. As a consequence, bidding the same for both units cannot happen in equilibrium.[71] This phenomenon has become known as 'demand reduction': bidders understate their true value for some units.

The above example might suggest that demand reduction is limited to a small number

[67] Engelbrecht-Wiggans and Kahn (1998b).

[68] Binmore and Swierzbinski (2000).

[69] Quotations from press articles can be found in Ausubel and Cramton (2002).

[70] Ausubel and Cramton (2002).

[71] Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998a) examine uniform-price auctions where each bidder desires up to two identical units. They find that a bidder generally has an incentive to bid sincerely on her first unit but to shade her bid on the second unit.

of bidders, or is mainly a theoretical concept. However, this is not the case as the following stylized example shows. Imagine that n bidders compete for n units, which are each worth $1 for every bidder. Suppose that each bidder bids $1 for one unit, and $0 for a second, third, etc. unit. The resulting price is $0 (the highest rejected bid is equal to $0).

While demand reduction implies that a bidder will not win some units that she would have liked to win, it is advantageous because it reduces the price which the bidder has to pay for all those units which she will win. Demand reduction may imply that the auction outcome is inefficient, and may also result in lower revenues than an efficient auction.[72]

Like in the pay-your-bid auction, if the number of bidders becomes very large, the outcome of the uniform-price auction is efficient. The intuition is simple. The probability that a bidder is the highest losing bidder is zero, and so is the probability that she determines the price. This may explain why the uniform-price auction is sometimes used in practice, for instance in auctions for treasury bills.[73]

The ranking of the uniform-price auction and the pay-your-bid auction in terms of revenue is ambiguous: depending on the circumstances, the expected revenue of one auction may be higher than the other.[74] Also in practice, like in treasury bill auctions, one auction does not dominate the other in terms of expected revenue.[75]

Multi-Unit Vickrey Auction

We have just observed that both the pay-your-bid auction and the uniform-price auction are inefficient in many circumstances. The multi-unit Vickrey auction (proposed by Vickrey, 1961) solves this problem. In fact, the multi-unit Vickrey auction is the correct generalization of the single unit Vickrey auction. The rules are the following. Bidders submit demand schedules. The auctioneer orders the bids from highest to lowest and awards the k highest bids. Each bidder pays for the j-th unit that she wins an amount equal to the j-th highest rejected bid of her opponents. In other words, each bidder should pay for the externality that she imposes on the other bidders by winning.

Let us discuss a simple example to clarify the rules of the multi-unit Vickrey auction.

Example 1.2 *Imagine that three bidders bid for three units. Their bids on the j-th unit are given in Table 1.5. Bidder 1 wins two units, as she submits the highest two bids, and bidder 2 wins a single unit, as her bid on the first unit is the third highest bid. Bidder 1 pays 10, as the highest two rejected bids from the other bidders are 6 and 4. Similarly, bidder 2's payment is 7.* ▲

[72] Ausubel and Cramton (2002) and Engelbrecht-Wiggans and Kahn (1998a).
[73] Binmore and Swierzbinski (2000).
[74] Ausubel and Cramton (2002).
[75] Binmore and Swierzbinski (2000).

Unit\Bidder	1	2	3
1st	10	8	6
2nd	9	4	3
3rd	7	3	3

Table 1.5: Bids on the j-th unit for each bidder.

The multi-unit Vickrey auction has an equilibrium in weakly dominant strategies, in which each bidder bids her value for each unit. In such an equilibrium, the outcome is efficient. Still, the multi-unit Vickrey auction is rarely used in practice, presumably for the very same reasons why the Vickrey auction is hardly ever applied: it may not be very obvious to bidders that it is optimal for them to reveal their demand, items may be sold far below the winner's willingness-to-pay, and bidders may not be willing to reveal information in the auction as the seller may use this information in later occasions. In the multi-unit case, there is another reason why Vickrey auctions are not very popular. As the above example shows, the per unit price may differ substantially, usually at the advantage of 'large' bidders, i.e., bidders who win many units in the auction. Politically, this may be hard to sell.

Ausubel Auction

The Ausubel auction (named after Lawrence Ausubel) is the dynamic version of the multi-unit Vickrey auction. In this sense, the Ausubel auction is the correct generalization of the single unit English auction. The rules are the following. The auction price starts at zero and then increases continuously. At any price, each bidder indicates how many units she wants. At a certain price, it will happen that one bidder is guaranteed to win one or more units. This is the case when the aggregate demand of the other bidders is smaller than the available number of units. The bidder then wins the residual supply at the current price. In Ausubel's words: the bidder 'clinches' these units. The auction then continues from this price with the remaining units, until the next unit is clinched. This process continues until all units are allocated. The Ausubel auction has roughly the same advantages over the multi-unit Vickrey auction as the English auction has over the Vickrey auction under single-unit supply.[76]

Simultaneous Ascending Auction

The SAA may also be implemented to allocate identical objects to bidders who demand more than one unit. However, the SAA loses many of the nice properties we discussed in the previous section. In particular, the SAA is sensitive to collusion, demand reduction, and the 'exposure problem'.

[76] Ausubel (2004).

The German GSM auction shows how bidders may be able to collude under the SAA. In 1999, the German government auctioned ten GSM licenses using the SAA with a 10% minimum bid increment. In the first round, Mannesmann made a jump-bid of 18.18 million Deutsche Mark (DM) per megahertz (MHz) on the first five licenses, and 20 million DM on the last five. Doing so, Mannesmann signalled to its main competitor T-Mobile that it would be happy to share the ten licenses equally. To see this, observe that if T-Mobile overbids the standing high bid on licenses 1-5 with 10%, the final price is almost exactly 20 million DM per MHz. T-Mobile indeed understood Mannesmann's signal, and the auction ended after just two rounds.[77]

This example also shows that the SAA is sensitive to demand reduction. If Mannesmann and T-Mobile had bid up to their willingness-to-pay for the ten licenses, the auction probably would have ended with a much higher price per MHz. Even ignoring the possibility to collude, bidders have a strong incentive to reduce their demand in order to avoid winning items at a high price. Demand reduction is also reported in US spectrum auctions.[78]

Finally, the SAA is sensitive to the so-called exposure problem. The exposure problem occurs if bidders face the risk of winning too few objects. This may result in low revenue and an inefficient allocation of the objects.[79] The 1998 GSM auction in the Netherlands is an example of a situation in which the exposure problem was present. The Dutch government had split up the spectrum in 18 licenses. Two of these licenses contained sufficient spectrum to operate a GSM network. The other 16 licenses, however, were so small that an entrant needed at least four of them to be able to operate a network. Bidders were discouraged to submit high bids on the small licenses, as they faced the risk of being overbid on a fraction of them, and winning insufficient spectrum. In fact, the per MHz price on the large licenses turned out to be about two and a half times as high as for the small licenses. This violation of 'the law of one price' may indicate that the allocation of the licenses was inefficient.[80]

1.4.4 Auctions of Multiple Non-Identical Objects with Multi-Object Demand

In the previous sections, we have observed that with single-object demand or identical objects, rather straightforward auctions generate an efficient allocation of the objects. In this section, we will see that in the case of multi-object demand and non-identical objects, there is still a mechanism that result in an efficient allocation, but that this mechanism has serious practical drawbacks. We will discuss innovative new designs that may partly mitigate these disadvantages.

[77] Jehiel and Moldovanu (2001) and Grimm et al. (2003).
[78] Weber (1997).
[79] Onderstal (2002b) and van Damme (1999).
[80] Onderstal (2002b) and van Damme (1999).

Vickrey-Clarke-Groves Mechanism

We start by describing the Vickrey-Clarke-Groves (VCG) mechanism, which is efficient under a large range of circumstances. The VCG mechanism is developed by Clarke (1971) and Groves (1973), and generalizes the (multi-unit) Vickrey auction.[81] The most important property of VCG mechanisms is that these are able to allocate objects efficiently under fairly general conditions. Let us study the following model to make this claim more precise.

Assume that a seller wishes to allocate k objects among n bidders, labeled $i = 1, ..., n$. Let Γ be the set of possible allocations of the k objects over the bidders, and t_i the monetary transfer by bidder i (where a negative number indicates a payment to bidder i). Bidder i has the following quasi-linear utility function:

$$u_i(\gamma, t_i, \theta_i) = v_i(\gamma, \theta_i) - t_i,$$

where $\gamma \in \Gamma$ is a feasible allocation, v_i a valuation function, and θ_i bidder i's type, which is private information to bidder i. Note that this model is very general: no assumptions are made with respect to the risk attitude of the bidders and the distribution of the types, no structure is assumed on the complementarity or substitutability of the objects, even allocative externalities are included in this model. The main restrictions are (1) that utility is additively separable in money and the allocation of the objects, (2) a bidder's utility does not depend on other bidders' types, and (3) the exclusion of financial externalities, i.e., a bidder's utility does not depend on how much other bidders pay. Note that in this model, an allocation γ^* of the objects over the bidders is ex post efficient if and only if

$$\sum_i v_i(\gamma^*, \theta_i) \geq \sum_i v_i(\gamma, \theta_i) \text{ for all } \gamma \in \Gamma. \tag{1.6}$$

A VCG mechanism has the following properties. All bidders are asked to announce a type $\tilde{\theta}_i$. Let $\tilde{\theta}$ be the vector of announcements, i.e., $\tilde{\theta} \equiv (\tilde{\theta}_1, ..., \tilde{\theta}_n)$. The objects are allocated efficiently under the assumption that the $\tilde{\theta}_i$'s are the true types. Let $\gamma^*(\tilde{\theta})$ denote such an allocation. Moreover, bidder i pays an amount $t_i(\tilde{\theta})$ equal to

$$t_i(\tilde{\theta}) = \sum_{j \neq i} v_j(\gamma^*_{-i}(\tilde{\theta}), \tilde{\theta}_j) - \sum_{j \neq i} v_j(\gamma^*(\tilde{\theta}), \tilde{\theta}_j), \tag{1.7}$$

with $\gamma^*_{-i}(\tilde{\theta})$ an allocation that would be efficient if there were only $n - 1$ bidders $j \neq i$ and the announced types were the true types. In words, bidder i pays an amount equal to the externality that she imposes on the other bidders.

[81] Clarke and Groves constructed this mechanism for a class of problems that is far more general than the allocation of objects: their mechanism applies to any public choice problem.

Lemma 1.3 *For each bidder, it is a weakly dominant strategy to announce her true type in a VCG mechanism.*

The following result then immediately follows from Lemma 1.3, as in equilibrium the allocation of the objects is efficient by (1.6).

Proposition 1.11 *The VCG mechanism has an efficient equilibrium in weakly dominant strategies.*

Unfortunately, the VCG mechanism has many practical disadvantages. We have elaborated on several of these disadvantages while discussing the Vickrey auction and the multi-unit Vickrey auction: (1) it is not straightforward for bidders to understand how to play the auction, (2) the outcome may be politically problematic as items may be sold far below the willingness-to-pay of the winner, (3) bidders may not be willing to reveal information in the auction as the seller may use this information on later occasions, and (4) the per unit price may differ substantially, usually at the advantage of 'large' bidders. In the case of heterogeneous objects, there are several additional drawbacks. First, bidding is complex as bidders have to specify bids on all packages they desire to win. If 10 objects are for sale, a bidder may specify a bid on $2^{10} - 1 = 1023$ packages. Second, more competition may lead to lower prices, which may be as low as zero even if competition is fierce. The VCG mechanism may thus result in an outcome in which the winning bidders may pay so little that a group of non-winning bidders is willing to bid more collectively.[82] Some argue this is not a fair allocation. Third, the VCG mechanism is sensitive to collusion. Fourth, the VCG mechanism is not robust against 'shill bidding'.

Milgrom (2004) constructs a very nice example to illustrate the last three problems. Suppose that two spectrum licenses (A and B) are put up for sale. Assume first that there are only two entrants interested in buying the licenses. For each of them the value of a single license is \$0. The pair of licenses is worth \$1 billion for bidder 1 and \$900 million for bidder 2. If these bidders are the only bidders in the auction, the VCG mechanism boils down to a Vickrey auction where the pair of licenses is for sale. Bidder 1 will win both licenses for a price of \$900 million.

Now, suppose that two incumbents also participate in the auction. Bidder 3 [Bidder 4] is already active in region B [region A] and is only interested in license A [license B], for which she is willing to pay \$1 billion. If all four bidders play their weakly dominant strategy in the VCG mechanism, then the licenses will be allocated to the incumbents (as this is the most efficient allocation). Surprisingly, both bidders get the licenses for free. Why? If one of the incumbents does not show up, the value to the other bidders is \$1 billion, which happens to

[82] In auction theoretic terms we say that such an outcome is not in the 'core'. When objects are substitutes [complements], the VCG mechanism produces [does not produce necessarily] core outcomes (see Ausubel and Milgrom, 2006).

be exactly the same value to the other bidders if she is present. In other words the externality she imposes on the other bidders is $0. To summarize, more intense competition may lead to lower revenue to the seller.

Next, imagine that bidders 3 and 4 only have a value for a license equal to $400. In the case that they play their weakly dominant strategies, they will not win a license. However, if they coordinate in such a way that both bid $1 billion on their license, they do win for a price of $0. In other words, the VCG mechanism is sensitive to collusion.

Third, consider another situation in which bidders 1 and 2 participate, together with a third entrant who values the two licenses at $800 million. If all players play their weakly dominant strategies, bidder 3 will not win. However, bidder 3 can hire a 'shill' bidder. If she and the shill bidder bid $1 billion on license A and B respectively, then they will win both licenses for a price equal to zero. Hence, the VCG mechanism is not robust against shill bidding.

Clock-Proxy Auction

In the previous section, we have seen that the VCG mechanism, though efficient in theory, has some serious practical drawbacks. Ausubel, Cramton, and Milgrom (2006) propose the clock-proxy auction (CPA) as an alternative to mitigate these disadvantages. The CPA consists of two stages: a clock phase and a proxy round.

In the clock phase, the auctioneer announces a price for each object put up for sale. The bidders express the quantities of each object desired at the specified prices. Then the prices are increased for objects in excess demand, while other prices remain unchanged. Next, the bidders express quantities at the new prices. This process is repeated until there is no object with excess demand. The market-clearing prices serve as a lower bound on the prices in the proxy phase.

The proxy phase consists of a single round in which each bidder reports her values to a proxy agent for all packages she is interested in. Budget constraints can also be reported. The proxy agent iteratively submits package bids in an ascending package auction on behalf of the real bidder, selecting these packages that would maximize the real bidder's profit given the bidder's reported values. After each round, the auctioneer selects the provisionally-winning bids that maximize revenues, also considering the bids submitted in the clock phase. This process continues until the proxy agents have no new bids to submit.[83] The winners pay an amount equal to the standing high bids.

There are several reasons why the CPA may be expected to result in desirable outcomes. First, when objects are substitutes, the CPA is efficient, just as the VCG mechanism.[84] Second,

[83] It would take very long for the process to complete if the auctioneer ran the proxy auction with negligible bid increments. However, the process can be accelerated by using various methods, see Day and Raghavan (2007), Hoffman et al. (2006), and Wurman et al. (2004).

[84] Ausubel and Milgrom (2002).

the auction ends at a 'core' allocation for the reported preferences when bidding is straightforward, implying seller revenues are competitive.[85] In other words, the seller will always generate sufficient revenue if competition is fierce. This is in contrast to the VCG mechanism, in which the revenue may decrease all the way down to zero if additional bidders enter the auction, as the example in the previous section showed. Third, the CPA is expected to handle pretty well other complications of the VCG mechanism (such as collusion), and of the SAA (collusion, demand reduction, and the exposure problem).[86] Fourth, the CPA has the advantage that in the clock phase, valuable information about the prices can be revealed, in contrast to sealed-bid formats such as the VCG mechanism.

Still, the CPA has at least three disadvantages. First, truthfull bidding is not optimal anymore.[87] Second, the proxy phase consists of a single round, so that there is no way for bidders to learn from each others' bid about the values of the packages. The practical reason for this is that the allocation problem is 'NP-hard': there are no general ways for solving the problem in reasonable time, so that it may take a long time for subsequent rounds to finish.[88] Third, if competition is strong and objects are mostly substitutes, then a clock auction without a proxy round may be a better approach, since it offers the greatest simplicity and transparency, while being highly efficient.[89] In fact, clock auctions have been implemented in the field for products like electricity in recent years with considerable success.[90] In this simple setting, the SAA also performs well. However, a clock auction is to be preferred as it has a couple of advantages over the SAA: (1) complex bid signalling and collusive strategies are eliminated, since the bidders cannot see individual bids, but only aggregate information, (2) the exposure problem is eliminated: bidders are free to reduce quantities on any object (as long as at least one price increases), and (3) clock auctions are faster, as the SAA is in particular slow near the end when there is little excess demand.

Hierarchical Package Bidding Auction

Another design that mitigates the practical disadvantages of the VCG mechanism is the hierarchical package bidding auction (HPBA) as proposed by Goeree and Holt (2010).[91] The HPBA is a package bidding variant of the SAA. In the HPBA bidders not only can bid on individual objects but also on predefined packages of objects. These packages have a hierarchical structure with a fixed number of levels or tiers. Within each level, packages do not overlap,

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[85] Ausubel and Milgrom (2002).

[86] Ausubel et al. (2006).

[87] See Goeree and Lien (2009).

[88] See, e.g., de Vries and Vohra (2003). Rothkopf et al. (1998) show that with "hierarchical" pre-packaging of objects computational issues can be avoided.

[89] Ausubel et al. (2006).

[90] Ausubel and Cramton (2004).

[91] This auction design has been used by the FCC in the 700MHz auction (FCC auction #73). For more information, see http://wireless.fcc.gov/auctions/default.htm?job=auction_summary&id=73.

and a package of a lower level fits in exactly one higher level package.[92] In the HPBA, the provisionally winning bids for individual objects or packages in a particular round are those that maximize seller revenue.[93] Prices for all individual objects and packages are determined such that they signal the bid amounts required to unseat the current provisional winners.[94]

The HPBA has two important advantages. First, the winner determination problem is recursive and can be solved in a linear manner, because with the hierarchical structure, revenue-maximizing "winners" at one level can be compared with those at the next level up in the hierarchy. Second, the HBPA provides a simple and intuitive pricing rule that indicates how high to bid in a subsequent round to beat the current provisionally winning bids. If a top level (e.g. nationwide) package bid is provisionally winning then the non-overlapping nature of the lower level packages together with the simple pricing feedback allows smaller (e.g. regional) bidders to avoid the threshold problem.[95]

Pre-packaging comes however at a cost. Without extensive knowledge of bidders' valuations, there will be some efficiency loss due to the fact that the predefined packages do not completely coincide with the bidders' preferred packages. If a bidder has super-additive values for multiple objects that are not spanned by a particular package definition, then he is not fully protected from exposure risk.

Goeree and Holt (2010) have tested the HPBA (using laboratory experiments based on two-layer and three-layer hierarchies) against the SAA and a flexible package bidding auction (FPBA). In the presence of value complementarities, the HPBA significantly outperforms the SAA due to the exposure problem in the latter auction.[96] HPBA also performed much better in terms of auction revenues and efficiencies than the FPBA even though the predefined packages allowed under the HPBA did not match the preferred packages for half of the regional bidders (non-top-layer bidders). One factor that contributed to this is that the custom packages constructed under the FPBA tended to overlap, causing a "fitting problem" that made it difficult for strong regional bidders to unseat a top-layer bid. Another factor is that in rounds when the national bidder won nothing, regional bidders were unable to coordinate their bids under the FPBA while under the HPBA there were almost no coordination problems.

[92]For instance, if there are only three levels then the lowest could contain individual licenses, the middle level could contain non-overlapping regional packages, and the highest level could contain the national package.

[93]The winning set of bids may consist of bids from various levels, as long as each object is included in only one winning bid.

[94]If a bid on an individual object is provisionally winning, then that bid becomes the price for the object in the next round, as is the case for SAA. If a bid on a package is provisionally winning, then the prices for the individual objects in this package will be scaled up by lump-sum "taxes" to share the burden of unseating a provisionally winning package bid.

[95]The threshold problem (sometimes called the free-rider problem) is a coordination problem between the smaller bidders. These bidders can outbid the larger bidder (the sum of their values is higher than the value of the package to the large bidder) by coordinating their actions, but the incentive to avoid a large payment may result in a coordination failure and they may not win the auction.

[96]Brunner et al. (2010) experimentally test several combinatorial auction formats against the SAA. All these combinatorial auction formats yield improved performance when value complementarities are present.

1.4.5 Summary

In this section, we have studied multi-object auctions, focussing on the trade-off between efficiency and complexity. Table 1.6 gives an overview of efficient auctions under different assumptions with respect to the bidders' demands. These auctions are 'simple', apart from the VCG mechanism when bidders have demand for several non-identical objects. We have argued that the CPA and the HPBA are reasonable substitutes for this mechanism.

	Identical Objects	Non-Identical Objects
Single-Object Demand	Pay-Your-Bid Auction Multi-Unit Dutch Auction Uniform-Price Auction Multi-Unit English Auction	SAA
Multi-Object Demand	Pay-Your-Bid Auction (for many bidders) Uniform-Price Auction (for many bidders) Multi-Unit Vickrey Auction Ausubel Auction	VCG mechanism

Table 1.6: Efficient auction formats.

1.5 Conclusions

In this paper, we have made a swift tour of auction theory and its applications. We have roughly followed the historical development of the field which started, in the 1960s, with the work of William Vickrey. For single-object auctions, we have observed how auction theorists have developed the celebrated revenue equivalence theorem, how they have constructed revenue maximizing auctions, and how they have shown that under various circumstances the revenue equivalence between commonly used auctions breaks down. One of the main findings has been the trade-off between efficiency and revenue. In addition, we have discussed how to optimally auction incentive contracts. For multi-object auctions, we have seen that a new trade-off enters the picture, namely the trade-off between efficiency and complexity.

We have also observed how auction-like games such as the all-pay auction and the war of attrition have been applied to a range of economic situations, such as lobbying and technology battles. In fact, any situation in which several 'agents' 'compete' for a 'prize' might be fruitfully modeled as an auction. Examples are tax competition (several jurisdictions offer advantages to attract new factories), the labor market (several firms make a job offer to a potential employee), and advertising (several firms spend money and effort in advertising to attract customers).[97]

Analysis of the highly important question whether, or under what circumstances, auctions are appropriate allocation mechanisms was beyond the scope of this paper. On this topic, a few good sources are already available.[98] One argument often heard against (high stake) auctions is that auctions may increase consumer prices. This argument is in contrast with the theory of sunk costs. Still, theorists and experimental economists have found some support.[99] Another counter argument is that, if firms - with hindsight - have paid too much (for example due to changing market developments), firms will invest less (in new technology) compared to a situation in which the firms had not overbid.[100]

1.5.1 Outline of the Thesis

Chapters 2 through 5 of this Ph.D. thesis contain four published papers in auction theory.[101] Each chapter can be read independently from the other chapters. Chapter 2 studies auctions with financial externalities. It underpins the revenue ranking result with respect to Assumption (A8) in Table 1.2. It also performs a study of the effect of a reserve price on equilibrium bidding. Chapter 3 analyzes fund-raising mechanisms. The American Association of Fundraising Counsel has estimated that the population in the USA donates yearly circa 250 billion dollar to charity. Although charity is big business, not much is known about what the most effective way is to raise money. This chapter ranks the revenues of standard winner-pay auctions, all-pay auctions and lotteries and characterizes the optimal mechanism. Chapter 4 studies simultaneous pooled auctions with multiple bids and preference lists. In these auctions single-object demand bidders submit bids for every object for sale, and a preference ordering over which object they would like to get if they have the highest bid on more than one object. This type of auction has been used in the Netherlands and in Ireland to auction available spectrum. The main target of this chapter is to examine whether this type of auction satisfies elementary desirable properties such as the existence of an efficient equilibrium. Chapter 5 analyzes how inefficient auction outcomes due to strong negative (informational) externalities (created by post-auction interactions) can be avoided.[102] As we will see, a surprisingly simple mechanism can do the job.

[97] First steps in this direction have been made by Menezes (2003) (tax competition), Julien et al. (2000) (labor market), and Onderstal (2002a) (advertising).

[98] See, e.g., Janssen (2004).

[99] See Janssen (2006), Janssen and Karamychev (2009), and Offerman and Potters (2006). Haan and Toolsema (2011), in contrast, show that prices may *decrease* when firms pay high prices in auctions if firms are limitedly liable.

[100] Zheng (2001) and Haan and Toolsema (2011) investigate what governments can do to prevent firms from overbidding.

[101] The references are, respectively: Maasland and Onderstal (2007); Goeree et al. (2005); Janssen et al. (2010); and Janssen et al. (2011).

[102] This negative externality reflects the fact that often in an auction where some after-auction interaction (such as after-market competition) takes place, a bidders' type (such as a measure of his cost efficiency) positively affects his own value but negatively inflicts upon the value of the competitor.

1.6 Appendix: Proofs

Proofs of Propositions 1.1, 1.5, 1.6, and 1.10, and Lemmas 1.1-1.3 follow.

Proof of Proposition 1.1. Two different techniques can be used to prove this proposition. The first is the 'direct' method, in which we assume that all bidders but bidder 1 use the same bidding strategy $b : [0, \bar{v}] \to [0, \infty)$. Then we construct the best response b_1 for bidder 1, and we derive conditions under which this best response is equal to $b(v_1)$. This yields us an educated guess of what may be a symmetric Bayesian-Nash equilibrium. Finally, we need to check whether this strategy indeed constitutes an equilibrium.

For the moment, assume that b is strictly increasing and differentiable. Let b^{-1} be the inverse function of b. Bidder 1's expected payoff from bidding b_1 is given by the difference between her value and her bid, multiplied by the probability that she wins:

$$\begin{aligned} U_1(v_1, b_1, b) &= (v_1 - b_1)\Pr(b_1 > b(v_2), ..., b_1 > b(v_n)) \\ &= (v_1 - b_1)F(b^{-1}(b_1))^{n-1}. \end{aligned}$$

When deriving bidder 1's best response, the first order condition is

$$\begin{aligned} \frac{\partial U_1}{\partial b_1} &= (v_1 - b_1)\frac{\partial F(b^{-1}(b_1))^{n-1}}{\partial b_1} - F(b^{-1}(b_1))^{n-1} \\ &= (v_1 - b_1)\frac{(n-1)F(b^{-1}(b_1))^{n-2}f(b^{-1}(b_1))}{b'\left(b^{-1}(b_1)\right)} - F(b^{-1}(b_1))^{n-1} = 0. \end{aligned}$$

We wish to construct a symmetric equilibrium, so that $b_1 = b(v_1)$, or $b^{-1}(b_1) = v_1$ for all v_1. The above first order condition then yields the differential equation

$$b'(v)\,F(v)^{n-1} = (v - b(v))(n-1)F(v)^{n-2}f(v),$$

or equivalently

$$b'(v)\,F(v)^{n-1} + b(v)(n-1)F(v)^{n-2}f(v) = v(n-1)F(v)^{n-2}f(v).$$

Solving for this differential equation, we get

$$b(v)F(v)^{n-1} = \int_0^v x(n-1)F(x)^{n-2}dF(x) + c,$$

where c is the constant of integration. As a bidder with value 0 will always bid 0 in equilibrium, the boundary condition is $b(0) = 0$, so that $c = 0$. Hence, a natural candidate for a symmetric

Bayesian-Nash equilibrium is

$$b(v) = \frac{\int_0^v x(n-1)F(x)^{n-2}dF(x)}{F(v)^{n-1}} = v - \frac{\int_0^v F(x)^{n-1}dx}{F(v)^{n-1}}, \tag{1.8}$$

where the second equality follows from integration by parts. It can be shown that b is indeed strictly increasing and differentiable, the assumptions we started with. An implication is that the equilibrium outcome is efficient, i.e., it is always the highest bidder who obtains the object.

Finally, we have to check that b is indeed an equilibrium. Assume that all bidders but bidder 1 submit a bid according to b. Is it optimal for bidder 1 to follow this strategy as well? As b is strictly increasing, the bidder with the highest value submits the highest bid and wins the auction. Obviously, bidder 1 does not wish to submit a bid $b_1 > b(\bar{v})$. As b is strictly increasing and continuous, a bid $0 \leq b_1 \leq b(\bar{v})$ corresponds to a unique value w for which $b(w) = b_1$. We can write bidder 1's expected profit from bidding b_1 as

$$\begin{aligned} U(w, v_1) &= F(w)^{n-1}(v_1 - b(w)) \\ &= F(w)^{n-1}(v_1 - w) + \int_0^w F(x)^{n-1}dx \\ &\leq \int_0^{v_1} F(x)^{n-1}dx \\ &= U(v_1, v_1), \end{aligned}$$

regardless of whether $w \geq v_1$ or $w \leq v_1$. Therefore, bidder 1 optimally chooses $b_1 = b(v_1)$ so that indeed b constitutes a symmetric Bayesian-Nash equilibrium.

The 'indirect' method is an alternative way to derive a symmetric Bayesian-Nash equilibrium. Let $b_i : [0, \bar{v}] \to [0, \infty)$ be the equilibrium bid function for bidder i. We assume that all bidders reveal a value to the auctioneer, and that the auctioneer calculates the equilibrium bids as if the signals were the true signal. To prove that the bidding function b_i constitutes an equilibrium, we must show that all bidders have an incentive to reveal their true value. Again, we start by assuming that all bidders but bidder 1 use the same bidding strategy $b : [0, \bar{v}] \to [0, \infty)$. Next, we define the utility $U(v, w)$ for bidder 1 having value v who misrepresents herself as having value w, whereas the other bidders report truthfully. Then we derive conditions under which bidder 1 wishes to honestly reveal her type. From these conditions, we are able to derive the equilibrium bidding strategy.

For the moment, assume that b is strictly increasing and differentiable. Then

$$U(v, w) = (v - b(w))F(w)^{n-1},$$

so that

$$\frac{\partial U(v,w)}{\partial w} = (v - b(w))\frac{\partial F(w)^{n-1}}{\partial w} - b'(w)F(w)^{n-1}. \tag{1.9}$$

For bidder 1, it should be optimal to reveal her true value, so that $U(v,w)$ is maximized at $w = v$. Hence, the first order condition of the equilibrium is

$$\left.\frac{\partial U(v,w)}{\partial w}\right|_{w=v} = 0.$$

The resulting differential equation turns out to be the same as under the 'direct' method, so that (1.8) is a solution.

We still need to verify whether the second order condition $sign(\frac{\partial U(v,w)}{\partial w}) = sign(v - w)$ holds true. Observe that (1.9) can be rewritten as:

$$\begin{aligned} \frac{\partial U(v,w)}{\partial w} &= \left.\frac{\partial U(y,w)}{\partial w}\right|_{y=w} + (v-w)(n-1)F(w)^{n-2}f(w) \\ &= (v-w)(n-1)F(w)^{n-2}f(w). \end{aligned} \tag{1.10}$$

The second equality follows from the observation that

$$\left.\frac{\partial U(x,w)}{\partial w}\right|_{x=w} = 0.$$

From (1.10), it immediately follows that the second order condition is satisfied.

As it is always the bidder with the highest value who submits the highest bid, the expected revenue R_{FPSB} equals the expected bid of the bidder with the highest value:

$$\begin{aligned} R_{FPSB} &= \int_0^{\overline{v}} B_{FPSB}(v)dF(v)^n \\ &= \int_0^{\overline{v}} vdF(v)^n - \int_0^{\overline{v}} nf(v)\int_0^{v} F(x)^{n-1}dxdv \\ &= \int_0^{\overline{v}} vdF(v)^n - \int_0^{\overline{v}}\int_x^{\overline{v}} nf(v)dvF(x)^{n-1}dx \\ &= \int_0^{\overline{v}} vdF(v)^n - n\int_0^{\overline{v}}(1-F(x))F(x)^{n-1}dx \\ &= \int_0^{\overline{v}} vd\left[F(v)^n + n(1-F(v))F(v)^{n-1}\right] \\ &= E\{Y_2^n\}, \end{aligned}$$

where the third equality is obtained by changing the order of integration and the fifth equality by integration by parts. ■

Proof of Proposition 1.5. We use the 'indirect' method to solve for the symmetric equilibrium bidding strategies. Let b be a strictly increasing and differentiable bidding function, and assume that all bidders but 1 use this strategy. Imagine that bidder 1 with value v wishes to act as if having value w. Let $U(v, w)$ be bidder 1's expected utility. Then,

$$U(v, w) = vF(w)^{n-1} - b(w).$$

Maximizing $U(v, w)$ with respect to w yields the first order condition of the equilibrium:

$$(n-1)vF(v)^{n-2}f(v) - b'(v) = 0.$$

Integrating over v and applying the boundary condition $b(0) = 0$ yields

$$b(v) = (n-1)\int_0^v xF(x)^{n-2}dF(x).$$

It is readily verified that b is indeed strictly increasing and continuous, and that the second order condition holds. As all bidders pay their bid, the seller expects to collect $nE[b(v)]$ which can be shown to be equal to $E\{Y_2^n\}$. ■

Proof of Proposition 1.6. We use the 'direct' method to derive the equilibrium. Assume that bidder 2 employs the strictly increasing and differentiable bidding function b. The expected utility for bidder 1 when bidding b_1 is equal to

$$U(v_1, b_1, b(\cdot)) = \int_0^{b^{-1}(b_1)} (v_1 - b(x))dF(x) - b_1(1 - F(b^{-1}(b_1))),$$

where the first [second] term on the right hand side indicates bidder 1's utility when she wins [loses]. The first order condition can be expressed as

$$\frac{\partial U}{\partial b_1} = (v_1 - b_1)f(b^{-1}(b_1))(b^{-1}(b_1))' - (1 - F(b^{-1}(b_1))) + b_1 f(b^{-1}(b_1))(b^{-1}(b_1))' = 0.$$

Substituting $b^{-1}(b_1) = v_1$ and some routine calculations yield

$$b'(v_1) = \frac{v_1 f(v_1)}{1 - F(v_1)}.$$

Solving for b with boundary condition $b(0) = 0$, we obtain

$$b(v) = \int_0^v \frac{xf(x)}{1-F(x)}dx \tag{1.11}$$

as a candidate for the symmetric Bayesian Nash equilibrium. It is then straightforwardly checked that b indeed constitutes an equilibrium. The expected equilibrium revenue R_{WoA} equals twice the bid of the lowest bidder, which, once again, is equal to $E\{Y_2^n\}$ (with $n = 2$). ■

Proof of Lemmas 1.1 and 1.2. Let us first introduce some additional notation and concepts. Define the sets $V \equiv [0, \bar{v}]^n$ and $V_{-i} \equiv [0, \bar{v}]^{n-1}$ with typical elements $\mathbf{v} \equiv (v_1, ..., v_n)$ and $\mathbf{v}_{-i} \equiv (v_1, ..., v_{i-1}, v_{i+1}, ..., v_n)$ respectively. Let $g(\mathbf{v}) \equiv f(v_1) \cdots f(v_n)$ be the joint density of $\mathbf{v}$, and let $g_{-i}(\mathbf{v}_{-i}) \equiv f(v_1) \cdots f(v_{i-1})f(v_{i+1}) \cdots f(v_n)$ be the joint density of $\mathbf{v}_{-i}$.

We define an auction as follows. In an auction, bidders are asked to simultaneously and independently choose a bid. Bidder i chooses a bid $b_i \in B_i$, where B_i is the set of possible bids for bidder i, $i = 1, ..., n$. Let $\mathbf{b} = (b_1, ..., b_n)$ be the vector of bids. An auction is characterized by its outcome functions $(\widehat{p}_i, \widehat{x}_i)_{i=1,...,n}$, where $\widehat{p}_i(\mathbf{b})$ is the probability that bidder i wins the object, and $\widehat{x}_i(\mathbf{b})$ is the expected payment of bidder i to the seller.

Lemma 1.1 follows from the following considerations. Consider an auction and the following direct revelation game. First, the seller asks each bidder to announce her value. Then, he determines the bid that each bidder would have chosen in the equilibrium of the auction given her announced value. Next, he implements the outcomes that would result in the auction from these bids. As the strategies form an equilibrium of the auction, it is an equilibrium for each bidder to announce her value truthfully in the direct revelation game. Therefore, the revelation game has the same outcome as the auction, so that both the seller and the bidders obtain the same expected utility as in the equilibrium of the auction.[103]

Bidder i's utility of direct revelation mechanism (p, x) given $\mathbf{v}$ equals $v_i p_i(\mathbf{v}) - x_i(\mathbf{v})$, so that if bidder i knows her value v_i, her expected utility from (p, x) can be written as

$$U_i(p, x, v_i) \equiv \int_{V_{-i}} \{v_i p_i(\mathbf{v}) - x_i(\mathbf{v})\} g_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}, \tag{1.12}$$

with $d\mathbf{v}_{-i} \equiv dv_1...dv_{i-1}dv_{i+1}...dv_n$.

The individual rationality constraint follows from the assumption that each bidder expects nonnegative expected utility, so that

$$U_i(p, x, v_i) \geq 0, \forall v_i, i. \tag{1.13}$$

[103] Note that we have already applied the revelation principle in the 'indirect method' for deriving an equilibrium for an auction (see the proof of Proposition 1.1).

The incentive compatibility constraint is imposed as we demand that each bidder has an incentive to announce her value truthfully. Thus,

$$U_i(p, x, v_i) \geq \int\limits_{V_{-i}} \{v_i p_i(\mathbf{v}_{-i}, w_i) - x_i(\mathbf{v}_{-i}, w_i)\} g_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}, \ \forall v_i, w_i, i,$$

where $(\mathbf{v}_{-i}, w_i) \equiv (v_1, ..., v_{i-1}, w_i, v_{i+1}, ..., v_n)$.

