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On Almost Distance-Regular Graphs *

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Abstract

Distance-regular graphs have been a key concept in Algebraic Combinatorics and have given place to several generalizations, such as association schemes. Motivated by spectral and other algebraic characterizations of distance-regular graphs, we study 'almost distance-regular graphs'. We use this name informally for graphs that share some regularity properties that are related to distance in the graph. For example, a known characterization of a distance-regular graph is the invariance of the number of walks of given length between vertices at a given distance, while a graph is called walk-regular if the number of closed walks of given length rooted at any given vertex is a constant. One of the concepts studied here is a generalization of both distance-regularity and walk-regularity called m-walk-regularity. Another studied concept is that of m-partial distance-regularity, or informally, distance-regularity up to distance m. Using eigenvalues of graphs and the predistance polynomials, we discuss and relate these and other concepts of almost distance-regularity, such as their common generalization of (ℓ, m)-walk-regularity. We introduce the concepts of punctual distance-regularity and punctual walk-regularity as a fundament upon which almost distance-regular graphs are built. We provide examples that are mostly taken from the Foster census, a collection of symmetric cubic graphs. Two problems are posed that are related to the question of when almost distance-regular becomes whole distance-regular. We also give several characterizations of punctually distance-regular graphs that are generalizations of the spectral excess theorem.

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1 Introduction

Distance-regular graphs [4] have been a key concept in Algebraic Combinatorics [17] and have given place to several generalizations, such as association schemes [21]. Motivated by spectral [8] and other algebraic [10] characterizations of distance-regular graphs, we study 'almost distance-regular graphs'. We use this name informally for graphs that share some regularity properties that are related to distance in the graph. For example, a known characterization (by Rowlinson [24]) of a distance-regular graph is the invariance of the number of walks of given length between vertices at a given distance. Godsil and McKay [18] called a graph walk-regular if the number of closed walks of given length rooted at any given vertex is a constant, cf. [17, p. 86]. One of the concepts studied here is a generalization of both distance-regularity and walk-regularity called m-walk-regularity, as introduced in [6]. Another studied concept is that of m-partial distance-regularity, or informally, distance-regularity up to distance m. Formally, it means that for $i \leq m$, the distance-i matrix can be expressed as a polynomial of degree i in the adjacency matrix. Related to this are two other generalizations of distance-regular graphs. Weichsel [27] introduced distance-polynomial graphs as those graphs for which each distance-i matrix can be expressed as a polynomial in the adjacency matrix. Such graphs were also studied by Beezer [2]. A graph is called distance degree regular if each distance-i graph is regular. Such graphs were studied by Bloom, Quintas, and Kennedy [3], Hilano and Nomura [19], and also by Weichsel [27] (as super-regular graphs).

This paper is organized as follows. In the next section we give the basic background for our paper. This includes our two main tools: eigenvalues of graphs and their predistance polynomials. In Section 3, we discuss several concepts of almost distance-regularity, such as partial distance-regularity in Section 3.2 and m-walk-regularity in Section 3.4. These concepts come together in Section 3.5, where we discuss (ℓ, m)-walk-regular graphs, as introduced in [7]. Sections 3.1 and 3.3 are used to introduce the concepts of punctual distance-regularity and punctual walk-regularity. These form the fundament upon which almost distance-regular graphs are built. Illustrating examples are mostly taken from the Foster census [25], a collection of symmetric cubic graphs that we checked by computer for almost distance-regularity. In Section 3 we also pose two problems. Both are related to the question of when almost distance-regular becomes whole distance-regular. The spectral excess theorem [12] is also of this type: it states that a graph is distance-regular if for each vertex, the number of vertices at extremal distance is the right one (i.e., some expression in terms of the eigenvalues), cf. [9, 14]. In Section 4 we give several characterizations of punctually distance-regular graphs that have the same flavor as the spectral excess theorem. We will show in Section 5 that these results are in fact generalizations of the spectral excess theorem. In this final section we focus on the case of graphs with spectrally maximum diameter (distance-regular graphs are such graphs).

2 Preliminaries

In this section we give the background on which our study is based. We would like to stress that in this paper we restrict to simple, connected, and regular graphs, unless we explicitly mention differently. First, let us recall some basic concepts and define our generic notation for graphs.

2.1 Spectra of graphs and walk-regularity

Throughout this paper, $G = (V, E)$ denotes a simple, connected, δ-regular graph, with order $n = |V|$ and adjacency matrix $\boldsymbol{A}$. The *distance* between two vertices u and v is denoted by $\partial(u, v)$, so that the *eccentricity* of a vertex u is $\mathrm{ecc}(u) = \max_{v\in V} \partial(u, v)$ and the *diameter* of the graph is $D = \max_{u\in V} \mathrm{ecc}(u)$. The set of vertices at distance i, from a given vertex $u \in V$ is denoted by $\Gamma_i(u)$, for $i = 0, 1, \ldots, D$. The degree of a vertex u is denoted by $\delta(u) = |\Gamma_1(u)|$. The *distance-$i$ graph* G_i is the graph with vertex set V and where two vertices u and v are adjacent if and only if $\partial(u, v) = i$ in G. Its adjacency matrix $\boldsymbol{A}_i$ is usually referred to as the *distance-i matrix* of G. The spectrum of G is denoted by

$$\mathrm{sp}\, G = \mathrm{sp}\, \boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\},$$

where the different eigenvalues of G are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$. In particular, note that $\lambda_0 = \delta$, $m_0 = 1$ (since G is connected and δ-regular) and $m_0 + m_1 + \cdots + m_d = n$.

For a given ordering of the vertices of G, the vector space of linear combinations (with real coefficients) of the vertices is identified with $\mathbb{R}^n$, with canonical basis $\{\boldsymbol{e}_u : u \in V\}$. Let $Z = \prod_{i=0}^{d}(x - \lambda_i)$ be the minimal polynomial of $\boldsymbol{A}$. The vector space $\mathbb{R}_d[x]$ of real polynomials of degree at most d is isomorphic to $\mathbb{R}[x]/(Z)$. For every $i = 0, 1, \ldots, d$, the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_i = \mathrm{Ker}(\boldsymbol{A} - \lambda_i \boldsymbol{I})$ is given by the Lagrange interpolating polynomial

$$\lambda_i^* = \frac{1}{\phi_i}\prod_{\substack{j=0\\ j\neq i}}^{d}(x - \lambda_j) = \frac{(-1)^i}{\pi_i}\prod_{\substack{j=0\\ j\neq i}}^{d}(x - \lambda_j)$$

of degree d, where $\phi_i = \prod_{j=0, j\neq i}^{d}(\lambda_i - \lambda_j)$ and $\pi_i = |\phi_i|$. These polynomials satisfy $\lambda_i^*(\lambda_j) = \delta_{ij}$. The matrices $\boldsymbol{E}_i = \lambda_i^*(\boldsymbol{A})$, corresponding to these orthogonal projections, are the (*principal*) *idempotents* of $\boldsymbol{A}$, and satisfy the known properties: $\boldsymbol{E}_i\boldsymbol{E}_j = \delta_{ij}\boldsymbol{E}_i$; $\boldsymbol{A}\boldsymbol{E}_i = \lambda_i\boldsymbol{E}_i$; and $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\lambda_i)\boldsymbol{E}_i$, for any polynomial $p \in \mathbb{R}[x]$ (see e.g. Godsil [17, p. 28]). The (u-)*local multiplicities* of the eigenvalue λ_i are defined as

$$m_u(\lambda_i) = \|\boldsymbol{E}_i\boldsymbol{e}_u\|^2 = \langle \boldsymbol{E}_i\boldsymbol{e}_u, \boldsymbol{e}_u\rangle = (\boldsymbol{E}_i)_{uu} \qquad (u \in V;\ i = 0, 1, \ldots, d),$$

and satisfy $\sum_{i=0}^{d} m_u(\lambda_i) = 1$ and $\sum_{u\in V} m_u(\lambda_i) = m_i$, $i = 0, 1, \ldots, d$ (see Fiol and Garriga [12]).

Related to this concept, we say that G is *spectrum-regular* if, for any $i = 0, 1, \ldots, d$, the u-local multiplicity of λ_i does not depend on the vertex u. Then, the above equations imply that the (standard) multiplicity 'splits' equitably among the n vertices, giving $m_u(\lambda_i) = m_i/n$.

By analogy with the local multiplicities, which correspond to the diagonal entries of the idempotents, Fiol, Garriga, and Yebra [16] defined the *crossed* (uv-)*local multiplicities* of the eigenvalue λ_i, denoted by $m_{uv}(\lambda_i)$, as

$$m_{uv}(\lambda_i) = \langle \boldsymbol{E}_i \boldsymbol{e}_u, \boldsymbol{E}_i \boldsymbol{e}_v \rangle = \langle \boldsymbol{E}_i \boldsymbol{e}_u, \boldsymbol{e}_v \rangle = (\boldsymbol{E}_i)_{uv} \qquad (u, v \in V;\ i = 0, 1, \ldots, d).$$

(Thus, in particular, $m_{uu}(\lambda_i) = m_u(\lambda_i)$.) These parameters allow us to compute the number of walks of length ℓ between two vertices u, v in the following way:

$$a_{uv}^{(\ell)} = (\boldsymbol{A}^\ell)_{uv} = \sum_{i=0}^{d} m_{uv}(\lambda_i)\lambda_i^\ell \qquad (\ell = 0, 1, \ldots). \tag{1}$$

Conversely, given the eigenvalues from which we compute the polynomials λ_i^*, and the tuple $\mathcal{C}_{uv} = (a_{uv}^{(0)}, a_{uv}^{(1)}, \ldots, a_{uv}^{(d)})$, we can obtain the crossed local multiplicities. With this aim, let us introduce the following notation: given a polynomial $p = \sum_{i=0}^{d} \zeta_i x^i$, let $p(\mathcal{C}_{uv}) = \sum_{i=0}^{d} \zeta_i a_{uv}^{(i)}$. Thus,

$$m_{uv}(\lambda_i) = (\boldsymbol{E}_i)_{uv} = (\lambda_i^*(\boldsymbol{A}))_{uv} = \lambda_i^*(\mathcal{C}_{uv}) \qquad (i = 0, 1, \ldots, d). \tag{2}$$

Let $a_u^{(\ell)}$ denote the number of closed walks of length ℓ rooted at vertex u, that is, $a_u^{(\ell)} = a_{uu}^{(\ell)}$. If these numbers only depend on ℓ, for each $\ell \geq 0$, then G is called *walk-regular* (a concept introduced by Godsil and McKay [18]). In this case we write $a_u^{(\ell)} = a^{(\ell)}$. Notice that, as $a_u^{(2)} = \delta(u)$, the degree of vertex u, a walk-regular graph is necessarily regular. By (1) and (2) it follows that spectrum-regularity and walk-regularity are equivalent concepts. It also shows that the existence of the constants $a^{(0)}, a^{(1)}, \ldots, a^{(d)}$ suffices to assure walk-regularity. As it is well known, any distance-regular graph as well as any vertex-transitive graph is walk-regular, but the converse is not true.

