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An odd characterization of the generalized odd graphs

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Abstract We show that any connected regular graph with $d + 1$ distinct eigenvalues and odd-girth $2d + 1$ is distance-regular, and in particular that it is a generalized odd graph.

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1 Introduction

A generalized odd graph is a distance-regular graph of diameter D, whose shortest odd cycles have length $2D + 1$ (the so-called odd-girth). It is also called an almost-bipartite distance-regular graph, or a regular thin near $(2D + 1)$-gon. Well-known examples of such graphs are the Odd graphs (also known as the Kneser graphs $K(2D+1, D)$), and the folded $(2D+1)$-cubes.

In this note, we shall characterize these graphs, by showing that any connected regular graph with $d + 1$ distinct eigenvalues and odd-girth (at least) $2d + 1$ is a distance-regular generalized odd graph. We remark that $D = d$ for distance-regular graphs, but in general we only have the inequality $D \leq d$. In general it is not true that any connected regular graph with diameter D and odd-girth $2D + 1$ is a generalized odd graph. Counterexamples can easily be found for the case $D = 2$ (among the triangle-free regular graphs with diameter two there are many graphs that are not strongly regular).

Huang and Liu [11] proved that any graph with the same spectrum as a generalized odd graph is such a graph. Because the odd-girth of a graph follows from the spectrum, our characterization is a generalization of this result.

For background on distance-regular graphs we refer the reader to [1], for eigenvalues of graphs to [2], for spectral characterizations of graphs to [5, 6], and for spectral and other algebraic characterizations of distance-regular graphs to [7] and [8], respectively. To show the claimed characterization, we shall use the so-called spectral excess theorem due to Fiol and Garriga [10]. Let Γ be a connected k-regular graph with $d+1$ distinct eigenvalues. The *excess* of a vertex u of Γ is the number of vertices at distance d from u. We also need the so-called predistance polynomial p_d of Γ, which will be explained in some detail in Section 3. The important property of p_d is that the value of $p_d(k)$ — the so-called *spectral excess* — only depends on the spectrum of Γ.

Spectral Excess Theorem. *Let Γ be a connected regular graph with $d+1$ distinct eigenvalues. Then Γ is distance-regular if and only if the average excess equals the spectral excess.*

For short proofs of this theorem we refer the reader to [3, 9].

2 The spectral characterization

Let Γ be a connected k-regular graph with adjacency matrix A having $d+1$ distinct eigenvalues $k = \lambda_0 > \lambda_1 > \cdots > \lambda_d$ and finite odd-girth at least $2d+1$. It follows that every vertex u has vertices at distance d, because otherwise the vertices at odd distance from u on one hand and the vertices at even distance from u on the other hand, would give a bipartition of the graph, contradicting that the odd-girth is finite. Because Γ has diameter D at most d, it follows that $D = d$, and that the odd-girth equals $2d+1$.

Because $(A^i)_{uv}$ counts the number of walks of length i in Γ from u to v, it follows that $p(A)$ has zero diagonal for any odd polynomial p of degree at most $2d-1$. Therefore also $\operatorname{tr} p(A) = 0$. Because the trace of $p(A)$ can also be expressed in terms of the spectrum of Γ, this also shows that the odd-girth condition on Γ is a condition on the spectrum of Γ. In the following, we make frequent use of polynomials. One of these is the Hoffman polynomial H defined by $H(x) = \frac{n}{\pi_0}\prod_{i=1}^{d}(x-\lambda_i)$, where n is the number of vertices and $\pi_0 = \prod_{i=1}^{d}(k-\lambda_i)$. This polynomial satisfies $H(A) = J$, the all-ones matrix.

Let us now consider two arbitrary vertices u, v at distance d. By considering the Hoffman polynomial, it follows that $(A^d)_{uv} = \frac{\pi_0}{n}$. By considering the minimal polynomial (or $(x-k)H$), it follows that $(A^{d+1})_{uv} - \tilde{a}_d(A^d)_{uv} = 0$, where $\tilde{a}_d = \sum_{i=0}^{d}\lambda_i$ is the coefficient of x^d in the minimal polynomial. Hence $(A^{d+1})_{uv} = \tilde{a}_d\frac{\pi_0}{n}$.

Lemma. *The average excess $\overline{k_d}$ of Γ equals $\frac{n}{\tilde{a}_d\pi_0^2}\operatorname{tr} A^{2d+1}$.*

Proof. For a vertex u, let $\Gamma_d(u)$ be the set of vertices at distance d from u. Then

$$(A^{2d+1})_{uu} = \sum_{v\in\Gamma_d(u)} (A^d)_{uv}(A^{d+1})_{vu} = k_d(u)\tilde{a}_d\pi_0^2/n^2,$$

where $k_d(u) = |\Gamma_d(u)|$ is the excess of u. Therefore $\overline{k_d}\tilde{a}_d\pi_0^2/n = \operatorname{tr} A^{2d+1}$ and $\tilde{a}_d \neq 0$. $\square$

In order to apply the spectral excess theorem, we have to ensure that $\overline{k_d} = p_d(k)$. However, $p_d(k)$ and $\frac{n}{\tilde{a}_d\pi_0^2}\operatorname{tr} A^{2d+1}$ only depend on the spectrum of Γ. Therefore, if Γ is cospectral with a distance-regular graph Γ', then the average $\overline{k_d}$ must equal $p_d(k)$, because it does so for Γ'. Hence we have:

Corollary. (Huang and Liu [11]) *Any graph cospectral with a generalized odd graph, is a generalized odd graph.*

3 The odd-girth characterization

Now let us show that $\overline{k_d} = p_d(k)$ for all graphs that we consider. To do this, we need some basic properties of the predistance polynomials; see also [9]. First, $\langle p, q\rangle = \frac{1}{n}\operatorname{tr}(p(A)q(A))$ defines an inner product (determined by the spectrum of Γ) on the space of polynomials modulo the minimal polynomial of Γ. Using this inner product, one can find an orthogonal system of so-called predistance polynomials $p_i, i = 0, 1, \ldots, d$, where p_i has degree i and is normalized such that $\langle p_i, p_i\rangle = p_i(k) \neq 0$. The predistance polynomials resemble the distance polynomials of a distance-regular graph; they also satisfy a three-term recurrence:

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1}, \quad i = 0, 1, \ldots, d,$$

where we let $\beta_{-1} = 0$ and $\gamma_{d+1}p_{d+1} = 0$ (the latter we may consider as a multiple of the minimal polynomial). A final property of these polynomials is that $\sum_{i=0}^{d} p_i$ equals the Hoffman polynomial H. This implies that the leading coefficient of p_d equals $\frac{n}{\pi_0}$ (the same as that of H).

For the graph Γ under consideration, specific properties hold. It is easy to show by induction that $\alpha_i = 0$ for $i < d$ and that p_i is an even or odd polynomial depending on whether i is even or odd, for all $i \leq d$. Indeed, it is clear that $p_0 = 1$ is even and $p_1 = x$ is odd, and hence that $\alpha_0 = 0$. Now suppose that $\alpha_i = 0$ for $i < j < d$ and that p_i is even or odd (depending on i) for $i \leq j$. Then the three-term recurrence implies that $\alpha_j p_j(k) = \langle xp_j, p_j\rangle = \frac{1}{n}\operatorname{tr}(Ap_j(A)^2) = 0$ because xp_j^2 is an odd polynomial of degree at most $2d-1$. Hence $\alpha_j = 0$ and then it follows from the recurrence that p_{j+1} is even or odd, which finishes the inductive argument.

What we shall use now is that xp_d^2 is an odd polynomial. It follows that

$$\alpha_d p_d(k) = \langle xp_d, p_d\rangle = \frac{1}{n}\operatorname{tr}(Ap_d(A)^2) = \frac{n}{\pi_0^2}\operatorname{tr} A^{2d+1}.$$

Thus, we have almost shown that the two expression in terms of the spectrum are the same; what remains is to show that $\alpha_d = \tilde{a}_d$. Therefore, consider again vertices u and v at distance d. Then

$$\alpha_d = \alpha_d(H(A))_{uv} = \alpha_d(p_d(A))_{uv} = (Ap_d(A))_{uv} = \frac{n}{\pi_0}(A^{d+1})_{uv} = \tilde{a}_d.$$

where the second last step follows because xp_d is odd or even, and therefore has no term of degree d. Thus, $\overline{k_d} = p_d(k)$ and by the spectral excess theorem we derive that Γ is distance-regular, which finishes the proof of our result.

Theorem. *Let Γ be a connected regular graph with $d+1$ distinct eigenvalues and finite odd-girth at least $2d+1$. Then Γ is a distance-regular generalized odd graph.*

It is unclear whether we can drop the regularity condition on Γ, or in other words, whether there exist nonregular graphs with $d+1$ distinct eigenvalues and odd-girth $2d+1$. For nonregular graphs it matters what matrix we consider (adjacency, Laplacian, etcetera). However, for $d = 2$ we know the following:

Proposition. *For the adjacency matrix, as well as for the Laplacian matrix, a connected graph with odd-girth five and three distinct eigenvalues is regular (and hence distance-regular).*

Proof. For the adjacency matrix A we consider the minimal polynomial m. Suppose $\lambda_0 > \lambda_1 > \lambda_2$ are the distinct eigenvalues of A. The diagonal of $m(A) = O$ gives that $(\lambda_0 + \lambda_1 + \lambda_2)k_u = -\lambda_0\lambda_1\lambda_2$, where k_u is the valency of vertex u. In case $\lambda_0 + \lambda_1 + \lambda_2 = \lambda_0\lambda_1\lambda_2 = 0$, if follows that $\lambda_0 = -\lambda_2$ and $\lambda_1 = 0$, so the graph would be bipartite, which is false. Thus k_u is constant.

For a graph whose Laplacian matrix has three distinct eigenvalues it is known that the number $\overline{\mu}$ of common nonneighbors of two adjacent vertices is constant (see [4]). Since there are no triangles, this implies that any two vertices at distance two have the same valency. The graph is connected with at least one odd cycle, hence there exists a walk of even length between any two vertices u and v. Because there are no triangles, every even vertex on that walk (which includes u and v) has the same valency. □

For the adjacency matrix we also managed to prove regularity for the analogous cases with four and five distinct eigenvalues, but we choose not to include the technical details.

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