The seller aims at finding an auction which gives him the highest possible expected revenue (*the seller's problem*). The seller's expected revenue of (p, x) is

$$U_0(p, x) \equiv \int\limits_V \sum_{i=1}^{n} x_i(\mathbf{v}) g(\mathbf{v}) d\mathbf{v}, \tag{1.14}$$

with $d\mathbf{v} \equiv dv_1...dv_n$.

Now, let

$$Q_i(p, v_i) \equiv E_{v_{-i}}\{p_i(\mathbf{v})\}$$

be the conditional probability that bidder i wins the object given her value v_i. Lemma 1.4 gives a characterization of direct revelation games that are individually rational and incentive compatible.

Lemma 1.4 *The direct revelation game (p, x) is individually rational and incentive compatible if and only if*

$$Q_i(p, w_i) \geq Q_i(p, v_i) \ \textit{if} \ w_i \geq v_i, \ \forall w_i, v_i, i, \tag{1.15}$$

$$U_i(p, x, v_i) = U_i(p, x, \underline{v}_i) + \int\limits_{\underline{v}_i}^{v_i} Q_i(p, y_i) dy_i, \ \forall v_i, i, \ \textit{and} \tag{1.16}$$

$$U_i(p, x, \underline{v}_i) \geq 0, \ \forall i. \tag{1.17}$$

Proof. Incentive compatibility implies

$$U_i(p, x, w_i) \geq U_i(p, x, v_i) + (w_i - v_i) Q_i(p, v_i), \tag{1.18}$$

so that (p, x) is individually rational and incentive compatible if and only if (1.13) and (1.18) hold. With (1.18),

$$(w_i - v_i) Q_i(p, w_i) \geq U_i(p, x, w_i) - U_i(p, x, v_i) \geq (w_i - v_i) Q_i(p, v_i),$$

from which (1.15) follows. Moreover, these inequalities imply

$$\frac{\partial U_i(p, x, v_i)}{\partial v_i} = Q_i(p, v_i), \tag{1.19}$$

at all points where p_i is differentiable in v_i. By integration of (1.19), (1.16) is obtained. Finally, with (1.13) and (1.16), individual rationality is equivalent to (1.17). ■

Now, with (1.12), the seller's expected revenue (1.14) can be rewritten as

$$U_0(p,x) = \sum_{i=1}^{n} \int_V v_i p_i(\mathbf{v}) g(\mathbf{v}) d\mathbf{v} - \sum_{i=1}^{n} \int_{\underline{v}_i}^{\bar{v}_i} U_i(p,x,v_i) f_i(v_i) dv_i. \tag{1.20}$$

Taking the expectation of (1.16) over v_i and using integration by parts, we obtain

$$E_{v_i}\{U_i(p,x,v_i)\} = U_i(p,x,\underline{v}_i) + E_{v_i}\left\{\frac{1-F_i(v_i)}{f_i(v_i)} p_i(\mathbf{v})\right\}. \tag{1.21}$$

Lemma 1.2 follows from (1.20) and (1.21). ■

Proof of Proposition 1.10. The proof consists of two parts: first we consider which mechanisms are incentive compatible by looking at firms' bidding behavior, after which we derive which incentive compatible mechanism maximizes the buyer's utility.

Firms' bidding behavior

If all firms bid truthfully, firm i's expected utility given its efficiency parameter α_i is equal to

$$U_i(\alpha_i) = E_{\alpha_{-i}}\{t_i(\boldsymbol{\alpha}) - x_i(\boldsymbol{\alpha})(\varphi(e_i(\boldsymbol{\alpha})) - \alpha_i e_i(\boldsymbol{\alpha}))\}, \tag{1.22}$$

where

$$\varphi(e) = \frac{1}{2}e^2 + e.$$

Let $U_i(\alpha_i, \tilde{\alpha}_i)$ be firm i's utility if it has efficiency parameter α_i, it announces $\tilde{\alpha}_i$, and all other firms truthfully reveal their type. Then

$$U_i(\alpha_i, \tilde{\alpha}_i) = E_{\alpha_{-i}}\{t_i(\alpha_{-i}, \tilde{\alpha}_i) - x_i(\alpha_{-i}, \tilde{\alpha}_i)\varphi(e_i(\alpha_{-i}, \tilde{\alpha}_i))\} + \alpha_i E_{\alpha_{-i}}\{e_i(\alpha_{-i}, \tilde{\alpha}_i) x_i(\alpha_{-i}, \tilde{\alpha}_i)\}. \tag{1.23}$$

Incentive compatibility requires that firm i optimally announces its own type, so that

$$\frac{\partial U_i(\alpha_i, \tilde{\alpha}_i)}{\partial \tilde{\alpha}_i} = 0 \tag{1.24}$$

at $\tilde{\alpha}_i = \alpha_i$. From (1.22), (1.23), and (1.24), it follows that

$$\begin{aligned} \frac{dU_i(\alpha_i)}{d\alpha_i} &= \left.\frac{\partial U_i(\alpha_i, \tilde{\alpha}_i)}{\partial \tilde{\alpha}_i}\right|_{\tilde{\alpha}_i = \alpha_i} + E_{\alpha_{-i}}\{e_i(\boldsymbol{\alpha}) x_i(\boldsymbol{\alpha})\} \\ &= E_{\alpha_{-i}}\{e_i(\boldsymbol{\alpha}) x_i(\boldsymbol{\alpha})\}. \end{aligned} \tag{1.25}$$

The participation constraint then immediately reduces to

$$U_i(0) \geq 0. \tag{1.26}$$

The buyer's problem

To maximize the buyer's utility given (1.25) and (1.26), we apply the Pontryagin principle. The buyer solves

$$\max_{(x_i(\cdot),e_i(\cdot),U_i(\cdot))} E_{\boldsymbol{\alpha}} \sum_i \{x_i(\boldsymbol{\alpha})(e_i(\boldsymbol{\alpha}) - C_i(e_i(\boldsymbol{\alpha}), \alpha_i)) - U_i(\alpha_i)\}$$

$$\text{s.t.} \quad \begin{array}{l} \dot{U}_i(\alpha_i) = E_{\alpha_{-i}}\{e_i(\boldsymbol{\alpha})x_i(\boldsymbol{\alpha})\} \\ U_i(0) \geq 0. \end{array}$$

This problem looks horrendously complicated, but we can rely on the following three tricks to make it solvable. First, it can be shown that $e_i(\boldsymbol{\alpha})$ only depends on α_i. We do not prove this formally, but the intuition is that e_i is a stochastic scheme when it depends on announcements other than α_i. As the firm's cost function is convex, there is a deterministic scheme that only depends on α_i which strictly improves the objective function of the buyer.

The second trick is to keep the x_i's fixed, and solve the problem. Let

$$X_i(\alpha_i) \equiv E_{\boldsymbol{\alpha}_{-i}} \{x_i(\boldsymbol{\alpha})\}.$$

For given $X_i(\alpha_i)$, the buyer's problem can be decomposed into the following n independent maximization problems:

$$\max_{(e_i(\cdot),U_i(\cdot))} \int_0^1 \{X_i(\alpha_i)(e_i(\alpha_i) - C_i(e_i(\boldsymbol{\alpha}), \alpha_i)) - U_i(\alpha_i)\} \, dF(\alpha_i)$$

$$\text{s.t.} \quad \begin{array}{l} \dot{U}_i(\alpha_i) = e_i(\alpha_i)X_i(\alpha_i) \\ U_i(0) \geq 0. \end{array}$$

The third trick in solving the buyer's problem is to realize that these problems amount to dynamic optimization programs where U_i is the state variable and e_i the control variable. We can now apply the Pontryagin principle to find a solution. The Hamiltonian H_i of each program is given by

$$H_i(\alpha_i, e_i, U_i, \mu_i) = \{X_i(\alpha_i)(e_i - C_i(e_i, \alpha_i)) - U_i\} \, f(\alpha_i) + \mu_i e_i X_i(\alpha_i).$$

Using the Pontryagin principle, we obtain the first-order condition of the programs:[104]

$$\begin{aligned}\dot{\mu}_i(\alpha_i) &= f(\alpha_i)\\ \mu_i(\alpha_i) &= \left(\frac{\partial C_i(e_i^*(\boldsymbol{\alpha}),\alpha_i)}{\partial e_i}-1\right) f(\alpha_i)\\ \mu_i(1) &= 0.\end{aligned}$$

Substituting $C_i(e_i,\alpha_i)=\frac{1}{2}e_i^2+e_i-\alpha_i e_i$ together with some straightforward calculations yields the optimal effort levels:

$$e_i^*(\boldsymbol{\alpha})=\begin{cases}\alpha_i-\frac{1-F(\alpha_i)}{f(\alpha_i)} & \text{if } \alpha_i\geq\underline{\alpha}\\ 0 & \text{if } \alpha_i<\underline{\alpha},\end{cases} \tag{1.27}$$

with $\underline{\alpha}$ the unique solution to $y=\frac{1-F(y)}{f(y)}$ with respect to y. Under these effort levels, the buyer's problem is reduced to

$$\max_{X_i(\cdot)}\sum_i E_{\alpha_i}\left\{X_i(\alpha_i)(e_i^*(\alpha_i)-C_i(e_i^*(\boldsymbol{\alpha}),\alpha_i))-\int_0^{\alpha_i} e_i^*(y)X_i(y)dy\right\},$$

which is equivalent to

$$\max_{X_i(\cdot)}\frac{1}{2}\sum_i E_{\alpha_i}\left\{X_i(\alpha_i)e_i^*(\alpha_i)^2\right\}. \tag{1.28}$$

From (1.28), it is straightforward to see to which firm i the project should be allocated. The buyer's expected utility is proportional to the sum of the firms' winning probability and the square of the optimal effort. By (1.3) and (1.27), $(e_i^*(\alpha_i))^2$ is strictly increasing in α_i for $\alpha_i\geq\underline{\alpha}$. Therefore, it is optimal for the buyer to maximize the winning probability of the firm with the highest α_i, i.e., to always allocate the project to the most efficient firm. ■

Proof of Lemma 1.3. The proof is by contradiction. Suppose that telling the truth is not a weakly dominant strategy for all bidders. Then for some bidder i there exist a θ_i, $\tilde{\theta}_i$, and $\theta_{-i}\equiv(\theta_1,\theta_{i-1},\theta_{i+1},\theta_n)$ such that

$$v_i(\gamma^*(\tilde{\theta}_i,\theta_{-i}),\theta_i)-t_i(\tilde{\theta}_i,\theta_{-i})>v_i(\gamma^*(\theta_i,\theta_{-i}),\theta_i)-t_i(\theta_i,\theta_{-i}).$$

Substituting t_i for (1.7) implies that

$$\sum_j v_j(\gamma^*(\tilde{\theta}_i,\theta_{-i}),\theta_j)>\sum_j v_j(\gamma^*(\theta_i,\theta_{-i}),\theta_j),$$

which contradicts (1.6). Thus it must be a weakly dominant strategy for each bidder to

[104] The second order conditions can be shown to hold as well.

announce her true type. ■

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Chapter 2

Auctions with Financial Externalities

2.1 Introduction

In this paper, we study sealed-bid auctions with financial externalities. Financial externalities arise when losers benefit directly or indirectly from a high price paid by the winner(s). In auction theory, it is generally assumed that losers are indifferent about how much the winner(s) pay(s). However, in real-life auctions, this assumption might be false. In reality, an auction is not an isolated game, as winners and losers also interact after the auction. Paying a high price in the auction could make a winner a weaker competitor later.

The series of UMTS auctions[1] that took place in Europe offers a concrete example of auctions in which losers benefit *indirectly* from a high price paid by the winners. In this context, there are at least three ways that firms that do not acquire a license may benefit from a winning firm that pays a high price. First, the share values of winning firms may drop, which makes the winner vulnerable to a hostile take-over by competing firms. For instance, the drop in share value of the Dutch telecom company KPN of about 95% is (according to telecom specialists) correlated with the huge sum of money the company spent to acquire British, Dutch and German UMTS licenses.[2] Second, if firms are budget constrained, a high payment in the first auction may give competing firms an advantage in later auctions. Third, high payments may force the winning firms to cut their budget for investment, which may be favorable for the losers' position in the telecommunications market, as the losing firms are not only competitors of the winning firms in the auction, but in the telecommunications market as well. Indeed, Klemperer (2002) argues, on the basis of a study by Börgers and Dustmann

[1] These are spectrum auctions. UMTS stands for Universal Mobile Telecommunications System: a third generation mobile telecommunication standard.

[2] In the UK, KPN bought part of the TIW license after the auction. In Germany, KPN has a majority share in E-Plus.

(2002), that financial externalities might (partly) explain seemingly irrational bidding in the British UMTS auction.[3]

Financial externalities occur *directly* when losing bidders receive money from the winner(s). For instance, this may happen in the case of bidding rings, in which a member of the ring receives money when she does not win the object. Also, partnerships are dissolved using an auction in which losing partners obtain part of the winner's bid. In takeover contests, losing bidders who have shares in the target firm receive payoffs proportional to the sales price. Furthermore, the owner of a large estate may specify in his last will that after his death, the estate should be sold to one of the heirs in an auction, where the auction revenue is divided among the losers. Finally, in some premium auctions, the runner-up receives a premium proportional to the difference between her bid and the minimum price (Goeree and Offerman, 2004).

In Section 2.2, we present a model of bidding in sealed-bid auctions with financial externalities. Either the first-price sealed-bid auction (FPSB) or the second-price sealed-bid auction (SPSB) is used to sell an indivisible object. We assume an independent private signals model, with private values models and common value models as special cases. Financial externalities are exogenously given and modelled by a parameter φ that is inserted in the bidders' utility functions. This is the simplest extension of the independent private signals model which incorporates financial externalities. Despite its simplicity, this model appears to be sufficiently rich to generate interesting insights.

In Section 2.3, we derive results for FPSB and SPSB without reserve price. We find a unique symmetric and efficient bid equilibrium for each of the two auction types. Equilibrium bids in FPSB decrease when φ increases. An intuition for this result is that larger financial externalities make losing more attractive for the bidders so that they submit lower bids. The effect of financial externalities on the equilibrium bids in SPSB is ambiguous. A possible explanation is that in SPSB, a bidder is not only inclined to bid less the higher φ is (as she gets positive utility from losing), she also has an incentive to bid higher, because, given that she loses, she is able to influence directly the level of payments made by the winner. Moreover, we construct an example in which the seller's revenue increases when φ increases. This finding suggests that the seller may gain more revenue by guaranteeing the losers a fraction of the auction revenue. This, however, turns out not to be the case: under no circumstances does revenue sharing increase revenue. Finally, we give a revenue comparison between FPSB and SPSB. We find that SPSB results in a higher expected revenue than FPSB unless a bidder's interest in her own payments is equal to the sum of the other bidders' interest in what she pays. In that case, FPSB and SPSB are revenue equivalent.

In Section 2.4, we characterize equilibrium bid strategies for the case that a reserve price is imposed in FPSB and SPSB. For simplicity, we assume a model with independent private

[3] Formal analysis is needed to get decisive answers.

values. In this section, we introduce the concept of a weakly separating Bayesian Nash equilibrium, which is an equilibrium in which all types below a threshold type abstain from bidding, and all types above this type submit a bid according to a strictly increasing bid function. We find that FPSB has no weakly separating Bayesian Nash equilibrium. However, we derive an equilibrium in which bidders with low values abstain from bidding, bidders with intermediate values pool at the reserve price, and bidders with high values submit a bid according to a strictly increasing bid function. SPSB has a weakly separating Bayesian Nash equilibrium if, and only if, the reserve price is sufficiently low (or so high that no bidder submits a bid). Otherwise, the equilibrium involves pooling at the reserve price.

These findings may shed new light on observations of identical bids in auctions. Theoretical work suggests that several bidders submitting a bid equal to the reserve price is a signal of collusive behavior.[4] Our finding indicates that these identical bids may not be explained by collusion but by the existence of financial externalities.

2.1.1 Related Literature

Our paper is related to a large range of papers in finance, industrial organization, and microeconomics that study similar models. In order to better link our paper to this literature, we anticipate some of the specifics of our model. We define the utility function of each bidder i as follows:

$$u_i(j,b) = \begin{cases} v_i - b & \text{if } j = i \\ \varphi b & \text{if } j \neq i, \end{cases} \tag{2.1}$$

where v_i is the value that i attaches to the auctioned object, j is the winner of the auction, b is the payment by the winner, and $\varphi > 0$ is the parameter indicating financial externalities.

Engers and McManus (2007) investigate a related model in the context of charity auctions, employing the following payoff structure:

$$u_i(j,b) = \begin{cases} v_i - b + \theta b & \text{if } j = i \\ \lambda b & \text{if } j \neq i. \end{cases} \tag{2.2}$$

The parameter $\lambda > 0$ can be interpreted as an altruistic feeling a person obtains if another bidder wins the auction (as the auction revenue is transferred to a charitable organization). $\theta \geq \lambda$ represents the utility the winner gets for her own contribution, where $\theta - \lambda$ is interpreted as the winner's 'warm glow' for giving to charity. Engers and McManus' model is a special case of ours in the sense that in the absence of a reserve price, the equilibrium bids in Engers

[4]McAfee and McMillan (1992) show that in first-price auctions it is optimal for a bidder who belongs to a weak cartel (that is, one that cannot make side payments) to submit a bid exactly equal to the reserve price (provided his value for the object exceeds the reserve price), and rely on the auctioneer's tie-breaking rule to randomly select a winner. Competitive bidders would submit identical bids with probability zero.

and McManus can be constructed from the equilibrium bids in our framework. To see this, rewrite (2.2) as

$$u_i(j,b) = \tilde{u}_i\left(j,\tilde{b}\right) = \begin{cases} v_i - \tilde{b} & \text{if } j = i \\ \varphi\tilde{b} & \text{if } j \neq i, \end{cases}$$

where $\tilde{b} = (1-\theta)\, b$, and $\varphi = \lambda\,(1-\theta)^{-1}$. Consequently, our model applies with the utility function $\tilde{u}$.[5] Observe that a strictly positive reserve price may imply that the candidate for the equilibrium bid in our model is below the reserve price, as this bid is constructed by multiplying the equilibrium bid in Engers and McManus by $1-\theta < 1$.

Quite some papers in the literature study special cases of Engers and McManus' model, and hence special cases of ours.[6] For instance, Engelbrecht-Wiggans (1994), Bulow et al. (1999), Ettinger (2009), and Goeree et al. (2005) [**Chapter 3 of this thesis**] assume $\lambda = \theta$, and apply this model to a large range of economic settings, including charity auctions and take-over battles in which the bidding firms own a toehold in the target.[7] Engelbrecht-Wiggans (1994) and Engers and McManus (2007) show that SPSB generates more revenue than FPSB, a result we show to hold true in our context as well. In contrast, Bulow et al. (1999) prove that small asymmetries in the toehold may imply that FPSB is superior to SPSB in terms of revenues raised.[8] Engers and McManus (2007) and Goeree et al. (2005) [**Chapter 3 of this thesis**] show that all-pay auctions dominate winner-pay auctions (such as FPSB and SPSB) as in the latter auctions, bidders forgo a positive externality if they top another's high bid.

A specific interpretation of our model is a situation in which all losing bidders equally share the payment by the winner, i.e., $\varphi = 1/(n-1)$, where n is the total number of bidders. This may be the case in situations of 'knock-out' auctions organized by bidding rings (Graham and Marshall, 1987, McAfee and McMillan, 1992, and Deltas, 2002), dissolving partnerships (Cramton et al., 1987, Angeles de Frutos, 2000, Kittsteiner, 2003, and Morgan, 2004), and heirs bidding for a family estate (Engelbrecht-Wiggans, 1994). If $n = 2$ and $\varphi = 1$, then FPSB and SPSB are special cases of the k-double auction with $k = 0$ and $k = 1$ respectively (Van Damme, 1992, and Kittsteiner, 2003).[9]

[5] Engers and McManus study their model in an independent private values setting, while we focus on a more general setting with independent private signals.

[6] Ettinger (2010) studies a more general model allowing for non-linear financial externalities, and compares FPSB and SPSB in a model with complete information.

[7] Engelbrecht-Wiggans (1994) analyzes a model with affiliated signals, while Bulow et al. (1999), Ettinger (2009) and Goeree et al. (2005) [**Chapter 3 of this thesis**] study special cases of this model: Bulow et al. (1999) focus on pure common values, Ettinger (2009) and Goeree et al. (2005) [**Chapter 3 of this thesis**] on independent private values.

[8] Bulow et al. (1999) show that a slight asymmetry in value functions may have dramatic effects on bidding behavior in the English auction in a common value setting, as the bidder with the lower value function faces a strong winner's curse, and therefore bids zero in equilibrium. This extreme outcome does not arise in FPSB.

[9] The k-double auction has the following rules. Both bidders submit a bid. The highest bidder wins the object, and pays the loser an amount equal to $kb_L + (1-k)b_W$, where b_L is the loser's bid, b_W the winner's

We add the following to the above mentioned papers. First, we derive comparative statics that are different from the ones in the other models. For instance, Engelbrecht-Wiggans (1994) shows that the equilibrium bid functions of FPSB and SPSB are increasing in θ if $\lambda = \theta$. In our model, the effect of φ on the equilibrium bids can be both increasing and decreasing.[10] Second, we examine the question whether the seller has an incentive to share a fraction of the auction revenue with the losing bidders. Bulow et al. (1999) show in their pure common value setting that the seller may increase the revenue of SPSB by giving a weak bidder a toehold,[11] in contrast to what we find in our model. Third, we analyze the effect of the reserve price on the equilibrium bids, which, as mentioned, may be different than in the above related settings since the one-to-one relationship between the equilibrium bid functions ceases to hold. Ettinger (2009) is one of the few papers that studies auctions with a positive reserve price. In his framework, he does not observe pooling at the reserve price, in contrast to what we find in our setting.

2.2 The Model

We consider a situation with $n \geq 2$ risk neutral bidders, numbered $1, 2, ..., n$, who bid for one indivisible object. The auction being used is either FPSB or SPSB.

We use Milgrom and Weber's (1982) model as a starting point with independent signals instead of affiliated ones. We assume that each bidder i receives a one-dimensional private signal t_i (we also say that bidder i is of type t_i). We will let $v_i(\mathbf{t})$ denote the value of the object for bidder i given the vector $\mathbf{t} \equiv (t_1, ..., t_n)$ of all signals. Special cases are independent private values models ($v_i(\mathbf{t})$ only depends on t_i), and common value models ($v_i(\mathbf{t}) = v_j(\mathbf{t})$ for all $i, j, \mathbf{t}$). Without loss of generality, we assume that the signals t_i are independently drawn from the uniform distribution on the interval $[0, 1]$.[12]

We make the following assumptions on the functions v_i.

- *Value Differentiability:* v_i is differentiable in all its arguments, for all $i, \mathbf{t}$.
- *Value Monotonicity:* $v_i(\mathbf{t}) \geq 0$, $\frac{\partial v_i(\mathbf{t})}{\partial t_i} > 0$, $\frac{\partial v_i(\mathbf{t})}{\partial t_j} \geq 0$, and $\frac{\partial v_i(\mathbf{t})}{\partial t_i} > \frac{\partial v_i(\mathbf{t})}{\partial t_j}$ for all $i, j, \mathbf{t}$.
- *Symmetry:* $v_i(..., t_i, ..., t_j, ...) = v_j(..., t_j, ..., t_i, ...)$ for all t_i, t_j, i, j.

bid, and $k \in [0, 1]$.

[10] Intuitively, if both the winner and the losers profit from the payment by the winner, bidders do have more incentive to bid higher than if only losers profit. This explains why the effect of θ and φ can be opposite.

[11] By leveling the playing field in this way, the seller induces a fairer contest. The likely higher auction price may more than outweigh the cost of giving away the toehold.

[12] Suppose the signals t_i are drawn from strictly increasing distribution functions F_i. Such a model is isomorphic to a model with uniformly distributed signals $\tilde{t}_i$ and value functions $\tilde{v}_i$, where $\tilde{t}_i \equiv F_i(t_i)$ and $\tilde{v}_i\left(\tilde{t}_1, ..., \tilde{t}_n\right) \equiv v_i(F_1^{-1}(\tilde{t}_1), ..., F_n^{-1}(\tilde{t}_n))$.

Value Differentiability is imposed to make the calculations on the equilibria tractable. *Value Monotonicity* indicates that all bidders are serious and that bidders' values are strictly increasing in their own signal and weakly in the signals of the others. Moreover, it includes a single crossing property. *Symmetry* may be crucial for the existence of efficient equilibria in standard auctions. *Value Differentiability, Value Monotonicity*, and *Symmetry* together ensure that the bidder with the highest signal is also the bidder with the highest value. These assumptions therefore imply that the seller assigns the object efficiently if and only if the bidder with the highest signal gets it.

Also, let us define $v(x, y)$ as the expected value that bidder i assigns to the object, given that her signal is x, and that the highest signal of all the other bidders is equal to y:

$$v(x, y) \equiv E\{v_i(\mathbf{t})|t_i = x, \max_{j \neq i} t_j = y\}.$$

By *Symmetry*, v does not depend on i.

The bidders are expected utility maximizers. Each bidder is risk neutral and cares about what other bidders pay in the auction. The utility of the bidders is as specified in (2.1):

$$u_i(j, b) = \begin{cases} v_i - b & \text{if } j = i \\ \varphi b & \text{if } j \neq i, \end{cases}$$

where v_i is the value that i attaches to the auctioned object, j is the winner of the auction, b is the payment by the winner, and $\varphi > 0$ is the parameter indicating financial externalities. It is a natural assumption to let a bidder's interest in her own payments be larger than or equal to the sum of the other bidders' interest in her payments, so that we assume $\varphi \leq 1/(n-1)$.

Financial externalities occur directly when the seller pays up to each loser. If the winner pays b to the seller, then each loser will get φb. The seller will end up with $(1 - (n-1)\varphi)b$. If the financial externalities occur indirectly, then the seller will get b. Since in the case of direct financial externalities, the seller does not receive the entire bid of the winner, we distinguish between the seller's expected revenues and the expected price, i.e. the total payment by the winner. Of course, in the case of indirect financial externalities, the seller's expected revenues and the expected price are the same.

2.3 Zero Reserve Price

Consider FPSB and SPSB with a zero reserve price. In Section 2.3.1 and Section 2.3.2 we characterize the equilibrium bid function for FPSB and SPSB, respectively. In Section 2.3.3 we give a revenue comparison between FPSB and SPSB.

2.3.1 First-Price Sealed-Bid Auction

The following proposition characterizes the equilibrium bid function for FPSB.[13] To derive equilibrium bidding, we suppose that, in equilibrium, all bidders use the same bid function. By a standard argument, this bid function must be strictly increasing and continuous. Let $U(t,s)$ be the utility for a bidder with signal t who behaves as if having signal s, whereas the other bidders play according to the equilibrium bid function. A necessary equilibrium condition is that

$$\frac{\partial U(t,s)}{\partial s} = 0$$

at $s = t$. From this condition, a differential equation can be derived from which the equilibrium bid function is uniquely determined (at least if we restrict our attention to differentiable bid functions). The auction outcome is efficient. Observe that in the case of independent private values ($v(x,y)$ only depends on x), the bid function is strictly increasing in n.

Proposition 2.1 *The unique differentiable symmetric Bayesian Nash equilibrium of FPSB is characterized by*

$$B_1(\varphi, t) = v(t,t) - \frac{\varphi}{1+\varphi} v(t,t) - \frac{1}{1+\varphi} \int_0^t \frac{dv(y,y)}{dy} \left(\frac{y}{t}\right)^{(n-1)(1+\varphi)} dy, \tag{2.3}$$

where $B_1(\varphi, t)$ is the bid of a bidder with signal t. The outcome of this auction is efficient.

Each of the terms of the right-hand side (RHS) of (2.3) has an attractive interpretation. The first term is the equilibrium bid for a bidder with type t in SPSB without financial externalities, as in the absence of financial externalities, in SPSB, a bidder will submit a bid equal to her maximal willingness to pay given that her strongest opponent has the same signal as she (Milgrom and Weber, 1982). The second term is the bid shading that would occur if all bidders attached the same value $v(t,t)$ to the object: in such a situation, if a bidder wins at a bid of b, her utility is $v(t,t) - b$, while if an opponent wins at the same bid, her utility is φb. Equating these utilities results in a bid of $\frac{1}{1+\varphi} v(t,t)$. The third term can be interpreted as the strategic bid shading because of private values. Note that if $\varphi = 0$, this term is equal to the standard strategic bid shading in FPSB.

This interpretation of the equilibrium bid function suggests that this function is decreasing in φ, which in fact follows directly from (2.3):

Corollary 2.1 *Increasing φ decreases $B_1(\varphi, t)$.*

Observe that Corollary 2.1 implies that the seller's revenue is decreasing in φ, both for direct and indirect financial externalities. The seller's revenue in the case of direct financial

[13] The proof of this and all other propositions are relegated to the appendix.

externalities is a fraction $(1-(n-1)\varphi)$ of the revenues under indirect financial externalities. Corollary 2.1 implies that revenue under indirect financial externalities is a decreasing function of φ, which also holds true for direct financial externalities as $(1-(n-1)\varphi)$ is decreasing in φ as well. Therefore, it is not attractive for the seller to have the losers share a fraction of the auction revenue.

2.3.2 Second-Price Sealed-Bid Auction

Equilibrium bids for SPSB are obtained using the same logic as for FPSB. The analysis reveals uniqueness and efficiency of the equilibrium bid function. Observe that in the case of independent private values, the bid function does not depend on n.[14]

Proposition 2.2 *The unique differentiable symmetric Bayesian Nash equilibrium of SPSB is characterized by*

$$B_2(\varphi,t)=v(t,t)-\frac{\varphi}{1+\varphi}v(t,t)+\frac{\varphi}{(1+\varphi)(1+2\varphi)}\int_t^1\frac{dv(y,y)}{dy}\left(\frac{1-y}{1-t}\right)^{\frac{1+\varphi}{\varphi}}dy, \tag{2.4}$$

where $B_2(\varphi,t)$ is the bid of a bidder with signal t. The outcome of this auction is efficient.

Each term of the RHS of (2.4) has its attractive interpretation. From the discussion of FPSB, it follows that the first term is the bid in SPSB in the absence of financial externalities, and the second term is the bid shading in the hypothetical situation that all bidders attach the same value $v(t,t)$ to the object. The third term increases the bid as a bidder of type t has an incentive to drive up the payment by types above t.

In contrast to FPSB, the effect of an increase in φ on the equilibrium bids in SPSB is dependent on a bidder's type. From (2.4), it is clear that the equilibrium bid of the highest type is decreasing in φ. This bidder does not have a type above her, so that she does not have an incentive to drive up the price. However, the effect of φ on the equilibrium bids of the other types is not clear. The effect of the second term of the RHS of (2.4) may be larger as well as smaller than the third term. The following example illustrates how equilibrium bidding is affected when φ is varied.

Example 2.1 ***(Effect of φ on equilibrium bidding)*** *Let $v(t,t)=t$ for all $t\in[0,1]$ and $n=2$. The equilibrium bid function is given by*

$$B_2(\varphi,t)=\frac{\varphi}{(1+\varphi)(1+2\varphi)}+\frac{1}{1+2\varphi}t,\ t\in[0,1].$$

[14] This is actually a quite subtle observation, as n does not appear in the expression for the equilibrium bid. However, in general, $v(t,t)$ depends on n.

It is readily verified that

$$\frac{\partial B_2(0,0)}{\partial \varphi} = 1 > 0,$$

and

$$\frac{\partial B_2(1,0)}{\partial \varphi} = -\frac{1}{36} < 0.$$

As B_2 is a continuous function in both φ and t, there is a strictly positive mass of types close to zero for which the effect of φ is ambiguous in the sense that for φ close to 0, an increase in φ leads to higher bids and for φ close to 1, an increase in φ leads to lower bids. Intuitively, if φ is large enough, $B_2(\varphi, t)$ decreases as for each bidder, losing becomes more interesting due to higher financial externalities. ▲

Also, the effect of φ on the expected price paid by the winner is ambiguous. This follows from Example 2.2, in which the expected price is increasing if φ is small, and decreasing if φ is large.

Example 2.2 ***(Effect of φ on the expected price)*** *Let $v(t,t) = t$ for all $t \in [0,1]$ and $n = 2$. The expected price is equal to the expectation of $B_2(\varphi, t^{(2)})$ with respect to the second highest signal $t^{(2)}$, which is given by*

$$E_{t^{(2)}}\{B_2(\varphi, t^{(2)})\} = \frac{1+4\varphi}{3(1+\varphi)(1+2\varphi)}.$$

This continuous function is increasing for φ close to 0 and decreasing for φ close to 1, as

$$\frac{\partial E_{t^{(2)}}\{B_2(0, t^{(2)})\}}{\partial \varphi} = \frac{1}{3} > 0,$$

and

$$\frac{\partial E_{t^{(2)}}\{B_2(1, t^{(2)})\}}{\partial \varphi} = -\frac{11}{108} < 0.$$

▲

Example 2.2 suggests that the seller might gain more revenue by guaranteeing the losers a fraction of the payment he receives from the winner. This, however, turns out not to be the case.

Proposition 2.3 *The seller cannot increase his revenue from SPSB by guaranteeing the losing bidders a share of the auction revenue.*

We prove this proposition by using the famous revenue equivalence theorem (Myerson, 1981), which states that the expected utility of the lowest type is a sufficient statistic for the ranking of efficient auctions: the higher the utility of the lowest type, the lower the expected revenue.[15] Consider the standard case in which the bidders obtain no indirect financial externalities. If the seller pays the losers a fraction of the auction revenue, the lowest type's expected utility goes up from zero to a strictly positive number because he gets a fraction of the second highest bid. As SPSB is efficient, the seller's expected revenue decreases. This result turns out to remain valid if the bidders do experience financial externalities, indirect or direct (e.g., the bidders own a share in the seller), as the lowest type's utility is strictly increasing in φ.

2.3.3 Revenue Comparison

Let us compare the expected revenue from FPSB and SPSB.[16] Observe that the revenue ranking of the two auctions is the same for direct and indirect financial externalities, since the revenue in the case of direct externalities is a fraction $(1-(n-1)\,\varphi)$ of the revenue under indirect externalities. As said, the auction for which the lowest type obtains the highest expected utility generates the lowest expected revenue. It turns out that if $\varphi < \frac{1}{n-1}$, SPSB generates a strictly higher expected revenue than FPSB.[17] For $\varphi = \frac{1}{n-1}$, both auctions are revenue equivalent.

Proposition 2.4 *For $\varphi < \frac{1}{n-1}$, SPSB generates a strictly higher expected revenue than FPSB. For $\varphi = \frac{1}{n-1}$, FPSB and SPSB are revenue equivalent.*

Intuitively, in SPSB, a bidder can increase the payment by the winner by submitting a higher bid, which is not the case for FPSB. However, this argument does not hold true if $\varphi = \frac{1}{n-1}$. Of course, when the financial externalities are direct, the winner's payment is entirely distributed among the losers, so that the seller's revenue in both auctions is zero. Note that this implies that the expected utility of the bidders is the same for both auctions because both auctions have an efficient allocation of the object. As bidding is not affected by whether the financial externalities are direct or indirect, FPSB and SPSB yield equal expected utility to the bidders in the case of indirect financial externalities. Then it immediately follows that also for indirect financial externalities, FPSB and SPSB are revenue equivalent.

2.4 Positive Reserve Price

Consider FPSB and SPSB with a reserve price $R > 0$. In order to keep the model tractable, we assume that the independent private values model holds. With some abuse of notation,

[15] Maasland and Onderstal (2006) show that the revenue equivalence theorem remains valid in our (more general) setting.

[16] The revenue comparison is mainly of interest for the case of *indirect* financial externalities. After all, in Sections 2.3.1 and 2.3.2, we have shown that in both FPSB and SPSB, the seller cannot gain by guaranteeing the losing bidders a share of the auction revenue, implying that it is optimal for the seller to set $\varphi = 0$ (in case of *direct* financial externalities).

[17] Engelbrecht-Wiggans (1994) claims the same result, but his proof is not correct. Engers and McManus (2007) derive this result in an independent private values model that is a special case of ours.

we write $v_i(\mathbf{t}) = v(t_i)$ for all i, $\mathbf{t}$, where v is a strictly increasing function. An additional assumption is that all $n-1$ losers receive financial externalities, irrespective of whether or not they submit a bid in the auction that is larger than or equal to the reserve price.

This section focuses on the existence of equilibria with pooling at the reserve price and of *weakly separating Bayesian Nash equilibria*, for which the following definition applies.

Definition 2.1 *A weakly separating Bayesian Nash equilibrium is a Bayesian Nash equilibrium in which all types below a threshold type abstain from bidding,*[18] *and all types above this type submit a bid according to a strictly increasing bid function.*

This weakly separating Bayesian Nash equilibrium is the equilibrium in the standard setting *without* financial externalities. It is interesting to see whether these type of equilibria also exist in the present model *with* financial externalities.

2.4.1 First-Price Sealed-Bid Auction

FPSB has a symmetric equilibrium that involves pooling at the reserve price. We assume that $R < v(1)$, as otherwise, none of the bidders have an incentive to submit a bid as their value would not be higher than the reserve price. Proposition 2.5 describes a Bayesian Nash equilibrium in which bidders with a type below a threshold type L do not bid, types in an interval $[L, H]$ submit a bid equal to R, and bidders with a type t above the threshold type H bid $g^R(t)$, where g^R is a strictly increasing function, defined as

$$g^R(t) \equiv (n-1)t^{-(n-1)(1+\varphi)}\left(\int_H^t y^{(n-1)\varphi+n-2}v(y)dy + v(H)H^{(n-1)(1+\varphi)}\right).$$

More specifically, let

$$H = \min\{1, v^{-1}((1+\varphi)\,R)\} \tag{2.5}$$

and L be the unique solution of

$$\left(p(L,H) - L^{n-1}\right)\varphi R = p(L,H)(v(L) - R), \tag{2.6}$$

where

$$p(L,H) = \frac{1}{n\,(H-L)}\,(H^n - L^n) \tag{2.7}$$

is the probability that a bidder wins given that she bids the reserve price.

