Representations of choice situations

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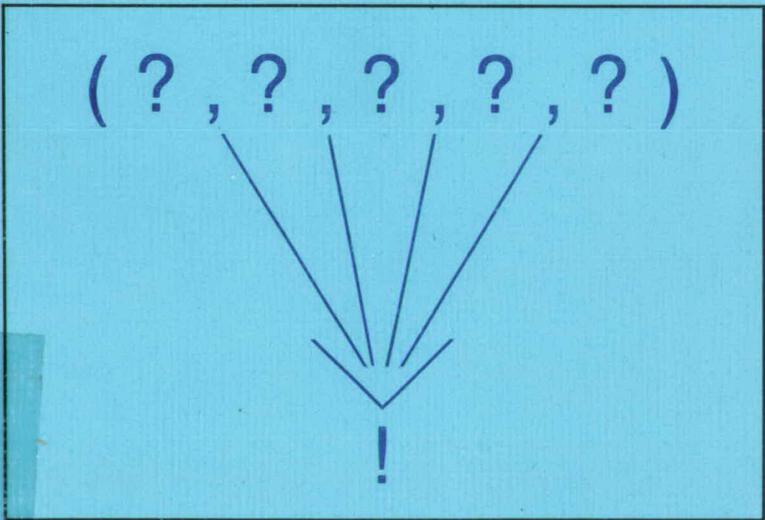

P.P. WAKKER

REPRESENTATIONS OF CHOICE SITUATIONS

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PROEFSCHRIFT

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Trees van der Eem did excellent typewriting, in an encouraging way. Bryan D. Williams corrected the English text.

CONTENTS

PREFACE AND SUMMARY

In decision theories the assumption is usually made that a decision maker maximizes some quantitative goal function. Depending on the context, such a function may be called a profit function, utility function, representing function (this will be our term), etc., and is usually assumed to possess certain desirable properties, such as continuity, concavity, etc.

The purpose of this monograph is to show a way to make the abovementioned assumption operational. To this end, choice behaviour of the decision maker is taken as observable primitive. In Chapter I we shall give the conditions under which choice behaviour can be represented by a preference relation. This preference relation then will be taken as primitive in the following chapters, in the formulation of the so-called "representation theorems" given there.

After specification of the presupposed context, these representation theorems will show equivalence of (usually) two statements. The first statement, numbered (i), says that a representing function, with certain desirable properties, exists. The second statement, numbered (ii), characterizes statement (i), i.e. gives the properties of the preference relation, necessary and sufficient for the truth of (i). Thus statement (ii) gives the criteria for verification/justification, or falsification/criticism, of the assumption that the desired representing function in (i) exists. At the end of the representation theorems usually so-called "uniqueness results" are listed, i.e. results which describe in how far a representing function in (i) is uniquely determined.

We have as much as possible formulated the representation theorems in such a way that the reader can understand them without consulting other parts of the text. The proofs of these representation theorems not only show the existence of representing functions, but they also indicate how to construct the (quantitative) representing functions from the (qualitative) information that is reflected by the preference relation.

The main subject of study in this monograph is subjective expected utility maximization, in the context of decision making under uncertainty. Subjective expected utility maximization is notorious for the many vivid discussions about its appropriateness. The first well-known representation theorems for (subjective) expected utility maximization, in von Neumann and Morgenstern (1944), and Savage (1954), have had great influence in economic literature, and have shocked the statistical literature because of their profound implications for the foundations of statistics. It should be emphasized that representation theorems as such are not only useful for advocates of the use of some special kind of representing function, but just as well give the operational tools for criticisms. The independence condition of von Neumann and Morgenstern, and the sure-thing principle of Savage, gave valuable tools to critics, see for instance Allais (1953, 1979).

The theorems of Savage, and von Neumann and Morgenstern, (and Anscombe and Aumann, 1963,) apply to special circumstances, where the state space is well structured, or where many lotteries are available. Such special circumstances are usually not present in economic contexts. The main purpose of this monograph is to provide representation theorems for subjective expected utility maximization, under special circumstances that are usually present in economic contexts.