2.2 The predistance polynomials and distance-regularity

A graph is called *distance-regular* if there are constants c_i, a_i, b_i such that for any $i = 0, 1, \ldots, D$, and any two vertices u and v at distance i, among the neighbours of v, there are c_i at distance $i-1$ from u, a_i at distance i, and b_i at distance $i+1$. In terms of the distance matrices $\boldsymbol{A}_i$ this is equivalent to

$$\boldsymbol{A}\boldsymbol{A}_i = b_{i-1}\boldsymbol{A}_{i-1} + a_i\boldsymbol{A}_i + c_{i+1}\boldsymbol{A}_{i+1} \qquad (i = 0, 1, \ldots, D)$$

(with $b_{-1} = c_{D+1} = 0$). From this recurrence relation, one can obtain the so-called *distance polynomials* p_i. These are such that $\deg p_i = i$ and $\boldsymbol{A}_i = p_i(\boldsymbol{A})$, $i = 0, 1, \ldots, D$.

From the spectrum of a given (arbitrary) graph, $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, one can generalize the distance polynomials of a distance-regular graph (to a certain extent) by considering the following scalar product in $\mathbb{R}_d[x]$:

$$\langle p, q\rangle = \frac{1}{n}\operatorname{tr}(p(\boldsymbol{A})q(\boldsymbol{A})) = \frac{1}{n}\sum_{i=0}^{d} m_i p(\lambda_i) q(\lambda_i). \tag{3}$$

Then, by using the Gram-Schmidt method and normalizing appropriately, it is immediate to prove the existence and uniqueness of an orthogonal system of so-called *predistance polynomials* $\{p_i\}_{0\le i\le d}$ satisfying $\deg p_i = i$ and $\langle p_i, p_j\rangle = \delta_{ij} p_i(\lambda_0)$ for any $i, j = 0, 1, \ldots d$. For details, see Fiol and Garriga [12, 13].

As every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1} \qquad (i = 0, 1, \ldots, d), \tag{4}$$

where the constants β_{i-1}, α_i, and γ_{i+1} are the Fourier coefficients of xp_i in terms of p_{i-1}, p_i, and p_{i+1}, respectively (and $\beta_{-1} = \gamma_{d+1} = 0$), initiated with $p_0 = 1$ and $p_1 = x$. Let ω_k be the leading coefficient of p_k. Then, from the above recurrence, it is immediate that

$$\omega_k = \frac{1}{\gamma_1\gamma_2\cdots\gamma_k}. \tag{5}$$

In general, we define the *preintersection numbers* ξ_{ij}^k, with $i, j, k = 0, 1, \ldots d$, as the Fourier coefficients of $p_i p_j$ in terms of the basis $\{p_k\}_{0\le k\le d}$; that is:

$$\xi_{ij}^k = \frac{\langle p_i p_j, p_k\rangle}{\|p_k\|^2} = \frac{1}{np_k(\lambda_0)}\sum_{l=0}^{d} m_l p_i(\lambda_l) p_j(\lambda_l) p_k(\lambda_l). \tag{6}$$

With this notation, notice that the constants in (4) correspond to the preintersection numbers $\alpha_i = \xi_{1,i}^i$, $\beta_i = \xi_{1,i+1}^i$, and $\gamma_i = \xi_{1,i-1}^i$. As expected, when G is distance-regular, the predistance polynomials and the preintersection numbers become the distance polynomials and the *intersection numbers* $p_{ij}^k = |\Gamma_i(u) \cap \Gamma_j(v)|$, $\partial(u, v) = k$, for $i, j, k = 0, 1, \ldots, D(= d)$. For an arbitrary graph we say that the intersection number p_{ij}^k is *well defined* if $|\Gamma_i(u) \cap \Gamma_j(v)|$ is the same for all vertices u, v at distance k, and we let $a_i = p_{1,i}^i$, $b_i = p_{1,i+1}^i$, and $c_i = p_{1,i-1}^i$. From a combinatorial point of view, we would like many of these intersection numbers to be well defined, in order to call a graph almost distance-regular.

Note that not all properties of the distance polynomials of distance-regular graphs hold for the predistance polynomials. The crucial property that is not satisfied in general is that of the equations $\boldsymbol{A}_i = p_i(\boldsymbol{A})$. In fact, informally speaking we will 'measure' almost distance-regularity by how much the matrices $\boldsymbol{A}_i$ look like the matrices $p_i(\boldsymbol{A})$. Walk-regular graphs, for example, were characterized by Dalfó, Fiol, and Garriga [6] as those graphs for which the matrices $p_i(\boldsymbol{A})$, $i = 1, \dots, d$, have null diagonals (as have the matrices $\boldsymbol{A}_i$, $i = 1, \dots, d$).

A property that holds for all graphs is that the sum of all predistance polynomials gives the Hoffman polynomial H:

$$H = \sum_{i=0}^{d} p_i = \frac{n}{\pi_0} \prod_{i=1}^{d} (x - \lambda_i) = n\, \lambda_0^*, \tag{7}$$

which characterizes regular graphs by the condition $H(\boldsymbol{A}) = \boldsymbol{J}$, the all-1 matrix [20]. Note that (7) implies that $\omega_d = \frac{n}{\pi_0}$. It can also be used to show that $\alpha_i + \beta_i + \gamma_i = \lambda_0 = \delta$ for all i.

For bipartite graphs we observe the following facts. Because the eigenvalues are symmetric about zero ($\lambda_i = -\lambda_{d-i}$ and $m_i = m_{d-i}$, $0 \le i \le d$), we have $\langle x p_i, p_i \rangle = 0$ from (3), and therefore $\alpha_i = 0$ for all i. It then follows from (4) that the predistance polynomials p_i are even for even i, and odd for odd i. Using (6), this implies among others that $\xi_{ij}^k = 0$ if $i + j + k$ is odd. It also follows that $\gamma_d = \lambda_0 = \delta$. Finally, the Hoffman polynomial splits into an even part $H_0 = \sum_i p_{2i}$ and an odd part $H_1 = H - H_0$, and these have the property that $(H_0)_{uv} = 1$ if u and v are in the same part of the bipartition, and $(H_1)_{uv} = 1$ if u and v are in different parts.

2.3 The adjacency algebra and the distance algebra

Given a graph G, the set $\mathcal{A} = \{p(\boldsymbol{A}) : p \in \mathbb{R}[x]\}$ is a vector space of dimension $d+1$ and also an algebra with the ordinary product of matrices, known as the *adjacency* or *Bose-Mesner algebra*, and $\{\boldsymbol{I}, \boldsymbol{A}, \dots, \boldsymbol{A}^d\}$ is a basis of $\mathcal{A}$. Since $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \dots, \boldsymbol{A}^D$ are linearly independent, we have that $\dim \mathcal{A} = d + 1 \ge D + 1$ and therefore the diameter is at most d. A natural question is to enhance the case when equality is attained; that is, $D = d$. In this case, we say that the graph G has *spectrally maximum* diameter.

Let $\mathcal{D}$ be the linear span of the set $\{\boldsymbol{A}_0, \boldsymbol{A}_1, \dots, \boldsymbol{A}_D\}$. The $(D+1)$-dimensional vector space $\mathcal{D}$ forms an algebra with the entrywise or Hadamard product of matrices, defined by $(\boldsymbol{X} \circ \boldsymbol{Y})_{uv} = \boldsymbol{X}_{uv} \boldsymbol{Y}_{uv}$. We call $\mathcal{D}$ the *distance* $\circ$*-algebra*.

In the following sections, we will work with the vector space $\mathcal{T} = \mathcal{A} + \mathcal{D}$, and relate the distance-i matrices $\boldsymbol{A}_i \in \mathcal{D}$ with the matrices $p_i(\boldsymbol{A}) \in \mathcal{A}$. Note that $\boldsymbol{I}$, $\boldsymbol{A}$, and $\boldsymbol{J}$ are matrices in $\mathcal{A} \cap \mathcal{D}$ since $\boldsymbol{J} = H(\boldsymbol{A}) \in \mathcal{A}$. Thus, $\dim(\mathcal{A} \cap \mathcal{D}) \ge 3$, if G is not a complete graph (in this exceptional case $\boldsymbol{J} = \boldsymbol{I} + \boldsymbol{A}$). Note that $\mathcal{A} = \mathcal{D}$ if and only if G is distance-regular, which is therefore equivalent to $\dim(\mathcal{A} \cap \mathcal{D}) = d + 1$. For this reason, the dimension of $\mathcal{A} \cap \mathcal{D}$ (compared to D and d) can also be seen as a measure of almost distance-regularity.

One concept of almost distance-regularity related to this was introduced by Weichsel [27]: a graph is called *distance-polynomial* if $\mathcal{D} \subset \mathcal{A}$, that is, if each distance matrix is a polynomial in $\boldsymbol{A}$. Hence a graph is distance-polynomial if and only if $\dim(\mathcal{A}\cap\mathcal{D}) = D+1$.

Note that for any pair of (symmetric) matrices $\boldsymbol{R}, \boldsymbol{S} \in \mathcal{T}$, we have

$$\operatorname{tr}(\boldsymbol{RS}) = \sum_{u\in V} (\boldsymbol{RS})_{uu} = \sum_{u\in V}\sum_{v\in V} \boldsymbol{R}_{uv}\boldsymbol{S}_{vu} = \operatorname{sum}(\boldsymbol{R}\circ\boldsymbol{S}).$$

Thus, we can define a scalar product in $\mathcal{T}$ in two equivalent forms:

$$\langle \boldsymbol{R}, \boldsymbol{S}\rangle = \frac{1}{n}\operatorname{tr}(\boldsymbol{RS}) = \frac{1}{n}\operatorname{sum}(\boldsymbol{R}\circ\boldsymbol{S}).$$

In $\mathcal{A}$, this scalar product coincides with the scalar product (3) in $\mathbb{R}[x]/(Z)$, in the sense that $\langle p(\boldsymbol{A}), q(\boldsymbol{A})\rangle = \langle p, q\rangle$. Observe that the factor $1/n$ assures that $\|\boldsymbol{I}\|^2 = \langle 1, 1\rangle = 1$. Note also that $\|\boldsymbol{A}_i\|^2 = \overline{\delta}_i$ (the *average degree* of G_i), whereas $\|p_i(\boldsymbol{A})\|^2 = p_i(\lambda_0)$.

Association schemes are generalizations of distance-regular graphs that will provide almost distance-regular graphs. A (symmetric) *association scheme* can be defined as a set of symmetric $(0,1)$-matrices (graphs) $\{\boldsymbol{B}_0 = \boldsymbol{I}, \boldsymbol{B}_1, \ldots, \boldsymbol{B}_e\}$ adding up to the all-1 matrix $\boldsymbol{J}$, and whose linear span is an algebra $\mathcal{B}$ (with the ordinary product), called the Bose-Mesner algebra. In the case of distance-regular graphs, the distance-matrices $\boldsymbol{A}_i$ form an association scheme. For more on association schemes, we refer to a recent survey by Martin and Tanaka [21].

3 Different concepts of almost distance-regularity

In this section we introduce some concepts of almost distance-regular graphs, together with some characterizations. We begin with some closely related 'local concepts' concerning distance-regular and distance-polynomial graphs.