For type H, the indifference relation (2.5) follows as she is indifferent between bidding R (and therefore pool with all types in the interval $[L, H]$) and bidding marginally higher than R. These two bids result in a different outcome if H, when bidding R, loses against another

[18] This is synonymous to submitting a bid below the reserve price.

bidder who also bids R. A bid just above R gives her utility $v(H)-R$, while bidding R results in φR. If there is no indifference type in the interval $[0,1]$, then H is set equal to 1.

The indifference relation (2.6) for type L is constructed as follows. L is indifferent between bidding R and abstaining from bidding. The outcome is different in two situations: first, if the bidder wins against another bidder who bids R (which occurs with probability $p(L,H)-L^{n-1}$), and second, if no other bidder bids (which happens with probability L^{n-1}). Note that $v(L) \geq R$: a bidder with a value below R will not submit a bid, as by bidding, she wins the object with a strictly positive probability, which gives a negative utility, while potentially forgoing a positive pay-off when another bidder submits a bid.

The incentive compatibility constraint for types above H results in the same differential equation as the bid function for FPSB without reserve price, of which g^R is the solution satisfying the boundary condition $g^R(H) = R$.

Proposition 2.5 *Assume independent private values and $R < v(1)$. Let $B_1^R(\varphi, t)$, the bid of a bidder with value t, be given by*

$$B_1^R(\varphi,t) = \begin{cases} g^R(t) & \text{if } t > H \\ R & \text{if } L \leq t \leq H \\ \text{“no bid”} & \text{if } t < L, \end{cases}$$

where H and L follow from (2.5) and (2.6) respectively. Then $B_1^R(\varphi, t)$ constitutes a symmetric Bayesian Nash equilibrium of FPSB if $R > 0$.[19]

To get an intuition why pooling at R occurs in equilibrium, consider a bidder with a value $v > R$. She prefers to win if none of the other bidders submit a bid. However, if someone does bid, she may wish to lose because the financial externalities are at least φR, which may be larger than $v - R$. If v is not much higher than R, then the first effect dominates and the bidder does not bid at all. If v is much higher than R, she has a good reason to raise her bid above R in order to beat some of the other bidders who also submit a bid. However, if she has an intermediate value, she wants to bid R so that in the absence of a competitor's bid, she will win the item, and in the presence of a competitor's bid she will win with as small probability as possible. Small increases in the value do not change the fact that the bidder prefers losing to winning in the presence of a competitor's bid, which implies that there is pooling at that price.

In contrast to a situation without financial externalities, there exists no weakly separating Bayesian Nash equilibrium for FPSB.

[19] Note that $B_1^R(\varphi, t)$ is continuous at H. This must be the case in equilibrium. Suppose, on the contrary, that the bid function has a jump at H. Then a bidder with a type slightly higher than H has an incentive to deviate from the bid strategy to a bid just above R.

Proposition 2.6 *Assume independent private values. FPSB has no weakly separating Bayesian Nash equilibrium if $R > 0$ and $\varphi > 0$.*

The proof of Proposition 2.6 is by contradiction. If a weakly separating Bayesian Nash equilibrium existed, all types exceeding $v^{-1}(R)$ (the type for whom the object is worth R) would submit a bid according to a strictly increasing equilibrium bid function. This function can be constructed in a similar way as the equilibrium bid function for FPSB without reserve price. A contradiction is established, as any strictly increasing equilibrium bid function requires a bidder with type $v^{-1}(R)$ to submit a bid below the reserve price.

2.4.2 Second-Price Sealed-Bid Auction

The shape of the equilibrium of SPSB when the seller imposes a reserve price $R > 0$ depends on the level of R. Let us start by observing that, regardless of R, all bidders who submit a bid above the reserve price do so according to the same bid function as in the absence of a reserve price. This implies that, in contrast to FPSB, SPSB has a weakly separating Bayesian Nash equilibrium given that the reserve price is not too high. This observation follows trivially when the reserve price is smaller than the lowest submitted equilibrium bid, which is strictly positive according to Proposition 2.2. However, in nontrivial cases weakly separating Bayesian Nash equilibria also exist. According to Proposition 2.7, for low R, types up to a threshold type $\widehat{t}$ abstain from bidding, and types above $\widehat{t}$ submit the same bid as in the case of no reserve price.

Proposition 2.7 *Assume independent private values. SPSB with reserve price R has a weakly separating Bayesian Nash equilibrium if and only if $R \leq B_2(\varphi, v^{-1}(R))$. If such an equilibrium exists, then it is given by:*

$$B_2^R(\varphi, t) = \begin{cases} B_2(\varphi, t) & \text{if } t \geq \widehat{t} \\ \text{“no bid”} & \text{if } t < \widehat{t}, \end{cases}$$

where $B_2^R(\varphi, t)$ is the bid of a bidder with value t. If $R < B_2(\varphi, 0)$, then $\widehat{t} = 0$, otherwise $\widehat{t}$ is the unique solution of

$$\varphi B_2(\varphi, \widehat{t})(1 - \widehat{t}^n) + \widehat{t}^n(v(\widehat{t}) - R) = \varphi(1 - \widehat{t})R. \tag{2.8}$$

The threshold type $\widehat{t}$ is indifferent between bidding $B_2(\varphi, \widehat{t})$ and abstaining from bidding, and hence follows from equation (2.8). Note that $\widehat{t}$ jumps to a bid strictly above the reserve price when $B_2(\varphi, \widehat{t}) > R$.

An intuition for the condition $R \leq B_2(\varphi, v^{-1}(R))$ being necessary is the following. In a weakly separating Bayesian Nash equilibrium, a bidder with type $v^{-1}(R)$ is always prepared to submit a bid of at least R. To see this, observe that for this bidder, in a weakly separating Bayesian Nash equilibrium, a bid equal to R yields the same revenue as abstaining from bidding. However, in equilibrium, each type that submits a bid does so according to the equilibrium bid function for the situation with no reserve price. This implies that if $B_2(\varphi, v^{-1}(R)) < R$, a bidder with type $v^{-1}(R)$ would submit a bid below the reserve price, which is not possible, and a contradiction is therefore established.

The condition $R \leq B_2(\varphi, v^{-1}(R))$ is sufficient for the following reason. As said, in a weakly separating Bayesian Nash equilibrium, each bidder who submits a bid does so as if there were no reserve price. Then, for the existence of a weakly separating equilibrium, it remains to be checked that $B_2(\varphi, \widehat{t}) \geq R$. If $B_2(\varphi, v^{-1}(R)) \geq R$, then there is a type $\widetilde{t} \leq v^{-1}(R)$ for which $B_2(\varphi, \widetilde{t}) = R$. As a reserve price does not affect equilibrium bidding of types that submit a bid, it follows that if type $\widetilde{t}$ would submit a bid in equilibrium, she would submit a bid equal to R. However, type $v^{-1}(R)$ is indifferent between bidding R and not submitting a bid, so that $\widetilde{t}$ prefers not to submit a bid. Therefore, $\widehat{t}$ must exceed $\widetilde{t}$, so that indeed $B_2(\varphi, \widehat{t}) \geq B_2(\varphi, \widetilde{t}) = R$.

The necessary and sufficient condition $R \leq B_2(\varphi, v^{-1}(R))$ implies that only for small R, a weakly separating Bayesian Nash equilibrium exists. As said, the existence of such an equilibrium is trivial in the case of small R. However, for large R, i.e., R close to $v(1)$, $R > B_2(\varphi, v^{-1}(R))$, as, by Proposition 2.2, $B_2(\varphi, 1) < v(1)$.

If the condition $R \leq B_2(\varphi, v^{-1}(R))$ is violated, then there may exist an equilibrium with pooling at R. If $\max\left\{B_2(\varphi, v^{-1}(R)), B_2(\varphi, 1)\right\} < R < v(1)$, all types above a threshold L submit a bid equal to the reserve price. Type L is indifferent between bidding and not bidding and follows uniquely from a similar condition as in FPSB:

$$\varphi R\left(p(L,1) - L^{n-1}\right) = p(L,1)(v(L) - R). \tag{2.9}$$

Moreover, if $R > v(1)$, none of the bidders submit a bid in equilibrium, as the value of winning for none of the bidders exceeds the reserve price.

Proposition 2.8 *Assume independent private values. Consider SPSB with reserve price R. If $R > \max\left\{B_2(\varphi, v^{-1}(R)), B_2(\varphi, 1)\right\}$, then the following bidding strategies constitute a Bayesian Nash equilibrium:*

$$B_2^R(\varphi, t) = \begin{cases} R & \text{if } L < t \leq 1 \\ \text{"no bid"} & \text{if } t \leq L, \end{cases}$$

where $L = 1$ if $R > v(1)$, and L follows from (2.9) otherwise.

The most involved case is $B_2(\varphi, v^{-1}(R)) < R \leq B_2(\varphi, 1)$. If this condition holds, then types below a certain type L abstain from bidding, bidders between types L and H bid the reserve price, and types $t > H$ bid $B_2(\varphi, t)$. The threshold types L and H respectively follow from the following indifference relations:

$$(p(L,H) - L^{n-1})\varphi R = p(L,H)\,(v(L) - R) \tag{2.10}$$

$$(q(H) + r(L,H))\varphi R = q(H)\varphi B_2(\varphi, H) + r(L,H)\,(v(H) - R)\,, \tag{2.11}$$

where $p(L,H)$ is the probability that a bidder wins given that she bids the reserve price, $q(H)$ is the probability that exactly one bidder has a type above H, and $r(L,H)$ is the probability that a bidder does *not* win given that she bids R and that the highest type of the other bidders does not exceed H. More specifically, $p(L,H)$ is defined in (2.7),

$$\begin{aligned} q(H) &= (n-1)\,(1-H)\,H^{n-2}, \text{ and} \\ r(L,H) &= H^{n-1} - p(L,H). \end{aligned}$$

The indifference relation (2.10) for type L is constructed as follows. L is indifferent between bidding R and abstaining from bidding. The outcome is different in two situations: first, if the bidder wins against another bidder who bids R (which occurs with probability $p(L,H) - L^{n-1}$), and if no other bidder bids (which happens with probability L^{n-1}).

For type H, the indifference relation (2.11) follows as she is indifferent between bidding R and submitting a bid $B_2(\varphi, H)$. These two bids result in a different outcome under the following two conditions: first, exactly one bid exceeds R, so that H determines the price of the winner (this event has probability $q(H)$), and second, H, when bidding R, loses against another bidder who also bids R (probability $r(L,H)$).

Proposition 2.9 *Assume independent private values. Consider SPSB with reserve price R. If $B_2(\varphi, v^{-1}(R)) < R \leq B_2(\varphi, 1)$, the following bidding strategies constitute a Bayesian Nash equilibrium:*

$$B_2^R(\varphi, t) = \begin{cases} B_2(\varphi, t) & \text{if } t > H \\ R & \text{if } L \leq t \leq H \\ \text{“no bid”} & \text{if } t < L, \end{cases}$$

where (L,H) is a solution of the system of equations (2.10) and (2.11).

To summarize: SPSB has no less than five types of equilibria if the seller requires a minimum bid R. First, if $R < B_2(\varphi, 0)$, all bidders submit a bid according to $B_2(\varphi, t)$. Second, if $R > v\,(1)$, no bidder bids. Third, if $B_2(\varphi, 1) < R < v\,(1)$, all bidders above a threshold bid exactly the reserve price. For $R \in (B_2(\varphi, 0), B_2(\varphi, 1))$, the condition $B_2(\varphi, v^{-1}(R)) \geq R$ becomes crucial. If this condition holds true, types up to a threshold do not bid, and types above this threshold submit bids according to $B_2(\varphi, t)$. Otherwise, low types abstain from bidding, intermediate types pool at R, and high types t bid $B_2(\varphi, t)$. All five types of equilibria are shown in Table 2.1.

Reserve Price	**Type of Equilibrium**
$R < B_2(\varphi, 0)$	Separating
$R \in (B_2(\varphi, 0), B_2(\varphi, 1))$ and $R \leq B_2(\varphi, v^{-1}(R))$	Weakly separating
$R \in (B_2(\varphi, 0), B_2(\varphi, 1))$ and $R > B_2(\varphi, v^{-1}(R))$	Pooling at R and high types bid above R
$R \in (B_2(\varphi, 1), v(1))$	Only bids at R
$R > v(1)$	No one bids

Table 2.1: Shape of the equilibrium bid function in SPSB depending on the level of the reserve price.

2.5 Concluding Remarks

We have studied auctions in which losing bidders obtain financial externalities from the winning bidder. We have derived bidding equilibria for FPSB and SPSB. In FPSB, larger financial externalities result in a lower expected price; in SPSB, the effect is ambiguous. Although the expected price in SPSB may increase if financial externalities increase, the seller cannot gain more revenue by guaranteeing the losing bidders a fraction of the auction revenue. Additionally, SPSB dominates FPSB in terms of expected auction revenue if $\varphi < \frac{1}{n-1}$ and both auctions are revenue equivalent if $\varphi = \frac{1}{n-1}$. Moreover, we have studied equilibrium bidding for FPSB and SPSB when a reserve price is imposed. We have observed pooling at the reserve price for FPSB, and for SPSB if the reserve price is sufficiently high. Pooling at the reserve price may thus arise naturally, and is therefore not something to be suspicious of.

Motivated by the observation that in SPSB, low signal bidders may increase their bids when φ is increased, a model with asymmetries in the valuation function may be fruitful to study. One may imagine that with one bidder with a low value and another with a high value, the price in SPSB may be higher with financial externalities than without financial externalities, as the bidder with the low value has an incentive to push up the price when φ is strictly positive. This indicates that the seller may have an incentive to promise low value bidders a share of the auction revenue. Indeed, Goeree and Offerman (2004) show that in asymmetric environments, the seller may obtain more revenue by rewarding one of the losing bidders a fraction of the auction revenue.

Our study does have practical relevance. First of all, because the model analyzed in this chapter is isomorphic to a general class of charity auction models, we are able to derive from our results that charities will raise more money if they use SPSB instead of FPSB. Carpenter et al. (2008) who run a field experiment found the opposite though. As a potential explanation for why their findings deviate from the theory, the authors argue that bidders were unfamiliar with the rules of SPSB, so that many were reluctant to participate in these auctions.

Second, the insight of our study that, if bidders meet each other in multiple markets, losers may have an incentive to drive up the price the winner pays now, in order to make him a less fierce competitor later, may help to explain seemingly irrational bidding in high stake spectrum auctions. In the British UMTS auction, for example, bidding by British Telecom (BT) was such that the prices bid for the large (2 x 15 MHz) "B" and small (2 x 10 MHz) "C", "D", and "E" licenses differed by roughly a constant in the early stages of the auction, and then switched to differing by roughly a fixed proportion (50% of the price level of the small licenses) in the later stages of the auction (after round 100). This unusual bidding pattern may be explained by the presence of financial externalities. According to Klemperer (2002), BT (who might have become confident during the auction that Vodafone valued a large license at 50% more than a small license) might have placed its bids for license B after round 100 not with the intention of winning the license, but with the intention of raising the price which Vodafone had to pay.[20] The rationale for this strategy is that BT might have believed that Vodafone had a limited budget for spectrum auctions, and that exhausting Vodafone's budget in the British UMTS auction would lead to advantages for BT in subsequent auctions (the British UMTS auction was the first of nine western European UMTS auctions, and was also followed by others elsewhere in the world).

Third, the study applies to dissolution of (business) partnerships. Dissolution often requires a change in property rights, from joint to single ownership, in the hands of one of the partners. FPSB or SPSB can be run to select the single owner. Our findings tell that FPSB and SPSB are revenue equivalent if the losers receive an equal share of the auction price as a transfer. If a professional auctioneer is hired to run the auction and is paid part of the auction price, then SPSB revenue dominates FPSB. A partner with a high value will consequently prefer FPSB and one with a low value will prefer SPSB.

2.6 Appendix: Proofs

Throughout the appendix, we let $F^{[1]}$ $[F^{[2]}]$ denote the cumulative distribution function of the first [second] order statistic of $n-1$ draws from the uniform distribution on $[0,1]$, and $f^{[1]}$ $[f^{[2]}]$ the corresponding density function.

Proof of Proposition 2.1. A higher type of bidder cannot submit a lower bid than a lower type of the same bidder. (If the low type gets the same expected surplus from strategies with two different probabilities of being the winner of the object, the high type strictly prefers the strategy with the highest probability of winning. Therefore, the high type will not submit a

[20] BT claimed after the auction it indeed had deliberately pushed up the price that Vodafone had paid (see Cane and Owen, 2000).

lower bid than the low type.) Also, $B_1(\varphi, t)$ cannot be constant on an interval $[t', t'']$. (By bidding slightly higher, a type t'' can largely improve her probability of winning, while only marginally influencing the payments by her and the other bidders.) Moreover, $B_1(\varphi, t)$ cannot be discontinuous at any t. (Suppose that $B_1(\varphi, t)$ makes a jump from $\underline{b}$ to $\overline{b}$ at t^*. A type just above t^* has an incentive to deviate to $\underline{b}$. Doing so, she is able to decrease the auction price, while just slightly affecting her probability of winning the object. As φ is small enough, she is able to improve her utility.) Hence, a symmetric equilibrium bid function must be strictly increasing and continuous. Then,

$$U(t,s) = \int_0^s v(t,y)dF^{[1]}(y) - F^{[1]}(s)B_1(\varphi,s) + \varphi \int_s^1 B_1(\varphi,y)dF^{[1]}(y).$$

The first two terms of the RHS of this expression refer to the case that this bidder wins the object. The third term refers to the case that she does not win. Assume that $B_1(\varphi, s)$ is differentiable in s. Maximizing $U(t, s)$ with respect to s and equating s to t gives the FOC of the equilibrium

$$f^{[1]}(t)v(t,t) - f^{[1]}(t)B_1(\varphi,t) - F^{[1]}(t)\frac{\partial B_1(\varphi,t)}{\partial t} - \varphi B_1(\varphi,t)f^{[1]}(t) = 0.$$

With some manipulation we get

$$F^{[1]}(t)^{\varphi} f^{[1]}(t)v(t,t) = (1+\varphi)B_1(\varphi,t)f^{[1]}(t)F^{[1]}(t)^{\varphi} + \frac{\partial B_1(\varphi,t)}{\partial t}F^{[1]}(t)^{1+\varphi}, \tag{2.12}$$

or, equivalently,

$$C_1 + \int_0^t F^{[1]}(y)^{\varphi} f^{[1]}(y)v(y,y)dy = F^{[1]}(t)^{1+\varphi}B_1(\varphi,t),$$

where C_1 is a constant. Substituting $t = 0$ gives $C_1 = 0$, so that the bid function is given by

$$\begin{aligned} B_1(\varphi,t) &= \frac{1}{F^{[1]}(t)} \int_0^t \left(\frac{F^{[1]}(y)}{F^{[1]}(t)}\right)^{\varphi} f^{[1]}(y)v(y,y)dy \\ &= (n-1)\, t^{-n+1-\varphi} \int_0^t y^{\varphi(n-1)+n-2} v(y,y)dy. \end{aligned} \tag{2.13}$$

It is readily checked that the second order condition $sign(\frac{\partial U(t,s)}{\partial s}) = sign(t-s)$ is fulfilled. Using integration by parts, (2.13) can be rewritten as (2.3). From (2.12), we infer that $\frac{\partial B_1(\varphi,t)}{\partial t} > 0$ if and only if $B_1(\varphi,t) < \frac{v(t,t)}{1+\varphi}$, which is the case for all $t < 1$. Then, by *Value Differentiability, Value Monotonicity*, and *Symmetry*, the efficiency of the auction outcome is established. ■

Proof of Proposition 2.2. Following the lines of the proof of Proposition 2.1 it can be established that a symmetric equilibrium function must be strictly increasing and continuous. The utility for a bidder with signal t acting as if she had signal s is given by

$$U(t,s) = \int_0^s (v(t,y) - B_2(\varphi,y))dF^{[1]}(y) + \varphi\pi(s)B_2(\varphi,s) + \varphi\int_s^1 B_2(\varphi,y)dF^{[2]}(y),$$

where $\pi(s) \equiv F^{[2]}(s) - F^{[1]}(s)$ denotes the probability that there is exactly one opponent with a signal larger than s. The first term of the RHS refers to the case that this bidder wins, the second term to the case that she submits the second highest bid, and the third term to her bid being the third or higher. Assume that $B_2(\varphi,s)$ is differentiable in s. The FOC of the equilibrium is

$$(v(t,t) - B_2(\varphi,t))f^{[1]}(t) + \varphi\frac{\partial\pi(t)B_2(\varphi,t)}{\partial t} - \varphi B_2(\varphi,t)f^{[2]}(t) = 0,$$

or, equivalently,

$$v(t,t)f^{[1]}(t) = -\frac{\partial B_2(\varphi,t)}{\partial t}\varphi\pi(t) + B_2(\varphi,t)[(1+\varphi)f^{[1]}(t)]. \tag{2.14}$$

The general solution to the above differential equation is equal to

$$B_2(\varphi,t)(1-t)^{\frac{1+\varphi}{\varphi}} = C_2 - \int_0^t (1-y)^{\frac{1}{\varphi}}v(y,y)dy,$$

where C_2 is a constant. Substituting $t = 1$ yields a unique solution for C_2:

$$C_2 = \int_0^1 (1-y)^{\frac{1}{\varphi}}v(y,y)dy.$$

The only possible differentiable bid function that may constitute a symmetric equilibrium is given by

$$B_2(\varphi,t) = \frac{1}{\varphi}(1-t)^{-1-\frac{1}{\varphi}}\int_t^1 (1-y)^{\frac{1}{\varphi}}v(y,y)dy. \tag{2.15}$$

It is readily checked that the second order condition $sign\left(\frac{\partial U(t,s)}{\partial s}\right) = sign(t-s)$ holds. Using integration by parts on $B_2(\varphi,t)$, we see that (2.15) can also be written as (2.4). To complete the proof, we must show that $B_2(\varphi, t)$ is indeed increasing in t. From (2.15), it follows that

$$B_2(\varphi, t) > \frac{v(t,t)\int_t^1 (1-y)^{\frac{1}{\varphi}} dy}{\varphi(1-t)^{\frac{1+\varphi}{\varphi}}} = \frac{v(t,t)}{1+\varphi}.$$

As (2.14) implies that $\frac{\partial B_2(\varphi,t)}{\partial t} > 0$ if and only if $B_2(\varphi, t) > \frac{v(t,t)}{1+\varphi}$, $B_2(\varphi, t)$ is indeed strictly increasing in t. Then, by *Value Differentiability, Value Monotonicity*, and *Symmetry*, it follows that the outcome of the auction is efficient. ■

Proof of Proposition 2.3. As stated in the text, it is sufficient to show that the expected utility $U_2(0)$ of the lowest type is increasing in φ. If $n = 2$ and if the lowest type is present, then the price paid is equal to her bid. Therefore,

$$U_2(0) = \varphi B_2(\varphi, 0) = \int_0^1 (1-t)^{\frac{1}{\varphi}} v(t,t)dt, \tag{2.16}$$

which is strictly increasing in φ. If $n > 2$, then the bidder receives financial externalities equal to φ times the second highest bid. Therefore, using the expression for the bid function in (2.15),

$$\begin{aligned} U_2(0) &= \varphi \int_0^1 B_2(\varphi, t) dF^{[2]}(t) \\ &= (n-1)(n-2)\int_0^1 t^{n-3}(1-t)^{-\frac{1}{\varphi}} \int_t^1 (1-y)^{\frac{1}{\varphi}} v(y,y)dydt \\ &= (n-1)(n-2)\int_0^1 t^{n-3} \int_t^1 \left(\frac{1-y}{1-t}\right)^{\frac{1}{\varphi}} v(y,y)dydt, \end{aligned} \tag{2.17}$$

which is strictly increasing in φ as in the inner integral, $y > t$. ■

Proof of Proposition 2.4. (The proof follows the same logic as the proof of Proposition 5 in Bulow et al. 1999.) Let $U_1(0)$ and $U_2(0)$ be the equilibrium utility of the lowest type in FPSB and SPSB respectively. According to Maasland and Onderstal (2006), for SPSB to generate higher [the same] expected revenue than [as] FPSB, it is sufficient to show that $U_1(0) > U_2(0)$ [$U_1(0) = U_2(0)$]. We split the proof in two cases: $n = 2$ and $n > 2$.

We start with the case $n = 2$. As the outcome of both auctions is efficient, a bidder with type 0 loses the auction with probability 1, and gets financial externalities as the other bidder has to pay. For FPSB, the expected price paid by the other bidder is the expectation of her bid (2.13). Hence,

$$\begin{aligned} U_1(0) &= \varphi \int_0^1 B_1(\varphi, t)dt \\ &= \varphi \int_0^1 t^{-1-\varphi} \int_0^t y^{\varphi} v(y,y)dydt \\ &= \varphi \int_0^1 y^{\varphi} v(y,y) \int_y^1 t^{-1-\varphi} dtdy \\ &= \int_0^1 v(y,y)(1-y^{\varphi})dy, \end{aligned}$$

where the third equality is obtained by changing the order of integration. With (2.16), we infer that

$$U_1(0) - U_2(0) = \int_0^1 \left\{1 - t^{\varphi} - (1-t)^{\frac{1}{\varphi}}\right\} v(t,t)dt.$$

For $\varphi < 1$ [$\varphi = 1$], $U_1(0) - U_2(0) > 0$ [$U_1(0) - U_2(0) = 0$], as the expression in curly brackets has expected value zero, and is strictly negative for all $t \in (0, \hat{t})$ and positive for all $t \in (\hat{t}, 1)$ for some $\hat{t}$ [is zero for all $t \in [0,1]$].

Now, suppose $n > 2$. Using the expression for the bid function in (2.13) and $U_2(0)$ in (2.17), we derive that

$$\begin{aligned} U_1(0) &= \varphi \int_0^1 B_1(\varphi, t)dF^{[1]}(t) \\ &= (n-1)^2 \varphi \int_0^1 t^{-1-\varphi(n-1)} \int_0^t y^{\varphi(n-1)+n-2} v(y,y)dydt \\ &= (n-1)^2 \varphi \int_0^1 y^{\varphi(n-1)+n-2} v(y,y) \int_y^1 t^{-1-\varphi(n-1)} dtdy \\ &= (n-1) \int_0^1 (y^{n-2} - y^{\varphi(n-1)+n-2}) v(y,y)dy \end{aligned}$$

and

$$\begin{aligned}
U_2(0) &= (n-1)(n-2)\int_0^1 t^{n-3}(1-t)^{-\frac{1}{\varphi}}\int_t^1 (1-y)^{\frac{1}{\varphi}}\, v(y,y)dydt \\
&= (n-1)(n-2)\int_0^1 (1-y)^{\frac{1}{\varphi}}\, v(y,y)\int_0^y t^{n-3}(1-t)^{-\frac{1}{\varphi}}dtdy \\
&= (n-1)\int_0^1 \left\{(1-y)^{\frac{1}{\varphi}}\int_0^y (1-t)^{-\frac{1}{\varphi}}dt^{n-2}\right\} v(y,y)dy \\
&= (n-1)\int_0^1 \left\{y^{n-2}-\frac{1}{\varphi}(1-y)^{\frac{1}{\varphi}}\int_0^y t^{n-2}(1-t)^{-\frac{1}{\varphi}-1}dt\right\} v(y,y)dy.
\end{aligned}$$

The difference between $U_1(0)$ and $U_2(0)$ can be expressed as

$$\frac{U_1(0)-U_2(0)}{n-1} = \int_0^1 \left\{-y^{\varphi(n-1)+n-2}+\frac{1}{\varphi}(1-y)^{\frac{1}{\varphi}}\int_0^y t^{n-2}(1-t)^{-\frac{1}{\varphi}-1}dt\right\} v(y,y)dy.$$

For $\varphi < \frac{1}{n-1}$ [$\varphi = \frac{1}{n-1}$], $U_1(0)-U_2(0) > 0$ [$U_1(0)-U_2(0) = 0$], as the expression in curly brackets has zero expected utility, and is strictly negative for all $y \in (0,\hat{y})$ and positive for all $y \in (\hat{y},1)$ for some $\hat{y}$ [is zero for all $y \in [0,1]$].

The following observations prove the last statement. Let

$$g(\varphi,n,y) \equiv -y^{\varphi(n-1)+n-2}+\frac{1}{\varphi}(1-y)^{\frac{1}{\varphi}}\int_0^y t^{n-2}(1-t)^{-\frac{1}{\varphi}-1}dt.$$

The expectation of $g(\varphi,n,y)$ with respect to y is

$$\int_0^1 g(\varphi,n,y)dy = -\frac{1}{(\varphi+1)(n-1)}+\frac{1}{\varphi}\int_0^1 t^{n-2}(1-t)^{-\frac{1}{\varphi}-1}\int_t^1 (1-y)^{\frac{1}{\varphi}}\, dydt = 0.$$

Define

$$\begin{aligned}
f(y) &\equiv \varphi(1-y)^{-\frac{1}{\varphi}}\, g(\varphi,n,y) \\
&= -\varphi(1-y)^{-\frac{1}{\varphi}}\, y^{\varphi(n-1)+n-2}+\int_0^y t^{n-2}(1-t)^{-\frac{1}{\varphi}-1}dt.
\end{aligned}$$

Note that $f(0)=0$, and $f(y)$ is negative for positive y close to 0, as the first [second] term on the RHS is of the order $y^{\varphi(n-1)+n-2}$ $[y^{n-1}]$ and $\varphi(n-1)+n-2<n-1$. Moreover,

$$f'(y) = -[\varphi(n-1)+n-2]\varphi(1-y)^{-\frac{1}{\varphi}} y^{\varphi(n-1)+n-3} + y^{n-2}\left(1-y^{\varphi(n-1)}\right)(1-y)^{-\frac{1}{\varphi}-1}.$$

Note that $\lim_{y\uparrow 1} f'(y) = +\infty$, and that for $y\in(0,1)$, $f'(y)=0$ implies

$$y + [\varphi(n-1)+n-2]\varphi(1-y) = y^{1-\varphi(n-1)}.$$

As the function on the LHS is linear, the one on the RHS concave, and the equality holds for $y=1$, there is at most one point $y\in(0,1)$ at which $f'(y)=0$. This implies that f is strictly negative for all $y\in(0,\hat{y})$ and positive for all $y\in(\hat{y},1)$ for some $\hat{y}$. Consequently, the same holds true for g.

For $\varphi=\frac{1}{n-1}$, we prove that $g(\varphi=\frac{1}{n-1},n,y)=0$ for all $y\in[0,1]$ by induction to n. It is straightforwardly checked that $g(\varphi=1,2,y)=0$. Suppose that $g(\varphi=\frac{1}{n-1},n,y)=0$ for some $n\geq 2$. We now show that this implies that $g\left(\varphi=\frac{1}{n},n+1,y\right)=0$:

$$\begin{aligned}
g(\frac{1}{n},n+1,y) &= -y^n + n(1-y)^n \int_0^y t^{n-1}(1-t)^{-n-1}dt \\
&= -y^n + y^{n-1} - (n-1)(1-y)^n \int_0^y t^{n-2}(1-t)^{-n}dt \\
&= -y^n + y^{n-1} - (1-y)\left\{g(\frac{1}{n-1},n,y) + y^{n-1}\right\} \\
&= 0.
\end{aligned}$$

■

Proof of Proposition 2.5. Assume that threshold types L and H exist such that in equilibrium all types $t<L$ abstain from bidding, all types $t\in[L,H]$ bid R, and all types $t>H$ bid according to a strictly increasing bid function g^R.

$p(L,H)$ is constructed as follows:

$$\begin{aligned}
p(L,H) &= \sum_{i=0}^{n-1} \frac{1}{i+1}\binom{n-1}{i} L^{n-1-i}(H-L)^i \\
&= \frac{1}{n(H-L)} \sum_{j=1}^{n} \binom{n}{j} L^{n-j}(H-L)^j \\
&= \frac{1}{n(H-L)}(H^n - L^n).
\end{aligned}$$

For type L, the indifference relation is

$$\begin{aligned}\left(p(L,H)-L^{n-1}\right)\varphi R &= p(L,H)(v(L)-R) \Longrightarrow \\ 1-\frac{L^{n-1}}{p(L,H)} &= \frac{v(L)-R}{\varphi R} \Longrightarrow \\ 1-\frac{n\left(HL^{n-1}-L^{n}\right)}{H^{n}-L^{n}} &= \frac{v(L)-R}{\varphi R} \Longrightarrow \\ 1-n-\frac{n\left(HL^{n-1}-H^{n}\right)}{H^{n}-L^{n}} &= \frac{v(L)-R}{\varphi R}. \end{aligned} \tag{2.18}$$

L is uniquely determined from (2.18) as the LHS of (2.18) is strictly decreasing in L and the RHS is strictly increasing in L for $L \geq 0$.

A type H is indifferent between bidding R and bidding an infinitesimal δ above R. These two bids only yield a different outcome if all other types are lower than H. The difference between bidding R and a bid just above R is that in the former case the bidder always wins and gains $v(H)-R$, whereas a bid equal to R may result in utility φR if another bidder also bids R. Hence, H is indifferent if and only if $v(H)=(1+\varphi)R$.

To complete the proof, we need to check whether types have no incentive to deviate from the proposed equilibrium. We only check whether a type $t>H$ has no incentive to mimic another type $t'>H$, as by a standard argument, other deviations are not profitable. Incentive compatibility of types $t>H$ implies that g^R is a solution to differential equation (2.12) with the boundary condition $g^R(H)=R$. This is indeed how g^R is constructed.

Finally, we should establish that $g^R(t)$ is strictly increasing for $t \geq H$. Analogous to the proof of Proposition 2.1, this is the case if and only if $g^R(t) < \frac{v(t)}{1+\varphi}$ for almost all $t \in [H,1]$. Now,

$$\begin{aligned} g^R(t) &= (n-1)t^{-(n-1)(1+\varphi)}\left\{\int_H^t y^{(n-1)\varphi+n-2}v(y)dy + v(H)H^{(n-1)(1+\varphi)}\right\} \\ &= \frac{v(t)}{1+\varphi} - \frac{t^{-(n-1)(1+\varphi)}}{1+\varphi}\int_H^t y^{(n-1)(\varphi+1)}v'(y)dy \\ &< \frac{v(t)}{1+\varphi}. \end{aligned}$$

■

Proof of Proposition 2.6. The proof is by contradiction. Suppose that a weakly separating equilibrium does exist. Then all types above a threshold type $\hat{t}$ submit a bid according to a strictly increasing bid function, which we denote by β. Then it must be the case that $\beta(\hat{t})=R$ (a bid strictly below R is not allowed, and it cannot be the case in equilibrium that $\beta(\hat{t})>R$, as $\hat{t}$ would strictly prefer to deviate to a bid of R). Moreover, type $\hat{t}$ is indifferent between bidding R and not bidding, which only makes a difference if no other bidder bids. Therefore, the indifference relation for $\hat{t}$ is $0 = v(\hat{t}) - R$, or $\hat{t} = v^{-1}(R)$. Now, analogous to the proof of Proposition 2.1, the utility of a type t that wishes to mimic a type $s > \widehat{t}$ is given by

$$U(t,s) = \int_{\widehat{t}}^{s} v(t)dF^{[1]}(y) - F^{[1]}(s)\beta(s) + \varphi \int_{s}^{1} \beta(y)dF^{[1]}(y).$$

The FOC of the equilibrium results in the following differential equation:

$$F^{[1]}(t)^{\varphi} f^{[1]}(t)v(t) = (1+\varphi)\beta(t)f^{[1]}(t)F^{[1]}(t)^{\varphi} + \beta'(t)F^{[1]}(t)^{1+\varphi}. \tag{2.19}$$

Then a necessary condition for β to be strictly increasing (the assumption we started with) is $\beta'(t) \geq 0$ for all $t \geq \tilde{t}$. (2.19) implies that this condition is equivalent to $\beta(t) \leq \frac{v(t)}{1+\varphi}$, so that it must be true that

$$\beta(\widehat{t}) \leq \frac{v(\widehat{t})}{1+\varphi} = \frac{R}{1+\varphi} < R.$$

In other words, a weakly separating equilibrium can only exist if type $\widehat{t}$ submits a bid strictly below R. However, this contradicts the fact that all submitted bids should exceed R. ■

Proofs of Propositions 2.7 - 2.9. Suppose that all types t above a threshold type $\widehat{t}$ submit a bid above R according to a strictly increasing bid function g. Analogous to the proof of Proposition 2.2, the utility of a type t that wishes to mimic a type $s > \widehat{t}$ is given by

$$U(t,s) = \int_{\widehat{t}}^{s} (v(t) - g(y))dF^{[1]}(y) + \varphi\pi(s)g(s) + \varphi \int_{s}^{1} g(y)dF^{[2]}(y).$$

The equilibrium bid function follows from the condition

$$\frac{\partial U(t,s)}{\partial s} = 0$$

at $s = t$. This immediately leads to differential equation (2.14) with the same boundary condition $g(1) = \frac{v(1)}{1+\varphi}$, so that $B_2^R(\varphi,t) = B_2(\varphi,t)$ is a solution for all R and $t \geq \widehat{t}$.