First, in Chapter 0, we give some elementary definitions.

In Chapter I we relate preference relations to choice behaviour by means of the "revealed preference" approach, which has originated from consumer demand theory. In order to achieve maximal operationality, we define our "revealed preference" relations slightly differently from the way most usual in literature; and we derive the characterizations with the aid of these. For intuitive purposes, choice behaviour in our view is a more appropriate primitive for decision theory, than a preference relation. Hence we discuss the "paradigm" of decision theory in terms of choice behaviour, in Chapter I.

In Chapter I we do not assume any structure (other than set-theoretic) on the set of alternatives. In the following chapters, more and more structure will be introduced on the set of alternatives. Then not only the preference relation, but also this structure on the set of alternatives, will be considered observable. No structure will ever be introduced which is not present in our main intended application: the one where the set of alternatives is a Euclidean space.

One reason to consider spaces, more general than Euclidean spaces, is to increase applicability. With the exception of section VII.6, all of our work is applicable to decision situations where no (physical) quantification of the alternatives is available. A second reason to consider general spaces is that, even if the only ultimate purpose is to obtain theorems for Euclidean spaces, then theorems for more general spaces may still have value as intermediate means. For example, if we would have formulated the main result of Chapter VI, Theorem VI.5.1, for Euclidean spaces only, then in its proof (Proposition VI.7.4, and subsection VI.7.2), we would still have needed the results of Chapter III for more general topological spaces.

In Chapter II the structure is introduced which will be the central subject of study of this monograph: the set of alternatives is assumed to be a cartesian product. Each coordinate of an alternative describes a relevant aspect. For making his decisions, the decision maker is to weigh the advantages and disadvantages of the several aspects against each other. The cartesian product structure plays a central role, and our work may find application, in very many fields of science. Section II.1 gives six economic examples, amongst them decision making under uncertainty.

In sections II.2 to II.5 we study monotonicity properties. An alternative which is best in each aspect, should be the best alternative, by monotonicity. In section II.6 we take up the approach, followed in the remainder of this monograph: the only preference relation, taken as observable, is the one on the set of alternatives. In section II.6 we then show that the only observable implication of the monotonicity properties is "coordinate independence".

Sections II.2 to II.6 are included, firstly because they contain new material that unifies the many versions of monotonicity occurring in literature; and secondly, because we think these sections give the most appropriate way to gain comprehension of coordinate independence, a property central for all of the remainder of this monograph.

In Chapter III topological structure is introduced. We assume that the set of alternatives is endowed with a connected product topology. From then on, in all our main theorems, the preference relation will be continuous and complete (, either as a presupposition, or as a consequence of other suppositions). Section III.1 gives some comments on the fact that the properties of continuity and completeness are of a technical nature, and are not fully operational.

In sections III.3 and III.4 we characterize the existence of continuous additively decomposable representing functions. Our results generalize some well-known theorems from literature.

In Chapter IV, a further structural assumption is added. It is assumed that all coordinate sets are identical. (This assumption will be dropped in sections VII.1 to VII.4 only.) Theorem IV.3.3 gives a main result of this monograph: a characterization of subjective expected utility maximization by means of a new property for preference relations: cardinal coordinate independence. Let us, for the moment, take for granted the, in economic contexts common, assumptions of continuity of the utility function, and continuity, completeness, and transitivity of the preference relation. Then Theorem IV.3.3 shows that subjective expected utility maximization can be justified (; or verified; or criticized; or falsified) if and only if cardinal coordinate independence of the preference relation can be. This is all done under the assumption that the state space is finite. The adaptation to infinite state spaces will be given in Chapter V.

In the remainder of Chapter IV, and in Chapters V and VI, many generalizations of Theorem IV.3.3 are obtained. Also applications to contexts other than decision making under uncertainty are given. For instance we give, for dynamic contexts, alternative characterizations of a representation, characterized before by Koopmans (1972).

