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Abstract

This paper introduces the class of $2 \times 2 \times 2$ trimatrix games with identical anonymous best-replies. For this class a complete classification on the basis of the Nash equilibrium set is provided.

Key words: trimatrix games, Nash equilibrium, best-reply correspondences, symmetric games
JEL code: C72

1 Introduction

This paper is about classifying a specific kind of strategic games on the basis of the equilibrium set of the games under consideration. For 2×2 bimatrix games, such a taxonomy was provided by Beniest (1964), Rapoport et al. (1976) and Borm (1987) in a cardinal setting. In an ordinal framework 2×2 bimatrix games were classified by Brams (1977), Fraser and Kilgour (1986) and Kilgour and Fraser (1988).

The aim of this paper is to classify a specific subclass of $2 \times 2 \times 2$ trimatrix games on the basis of their equilibrium set in a cardinal setting. As already indicated in González-Alcón et al. (2007), each player's best reply correspondence in a $2 \times 2 \times 2$ trimatrix games can be of 13 essentially different types,

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in principle leading to 2197 combinations. These could possibly be combined on the basis of symmetry arguments or on the basis of the structure of the set of Nash equilibria, but for the remaining classes possibly also a further subdistinction could be necessary.

We restrict attention to games with identical anonymous best replies (IABR). IABR is a symmetry property of the joint best reply correspondences. In particular, every symmetric game is IABR. We provide a taxonomy of IABR games into 17 classes on the basis of their equilibrium set.

2 $2 \times 2 \times 2$ trimatrix games and best-replies

We denote a $2 \times 2 \times 2$ trimatrix game by $(N, \{S_i\}_{i \in N}, \{P_i\}_{i \in N})$. Here $N = \{1, 2, 3\}$ is the player set and for all $i \in N$ we denote by 0 and 1 the two pure strategies of player i. The set of mixed strategies S_i is the set of probability distributions on the set $\{0, 1\}$. A mixed strategy can be identified with a real number $s \in [0, 1]$, which is interpreted as the strategy using the pure strategy 1 with probability s and the pure strategy 0 with probability $1 - s$. In this way we set $S_i = [0, 1]$ for all $i \in N$. We define $S = \prod_{i \in N} S_i = [0, 1]^N$ and for any $s \in S$, $s_{-i} = \{s_j\}_{j \in N \setminus \{i\}}$ is the restriction of s to i's opponents. Finally, $P_i : S \to I\!R$ is the payoff function to player i, where for strategy profile $s = (s_1, s_2, s_3) \in S$, player i's expected payoff equals

$$P_i(s) = \sum_{T \subset N} \Big[\prod_{j \in T} s_j \prod_{j \in N \setminus T} (1 - s_j) P_i(1_T, 0_{N \setminus T}) \Big],$$

where $(1_T, 0_{N \setminus T})$ denotes the strategy profile where the players in T play their pure strategy 1 and the remaining players play 0. For player $i \in N$, the best-reply correspondence $B_i : S \twoheadrightarrow S_i$ is defined by

$$B_i(s) = \{t \in S_i \,|\, P_i(s_{-i}, t) \geq P_i(s_{-i}, t') \text{ for all } t' \in S_i\}.$$

Because P_i is multilinear, for any $s \in S$, $B_i(s)$ can take only three possible outcomes: $\{0\}$, $\{1\}$ and $[0, 1]$. Note moreover that $B_i(s)$ does not depend on s_i.

A strategy profile consisting of best replies against itself is a Nash equilibrium:

$$NE(N, S, P) = \{s \in S \,|\, \forall i \in N : s_i \in B_i(s)\}.$$

So, a Nash equilibrium is a strategy profile in which a unilateral deviation of one player cannot be profitable to this player.

In this paper we focus on games with all the players having "identical" best-reply correspondences. As a first step we focus on a player's opponents.

We say that in a game the best-reply correspondence of a player $i \in N$ exhibits *opponent anonymous best-replies* (OABR) if for every strategy profile $s \in S$ and bijection $\pi : N \to N$ such that $\pi(i) = i$,

$$B_i(s) = B_i(\pi(s)),$$

where $\pi(s) = (s_{\pi(j)})_{j \in N}$. That is, the best response of player i depends on the strategies used by the other players as a whole, but not on which player uses what strategy.