3.1 Punctually distance-polynomial and punctually distance-regular graphs

We say that a graph G is *h-punctually distance-polynomial* for some integer $h \le D$, if $\boldsymbol{A}_h \in \mathcal{A}$; that is, there exists a polynomial $q_h \in \mathbb{R}_d[x]$ such that $q_h(\boldsymbol{A}) = \boldsymbol{A}_h$. Obviously, $\deg q_h \ge h$. In case of equality, i.e., if $\deg q_h = h$, we call the graph *h-punctually distance-regular*. Notice that, since $\boldsymbol{A}_0 = \boldsymbol{I}$ and $\boldsymbol{A}_1 = \boldsymbol{A}$, every graph is 0-punctually distance-regular ($q_0 = 1$) and 1-punctually distance-regular ($q_1 = x$). In general, we have the following result.

Lemma 3.1 *Let $h \le D$ and let G be h-punctually distance-polynomial, with $\boldsymbol{A}_h = q_h(\boldsymbol{A})$. Then the distance-$h$ graph G_h is regular of degree $q_h(\lambda_0) = \|q_h\|^2$. If $\deg q_h = h$ (G is h-punctually distance-regular), then $q_h = p_h$, the predistance polynomial of degree h. If $\deg q_h > h$, then $\deg q_h > D$.*

Proof. Let $\boldsymbol{j}$ denote the all-1 vector. Because $\boldsymbol{A}_h\boldsymbol{j} = q_h(\boldsymbol{A})\boldsymbol{j} = q_h(\lambda_0)\boldsymbol{j}$, the graph G_h is regular with degree $q_h(\lambda_0) = \frac{1}{n}\operatorname{tr}(\boldsymbol{A}_h^2) = \|\boldsymbol{A}_h\|^2 = \|q_h\|^2$. Moreover, for every polynomial $p \in \mathbb{R}_{h-1}[x]$, we have $\langle q_h, p\rangle = \langle \boldsymbol{A}_h, p(\boldsymbol{A})\rangle = 0$. Thus, if $\deg q_h = h$, we must have $q_h = p_h$ by the uniqueness of the predistance polynomials. If $h < \deg q_h = i \leq D$ and q_h has leading coefficient ς_i then we would have $(q_h(\boldsymbol{A}))_{uv} = \varsigma_i a_{uv}^{(i)} \neq 0$ for any two vertices u, v at distance i, which contradicts $(q_h(\boldsymbol{A}))_{uv} = (\boldsymbol{A}_h)_{uv} = 0$. □

This lemma implies that the concepts of h-punctually distance-polynomial and h-punctually distance-regular are the same for graphs with spectrally maximum diameter $D = d$. We will consider such graphs in more detail in Section 5.

Any polynomial of degree at most d is a linear combination of the polynomials $p_0, \ldots, p_d$. If $\boldsymbol{A}_h = q_h(\boldsymbol{A})$, then clearly q_h is a linear combination of the polynomials $p_h, \ldots, p_d$. For example, in the case of a graph with $D = 2$ (which is always distance-polynomial; see the next section), we have $\boldsymbol{A}_2 = q_2(\boldsymbol{A})$, with $q_2 = p_2 + \cdots + p_d$.

On the other hand, if $p_h(\boldsymbol{A})$ is a linear combination of the distance-matrices $\boldsymbol{A}_i, i = 0, 1, \ldots, D$, then we have the following.

Lemma 3.2 *Let $h \leq d$. If $p_h(\boldsymbol{A}) \in \mathcal{D}$, then $h \leq D$ and G is h-punctually distance-regular.*

Proof. If $p_h(\boldsymbol{A}) \in \mathcal{D}$, then $p_h(\boldsymbol{A}) = \sum_{i=0}^{h} \zeta_i \boldsymbol{A}_i$ for some ζ_i, $i = 0, 1, \ldots, h$. Note first that $\langle \boldsymbol{A}_i, p_i(\boldsymbol{A})\rangle = \frac{1}{n}\sum_{\partial(u,v)=i}(p_i(\boldsymbol{A}))_{uv} = \frac{\omega_i}{n}\sum_{\partial(u,v)=i}(\boldsymbol{A}^i)_{uv} \neq 0$ for $i \leq D$. Now it follows that $0 = \langle p_h(\boldsymbol{A}), p_0(\boldsymbol{A})\rangle = \zeta_0\langle \boldsymbol{A}_0, p_0(\boldsymbol{A})\rangle$ and hence that $\zeta_0 = 0$. By using that $0 = \langle p_h(\boldsymbol{A}), p_i(\boldsymbol{A})\rangle$ one can similarly show by induction that $\zeta_i = 0$ for $i < h$. If $h > D$, then this implies that $p_h(\boldsymbol{A}) = \boldsymbol{O}$, which is a contradiction. Hence $h \leq D$ and $\boldsymbol{A}_h = \frac{1}{\zeta_h} p_h(\boldsymbol{A})$. By Lemma 3.1 it then follows that $\boldsymbol{A}_h = p_h(\boldsymbol{A})$, i.e., that G is h-punctually distance-regular. □

Graph F026A from the Foster Census [25] is an example of a (bipartite) graph with $D = d = 5$, that is h-punctually distance-regular for $h = 2$ and 4, but not for $h = 3$ and 5. It is interesting to observe, however, that the intersection number $c_5 = 3$ is well defined, whereas $|\Gamma_1(u) \cap \Gamma_3(v)| = 2$ or 3 for $\partial(u, v) = 4$, so c_4 is not well defined. Thus, there does not seem to be a combinatorial interpretation in terms of intersection numbers of the algebraic definition of punctual distance-regularity. In the next section, the combinatorics will return.

3.2 Partially distance-polynomial and partially distance-regular graphs

A graph G is called *m-partially distance-polynomial* if $\boldsymbol{A}_h = q_h(\boldsymbol{A}) \in \mathcal{A}$ for every $h \leq m$ (that is, G is h-punctually distance-polynomial for every $h \leq m$). If each polynomial q_h has degree h, for $h \leq m$, we call the graph *m-partially distance-regular* (that is, G is h-punctually distance-regular for every $h \leq m$). In this case, $\boldsymbol{A}_h = p_h(\boldsymbol{A})$ for $h \leq m$, by Lemma 3.1.

Alternatively, and recalling the combinatorial properties of distance-regular graphs, we can say that a graph is m-partially distance-regular when the intersection numbers c_i, a_i, b_i up to c_m are well defined, i.e., the distance matrices satisfy the recurrence

$$\boldsymbol{A}\boldsymbol{A}_i = b_{i-1}\boldsymbol{A}_{i-1} + a_i\boldsymbol{A}_i + c_{i+1}\boldsymbol{A}_{i+1} \qquad (i = 0, 1, \ldots, m-1).$$

From this we have the following lemma, which may be useful in finding examples of m-partially distance-regular graphs with large m.

Lemma 3.3 *If G has girth g, then G is m-partially distance-regular with $m = \lfloor \frac{g-1}{2} \rfloor$.*

Proof. Just note that if the girth is g then there is a unique shortest path between any two vertices at distance at most $m = \lfloor \frac{g-1}{2} \rfloor$. Hence the intersection parameters c_i, b_i, and a_i up to c_m are well defined; indeed, if G has degree δ, then $c_i = 1$, $1 \le i \le m$; $a_i = 0$, $0 \le i \le m-1$; and $b_0 = \delta$, $b_i = \delta - 1$, $1 \le i \le m-1$. $\square$

Generalized Moore graphs are regular graphs with girth at least $2D-1$, cf. [22, 26]. By Lemma 3.3, such graphs are $(D-1)$-partially distance-regular. Only few examples of generalized Moore graphs that are not distance-regular are known.

It is clear that every D-partially distance-polynomial graph is distance-polynomial, and every D-partially distance-regular graph is distance-regular (in which case $d = D$). In fact, the conditions can be slightly relaxed as follows.

Proposition 3.4 *If G is $(D-1)$-partially distance-polynomial, then G is distance-polynomial. If G is $(d-1)$-partially distance-regular, then G is distance-regular.*

Proof. Let G be $(D-1)$-partially distance-polynomial, with $\boldsymbol{A}_h = q_h(\boldsymbol{A})$, $h \le D-1$. Then by using the expression for the Hoffman polynomial in (7), we have:

$$\boldsymbol{A}_D + \sum_{h=0}^{D-1} q_h(\boldsymbol{A}) = \sum_{h=0}^{D} \boldsymbol{A}_h = \boldsymbol{J} = H(\boldsymbol{A}),$$

so that $\boldsymbol{A}_D = q_D(\boldsymbol{A})$, where $q_D = H - \sum_{h=0}^{D-1} q_h$, and G is distance-polynomial.

Similarly, if G is $(d-1)$-partially distance-regular, then from $\boldsymbol{A}_d + \sum_{i=0}^{d-1} p_i(\boldsymbol{A}) = \sum_{i=0}^{d} \boldsymbol{A}_i = H(\boldsymbol{A})$, we get $\boldsymbol{A}_d = p_d(\boldsymbol{A})$, and G is distance-regular. $\square$

In particular, Proposition 3.4 implies the observation by Weichsel [27] that every (regular) graph with diameter two is distance-polynomial.

The distinction between D and d in Proposition 3.4 is essential. A $(D-1)$-partially distance-regular graph is not necessarily distance-regular. In fact, Koolen and Van Dam [private communication] observed that the direct product of the folded $(2D-1)$-cube [4, p. 264] and K_2 is $(D-1)$-partially distance-regular with diameter D, but a_{D-1} is not well defined. Note that these graphs also occur as so-called boundary graphs in related work [16].

It would also be interesting to find examples of m-partially distance-regular graphs with m equal (or close) to $d-2$ that are not distance-regular (for all d), is these exist. More specifically, we pose the following problem.

Problem 1 *Determine the smallest $m = m_{pdr}(d)$ such that every m-partially distance-regular graph with $d+1$ distinct eigenvalues is distance-regular.*

For bipartite graphs, the result in Proposition 3.4 can be improved as follows.

Proposition 3.5 *Let G be bipartite. If G is $(D-2)$-partially distance-polynomial, then G is distance-polynomial. If G is $(d-2)$-partially distance-regular, then G is distance-regular.*

Proof. Similar as the proof of Proposition 3.4; instead of the Hoffman polynomial, one should use its even and odd parts H_0 and H_1. $\square$

It is interesting to note that a graph with $D = d$ that is D-punctually distance-regular must be distance-regular. This result is a small part in the proof of the spectral excess theorem, cf. [9, 14]. We will generalize this in Proposition 3.7 by showing that we do not need to have h-punctual distance-regularity for all $h \le m$ to obtain m-partial distance-regularity. The following lemma is a first step in this direction.