The main result of Chapter V, Theorem V.6.1, adapts the results of Chapter IV to infinite state spaces. Thus it provides the most general characterization of subjective expected utility maximization with continuous utility, presently available in literature. This is done both for finitely additive, and for countably additive, probability measures.

Chapter VI extends Theorem IV.3.3 to "capacities", i.e. "nonadditive probability measures". The use of nonadditive measures has been initiated by Schmeidler (1984 a,b), where motivations concerning decision making under uncertainty are also given. Further the applicability to welfare theory has given motivation. Our contribution to Schmeidler's work is like the contribution of our Theorem IV.3.3 to a theorem of Anscombe and Aumann (1963): we replace the restrictive assumption that many lotteries are available, by the restrictive assumption that utility is continuous. Section VI.11 characterizes strong sub- or superadditivity of the involved capacities.

In Chapter VII a further structure on the set of alternatives is added: a mixture-space-structure. Again, the most well-known examples are convex subsets of linear spaces. We use this structure to define, and characterize, *concave* additively decomposable representing functions, by means of the "concavity assumption". Such (representing) functions are frequently used in mathematical programming, consumer and producers theory, and decision making under uncertainty (to characterize risk aversion). Still no characterization of them was yet available in literature.

In section VII.6 we assume that the coordinate sets are convex subsets of the set of real numbers. Thus here the alternative sets of this monograph, endowed with most structure, are dealt with. In section VII.6 it is then shown that assumptions on (nonincreasing) risk aversion, current in economic literature, simplify in a surprising way the characterization of subjective expected utility maximization.

Finally, Chapter VIII gives some mathematical results on functions on intervals, used, and referred to, at many places in this monograph.

For the most part, for the understanding of chapters, consultation of elementary definitions in previous chapters is sufficient. Only Sections III.2, III.3, IV.2, IV.3, and perhaps II.1, are needed for understanding of the sequel.

CHAPTER 0

ELEMENTARY DEFINITIONS AND NOTATIONS

In this chapter we give elementary definitions and notations. The reader familiar with them may wish to skip this chapter, or only look at the standard notations at the end, and may in case of doubt consult this chapter by way of the subject index.

A *binary relation* on a set X is a subset of $X \times X$. For a binary relation $\succcurlyeq$ on X we usually write $x \succcurlyeq y$ instead of $(x,y) \in \succcurlyeq$. One binary relation $\succcurlyeq$ *extends* another binary relation $\succcurlyeq'$, if $\succcurlyeq \supset \succcurlyeq'$.

A binary relation $\succcurlyeq$ on X is:

(a) *transitive* if $[x \succcurlyeq y$ and $y \succcurlyeq z] \Rightarrow [x \succcurlyeq z]$ for all $x,y,z \in X$.

(b) *complete* if $x \succcurlyeq y$ or $y \succcurlyeq x$ for all $x,y \in X$.

(c) *reflexive* if $x \succcurlyeq x$ for all $x \in X$.

(d) *irreflexive* if not $[x \succcurlyeq x]$ for all $x \in X$.

(e) *symmetric* if $[x \succcurlyeq y] \Rightarrow [y \succcurlyeq x]$ for all $x,y \in X$.

(f) *asymmetric* if $[x \succcurlyeq y] \Rightarrow$ not $[y \succcurlyeq x]$ for all $x,y \in X$.

(g) *antisymmetric* if $[x \succcurlyeq y$ and $y \succcurlyeq x] \Rightarrow [x = y]$ for all $x,y \in X$.

Throughout this monograph $\succ$ denotes the *asymmetric part* of $\succcurlyeq$ (i.e. $x \succ y$ iff[1] $x \succcurlyeq y$ and not $y \succcurlyeq x$) and $\approx$ is the *symmetric part* of $\succcurlyeq$ (i.e. $x \approx y$ iff $x \succcurlyeq y$ and $y \succcurlyeq x$). Further notations are $x \preccurlyeq y$ for $y \succcurlyeq x$, and $x \prec y$ for $y \succ x$. If a binary relation $\succcurlyeq$ is endowed with

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1) Iff: if and only if.

indexes, then without further mention $>$, $\approx$, $\prec$, and $<$, when endowed with the same indexes, are defined analogously.