Example 1 *Consider the $2 \times 2 \times 2$ trimatrix game in which the payoffs corresponding to the pure strategy profiles are represented below:*

		player 2	
		1	*0*
player 1	*1*	*1,1,1*	*1,0,1*
	0	*0,1,0*	*0,0,0*

player 3: 1

	player 2	
	1	*0*
1	*1,0,0*	*0,0,0*
0	*0,0,1*	*1,1,1*

player 3: 0

The best-replies of the players are, for any $s \in S$, given by

$$B_1(s) = \begin{cases} \{0\} & \text{if } s_2 + s_3 - s_2 s_3 < \frac{1}{2}, \\ [0,1] & \text{if } s_2 + s_3 - s_2 s_3 = \frac{1}{2}, \\ \{1\} & \text{if } s_2 + s_3 - s_2 s_3 > \frac{1}{2}; \end{cases}$$

$$B_2(s) = \begin{cases} \{0\} & \text{if } s_1 + s_3 - s_1 s_3 < \frac{1}{2}, \\ [0,1] & \text{if } s_1 + s_3 - s_1 s_3 = \frac{1}{2}, \\ \{1\} & \text{if } s_1 + s_3 - s_1 s_3 > \frac{1}{2}; \end{cases}$$

$$B_3(s) = \begin{cases} \{0\} & \text{if } s_1 < \frac{1}{2}, \\ [0,1] & \text{if } s_1 = \frac{1}{2}, \\ \{1\} & \text{if } s_1 > \frac{1}{2}. \end{cases}$$

The best-reply correspondence B_1 is OABR: the roles of players 2 and 3 can be interchanged. Similarly, B_2 is OABR, but B_3 is not.

In this paper we consider games with *identical anonymous best-replies* (IABR), that is, for each strategy profile $s \in S$ and each bijection $\pi : N \to N$,

$$B_i(\pi(s)) = B_{\pi(i)}(s),$$

for all $i \in N$. Note that whereas OABR is a property of a single best-reply correspondence, IABR relates to all best-reply correspondences in a game simultaneously. Obviously, if the best-reply correspondences in a game are IABR, then each of them separately is OABR.

Example 2 *Consider the $2 \times 2 \times 2$ trimatrix game, which only differs from Example 1 in P_3, represented by*

		player 2: 1	player 2: 0
player 3: 1	player 1: 1	1,1,1	1,0,1
	player 1: 0	0,1,1	0,0,0
player 3: 0	player 1: 1	1,0,0	0,0,0
	player 1: 0	0,0,0	1,1,1

For all $s \in S$, $B_1(s)$ and $B_2(s)$ are as given in Example 1 and

$$B_3(s) = \begin{cases} \{0\} & \text{if } s_1 + s_2 - s_1 s_2 < \frac{1}{2}, \\ [0,1] & \text{if } s_1 + s_2 - s_1 s_2 = \frac{1}{2}, \\ \{1\} & \text{if } s_1 + s_2 - s_1 s_2 > \frac{1}{2}. \end{cases}$$

Hence, this is a game with IABR.

IABR is a property of best-reply correspondences and does not necessarily imply symmetry in the corresponding payoffs[1]. This is illustrated in the following example.

Example 3 *Consider the $2 \times 2 \times 2$ trimatrix game represented by*

		player 2: 1	player 2: 0
player 3: 1	player 1: 1	5,4,4	3,1,2
	player 1: 0	2,2,1	3,2,0
player 3: 0	player 1: 1	2,2,1	1,2,2
	player 1: 0	2,1,1	4,4,3

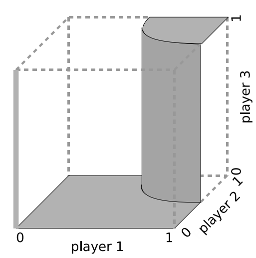
This game is not symmetric in terms of payoffs, but the best-reply correspondences exhibit IABR.

[1] So IABR does not imply $P_i(\pi(s)) = P_{\pi(i)}(\pi(s))$ for all $i \in N$. Obviously, every symmetric game is a game with IABR. In fact, it is not difficult to show that also every weighted potential game with a symmetric potential function is a game with IABR.