Now, suppose there is an R for which a weakly separating equilibrium exists. Then there is an indifference type $\widehat{t}$ such that

$$B_2^R(\varphi,t) = \begin{cases} B_2(\varphi,t) & \text{if } t \geq \widehat{t} \\ \text{“no bid”} & \text{if } t < \widehat{t} \end{cases}$$

is an equilibrium. $\widehat{t}$ is indifferent between submitting no bid, and submitting a bid equal to $B_2(\varphi, \widehat{t})$, so that indeed $\widehat{t}$ follows from (2.8). By the intermediate value theorem, (2.8) has a solution as for $\widehat{t} = 0$ [$\widehat{t} = 1$], the LHS is smaller [larger] than the RHS. Moreover, observe that (2.8) can be rewritten as

$$\varphi B_2(\varphi, \widehat{t}) + \frac{\widehat{t}^n}{1 - \widehat{t}^n}(v(\widehat{t}) - R) = \frac{\varphi(1 - \widehat{t})R}{1 - \widehat{t}^n}. \tag{2.20}$$

(2.20) has a unique solution as the RHS [LHS] is strictly increasing [decreasing] in $\widehat{t}$.

A weakly separating equilibrium exists if and only if $B_2(\varphi, \widehat{t}) \geq R$, because all bids should be above R. We will now show that $B_2(\varphi, \widehat{t}) \geq R$ is equivalent to the condition $B_2(\varphi, v^{-1}(R)) \geq R$. Define $\widetilde{t}$ such that $B_2(\varphi, \widetilde{t}) = R$. As $B_2(\varphi, t)$ is strictly increasing in t, $\widetilde{t}$ is uniquely determined. Consider the function h with

$$h(t) \equiv \varphi B_2(\varphi, t) + \frac{t^n(v(t) - R)}{1 - t^n}$$

for all t. Note that h is a strictly increasing function, with $h(\widehat{t}) = \varphi R$ (which follows from (2.8)), and

$$h(\widetilde{t}) = \varphi R + \frac{\widetilde{t}^n(v(\widetilde{t}) - R)}{1 - \widetilde{t}^n}.$$

Now, as $B_2(\varphi, t)$, $h(t)$, and $v(t)$ are strictly increasing in t,

$$\begin{aligned} B_2(\varphi, \widehat{t}) \quad &\geq \quad R = B_2(\varphi, \widetilde{t}) \Longleftrightarrow \widehat{t} \geq \widetilde{t} \Longleftrightarrow h(\widehat{t}) \geq h(\widetilde{t}) \Longleftrightarrow R \geq v(\widetilde{t}) \\ &\Longleftrightarrow \; v^{-1}(R) \geq \widetilde{t} \Longleftrightarrow B_2(\varphi, v^{-1}(R)) \geq B_2(\varphi, \widetilde{t}) = R. \end{aligned}$$

Finally, if $B_2(\varphi, v^{-1}(R)) < R$, the pooling equilibrium is straightforwardly established. What remains to be checked is whether the system of equations (2.10) and (2.11) has a solution. First, we fix $H \geq v^{-1}(R)$, so that L is a solution of

$$\Phi(x) \equiv (p(x, H) - x^{n-1})\varphi R - p(x, H)\,(v(x) - R) = 0. \tag{2.21}$$

Note that Φ is strictly decreasing. Moreover, $\Phi(0) > 0$ and $\Phi(H) < 0$, so that (2.21) has a unique solution $L(H) \in (0, H)$. By the implicit function theorem, $L(H)$ is continuous.

Now, we check that the following equation has a solution $y \geq v^{-1}(R)$:

$$\Psi(y) \equiv [q(y) + r(L(y), y)]\,\varphi R - q(y)\varphi B_2(\varphi, y) - r(L(y), y)\,(v(y) - R) = 0.$$

Observe that $L(v^{-1}(R)) = v^{-1}(R)$, so that $\Psi(v^{-1}(R)) > 0$ (as $r(H, H) = 0$ for all H, and $B_2(\varphi, v^{-1}(R)) < v^{-1}(R)$). Furthermore, let $\tilde{x} \equiv v^{-1}((1 + \varphi)\,R)$. Note that $\tilde{x} \in (v^{-1}(R), 1)$

and $B_2(\varphi, \tilde{x}) \geq \frac{1}{1+\varphi} v(\tilde{x}) = R$.[21] Then, $\Psi(\tilde{x}) = q(\tilde{x})\varphi(R - B_2(\varphi, \tilde{x})) \leq 0$. Hence, by the intermediate value theorem, there is a $y \in [v^{-1}(R), \tilde{x}]$ for which $\Psi(y) = 0$.[22] ■

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[21] $\tilde{x} < 1$ because $v(\tilde{x}) = (1+\varphi)\,R < (1+\varphi)\,B_2(\varphi, 1) = v(1)$.

[22] The solution is not necessarily unique.

Goeree, J.K., E. Maasland, S. Onderstal, and J.L. Turner (2005) "How (Not) to Raise Money," *Journal of Political Economy*, 113(4), 897-918. **[Chapter 3 of this thesis]**

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Chapter 3

How (Not) to Raise Money

3.1 Introduction

It is well known that mechanisms used to finance public goods may yield disappointing revenues because they suffer from a free-rider problem. For example, simply asking for voluntary contributions generally results in underprovision of the public good (e.g. Bergstrom et al., 1986). From a theoretical viewpoint, Groves and Ledyard (1977) solved the decentralized public-goods provision problem by identifying an optimal tax mechanism that overcomes the free-rider problem. This mechanism, however, is mainly of theoretical interest. In contrast, lotteries and auctions are frequently employed as practical means to raise money for a public good. Even the voluntary contribution method is commonly observed in practice, despite its inferior theoretical properties. The co-existence of these alternative formats raises the obvious question: "which method is superior at raising money?"

Morgan's (2000) work constitutes an important first step in answering this question. He studies the fund-raising properties of lotteries and makes the point that the public-good free-rider problem is mitigated by the negative externality present in lotteries. This negative externality occurs because an increase in the number of lottery tickets that one person buys lowers others' chances. As a result, lotteries have a net positive effect on the amount of money raised vis-à-vis voluntary contributions. A similar negative externality emerges in auctions, where a bidder's probability of winning is negatively affected by more aggressive bidding behavior of others.

A priori, most economists would probably expect that auctions are superior to lotteries in terms of raising money. Unlike lotteries, auctions are efficient; in equilibrium, the bidder with the highest value for the object places the highest bid and wins. This efficiency property promotes aggressive bidding and boosts revenue, suggesting that lotteries are suboptimal. However, fund-raisers that use lotteries, or "raffles," are quite prevalent, which casts doubt on the empirical validity of this conclusion.

The flaw in the above argument stems from a separate problem that emerges in auctions where only the winner pays. When a bidder tops the highest bid of others, she wins the object but concurrently eliminates the benefit she would have derived from free-riding off that (previously highest) bid. The possible elimination of positive externalities associated with others' high bids exerts downward pressure on equilibrium bids in winner-pay auctions. Notice that this feature does not occur in lotteries where all non-winning tickets are paid.

In this paper we determine the extent to which bids are suppressed in winner-pay auctions and find that these formats yield dramatically low revenues. Even when bidders value $1 given to the public good the same as $1 for themselves, revenues are finite. In contrast, lotteries generate infinite revenue in this case, notwithstanding their inefficiency. Though extreme, this example suggests that it may make sense to use lotteries instead of winner-pay auctions to raise money.

The main virtue of lotteries in the above example, i.e. that all tickets are paid, can be incorporated into an efficient mechanism. "All-pay" auctions, where everyone pays irrespective of whether they win or lose, avoid the problems inherent in winner-pay auctions. Since they are also efficient, they are prime candidates for superior fund-raising mechanisms. In this paper, we prove this intuition correct. We introduce a general class of all-pay auctions, rank their revenues, and illustrate the extent to which they dominate winner-pay auctions and lotteries. Furthermore, we show that the optimal fund-raising mechanism is among the all-pay formats we consider.

Adding an all-pay element to fund-raisers seems very natural. Indeed, the popularity of lotteries as means to finance public goods indicates that people are willing to accept the obligation to pay even though they may lose. Presumably, the costs of losing the lottery are softened because they benefit a good cause. In some cases, it may even be awkward to not collect all bids. Suppose, for instance, that a group of parents submit sealed bids for a set of prizes that are auctioned, knowing that the proceeds benefit their children's school. Some parents may be offended when told they contributed nothing because they lost the auction, or, in other words, because their contributions were not high enough.

This paper is organized as follows. In the next section, we consider winner-pay auctions where bidders derive utility from the revenue they generate. We build on the work of Engelbrecht-Wiggans (1994), who studies such auctions for the two-bidder case. We extend his finding that second-price auctions revenue dominate first-price auctions by showing that both auctions may be dominated by a third-price auction. The main point of Section 3.2, however, is punctuated by a novel revenue equivalence result for the case when people are indifferent between a dollar donated and a dollar kept. We show that the amount of money generated in this case is identical for all winner-pay formats and surprisingly low.

In Section 3.3 we introduce a general class of all-pay auctions. We show how these formats avoid the shortcomings of winner-pay auctions and we rank their revenues.[1] We demonstrate

[1] A related paper is that of Krishna and Morgan (1997) who study first-price and second-price all-pay auctions. They show that when bidders' values are affiliated, revenue equivalence does not hold. Baye et al.

that an increase in the number of bidders may decrease revenues as low bids more and more resemble voluntary contributions. Fund-raisers can therefore benefit from limiting the number of contestants. In Section 3.4 we derive the optimal fund-raising mechanism, which involves both an entry fee and a reserve price.

Our work is related to several papers that consider auctions in which losing bidders gain by driving up the winner's price. In takeover situations, for example, losing bidders who own some of the target's shares ("toeholds") receive payoffs proportional to the sales price (e.g. Singh, 1998; Bulow et al., 1999). A related topic is the dissolution of a partnership, as analyzed by Cramton et al. (1987). Graham and Marshall (1987) and McAfee and McMillan (1992) study "knockout auctions" where every member of a bidding ring receives a payment proportional to the winning bid. Other examples include creditors bidding in bankruptcy auctions (Burkart, 1995) and heirs bidding for a family estate (Engelbrecht-Wiggans, 1994). These papers restrict attention to standard winner-pay auctions, i.e. first-price, second-price, and English auctions. Another important difference is our assumption of a public-good setting: one bidder's benefit from the auction's revenue does not diminish its value to others.

The paper most closely related to ours is Engers and McManus (2007), who consider "charity auctions."[2] They consider first-price and second-price auctions and extend Engelbrecht-Wiggans's (1994) ranking to the n-bidder case. Our results, however, demonstrate that (i) there exist other winner-pay formats that revenue-dominate the second-price auction, and (ii) all winner-pay formats are poor fund-raisers. Engers and McManus (2007) find that a first-price all-pay auction yields a higher revenue than a first-price auction, but that its revenue may be more or less than that of a second-price auction. Our paper provides a framework to explain these results and gives a more general ranking of all-pay revenues. In addition, we prove that the lowest-price all-pay auction augmented with an entry fee and reserve price is the optimal fund-raising mechanism.

Finally, our work is related to that of Jehiel et al. (1996) who consider auctions in which the winning bidder imposes an individual-specific negative externality on the losers. One important difference is that the magnitudes of the externalities that occur in fund-raisers are endogenously determined, whereas those considered by Jehiel et al. (1996) are fixed.

3.2 Winner-Pay Auctions

In this section we consider "standard" auctions in which only the winner has to pay. We start with a simple three-bidder example to illustrate and extend previous results in the literature and, more importantly, to demonstrate that winner-pay auctions are poor at raising money. We underscore our point by proving a novel revenue equivalence result: when bidders value \$1

(1998, 2005) also study these all-pay formats with affiliated values and consider their applications in a wide variety of two-person contests, including patent races, lobbying, and litigation.

[2] See Ledyard (1978) for an early evaluation of the use of auctions to raise money for a public good.

given to the public good the same as \$1 for themselves, the revenue generated is identical for all winner-pay auctions. Most importantly, however, revenue in this case is only the expected value of the highest order statistic.

Consider three bidders who compete for a single indivisible object. Suppose bidders' values are independently and uniformly distributed on $[0, 1]$ and the auction's proceeds accrue to a public good that benefits the bidders. We assume a particularly simple linear "production technology" where every bidder receives $\$\alpha$ from \$1 spent on the public good. Hence, bidders in the auction receive αR in addition to their usual payoffs, where R is the auction's revenue. Engelbrecht-Wiggans (1994) first studied auctions where bidders benefit from the auction's revenue. He derived the optimal bids for the first-price and second-price auctions when there are two bidders. His answers, however, can easily be extrapolated to our three-bidder example. In the first-price auction, equilibrium bids are[3]

$$B_{1,3}(v) = \frac{2v}{3-\alpha}, \tag{3.1}$$

where the first subscript indicates the auction format and the second the number of bidders. Similarly, equilibrium bids in the second-price auction are

$$B_{2,3}(v) = \frac{v+\alpha}{1+\alpha}. \tag{3.2}$$

Since the bidding functions are linear, revenues follow by evaluating (3.1) and (3.2) at the expected value of the highest and second-highest of three draws: $R_{1,3} = 3/(6-2\alpha)$ and $R_{2,3} = (1+2\alpha)/(2+2\alpha)$. Note that $R_{1,3} = R_{2,3} = 1/2$ when $\alpha = 0$, which is the usual revenue equivalence result, and $R_{1,3} = R_{2,3} = 3/4$ when $\alpha = 1$. For intermediate values of α we have $R_{2,3} > R_{1,3}$, a result first shown by Engelbrecht-Wiggans (1994) for the case of two bidders.

This suggests that the second-price auction should be preferred for fund-raising. The result is of limited interest, however, as it is easy to find other formats that revenue dominate the second-price auction. Consider, for instance, a third-price auction in which the winner has to pay the third-highest price. Equilibrium bids for this format are given by

$$B_{3,3}(v) = \frac{2(v-\alpha)}{1-\alpha} + \frac{\alpha}{2(1-\alpha)}\left(1+\sqrt{1+8/\alpha}\right)(1-v)^{\frac{1}{2}(\sqrt{1+8/\alpha}-1)} \tag{3.3}$$

with corresponding revenue

$$R_{3,3} = \frac{1-\alpha+3\alpha^2\left(3-\sqrt{1+8/\alpha}\right)}{2(1-\alpha)(1-3\alpha)}. \tag{3.4}$$

[3] Consider a bidder with value v who bids as if he has value w and faces rivals who bid according to $B_{1,3}(\cdot)$. The expected payoff is: $\pi^e(B_{1,3}(w)|v) = [v-(1-\alpha)B_{1,3}(w)]\,w^2 + \alpha\int_w^1 B_{1,3}(z)dz^2$. It is easy to verify that the first-order condition for profit maximization is: $\partial_w \pi^e(B_{1,3}(w)|v) = 2(v-w)w$, so it is optimal for a bidder with value v to bid $B_{1,3}(v)$.

Also the third-price auction yields revenue 1/2 when $\alpha = 0$ as dictated by the Revenue Equivalence Theorem, and 3/4 when $\alpha = 1$. For intermediate values of α, the third-price auction results in higher revenues than the other two formats, as shown in Figure 3.1.

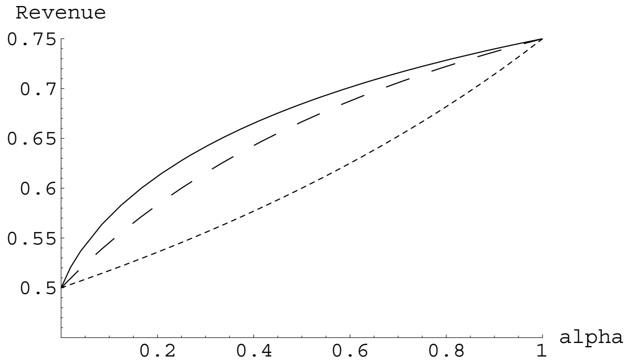

Figure 3.1: *Revenues from a first-price (short dashes), second-price (long dashes), and third-price (solid line) auction with three bidders for* $0 \leq \alpha \leq 1$.

The revenue equivalence result for $\alpha = 1$ holds quite generally. Consider a setting with n bidders whose values are identically and independently distributed on $[0, 1]$ according to a distribution $F(\cdot)$.[4] To derive the amount of money raised when $\alpha = 1$, we focus on the first-price auction, for which it is a weakly dominant strategy to bid one's value. To verify this claim, consider bidder 1 and let $b_{-1} = \max_{i=2,\ldots,n}\{b_i\}$ denote the highest of the others' bids. When $v_1 \geq b_{-1}$, bidder 1's expected payoff when she bids her value is v_1, and she gets the same payoff for all bids with which she wins. When she bids too low and loses the auction, however, her expected payoff becomes $b_{-1} < v_1$. In other words, bidder 1 never gains but may lose when choosing a bid different from her value. Similarly, when $v_1 < b_{-1}$, bidder 1's expected payoff when she bids her value is b_{-1}. This payoff is the same for all bids with which she loses, but a bid that would lead her to win the auction yields a lower expected payoff equal to v_1. So it is optimal to bid one's value and the auction's revenue is simply the expected value of the highest order statistic. We next show that other winner-pay formats yield the same revenue (see the appendix for a proof). Let Y_k^n denote the k^{th}-highest order statistic from n value draws.

Proposition 3.1 *The revenue of any winner-pay auction is* $E(Y_2^n)$ *for* $\alpha = 0$ *and* $E(Y_1^n)$ *for* $\alpha = 1$.

This revenue equivalence result is somewhat interesting in its own right, but the main point

[4] Throughout this paper we assume that the corresponding probability density function $f(\cdot)$ is positive and continuous.

is that winner-pay auctions are ineffective at raising money. Revenues are increasing with α (see Figure 3.1), so the highest revenue should be expected for $\alpha = 1$. In this extreme case bidders are indifferent between keeping \$1 for themselves or giving it to the public good, yet revenues are only $E(Y_1^n)$ in a winner-pay auction. We show below that bidders would spend their entire budgets if a lottery or all-pay auction were used.

3.3 All-Pay Auctions

The problem with winner-pay auctions is one of opportunity costs. A high bid by one bidder imposes a positive externality on all others, who forgo this positive externality if they top the high bid. Bids are suppressed as a result, and so are revenues. This would not occur in situations where every bidder pays, regardless of whether they win or lose.[5] In this section, we introduce k^{th}-price all-pay auctions where the highest bidder wins, the $n-k$ lowest bidders pay their bids, and the k highest bidders pay the k^{th}-highest bid.

To derive the bidding functions, consider the marginal benefits and costs of raising one's bid. The positive effects of increasing one's bid from $B(v)$ to $B(v+\epsilon) \approx B(v) + \epsilon B'(v)$ are twofold. First, it might lead one to win the auction that otherwise would have been lost. This occurs when the highest of the others' values falls between v and $v+\epsilon$, which happens with probability $(n-1)\epsilon f(v)F(v)^{n-2}$. Second, an increase in one's bid raises revenue by $\epsilon B'(v)$ if there are at least $k-1$ higher bids and by an additional $\epsilon(k-1)B'(v)$ if there are exactly $k-1$ higher bids. Let $F_{Y_{k-1}^{n-1}}$ denote the distribution function of the $(k-1)^{th}$ order statistic from $n-1$ draws with the convention $F_{Y_0^{n-1}}(v) = 0$ and $F_{Y_n^{n-1}}(v) = 1$. The probability that there are at least $k-1$ bidders with values higher than v is $1 - F_{Y_{k-1}^{n-1}}(v)$. Similarly, the probability that there are exactly $k-1$ such bidders is $(1 - F_{Y_{k-1}^{n-1}}(v)) - (1 - F_{Y_k^{n-1}}(v)) = F_{Y_k^{n-1}}(v) - F_{Y_{k-1}^{n-1}}(v)$. Combining the different terms, the expected marginal benefit can be written as ϵ times

$$(n-1)\, vf(v)F(v)^{n-2} + \alpha B'(v)\{(1 - F_{Y_{k-1}^{n-1}}(v)) + (k-1)(F_{Y_k^{n-1}}(v) - F_{Y_{k-1}^{n-1}}(v))\}. \tag{3.5}$$

Likewise, the marginal cost is $\epsilon B'(v)$ when there are at least $k-1$ higher bids, and the expected marginal cost is therefore ϵ times

$$B'(v)(1 - F_{Y_{k-1}^{n-1}}(v)). \tag{3.6}$$

[5] Morgan (2000) considers lotteries as ways to fund public goods. Lotteries have an "all-pay" element in that losing tickets are not reimbursed. A major difference is that lotteries are not, in general, efficient, i.e. they do not necessarily assign the object for sale to the bidder that values it the most. Indeed, even in symmetric complete information environments where efficiency plays no role, lotteries tend to generate less revenues because the highest bidder is not necessarily the winner. To see this, suppose the prize is worth V to all bidders. In a lottery the optimal number of tickets to buy is $(n-1)V/(n^2(1-\alpha))$, resulting in a revenue of $(n-1)V/(n(1-\alpha))$. In the first-price all-pay auction, the symmetric Nash equilibrium is in mixed-strategies. The equilibrium distribution of bids is $F(b) = (b/((1-\alpha)V))^{1/(n-1)}$, and the resulting revenue is $V/(1-\alpha)$, which exceeds that of a lottery for all n. Note, however, that the revenue of a lottery may exceed that of a first-price winner-pay auction, for instance, where the unique symmetric equilibrium entails bidding V, and hence revenue is V, for all $\alpha \leq 1$.

The optimal bids can be derived by equating benefits to costs. The resulting differential equation has a well-defined solution for $\alpha < 1/k$. This case is studied in the next proposition, which also compares the resulting revenues to that of a lottery (R^{LOT}).

Proposition 3.2 *When $\alpha < 1/k$, the equilibrium bids of the k^{th}-price all-pay auction are*

$$B_{k,n}^{AP}(v) = \int_0^v \frac{(n-1)\, zf(z)F(z)^{n-2}}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))}\, dz, \tag{3.7}$$

and revenues are

$$R_{k,n}^{AP} = \int_0^1 \frac{z\,(1-F_{Y_{k-1}^{n-1}}(z))}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))}\, dF_{Y_2^n}(z). \tag{3.8}$$

Revenues of the k^{th}-price all-pay auction are increasing in α but may decrease with n, and $R^{LOT} < R_{k-1,n}^{AP} < R_{k,n}^{AP}$ for $2 \le k \le n$ and $\alpha > 0$.

Thus, the all-pay formats revenue dominate the lottery and, most importantly, the lowest-price all-pay auction revenue dominates all other all-pay formats. Not surprisingly, it also revenue dominates all winner-pay auctions. This latter result, which we prove in the next section, is foreshadowed by Figure 3.2. This figure shows the revenues of a first-price, second-price, and third-price all-pay auction when there are three bidders whose values are uniformly distributed. Comparing Figures 3.1 and 3.2 illustrates clearly the extent to which revenues are suppressed in winner-pay auctions.

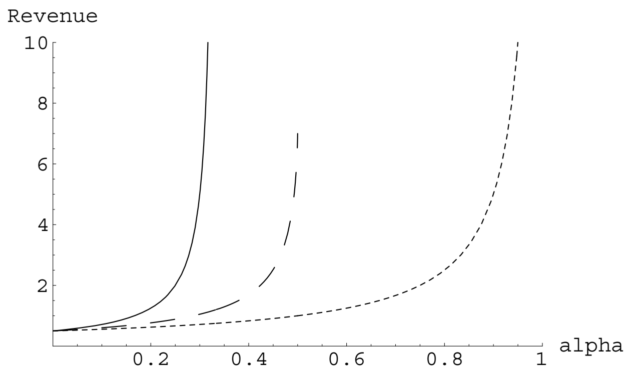

Figure 3.2: *Revenues from a first-price (short dashes), second-price (long dashes), and third-price (solid line) all-pay auction with three bidders for $0 \le \alpha \le 1$.*

Unlike winner-pay formats where revenues are increasing in both α and n,[6] all-pay formats may yield lower revenues when there are more bidders. The intuition behind this result can be made clear by considering the second-price all-pay auction. With two bidders, the loser knows that her bid determines the price paid by the winner, which provides the loser with an incentive to drive up the price. This is not true with three or more bidders, however, in which case the $n-2$ lowest bids are paid only by the losers. Hence there are no positive externalities associated with such bids, which become like voluntary contributions to the public good. This suppresses bids of low-value bidders, who free-ride on the revenues generated by the bidders with higher values. Fund-raisers may thus benefit from limiting competition and restricting access to "a happy few."

When $\alpha > 1/k$ the equilibrium bidding function in (3.7) breaks down and revenues diverge. This divergence is, of course, a consequence of our assumption of a linear production technology for the public good. If the marginal benefit of the public good is sufficiently decreasing (instead of being constant), revenues are finite. We keep the constant marginal benefit assumption because it provides a tractable model to show how much worse winner-pay auctions are in terms of raising money compared to all-pay formats.

To deal with the case $\alpha > 1/k$, we assume that bidders have a finite budget M, where M is much larger than 1. Recall from Proposition 3.1 that revenues of winner-pay auctions are bounded above by 1 whenever $\alpha \leq 1$, and they are bounded by M when $\alpha > 1$ since only a single bidder pays. In contrast, we next show that the lowest-price all-pay auction raises the maximum amount nM when $\alpha > 1/n$.

Proposition 3.3 *When $\alpha > 1/k$ and bidders face a budget constraint M, the equilibrium bids of the k^{th}-price all-pay auction are*[7]

$$B_{k,n}^{AP}(v, M) = \begin{cases} B_{k,n}^{AP}(v) & \text{for } v < v^* \\ \\ M & \text{for } v \geq v^* \end{cases} \tag{3.9}$$

with $B_{k,n}^{AP}(v)$ given by (3.7). The cut-point, v^*, satisfies $0 < v^* < 1$ when $k < n$ and $1/n < \alpha < 1$, in which case $R^{LOT} < R_{1,n}^{AP} < nM$ and $R_{k,n}^{AP} < R_{n,n}^{AP} = nM$. When $\alpha \geq 1$, $v^* = 0$ and $R^{LOT} = R_{k,n}^{AP} = nM$ for all k.

In particular, when bidders value \$1 for the public good the same as \$1 kept, revenues

[6] Equilibrium bids are $B_{1,n}(v) = \int_0^v z\, dF_\alpha^n(z|v)$, where $F_\alpha^n(z|v) \equiv (F(z)/F(v))^{\frac{n-1}{1-\alpha}}$, for the first-price auction. Note that F_α^n first-order stochastically dominates $F_{\alpha'}^n$ for all $\alpha \geq \alpha'$, and F_α^n first-order stochastically dominates $F_\alpha^{n'}$ for all $n \geq n'$. Hence an increase in α or n raises bids and revenues. Equilibrium bids for the second-price auction are $B_{2,n}(v) = \int_v^1 z\, dG_\alpha(z|v)$, where $G_\alpha(z|v) \equiv 1 - (\frac{1-F(z)}{1-F(v)})^{\frac{1}{\alpha}}$, independent of n. G_α first-order stochastically dominates $G_{\alpha'}$ for all $\alpha \geq \alpha'$, and an increase in α results in higher bids and higher revenues. Bids in the second-price auction are independent of the number of bidders, but the expected value of the second-highest order statistic increases with n and, hence, so does revenue.

[7] See Gavious et al. (2002) for a similar analysis and results.

of a lottery or any of the all-pay auctions are equal to the sum of the bidders' budgets, nM. This maximum possible revenue contrasts with the expected revenue of a winner-pay auction, $E(Y_1^n) < 1$, see Proposition 3.1. When $\alpha < 1$ and $k < n$, bidders with sufficiently small values continue to bid according to (3.7) in the k^{th}-price all-pay auction because the value from possibly winning the item is too small to justify the increased cost of a jump to M. Revenues are strictly smaller than nM in this case unless the lowest-price auction is used for which this maximum amount is guaranteed whenever $\alpha > 1/n$.

3.4 Optimal Fund-Raising Mechanisms

In the previous section we showed that when $\alpha > 1/n$, the lowest-price all-pay auction raises the maximum possible revenue. Here we prove that the lowest-price all-pay auction is the optimal fund-raising mechanism for $\alpha < 1/n$ and, hence, is optimal generally. Consider first the case where the seller cannot commit to keeping the good, so that he cannot use an entry fee or a reserve price. Note that this assumption is closely related to the assumption in the Coase Conjecture that a seller of a durable good cannot commit to selling the good for the monopoly price (Coase, 1972).

Proposition 3.4 *When the seller cannot commit to keeping the good, the lowest-price all-pay auction is revenue maximizing. The total amount raised is increasing in α.*

The intuition for this result is as follows. The total surplus generated by the auction is maximized when the auction outcome is efficient. This surplus is divided between the bidders and the seller: the bidders' share is minimized, and, hence, revenues are maximized, when the lowest-value bidder has zero expected payoffs (see the proof in the appendix). The lowest-price all-pay auction maximizes total surplus because it assigns the object to the highest-value bidder. In addition, the zero-value bidder who loses for sure also determines the price paid in the auction. Hence, the zero-value bidder's expected payoff is $n\alpha B_{n,n}^{AP}(0)$, which is zero by (3.7) for all $\alpha < 1/n$.[8]

For the standard case without a public good ($\alpha = 0$), it is well known that the seller can obtain higher revenues by screening out low-value bidders. Myerson (1981) and Riley and Samuelson (1981) prove that it is revenue maximizing to screen out all bidders with values less than the cut-off value, $\widehat{v}$, that satisfies[9]

$$\widehat{v} - \frac{1 - F(\widehat{v})}{f(\widehat{v})} = 0. \tag{3.10}$$

[8] Indeed, a strictly positive bid by the zero-value bidder implies that the expected lowest bid is strictly positive, and since the zero-value bidder's expected profit is $n\alpha - 1 < 0$ times the expected lowest bid, she is better off bidding zero.

[9] We make the common assumption that the marginal revenue $MR(v) \equiv v - (1 - F(v))/f(v)$ is strictly increasing in v, see Myerson (1981). Under this assumption there is a unique solution to (3.10).

Screening can be implemented, for instance, by imposing a "minimum bid" or reserve price. By using a reserve price, the seller lowers the expected payoffs of bidders with values between 0 and $\widehat{v}$ to zero, thus capturing part of the bidders' rents.

When $\alpha > 0$, however, the optimal mechanism cannot be implemented with a reserve price only since low-value bidders that abstain from bidding would still get utility from the amount raised for the public good. Consider instead the following two-stage auction mechanism, $\Gamma(r, \varphi)$, that involves both a reserve price, r, and an entry fee, φ. In the first stage, bidders are asked whether or not they want to participate. If at least one of the bidders refuses to participate, the game ends and the seller keeps the object. Otherwise, each bidder pays the seller the entry fee φ. Then bidders enter the second stage and play the lowest-price all-pay auction with reserve price r. In this auction, each bidder either submits a bid of at least r or abstains from bidding. If all bidders abstain, the object remains in the hands of the seller; otherwise it will be sold to the bidder with the highest bid. All bidders who submitted a bid pay the auction price, which equals the lowest submitted bid when all bidders submitted a bid and equals r otherwise.

The equilibrium strategy for the lowest-price all-pay auction in the presence of a reserve price r changes as follows:

$$B_r(v) \equiv \begin{cases} B(v, \widehat{v}) & \text{for } v \geq \widehat{v} \\ \\ \text{“no bid”} & \text{for } v < \widehat{v}, \end{cases} \tag{3.11}$$

where

$$B(v, \widehat{v}) \equiv r + \frac{n-1}{1-n\alpha} \int_{\widehat{v}}^{v} \frac{z f(z) F(z)^{n-2}}{1 - F(z)^{n-1}} \, dz, \tag{3.12}$$

the threshold $\widehat{v}$ satisfies (3.10) and r is the unique solution to $\widehat{v}F(\widehat{v})^{n-1} = (1-\alpha)r$.[10] Note that (3.12) has a similar structure to the solution derived in Proposition 3.2 for $k = n$. The reason is that for bidders with values $v > \widehat{v}$, the optimal bids again follow by equating the expected marginal benefits and costs in (3.5) and (3.6) respectively. The only difference is that the boundary condition is now given by $B_r(\widehat{v}) = r$ instead of $B(0) = 0$.

Proposition 3.5 *The optimal fund-raising mechanism is given by the two-stage mechanism $\Gamma(r, \varphi)$, in which bidders first decide whether or not to pay an entry fee φ and then compete in a lowest-price all-pay auction with reserve price r, where*

$$\begin{aligned} (1-\alpha)\, r &= \widehat{v}F(\widehat{v})^{n-1}, \\ (1-n\alpha)\, \varphi &= \alpha r (n-1)(1 - F(\widehat{v})), \end{aligned}$$

[10] The reserve price r can be derived by noting that, in equilibrium, a bidder with value $\widehat{v}$ must be indifferent between abstaining and bidding r. Hence $\alpha(n-1)r(1-F(\widehat{v})) = -r + \alpha r \{1 + (n-1)(1-F(\widehat{v}))\} + \widehat{v}F(\widehat{v})^{n-1}$, where $(n-1)(1-F(\widehat{v}))$ is the expected number of other bidders that submit a bid in excess of r.

and $\widehat{v}$ satisfies (3.10). In equilibrium, all bidders participate and play according to (3.11). The total amount raised is increasing in α.

In practice, fund-raising events frequently employ structures with some of the characteristics of Γ. Under one common structure, attendees are charged a fee for dinner and drinks, then are allowed to bid in auctions later in the event. However, these fund-raisers usually use winner-pay auctions and thus do not maximize revenue. Indeed, an easy corollary to Proposition 3.5 is that the formats most commonly employed in practice are non-optimal.

Corollary 3.1 *Lotteries and winner-pay auctions (with or without reserve prices) are non-optimal.*

The intuition is that a lottery does not maximize revenues because the expected payoff of the lowest-value bidder is strictly positive, and the object is not allocated to the bidder with the highest value. A winner-pay auction does not maximize revenues as the lowest-value bidder expects strictly positive utility from the winner's payment.

3.5 Conclusion

Large voluntary contributions such as the recent $24 billion committed by Bill Gates to the *Bill and Melinda Gates Foundation*, make up a substantial part of total fund-raising revenue today.[11] Not surprisingly, such gifts garner significant attention in the popular media.[12] The vast majority of fund-raising organizations, however, seek small contributions from a large number of donors. These organizations frequently prefer lotteries and auctions over the solicitation of voluntary contributions.[13]

Moreover, as electronic commerce on the Internet has grown, web sites offering charity auctions have proliferated. Electronic auction leaders such as *Ebay* and *Yahoo!* have specific sites for charity auctions where dozens of items are sold each day. The established fund-raising community has taken notice of these developments. In a recent report for the *W.K. Kellogg Foundation*, Reis and Clohesy (2000) identified auctions as one of the most important, and fastest growing, options that fund-raisers use to leverage the power of the Internet. Given these trends, it is clear that professional fund-raisers can profit from an improved auction design.

Currently, most fund-raisers employ standard auctions where only the winner pays. These familiar formats have long been applied in the sales of a variety of goods, and their revenue-generating virtues are well established, both in theory and practice (e.g. Klemperer, 1999).

[11] Total giving was an estimated $190 billion in 1999, according to *Giving USA*.

[12] "Bill's Biggest Bet Yet," *Newsweek*, February 4, 2002, p. 46.

[13] For example, in the year 2000, *Ducks Unlimited* raised a total of $75 million from special events organized by its 3,300 local chapters, with over 50% of the revenue coming from auctions.

We show, however, that they are ill suited for fund-raising. The problem with winner-pay auctions in this context is one of opportunity costs. A high bid by one bidder imposes a positive externality on all others, which they forgo if they top the high bid. Bids are suppressed as a result, and so are revenues. We show that the amount raised by winner-pay auctions is surprisingly low even when people are indifferent between a dollar donated and a dollar kept.

The elimination of positive externalities associated with others' bids does not occur when bidders have to pay irrespective of whether they win or lose. Many fund-raisers employ lotteries, for example, where losing tickets are not reimbursed (see Morgan, 2000). Lotteries are generally not efficient, however, which negatively affects revenues. We introduce a novel class of all-pay auctions, which are efficient while avoiding the shortcomings of winner-pay formats. We rank the different all-pay formats and demonstrate their superiority in terms of raising money (see Figures 3.1 and 3.2). We prove that the lowest-price all-pay auction augmented with a reserve price and an entry fee is the optimal fund-raising mechanism.

Our findings are not just of theoretical interest. The frequent use of lotteries as fund-raisers indicates that people are willing to accept an obligation to pay even though they may lose. The all-pay formats studied here may be characterized as incorporating "voluntary contributions" into an efficient mechanism. They are easy to implement and may revolutionize the way in which money is raised.

At the end of this chapter, we want to stress the limitations of this paper. First, the reader should be aware that this is a theory paper. Theory should be augmented with empirical analysis (of experimental data, or field data) before policy recommendations can be advocated. Quite recently, the theory proposed in this chapter has been tested in a laboratory experiment and some field experiments. Onderstal and Schram (2009) ran a laboratory experiment that was a straightforward implementation of our theory and found support for the theorical predictions. In a comparison of first-price winner-pay auctions, first-price all-pay auctions and lotteries, they found that the all-pay format raises substantially higher revenue than the other mechanisms. Carpenter et al. (2008) who ran a field experiment (during fundraising festivals organized by preschools in Addison County) found, in contrast, that the first-price winner-pay auction outperforms the first-price all-pay auction.[14] These authors attribute this result to differences in the participation rates across the mechanisms. Many people seemed to be reluctant to participate in the all-pay auction because of the unfamiliarity with the auction rules. A similar finding was obtained in a door-to-door fundraising field experiment in the Netherlands by Onderstal et al. (2011). New experiments, both in the lab and in the field, should reveal under which circumstances the all-pay auction is the preferred fundraising mechanism.