A *weak order* $\succcurlyeq$ is transitive and complete. Hence it is also reflexive, its symmetric part is an *equivalence relation* (i.e. is transitive, reflexive, and symmetric), its asymmetric part is transitive, irreflexive, asymmetric, and we have $x \succ y$ iff not $y \succcurlyeq x$.

Some further terminology: $\succcurlyeq$ is *trivial* if $x \succcurlyeq y$ for all x,y, it is the *identity(relation)* if $x \succcurlyeq y \Leftrightarrow x = y$. The identity can of course also be considered as a *function*. A pair of elements x,y of X is *incomparable* (w.r.t. $\succcurlyeq$) if neither $x \succcurlyeq y$ nor $y \succcurlyeq x$.

In Chapter I we shall deal with <u>choice functions</u>. A *choice function* C is a function from a collection $\mathcal{D}$ of subsets of a set X, to 2^X, such that $\emptyset \neq C(D) \subset D$ for all $D \in \mathcal{D}$. Of course this implies $\emptyset \notin \mathcal{D}$.

An <u>*interval*</u> V is a subset of $\mathbb{R}$ that is convex ($\mu,\nu \in V$, $0 \leq \lambda \leq 1$, then $\lambda\mu + (1-\lambda)\nu \in V$), and that may be open, closed, or half-open, and bounded or unbounded, both from the left and the right. By $[\mu,\nu]$ we denote the interval $\{\lambda \in \mathbb{R} : \mu \leq \lambda \leq \nu\}$, by $]\mu,\nu[$ the interval $\{\lambda \in \mathbb{R} : \mu < \lambda < \nu\}$. A *nondegenerate interval* is an interval with more than one (so infinitely many) elements. $\mathbb{R}_+ = \{\mu \in \mathbb{R} : \mu \geq 0\}$, $\mathbb{R}_{++} = \{\mu \in \mathbb{R} : \mu > 0\}$.

Now let V be an arbitrary subset of $\mathbb{R}$, and let $\phi : V \to \mathbb{R}$. Then ϕ is *strictly increasing* if, for all $\mu > \nu$ in V, $\phi(\mu) > \phi(\nu)$; it is *strictly decreasing* if, for all $\mu > \nu$ in V, $\phi(\mu) < \phi(\nu)$. Furthermore ϕ is *nondecreasing* if, for all $\mu > \nu$ in V, $\phi(\mu) \geq \phi(\nu)$; ϕ is *nonincreasing* if, for all $\mu > \nu$ in V, $\phi(\mu) \leq \phi(\nu)$.

The function ϕ as above is *convex* if, for all $0 \leq \lambda \leq 1$, and μ,ν, and $\lambda\mu + (1-\lambda)\nu$ in V, $\phi(\lambda\mu + (1-\lambda)\nu) \leq \lambda\phi(\mu) + (1-\lambda)\phi(\nu)$; ϕ is *concave* if $-\phi$ is convex; and ϕ is *affine* if it is both convex and concave. The function ϕ is affine iff there exist real σ,τ such that $\phi : \mu \mapsto \sigma\mu + \tau$. Note that we also allow $\sigma = 0$. Further ϕ is *positive affine* if σ above is positive. The function ϕ is *quasiconvex* if, for all $0 \leq \lambda \leq 1$, and μ, ν, and $\lambda\mu + (1-\lambda)\nu$ in V, $\phi(\lambda\mu + (1-\lambda)\nu) \leq \max\{\phi(\mu), \phi(\nu)\}$; ϕ is *quasiconcave* if $-\phi$ is quasiconvex. A convex function is quasiconvex, a concave function is quasiconcave.