Lemma 3.6 *Let $d - m < s \le m \le D$ and let G be h-punctually distance-regular for $h = m-s+1, \dots, m$. Then G is $(m-s)$-punctually distance-regular.*

Proof. By the assumption, we have $\boldsymbol{A}_{m-s+1} = p_{m-s+1}(\boldsymbol{A})$, ... , $\boldsymbol{A}_m = p_m(\boldsymbol{A})$, and we want to show that $p_{m-s}(\boldsymbol{A}) = \boldsymbol{A}_{m-s}$. We therefore check the entry uv in $p_{m-s}(\boldsymbol{A})$, and distinguish the following three cases:

- (a) For $\partial(u,v) > m-s$, we have $(p_{m-s}(\boldsymbol{A}))_{uv} = 0$.
- (b) For $\partial(u,v) < m-s$, we use the equation $xp_{m-s+1} = \beta_{m-s}p_{m-s} + \alpha_{m-s+1}p_{m-s+1} + \gamma_{m-s+2}p_{m-s+2}$, which gives us $\boldsymbol{A}\boldsymbol{A}_{m-s+1} = \beta_{m-s}p_{m-s}(\boldsymbol{A}) + \alpha_{m-s+1}\boldsymbol{A}_{m-s+1} + \gamma_{m-s+2}\boldsymbol{A}_{m-s+2}$ (in case $s = 1$ we have $m = d$ and then the last term vanishes), and hence that

$$(p_{m-s}(\boldsymbol{A}))_{uv} = \frac{1}{\beta_{m-s}}(\boldsymbol{A}\boldsymbol{A}_{m-s+1})_{uv} = \frac{1}{\beta_{m-s}} \sum_{w \in \Gamma_1(u)} (\boldsymbol{A}_{m-s+1})_{wv} = 0,$$

since $\partial(v,w) \le \partial(v,u) + \partial(u,w) < m-s+1$ for the relevant w.

(*c*) For $\partial(u,v) = m-s$, we claim that $(p_i(\boldsymbol{A}))_{uv} = 0$ for $i \neq m-s$. This is clear if $i < m-s$ and also if $m-s+1 \leq i \leq m$, because then $(p_i(\boldsymbol{A}))_{uv} = (\boldsymbol{A}_i)_{uv} = 0$. So, we only need to check that the entries $(p_{m+1}(\boldsymbol{A}))_{uv}, (p_{m+2}(\boldsymbol{A}))_{uv}, \ldots, (p_d(\boldsymbol{A}))_{uv}$ are zero. To do this, we will show by induction that $(p_{m+i}(A))_{yz} = 0$ if $\partial(y,z) < m-i$ and $i = 0, \ldots, d-m$. For $i = 0$ this is clear. For $i = 1$, this follows from the equation $\boldsymbol{A}\boldsymbol{A}_m = \beta_{m-1}\boldsymbol{A}_{m-1} + \alpha_m \boldsymbol{A}_m + \gamma_{m+1}p_{m+1}(\boldsymbol{A})$ and a similar argument as in case (*b*). The induction step then follows similarly: if $\partial(y,z) < m-i-1$, then the equation

$$\gamma_{m+i+1}p_{m+i+1}(\boldsymbol{A}) = \boldsymbol{A}p_{m+i}(\boldsymbol{A}) - \alpha_{m+i}p_{m+i}(\boldsymbol{A}) - \beta_{m+i-1}p_{m+i-1}(\boldsymbol{A})$$

and induction show that $(p_{m+i+1}(\boldsymbol{A}))_{yz} = 0$.

Thus our claim is proven, and by taking the entry uv in the equation

$$p_{m-s}(\boldsymbol{A}) = \boldsymbol{J} - \sum_{i \neq m-s} p_i(\boldsymbol{A}),$$

we have $(p_{m-s}(\boldsymbol{A}))_{uv} = 1$.

Joining (*a*), (*b*), and (*c*), we obtain that $p_{m-s}(\boldsymbol{A}) = \boldsymbol{A}_{m-s}$. □

Proposition 3.7 *Let $\lceil d/2 \rceil \leq m \leq D$. Then G is m-partially distance-regular if and only if G is h-punctually distance-regular for $h = 2m-d, \ldots, m$.*

Proof. This follows from applying Lemma 3.6 repeatedly for $s = d-m+1, \ldots, m$. □

As mentioned, this is a generalization of the following, which follows by taking $m = D = d$.

Corollary 3.8 *[15] Let G be a graph with spectrally maximum diameter $D = d$. Then G is distance-regular if and only if it it D-punctually distance-regular.*

The following is a new variation on this theme. Note that we will get back to the case $D = d$ in Section 5.

Corollary 3.9 *Let G be a graph with spectrally maximum diameter $D = d$. Then G is distance-regular if and only if it it $(D-1)$-punctually distance-regular and $(D-2)$-punctually distance-regular.*

3.3 Punctually walk-regular and punctually spectrum-regular graphs

In a similar way as in the previous sections, we will now generalize the concept of walk-regularity. We say that a graph G is *h-punctually walk-regular*, for some $h \leq D$, if for every $\ell \geq 0$ the number of walks of length ℓ between a pair of vertices u, v at distance h does not depend on u, v. If this is the case, we write $a_{uv}^{(\ell)} = (\boldsymbol{A}^\ell)_{uv} = a_h^{(\ell)}$.

Similarly, we say that a graph G is *h-punctually spectrum-regular* for a given $h \leq D$ if, for any $i \leq d$, the crossed uv-local multiplicities of λ_i are the same for all vertices u, v at distance h. In this case, we write $m_{uv}(\lambda_i) = m_{hi}$. Notice that, for $h = 0$, these concepts are equivalent, respectively, to walk-regularity and spectrum-regularity. As we saw, the latter two are also equivalent to each other. In fact, as an immediate consequence of (1) and (2), the analogous result holds for any given value of h.

Lemma 3.10 *Let $h \leq D$. Then G is h-punctually walk-regular if and only if it is h-punctually spectrum-regular.*

The following lemma turns out to be very useful for checking punctual walk-regularity; we will use this in the proofs of Propositions 3.21 and 5.4.

Lemma 3.11 *Let $h \leq D$. If the number of walks in G of length ℓ between vertices u and v depends only on $\partial(u, v) = h$, for each $\ell \leq d-1$, or, if G is bipartite, for each $\ell \leq d-2$, then G is h-punctually walk-regular.*

Proof. By using the Hoffman polynomial H we know that

$$\frac{\pi_0}{n} H(\boldsymbol{A}) = \boldsymbol{A}^d + \eta_{d-1}\boldsymbol{A}^{d-1} + \cdots + \eta_0 \boldsymbol{I} = \frac{\pi_0}{n}\boldsymbol{J}. \tag{8}$$

Let u, v be vertices at distance h. Then the existence of the constants $a_h^{(\ell)}$, $\ell \leq d-1$, assures that

$$a_{uv}^{(d)} = (\boldsymbol{A}^d)_{uv} = \frac{\pi_0}{n} - \eta_{d-1} a_h^{(d-1)} - \cdots - \eta_0 a_h^{(0)}$$

is also constant. From the fact that $\{\boldsymbol{I}, \boldsymbol{A}, \ldots, \boldsymbol{A}^d\}$ is a basis of $\mathcal{A}$, it then follows that G is h-punctually distance-regular. Now let G be bipartite. If h and d have the same parity, then $a_h^{(d-1)} = 0$, and the result follows as in the general case. If h and d have different parities, then $a_h^{(d)} = 0$. Now it follows from (8) that if $a_{uv}^{(\ell)}$ is a constant for $\ell \leq d-2$, then $a_{uv}^{(d-1)}$ also is. Here we use that $\eta_{d-1} = \delta \neq 0$ because G is bipartite (and hence $\lambda_i = -\lambda_{d-i}$, $0 \leq i \leq d$). Hence G is h-punctually distance-regular. □

Next we will show that 1-punctual walk-regularity implies walk-regularity. Later we will generalize this result in Proposition 3.24.

Proposition 3.12 *Let G be* 1*-punctually walk-regular. Then G is walk-regular (and spectrum-regular) with $a_0^{(\ell)} = \delta a_1^{(\ell-1)}$ for $\ell > 1$, and $m_{1i} = \frac{\lambda_i}{\lambda_0}\frac{m_i}{n}$ for $i = 0, 1, \ldots, d$.*

Proof. For a vertex u and $\ell > 0$ we have that $a_{uu}^{(\ell)} = (\boldsymbol{A}^\ell)_{uu} = \sum_{v \in \Gamma_1(u)} (\boldsymbol{A}^{\ell-1})_{uv} = \delta a_1^{(\ell-1)}$, which shows that G is walk-regular with $a_0^{(\ell)} = \delta a_1^{(\ell-1)}$. Then G is also 1-punctually spectrum-regular and spectrum-regular by Lemma 3.1, and then $\lambda_0 m_{1i} = \sum_{v \in \Gamma_1(u)} (\boldsymbol{E}_i)_{vu} = (\boldsymbol{A}\boldsymbol{E}_i)_{uu} = \lambda_i (\boldsymbol{E}_i)_{uu} = \lambda_i \frac{m_i}{n}$, which finishes the proof. □

Interesting examples of punctually walk-regular graphs can be obtained from association schemes.

Proposition 3.13 *Let $\{\boldsymbol{B}_0 = \boldsymbol{I}, \boldsymbol{B}_1, \ldots, \boldsymbol{B}_e\}$ be an association scheme and let G be one of the graphs in this scheme. If also its distance-h graph G_h is in the scheme, then G is h-punctually walk-regular.*

Proof. By the assumption there are i, k such that $\boldsymbol{A} = \boldsymbol{B}_i$ and $\boldsymbol{A}_h = \boldsymbol{B}_k$. Let u, v be vertices at distance h in G. Because the Bose-Mesner algebra $\mathcal{B}$ is closed under ordinary product, there are constants $c_{j\ell}$ such that

$$(\boldsymbol{A}^\ell)_{uv} = (\boldsymbol{B}_i^\ell)_{uv} = (\sum_{j=0}^{e} c_{j\ell}\boldsymbol{B}_j)_{uv} = c_{k\ell}.$$

So G is h-punctually walk-regular. □

In fact, this proposition shows that any graph in an association scheme is h-punctually walk-regular for $h = 0$ ($\boldsymbol{A}_0 = \boldsymbol{B}_0$) and $h = 1$ ($\boldsymbol{A}_1 = \boldsymbol{B}_i$). Note that because of our restriction in this paper to connected graphs, we should (formally speaking) say that each of the connected components of a graph in an association scheme is h-punctually walk-regular for $h = 0, 1$. Specific examples with other h will show up in the next section.

3.4 m-Walk-regular graphs

In [6], the concept of m-walk-regularity was introduced: for a given integer $m \leq D$, we say that G is *m-walk-regular* if the number of walks $a_{uv}^{(\ell)}$ of length ℓ between vertices u and v only depends on their distance h, provided that $h \leq m$. In other words, G is m-walk-regular if it is h-punctually walk-regular for every $h \leq m$. Obviously, 0-walk-regularity is the same concept as walk-regularity.

Similarly, a graph is called m-spectrum-regular graph if it is h-punctually spectrum-regular for all $h \leq m$. By Lemma 3.10, this is equivalent to m-walk-regularity. Moreover, in [6], m-walk-regular graphs were characterized as those graphs for which $\boldsymbol{A}_i$ looks the same as $p_i(\boldsymbol{A})$ for every i when looking through the 'window' defined by the matrix $\boldsymbol{A}_0 + \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_m$. A generalization of this will be proved in the next section.