Second, the reader should note that the lowest-price all-pay auction has, apart from the efficient equilibrium, highly inefficient equilibria in the case of two bidders. It is easily verified

[14] In contrast to theory, they also found that the first-price winner-pay auction revenue dominates the second-price winner-pay auction.

that there is an equilibrium in which one bidder submits a very high bid, and the other bids zero.[15] In addition, the lowest-price all-pay auction is prone to 'shill bidding', i.e., a bidder has an incentive to hire someone to bid zero in the auction while she submits a very high bid. Doing so, she wins the object for a price of zero. Orzen (2008) has tested the lowest-price all-pay auction in the lab. He observes that the lowest-price all-pay auction is a better fund-raising mechanism than the first-price all-pay auction, a lottery, and a voluntary contribution mechanism. Orzen's experimental results are thus in line with our theoretical results. It should in fairness be said however, that Orzen considers a situation where $\alpha > 1/n$, in which shill bidding is not attractive.

3.6 Appendix: Proofs

Proof of Proposition 3.1. Consider a standard auction format in which the highest bidder wins and only the winner pays. In an efficient auction, the surplus generated is $S = E(Y_1^n) + n\alpha R$, with R the auction's revenue. This surplus is divided between the seller and the bidders: $S = R + U_{bidders}$, where $U_{bidders}$ denotes the *ex ante* expected payoffs for the group of bidders. Solving for R we derive

$$R = \frac{E(Y_1^n) - U_{bidders}}{1 - n\alpha}. \tag{3.13}$$

The revenue equivalence result for $\alpha = 0$ is standard. When $\alpha = 1$, the winning bidder's *net* payment is zero. A bidder with a value of 1, who wins for sure, therefore has an expected payoff of 1. A simple Envelope Theorem argument shows that the expected rents for a bidder with value v are given by

$$U(v) = U(0) + \int_0^v F^{n-1}(z)\, dz,$$

(see also Lemma 3.1 in the proofs of Propositions 3.4 and 3.5) from which we derive

$$\begin{aligned} U(0) &= 1 - \int_0^1 F(z)^{n-1}\, dz \\ &= (n-1) \int_0^1 z\, f(z)\, F(z)^{n-2}\, dz \\ &= (n-1) \int_0^1 z\, f(z) \left\{ F(z)^{n-1} + F(z)^{n-2}(1 - F(z)) \right\} dz \\ &= \frac{1}{n} \left[(n-1)\, E(Y_1^n) + E(Y_2^n) \right]. \end{aligned}$$

[15] For three or more bidders, there is no equilibrium in which one bidder bids very high and the other bidders bid zero, because in such an equilibrium one of the low bidders has an incentive to overbid the high bidder.

Moreover $U_{bidders} = n \int_0^1 U(v)\, dF(v)$, so:

$$
\begin{aligned}
U_{bidders} &= nU(0) + n \int_0^1 \int_0^v F(z)^{n-1}\, dz\, dF(v) \\
&= nU(0) + n \int_0^1 \int_z^1 dF(v)\, F(z)^{n-1} dz \\
&= nU(0) + E(Y_1^n) - E(Y_2^n) \\
&= nE(Y_1^n).
\end{aligned}
$$

From the last line and (3.13) we derive

$$
R = \frac{E(Y_1^n) - nE(Y_1^n)}{1-n} = E(Y_1^n),
$$

which completes the proof. ■

Proof of Proposition 3.2. Let $B(\cdot)$ denote the bidding function given in (3.7). Since the denominator in (3.7) is bounded away from 0 for all $v < 1$ when $\alpha < 1/k$, the bidding function is well defined for all $v < 1$ and possibly diverges in the limit $v \to 1$. The derivative of the expected profit of a bidder with value v who bids as if of type w and who faces rivals bidding according to $B(\cdot)$ is

$$
\begin{aligned}
\partial_w \pi^e(B(w)|v) \;=\; & (n-1)vf(w)F(w)^{n-2} - (1-\alpha)B'(w)(1 - F_{Y_{k-1}^{n-1}}(w)) \\
& + \alpha(k-1)B'(w)(F_{Y_k^{n-1}}(w) - F_{Y_{k-1}^{n-1}}(w)).
\end{aligned}
$$

Using the expression for $B(\cdot)$ given by (3.7), the marginal expected profits can be rewritten as

$$
\partial_w \pi^e(B(w)|v) = (n-1)(v-w)f(w)F(w)^{n-2},
$$

and it is therefore optimal for a bidder with value v to bid $B(v)$. The revenue of the k^{th}-price all-pay auction equals

$$
\begin{aligned}
R &= \sum_{i=k+1}^{n} \int_0^1 B(v)\, dF_{Y_i^n}(v) + k \int_0^1 B(v)\, dF_{Y_k^n}(v) \\
&= n \int_0^1 B(v)\, dG(v),
\end{aligned}
$$

where

$$G(v) \equiv \frac{1}{n}\sum_{i=k+1}^{n} F_{Y_i^n}(v) + \frac{k}{n}F_{Y_k^n}(v).$$

Note that $G(\cdot)$ is increasing with $G(0) = 0$ and $G(1) = 1$. Using $\frac{1}{n}\sum_{i=1}^{n} F_{Y_i^n} = F$, the distribution $G(\cdot)$ can be rewritten as

$$\begin{aligned}
G(v) &= F(v) + \frac{1}{n}\sum_{i=1}^{k-1}(F_{Y_k^n}(v) - F_{Y_i^n}(v)) \\
&= F(v) + \frac{1}{n}\sum_{i=1}^{k-1}\left\{\sum_{j=n+1-k}^{n}\binom{n}{j}F(v)^j(1-F(v))^{n-j} - \sum_{j=n+1-i}^{n}\binom{n}{j}F(v)^j(1-F(v))^{n-j}\right\} \\
&= F(v) + \frac{1}{n}\sum_{i=1}^{k-1}\sum_{j=n+1-k}^{n-i}\binom{n}{j}F(v)^j(1-F(v))^{n-j} \\
&= F(v) + \frac{1}{n}\sum_{j=n+1-k}^{n-1}\sum_{i=1}^{n-j}\binom{n}{j}F(v)^j(1-F(v))^{n-j} \\
&= F(v) + (1-F(v))F_{Y_{k-1}^{n-1}}(v),
\end{aligned}$$

where we used some basic properties of order statistics.[16] The revenue of the k^{th}-price all-pay auction thus becomes

$$\begin{aligned}
R &= n\int_0^1\int_0^v \frac{(n-1)\,zf(z)F(z)^{n-2}}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))}\,dz\,dG(v) \\
&= n\int_0^1\left[\int_z^1 dG(v)\right]\frac{(n-1)\,zf(z)F(z)^{n-2}}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))}\,dz \\
&= \int_0^1 \frac{z\,(1-F_{Y_{k-1}^{n-1}}(z))}{(1-k\alpha)(1-F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1-F_{Y_k^{n-1}}(z))}\,dF_{Y_2^n}(z),
\end{aligned}$$

where we used $G(1) - G(z) = (1-F(z))(1-F_{Y_{k-1}^{n-1}}(z))$.

The derivative of (3.8) with respect to α is the integral of a strictly positive function times

$$\begin{aligned}
&k(1-F_{Y_{k-1}^{n-1}}(z)) - (k-1)(1-F_{Y_k^{n-1}}(z)) \\
&= (1-F_{Y_{k-1}^{n-1}}(z)) + (k-1)(F_{Y_k^{n-1}}(z) - F_{Y_{k-1}^{n-1}}(z)) \;>\; 0,
\end{aligned}$$

for all $z < 1$. Hence revenues are increasing in α. Note that the revenue of the k^{th}-price all-pay

[16] See, e.g., Mood et al. (1963).

auction (3.8) can be written as

$$R^{AP}_{k,n} = \int_0^1 z \left((1-\alpha) - (k-1)\alpha \left\{ \frac{F_{Y^{n-1}_k}(z) - F_{Y^{n-1}_{k-1}}(z)}{1 - F_{Y^{n-1}_{k-1}}(z)} \right\} \right)^{-1} dF_{Y^n_2}(z).$$

A sufficient condition for revenues to be increasing in k is that the term between the curly brackets is increasing in k for all $z \neq 0, 1$. We first make this condition somewhat more intuitive. Consider an urn filled with red and blue balls and let $q = 1 - F(z)$ be the chance of drawing a blue ball, where $0 < q < 1$. Suppose we draw $n-1$ times with replacement. The above condition can then be rephrased as follows: the chance of drawing exactly $k-1$ blue balls, given that at least $k-1$ blue balls were drawn, is increasing in k. Hence, for all k is has to be true that

$$\frac{\binom{n-1}{k-1} q^{k-1}(1-q)^{n-k}}{\sum\limits_{j=k-1}^{n-1} \binom{n-1}{j} q^j (1-q)^{n-j-1}} < \frac{\binom{n-1}{k} q^k (1-q)^{n-k-1}}{\sum\limits_{j=k}^{n-1} \binom{n-1}{j} q^j (1-q)^{n-j-1}}.$$

Introducing $x \equiv q/(1-q) > 0$, the above inequality can be rearranged as:

$$\left(1 - \frac{\binom{n-1}{k}}{\binom{n-1}{k-1}} x \right) \left(1 + \sum_{j=k+1}^{n-1} \frac{\binom{n-1}{j}}{\binom{n-1}{k}} x^{j-k} \right) < 1.$$

The left side of this inequality can be expanded as $1 + \sum\limits_{i=1}^{n-k} a_i \, x^i$ where

$$a_i = \frac{\binom{n-1}{k+i}}{\binom{n-1}{k}} - \frac{\binom{n-1}{k+i-1}}{\binom{n-1}{k-1}} = -\frac{\binom{n-1}{k+i-1}}{\binom{n-1}{k-1}} \frac{n\,i}{(n-k)(k+i)} < 0,$$

which shows that revenues increase in k.

To prove that a lottery yields less revenue than all-pay auctions, it is sufficient to show that the first-price all-pay auction revenue dominates a lottery. Let $R^{LOT,\alpha}$ and $R^{AP,\alpha}$ denote the expected revenue from a lottery and and an all-pay auction, respectively, given $\alpha > 0$. In both formats, bidders' payments are equal to their bids and $\alpha > 0$ acts as a simple rebate. Hence,

$$R^{LOT,\alpha} = \frac{R^{LOT,0}}{1-\alpha}$$

as in equilibrium, each bidder submits a bid equal to $\frac{1}{1-\alpha}$ times the equilibrium bid for $\alpha = 0$. Likewise,

$$R^{AP,\alpha} = \frac{R^{AP,0}}{1-\alpha}.$$

The first-price all-pay auction is efficient, i.e., the object is always allocated to the bidder with the highest value, whereas a lottery is not. As, by assumption, the bidder with the highest value is the bidder with the highest marginal revenue, it follows from Lemma 3.2 (see the proofs of Propositions 3.4 and 3.5) that

$$R^{LOT,0} < R^{AP,0}$$

as for $\alpha = 0$, the utility for the lowest type equals 0 for both the lottery and the all-pay auction. Therefore,

$$R^{LOT,\alpha} < R^{AP,\alpha}$$

for all $0 \leq \alpha < 1$.

Finally, to prove that revenue may decrease with n, we compare the revenues of the $(n-1)^{th}$-price all-pay auction with $n-1$ and n players when $\alpha \to \frac{1}{n-1}$. In this limit, the revenue with $n-1$ players tends to

$$\lim_{\alpha \to \frac{1}{n-1}} R^{AP}_{n-1,n-1} = \frac{n-1}{n-2} \int_0^1 z \left(\frac{1 - F_{Y^{n-2}_{n-2}}(z)}{1 - F_{Y^{n-2}_{n-1}}(z)} \right) dF_{Y^{n-1}_2}(z),$$

which diverges to infinity as $1 - F_{Y^{n-2}_{n-1}}(z) = 0$ for all z. The revenue with n players is equal to

$$\lim_{\alpha \to \frac{1}{n-1}} R^{AP}_{n-1,n} = \frac{n-1}{n-2} \int_0^1 z \left(\frac{1 - F_{Y^{n-1}_{n-2}}(z)}{1 - F_{Y^{n-1}_{n-1}}(z)} \right) dF_{Y^n_2}(z),$$

which is finite. Hence, for α close to $1/(n-1)$, revenues are higher with $n-1$ bidders than with n bidders. ■

Proof of Proposition 3.3. First, consider the two cases (i) $\alpha \geq 1$ and (ii) $\alpha > 1/n$ and $k = n$. When $\alpha \geq 1$, any contribution to the public good returns at least as much as it costs, and it is optimal to bid M. This is also true for the zero-value bidder in the lowest-price all-pay auction when $\alpha > \frac{1}{n}$. Thus, in both cases (i) and (ii), $v^* = 0$ and revenue equals nM.

Next, consider the case $k < n$ and $\alpha < 1$. The condition determining the cut-point v^* is that the difference between the expected payoff of bidding M and $B^{AP}_{k,n}(v^*)$ is zero:

$$\begin{aligned} 0 = & \sum_{j=1}^{n-1} \frac{v^*}{j+1} \left(F_{Y^{n-1}_{j+1}}(v^*) - F_{Y^{n-1}_j}(v^*) \right) \\ & + \left(M - B^{AP}_{k,n}(v^*) \right) (\alpha k - 1) \left(F_{Y^{n-1}_k}(v^*) - F_{Y^{n-1}_{k-1}}(v^*) \right) \\ & - \left(M - B^{AP}_{k,n}(v^*) \right) (1 - \alpha) \left(1 - F_{Y^{n-1}_k}(v^*) \right). \end{aligned} \quad (3.14)$$

To understand the right-hand side of (3.14), consider a bidder with value v^* and assume all others bid according to $B_{k,n}^{AP}(v, M)$ in (3.9). The expression in the top line captures the bidder's increased chance of winning the object worth v^* when she raises her bid from $B_{k,n}^{AP}(v^*)$ to M. The expression in the second line pertains to the case in which she is the k^{th}-highest bidder and, by bidding M, she increases the price she and the $k-1$ highest bidders pay from $B_{k,n}^{AP}(v^*)$ to M. The third line applies when $B_{k,n}^{AP}(v^*)$ is not among the k highest bids and by bidding M the bidder increases only her own price. When these benefits and costs balance, the bidder is indifferent between bidding $B_{k,n}^{AP}(v^*)$ and M.

To show there exists an interior solution $0 < v^* < 1$ to (3.14), we define

$$\zeta(v) \equiv (\alpha k - 1)\left(F_{Y_k^{n-1}}(v) - F_{Y_{k-1}^{n-1}}(v)\right) - (1-\alpha)\left(1 - F_{Y_k^{n-1}}(v)\right),$$

and

$$\psi(v) \equiv (M - B_{k,n}^{AP}(v))\zeta(v) + \sum_{j=1}^{n-1} \frac{v}{j+1}\left(F_{Y_{j+1}^{n-1}}(v) - F_{Y_j^{n-1}}(v)\right).$$

The indifference condition (3.14) then becomes $\psi(v^*) = 0$. First note that $\psi(0) < 0$ as $\zeta(0) < 0$. Since $B_{k,n}^{AP}(v)$ diverges when v tends to 1 (if not before), there must be some value $\tilde{v} < 1$ for which $B_{k,n}^{AP}(\tilde{v}) = M$. At that value, $\psi(\tilde{v}) > 0$. Hence, continuity of $\psi(\cdot)$ implies there exists an interior value where $\psi(\cdot)$ vanishes. Let $0 < v^* < 1$ denote the smallest v for which $\psi(v) = 0$.

To prove that (3.9) constitutes an equilibrium we need to show: (i) bidders with values $v < v^*$ submit bids according to a well-defined increasing bid function, (ii) this bid function follows from the same equilibrium differential equation as in Proposition 3.2, and (iii) bidders with values $v \geq v^*$ strictly prefer to bid M given others' equilibrium bids. Condition (ii) is readily checked. Condition (iii) follows immediately from (3.14), since for bidders with values $v > v^*$ only the potential gain in the top line changes. Hence, if a bidder with $v = v^*$ is indifferent between bidding M and $B_{k,n}^{AP}(v^*)$, bidders with types $v > v^*$ strictly prefer to bid M. The only condition that remains to be checked is (i).

Let v^{**} be the smallest v for which $\zeta(v) = 0$. We first show that $B_{k,n}^{AP}(v)$ is well-defined and increasing in v for all $v < v^{**}$ and then show that $v^* \leq v^{**}$. Recall from (3.7) that

$$B_{k,n}^{AP}(v) = \int_0^v \frac{(n-1)zf(z)F(z)^{n-2}}{-(\alpha k - 1)(1 - F_{Y_{k-1}^{n-1}}(z)) + \alpha(k-1)(1 - F_{Y_k^{n-1}}(z))} dz,$$

which is well-defined and strictly increasing as long as the denominator is positive, i.e. when $\zeta(v) < 0$. As $\zeta(\cdot)$ is continuous and $\zeta(0) < 0$, $B_{k,n}^{AP}(v)$ is strictly increasing in v for all $v < v^{**}$. (Note that by the definition of v^{**}, the derivative of $B_{k,n}^{AP}(v)$ with respect to v approaches infinity as v approaches v^{**}.) Clearly $\psi(v^{**}) \geq 0$, and since $\psi(0) < 0$, continuity of $\psi(\cdot)$ implies that $\psi(v) = 0$ for some $v \leq v^{**}$: the smallest v for which this holds is v^*, so $v^* \leq v^{**}$.

Finally, $R_{k,n}^{AP} < nM$ for all $k < n$ when $\alpha < 1$ since the cut-point satisfies $v^* > 0$ in this case. Thus, there is positive probability that at least one bidder bids less than M, and expected revenue is thus less than nM. From the proof of Proposition 3.2 we know that $R^{LOT} < R^{AP}_{1,n}$ when $\alpha < 1$, so $R^{LOT} < nM$ when $\alpha < 1$. ■

Proofs of Propositions 3.4 and 3.5. To prove that the lowest-price all-pay auction is optimal, we consider more general mechanisms and derive their revenue properties in Lemmas 3.1 and 3.2. First, some notation:

$$V \equiv [0,1]^n,$$

and

$$V_{-i} \equiv [0,1]^{n-1},$$

with typical elements $\mathbf{v} \equiv (v_1, \cdots, v_n)$ and $\mathbf{v}_{-i} \equiv (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$ respectively. Let

$$g(\mathbf{v}) \equiv \prod_j f(v_j)$$

be the joint density of $\mathbf{v}$, and let

$$g_{-i}(\mathbf{v}_{-i}) \equiv \prod_{j \neq i} f(v_j)$$

be the joint density of $\mathbf{v}_{-i}$. We define the marginal revenue $MR_i(v_i) \equiv v_i - (1 - F(v_i))/f(v_i)$ and assume it is strictly increasing in v_i.

We follow Myerson (1981) closely. Using the Revelation Principle, we may assume, without loss of generality, that the seller considers feasible direct mechanisms only.[17] Let (p, x) denote a feasible direct mechanism, where $p : V \to [0,1]^n$ with $\sum_j p_j(\mathbf{v}) \leq 1$, and $x : V \to \Re^n$. We interpret $p_i(\mathbf{v})$ as the probability that bidder i wins and $x_i(\mathbf{v})$ as the expected payments by i to the seller when the vector of values $\mathbf{v} = (v_1, \cdots, v_n)$ is truthfully announced. Given v_i, bidder i's interim utility under (p, x) is

$$U_i(p, x, v_i) \equiv \int_{V_{-i}} \left[v_i p_i(\mathbf{v}) - x_i(\mathbf{v}) + \alpha \sum_{j=1}^{n} x_j(\mathbf{v}) \right] g_{-i}(\mathbf{v}_{-i}) d\mathbf{v}_{-i}. \quad (3.15)$$

Similarly, the seller's expected utility is

$$U_0(p, x) \equiv \int_V \sum_{i=1}^{n} x_i(\mathbf{v}) g(\mathbf{v}) d\mathbf{v}.$$

[17] A direct mechanism is a mechanism where bidders are simply asked to announce their values. We say that a mechanism is feasible if it satisfies individual rationality conditions, incentive compatibility conditions, and straightforward restrictions on the allocation rule.

The following two lemmas will be used to solve the seller's problem.

Lemma 3.1 *Let (p,x) be a feasible direct revelation mechanism. Then the interim utility of (p,x) for bidder i is given by*

$$U_i(p,x,v_i) = U_i(p,x,0) + \int_0^{v_i} Q_i(w)dw, \tag{3.16}$$

with $Q_i(v_i) \equiv E_{\mathbf{v}_{-i}}\{p_i(v)\}$.

Proof. The proof follows in a straightforward manner from the incentive compatibility constraints, see Myerson (1981). ■

Lemma 3.2 *Let (p,x) be a feasible direct revelation mechanism. The seller's expected revenue from (p,x) is given by*

$$U_0(p,x) = \frac{E_{\mathbf{v}}\{\sum_{i=1}^n MR_i(v_i)p_i(\mathbf{v})\} - \sum_{i=1}^n U_i(p,x,0)}{1-n\alpha}. \tag{3.17}$$

Proof. Define $X_i = \int_V x_i(\mathbf{v})g(\mathbf{v})d\mathbf{v}$, $W_i = \int_V v_i p_i(\mathbf{v})g(\mathbf{v})d\mathbf{v}$, and $Y_i = \int_{V_i} U_i(p,x,v_i)f(v_i)dv_i$. By (3.15), we have, for all i,

$$Y_i = W_i - X_i + \alpha \sum_{j=1}^n X_j. \tag{3.18}$$

Summing over i in (3.18) and rearranging shows that the seller's expected revenue from a feasible direct revelation mechanism (p,x) is given by

$$U_0(p,x) = \sum_{i=1}^n X_i = \frac{\sum_{i=1}^n W_i - \sum_{i=1}^n Y_i}{1-n\alpha}. \tag{3.19}$$

Taking the expectation of (3.16) over v_i and using integration by parts, we obtain

$$Y_i = U_i(p,x,0) + E_{v_i}\left\{\frac{1-F(v_i)}{f(v_i)}Q_i(v_i)\right\},$$

so that (3.17) follows. ■

Now, using Lemma 3.2, we prove Propositions 3.4 and 3.5. From (3.17), it is clear that a feasible auction mechanism is revenue maximizing if it: (1) assigns the object to the bidder with the highest marginal revenue if the highest marginal revenue is positive and leaves the object in the hands of the seller otherwise, and (2) gives the lowest type zero expected utility.

We first prove Proposition 3.4. Under the restriction that the seller cannot commit to keeping the good, a feasible auction mechanism is revenue maximizing if it assigns the good to the bidder with the highest marginal revenue (even if negative) and guarantees the lowest-type bidder zero expected utility. It is clear that $B_{n,n}^{AP}(v)$ in (3.7) is strictly increasing in v, as the denominator of the integrand in (3.7) is strictly positive when $\alpha < 1/n$. So the lowest-price auction assigns the object to the bidder with the highest value and, hence, to the bidder with the highest marginal revenue. Furthermore, a zero-type bidder bids zero according to (3.7), which sets the auction's revenue at zero, leaving the lowest-type bidder with zero expected utility. Hence, the lowest-price all-pay auction is revenue maximizing. (We have already shown that revenue increases with α in the proof of Proposition 3.2.)

Next we turn to Proposition 3.5. In the equilibrium defined by (3.11), only bidders with values $v > \widehat{v}$ submit a bid according to a strictly increasing bid function whereas bidders with values $v < \widehat{v}$ abstain from bidding. Hence, Γ assigns the good only to bidders with positive marginal revenues (if at all). Moreover, the bidding function in (3.11) is strictly increasing in v so that the bidder with the highest marginal revenue receives the object. Finally, the expected utility of a bidder with the lowest type equals zero over both stages of Γ, as

$$U(p, x, 0) = (n\alpha - 1)\varphi + \alpha r(n-1)(1 - F(\widehat{v})) = 0,$$

by the definition of φ. The given strategies constitute a Bayesian Nash equilibrium, and when these are played, Γ maximizes (3.17) and is thus optimal. Revenues increase with α, as the denominator of (3.17) is decreasing in α. ■

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Chapter 4

Simultaneous Pooled Auctions with Multiple Bids and Preference Lists

4.1 Introduction

In the past few decades, it has become more and more common that governments use auctions to allocate scarce resources such as spectrum for mobile communication or radio broadcasting, petrol station locations, telephone numbers, etc. Given the official goals of various allocation procedures, governments not always had a "lucky hand" in choosing the right auction design.[1] This paper adds to the list of unfortunate auction designs by analyzing the theoretical properties of an auction whose properties were not yet known.

The allocation mechanism we study has been used in practice at least twice. The first time, it was used for allocating licenses for commercial radio stations in The Netherlands in 2003.[2] In 2005, it was used in Ireland to allocate licenses for wideband digital mobile data services.[3]

In the two auctions, multiple (possibly heterogeneous) licenses were allocated (in the Netherlands, licenses differed in their coverage). Each firm was allowed to acquire at most one license. The auction format was sealed-bid, and firms could express different bids for different licenses. Each firm also had to submit a list specifying its preferences over the licenses on which it bids. These preference lists played a role when a firm had submitted highest bids for several licenses. Each winning firm paid its own bid for the license it acquired.[4] This allocation mechanism can be best described as a simultaneous pooled auction with multiple

[1] See, e.g., Klemperer (2002) for a review.

[2] See Staatscourant 26 February 2003, No. 40, p. 19, for the precise rules of the allocation mechanism used in The Netherlands, available at https://zoek.officielebekendmakingen.nl/stcrt-2003-40-p19-SC38745.html.

[3] The media release of the outcome of the auction, reference number PR211205, can be found on the website of ComReg: http://www.comreg.ie/_fileupload/publications/PR211205.pdf. The auction documents of the Commission for Communications Regulation (ComReg) are confidential; the detailed information over the proceeding of the Irish auction has not yet been made public.

[4] The details of the allocation procedure are described in Section 4.2.

bids and preference lists.

In this paper, we show that this auction format fails to produce one of the most basic and desirable properties of an allocation mechanism, namely that it has an efficient equilibrium. In other words, the licenses do not always end up in the hands of those who value them the most. The reason for this result is as follows. Allocation efficiency requires that all bidders follow the same (symmetric) monotonically increasing (pure) bidding strategy. This implies that if an efficient equilibrium exists, the bidder with the highest possible valuation must submit the highest bids for all objects, and he takes the most preferred one. However, this bidder can potentially increase his expected profit by changing his most preferred object and, at the same time, significantly reducing the bid for that object. In this deviation, the bidder's equilibrium (high) bid for his equilibrium (old) most preferred object remains the highest and, therefore, guarantees him his equilibrium pay-off. The bidder will obtain his equilibrium pay-off if the reduced bid for the new most preferred object is not the highest. However, if the reduced bid turns out to be the highest bid, the bidder obtains his new most preferred object for a very low price.

Of course, it is difficult to assure that the auction was indeed inefficient as the presence of economic inefficiency is difficult to test statistically given the data available. However, there are some indications that the outcome of the Dutch allocation mechanism was inefficient - at least, *ex-post*. A first indication is that not long after the auction, quite a few licenses were resold to third parties or retraded between the parties.[5] Had the licenses ended up in the hands of those parties that valued them the most, reselling/retrading just after the auction should not have taken place.[6] A second indication is that one of the licenses with a specific format requirement (these licenses were auctioned separately from the licenses for unrestricted programming at the same moment in time), was sold for a higher amount than the cheapest license for unrestricted programming (presumably, a more valuable license).

This paper fits into the literature that deals with the question how to auction a set of heterogeneous licenses. The literature distinguishes between static auction mechanisms, such as the Vickrey-Clarke-Groves (VCG) mechanism, and dynamic auction mechanisms, such as the simultaneous ascending auction. The VCG mechanism is developed by Clarke (1971) and Groves (1973), and generalizes the (multi-unit) Vickrey auction.[7] In this mechanism

[5] In particular, Noordzee FM (Talpa Radio International) was sold to De Persgroep on 31 May 2005, Radio 538 (Advent International Corporation) to Talpa Radio International on 31 May 2005, Yorin FM (RTL Nederland) to SBS Broadcasting on 4 January 2006, and Sky Radio (News Corporation) to TMG (Telegraaf Media Groep) on 1 February 2006.

[6] Formally, we do not allow for resale after the auction in our model. However, had an equilibrium existed in a model with resale where the auction outcome had been efficient, the resale option would not have been used. But then there must also have been an efficient equilibrium in the model without resale that we analyze in the current paper. Since we show that there is no such equilibrium, there cannot be an equilibrium with an efficient auction outcome in an auction with resale opportunities. Our results thus imply that only equilibria with a possibility of inefficient outcomes can arise in a model where resale is explicitly taken into account.

[7] Clarke and Groves constructed this mechanism for a class of problems that is far more general than the allocation of objects: their mechanism applies to any public choice problem.

all bidders simultaneously submit sealed-bids on all objects. The auctioneer determines an efficient assignment of the objects based on the bids. Payments are determined so as to allow each bidder a payoff equaling the incremental surplus that he brings to the auction. In simultaneous ascending auctions, all objects are sold simultaneously using an English auction procedure in which prices on each object are increased until there is no more bidding for any of the objects. At that point, the auction ends and the bidders that have made the highest bids receive the objects. Leonard (1983) and Demange et al. (1986) show that in a simultaneous ascending auction, an efficient equilibrium is established when bidders bid "straightforwardly," i.e., in each round, each bidder who currently does not have a standing high bid, bids for the object that currently offers him the highest surplus; and they drop out once the highest available surplus becomes negative.

Other related auction mechanisms are static and dynamic right-to-choose auctions, in which the right-to-choose is auctioned rather than the objects themselves. An example of a static right-to-choose auction is the simultaneous pooled auction (see, e.g., Menezes and Monteiro, 1998). In such an auction bidders simultaneously submit a nonearmarked single bid in a sealed envelope for one of the objects in a pool. The highest bidder chooses his most preferred object; the second highest bidder chooses among the remaining objects; and this process continues until all objects are sold. Each winner pays his own bid. Because bidders are uncertain about which object from the pool they are going to win, and are only allowed to submit a single bid, they may fall prey to some sort of winner's curse. Salmon and Iachini (2007) experimentally show that bidders often overbid and incur losses because they are forced to buy objects that are not their most preferred ones. In the mechanism analyzed in the present paper, bidders do not suffer from this unexpected loss because they are allowed to submit as many bids as objects. Menezes and Monteiro (1998) show that in the homogeneous private-value case with risk-neutral bidders, simultaneous pooled auctions are revenue equivalent to a sequential first-price sealed-bid auction.

Dynamic right-to-choose auctions, also referred to as a sequential pooled auctions or condo auctions (because of its use in selling condominiums in the United States)[8], consist of a sequence of regular auctions in which bidders bid for the right to choose any object among the objects not yet sold. Burguet (2005) shows that ascending right-to-choose auctions, i.e., right-to-choose auctions that consist of a sequence of regular English auctions, are efficient for two *ex-ante* symmetric objects. Gale and Hausch (1994) derive the same conclusion for a two-bidder model with more general preferences than in Burguet (2005). Goeree et al. (2004) introduce bidders' risk aversion into Burguet's (2005) model. They show that ascending right-to-choose auctions raise more revenue than standard simultaneous ascending auctions. Eliaz et al. (2008) examine second-price sealed-bid right-to-choose auctions, i.e., right-to-choose auctions that consist of a sequence of second-price sealed-bid auctions. They show,

[8] See Ashenfelter and Genesove (1992).

both theoretically and experimentally, that in thin markets where there is little interest per object, the second-price sealed-bid right-to-choose auction raises more revenue than sequential auctions for the individual objects. They also provide experimental evidence that a right-to-choose auction can generate even more revenue than a theoretically optimal auction. Moreover, in contrast to the optimal auction, the right-to-choose auction is approximately efficient in the sense that the surplus it generates is close to the maximal one.

This paper is also related to the literature on the efficiency properties of auction mechanisms. Moldovanu and Sela (2003) who analyze aftermarket Bertrand competition where cost is private information at the auction stage (and statistically independent of rivals' costs), show that standard auction mechanisms may lead to inefficient allocations if firms' values of a patent for a cost-reducing technology are strongly interdependent.[9] The reason is that, if a symmetric separating equilibrium is played, then, on the margin, due to the strong negative externality, a firm that is more cost efficient has a lower willingness to pay for a license than a firm that is less cost efficient. Janssen and Karamychev (2010) show that even if the informational externality (interdependence) is weak, efficient equilibria may fail to exist if the bidders' types, i.e., bidders' cost efficiency parameters are strongly correlated ex ante. The present paper, in contrast, shows that simultaneous pooled auctions with multiple bids and preference lists are inefficient even in the independent private valuation setting.

Finally, the paper can be related to the literature on price dispersion.[10] We show that even if objects are perfect substitutes, there can be an equilibrium in which firms bid differently for the objects.

The rest of the paper is organized as follows. In Section 4.2, we set up the model, which contains the key features of the design of auctions held in The Netherlands and in Ireland. In Section 4.3, we look at efficient Nash equilibria of the model and analyze their existence conditions. Section 4.4 concludes.

4.2 The Model

There are two objects for sale and $N > 2$ bidders.[11] Bidders are allowed to win at most one object.[12] Bidder i assigns a value v_i to object 1 and a value αv_i to object 2, where $\alpha \in (0, 1]$ is common to all bidders. The assumption of linear dependence of the valuations is made for simplicity; a common monotonically increasing scalar function would yield the same result but add to the notational complexity. Valuations v_i are independently and identically distributed

[9] The value of the patent for a firm is the difference between the profit it makes in case it acquires the patent, and the profit in case it does not. An auction is said to be efficient if it awards the patent to the firm with the highest value, i.e., the firm with the ex-ante lowest cost.

[10] For an overview of this literature, see Baye et al. (2006).

[11] As the analysis for more than two objects is very similar, for simplicity of notation we concentrate on the two-object case. The two-object case is enough to prove inefficiency.

[12] It may well be that the firms value more than one license, but the Dutch and Irish auction rules prevented the bidders from getting more than one license.

over the unit interval [0, 1] according to the distribution function F. The value of v_i is private information to bidder i. The values of α and N and the distribution function F are common knowledge.

If $\alpha < 1$, the objects are heterogeneous and the first object is preferred by all bidders to the second.[13] The ratio of valuations for the two objects is identical for all bidders. This seems to capture the essence of the Dutch radio-frequency auction quite well, where the value of a license is directly related to its demographic coverage. The licenses in the Dutch radio-frequency auction differed in their demographic coverage. If the coverage of the license increases and a firm attracts a certain percentage of the population, then the total number of listeners (hence, the firm's valuations of licenses) is proportional to that coverage. If licenses are *ex-ante* identical in their demographic coverage, they can be analyzed by the model with homogeneous objects, where $\alpha = 1$. In what follows, we do not consider asymmetric auctions where different bidders are characterized by different values of α as in case of asymmetries a general argument can be easily invoked to establish the inefficiency of the auction.[14]

In order to be more precise about modeling the role of the preference list, we first describe the specific allocation rule used in The Netherlands. The allocation rule determines which bidder wins which object. In the first iteration of the allocation process, it is determined whether the highest bidder for an object has given his first preference to this object. If this is the case, then all such bidders are considered to be the winners of the respective objects. Those bidders who have obtained an object in the first iteration are excluded along with the object that they have got, and the second iteration starts (if there are objects left). In this second iteration, for each remaining object it is determined who of the remaining bidders have submitted the highest bid. These bidders then win these objects provided they have given their first or second preference to these objects. These winning bidders and objects are also removed from the procedure. In the third iteration, the highest bidders with up to the third preference to the corresponding objects are admitted to be the winners and are eliminated along with the objects. This process iterates until all objects are allocated. In each iteration, the minimal required preference is increased with one.

The allocation rule used in Ireland is slightly different from the Dutch rule. Objects are assigned to the highest bidder unless this highest bidder has submitted multiple highest bids. In the latter case, he wins the object that is higher on his preference list. The winners and the corresponding objects are removed. The other objects are assigned in the next iteration(s) in exactly the same fashion. Thus, preferences in the Irish auction only play a role if a bidder

[13] Even though a bidder has a higher valuation for object 1 he might still put his preference on object 2 together with significantly lowering his bid for object 2.

[14] If bidders are asymmetric, their types are either drawn from different distributions or the ratio of their valuations for the two objects is different. Efficient equilibria do not exist in either one of these cases, as efficiency requires that bidding functions for different bidders must be identical (players with higher valuations must bid higher), while asymmetry requires different bidders to use different bidding functions (as the distribution of valuations of a bidder's competitors has an impact on the bidder's equilibrium bidding functions).

has submitted multiple highest bids. Note that, despite the Irish and the Dutch allocation rules are formally different, they yield the same allocation if there are only two objects.

We capture this preference list in our two objects model in the following way. Every bidder i submits two bids, b_i^1 and b_i^2, in a sealed envelope, one for each object, and states his preference over the two objects. The preference is expressed in terms of a probability distribution over the two objects and is represented by p_i, the probability of taking object 1.

The auctioneer collects all triples $\left(b_i^1, b_i^2, p_i\right)$ from all bidders and determines the highest bids for every object. If these highest bids belong to different bidders, these bidders win the objects for which they are the highest bidders, and they pay their winning bid as a price. If, however, it is one and the same bidder j who has submitted the highest bids for both objects, then this bidder gets object 1 with probability p_j and object 2 with probability $(1 - p_j)$. Bidder j pays his bid for the object that he gets. The other object goes to the bidder who has submitted the second highest bid for that object. This bidder also pays his bid as a price.