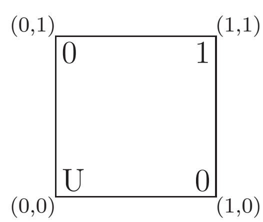
3 Nash equilibria of IABR $2 \times 2 \times 2$ games

In this section we provide a full description of the set of Nash equilibria in $2{\times}2{\times}2$ trimatrix games satifying IABR. To do this, we classify all such games on the basis of the shape of their (identical) best-reply correspondences. Our classification differs from the one in González-Alcón et al. (2007) because, on the one hand they do not impose IABR, and on the other hand symmetries in terms of only a single best-reply correspondence may break down when considering the interaction between all best replies.

To illustrate our classification, consider player 3. His best reply correspondence may look as follows:

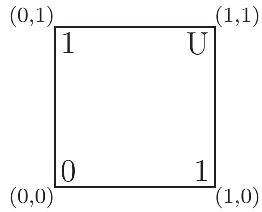


Note again that against any strategy profile of players 1 and 2, player 3's best response is either $\{0\}$, $\{1\}$, or $[0,1]$. Moreover, as a result of IABR, it is symmetric around the diagonal $(0,0)-(1,1) \subset S_1 \times S_2$. To get a more compact overview, we consider the projection of the picture above on the opponents' joint strategy space $S_1 \times S_2$, where in the four extreme points player 3's best reply is indicated (with $U = [0,1]$):



Note that as a result of IABR, the best reply against $(1,0)$ and $(0,1)$ must be the same.

If we relabel the pure strategies of *all* players *simultaneously* by switching 0 and 1, the best-reply correspondences still exhibit IABR. Basically the only thing that changes is that all best-reply correspondences (and hence, the set of Nash equilibria) are inverted trough point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Applying this to the best-reply correspondence depicted above yields:



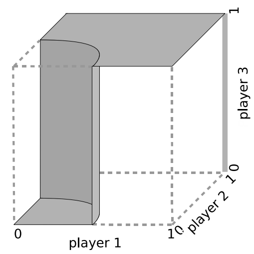


Table 1 shows our classification of IABR $2 \times 2 \times 2$ games. Column 2 depicts the best replies projected onto the opponents' joint strategy space. Columns 3, 4 and 5 depicts the whole strategy space S with in it the graphs of B_1, B_2 and B_3, respectively. Their respective intersection is depicted in black, so the black points in the final column are all Nash equilibria. An isolated Nash equilibrium is depicted as a square when it is in pure strategies and as a circle if it is mixed. Equilibria "hidden" by Player 3's best-reply correspondence are hollow circles or squares, or thick dashed lines.

In various cases an indication of the different shapes that the indifference curve (the locus of U) can have is given. This shape only affects the equilibrium set in the exact location of some of the mixed-strategy equilibria, except for classes 14, 15 and 16 where because of its further-reaching implications we indeed distinguish between three separate classes.