Proposition 3.14 *[6] Let $m \leq D$. Then G is m-walk-regular (and m-spectrum-regular) if and only if $p_i(\boldsymbol{A}) \circ \boldsymbol{A}_j = \delta_{ij}\boldsymbol{A}_i$ for $i = 0, 1, \ldots, d$ and $j = 0, 1, \ldots, m$.*

This result implies the following connection with partial distance-regularity.

Proposition 3.15 *Let $m \leq D$ and let G be m-walk-regular. Then G is m-partially distance-regular and a_m (and hence b_m) is well-defined.*

Proof. Proposition 3.14 implies that $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ for $i \leq m$, and hence that G is m-partially distance-regular, and that $\boldsymbol{p}_{m+1}(\boldsymbol{A}) \circ \boldsymbol{A}_m = \boldsymbol{O}$. It follows that

$$(\boldsymbol{A}\boldsymbol{A}_m) \circ \boldsymbol{A}_m = (\boldsymbol{A}p_m(\boldsymbol{A})) \circ \boldsymbol{A}_m = (\beta_{m-1}\boldsymbol{A}_{m-1} + \alpha_m \boldsymbol{A}_m + \gamma_{m+1}p_{m+1}(\boldsymbol{A})) \circ \boldsymbol{A}_m = \alpha_m \boldsymbol{A}_m,$$

which shows that $a_m = \alpha_m$ is well defined, and hence also b_m is well defined. □

It turns out though that much weaker conditions on the number of walks are sufficient to show m-partial distance-regularity.

Proposition 3.16 *Let $m \leq D$. If the number of walks in G of length ℓ between vertices u and v depends only on $\partial(u,v) = h$ for each $h < m$, $\ell = h, h+1$, and $h = \ell = m$, then G is m-partially distance-regular.*

Proof. If $\partial(u,v) = h \leq m$, then $a_h^{(h)} = |\Gamma_1(u) \cap \Gamma_{h-1}(v)| a_{h-1}^{(h-1)}$ assures that c_h is well defined. If $\partial(u,v) = h < m$, then similarly $a_h^{(h+1)} = |\Gamma_1(u) \cap \Gamma_h(v)| a_h^{(h)} + c_h a_{h-1}^{(h)}$ assures that a_h is well defined. □

In the next section, we shall further work out the difference between m-partial distance-regularity and m-walk-regularity. The following characterization by Rowlinson [24] (see also Fiol [10]) follows immediately from Proposition 3.14.

Proposition 3.17 *[24] A graph is D-walk-regular if and only if it is distance-regular.*

In the previous section we showed that any graph G in an association scheme is 1-walk-regular. In case the distance-matrices $\boldsymbol{A}_h$ of G are in the association scheme for all $h \leq m$, then the graph is clearly m-walk-regular by Proposition 3.13. Such graphs are examples of so-called distance(m)-regular graphs, as introduced by Powers [23]. A graph is called *distance(m)-regular* if for every vertex u there is an equitable partition $\{\{u\}, \Gamma_1(u), \ldots, \Gamma_m(u), V_{m+1}(u), \ldots, V_e(u)\}$ of the vertices, with quotient matrix being the same for every u (we refer the reader who is unfamiliar with equitable partitions to [17, p. 79]). We observe that this is equivalent to the existence of $(0,1)$-matrices $\boldsymbol{B}_{m+1}, \ldots, \boldsymbol{B}_e$ that add up to $\boldsymbol{A}_{m+1} + \cdots + \boldsymbol{A}_D$, such that the linear span of the set $\{\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_m, \boldsymbol{B}_{m+1}, \ldots, \boldsymbol{B}_e\}$ is closed under left multiplication by $\boldsymbol{A}$. Consequently, a distance(m)-regular graph is m-walk-regular (the same argument as in the proof of Proposition 3.13 applies). We now present some interesting examples of distance(m)-regular graphs (mostly coming from association schemes).

The bipartite incidence graph of a square divisible design with the dual property (i.e., such that the dual design is also divisible with the same parameters as the design itself) is a distance(2)-regular graph with $D = 4$ (and in general $d = 5$). This follows for example from the distance-distribution diagram (see [4, p. 24]); hence these graphs are 2-walk-regular.

The distance-4 graph of the distance-regular Livingstone graph is a distance(2)-regular graph with $D = 3$ (and $d = 4$); again, see the distribution diagram [4, p. 407].

The graph defined on the 55 flags of the symmetric 2-$(11, 5, 2)$ design, with flags (p, b) and (p', b') being adjacent if also (p, b') and (p', b) are flags is distance(3)-regular with $D = 4$ and $d = 5$; see the distribution diagram in Figure 1.

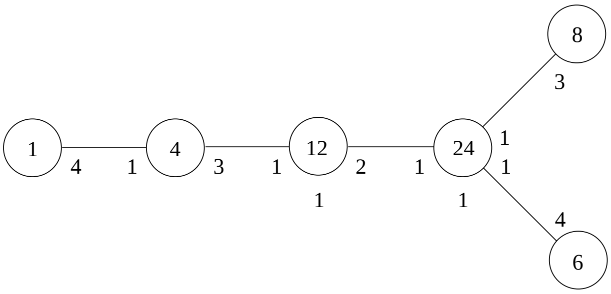

Figure 1: Distance distribution diagram of the flag graph

The above examples show that there are $(D-1)$-walk-regular graphs with diameter D that are not distance-regular, for small D. For larger D, we do not have such examples however, so the question arises if these exist at all.

Problem 2 *(a) Determine the smallest* $m = m_{wr,D}(D)$ *such that every* m*-walk-regular graph with diameter* D *is distance-regular.*

(b) Determine the smallest $m = m_{wr,d}(d)$ *such that every* m*-walk-regular graph with* $d + 1$ *distinct eigenvalues is distance-regular.*

Note that a $(d-1)$-walk-regular graph (with $d - 1 \leq D$) is distance-regular by Propositions 3.15 and 3.4.

Another interesting example related to this problem is the graph F234B from the Foster Census [25]. This graph has $D = 8$, $d = 11$, it is 5-arc-transitive, and hence 5-walk-regular. The vertices correspond to the 234 triangles in $PG(2, 3)$ with two vertices being adjacent whenever the corresponding triangles have one common point and their remaining four points are distinct and collinear [1, p. 125]. This and the above examples suggest that $m_{wr,D}(D) > \frac{D}{2} + 1$.

3.5 (ℓ, m)-Walk-regular graphs

In order to understand the difference between m-partial distance-regularity and m-walk-regularity, the following generalization of the latter is useful. As before, let G be a graph with diameter D and $d+1$ eigenvalues. Given two integers $\ell \leq d$ and $m \leq D$ satisfying $\ell \geq m$, we say that is G is *ℓ-partially m-walk-regular*, or *(ℓ, m)-walk-regular* for short, if the number of walks of length $\ell' \leq \ell$ between any pair of vertices u, v at distance $m' \leq m$ does not depend on such vertices but depends only on ℓ' and m'. The concept of (ℓ, m)-walk-regularity was introduced in [7], and generalizes some of the concepts from the previous sections. In fact, the following equivalences follow immediately:

- $(d, 0)$-walk-regular graph $\equiv$ walk-regular graph
- (d, m)-walk-regular graph $\equiv$ m-walk-regular graph
- (d, D)-walk-regular graph $\equiv$ distance-regular graph

We also note that $(\ell, 0)$-walk-regular graphs were introduced in [11] under the name of ℓ-partially walk-regular graphs, and they were also studied by Huang et al. [5]. More relations follow from the following generalization of Proposition 3.14. Here we will give a new (and shorter) proof.

Proposition 3.18 *[7] Let $d \geq \ell \geq m \leq D$. Then G is (ℓ, m)-walk-regular if and only if $p_i(\boldsymbol{A}) \circ \boldsymbol{A}_j = \delta_{ij}\boldsymbol{A}_i$ for $i = 0, 1, \ldots, \ell$ and $j = 0, 1, \ldots, m$.*

Proof. Assume the latter. Let $x^h = \sum_{i=0}^{h} \eta_{ih} p_i$ for $h \leq \ell$. Then for each pair of vertices u, v at distance $j \leq m$, and $h \leq \ell$, we have:

$$(\boldsymbol{A}^h)_{uv} = (\boldsymbol{A}^h \circ \boldsymbol{A}_j)_{uv} = \sum_{i=0}^{h} \eta_{ih} \left(p_i(\boldsymbol{A}) \circ \boldsymbol{A}_j\right)_{uv} = \eta_{jh}.$$

Consequently, G is (ℓ, m)-walk-regular. Conversely, consider the mapping $\Phi : \mathbb{R}_\ell[x] \to \mathbb{R}^{m+1}$ defined by $\Phi(p) = (\varphi_0(p), \ldots, \varphi_m(p))$, with $p(\boldsymbol{A}) \circ \boldsymbol{A}_j = \varphi_j(p)\boldsymbol{A}_j$. This mapping is linear and $\Phi(x^j) = (\varphi_0(x^j), \ldots, \varphi_j(x^j), 0, \ldots, 0)$ with $\varphi_j(x^j) \neq 0$, for $j = 0, 1, \ldots, m$. Therefore the restriction $\widetilde{\Phi}$ of Φ to $\mathbb{R}_m[x]$, is one-to-one. Now, let $r_i = \widetilde{\Phi}^{-1}(0, \ldots, 1, \ldots, 0)$, with the 1 in the i-th position, for $i \leq m$. In other words, $r_i(\boldsymbol{A}) \circ \boldsymbol{A}_j = \delta_{ij}\boldsymbol{A}_i$ for $i, j \leq m$. Each polynomial r_i satisfies $r_i(\boldsymbol{A}) = \sum_{j=0}^{m} r_i(\boldsymbol{A}) \circ \boldsymbol{A}_j = \boldsymbol{A}_i$, and therefore $r_i = p_i$ by Lemma 3.1. Thus, $p_i(\boldsymbol{A}) \circ \boldsymbol{A}_j = \delta_{ij}\boldsymbol{A}_i$ for $i, j \leq m$.

Now let $m+1 \leq i \leq \ell$ and $j \leq m$. Then $p_i(\boldsymbol{A}) \circ p_j(\boldsymbol{A}) = p_i(\boldsymbol{A}) \circ \boldsymbol{A}_j = \varphi_j(p_i)\boldsymbol{A}_j$. From this equation, we find that $\varphi_j(p_i)p_j(\lambda_0) = \varphi_j(p_i)\frac{1}{n}\,\mathrm{sum}(\boldsymbol{A}_j) = \frac{1}{n}\,\mathrm{sum}(p_i(\boldsymbol{A}) \circ p_j(\boldsymbol{A})) = \langle p_i, p_j \rangle = 0$. Thus, $\varphi_j(p_i) = 0$ and $p_i(\boldsymbol{A}) \circ \boldsymbol{A}_j = \boldsymbol{O}$, which completes the proof. $\square$

The following equivalences now follow; see also the proof of Proposition 3.15.

- (m, m)-walk-regular graph $\equiv$ m-partially distance-regular graph
- $(m+1, m)$-walk-regular graph $\equiv$ m-partially distance-regular graph with a_m (and hence b_m) well defined

We have seen in Proposition 3.16 though that weaker conditions on the number of walks are sufficient to show m-partial distance-regularity.