4.3 Analysis

We will search for efficient Nash equilibria of this game, i.e., equilibria in which the two bidders with the highest two valuations win the objects, and, furthermore, in case $\alpha < 1$, the bidder with the highest value wins object 1 (the most valuable object) and the bidder with the second highest value wins object 2. As allocation efficiency[15] requires that bidders follow a symmetric monotonically increasing bidding strategy, we focus on such equilibria. In an efficient equilibrium, if it exists, each bidder i with valuation v_i submits a bid $b_i^1 = b^1(v_i)$ for object 1 and a bid $b_i^2 = b^2(v_i)$ for object 2, and sets his preferences for object 1, i.e., $p_i = 1$.

In case $\alpha = 1$, efficiency does not require $p_i = 1$. For example, the strategy $b_i^1 = b_i^2 = b(v_i)$ with $b'(v_i) > 0$ and an arbitrarily distributed $p_i \in [0, 1]$ always yields an efficient allocation. As we will see in Proposition 4.2, these strategies do not form an equilibrium.

We first show that for any $\alpha < 1$ there is no efficient Nash equilibrium if the number of bidders, N, is sufficiently large. The intuition is as follows. In an efficient equilibrium, a bidder i with the highest possible valuation $v_i = 1$ submits the highest bid on both objects with certainty. Efficiency requires that $p_i = 1$, and bidder i wins object 1. By significantly reducing his bid on object 2 and making this object his most preferred choice by submitting $p_i = 0$, he can increase his expected payoff (due to a higher surplus in the event he is still the highest bidder on object 2), which constitutes a profitable deviation. As the realized profit from such a deviation is strictly positive and independent of N, whereas the profit in the proposed efficient equilibrium asymptotically decreases to zero, a larger number of bidders, N, makes the deviation more profitable.

[15] Allocation efficiency is obtained if the sum of the valuations of the winning bidders is maximized (where the valuation of the second object in the hands of the same bidder is zero).

Proposition 4.1 *For any $\alpha \in (0,1)$, there exists a number $\hat{N}(\alpha)$ such that for all $N > \hat{N}$, any equilibrium of the auction is inefficient, i.e., the outcome of the auction is inefficient with positive probability.*

Proof. We start this proof by assuming that there exists an efficient (i.e., a monotone symmetric pure-strategy) bidding equilibrium. In such an equilibrium, bids b_i^1 and b_i^2 are monotonically increasing functions of types, the distribution of bids has no mass points, and the expected profit of a bidder is a continuous and monotonically increasing function of his type.[16] Below we show that under the condition of the proposition this assumption leads to a contradiction.

First of all, in any efficient equilibrium, the surplus of every type v_j must converge to zero when $N \to \infty$. This can be seen as follows. In equilibrium, a bidder has the highest bid on both objects or on neither object. If a bidder j has a valuation $v_j < 1$, then the probability that he wins any of the objects converges to zero when $N \to \infty$. Consequently, the *ex-ante* expected surplus of bidder j also converges to zero with N. If, however, $v_j = 1$, then the winning probability is equal to 1 for any number of bidders, and bidder j's expected surplus is equal to the surplus of his most preferred object revealed by p_i, i.e., the larger of the two surpluses $1 - b_j^1$ and $\alpha - b_j^2$. If it were true that $1 - b_j^1 > \varepsilon > 0$ for all N, then bidder k with value $v_k = 1 - \varepsilon/3$ (who is receiving asymptotically zero expected surplus in equilibrium, as we explained above) would have got, by bidding $b_k^1 = b_j^1 + \varepsilon/3$, a strictly positive surplus of

$$v_k - b_k^1 = \left(1 - \frac{\varepsilon}{3}\right) - \left(b_j^1 + \frac{\varepsilon}{3}\right) > \frac{\varepsilon}{3} > 0.$$

If, on the other hand, it were true that $\alpha - b_j^2 > \varepsilon > 0$ for all N, then bidder k with value $v_k = 1 - \varepsilon/(3\alpha)$ would have got, by bidding $b_k^2 = b_j^2 + \varepsilon/3$, a strictly positive surplus of

$$\alpha v_k - b_k^2 = \alpha\left(1 - \frac{\varepsilon}{3\alpha}\right) - \left(b_j^2 + \frac{\varepsilon}{3}\right) > \frac{\varepsilon}{3} > 0.$$

In both cases, there is a bidder k who can profitably deviate. Hence, for any $\varepsilon > 0$, $1 - b_j^1 < \varepsilon$ and $\alpha - b_j^2 < \varepsilon$ if N is taken to be large enough.

Next, we consider a bidder j with valuation $v_j = 1$, who submits the highest bid on both objects and surely wins his most preferred object, i.e., object 1 (as efficiency requires $p_i = 1$). His surplus from this object converges to zero when $N \to \infty$ (as shown above). Hence, there exists a number $\hat{N}(\alpha)$ such that $1 - b_j^1 < \alpha/2$ for all $N > \hat{N}(\alpha)$. Bidder j, though, can profitably deviate by bidding $\gamma < \alpha/2$ for object 2 and submitting $p_i = 0$. He is then still the bidder with the highest bid on object 1, which assures him his equilibrium profit. In addition, with a small probability, he has the highest bid on object 2 as well. In that case he wins object 2 at price γ. For all $N > \hat{N}(\alpha)$ this deviation is profitable because $\alpha - \gamma > \alpha/2 > 1 - b_j^1$.

[16] Monotonicity follows from the following simple consideration: by submitting the same bids as type x', any higher type $x'' > x'$ has the same winning probability as type x' but gets a strictly higher surplus conditional on winning. Continuity follows from the fact that in equilibrium, the distribution of bids has no mass points.

Since the expected profit of a bidder in equilibrium is a continuous function, similar arguments apply to all types that are sufficiently close to type $v_j = 1$. Hence, the game has no efficient equilibrium, i.e., no monotone symmetric pure-strategy bidding equilibrium. Consequently, the auction is inefficient with positive probability. ■

Proposition 4.1 thus provides a condition under which bidders with the highest values leave the auction empty-handed with strictly positive probability. The reason for this type of outcome inefficiency applies quite generally. It does not matter whether firms' values for both objects are perfectly correlated (as in our model) or imperfectly correlated; the argument is even applicable when values are negatively (imperfectly) correlated. We illustrate this point as follows.

Suppose that the values v^1 (for object 1) and v^2 (for object 2) are continuously distributed over the support $[0, 1]^2$ with an arbitrary distribution function. Then, if an efficient equilibrium exists, equilibrium bids of the highest type $\left(v^1, v^2\right) = (1, 1)$ for both objects must be the highest bids for both objects, due to the monotonicity of the equilibrium bidding functions. This, however, is exactly the source of a profitable deviation for the bidder: to reduce one bid and to make the corresponding object his preferred object. Following the logic of Proposition 4.1, this deviation is profitable as long as the number of bidders is large. What is crucial for this argument to work is that the highest type belongs to the support of the distribution.

In accordance with Proposition 4.1, if the number of bidders is sufficiently large, only inefficient equilibria may exist. This inefficiency result might not really be a problem in practical situations if the game has efficient equilibria for small N. The following proposition, however, shows that the nonexistence of efficient equilibria may even occur for $N = 3$ if α is sufficiently large, i.e., the objects are sufficiently close to each other in value. This result is intuitive. Fix N. Consider bidder i with the highest possible valuation $v_i = 1$. For small N, bidder i's profit in the efficient equilibrium is relatively large. The profit from following the deviation strategy (see above) differs from the equilibrium profit in the event bidder i is still the highest bidder on object 2 when he deviates. Bidder i's profit from deviating is in this event at most α (the value of object 2 to bidder i). If α is small, i.e., the objects are very heterogeneous, the equilibrium profit is larger than the profit from deviating. Bidder i then does not have an incentive to deviate from the equilibrium strategy.

Proposition 4.2 *For any $N \geq 3$, there exists a number $\hat{\alpha}(N) \in (0,1)$ such that for all $\alpha \in (\hat{\alpha}(N), 1]$, any equilibrium of the auction is inefficient, i.e., the outcome of the auction is inefficient with positive probability.*

Proof. Suppose that there exists an efficient (i.e., a monotone symmetric pure-strategy) bidding equilibrium, and $s(\alpha, N, v)$ denotes the equilibrium expected surplus of the bidder of type v. Define

$$\hat{s}(N, v) = \sup_{\alpha \in (0,1)} s(\alpha, N, v).$$

We first show that $\hat{s}(N,1) < 1$. Subsequently, we show that for all $\alpha \in (\hat{s}(N,1), 1)$ there are types that have a profitable deviation from the proposed equilibrium. Finally, we show that these same types also have a profitable deviation for $\alpha = 1$. Thus, for all $\alpha \in (\hat{\alpha}(N), 1]$, where $\hat{\alpha}(N) = \hat{s}(N,1)$, the game has no efficient equilibrium. Consequently, the auction is inefficient with positive probability.

Suppose, to the contrary, that $\hat{s}(N,1) = 1$. This implies that for any $\varepsilon > 0$ there exists some $\tilde{\alpha} < 1$ such that $s(\tilde{\alpha}, N, 1) > 1 - \varepsilon$. As efficiency requires $p_i = 1$, the equilibrium bid b^1 of type $v = 1$ must satisfy $b^1 < \varepsilon$, and, due to monotonicity, all other types also bid below ε. As ε is taken arbitrarily small, this leads to $b^1 = 0$ for all types, which is clearly impossible in equilibrium. Hence, $\hat{s}(N,1) < 1$.

Let us consider now bidder i of type $v_i = 1$ (the same argument applies to types in a small neighborhood of $v_i = 1$). The deviation by bidding $b_i^2 = \gamma$ for object 2 and submitting $p_i = 0$ is a profitable deviation provided that $\alpha - \gamma > s(\alpha, N, 1)$. Hence, as long as $\alpha > \hat{s}(N,1) \geq s(\alpha, N, 1)$ and $\alpha < 1$, there exists a sufficiently small γ, $0 < \gamma < \alpha - \hat{s}(N,1)$, so that the deviation is indeed profitable. On the other hand, it must be that $s(1,N,1) < 1$. If this were not the case, it would have been the case that $s(1,N,1) = 1$, type $v_i = 1$ would have bid zero for one of the objects, and hence all types would have bid zero for that object due to monotonicity. This, however, can never happen in equilibrium. Hence, also for $\alpha = 1$, type $v_i = 1$ has a profitable deviation by bidding γ, $0 < \gamma < s(1,N,1)$. ■

The conditions in Proposition 4.1 and Proposition 4.2 can be made more precise if we make an extra assumption about the distribution function F. In the following example we show that when valuations are uniformly distributed, the outcome is inefficient even if the objects are quite different, *i.e.*, even if α is relatively small (but larger than $1/(N-1)$).

Example 4.1 Let $N \geq 3$ and the valuations be uniformly distributed over $[0,\ 1]$. The efficiency criterion requires $p_j = 1$. Suppose all bidders $j \neq i$ follow a symmetric bidding strategy and bid $\left(b_j^1, b_j^2, p_j\right) = \left(b^1(v_j), b^2(v_j), 1\right)$. If bidder i bids $\left(b_i^1, b_i^2, p_i\right) = \left(b^1(x^1), b^2(x^2), 1\right)$, where $b^1(v)$ and $b^2(v)$ are assumed to be strictly increasing and continuously differentiable bidding functions, he gets the following expected payoff:

$$\pi_i = \left(v_i - b_i^1\right) \Pr\left(b_i^1 > b_{-i}^1\right) + \left(\alpha v_i - b_i^2\right) \Pr\left(b_i^2 > b_{-i}^2 \left| \exists j : b_i^1 < b_{-i}^1\right.\right) \left(1 - \Pr\left(b_i^1 > b_{-i}^1\right)\right).$$

Let us consider the following two cases, in which we denote the first and the second order statistics of $N-1$ competitors' valuations v_{-i} by y and z respectively:

(a) If $x^1 \geq x^2$, then $\Pr\left(b_i^1 > b_{-i}^1\right) = \Pr\left(x^1 > y\right) = \left(x^1\right)^{N-1}$ and

$$\begin{aligned} & \Pr\left(b_i^2 > b_{-i}^2 \left| \exists j : b_i^1 < b_j^1\right.\right) \left(1 - \Pr\left(b_i^1 > b_{-i}^1\right)\right) \\ = & \Pr\left(y > x^1 \geq x^2 > z\right) \\ = & (N-1)\left(1 - x^1\right)\left(x^2\right)^{N-2}, \end{aligned}$$

so that

$$\pi_i = \left(v_i - b_i^1\right)\left(x^1\right)^{N-1} + \left(\alpha v_i - b_i^2\right)(N-1)\left(1-x^1\right)\left(x^2\right)^{N-2}.$$

(b) If $x^1 \leq x^2$, then, again, $\Pr\left(b_i^1 > b_{-i}^1\right) = \Pr\left(x^1 \geq y\right) = \left(x^1\right)^{N-1}$ and

$$\begin{aligned}
&\Pr\left(b_i^2 > b_{-i}^2 \left| \exists j : b_i^1 < b_j^1\right.\right)\left(1 - \Pr\left(b_i^1 > b_{-i}^1\right)\right) \\
&= \Pr\left(y > x^1, x^2 > z\right) \\
&= \Pr\left(x^2 > y > x^1\right) + \Pr\left(y > x^2 > z\right) \\
&= \left(x^2\right)^{N-1} - \left(x^1\right)^{N-1} + (N-1)\left(1-x^2\right)\left(x^2\right)^{N-2},
\end{aligned}$$

so that

$$\pi_i = \left(v_i - b_i^1\right)\left(x^1\right)^{N-1} + \left(\alpha v_i - b_i^2\right)\left(\left(x^2\right)^{N-1} - \left(x^1\right)^{N-1} + (N-1)\left(1-x^2\right)\left(x^2\right)^{N-2}\right).$$

Combining both cases, we can rewrite π_i as follows:

$$\begin{aligned}
\pi_i\left(v_i, x^1, x^2\right) &= \left(v_i - b^1\left(x^1\right)\right)\left(x^1\right)^{N-1} + \left(\alpha v_i - b^2\left(x^2\right)\right)(N-1)\left(1-x^1\right)\left(x^2\right)^{N-2} \\
&+ \left(\alpha v_i - b^2\left(x^2\right)\right)\left(\left(x^2\right)^{N-1} - \left(x^1\right)^{N-1} - (N-1)\left(x^2 - x^1\right)\left(x^2\right)^{N-2}\right) 1_{\geq 0}\left(x^2 - x^1\right),
\end{aligned}$$

where $1_{\geq 0}(x)$ is the indicator function of the subset of nonnegative real numbers:

$$1_{\geq 0}(x) \equiv \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Bidder i maximizes π_i with respect to x^1 and x^2, and the maximum must be attained at $v_i = x^1 = x^2 = v$, which is the truth-telling condition for the mechanism. The first-order conditions are:

$$\begin{cases} 0 = \frac{\partial \pi_i}{\partial x^1}(v,v,v) = -\left(v\frac{db^1}{dv}(v) - (N-1)\left(v - b^1(v)\right) + (N-1)\left(\alpha v - b^2(v)\right)\right)v^{N-2} \\ 0 = \frac{\partial \pi_i}{\partial x^2}(v,v,v) = -(N-1)\left(v\frac{db^2}{dv}(v) - (N-2)\left(\alpha v - b^2(v)\right)\right)(1-v)\,v^{N-3}. \end{cases}$$

Solving this system of differential equations yields the following unique candidate bidding functions:

$$\begin{cases} b^1(v) = \frac{N-(1+\alpha)}{N}v \\ b^2(v) = \frac{N-2}{N-1}\alpha v. \end{cases}$$

Thus, if an efficient Nash equilibrium exists, it must be given by the above bidding functions.

Let us consider a bidder j with valuation $v_j = 1$. His equilibrium payoff is

$$\pi_j(1,1,1) = 1 - \frac{N-(1+\alpha)}{N} = \frac{1+\alpha}{N}.$$

Deviating by bidding $b_j^2 = b^2(\varepsilon) = \alpha\varepsilon(N-2)/(N-1)$, where ε is arbitrarily small, bidder j has still the highest bid on object 1, but with a small probability he is also the highest bidder for object 2. Stating his preference as $p_j = 0$ yields him on such rare occasions a payoff of $\alpha - b^2(\varepsilon)$. Thus, the payoff $\tilde{\pi}_j$ of bidder j from such a deviation is:

$$\begin{aligned}
\tilde{\pi}_j &\equiv \left(\alpha v_j - b_j^2\right)\Pr\left(b^2(\varepsilon) > b^2(v_{-j})\right) \\
&\quad + \left(v_j - b_j^1\right)\Pr\left(b_j^1 > b_{-j}^1 \left| \exists i : b^2(\varepsilon) < b^2(v_i)\right.\right)\left(1 - \Pr\left(b^2(\varepsilon) > b^2(v_{-j})\right)\right) \\
&= \left(\alpha v_j - b^2(\varepsilon)\right)\Pr\left(b^2(\varepsilon) > b^2(v_{-j})\right) + \left(v_j - b_j^1\right)\left(1 - \Pr\left(b^2(\varepsilon) > b^2(v_{-j})\right)\right) \\
&= \left(\alpha v_j - b^2(\varepsilon)\right)\Pr(v_{-j} < \varepsilon) + \left(v_j - b_j^1\right)\left(1 - \Pr(v_{-j} < \varepsilon)\right) \\
&= \left(\alpha - \frac{N-2}{N-1}\alpha\varepsilon\right)\varepsilon^{N-1} + \left(1 - \frac{N-(1+\alpha)}{N}\right)\left(1 - \varepsilon^{N-1}\right) \\
&= \frac{1+\alpha}{N} + \left(\alpha - \frac{1+\alpha}{N} - \frac{N-2}{N-1}\alpha\varepsilon\right)\varepsilon^{N-1}.
\end{aligned}$$

This implies that if $\alpha > (1+\alpha)/N$, there exists an ε,

$$\varepsilon < \frac{\alpha - \frac{1+\alpha}{N}}{\frac{N-2}{N-1}\alpha} = \frac{(N-1)((N-1)\alpha - 1)}{(N-2)N\alpha},$$

such that $\tilde{\pi}_j\left(v_j, b^1(v_j), b^2(\varepsilon)\right) > (1+\alpha)/N = \pi_j(1,1,1)$, so that the deviation is profitable. Hence, an efficient equilibrium does not exist for $\alpha > 1/(N-1)$. ▲

In summary, Proposition 4.2 shows that the non-existence of efficient Nash equilibria is not only an asymptotic property of the game, as established in Proposition 4.1. Efficient equilibria fail to exist also for small N, as long as the objects are sufficiently close in value. For the uniform distribution (see Example 4.1), efficient Bayes-Nash equilibria fail to exist for any $N \geq 3$ provided $\alpha > 0.5$. On the other hand, efficient equilibria fail to exist even for small values of α, i.e., when objects are very different in valuations, as long as the number of bidders is large. For the uniform distribution, efficient Bayes-Nash equilibria fail to exist for any $\alpha > 0$ provided $N > 1 + 1/\alpha$.

As an efficient equilibrium does not exist, an equilibrium must be non-monotone, asymmetric, or in mixed strategies. In either of these three cases, there is a strictly positive probability that the less valued object is sold for more than the more valued object. In this sense, the paper may give an explanation for the observation in the Dutch auction that less valuable spectrum is sold for a higher price than more valuable spectrum.

When $\alpha = 1$, efficiency does not require $p_i = 1$ anymore (because both objects are homogeneous). This may lead to another type of equilibrium, where bidders do not coordinate on bidding $b^1(v_i)$ for object 1 and $b^2(v_i)$ for object 2. Bidders simply submit two bids for two objects, and they do not pay any attention whether the bid $b^1(v_i)$ is placed on object 1 and the bid $b^2(v_i)$ is placed on object 2, or the other way around. From the point of view of one bidder, any other bidder puts $b^1(v_i)$ and $b^2(v_i)$ on object 1 with equal probability.[17] In other words, bidders put their (deterministic) bids randomly on both objects. In order to get the highest possible expected surplus, every bidder will set his preferences for the object on which he submits the lowest bid. As all N bidders place their bids on both objects independently of each other, each of 2^N possible distributions of $2N$ bids across two objects occurs with equal probability. This equilibrium can alternatively be viewed as a symmetric mixed-strategy bidding equilibrium.

Interestingly, Proposition 4.2 is also applicable for $\alpha = 1$, as the following corollary states.

Corollary 4.1 *For $\alpha = 1$ and $N \geq 3$, any equilibrium of the auction is inefficient, i.e., the outcome of the auction is inefficient with positive probability.*

As noted above, efficiency does not require $p_i = 1$ anymore when $\alpha = 1$. Nevertheless, efficiency still requires the bidding functions $b^1(v)$ and $b^2(v)$ to be monotonically increasing and symmetric. The highest type must therefore be the highest bidder for both objects. As in the case $\alpha < 1$, this requirement leads to the existence of a profitable deviation by type $v_j = 1$.

So far, we have argued that the mechanism we discuss does not have an efficient equilibrium. One may wonder what an inefficient equilibrium looks like. This is, however, extremely difficult to analyze in general, because equilibria are either non-monotone, asymmetric, or in mixed strategies. In the following example, we have numerically computed equilibrium bids in an inefficient equilibrium when valuations are uniformly distributed and objects are homogeneous, i.e., $\alpha = 1$. The example also shows in more detail why, even when $\alpha = 1$, the auction outcome may be inefficient. The main idea is that the bidder with the second-highest valuation does not need to win the second object.

Example 4.2 Let $N = 3$, $\alpha = 1$, and the valuations be uniformly distributed over $[0, 1]$. Each bidder j of type v_j follows the following strategy. First, he computes two bids $b^H(v_j)$ and $b^L(v_j)$; see Figure 4.1. Then, with probability 50%, the bidder places his high bid $b^H(v_j)$ on object 1 and his low bid on object 2, and sets $p_j = 0$. With the remaining probability of 50%, he places his high bid $b^H(v_j)$ on object 2 and his low bid on object 1, and sets $p_j = 1$.

[17] The superscripts of the bidding functions $b^1(v_i)$ and $b^2(v_i)$ do not refer anymore to the objects and are only used to make a distinction between the bids.

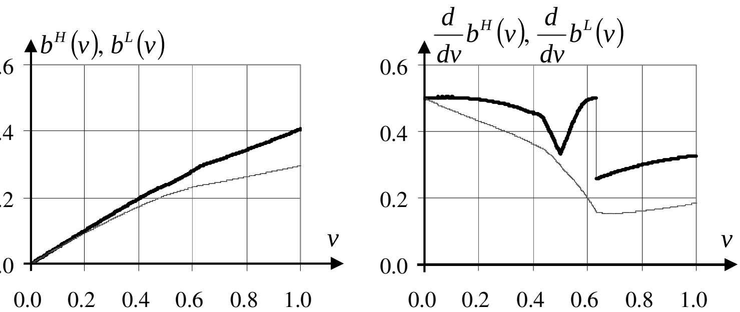

Figure 4.1: *Equilibrium bidding functions* $b^H(v)$ *(bold line) and* $b^L(v)$ *(dotted line), left graph, and their derivatives, right graph, as functions of* v.

Confronted with such a behavior by his competitors, bidder j faces equal distributions of competitors' bids for objects 1 and 2. As a result, he is indifferent between the objects, and mixes his own bids as well. The equilibrium is in mixed strategies, and the functions $b^H(v_j)$ and $b^L(v_j)$ are computed in such a way that they form an ε-equilibrium[18] with $\varepsilon = 10^{-6}$.

Bidding behavior by a bidder with a value close to zero can be described as follows. The probability that he outbids two competitors is negligibly small compared to outbidding only one. Therefore, his bidding strategy is based on outcompeting only one bidder. Consequently, the auction game for a low-valuation bidder is like a single-object first-price sealed-bid auction with one competitor, where bidding half of his value is an equilibrium strategy. Bidding behavior by a bidder with a value close to one is more interesting. He is very likely the highest bidder for the object for which he bids $b^H(1) \approx 0.4$, and this ensures him a relatively large surplus. Nevertheless, he submits a significantly lower bid $b^L(1) \approx 0.3$ for the other object in the hope of getting it cheaper.

Observe that $b^H(v_j)$ contains a kink at $v \approx 0.633$. The reason that there is a kink in $b^H(v_j)$ is the following. Let us have a look at a bidder with $v = 0.80$. His high bid is always higher than the low bid of the highest value bidder ($v = 1$). For a bidder with for example $v = 0.40$ this is not the case. The kink is exactly at the level v for which $b^H(v) = b^L(1)$. This kink of $b^H(v_j)$ causes a kink in the derivative of $b^L(v_j)$ and, therefore, it causes a discontinuity in its second-order derivative. This discontinuity, in turn, causes a kink in the derivative of $b^H(v_j)$, and, therefore, it causes a kink in the second derivative of $b^L(v_j)$. This latter kink, in turn, causes a discontinuity in the third-order derivative of $b^L(v_j)$, and so on, *ad infinitum.* It makes numerical calculations of the bidding functions unstable and complex.

Let the bidders' values be $v_1 > v_2 > v_3$. In equilibrium, due to bids' monotonicity, the highest-value bidder always gets an object, as $b^H(v_1) > \max\left(b^H(v_2), b^H(v_3)\right)$. The second-

[18] In an ε-equilibrium, no player can unilaterally increase his profit by more than ε; see Radner (1980).

highest-value bidder, however, gets the other object only if his bid for this (second) object is higher than the bid of the third, the least-value bidder. With probability 1/8, bids $b^L(v_1)$, $b^L(v_2)$, $b^H(v_3)$ will be placed on the second object. With probability approximately 34.5%, bidders' values satisfy $b^H(v_3) > b^L(v_1) > b^L(v_2)$. Thus, with probability approximately 4.3%, the third bidder gets the second object, which is inefficient.[19] ▲

4.4 Concluding Remarks

In this paper, we show that simultaneous pooled auctions with multiple bids and preference lists, where single-object demand bidders are allowed to make separate bids for each object and submit a preference list to rank these objects, never have efficient equilibria unless objects are sufficiently heterogeneous and the number of bidders is small. In so far as efficiency of auctions' outcome is an important consideration for governments – and which government would ever want to openly deny that this is the case? – the paper shows that this type of auction format, i.e., a multiobject sealed-bid auction with right-to-choose ingredients, should not be used (anymore).[20] Other mechanisms exist that exhibit these efficiency properties (under fairly general conditions), such as the Vickrey-Clarke-Groves mechanism, the simultaneous ascending auction, and the ascending right-to-choose auction. In laboratory experiments, Goeree et al. (2006) show that, when there is single object demand, the simultaneous ascending auction performs better with respect to efficiency than auctions with a first-price element (like the simultaneous first-price auction, the sequential first-price auction, and the simultaneous descending auction). As other auction formats perform better, we do not see good economic arguments why the auctions analyzed in this paper should be used in future allocation processes.

An interesting question about the auction format used in this paper is whether it is a revenue maximizing (i.e., optimal) mechanism or not. In a different context where players can get multiple objects, Armstrong (2000) shows that the revenue maximizing auction is not efficient.

The focus of this paper has been on the efficiency of the allocation mechanism rather than on the outcome of the allocation procedure.[21] It is important to note that assigning licenses to those who value them the most may not lead to a welfare maximizing outcome. In the radio broadcasting context, for example, this is true when the advertising value of a listener does not match the social value (see Anderson and Coate, 2005). Radio formats that appeal

[19] If the number of bidders is larger than 3, then this probability is larger too.

[20] Unfortunately, the Dutch government has used this auction format again in 2007 to allocate licenses for commercial radio stations (see https://zoek.officielebekendmakingen.nl/stcrt-2007-220-p8-SC82971.html) in spite of popular scientific journal articles conveying the message of the present paper (see, e.g., Janssen and Maasland, 2003). Also the Taiwanese government used (a multi-round version of) this auction format in 2007 to allocate Wimax licenses (see Fan, 2011).

[21] For an analysis to what extent the auction for commercial radio frequencies in the Netherlands in 2003 has helped to safeguard public interests, see Maasland et al. (2005).

only to a group of people who are not an interesting target for advertisers will not be offered when a regular (efficient) auction is used, although from a welfare point of view it may be desirable. Colored auctions, where a certain number of licenses is set aside for specific type of broadcasting programs, may therefore be necessary to reach public interests.

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Chapter 5

Auctions with Flexible Entry Fees

5.1 Introduction

A relatively recent literature has studied how post-auction interactions, such as resale or future market competition, affect bidders' bidding behavior in auctions. One application that has received particular attention, both in the theoretical literature and in the popular press, are the auctions for third generation mobile telecommunication licenses around the world, which enable the winning bidders to compete with each other by offering telecommunication services to final consumers.[1] There are now many papers studying this framework, and most of them emphasize that common auction properties do not hold in such an environment and that inefficient outcomes are likely to emerge.

The literature the present paper builds on studies auctions in the presence of negative informational externalities due to future market interactions (see Jehiel and Moldovanu, 2006, for an overview). One application of this literature (see Goeree, 2003, Das Varma, 2003, and Moldovanu and Sela, 2003) is a single-unit auction where one object, namely a patent for a cost reduction, is auctioned and the winner competes in the market after the auction with all non-winners. Moldovanu and Sela (2003) who analyze aftermarket Bertrand competition where a firm's marginal cost is private information at the auction stage (and statistically independent of its rivals' marginal costs), show that standard auctions lead to inefficient allocations when bidders' values are strongly and negatively interdependent.[2] The reasoning why efficient equilibria do not exist is as follows. If a symmetric separating equilibrium is played, then, on the margin, a low-cost firm expects to outbid a rival who also has a low cost, implying that, conditional on winning, the price this firm will be able to charge in the market (and hence its market profit) will be low. Therefore, on the margin, a low cost firm has a lower

[1] For easily accessible theoretical articles on the main issues involved, see Binmore and Klemperer (2002), Börgers and Dustmann (2003), Klemperer (2002a), Klemperer (2002b), and van Damme (2002). For a popular press article, see Klemperer (2000).

[2] See also Jehiel et al. (1996) and Jehiel and Moldovanu (2000) for related papers where an (informational) externality may lead to inefficiency in standard single-unit auctions.

willingness to pay than a firm with high cost, which can expect to charge a high market price. Clearly, this argument cannot work in a private values model where the value conditional on winning does not depend on rivals' characteristics.

Goeree (2003) and Das Varma (2003) analyze a similar setting but allow for signaling private information through the auction bid. The reason why an efficient equilibrium may not exist in these papers is that under strategic complementarity, a more efficient firm may understate its private information by shading its bid in order to relax market competition. This phenomenon has been documented as a "fat cat" business strategy in Fudenberg and Tirole (1984). Das Varma (2003) shows that the inefficiency disappears when the downstream market becomes perfectly competitive. Katzman and Rhodes-Kropf (2008) analyze how different bid-announcement policies affect the efficiency and revenue of an auction (see also Molnár and Virág, 2008). In particular, Katzman and Rhodes-Kropf (2008) show that when signaling reduces the revenue and threatens the efficiency due to, e.g., strategic complementarity, auctioneers prefer auction formats that do not reveal the winning bid.

Another instance of this literature studies auctioning of multiple objects. Hoppe et al. (2006) concentrate on auctions where bidders are *ex-ante* asymmetric such as in markets with incumbents and entrants. The main insight in Hoppe et al. (2006) is that auctioning more licenses does not necessarily induce a higher degree of competitiveness, i.e., higher market efficiency. Janssen and Karamychev (2009) show that a negative externality (and associated with it potential allocative inefficiency) may appear when firms differ in their attitudes toward risk. Janssen and Karamychev (2010) show that when bidders' types are *ex-ante* correlated, efficient equilibria may fail to exist even when the negative externality is weak. The main reason for this is that the correlation and the externality are, to a certain degree, alternative ways to create conditions for the non-existence of monotone equilibria.

All these papers differ in many details, such as whether one or multiple objects are auctioned, whether bidders are *ex-ante* symmetric or not, whether market demand is certain or not, whether risk attitude plays a role, which auction format is used, etc. In all these kinds of environments, efficient equilibria may fail to exist. In the present paper, we study a general model that encompasses many of the environments studied in the literature. We show that in all such environments a mechanism exists that possesses an efficient equilibrium. In this particular mechanism, *prior* to the auction, bidders are asked to pay *any* publicly observable sum of money *they would like*. We call these voluntary payments "flexible entry fees".

The idea of a voluntary entry fee could be traced back to Maskin and Riley (1981). The voluntary entry fee in that paper, however, is very different from our flexible entry fee. In Maskin and Riley (1981), the auctioneer sets a fixed (inflexible) entry fee and bidders can decide whether to pay that fee or not. The object is then allocated to the highest bidder who has paid the entry fee, if any, and if no bidder has paid the entry fee the object is allocated to the highest bidder. In our paper, bidders decide themselves on the amount of the entry fee they pay (flexibility), and the only thing the auctioneer does is that he collects and announces the entry fees individual bidders have paid. Independent of the chosen entry fee, *all* bidders participate in the auction.

The flexible entry fee gives bidders a possibility to signal their type. The incentive to reveal their type is exactly the reason why bidders may pay a positive entry fee. Signaling types through bidding behavior during an auction is usually detrimental to the efficiency of the auction (see, e.g., Goeree, 2003, and Das Varma, 2003). Signaling prior to the auction, as in our paper, turns out to have the opposite effect.

The intuition is as follows. Due to negative interdependencies, firms' values are negatively related to the types of other firms. For example, if firms compete à la Bertrand or Cournot in a market after the auction and a firm's type is its cost efficiency, a firm's valuation for the license, i.e., its market profit, negatively depends on the types of the firm's competitors. This negative interdependency creates an incentive for a firm to signal its high efficiency level in the pre-auction signaling stage so that competitors bid lower in the auction. The more efficient the firm, the larger its incentive to signal, as a more efficient firm wins with a higher probability and, therefore, is willing to spend a larger part of its market profit on signaling its type through the entry fee. Together with the fact that the market profit of this more efficient firm is higher, this implies that the more efficient firm sets a higher entry fee, and the equilibrium is perfectly separating. This information revelation before the auction makes the auction efficient.[3]

In the main body of the paper, we show how this auction works in detail for a second-price sealed-bid auction where bidders' valuations negatively depend on the types of the other bidders. For a second-price sealed-bid auction with independently distributed types, we show that if the negative interdependencies are relatively weak, the auction with flexible entry fees is revenue equivalent to and yields the same (efficient) allocation as the standard second-price sealed-bid auction. If the negative interdependencies are relatively strong, the auction with flexible entry fees remains efficient whereas the standard second-price sealed-bid auction is known to be inefficient. When types are *ex-ante* affiliated and the affiliation is not too strong, a similar result holds true, but revenue equivalence fails. It turns out that when both auctions have an efficient equilibrium, the auction with flexible entry fees performs better in terms of revenue. In the concluding section, we discuss whether the argument also holds true when the externality is positive.

Another interpretation of the auction with flexible entry fees (where the monetary fees are collected by the auctioneer) is that firms burn money or hire expensive auction experts to signal their strength. As long as the amount of money burnt (the cost of hiring auction experts) is either visible or made public, then this will have the same effect as flexible entry

[3] The importance of costly signaling to restore auction efficiency is also studied, although in a very different context, by Schwarz and Sonin (2005).

fees.[4] Again, contrary to the common idea that exchanging information is bad (as it may lead to collusion)[5], making this kind of information public will improve the efficiency of the auction. This signaling resolves the uncertainty firms have about each other's signals. In this interpretation, however, the revenue collected by the auctioneer is lower than in the standard second-price sealed-bid auction, because a part of the revenue is either burnt or spent on experts.

Apart from the above-mentioned literature, the paper is also related to the literature on auctions with entry fees. Milgrom and Weber (1982) show that entry fees may lead to problems with the existence of monotonic equilibria, and Landsberger and Tsirelson (2000) show that with entry fees or other participation costs, monotonic equilibria become increasingly unlikely once the number of bidders is large. These sources of inefficiency do not arise here, however, as entry fees in this paper are flexible so that bidders can decide on the size of the fee they would like to pay. Perry et al. (2000) analyze a two stage sealed-bid auction for a single object where the two highest bidders of the first stage proceed to the second stage and all loosing bids are revealed.

The rest of the paper is organized as follows. In Section 5.2, we describe the basic model with negative externalities by means of an example. In this example, there is one object to be auctioned, there are two bidders with private information about their types, the types are identically and independently distributed, and the bidders have additively separable linear valuation functions with negative interdependencies. In this basic set-up, we show that when the negative interdependency is strong, the auction with flexible entry fees, contrary to a second-price sealed-bid auction, always has an efficient equilibrium. Moreover, we show that in terms of revenues the two mechanisms are equivalent when both have an efficient equilibrium. In Section 5.3, we show that these results hold in a general setting with independent types. Section 5.4 analyzes the case of correlated types and Section 5.5 concludes with a discussion. Proofs are in the appendix.

5.2 The Basic Model

As an example, we consider a standard symmetric single-object second-price sealed-bid auction, from now on termed the SP-auction, where two bidders, denoted by $i, j \in \{1, 2\}$, $j \neq i$, have interdependent valuations. Bidder i's type and his value for the object are denoted by X_i and V_i respectively. The types of the bidders are identically and independently distributed over the interval $[0, 1]$ in accordance with a distribution function $F(x) \equiv \Pr(X_i \leq x)$. The values of the bidders for the object are interdependent and given by the following linear

[4] Klemperer (2002a) observes that in the mid nineties Pacific Telephone paid for full page ads in newspapers and hired one of the most prominent auction theorists to give seminars, signaling that the California license was of utmost importance to them.

[5] See, e.g., Grimm et al. (2003) on the role of information provision in facilitating collusion.

valuation function:

$$V_i = v(X_i, X_j) = aX_i - bX_j + c,$$

where $a > 0$ and $c > b > 0$, which ensures that values are positive. It is easily seen that the values are negatively interdependent: a bidder i's own type affects his value positively whereas the type of his competitor has a negative effect on his value. This negative dependence reflects the fact that often in an auction where some post-auction interaction (such as future market competition) takes place, a bidders' type (such as a measure of his cost efficiency) positively affects his own value but negatively inflicts upon the value of the competitor. Moreover, it is readily checked that auction efficiency requires the bidder with the highest type to win the auction.