Table 1: Classification of IABR $2 \times 2 \times 2$ games

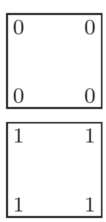
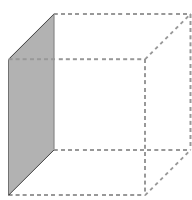
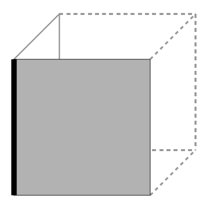
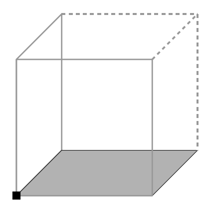
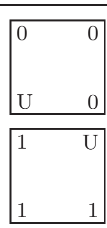
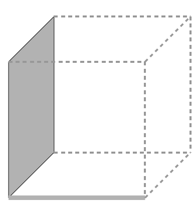
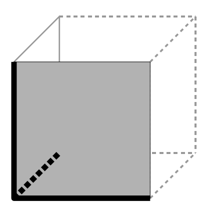
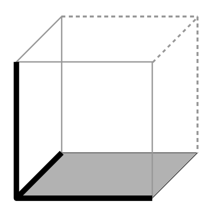
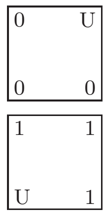
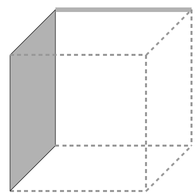
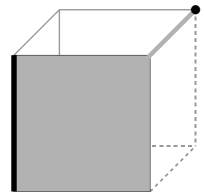
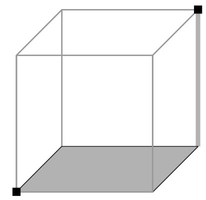
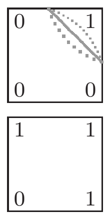
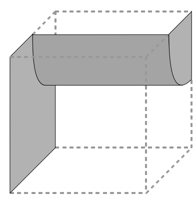
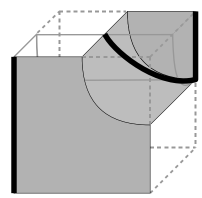
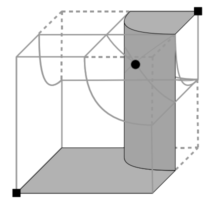
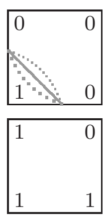
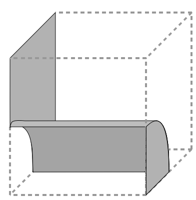
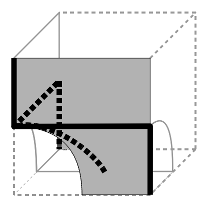
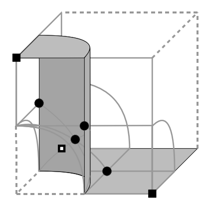
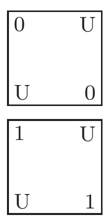
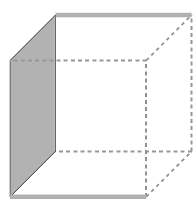
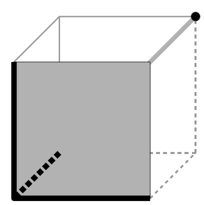
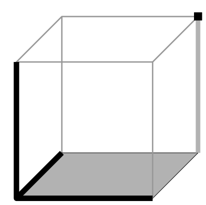
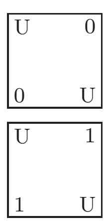
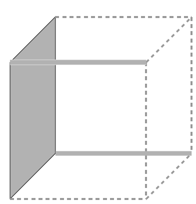
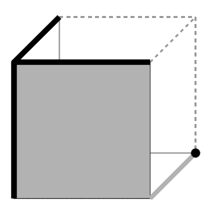
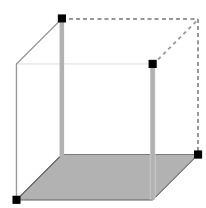
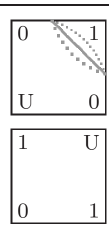
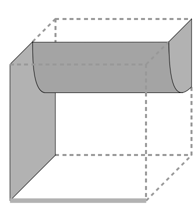
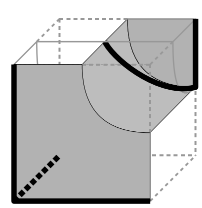
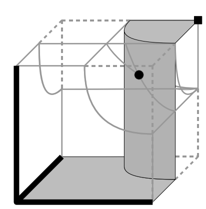
class	best replies	player 1's best replies	player 2's best replies and interaction with 1's	player 3's best replies and equilibria
1				
2				

Table 1: Continuation

class	best replies	player 1's best replies	player 2's best replies and interaction with 1's	player 3's best replies and equilibria
3				
4				
5				
6				
7				
8				