The next proposition follows from the characterization in Proposition 3.18. It clarifies the role of the preintersection numbers given by the expressions in (6).

Proposition 3.19 *[7] Let $d \geq \ell \geq m \leq D$, let G be (ℓ, m)-walk-regular, and let $i, j, k \leq m$. If $i + j \leq \ell$, then the preintersection number ξ_{ij}^{k} equals the well defined intersection number p_{ij}^{k}. If $i + j \geq \ell + 1$, then the preintersection number ξ_{ij}^{k} equals the average $\overline{p}_{ij}^{k}$ of the values $p_{ij}^{k}(u, v) = |\Gamma_i(u) \cap \Gamma_j(v)|$ over all vertices u, v at distance k.*

The graph F084A from the Foster Census [25] has $D = 7$ and $d = 10$. It is 2-walk-regular, 3-partially distance-regular, and all intersection numbers $c_i, i = 1, 2, \ldots, 7$ are well defined. This implies that the number of walks of length ℓ between vertices at distance ℓ depends only on ℓ. Still, this graph is not even $(4, 3)$-walk-regular, because a_3 is not well-defined.

We will now obtain relations between various kinds of partial walk-regularity.

Proposition 3.20 *Let $d - 1 \geq \ell \geq m \geq 1$, $m \leq D$, and let G be (ℓ, m)-walk-regular. Then G is $(\ell + 1, m - 1)$-walk-regular.*

Proof. Let u, v be two vertices of G at distance $j \leq m - 1$, with $j < \ell - 1$ (if $m = \ell$). From $\gamma_{\ell+1} p_{\ell+1} = x p_\ell - \beta_{\ell-1} p_{\ell-1} - \alpha_\ell p_\ell$ we have:

$$\begin{aligned} \gamma_{\ell+1}(p_{\ell+1}(\boldsymbol{A}) \circ \boldsymbol{A}_j)_{uv} &= (\boldsymbol{A} p_\ell(\boldsymbol{A}) \circ \boldsymbol{A}_j)_{uv} = (\boldsymbol{A} p_\ell(\boldsymbol{A}))_{uv} = \\ & \sum_w \boldsymbol{A}_{uw}(p_\ell(\boldsymbol{A}))_{wv} = \sum_{\partial(w,u)=1} (p_\ell(\boldsymbol{A}))_{wv} = 0\,, \end{aligned}$$

since $\partial(w, v) \leq j + 1 \leq m$, $\partial(w, v) < \ell$ if $m = \ell$, and $p_\ell(\boldsymbol{A}) \circ \boldsymbol{A}_i = \boldsymbol{O}$ for $i \leq m < \ell$. Moreover, if $m = \ell$ and $j = \ell - 1$ then G is ℓ-partially distance-regular. Thus, we get

$$\gamma_{\ell+1}(p_{\ell+1}(\boldsymbol{A}) \circ \boldsymbol{A}_{\ell-1})_{uv} = (\boldsymbol{A}\boldsymbol{A}_\ell)_{uv} - b_{\ell-1}(\boldsymbol{A}_{\ell-1})_{uv} = 0,$$

since $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$, $0 \leq i \leq \ell$, and $b_{\ell-1} = \beta_{\ell-1} = (\boldsymbol{A}\boldsymbol{A}_\ell)_{uv}$ is well defined. Therefore, $p_{\ell+1}(\boldsymbol{A}) \circ \boldsymbol{A}_j = \boldsymbol{O}$ for every $j \leq m - 1$, and Proposition 3.18 yields the result.

Alternatively, notice that, if G is (ℓ, m)-walk-regular, then the number of walks of length $\ell + 1$ between vertices u, v at distance $j < m$ equals

$$a_{uv}^{(\ell+1)} = c_j a_{j-1}^{(\ell)} + a_j a_j^{(\ell)} + b_j a_{j+1}^{(\ell)}$$

and hence is a constant $a_j^{(\ell+1)}$. $\square$

As a direct consequence of this last result, we have that (ℓ, m)-walk-regularity implies $(\ell + r, m - r)$-walk-regularity for every integer $r \leq d - \ell$ and $1 \leq r \leq m$. In particular, every (ℓ, m)-walk-regular graph with $\ell \geq d - m$ is also walk-regular. Also the following connections between partial distance-regularity and m-walk-regularity follow.

Proposition 3.21 *Let $m \leq D$ and let G be m-partially distance-regular. If $m \geq \frac{d-1}{2}$, then G is $(2m+1-d)$-walk-regular. If $m \geq \frac{d-2}{2}$ and a_m is well defined, then G is $(2m+2-d)$-walk-regular. If $m \geq \frac{d-3}{2}$ and G is bipartite, then G is $(2m+3-d)$-walk-regular.*

Proof. For the first statement, observe that G is (m,m)-walk-regular, so by Proposition 3.20 it is $(d-1, 2m+1-d)$-walk-regular. By Lemma 3.11, G is therefore $(2m+1-d)$-walk-regular. The proof of the second statement is similar, starting from $(m+1,m)$-walk-regularity. Also for the third statement we can start from $(m+1,m)$-walk-regularity, because $a_m = 0$ is well defined for a bipartite graph. Now it follows that G is $(d-2, 2m+3-d)$-walk-regular, and by Lemma 3.11, G is $(2m+3-d)$-walk-regular. □

Note that this proposition also relates Problems 1 and 2. For example, if $m_{pdr}(d) = d-1$ (for some d), then there is a $(d-2)$-partially distance-regular graph that is not distance-regular. This graph would be $(d-3)$-walk-regular by the proposition, which would imply that $m_{wr,d}(d) \geq d-2$. In general it shows that $m_{wr,d}(d) \geq 2m_{pdr}(d) - d$.

As it is known, graphs with few distinct eigenvalues have many regularity features. For instance, every (regular, connected) graph with three distinct eigenvalues is strongly regular (that is, distance-regular with diameter two). Any graph with four distinct eigenvalues is known to be walk-regular, and the bipartite ones with four distinct eigenvalues are always distance-regular. This also follows from Propositions 3.21 ($d=3, m=1$) and 3.4. It also follows that if G has four distinct eigenvalues and a_1 is well defined, then it is 1-walk-regular. If moreover c_2 is well defined, then the graph is distance-regular by Proposition 3.4. Similarly, if G is a bipartite graph with five distinct eigenvalues then G is 1-walk-regular. Moreover, if c_2 is well defined, then G is distance-regular.

A natural question would be to find out when the converse of Proposition 3.20 is true. At least the following can be said (we omit the proofs):

Proposition 3.22 *Let $m \leq D, m \leq d-1$. Then G is (m,m)-walk-regular if and only if it is $(m+1, m-1)$-walk-regular and the intersection number c_m is well defined.*

Proposition 3.23 *Let $m \leq D, m \leq d-2$. Then G is $(m+1,m)$-walk-regular if and only if it is $(m+2, m-1)$-walk-regular and the intersection numbers c_m, a_m, and b_m are well defined.*

It seems complicated to extend this further; for example, $(m+2,m)$-walk-regularity implies $(m+3, m-1)$-walk-regularity, but for the reverse we need (besides c_m, a_m, b_m) that c_{m+1} is well-defined. But $(m+2,m)$-walk-regularity does not necessarily imply that c_{m+1} is well-defined.

An interesting example is the graph F168F from the Foster Census [25]; it is a (bipartite) graph with $D=8$ and $d=20$. The intersection numbers are well defined up to

b_5, so the graph is $(6,5)$-walk-regular, and hence also $(7,4)$-walk-regular. Moreover, it is $(10,3)$-walk-regular, and 2-walk-regular.

As a final result in this section, we generalize Proposition 3.12. Note that every (regular) graph is $(\ell,0)$-walk-regular for $\ell \leq 2$, and that $q_h = x$ for $h = 1$.

Proposition 3.24 *Let $h \leq D$ and let G be h-punctually distance-polynomial, with $\boldsymbol{A}_h = q_h(\boldsymbol{A})$. Let $\ell+1$ be the number of distinct eigenvalues λ_i for which $q_h(\lambda_i) = 0$. If G is h-punctually spectrum-regular and $(\ell,0)$-walk-regular, then it is walk-regular (and spectrum-regular) and*

$$m_{hi} = \frac{q_h(\lambda_i)}{q_h(\lambda_0)} \frac{m_i}{n} \qquad (i = 0, 1, \ldots, d). \tag{9}$$

Proof. Let $\mathcal{I}$ denote the set of indices i such that $q_h(\lambda_i) = 0$, so $|\mathcal{I}| = \ell + 1$. If G is h-punctually spectrum-regular then

$$q_h(\lambda_0) m_{hi} = \sum_{v \in \Gamma_h(u)} (\boldsymbol{E}_i)_{vu} = (\boldsymbol{A}_h \boldsymbol{E}_i)_{uu} = (q_h(\boldsymbol{A}) \boldsymbol{E}_i)_{uu} = q_h(\lambda_i) (\boldsymbol{E}_i)_{uu} \qquad (u \in V),$$

which shows that $m_u(\lambda_i) = (\boldsymbol{E}_i)_{uu}$ is a constant, and $m_{0i} = \frac{q_h(\lambda_0)}{q_h(\lambda_i)} m_{hi}$, for every $i \notin \mathcal{I}$. Moreover, if G is $(\ell,0)$-walk-regular, then (1) yields:

$$\sum_{i \in \mathcal{I}} m_u(\lambda_i) \lambda_i^{\ell'} = a^{(\ell')} - \sum_{i \notin \mathcal{I}} m_{0i} \lambda_i^{\ell'} \qquad (0 \leq \ell' \leq \ell).$$

This is a linear system of $\ell+1$ equations with $\ell+1$ unknowns $m_u(\lambda_i)$, and this system has a unique solution as it has a Vandermonde matrix of coefficients. Hence $m_u(\lambda_i) = \frac{m_i}{n}$ for all $0 \leq i \leq d$ and we get (9). $\square$

With reference to (9), we note that the multiplicities m_i can be computed from the highest degree predistance polynomial as $m_i = (-1)^i \frac{\pi_0 p_d(\lambda_0)}{\pi_i p_d(\lambda_i)}$, cf. [12].