We make the usual assumptions of a game with private information, namely that a bidder's type is private information, i.e., bidder i knows the realization x_i of X_i, the other bidder j does not know x_i, but he knows v and F, and all this is common knowledge. Invoking a standard procedure, one can easily verify that

$$\beta^{SP}(x) = (a - b)x + c$$

constitutes a unique symmetric equilibrium bidding function of the SP-auction provided that $a > b$. This equilibrium is efficient and ensures that the bidder with the highest value makes the highest bid and wins the object. If, on the other hand, $a \leq b$, the function $\beta^{SP}(x)$ decreases so that the SP-auction does not have a monotone symmetric equilibrium. Therefore, there is a strictly positive probability that in this case the SP-auction results in an allocation that is inefficient.

The non-standard feature of the mechanism that we consider is that prior to the auction, each bidder decides on an amount $e_i \geq 0$ that he voluntarily pays to the seller before participating in the auction. We call this e_i a flexible entry fee and it is important that this e_i is public information before the auction takes place so that bidders can update their beliefs about their competitor's type before the auction starts. We refer to the second-price sealed-bid auction with flexible entry fees as the FEF-auction.

The timing of the game is as follows. In the first stage, after nature assigns types to bidders, both bidders simultaneously submit payments $e_i \geq 0$ to the auctioneer. These payments are publicly observed. In the second stage, the bidders participate in a second-price sealed-bid auction where they submit bids $\beta_i \geq 0$, and the bidder with the highest bid gets the object and pays a price that is equal to the second highest bid, which in the case of $N = 2$ is also the lowest bid. In case of a tie (which will not happen in equilibrium with positive probability), an arbitrary tie-breaking rule applies. Importantly, only the bids made during the auction (and not the entry fees paid) determine the allocation of the object.

We use Weak Perfect Bayesian Equilibrium, WPBE hereinafter, as the equilibrium concept.[6]

Definition 5.1 *A symmetric WPBE of the FEF-auction consists of a strategy and beliefs such that:*

1. the strategy of bidder i of type x_i is a pair of functions, $e_i = e^* (x_i)$ and $\beta_i = \beta^* (x_i, e_i, e_j)$, where $e_i = e^* (x_i)$ is the entry fee chosen, and $\beta^* (x_i, e_i, e_j)$ is the bid when e_i and e_j are the chosen entry fees;

2. the belief of bidder i is the conditional probability distribution
   $B^* (y \,|x_i, e_i, e_j\,) \equiv \Pr (X_j \leq y \,|X_i = x_i, e_i, e_j\,)$ of the competitor's type X_j, conditional on x_i, e_i, and e_j;

3. strategies are optimal given the strategy of the other bidders and beliefs;

4. beliefs are generated by Bayes' rule on-the-equilibrium path.

It is easy to see that $\beta^{SP} (x)$ is also an equilibrium of the FEF-auction, i.e., if $a > b$, then $e^* (x) = 0$, $\beta^* (x, e_i, e_j) = \beta^{SP} (x)$, and naive beliefs $B^* (y \,|x, e_i, e_j\,) = F (y)$ on- and off-the-equilibrium path, constitute a symmetric equilibrium of the FEF-auction. The reason is as follows. In such a pooling equilibrium, the entry fees that bidders choose are all zero and do not contain information on their types. Consequently, the bids are solely based on a bidder's own type. On the other hand, if the bidders anyway do not adjust their bids depending on which entry fees are paid, then there is no point in paying a positive entry fee.

In addition to the pooling equilibrium (which only exists if $a > b$), there is another symmetric WPBE of the FEF-auction, which is perfectly separating and which always exists. In this equilibrium, bidders choose positive entry fees in accordance with the following increasing and continuously differentiable function:[7]

$$e^{FEF} (x) = b \int_0^x F(z)dz = b \left[x - E(z|z < x)\right] F(x). \tag{5.1}$$

For convenience, we define its generalized inverse function for non-negative values of e as follows:

$$h(e) = \max\{x : e^{FEF} (x) \leq e\}. \tag{5.2}$$

In other words, if $e \in \left[0, e^{FEF} (1)\right]$ then $h(e)$ is the type which pays entry fee $e : e^{FEF} (h(e)) = e$. If, however, $e > e^{FEF} (1)$ then $h(e)$ is defined by $h(e) = 1$. The function $h(e)$ represents

[6] A WPBE is the same as a perfect Bayesian equilibrium, with this difference that it does not impose restrictions on out-of-equilibrium beliefs. See, e.g., Mas-Colell et al. (1995), p. 285.

[7] In the next section, we comment on the interpretation of this expression for the entry fee.

bidders' beliefs:

$$B^{FEF}(y|x, e_i, e_j) = \begin{cases} 1 \text{ if } & y \geq h(e) \\ \\ 0 \text{ if } & y < h(e). \end{cases} \tag{5.3}$$

In other words, having observed an entry fee e_j of bidder j, bidder i believes that bidder j is of type $x_j = h(e_j)$ with probability one:

$$\Pr(X_j = h(e)|e_j = e) = 1.$$

It is easy to see that along the equilibrium path, the belief satisfies Bayes' rule.

In the second stage, bidder i bids his value given his belief:

$$\beta_i = \beta^{FEF}(x_i, e_i, e_j) = ax_i - bh(e_j) + c. \tag{5.4}$$

We will now argue that (5.1), (5.2), (5.3), and (5.4) constitute a WPBE. It is clear that if it is an equilibrium, it is efficient as the highest type bidder submits the highest bid and gets the object. Due to full information revelation in the first stage, beliefs are degenerate in the second stage, and bidding one's own valuation is an optimal action in the second-price sealed-bid auction. Hence, bidders do not have a profitable deviation away from bidding $\beta^{FEF}(x_i, e_i, e_j)$.

We now concentrate on the optimality of paying the flexible entry fee specified in (5.1). If a bidder i of type x sets entry fee $e^{FEF}(y)$, as if he were of type y, then he wins the object (neglecting ties) if the type of the other bidder z satisfies

$$\beta^{FEF}\left(x, e^{FEF}(y), e^{FEF}(z)\right) > \beta^{FEF}\left(z, e^{FEF}(z), e^{FEF}(y)\right),$$

which can be written as $ax - bz + c > az - by + c$ or

$$z < \frac{ax + by}{a + b}.$$

If this is the case, bidder i wins the object at auction price

$$\beta_j = \beta^{FEF}\left(z, e^{FEF}(z), e^{FEF}(y)\right) = az - by + c,$$

and has valuation $v(x, z) = \beta_i = ax - bz + c$. The expected profit of bidder i is therefore given by

$$\pi(x, y) \equiv -e^{FEF}(y) + \int_0^{(ax+by)/(a+b)} ((ax - bz + c) - (az - by + c))\, dF(z),$$

which can be rewritten as follows, by using (5.1) and integrating by parts:

$$\begin{aligned}
\pi(x,y) &= -b\int_0^y F(z)\,dz + \int_0^{(ax+by)/(a+b)} ((ax-bz+c)-(az-by+c))\,dF(z) \\
&= -b\int_0^y (y-z)\,dF(z) + b\int_0^{(ax+by)/(a+b)} (y-z)\,dF(z) + a\int_0^{(ax+by)/(a+b)} (x-z)\,dF(z) \\
&= b\int_y^{(ax+by)/(a+b)} (y-z)\,dF(z) + a\int_0^{(ax+by)/(a+b)} (x-z)\,dF(z).
\end{aligned}$$

Setting the entry fee equal to $e^{FEF}(x)$ is optimal because

$$\pi(x,x)-\pi(x,y) = a\int_{x-b(x-y)/(a+b)}^{x} (x-z)\,dF(z) + b\int_y^{y+a(x-y)/(a+b)} (z-y)\,dF(z) > 0$$

for all $y \neq x$. Thus, there is no profitable deviation from $e^{FEF}(x)$, which is lower than $e^{FEF}(1)$. Also, choosing a fee above $e^{FEF}(1)$ is strictly suboptimal, given the proposed beliefs off-the-equilibrium path. Indeed, setting any entry fee $e_i > e^{FEF}(1)$ induces the same belief of bidder j as entry fee $e_i = e^{FEF}(1)$: $h(e_i) = 1$. Thus, raising the entry fee above $e^{FEF}(1)$ neither affects the bid of bidder j (and thus the price to be paid if bidder i wins), nor the winning probability of bidder i. It only increases his own expenses and is, therefore, strictly suboptimal. Thus, no bidder has an incentive to deviate from $e^{FEF}(x)$.

In fact, any off-the-equilibrium belief supports the equilibrium strategy $e^{FEF}(x)$ and $\beta^{FEF}(x_i, e_i, e_j)$. Indeed, with any other off-the-equilibrium path beliefs different from B^{FEF}, bidder j puts some positive probability that the deviating bidder i is of type $x < 1$, whereas B^{FEF} puts zero probability on this event. As a result, the expected value of bidder j and, therefore, his bid, is strictly higher than with the equilibrium belief B^{FEF}, which assigns $x = 1$ with probability one. This implies that setting $e_i > e^{FEF}(1)$ is even less attractive for bidder i if the belief of bidder j differs from B^{FEF}. Therefore, *any* belief supports the strategy $e^{FEF}(x)$ and $\beta^{FEF}(x_i, e_i, e_j)$ as a WPBE. The fact that off-the-equilbrium path beliefs may also be determined by Bayes' rule implies that the equilibrium strategies of the separating WPBE form a perfect Bayesian equilibrium.

Thus, and this is the main point of the example, the FEF-auction has an efficient equilibrium for all values of the parameters a, b and c. Moreover, the FEF-auction is never worse (in terms of efficiency) than the SP-auction and is strictly better for some values of the parameters (in particular when $a \leq b$ and the SP-auction is inefficient).

Interestingly, for the case where $a > b$ and both the SP-auction and the FEF-auction have efficient equilibria, they generate equal revenues. In the SP-auction, the revenue comes solely from bids:

$$\begin{aligned} R^{SP} &= E\left(\beta^{SP}\left(\min\left\{X_1, X_2\right\}\right)\right) \\ &= E\left(v\left(z,z\right)|z = \min\left\{X_1, X_2\right\}\right) \\ &= 2\int_0^1\int_0^x \left(az - bz + c\right)\, dF\left(z\right) dF\left(x\right) \\ &= 2\left(a-b\right)\int_0^1\int_0^x z\, dF\left(z\right) dF\left(x\right) + c. \end{aligned}$$

In the FEF-auction, to the contrary, a part $R^{FEF,e}$ of the revenue comes from collecting the entry fees:

$$R^{FEF,e} = 2E\left(e^{FEF}\left(X_i\right)\right) = 2\int_0^1 e^{FEF}\left(x\right)\, dF\left(x\right) = 2b\int_0^1\int_0^x F\left(z\right) dz\, dF\left(x\right),$$

and integrating by parts yields:

$$R^{FEF,e} = 2b\int_0^1\int_0^x F\left(z\right) dz\, dF\left(x\right) = 2\int_0^1\int_0^x b\left(x-z\right)\, dF\left(z\right) dF\left(x\right).$$

The remaining part $R^{FEF,\beta}$ of the revenue stems from the bids made:

$$\begin{aligned} R^{FEF,\beta} &= E\left(\beta^{FEF}\left(z, e^{FEF}\left(z\right), e^{FEF}\left(x\right)\right)|z = \min\left\{X_1, X_2\right\},\, x = \max\left\{X_1, X_2\right\}\right) \\ &= 2\int_0^1\int_0^x v\left(z,x\right)\, dF\left(z\right) dF\left(x\right) \\ &= 2\int_0^1\int_0^x \left(az - bx\right)\, dF\left(z\right) dF\left(x\right) + c. \end{aligned}$$

It can easily be verified that $R^{FEF,e} + R^{FEF,\beta} = R^{SP}$, so that revenue equivalence holds. The intuition for this revenue equivalence is that both auctions are efficient, the lowest type gets zero expected profit, and types are statistically independent, and, consequently, the expressions for revenues in these two auctions are identical.

The fact that the second highest bidder, i.e., the bidder whose bid is relevant for the auction payment, in the FEF-auction shades his bid relative to the bid he would make in the SP-auction follows from (5.4). Knowing that his competitor has a higher type (as $h\left(e_j\right) > x_i$) makes bidder i bidding less ($\beta^{FEF} = ax_i - bh\left(e_j\right) + c$) than he would have bid in the SP-auction ($\beta^{SP} = ax_i - bx_i + c$) where he would have bid an amount as if his competitor were of the same type x_i.

As all bidders have to pay the entry fee they have proposed, it is clear that the winner of the auction is better off in the FEF-auction and all non-winners are worse off. In other words, from the perspective of a bidder of a given type, the FEF-auction provides higher pay-off in case he wins, and lower pay-off in case he loses, than the SP-auction. At the same time, both auctions yield equal *expected* surplus to the bidder. This implies that, from the bidders' perspective, the FEF-auction is riskier than the SP-auction. This consequently suggests that, with risk-averse bidders, the FEF-auction raises higher revenue than the SP-auction (as risk-averse bidders are willing to pay a higher risk premium in a riskier auction).[8]

A natural question that arises is why bidders want to pay a positive entry fee. The reason is that, although the entry fee is sunk at the moment of the auction, entry fees signal bidders' types thereby affecting each other's bids in a desirable way: bids get lower when entry fees increase. By raising the entry fee, a bidder reduces the bid of his competitor and, therefore, lowers the price to be paid in case he wins the object.

We conclude that the FEF-auction yields the same outcome in terms of efficiency and revenue as the SP-auction when the latter has an efficient equilibrium, but retains the property of efficiency for parameter values where the SP-auction does not have an efficient equilibrium.

5.3 The General Model with Independent Types

The example in the previous section was special in a number of ways: the valuation function was supposed to be linear in the bidders' types, the analysis was restricted to two bidders and one object, and the types were supposed to be independently distributed. In this section, we first relax the first two assumptions and show that they are not essential to the argument. Section 5.4 analyzes the effect of allowing bidders' types to be correlated. We will see that in that case the argument can only be extended by allowing weak forms of affiliation.

Consider a standard symmetric multi-unit uniform-price auction where $N \geq 2$ bidders with unit demand, denoted by subscript i, compete for $n \geq 1$ homogeneous objects, $n < N$. Bidders' types X_i are identically and independently distributed over the interval [0, 1] in accordance with a distribution function $F(x)$. The values of the bidders for the objects are interdependent and given by the following valuation function:

$$V_i = v(X_i, \mathbf{X}_{-i}),$$

where $\mathbf{X}_{-i}$ is a collection of types of all bidders other than i. We assume that $v(X_i, \mathbf{X}_{-i})$ is symmetric in all $X_j \in \mathbf{X}_{-i}$, differentiable on $[0,1]^N$, and $\partial v/\partial X_i > 0 > \partial v/\partial X_j$, *i.e.*, there is

[8] We like to stress here that this is just a conjecture.

a negative externality.[9]

In the FEF-auction, bidders simultaneously choose and publicly pay entry fees e_i and then simultaneously submit auction bids β_i. The n bidders who have submitted the n highest bids get the objects and pay the auction price, which is equal to the $(n+1)^{st}$ highest non-winning bid. We denote the equilibrium bidding function by $\beta_i = \beta^{FEF}(x_i, e_i, \mathbf{e}_{-i})$, where $\mathbf{e}_{-i}$ is a collection of entry fees chosen by all bidders other than i, and consider symmetric equilibria where $\beta^{FEF}(x_i, e_i, \mathbf{e}_{-i})$ is symmetric in all $e_j \in \mathbf{e}_{-i}$. The belief of bidder i is the joint probability distribution of its competitors's types $\mathbf{X}_{-i}$ conditional on the information available to bidder i: $B^*(\mathbf{y}_{-i}|x_i, e_i, \mathbf{e}_{-i}) \equiv \Pr\left(X_j \leq y_j | X_i = x_i, e_i, \mathbf{e}_{-i}\right)$.

Suppose that a bidder i's type X_i takes a value x_i. We denote the bidder with the n^{th} highest type amongst all $N-1$ other bidders (all bidders except bidder i) by k so that his type is x_k. Excluding bidders i and k, we refer to all $n-1$ remaining bidders of types $x_j > x_k$ by subscript $w \in \mathbf{W}$ (they all win the auction and get objects), and we refer to all $N-n-1$ remaining bidders of types $x_j < x_k$ by subscript $l \in \mathbf{L}$ (they all lose the auction). If $n=1$ then $\mathbf{W} = \emptyset$, and if $N = n+1$ then $\mathbf{L} = \emptyset$.

We define a function $\hat{v}(x, y, z)$ as the expected value of a bidder i of type $x_i = x$ conditional on (i) one of his competitors, bidder j, being of type $x_j = y$, (ii) $n-1$ bidders $w \in \mathbf{W}$ being of type $x_w > z$, and (iii) $N-n-1$ bidders $l \in \mathbf{L}$ being of type $x_l < z$:

$$\hat{v}(x, y, z) \equiv E\left(v(X_i, \mathbf{X}_{-i}) | X_i = x, X_j = y, X_w > z, X_l < z\right),$$

where the expectation is taken with respect to $n-1$ random variables X_w and $N-n-1$ random variables X_l.

The main proposition of this paper demonstrates that the FEF-auction always has an efficient, i.e., a perfectly separating, monotone, and symmetric, WPBE.

Proposition 5.1 *There exists an efficient WPBE of the FEF-auction, where bidders choose an entry fee according to*

$$e^{FEF}(x) = \int_0^x \left(\hat{v}(z, z, z) - \hat{v}(z, x, z)\right) dG(z),$$

and bid according to

$$\beta^{FEF}(x_i, e_i, \mathbf{e}_{-i}) = v(x_i, \mathbf{h}(\mathbf{e}_{-i})),$$

where the components of $\mathbf{h}(\mathbf{e}_{-i})$ *are* $h(e_j) = \max\left\{x : e^{FEF}(x) \leq e_j\right\}$, *and beliefs are given*

[9] In some settings, it is more realistic to assume that values only depend on the types of the winning bidders, e.g., when auction winners compete with each other in an after-market. The analysis tolerates such a setting quite easily.

by

$$B^{FEF}\left(\mathbf{y}_{-i}|x,e_i,\mathbf{e}_{-i}\right)=\begin{cases}1 \;\; \textit{if} \;\; y_j \geq h\left(e_j\right) \;\; \textit{for all } j\neq i\\ \\ 0 \;\; \textit{otherwise.}\end{cases}$$

In equilibrium, higher types pay a higher entry fee in the first stage. By inverting the entry fee function, the bidders hold degenerate beliefs: they know exactly the types of their competitors. In the second stage, all bidders bid their expected values given beliefs, i.e., they bid their true values on the equilibrium path. The inverse $\mathbf{h}\left(\mathbf{e}_{-i}\right)$ is constructed in such a way that when bidders observe a too high entry fee $e_i > e^{FEF}(1)$, they belief that $x_i = 1$ with probability one. Proposition 5.1 shows that all the properties of the unique monotone symmetric perfectly separating WPBE presented in the example of Section 5.2 continue to hold for an arbitrary valuation function which exhibits a negative externality, and for an arbitrary number of objects and bidders. As in the example, bidders' off-the-equilibrium path beliefs do not play an important role here as any belief supports the separating Bayesian equilibrium strategies as WPBE.

It is interesting to interpret the entry fee. Let bidder i have the same type as bidder k, i.e., $x_i = x_k = z$. Then, the valuation of bidder k is $\hat{v}\left(z,z,z\right)$. Hence, by having value $x_i = x$, bidder i imposes a negative externality of size $\hat{v}\left(z,z,z\right)-\hat{v}\left(z,x,z\right)$ on bidder k by reducing his value by that amount. This externality only realizes when bidder i wins and bidder k does not win, i.e., when $z < x$. Therefore, from bidder i's perspective, the entry fee $e^{FEF}\left(x\right)$ he pays is the expected externality he imposes on the marginal bidder k. Thus, flexible entry fees allow bidders to internalize the negative externality they impose on each other so that the externality does not affect the monotonicity property of bidders' bids. Hence, an efficient equilibrium always exists. This is the crucial difference between the entry fee chosen by the auctioneer (including the case of a voluntary entry fee à la Maskin and Riley, 1981) and flexible entry fees chosen by bidders themselves.

In case $\hat{v}\left(x,x,x\right)$ is an increasing function of x, the FEF-auction also has a pooling equilibrium that coincides with the equilibrium of the SP-auction, where all bidders choose an entry fee of zero and bid their expected valuation in case they are uncertain about their competitors' types, i.e., $e^*\left(x\right)=0$ and $\beta^*\left(x,e_i,\mathbf{e}_{-i}\right)=\beta^{SP}\left(x\right)=\hat{v}\left(x,x,x\right)$, and beliefs are the prior beliefs $B^*\left(\mathbf{Y}_{-i}|x_i,e_i,\mathbf{e}_{-i}\right)=\prod_{j\neq i}F\left(y_j\right)$. Contrary to the separating equilibrium, the pooling equilibrium requires specific beliefs off-the-equilibrium path. The revenue generated in the separating equilibrium is equal to the revenue generated in the SP-auction as revenue-equivalence holds. In case $\hat{v}\left(x,x,x\right)$ is not monotonically increasing, the SP-auction does not have an efficient equilibrium and, therefore, its outcome is inefficient with positive probability. These results are summarized in the next proposition.

Proposition 5.2 *The SP-auction has an efficient equilibrium if and only if $\hat{v}\left(x,x,x\right)$ is an increasing function of x. If $\hat{v}\left(x,x,x\right)$ is an increasing function of x, $\beta^{SP}\left(x\right)=\hat{v}\left(x,x,x\right)$, and* *the SP-auction raises the same expected revenue as the separating equilibrium of the FEF-auction.*

5.4 Correlated Types

We now generalize our example of Section 5.2 to an environment where bidders' types are positively correlated, provided the correlation is not too strong. It is clear that positive correlation of types reinforces the negative externality so that an efficient equilibrium of the SP-auction is even less likely to exist (see also, e.g., Janssen and Karamychev, 2010). The reason is as follows. A first effect of a bidder's type is that a high type bidder has a higher value than a low type bidder, for the same fixed types of their competitors. This first, direct effect is positive. However, due to positive correlation, a high type bidder expects competitors to be of higher types than a low type bidder expects them to be. This creates a second, indirect effect, on the value, which is negative. When the correlation is strong, the second effect dominates the first one so that the *ex-ante* expected value of a bidder conditional on winning is not a monotonically increasing function of his type. Consequently, as his bid in the SP-auction is his expected value, a monotone bidding equilibrium fails to exist.

In our model where signaling is allowed, if bidders' types are strongly correlated bidders do not have an incentive to signal their types by paying a (high) entry fee as the other bidders can anyway infer someone's type once they have observed their own type. Therefore, an efficient equilibrium of the FEF-auction only exists if the correlation is not too strong.

To study the effects of correlation we consider for simplicity the two-bidder setting of Section 5.2. Suppose bidders' types are weakly affiliated and the distribution function of X_j conditional on $X_i = x$ is $F(z|x) \equiv \Pr(X_j \leq z|X_i = x)$, the density is $f(z|x)$, and $F_x(z|x) \equiv \partial F(z|x)/\partial x \leq 0$, i.e., there is affiliation. Let the value function be $v(X_i, X_j) = aX_i - bX_j + c$. We consider situations where

$$F(y|y) \geq F\left(\left.\frac{ax+by}{a+b}\right| x\right) \text{ for } y > x, \text{ and } F(y|y) \leq F\left(\left.\frac{ax+by}{a+b}\right| x\right) \text{ for } y < x, \tag{5.5}$$

for all x, $y \in [0,1]$. When types are independent, condition (5.5) is always satisfied. When X_j and X_i are affiliated and $y > x$, the distribution $F(...|y)$ stochastically dominates $F(...|x)$ so that $F(y|y) < F(y|x)$. Hence, (5.5) is only satisfied when the affiliation is weak. On the other hand, for a given distribution with affiliation, (5.5) is never satisfied for $a = 0$, which corresponds to the limiting case of an externality that is extremely strong. Hence, (5.5) assumes both a relatively weak affiliation of types and a weak externality.

The following proposition shows that condition (5.5) guarantees the existence of an efficient WPBE of the FEF-auction.

Proposition 5.3 *Consider the case where $N = 2$ and the value function is linear and given by $v(X_i, X_j) = aX_i - bX_j + c$. If (5.5) holds (sufficient condition), then the strategy consisting of an entry fee $e^{FEF}(x)$ and bids $\beta_i = \beta^{FEF}(x_i, e_i, e_j)$ with $e^{FEF}(x) = b\int_0^x F(z|\, z)\, dz$ and $\beta^{FEF}(x_i, e_i, e_j) = ax_i - bh(e_j) + c$, where $h(e_j) = \max\{x : e^{FEF}(x) \le e_j\}$ and beliefs*

$$B^{FEF}(y\,|x, e_i, e_j) = \begin{cases} 1 \ \ if \ \ y \ge h(e_j) \\ \\ 0 \ \ if \ \ y < h(e_j), \end{cases}$$

constitute an efficient WPBE of the FEF-auction. This WPBE only exists if (necessary condition)

$$\frac{a}{a+b} f(x|\, x) + F_x(x|\, x) \ge 0.$$

The argument made in the proof of Proposition 5.3 is similar to the one made in Section 5.2 and replaces the unconditional distribution function used there by the conditional distribution function and then shows that the argument can be extended by allowing weak forms of affiliation of the type that satisfies (5.5).

In equilibrium, bidders bid their values in the second stage. In the first stage, they pay entry fees that are increasing in types. Condition (5.5) guarantees that there is no profitable deviation from $e^{FEF}(x)$, i.e., it is essentially a (global) sufficient second-order condition that ensures that further deviations are even less profitable than smaller deviations. The necessary (local) second-order condition for the FEF-auction to have an efficient equilibrium can be obtained from (5.5) by taking a limit when x and y converge to each other.

In accordance with Proposition 5.3, the FEF-auction does not have an efficient *separating* equilibrium when the correlation and the externality are strong. Nevertheless, the FEF-auction can have an efficient *pooling* equilibrium, which is an efficient equilibrium of the SP-auction if it exists. Thus, the FEF-auction is at least as efficient as the SP-auction, and sometimes it is strictly more efficient.

In the case of affiliation, we may also wonder how revenues under the efficient equilibrium of the SP-auction and the efficient equilibrium of the FEF-auction compare. To make the comparison useful, we have to consider situations where both equilibria exist, and therefore we restrict the analysis to the case where (5.5) holds and, in addition, $a > b$ so that the SP-auction has an efficient equilibrium. It is straightforward to show that the efficient equilibrium of the SP-auction is given by $\beta^{SP}(x) = v(x, x) = (a - b)x + c$. The next proposition shows that the FEF-auction generates larger revenues than the SP-auction.

Proposition 5.4 *Consider the case where $N = 2$ and the value function is linear and given by $v(X_i, X_j) = aX_i - bX_j + c$, $a > b$, X_i and X_j are affiliated, and (5.5) holds. Then, revenue in the FEF-auction is strictly higher than in the SP-auction.*

In the FEF-auction, the revenue does not only come from the winning bid and the entry fee paid by the winning bidder, but also from the entry fees paid by all other bidders. Therefore, the Linkage (Revenue Ranking) Principle (*cf.* Krishna, 2002, p. 103) cannot be applied, and the revenue has to be computed and compared with the revenue in the SP-auction directly. It turns out that the FEF-auction better exploits the correlation of bidders' types from the perspective of the auctioneer and results in a higher revenue.

5.5 Discussion and Conclusion

In this paper, we have argued that by allowing bidders to make flexible, publicly observable payments before they enter an auction, the effect of negative externalities, which have played an important role in the recent literature on auctions with post-auction interactions, can be mitigated. The literature has stressed that if the negative externalities are strong enough, auctions may not yield an efficient allocation of the object(s). We have argued that asking for a flexible entry fee restores efficiency, and in case of affiliated types brings about a higher revenue. Important to note here is that this argument can be generalized to settings with asymmetric bidders, e.g., to bidders with different valuation functions and different distributions of bidders' types. The reason is that in a separating equilibrium bidders just bid their true values irrespective of whether there are asymmetries between them or not.

We have considered second-price sealed-bid auctions (and auctions that are strategically equivalent) and one may wonder what the results may be if a first-price sealed-bid auction is considered. It is easy to see that a perfectly revealing equilibrium may not exist in a first-price sealed-bid auction with negative externalities. This can best be seen in the two bidder case. If such an equilibrium had existed, the bidder with the highest type would have bid marginally higher than the other bidder who, in turn, would have bid his true value. This outcome is, however, prone to the following deviation. By setting his entry fee equal to zero, bidder one ensures that the other bidder believes in winning the auction. As a result, bidder two will not bid his own value but will bid the value of the first bidder conditional on his type being zero, which is lower. Hence, by setting his entry fee equal to zero, bidder one lowers the bid function of bidder two and wins the object with certainty. Besides, he saves on the entry fee. This makes the deviation profitable for highest types, and the corresponding separating equilibrium fails to exist. Thus, the second-price feature is important for obtaining the separating result we emphasize in this paper.

We have made the argument in this paper by considering negative externalities. It is easy to see that a perfectly revealing equilibrium never exists in a second-price sealed-bid auction with positive externalities. The reason bidders are willing to pay an entry fee in a setting where externalities are negative is that this has a negative impact on the expected valuation of the other bidders, hence on their bids, and thereby on the price that bidders have to pay for the object in case they win the auction. Under positive externalities, to the contrary, bidders are willing to signal that their types are the lowest possible types as this has a negative impact on the expected valuation of the other bidders, hence on their bids, and thereby on the price that bidders have to pay for the object in case they win the auction. Consequently, if the externality is positive only pooling equilibria exist, in which bidders do not pay an entry fee and then play the standard second-price sealed-bid auction. The type of applications that motivate this paper, however, like auctions with Cournot and Bertrand type of competition in the downstream market are all examples where negative externalities are present.

In summary, we have shown that if the auction itself is efficient, the efficiency of the auction will not be affected (degraded) by the introduction of flexible entry fees. If the auction is, to the contrary, inefficient, then the introduction of flexible entry fees might restore the efficiency of the auction by allowing bidders to signal their type prior to the auction.

To our knowledge, auctions with flexible entry fees have not yet been used in practice. Firms hiring expensive auction consultants or burning money in public (such as large and seemingly useless advertisement campaigns) to signal their strength does have the same effect as flexible entry fees though. If we reinterpret entry fees in this way, then this 'new' mechanism has been in place for a long time.

The reader should be aware that no policy recommendations can be derived on the basis of this paper alone, because the theory exposited in this chapter may not capture practice completely. One crucial assumption of our model is that firms behave competitively (do not collude). In practice, however, pre-auction signaling may encourage collusion. For instance, a firm may burn a lot of money hiring a very expensive consultant not (primarily) to signal its type but to signal that it cannot easily be outcompeted, and that it may be a better idea for every firm involved to split the market at low prices. Another assumption of our model is that bidder participation is independent of the auction format. In practice, this might not be the case.[10] An auction with a pre-auction signaling phase may, for example, encourage predatory behavior and/or joint bidding resulting in a reduction in the number of bidders. Theory should therefore be combined with empirical analysis (of field data, or experimental data) before judgments can be made which auction mechanism performs best under what circumstances.

An important message we want to convey with this chapter is that the auction itself is part of a larger game. The auction does not start in the auction room but way earlier. Prior to the auction, firms may have an incentive (due to what will happen after the auction) to burn money to signal their strength. The amount of money spent by firms to acquire licenses might thus be far larger than only the amount of money paid to the auctioneer. Thus by focussing only on the auction one may draw the wrong conclusions.

This insight might have huge consequences for structural econometric research in auc-

[10] See, e.g., Athey et al. (2011), for an example that the choice of the auction format can have significant effects on bidder participation.

tions.[11] Structural econometricians estimate causal relations between signals (a bidder's type) and bids, and therefore between the distributions of signals and bids. The bid data used are the data from the auction itself. Bids made in a possible pre-auction phase are not taken into account. By discarding the bids from the pre-auction phase, the estimation of the functional relationship will be incorrect.

Several lines of further research can be suggested. First, we may allow for a timing aspect in the pre-auction signaling phase (thus dropping the assumption that firms burn money simultaneously). Asymmetric equilibria should in that case be considered. We expect the bidder with the highest type to signal first (as he has the largest incentive). It will be interesting to see whether all bidders will have an incentive to signal. After having seen the amount of money burnt by the higher type bidders, lower type bidders may not have an incentive anymore to burn money themselves. Second, as signaling prior to the auction may raise credibility issues in certain instances (in practice), it is interesting to examine whether qualitatively similar results will be obtained when the pre-auction signaling phase is charaterized by a cheap talk model (instead of a money burning model).

5.6 Appendix: Proofs

Proof of Proposition 5.1. First, it is easy to see that $e^{FEF}(x)$ is a strictly increasing function so that the proposed WPBE is perfectly separating:

$$\frac{d}{dx}e^{FEF}(x) = -\int_0^x \left(\frac{d}{dx}\hat{v}(z,x,z)\right) dG(z) = -\int_0^x \left(\frac{\partial \hat{v}}{\partial y}(z,x,z)\right) dG(z) > 0$$

due to $\partial\hat{v}/\partial y < 0$ as being the conditional expectation of $\partial v/\partial X_j < 0$.

Suppose all bidders except bidder i have beliefs B^{FEF} and follow the proposed WPBE strategy. In this case, each bidder j sets entry fee $e_j = e^{FEF}(x_j) \leq e^{FEF}(1)$ and, therefore, bidder i correctly infers the type of all other bidders on-the-equilibrium path by using the inverse function $h(e_j) = x_j$. Irrespective of the chosen entry fee e_i, bidding his exact value $v_i = v(x_i, \mathbf{h}(\mathbf{e}_{-i}))$ in the second stage is optimal for bidder i, just like in the SP-auction. Thus, bidder i has no profitable deviation from bidding $\beta^{FEF}(x_i, e_i, \mathbf{e}_{-i}) = v(x, \mathbf{h}(\mathbf{e}_{-i}))$. In the rest of the proof, we show that bidders do not benefit by deviating from $e^{FEF}(x)$ for the proposed on- and off-the-equilibrium path beliefs.

For notational convenience, we will write the collection of types other than X_i as $\mathbf{X}_{-i} = (x_k, \mathbf{X}_{-ik})$ referring to the type of a given bidder k. By $\hat{F}(\mathbf{x}_{-ik}|z)$ we denote the joint distribution function of $\mathbf{X}_{-ik}$ conditional on the event $X_w > z > X_l$ for all other winning bidders $w \in \mathbf{W}$ and losing bidders $l \in \mathbf{L}$. Using this notation, $\hat{v}(x,y,z)$ can be written as

[11] See, e.g., Hendricks and Porter (2007) for a survey on structural econometric methods.

follows:

$$\hat{v}(x,y,z)=\frac{\int\limits_{\substack{X_w>z,X_l<z\\ X_i=x,X_k=y}} v(X_i,\mathbf{X}_{-i})\prod\limits_{\substack{j\neq i\\ j\neq k}} dF(x_j)}{\int\limits_{\substack{X_w>z,X_l<z\\ X_i=x,X_k=y}}\prod\limits_{\substack{j\neq i\\ j\neq k}} dF(x_j)}=\int\limits_{X_l<z<X_w} v(x,(y,\mathbf{X}_{-ik}))\, d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z).$$

Suppose bidder i of type x sets entry fee $e_i=e^{FEF}(y)$ as if he were of type $y\in[0,1]$, and all other bidders j follow the equilibrium strategy. Bidder i wins and gets the object if and only if his bid β_i is higher than bid β_k of bidder k. Denoting $x_k=z$ and taking into account that $\beta_i=v(x,(z,\mathbf{X}_{-ik}))$ and $\beta_k=v(z,(y,\mathbf{X}_{-ik}))$, we write $\beta_i>\beta_k$ as $v(x,(z,\mathbf{X}_{-ik}))>v(z,(y,\mathbf{X}_{-ik}))$. If $v(x,(z,\mathbf{X}_{-ik}))>v(z,(y,\mathbf{X}_{-ik}))$, bidder i pays the auction price β_k and gets surplus

$$s(x,y,z,\mathbf{X}_{-ik})\equiv\beta_i-\beta_k=v(x,(z,\mathbf{X}_{-ik}))-v(z,(y,\mathbf{X}_{-ik})).$$

If $v(x,(z,\mathbf{X}_{-ik}))<v(z,(y,\mathbf{X}_{-ik}))$, bidder i does not get the object. The expected surplus $\hat{s}(x,y)$ of bidder i conditional on winning is

$$\hat{s}(x,y)\equiv E\left(s(x,y,z,\mathbf{X}_{-ik})\,|s(x,y,z,\mathbf{X}_{-ik})>0\right),$$

where the expectation is taken with respect to z and $\mathbf{X}_{-ik}$. Bidder i's *ex-ante* surplus is, therefore,

$$\pi(x,y)\equiv-e^{FEF}(y)+\Pr\left(s(x,y,z,\mathbf{X}_{-ik})>0\right)\cdot\hat{s}(x,y).$$

We will show that $\pi_y(x,y)<0$ for all $y>x$ and $\pi_y(x,y)>0$ for all $y<x$, which implies that $\pi(x,y)$ attains its unique global maximum w.r.t. y at $y=x$.