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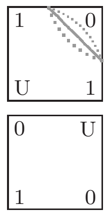
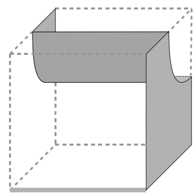
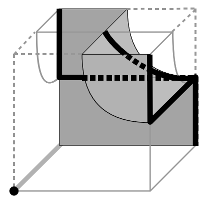
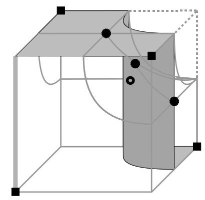
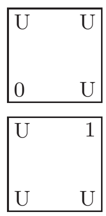
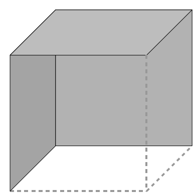
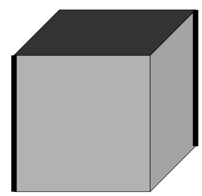
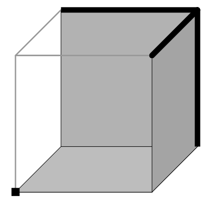
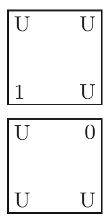
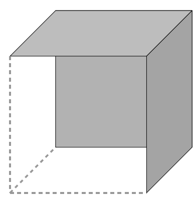
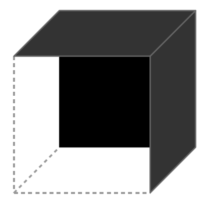
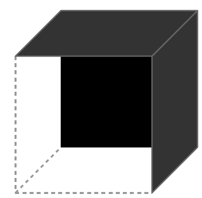
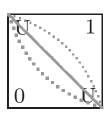
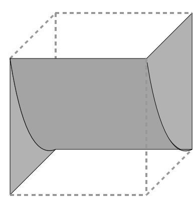
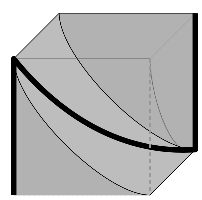
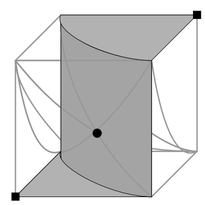
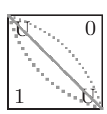
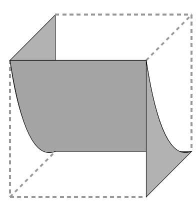
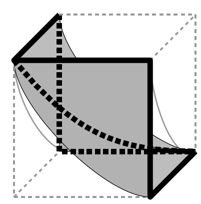
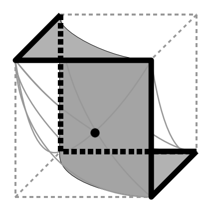
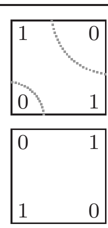
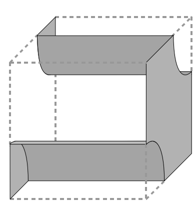
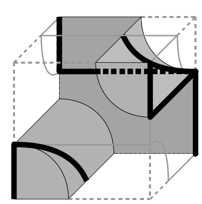
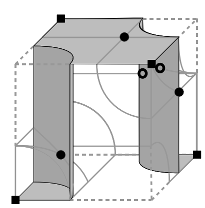
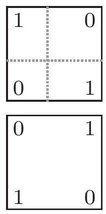
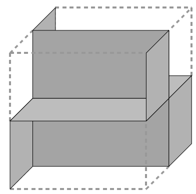
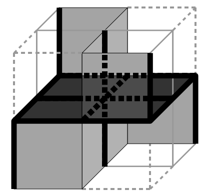
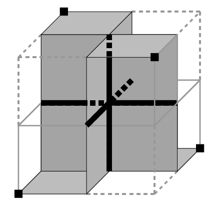
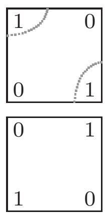
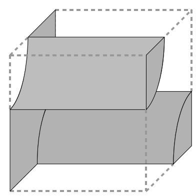
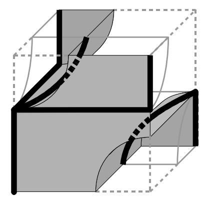
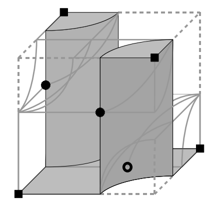
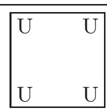
9				
10				
11				
12				
13				
14				

Table 1: Continuation

class	best replies	player 1's best replies	player 2's best replies and interaction with 1's	player 3's best replies and equilibria
15				
16				
17				

The equilibrium sets present several structures: some with only pure strategy equilibria (classes 1, 3 and 7), other with pure and mixed but always isolated equilibria (4, 5, 9, 12, 14, 16). In some cases there appear a continuum of equilibria in lines (2, 6, 8, 10, 13, 15), in surfaces (11), apart of the degenerate case of class 17. Moreover, note that all the classes are different in the composition of their Nash equilibria set except classes 4 and 12 (two pure, one mixed isolated equilibria) and 6 and 10 (a pure isolated plus the three edges with common vertex the opposite point).

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