4 Spectral distance-degree characterizations

In this section we will obtain results that have the same flavor as the spectral excess theorem [12]. This theorem states that the average degree $\overline{\delta}_d$ of the distance-d graph is at most $p_d(\lambda_0)$ with equality if and only if the graph is distance-regular (for short proofs of this theorem, see [9, 14]). The following result gives a quasi-spectral characterization of punctually distance-polynomial graphs, in terms of the average degree $\overline{\delta}_h = \frac{1}{n}\operatorname{sum}(\boldsymbol{A}_h)$ of the distance-h graph G_h and the *average crossed local multiplicities*

$$\overline{m}_{hi} = \frac{1}{n\overline{\delta}_h} \sum_{\partial(u,v)=h} m_{uv}(\lambda_i).$$

Proposition 4.1 *Let $h \le D$. Then*

$$\overline{\delta}_h \le \frac{1}{n}\left(\sum_{i=0}^{d} \frac{\overline{m}_{hi}^2}{m_i}\right)^{-1}$$

with equality if and only if G is h-punctually distance-polynomial. If $\boldsymbol{A}_h = q_h(\boldsymbol{A})$, then

$$\delta_h = q_h(\lambda_0) \qquad \text{and} \qquad \overline{m}_{hi} = \frac{q_h(\lambda_i)}{q_h(\lambda_0)}\frac{m_i}{n} \qquad (i = 0, 1, \ldots, d).$$

Proof. We denote by $\widetilde{\boldsymbol{A}_h}$ the orthogonal projection of $\boldsymbol{A}_h$ onto $\mathcal{A}$. By using the orthogonal basis consisting of the matrices $\boldsymbol{E}_i = \lambda_i^*(\boldsymbol{A})$, $i = 0, 1, \ldots, d$, we have

$$\widetilde{\boldsymbol{A}_h} = \sum_{i=0}^{d} \frac{\langle \boldsymbol{A}_h, \boldsymbol{E}_i\rangle}{\|\boldsymbol{E}_i\|^2}\boldsymbol{E}_i = \sum_{i=0}^{d} \frac{1}{m_i}\left(\sum_{\partial(u,v)=h} (\boldsymbol{E}_i)_{uv}\right)\boldsymbol{E}_i = n\overline{\delta}_h \sum_{i=0}^{d} \frac{\overline{m}_{hi}}{m_i}\boldsymbol{E}_i.$$

Hence the orthogonal projection of $\boldsymbol{A}_h$ onto $\mathcal{A}$ is the matrix $q_h(\boldsymbol{A})$, where

$$q_h = n\overline{\delta}_h \sum_{i=0}^{d} \frac{\overline{m}_{hi}}{m_i}\lambda_i^*. \tag{10}$$

Since

$$\|\widetilde{\boldsymbol{A}_h}\|^2 = \langle q_h, q_h\rangle = n^2\overline{\delta}_h^2 \sum_{i=0}^{d} \frac{\overline{m}_{hi}^2}{m_i^2}\frac{m_i}{n} = n\overline{\delta}_h^2 \sum_{i=0}^{d} \frac{\overline{m}_{hi}^2}{m_i}$$

and $\|\boldsymbol{A}_h\|^2 = \overline{\delta}_h$, the upper bound on $\overline{\delta}_h$ follows from $\|\widetilde{\boldsymbol{A}_h}\| \le \|\boldsymbol{A}_h\|$. Moreover, Pythagoras's theorem says that the scalar condition $\|\widetilde{\boldsymbol{A}_h}\| = \|\boldsymbol{A}_h\|$ is equivalent to $\boldsymbol{A}_h \in \mathcal{A}$ and hence to G being h-punctually distance-polynomial. Moreover, it shows that if G is punctually distance-polynomial, then $\boldsymbol{A}_h = q_h(\boldsymbol{A})$, with q_h as given in (10). It follows from Lemma 3.1 that G_h is regular of degree $\overline{\delta}_h = \delta_h = q_h(\lambda_0)$. Moreover, from (10) it follows that $q_h(\lambda_i) = n\overline{\delta}_h\frac{\overline{m}_{hi}}{m_i}$, and this gives the required expression for $\overline{m}_{hi}$. $\square$

Let $\overline{a}_h^{(\ell)}$ be the average number of walks of length ℓ between vertices at distance $h \le D$, and recall from (5) that the leading coefficient ω_h of p_h satisfies $\omega_h^{-1} = \gamma_1\gamma_2\cdots\gamma_h$. Now the following results are variations of Proposition 4.1 for punctual distance-regularity.

Proposition 4.2 *Let $h \le D$. Then*

$$\overline{\delta}_h \le \frac{p_h(\lambda_0)}{[\omega_h \overline{a}_h^{(h)}]^2}$$

with equality if and only if G is h-punctually distance-regular, which is the case if and only if $\overline{a}_h^{(h)} = \gamma_1\gamma_2\cdots\gamma_h$ and $\overline{\delta}_h = p_h(\lambda_0)$.

Proof. First, observe that

$$\langle \boldsymbol{A}_h, p_h(\boldsymbol{A})\rangle = \frac{1}{n}\sum_{\partial(u,v)=h}(p_h(\boldsymbol{A}))_{uv} = \frac{\omega_h}{n}\sum_{\partial(u,v)=h} a_{uv}^{(h)} = \omega_h\overline{\delta}_h\overline{a}_h^{(h)}.$$

Thus, the orthogonal projection of $\boldsymbol{A}_h$ onto $\langle p_h(\boldsymbol{A})\rangle$ is $\breve{\boldsymbol{A}}_h = \frac{\omega_h\overline{\delta}_h\overline{a}_h^{(h)}}{p_h(\lambda_0)}\boldsymbol{p}_h(\boldsymbol{A})$, and

$$\frac{[\omega_h\overline{\delta}_h\overline{a}_h^{(h)}]^2}{p_h(\lambda_0)} = \|\breve{\boldsymbol{A}}_h\|^2 \le \|\boldsymbol{A}_h\|^2 = \overline{\delta}_h$$

gives the claimed inequality for $\overline{\delta}_h$ (alternatively, it follows from Cauchy-Schwarz). As before, it is clear that equality holds if and only if $\boldsymbol{A}_h = \breve{\boldsymbol{A}}_h$. Using Lemma 3.1, this is equivalent to $\boldsymbol{A}_h = p_h(\boldsymbol{A})$ (G being h-punctually distance-regular). Equality thus implies that $\overline{\delta}_h = p_h(\lambda_0)$ and hence that $\overline{a}_h^{(h)} = \omega_h^{-1} = \gamma_1\gamma_2\cdots\gamma_h$. To complete the argument, note that the latter implies that equality holds in the inequality. □

The bound of Proposition 4.1 is more restrictive than that of Proposition 4.2. This follows from the fact that $\boldsymbol{A}_h$ and $\widetilde{\boldsymbol{A}_h}$ have the same projection $\breve{\boldsymbol{A}}_h$ onto $\langle p_h(\boldsymbol{A})\rangle$, and hence that $\|\breve{\boldsymbol{A}}_h\| \le \|\widetilde{\boldsymbol{A}_h}\| \le \|\boldsymbol{A}_h\|$. This means that the bound of Proposition 4.1 is sandwiched between the average degree of G_h and the bound of Proposition 4.2. Thus, the tighter the latter bound is, the tighter the first one is. For a better comparison of the bounds, notice that a simple computation gives that

$$\overline{a}_h^{(h)} = \sum_{i=0}^{d}\overline{m}_{hi}\lambda_i^h = \frac{1}{\omega_h}\sum_{i=0}^{d}\overline{m}_{hi}p_h(\lambda_i) \qquad (i = 0, 1, \dots, d).$$

We thus find that

$$\overline{\delta}_h \le \frac{1}{n}\left(\sum_{i=0}^{d}\frac{\overline{m}_{hi}^2}{m_i}\right)^{-1} \le \frac{p_h(\lambda_0)}{\omega_h^2}\left(\sum_{i=0}^{d}\overline{m}_{hi}\lambda_i^h\right)^{-2} = p_h(\lambda_0)\left(\sum_{i=0}^{d}\overline{m}_{hi}p_h(\lambda_i)\right)^{-2}.$$

As we shall see in more detail in the next section, Proposition 4.2 is a generalization of the spectral excess theorem, at least if we combine it with Corollary 3.8. For the next proposition this is also the case; by considering the case $h = D = d$.

Proposition 4.3 *Let $h \le D$ and let G be such that $\langle p_i(\boldsymbol{A}), \boldsymbol{A}_h\rangle = 0$ for $i = h+1, \dots, d$. Then $\overline{\delta}_h \le p_h(\lambda_0)$ with equality if and only if G is h-punctually distance-regular.*

Proof. The orthogonal projection of $\boldsymbol{A}_h$ onto $\mathcal{A}$ is

$$\begin{aligned}\widetilde{\boldsymbol{A}_h} &= \sum_{i=0}^{d}\frac{\langle \boldsymbol{A}_h, p_i(\boldsymbol{A})\rangle}{\|p_i(\boldsymbol{A})\|^2}p_i(\boldsymbol{A}) = \frac{\langle \boldsymbol{A}_h, p_h(\boldsymbol{A})\rangle}{\|p_h(\boldsymbol{A})\|^2}p_h(\boldsymbol{A}) = \frac{\langle \boldsymbol{A}_h, H(\boldsymbol{A})\rangle}{\|p_h(\boldsymbol{A})\|^2}p_h(\boldsymbol{A}) \\ &= \frac{\langle \boldsymbol{A}_h, \boldsymbol{J}\rangle}{p_h(\lambda_0)}p_h(\boldsymbol{A}) = \frac{\langle \boldsymbol{A}_h, \boldsymbol{A}_h\rangle}{p_h(\lambda_0)}p_h(\boldsymbol{A}) = \frac{\overline{\delta}_h}{p_h(\lambda_0)}p_h(\boldsymbol{A}).\end{aligned}$$

We have $\|\boldsymbol{A}_h\|^2 = \overline{\delta}_h$ and $\|\widetilde{\boldsymbol{A}_h}\|^2 = \dfrac{\overline{\delta}_h^2}{p_h(\lambda_0)}$. From $\|\widetilde{\boldsymbol{A}_h}\| \leq \|\boldsymbol{A}_h\|$, we obtain $\overline{\delta}_h \leq p_h(\lambda_0)$. From Pythagoras's theorem, equality gives $\boldsymbol{A}_h = \widetilde{\boldsymbol{A}_h} = p_h(\boldsymbol{A})$. □

By projection onto $\mathcal{D}$ we obtain the following 'dual' result.

Proposition 4.4 *Let $h \leq D$ and let G be such that $\langle p_h(\boldsymbol{A}), \boldsymbol{A}_i\rangle = 0$ for $i = 0, \ldots, h-1$. Then $\overline{\delta}_h \geq p_h(\lambda_0)$ with equality if and only if G is h-punctually distance-regular.*

Proof. We now consider the orthogonal projection $\widehat{p_h(\boldsymbol{A})}$ of $p_h(\boldsymbol{A})$ onto $\mathcal{D}$:

$$\begin{aligned}\widehat{p_h(\boldsymbol{A})} &= \sum_{i=0}^{D} \frac{\langle p_h(\boldsymbol{A}), \boldsymbol{A}_i\rangle}{\|\boldsymbol{A}_i\|^2}\boldsymbol{A}_i = \sum_{i=0}^{h} \frac{\langle p_h(\boldsymbol{A}), \boldsymbol{A}_i\rangle}{\|\boldsymbol{A}_i\|^2}\boldsymbol{A}_i = \frac{\langle p_h(\boldsymbol{A}), \boldsymbol{A}_h\rangle}{\|\boldsymbol{A}_h\|^2}\boldsymbol{A}_h \\ &= \frac{\langle p_h(\boldsymbol{A}), \boldsymbol{J}\rangle}{\overline{\delta}_h}\boldsymbol{A}_h = \frac{\langle p_h(\boldsymbol{A}), p_h(\boldsymbol{A})\rangle}{\overline{\delta}_h}\boldsymbol{A}_h = \frac{p_h(\lambda_0)}{\overline{\delta}_h}\boldsymbol{A}_h.\end{aligned}$$

From this we now obtain that $\frac{(p_h(\lambda_0))^2}{\overline{\delta}_h} = \|\widehat{p_h(\boldsymbol{A})}\|^2 \leq \|p_h(\boldsymbol{A})\|^2 = p_h(\lambda_0)$, and hence that $\overline{\delta}_h \geq p_h(\lambda_0)$. Moreover, equality gives $\boldsymbol{A}_h = \widehat{p_h(\boldsymbol{A})} = p_h(\boldsymbol{A})$. □

From the latter two propositions, we obtain the following result.