First, we note that $z<x$ is equivalent to $v(z,(z,\mathbf{X}_{-ik}))>v(z,(x,\mathbf{X}_{-ik}))$ for any realization of $\mathbf{X}_{-ik}$. This is so because the right-hand side is strictly decreasing in x and equals the left-hand side at $x=z$. This allows us to rewrite $e^{FEF}(x)$ as follows:

$$\begin{aligned}e^{FEF}(x)&=\int\limits_0^x(\hat{v}(z,z,z)-\hat{v}(z,x,z))\,dG(z)\\&=\int\limits_{v(z,(z:\mathbf{X}_{-ik}))>v(z,(x:\mathbf{X}_{-ik}))}(\hat{v}(z,z,z)-\hat{v}(z,x,z))\,dG(z).\end{aligned}$$

Next, we write

$$\hat{v}(z,z,z)=\int\limits_{X_l<z<X_w} v(x,(y,\mathbf{X}_{-ik}))\,d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z),$$

and

$$\hat{v}(z,x,z) = \int\limits_{X_l<z<X_w} v(z,(x,\mathbf{X}_{-ik}))\ d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z).$$

Hence,

$$\begin{aligned} e^{FEF}(x) &= \int\limits_{\substack{v(z,(z:\mathbf{X}_{-ik}))>v(z,(x:\mathbf{X}_{-ik}))\\ X_l<z<X_w}} [v(z,(z,\mathbf{X}_{-ik}))-v(z,(x,\mathbf{X}_{-ik}))]\ d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z)\,dG(z) \\ &= \int\limits_{\substack{s(z,x,z,\mathbf{x}_{-ik})>0\\ X_l<z<X_w}} s(z,x,z,\mathbf{x}_{-ik})\ d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z)\ dG(z)\,. \end{aligned}$$

In a similar fashion, we rewrite $\pi(x,y)$:

$$\begin{aligned} \pi(x,y) &= -\int\limits_{\substack{s(z,y,z,\mathbf{x}_{-ik})>0\\ X_l<z<X_w}} s(z,y,z,\mathbf{x}_{-ik})\,dG(z)\,d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z) \\ &\quad+\int\limits_{\substack{s(x,y,z,\mathbf{X}_{-ik})>0\\ X_l<z<X_w}} s(x,y,z,\mathbf{x}_{-ik})\,dG(z)\,d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z)\,. \end{aligned}$$

Under our assumptions on v and F, $\pi(x,y)$ is differentiable. By taking the partial derivative of $\pi(x,y)$ w.r.t. y, we have to consider variations of the integrands and of the domains of integration, i.e., variations of the sets of the values of z and $\mathbf{X}_{-ik}$ where $s(z,y,z,\mathbf{x}_{-ik})>0$ and $s(x,y,z,\mathbf{x}_{-ik})>0$. Due to the continuity of $s(x,y,z,\mathbf{X}_{-ik})$, all variations of domains happen at $s=0$ and do not contribute to $\pi_y(x,y)$. Thus,

$$\begin{aligned} \pi_y(x,y) &= -\int\limits_{\substack{s(z,y,z,\mathbf{x}_{-ik})>0\\ X_l<z<X_w}} \frac{\partial}{\partial y}s(z,y,z,\mathbf{x}_{-ik})\,dG(z)\,d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z) \\ &\quad+\int\limits_{\substack{s(x,y,z,\mathbf{X}_{-ik})>0\\ X_l<z<X_w}} \frac{\partial}{\partial y}s(x,y,z,\mathbf{x}_{-ik})\,dG(z)\,d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z). \end{aligned}$$

Next, using the definition of $s(x,y,z,\mathbf{X}_{-ik})$ and writing $v_k\equiv\partial v/\partial X_k$ yields:

$$\begin{aligned} \pi_y(x,y) &= \int\limits_{\substack{s(z,y,z,\mathbf{x}_{-ik})>0\\ X_l<z<X_w}} v_k(z,(y,\mathbf{x}_{-ik}))\,dG(z)\,d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z) \\ &\quad-\int\limits_{\substack{s(x,y,z,\mathbf{X}_{-ik})>0\\ X_l<z<X_w}} v_k(z,(y,\mathbf{x}_{-ik}))\,dG(z)\,d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z). \end{aligned}$$

Then, using the following chain of the equivalence relations

$$\begin{aligned} s(z,y,z,\mathbf{x}_{-ik}) &> 0 \Leftrightarrow v(z,(z,\mathbf{X}_{-ik})) > v(z,(y,\mathbf{X}_{-ik})) \Leftrightarrow z < y \\ &\Leftrightarrow v(y,(z,\mathbf{X}_{-ik})) > v(z,(y,\mathbf{X}_{-ik})), \end{aligned}$$

we rewrite $\pi_y(x,y)$ as follows:

$$\begin{aligned} \pi_y(x,y) &= \int\limits_{\substack{v(y,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik})) \\ X_l<z<X_w}} v_k(z,(y,\mathbf{x}_{-ik}))\, dG(z)\, d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z) \\ &- \int\limits_{\substack{v(x,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik})) \\ X_l<z<X_w}} v_k(z,(y,\mathbf{x}_{-ik}))\, dG(z)\, d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z). \end{aligned}$$

Suppose now that $y > x$. This implies $v(y,(z,\mathbf{X}_{-ik})) > v(x,(z,\mathbf{X}_{-ik}))$ so that we can rewrite $\pi_y(x,y)$ as follows:

$$\begin{aligned} \pi_y(x,y) &= \int\limits_{\substack{v(y,(z:\mathbf{X}_{-ik}))>v(x,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik})) \\ X_l<z<X_w}} v_k(z,(y,\mathbf{x}_{-ik}))\, dG(z)\, d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z) \\ &+ \int\limits_{\substack{v(y,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik}))>v(x,(z:\mathbf{X}_{-ik})) \\ X_l<z<X_w}} v_k(z,(y,\mathbf{x}_{-ik}))\, dG(z)\, d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z) \\ &- \int\limits_{\substack{v(x,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik})) \\ X_l<z<X_w}} v_k(z,(y,\mathbf{x}_{-ik}))\, dG(z)\, d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z). \end{aligned}$$

The first term cancels the third term so that

$$\pi_y(x,y) = \int\limits_{\substack{v(y,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik}))>v(x,(z:\mathbf{X}_{-ik})) \\ X_l<z<X_w}} v_k(z,(y,\mathbf{x}_{-ik}))\, dG(z)\, d^{n-2}\hat{F}(\mathbf{x}_{-ik}\,|z) < 0.$$

The last inequality follows from (i) the integrand which is strictly negative ($v_k < 0$) and (ii) from the domain of integration which is not empty. It is not empty because for $z = y > x$:

$$v(y,(z,\mathbf{x}_{-ik})) = v(z,(y,\mathbf{x}_{-ik})) > v(x,(z,\mathbf{x}_{-ik})),$$

and for z marginally lower than y:

$$v(y,(z,\mathbf{x}_{-ik})) > v(z,(y,\mathbf{x}_{-ik})).$$

Since $\pi_y(x,y) < 0$, $\pi(x,y) < \pi(x,x)$ for all $y > x$. Thus, choosing $y > x$ is not a profitable deviation.

Similarly, $y < x$ implies $v\left(y,\left(z,\mathbf{X}_{-ik}\right)\right) < v\left(x,\left(z,\mathbf{X}_{-ik}\right)\right)$ so that we rewrite $\pi_y\left(x,y\right)$ as follows:

$$\begin{aligned}\pi_y\left(x,y\right) &= \int\limits_{\substack{v(y,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik}))\\ X_l<z<X_w}} v_k\left(z,\left(y,\mathbf{x}_{-ik}\right)\right)dG\left(z\right)d^{n-2}\hat{F}\left(\mathbf{x}_{-ik}\left|z\right.\right)\\ &- \int\limits_{\substack{v(x,(z:\mathbf{X}_{-ik}))>v(y,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik}))\\ X_l<z<X_w}} v_k\left(z,\left(y,\mathbf{x}_{-ik}\right)\right)dG\left(z\right)d^{n-2}\hat{F}\left(\mathbf{x}_{-ik}\left|z\right.\right)\\ &- \int\limits_{\substack{v(x,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik}))>v(y,(z:\mathbf{X}_{-ik}))\\ X_l<z<X_w}} v_k\left(z,\left(y,\mathbf{x}_{-ik}\right)\right)dG\left(z\right)d^{n-2}\hat{F}\left(\mathbf{x}_{-ik}\left|z\right.\right),\end{aligned}$$

which yields:

$$\pi_y\left(x,y\right) = -\int\limits_{\substack{v(x,(z:\mathbf{X}_{-ik}))>v(z,(y:\mathbf{X}_{-ik}))>v(y,(z:\mathbf{X}_{-ik}))\\ X_l<z<X_w}} v_k\left(z,\left(y,\mathbf{x}_{-ik}\right)\right)dG\left(z\right)d^{n-2}\hat{F}\left(\mathbf{x}_{-ik}\left|z\right.\right) > 0.$$

Hence, neither $y < x$ is a profitable deviation.

On the other hand, setting fee e_i above $e^{FEF}\left(1\right)$ is strictly suboptimal for the given off-the-equilibrium path beliefs as it affects neither the bid of the other bidders, nor the winning probability of bidder i, nor the auction price bidder i pays if he wins. It only increases expenses and, therefore, is strictly suboptimal. Thus, no bidder has incentives to deviate from equilibrium fee $e^{FEF}\left(x\right)$. ■

Proof of Proposition 5.2. Let bidder i of type $X_i = x$ bid $\beta^{SP}\left(x\right)$ in the SP-auction, where $\beta^{SP}\left(x\right)$ is a monotonically increasing symmetric equilibrium bidding function. Then, bidder i has expected value $\hat{v}\left(x,z,z\right)$ and wins if and only if he outbids the bid $\beta^{SP}\left(z\right)$ of bidder k with type $X_k = z$. The expected profit of bidder i is, therefore,

$$\pi\left(x,y\right) = \int\limits_0^y \left(\hat{v}\left(x,z,z\right) - \beta^{SP}\left(z\right)\right) dG\left(z\right).$$

Maximizing $\pi\left(x,y\right)$ w.r.t. y yields the necessary first-order condition:

$$\beta^{SP}\left(y\right) = \hat{v}\left(x,y,y\right),$$

which must hold for $y = x$. Thus, if such an equilibrium does exist, it must be $\beta^{SP}\left(x\right) = \hat{v}\left(x,x,x\right)$.

Suppose now that all bidders follow $\beta^{SP}(x) = \hat{v}(x,x,x)$. Then

$$\pi(x,y) = \int_0^y (\hat{v}(x,z,z) - \hat{v}(z,z,z))\, dG(z).$$

It can be easily seen that $y = x$ is a global maximum of $\pi(x,y)$ w.r.t. y because

$$\pi_y(x,y) = (\hat{v}(x,y,y) - \hat{v}(y,y,y))\, g(y) < 0$$

for $y > x$ and $\pi_y(x,y) > 0$ for $y < x$. Hence, $\beta^{SP}(x) = \hat{v}(x,x,x)$ is a unique monotonically increasing symmetric equilibrium, provided $\hat{v}(x,x,x)$ monotonically increases in x.

The revenue raised in the SP-auction can be written as follows:

$$\begin{aligned} R^{SP} &= E\left(\beta^{SP}(X_k) \,|X_i > X_k, X_w > X_k, X_l < X_k\right) \\ &= N \int_0^1 \left(\int_0^x \beta^{SP}(z)\, dG(z) \right) dF(x) \\ &= N \int_0^1 \int_0^x \hat{v}(z,z,z)\, dG(z)\, dF(x). \end{aligned}$$

In the FEF-auction, a part $R^{FEF,e}$ of the revenue comes from collecting the entry fees:

$$R^{FEF,e} = N \cdot E\left(e^{FEF}(X_i)\right) = N \int_0^1 \int_0^x (\hat{v}(z,z,z) - \hat{v}(z,x,z))\, dG(z)\, dF(x).$$

The remaining part $R^{FEF,\beta}$ of the revenue stems from the bids made:

$$\begin{aligned} R^{FEF,\beta} &= E\left(\beta^{FEF}(X_k, e_k, \mathbf{e}_{-k}) \,|X_w > X_k > X_l, X_i > X_k\right) \\ &= E\left(v(X_k, (X_i, \mathbf{X}_{-ik})) \,|X_w > X_k > X_l, X_i > X_k\right) \\ &= N \int_0^1 \int_0^x E\left(v(X_k, (X_i, \mathbf{X}_{-ik})) \,|X_w > X_k = z > X_l, X_i = x > X_k\right) dG(z)\, dF(x) \\ &= N \int_0^1 \int_0^x \hat{v}(z,x,z)\, dG(z)\, dF(x). \end{aligned}$$

As $R^{FEF,e} + R^{FEF,\beta} = R^{SP}$, revenue equivalence holds. ■

Proof of Proposition 5.3. First, it is easy to check that $e^{FEF}(x)$ strictly increases:

$$\frac{d}{dx} e^{FEF}(x) = bF(x|\, x) > 0.$$

This implies that $h(e) = \max\left\{x : e^{FEF}(x) \leq e\right\}$ is a proper inverse function, which leads to the Bayesian beliefs $B^{FEF}\left(x\left|e_j\right.\right)$ on-the-equilibrium path.

Second, bidding one's own valuation is optimal given beliefs. In the rest of the proof, we show that bidders do not benefit by deviating from $e^{FEF}(x)$ for the proposed on- and off-the-equilibrium path beliefs.

If bidder i of type x sets entry fee $e^{FEF}(y)$, his expected profit $\pi(x,y)$ is

$$\begin{aligned}\pi(x,y) &\equiv -e^{FEF}(y) + \int_0^{(ax+by)/(a+b)} (ax - bz - az + by)\, dF(z|\,x) \\ &= -b\int_0^y F(z|\,z)\, dz + (a+b)\int_0^{(ax+by)/(a+b)} F(z|\,x)\, dz\,.\end{aligned}$$

Hence, if $\pi(x,y) \leq \pi(x,x)$ for all y bidder i has no incentives to deviate from $e^{FEF}(x)$. Differentiating $\pi(x,y)$ w.r.t. y yields:

$$\pi_y(x,y) = -b\left(F(y|\,y) - F\left(\left.\frac{ax+by}{a+b}\right|x\right)\right).$$

Under the assumption of the proposition, $\pi_y(x,y) \leq 0$ for $y > x$ and $\pi_y(x,y) \geq 0$ for $y \leq x$, so that $y = x$ is a global maximum. Hence, there are no profitable deviations from $e^{FEF}(x)$ below $e^{FEF}(1)$. For deviations above $e^{FEF}(1)$, the same argument as in the proof of Proposition 5.1 applies.

In order to derive the necessary condition for an equilibrium to exist we note that the global maximum $y = x$ of $\pi(x,y)$ w.r.t. y must necessarily be a local maximum. The second-order condition for the local maximum is $\pi_{y,y}(x,x) \leq 0$:

$$\pi_{y,y}(x,x) = -b\left(\frac{d}{dx}F(x|\,x) - \frac{b}{a+b}f(x|\,x)\right) \leq 0,$$

and the necessary condition of Proposition 5.3 follows. ■

Proof of Proposition 5.4. Expected payment $M^{SP}(x)$ of bidder i of type x in the SP-auction is:

$$\begin{aligned}M^{SP}(x) &= G(x\,|x)\,E\left((a-b)X_j + c\,|X_i = x,\, X_j \leq x\right) \\ &= (a-b)\int_0^x z\,dG(z\,|x) + cG(x\,|x) \\ &= ((a-b)x + c)\,G(x\,|x) - (a-b)\int_0^x G(z\,|x)\, dz.\end{aligned}$$

His expected payment in the FEF-auction is:

$$\begin{aligned} M^{FEF}(x) &= e^{FEF}(x) + G(x|x)\, E(aX_j - bX_i + c\,|X_i = x, X_j < x) \\ &= b\int_0^x G(z|z)\,dz + \int_0^x (az - bx)\,dG(z|x) + cG(x|x) \\ &= ((a-b)x + c)\,G(x|x) + b\int_0^x G(z|z)\,dz - a\int_0^x G(z|x)\,dz. \end{aligned}$$

The difference $M^{FEF}(x) - M^{SP}(x)$ is

$$M^{FEF}(x) - M^{SP}(x) = b\int_0^x (G(z|z) - G(z|x))\,dz > 0$$

for all $x > 0$ because, due to affiliation, $G(z|z) > G(z|x)$ for all $z < x$. Therefore, the *ex-ante* payment of any bidder, hence the auction revenue as well, is strictly higher in the FEF-auction than in the SP-auction. ■

5.7 References

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Summary

Auctions have been widely used over thousands of years. The Babylonians auctioned wives, the ancient Greeks sold mine concessions in auctions, and the Romans put slaves, war booty, and debtors property up for auction, among many other things. Nowadays, the use of auctions is also widespread. There are auctions for art, fish, flowers, and wine, but also for more abstract objects like treasury bills, radio frequency spectrum, and electricity distribution contracts. In some of these auctions, the amount of money raised is almost beyond imagination. In the 1990s, the US government collected tens of billions of dollars in auctions for licenses for second generation mobile telecommunication, and in 2000, both the British and German governments raised tens of billions of euros in auctions for license for third generation mobile telecommunication. Auction theorists were closely involved in several of these auctions, both consulting governments on the designs of these auctions and advising bidders on their bidding strategies. This has generated a burst of auction theory.

Auction theory is a collection of game-theoretic models related to the interaction of bidders in auctions, and was pioneered by William Vickrey in 1961. Vickrey, an economist from the Columbia University in New York, studied private value auctions, in which each bidder's value for the object for sale is independent of the values of the other bidders. After Vickrey's seminal work, auction theory was mainly developed in the 1980s. Although several issues were touched upon, such as the effects of risk aversion, correlation of information, budget constraints, asymmetries, and so forth, these were not felt as being the main issues in auction design in practice. In the 1990s, new models were developed that focused upon practical issues. Today, many economists regard auction theory as the best application of game theory to economics.

Auction theory is an important theory to study for several reasons. First, as many objects are being sold in auctions, it is important to understand how auctions work, and which auctions perform best, for instance in terms of generating revenues or in terms of efficiency. Second, auction theory is a fundamental tool in economic theory. It provides a price formation model, whereas the widely used Arrow-Debreu model from general equilibrium theory is not explicit in how prices form. Also, the insights generated by auction theory can be useful when studying several other phenomena which have structures that resemble auctions, like lobbying contests, queues, war of attritions, and monopolist's market behavior. For instance, the theory of monopoly pricing is mathematically the same as the theory of revenue maximizing auctions. Reflecting its importance, auction theory has become a substantial field in economic theory.

This Ph.D. thesis is a collection of five published papers in auction theory. Each chapter can be read independently from the other chapters. Chapter 1 provides a swift tour of auction theory and its applications. Chapter 2 studies auctions with financial externalities, i.e., auctions in which bidders prefer their opponents to pay the highest amount of money to the seller. These kind of externalities might have played a role in high stake spectrum auctions like the UMTS-auctions in Europe. Chapter 3 analyzes fund-raising mechanisms. The American Association of Fundraising Counsel has estimated that the population in the USA donates yearly circa 250 billion dollar to charity. Although charity is big business, not much is known about what the most effective way is to raise money. Chapter 4 studies simultaneous pooled auctions with multiple bids and preference lists. In these auctions single-object demand bidders submit bids for every object for sale, and a preference ordering over which object they would like to get if they have the highest bid on more than one object. This type of auction has been used in the Netherlands and in Ireland to auction available spectrum. The results in this chapter should convince governments not to use this type of format anymore in future allocations. Finally, Chapter 5 gives an answer to the question how inefficient auction outcomes due to strong negative (informational) externalities (created by post-auction interactions) can be avoided. These negative externalities arise when a bidders' type (such as a measure of his cost efficiency) positively affects his own value but negatively inflicts upon the value of the competitor. For each chapter we will give a short summary.

Chapter 1 This introductory chapter gives an easily accessible overview of the most important insights of auction theory. Among the questions it considers are: How much do bidders bid in commonly studied single-object auctions? How efficient are these auctions? How much revenue do they generate? Which single-object auction maximizes the seller's expected revenue? What is the best way to auction incentive contracts? And, how efficient and complex are multi-object auctions?

In this chapter, we show that in the symmetric independent private values (SIPV) model all efficient auctions yield the same revenue to the seller as long as the bidder with the lowest possible value obtains zero expected utility. All four standard auctions (first-price sealed-bid auction, Dutch auction, Vickrey auction, and English auction) are therefore revenue equivalent. Relaxing the (strict) assumptions of the SIPV model destroys the revenue equivalence result though. We also show that the seller faces a trade-off between efficiency and revenue. When selling a single object, the seller maximizes his revenue by imposing a reserve price. This causes inefficiency as the object remains unsold when none of the bidders turns out to be willing to pay the reserve price, while they may assign a positive value to it. Equivalently, in auctions of incentive contracts, the revenue maximizing buyer only assigns the incentive contract if a sufficiently efficient firm enters the auction.

In multi-object auctions, a new trade-off enters the picture: the trade-off between efficiency and complexity. If each bidder in the auction only demands one object, and if the seller offers homogeneous objects, the main results from the single-object case carry over: straightforward generalizations of the standard auctions are efficient (and revenue equivalent). However, as soon as objects are heterogeneous, or when bidders demand more than one object, an efficient outcome is no longer guaranteed. Luckily, rather simple efficient auctions can be constructed with multi-object demand if objects are homogeneous and with heterogeneous objects if there is single-object demand. In the general case, with multi-object demand and heterogeneous objects, the Vickrey-Clarke-Groves mechanism is efficient. However, this auction has several practical drawbacks, for instance that it is complex as bids are needed on a large range of packages. The disadvantages are only partially mitigated in innovative new designs that have been recently proposed in the literature, such as Ausubel, Cramton, and Milgrom's clock-proxy auction and Goeree and Holt's hierarchical package bidding auction.

Published: Going, Going, Gone! A Swift Tour of Auction Theory and Its Applications, De Economist, 154(2), 2006, pp. 197-249 (with Sander Onderstal).

Chapter 2 This chapter studies sealed-bid auctions in environments with financial externalities, i.e. environments in which losers' utilities depend on how much the winner pays. Standard auction models usually assume that losers of the auction are indifferent about how much the winner has paid. In reality, this does not need to be the case. In the series of UMTS auctions in Europe bidders might have seeked to raise the prices paid by their competitors. One purpose might have been to reduce competitors ability to finance the investment necessary to develop a third generation mobile phone network. Another purpose might have been to reduce competitors ability to bid in subsequent auctions.

In this chapter, we show that in the first-price auction, larger financial externalities result in lower bids and therefore lead to a lower expected price. An intuition for this result is that larger financial externalities make losing more attractive for the bidders. In the second-price auction, the effect of financial externalities on both bids and expected price is ambiguous. A possible explanation for this result is that in a second-price auction a bidder is not only inclined to bid less the higher the financial externality (as he gets positive utility from losing), he also has an incentive to bid higher, because, given that he loses, he is able to influence directly the level of payments made by the winner. Although the expected price in the second-price auction may increase if financial externalities increase, we show that the seller is not able to gain more revenue by guaranteeing the losers a fraction of the auction revenue. We also give a revenue comparison between the first-price and second-price auction. We find that the second-price auction results in a higher expected revenue than the first-price auction unless a bidder's interest in his own payments is equal to the sum of the other bidders' interest in what he pays. In that case, both auctions are revenue equivalent.

We also perform a study of the effect of a reserve price on equilibrium bidding. With a reserve price, we find that both auctions may have pooling at the reserve price, i.e. for a range of bidders it is optimal to submit a bid equal to the reserve price. To get an intuition why pooling at the reserve price occurs in equilibrium, consider a bidder with a valuation larger than the reserve price. He prefers to win if none of the other bidders submit a bid. However, if someone does bid, he may wish to lose because the financial externalities may be larger than the surplus he gets if he wins. If his valuation is not much higher than the reserve price, then the tiny surplus he gets by bidding in the absence of a competitor's bid does not balance the risk of winning the auction in the presence of a competitor's bid (and missing the relatively large financial externalities); he therefore does not bid at all. If his valuation is higher than a critical value, he submits a bid equal to the reserve price so that in the absence of a competitor's bid, he will win the object, and in the presence of a competitor's bid he will win with as small probability as possible. Small increases in the valuation do not change the fact that the bidder prefers losing to winning in the presence of a competitor's bid, which implies that there is pooling at that price. Only if his valuation is much higher than the reserve price, so that he would prefer winning to losing against another bidder who submits a bid, would he raise his bid above the reserve price. Pooling at the reserve price may thus arise naturally, and is therefore not something to be suspicious of.

In contrast to an environment without financial externalities, there exists no weakly separating Bayesian Nash equilibrium for the first-price auction. The second-price auction has a weakly separating Bayesian Nash equilibrium if, and only if, the reserve price is sufficiently low.

Published: Auctions with Financial Externalities, Economic Theory, 32(3), 2007, pp. 551-574 (with Sander Onderstal).

Chapter 3 What do Eric Clapton's guitar, Margaret Thatcher's handbag, and Britney Spears' pregnancy testing kit have in common? They were all auctioned for the benefit of charity. Besides auctions, charities also organize lotteries and voluntary contributions to collect money. The co-existence of these mechanisms raises the obvious question: "Which mechanism is superior at raising money?" We answer this question in this chapter. We show that "all-pay" auctions are better fundraising mechanisms than "standard" auctions, lotteries, and voluntary contributions.

In this paper, we assume that bidders obtain extra utility ($\$\alpha$) for each dollar that is transferred to a charitable organization. We find that in standard auctions (in which only the winner pays), revenues are relatively low. The reason is that all bidders forgo the extra utility they obtain from a high bid by one bidder if they top this bid. Bids are suppressed as a result, and so are revenues. This problem does not occur in lotteries and all-pay auctions where bidders pay irrespective of whether they win or lose. Moreover, bidders are willing to bid more in all-pay auctions than in lotteries because in all-pay auctions, the highest bidder always wins (in contrast to lotteries). We introduce a general class of all-pay auctions, rank their revenues, and illustrate how they dominate lotteries, standard auctions, and voluntary contributions. The optimal fund-raising mechanism is the lowest-price all-pay auction augmented with an entry fee and reserve price. If the charity cannot commit to keep the good (so that it cannot use an entry fee or a reserve price), the lowest-price all-pay auction raises the maximum possible revenue. The intuition for the fact that the lowest-price all-pay auction generates the highest revenue is that this auction is the only auction in which the lowest bidder not only influences his own payment but also the payments of the other bidders. It is therefore not attractive to bid low.

We demonstrate that an increase in the number of bidders may decrease revenues in all-pay formats. Fund-raisers may therefore benefit from limiting the number of contestants. The intuition behind this result can be made clear by considering the second-price all-pay auction. With two bidders ($n = 2$), the loser knows that his bid determines the price paid by the winner, which provides the loser with an incentive to drive up the price. This is not true with three or more bidders, however, in which case the $n - 2$ lowest bids are paid only by the losers. Hence there are no positive externalities associated with such bids, which become like voluntary contributions to the charity. This suppresses bids of low-value bidders, who free-ride on the revenues generated by the bidders with higher values.

The total amount raised is increasing with α. If α is large, then bidders may bid infinite amounts and, to deal with this, a budget constraint, M, is assumed. If bidders value \$1 for the charity the same as \$1 kept, revenues of a lottery or any of the all-pay auctions are equal to the sum of the bidders' budgets, nM. This maximum possible revenue contrasts with the expected revenue of a standard auction (since only a single bidder pays).

Our findings are not just of theoretical interest. The frequent use of lotteries as fund-raisers indicates that people are willing to accept an obligation to pay even though they may lose. All-pay auctions may be characterized as incorporating "voluntary contributions" into a standard auction. They are easy to implement and may revolutionize the way in which money is raised. If a charity wants to stick to standard auctions (for whatever reason), then it is better to use higher-price auctions, because, as we show in our paper, third-price auctions revenue dominate second-price auctions, which in turn revenue dominate first-price auctions. Only in case bidders are indifferent between a dollar donated and a dollar kept, the amount of money generated is identical for all standard auctions. The revenue is then equal to the expected value of the highest order statistic. Unlike all-pay auctions which may yield lower revenues when there are more bidders, standard auctions yield higher revenues when there are more bidders. It is therefore not optimal to limit competition and to restrict access to "a happy few."

Published: How (Not) to Raise Money, Journal of Political Economy, 113(4), 2005, pp. 897-918

(with Jacob Goeree, Sander Onderstal, and John Turner).

Chapter 4 This chapter analyzes the multi-object sealed-bid auction used to allocate Dutch commercial radio station licenses and Irish digital mobile data service licenses. In this particular auction, bidders submit bids for every object, and state a preference ordering over which object they would like to get if they have the highest bid on more than one object. This ordering decides which of those objects will be given to the bidder, who can only win a single object. In the two-object case, the allocation rule is as follows. The auctioneer opens all bids. If a bidder has only one highest bid, he gets the corresponding object, independent of his preference. If he has two highest bids, he gets the object that is highest on his preference list. The other object goes to the bidder who has submitted the second highest bid for that object. All winners pay their winning bid as a price.

In our model, bidders' values for object 1 are determined by independent draws from a fixed distribution. Bidders' values for object 2 are simply equal to $0 < \alpha \leq 1$ times the value of the first object. Each bidder knows his own value, but not the values of the other bidders. The ratio of valuations for the two objects is assumed to be identical for all bidders, because in the Dutch auction the value of a license was likely to be proportionally related to its demographic coverage. The preference ordering is modeled as a probability of taking object 1.

In the paper, we focus on efficient Bayes-Nash equilibria. In our setup, an efficient auction would allocate object 1 to the bidder with the highest draw of the random variable and object 2 to the bidder with the second highest draw of the random variable (when objects are heterogeneous). There are two main results. The first result establishes that an efficient equilibrium fails to exist when the number of bidders is sufficiently large. The second result establishes that inefficiencies also arise for a small number of bidders, as long as the objects are sufficiently close in value.

The intuition for the first result is as follows. In an efficient equilibrium, bidders use increasing bidding strategies, and express a preference for object 1. A bidder with a high valuation then has a profitable deviation: he wishes to lower his bid on object 2, and express a preference for the lower valued object. Since he is almost certain to win the auction for object 1, his bid on object 2 in the efficient equilibrium is wasted. The bidder should thus lower his bid on object 2 until the profit margin exceeds that on object 1. While he is unlikely to win object 2, he increases his profits in the states he does win and does not affect his profits in the states he loses. As the realized profit from such a deviation is strictly positive and independent of the number of bidders, whereas the profit in the efficient equilibrium asymptotically decreases to zero, a larger number of bidders makes the deviation more profitable.

The intuition for the second result is as follows. Fix the number of bidders. Consider the bidder with the highest possible valuation. For a small number of bidders, this bidder's profit in the efficient equilibrium is relatively large. As noted above, the profit from following the deviation strategy differs from the equilibrium profit only in the event this bidder is still the highest bidder on object 2 when deviating. The deviation profit is in this event at most α (the value of object 2 to this bidder). If α is small, i.e., if the objects are very heterogeneous, the equilibrium profit is larger than the deviation profit. For sufficiently large α, this bidder deviates though.

As an efficient equilibrium does not exist, an equilibrium must be non-monotone, asymmetric, or in mixed strategies. In either of these three cases, there is a strictly positive probability that the less valued object is sold for more than the more valued object. In this sense, the paper may give an explanation for the observation in the Dutch auction that less valuable spectrum is sold for a higher price than more valuable spectrum.

Also when objects are homogeneous, this auction leads to inefficiencies (with positive probability). The argument is as follows. As in the heterogeneous object case, the bidder with the highest possible valuation has an incentive to submit a high bid on one object and a low bid on the other object, and to express his preference for the object on which he submits the lowest bid. Homogeneity of the objects leads bidders to put their (deterministic) high and low bids randomly on both objects. If the high bid of the bidder with the third highest valuation is matched with the low bid of the bidder with the second highest valuation, then the bidder with the second highest valuation may leave the auction empty handed.

In this chapter, we therefore argue that this type of auction format, i.e., a multi-object sealed-bid auction with right-to-choose ingredients, should not be used (anymore). Other mechanisms exist that do have efficient equilibria (under fairly general conditions).

Published: Simultaneous Pooled Auctions with Multiple Bids and Preference Lists, Journal of Institutional and Theoretical Economics, 166(2), 2010, pp. 286-298 (with Maarten Janssen and Vladimir Karamychev).

Chapter 5 This chapter introduces a new class of selling mechanisms: auctions with flexible entry fees. The mechanism is a two stage multi-agent signaling game in which bidders observe their private information, i.e. their type, and then participate in a two stage game. In the first stage (the signaling stage), bidders simultaneously make publicly observable voluntary payments (so-called flexible entry fees) to an auctioneer. In the second stage (the auction stage), bidders participate in a second-price sealed-bid auction where bidders' valuations negatively depend on the types of the other bidders. We show that, given certain assumptions, there exists an equilibrium of this mechanism in which, in the first stage, the payment is monotonically increasing in the bidder's type, while in the second stage, each bidder bids his valuation, conditional on everything that is revealed in the first stage. Because of monotonicity, this equilibrium is efficient, that is, the object ends up with the bidder that values it most. The extension of this result to n identical objects with the $(n+1)$-bid price rule is immediate.

The reason why bidders want to pay a positive entry fee is that, although the entry fee is sunk at the moment of the auction, entry fees signal bidders' types thereby affecting each other's bids in a desirable way (due to the negative interdependency): bids in the second stage get lower when entry fees increase. By raising the entry fee, a bidder reduces the bid of his competitor and, therefore, lowers the price to be paid in case he wins the object. The larger the bidder's type, the larger his incentive to signal, because a bidder of a higher type wins with a higher probability and, therefore, is willing to spend a larger part of his maximum willingness to pay on signaling his type through the entry fee. Together with the fact that the maximum willingness to pay of this higher type is higher (for the same fixed types of competitors), this implies that the higher type sets a higher entry fee.

For a second-price sealed-bid auction with independently distributed types, we show that if the negative interdependencies are relatively weak, the auction with flexible entry fees is revenue equivalent to and yields the same (efficient) allocation as the standard second-price sealed-bid auction. If the negative interdependencies are relatively strong, the auction with flexible entry fees remains efficient whereas the standard second-price sealed-bid auction is known to be inefficient. When types are ex-ante positively correlated and the correlation is not too strong, a similar result holds true, but revenue equivalence fails. It turns out that when both auctions have an efficient equilibrium, the auction with flexible entry fees performs better in terms of revenue.

When bidders' types are strongly positively correlated bidders do not have an incentive to signal their types by paying a (high) entry fee as the other bidders can anyway infer someone's type once they have observed their own type. Therefore, an efficient equilibrium of the auction with flexible entry fees only exists if the correlation is not too strong. The good news, however, is that the separating equilibrium of the auction with flexible entry fees generates strictly higher revenue than the standard second-price sealed-bid auction.

The main message of this chapter is that inefficient auction outcomes due to strong negative externalities (which are often created by post-auction interactions such as resale or future market competition) can be avoided by asking bidders prior to the auction to submit any publicly observable payment they would like to make. In case of positively correlated types, the possibility of pre-auction payments also brings about a higher revenue to the auctioneer. Giving bidders a possibility to signal their type prior to the auction (through the flexible entry) might thus have a positive effect, in contrast to signaling types through bidding behavior during an auction.

Another interpretation of the auction with flexible entry fees (where the monetary fees are collected by the auctioneer) is that firms burn money or hire expensive auction experts to signal their strength. As long as the amount of money burnt (the cost of hiring auction experts) is either visible or made public, then this will have the same effect as flexible entry fees. Again, contrary to the common idea that exchanging information is bad (as it may lead to collusion), making this kind of information public will improve the efficiency of the auction. In this interpretation, however, the revenue collected by the auctioneer is lower than in the standard second-price sealed-bid auction, because a part of the revenue is either burnt or spent on experts.

Published (shortened version): Auctions with Flexible Entry Fees: A Note, Games and Economic Behavior, 72(2), 2011, pp. 594-601 (with Maarten Janssen and Vladimir Karamychev).

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EMIEL MAASLAND graduated in Economics from Erasmus University Rotterdam in 1996. He carried out his Ph.D. research at the CentER for Economic Research (Tilburg University) from 1996 to 2000. His research interests include auctions, charitable fundraising, industrial organization, and competition policy. He has published in international academic journals, such as Journal of Political Economy, Games and Economic Behavior, Economic Theory, Journal of Institutional and Theoretical Economics, and Telecommunications Policy. Currently, he works at SEOR, an applied economic research institute that operates independently under the umbrella of Erasmus University Rotterdam.

Auction theory is a branch of game theory that considers human behavior in auction markets and the ensuing market outcomes. It is also successfully used as a tool to design real-life auctions. This thesis contains five essays addressing a variety of topics within the realm of auction theory. The first essay gives an easily accessible overview of the most important insights of auction theory. The second essay, motivated by the UMTS-auctions that took place in Europe, studies auctions in which, in contrast to standard auction theory, losing bidders benefit from a high price paid by the winner(s). Under this assumption, the first-price sealed-bid auction and the second-price sealed-bid auction are no longer revenue equivalent. The third essay analyzes how well different kinds of auctions are able to raise money for charity. It turns out that standard winner-pay auctions are inept fund-raising mechanisms because of the positive externality bidders forgo if they top another's high bid. As this problem does not occur in all-pay auctions, where bidders pay irrespective of whether they win or lose, all-pay auctions are more effective in raising money. The fourth essay studies a particular auction type, a so-called simultaneous pooled auction with multiple bids and preference lists, that has been used for example in the Netherlands and Ireland to auction available spectrum. The results in this essay show that this type of auction does not satisfy elementary desirable properties such as the existence of an efficient equilibrium. The fifth essay argues that inefficient auction outcomes due to strong negative (informational) externalities (created by post-auction interactions) can be avoided by asking bidders prior to the auction to submit any publicly observable payment they would like to make.

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