Corollary 4.5 *Let $h \leq D$. Then G is h-punctually distance-regular if and only if $\langle p_h(\boldsymbol{A}), \boldsymbol{A}_i\rangle = 0$ for $i = 0, \ldots, h-1$ and $\langle p_i(\boldsymbol{A}), \boldsymbol{A}_h\rangle = 0$ for $i = h+1, \ldots, d$.*

5 Graphs with spectrally maximum diameter

In this section we focus on the important case of graphs with spectrally maximum diameter $D = d$. Distance-regular graphs are examples of such graphs. In this context, we first recall the following characterizations of distance-regularity. We include a new proof for completeness.

Proposition 5.1 (Folklore) *The following statements are equivalent:*

(i) *G is distance-regular,*

(ii) *$\mathcal{D}$ is an algebra with the ordinary product,*

(iii) *$\mathcal{A}$ is an algebra with the Hadamard product,*

(iv) *$\mathcal{A} = \mathcal{D}$.*

Proof. We already observed in Section 2.3 that (i) and (iv) are equivalent, and that these imply (ii) and (iii). So we only need to prove that both (ii) and (iii) imply (iv).
(ii) $\Rightarrow$ (iv): As $\boldsymbol{A} = \boldsymbol{A}_1 \in \mathcal{D}$, we have that $\boldsymbol{A}^k \in \mathcal{D}$ for any $k \geq 0$. Thus, $\mathcal{A} \subset \mathcal{D}$ and, since $\dim \mathcal{A} = d+1 \geq D+1 = \dim \mathcal{D}$, we get $\mathcal{A} = \mathcal{D}$.
(iii) $\Rightarrow$ (iv): As $\boldsymbol{E}_i \circ \boldsymbol{A}^j \in \mathcal{A}$, we have that $\boldsymbol{E}_i \circ \boldsymbol{A}^j = q_{ji}(\boldsymbol{A})$ for some polynomial q_{ji}, and this polynomial clearly has degree at most j. Let ψ_{ji} be the coefficient of x^j in q_{ji}, then it follows that $(\boldsymbol{E}_i)_{uv}(\boldsymbol{A}^j)_{uv} = \psi_{ji}(\boldsymbol{A}^j)_{uv}$ for vertices u, v at distance j, and hence that $(\boldsymbol{E}_i)_{uv} = \psi_{ji}$. It thus follows that $\boldsymbol{E}_i = \sum_j \psi_{ji}\boldsymbol{A}_j \in \mathcal{D}$. Therefore $\mathcal{A} \subset \mathcal{D}$ and, as before, we obtain $\mathcal{A} = \mathcal{D}$. □

5.1 Partially distance-regular graphs

We already observed in Section 3.1 that if a graph with $D = d$ is h-punctually distance-polynomial, then it is h-punctually distance-regular. The following, which is a bit stronger, is an immediate consequence of Lemmas 3.1 and 3.2.

Corollary 5.2 *Let $h \leq D$ and let G have spectrally maximum diameter $D = d$. Then $\boldsymbol{A}_h \in \mathcal{A}$ if and only if $p_h(\boldsymbol{A}) \in \mathcal{D}$, in which case $\boldsymbol{A}_h = p_h(\boldsymbol{A})$.*

It is also clear that if a graph with $D = d$ is m-partially distance-polynomial, then it is m-partially distance-regular. If we let $\mathcal{A}_m = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^m\}$ and $\mathcal{D}_m = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m\}$, then we obtain the following by extending the previous corollary.

Corollary 5.3 *Let $m \leq D$ and let G have spectrally maximum diameter $D = d$. Then the following statements are equivalent: G is m-partially distance-regular, $\mathcal{D}_m \subset \mathcal{A}$, $\mathcal{A}_m \subset \mathcal{D}$, and $\mathcal{A}_m = \mathcal{D}_m$.*

5.2 Punctually walk-regular graphs

Graphs with spectrally maximum diameter turn out to be d-punctually walk-regular. This will be used in the next section to show the relation of Propositions 4.1 and 4.2 to the spectral excess theorem.

Proposition 5.4 *Let G have spectrally maximum diameter $D = d$. Then it is both d-punctually walk-regular and d-punctually spectrum-regular with parameters*

$$a_d^{(d)} = \frac{\pi_0}{n} = \gamma_1\gamma_2\cdots\gamma_d, \qquad m_{di} = (-1)^i \frac{\pi_0}{n\pi_i} \qquad (i = 0, \ldots, d).$$

If G is bipartite, then it is both $(d-1)$-punctually walk-regular and $(d-1)$-punctually spectrum-regular with parameters

$$a_{d-1}^{(d-1)} = \frac{\pi_0}{n\delta} = \gamma_1\gamma_2\cdots\gamma_{d-1}, \qquad m_{d-1,i} = (-1)^i \frac{\pi_0}{n\pi_i}\frac{\lambda_i}{\delta} \qquad (i = 0, \ldots, d).$$

Proof. It follows from Lemma 3.11 and its proof that G is d-punctually walk-regular with $a_d^{(d)} = \frac{\pi_0}{n}$. The latter equals $\gamma_1\gamma_2\cdots\gamma_d$ by (5) and (7). Then by Lemma 3.10, G is also d-punctually spectrum-regular. Now observe that if u, v are vertices at distance d, then $m_{di} = (\boldsymbol{E}_i)_{uv} = \lambda_i^*(\boldsymbol{A})_{uv} = \frac{(-1)^i}{\pi_i} a_d^{(d)} = (-1)^i \frac{\pi_0}{n\pi_i}$.

If G is bipartite, then it follows from Lemmas 3.11 and 3.10 that G is $(d-1)$-punctually walk-regular and $(d-1)$-punctually spectrum-regular. Moreover, it is clear that $a_d^{(d)} = \delta a_{d-1}^{(d-1)}$, hence $a_{d-1}^{(d-1)} = \frac{\pi_0}{n\delta} = \gamma_1\gamma_2\cdots\gamma_{d-1}$ (because $\gamma_d = \delta$ for a bipartite graph). If u, v are vertices at distance d, then $\lambda_i m_{di} = (\lambda_i \boldsymbol{E}_i)_{uv} = (\boldsymbol{A}\boldsymbol{E}_i)_{uv} = \sum_{w\in\Gamma_1(u)\cap\Gamma_{d-1}(v)} (\boldsymbol{E}_i)_{wv} = \delta m_{d-1,i}$, hence $m_{d-1,i} = (-1)^i \frac{\pi_0}{n\pi_i}\frac{\lambda_i}{\delta}$. □

An example of an almost distance-regular graph that illustrates this proposition is the earlier mentioned graph F026A. It is bipartite with $D = d = 5$, hence it is h-punctually walk-regular for $h = 4, 5$. Moreover, this graph is 2-arc transitive, hence it is also 2-walk-regular (h-punctually walk-regular for $h = 0, 1, 2$). The intersection number c_3 is not well defined however, so the number of walks of length 3 between vertices at distance 3 is not constant either, and therefore the graph is not 3-punctually walk-regular.

5.3 From punctual to whole distance-regularity

We already observed that Proposition 4.3 and Corollary 3.8 together imply the spectral excess theorem. Proposition 5.4 shows that $\omega_d a_d^{(d)} = 1$, hence also Proposition 4.2 implies the spectral excess theorem (again, with Corollary 3.8). Finally, we will also show the connection of Proposition 4.1 to this theorem. To do this, we first restrict it to h-punctually spectrum-regular graphs with spectrally maximum diameter.

Proposition 5.5 *Let $h \le D$ and let G be h-punctually spectrum-regular with spectrally maximum diameter $D = d$. Then*

$$\bar{\delta}_h \le \frac{1}{n}\left(\sum_{i=0}^{d} \frac{m_{hi}^2}{m_i}\right)^{-1}$$

with equality if and only if G is h-punctually distance-regular, in which case the crossed local multiplicities are $m_{hi} = \frac{p_h(\lambda_i)}{p_h(\lambda_0)}\frac{m_i}{n}$, $i = 0, \ldots, d$.

Notice that every (not necessarily regular) graph is 0-punctually distance-regular and 1-punctually distance-regular, because $\boldsymbol{A}_0 = \boldsymbol{I} \in \mathcal{A}$ and $\boldsymbol{A}_1 = \boldsymbol{A} \in \mathcal{A}$. However, in general a graph is neither 0-punctually spectrum-regular nor 1-punctually spectrum-regular. If we apply Proposition 5.5 for $h = 0, 1$ though, then we obtain reassuring results. Indeed, if G is 0-punctually spectrum-regular then $m_{0i} = \frac{m_i}{n}$, and

$$\bar{\delta}_0 = \frac{1}{n}\left(\sum_{i=0}^{d} \frac{m_{0i}^2}{m_i}\right)^{-1} = \frac{1}{n}\left(\sum_{i=0}^{d} \frac{m_i}{n^2}\right)^{-1} = n\left(\sum_{i=0}^{d} m_i\right)^{-1} = 1.$$

If G is 1-punctually spectrum-regular then $m_{1i} = \frac{\lambda_i}{\lambda_0}\frac{m_i}{n}$ by Proposition 3.12, and indeed

$$\overline{\delta}_1 = \frac{1}{n}\left(\sum_{i=0}^{d}\frac{m_i\lambda_i^2}{n^2\lambda_0^2}\right)^{-1} = n\lambda_0^2\left(\sum_{i=0}^{d}m_i\lambda_i^2\right)^{-1} = n\lambda_0^2\,(n\lambda_0)^{-1} = \lambda_0.$$

The most interesting result we obtain of course for $h = d$ $(= D)$. By Proposition 5.4, G is d-punctually spectrum-regular with $m_{di} = (-1)^i\frac{\pi_0}{n\pi_i}$. Then the condition of Proposition 5.5 for d-punctual distance-regularity (and hence distance-regularity; we again use Corollary 3.8) becomes

$$\overline{\delta}_d = \frac{1}{n}\left(\sum_{i=0}^{d}\frac{m_{di}^2}{m_i}\right)^{-1} = \frac{1}{n}\left(\sum_{i=0}^{d}\frac{\pi_0^2}{n^2\pi_i^2 m_i}\right)^{-1} = \frac{n}{\pi_0^2}\left(\sum_{i=0}^{d}\frac{1}{m_i\pi_i^2}\right)^{-1},$$

which corresponds to the condition of the spectral excess theorem for a (regular) graph to be distance-regular, as the right hand side of the equation is known as an easy expression for $p_d(\lambda_0)$ in terms of the eigenvalues.

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