Let (X,T) be a topological space. X is *(topologically) separable* if there exists a countable dense subset of X. For $E \subset X$, $int(E)$ is the topological interior of E. X is *connected* if there do not exist open nonempty subsets V, W of X such that $V \cap W = \emptyset$, $V \cup W = X$. This is iff no closed nonempty subsets V, W of X exist such that $V \cap W = \emptyset$, $V \cup W = X$, i.e. iff the only subsets of X, which are both open and closed, are $\emptyset$ and X. If X is connected, and g is a continuous function from X to another topological space, then $g(X)$ is connected too. X is *arcwise connected*, or *arcconnected* as we shall usually write, if, for every $x,y \in X$, there exists an *arc* from x to y, i.e. a continuous function $\phi : [0,1] \to X$ with $\phi(0) = x$, $\phi(1) = y$. If X is arcconnected, then it is connected.

If X is a cartesian product $\times_{i\in I}C_i$, where every C_i is endowed with a topology T_i, then the *product topology* on X is the smallest topology containing all subsets of X of the form $E_i \times (\times_{j\neq i}C_j)$ with $i \in I$, $E_i \in T_i$. An elementary result for this:

LEMMA 0.1. *Let* $E \subset X = \times_{i\in I}C_i$ *be open [respectively closed] with respect to the product topology on* X. *Let* $A \subset I$, $z \in X$. *Then*

$$V := \{x_A \in \times_{i\in A}C_i : E \text{ contains the element } v \text{ of } X \text{ which has } v_i = x_i \text{ for all } i \in A,\ v_i = z_i \text{ for all } i \notin A\}$$

is open [respectively closed] with respect to the product topology on $\times_{i\in A}C_i$.

PROOF. Let $x_A \in V$. There must exist open E_i, for all $i \in I$, with $E_i \neq C_i$ for only finite many i, such that the v, as defined above, is in $\times_{i\in I}E_i$, and such that the latter is a subset of E. We see that $x_A \in \times_{i\in A}E_i \subset V$. Only finite many E_i's being different from C_i, $\times_{i\in A}E_i$ is an open neighbourhood of x_A within V.

□

Next we give some measure-theoretic definitions. A collection $\mathcal{A}$ of subsets of a set I is an *algebra* if $I \in \mathcal{A}$, and for all $A,B \in \mathcal{A}$ also A^c and $A \cup B \in \mathcal{A}$. Then $\emptyset \in \mathcal{A}$, and for all $A_1,\ldots,A_m \in \mathcal{A}$ also $\cup_{j=1}^m A_j$ and $\cap_{j=1}^m A_j$ are in $\mathcal{A}$. $\mathcal{A}$ is a *σ-algebra* if furthermore, for all $(A_j)_{j=1}^\infty \in \mathcal{A}$, $\cup_{j=1}^\infty A_j$ is in $\mathcal{A}$. A function P on an algebra $\mathcal{A}$ is a *probability measure* if $P(I) = 1$, and if furthermore P is *finitely additive*, i.e. for all disjoint $A, B \in \mathcal{A}$, $P(A \cup B) = P(A) + P(B)$. Note that we do not assume "σ-additivity". P is *σ-additive* (, or *countably additive*,) if, for any $(A_j)_{j=1}^{\infty} \in \mathcal{A}$ with $A_{m+1} \subset A_m$ for all m and $\cap_{m=1}^{\infty} A_m = \emptyset$, we have $\lim_{m\to\infty} P(A_m) = 0$. If I is finite, say $I = \{1,\ldots,n\}$, then we usually assume, without further mention, that $\mathcal{A} = 2^I$. Note that then the probability measure P is completely determined by $(p_j)_{j=1}^{n}$, with $p_j := P(\{j\})$ for all j. Finally, a *partition* $\mathcal{P} = (A_1,\ldots,A_m)$ of a set I is a sequence of disjoint subsets of I, with union I. We do not exclude $A_j = \emptyset$ for some j's.

In a <u>cartesian product</u> $X_{i\in I}C_i$, I is called the *index set*, and the C_i's are *coordinate sets*. For an element $(x_i)_{i\in I}$ of such a cartesian product, x_i is the i-th *coordinate* of x. Other indexes than those referring to coordinates are usually indicated by superscripts.

Some <u>standard notations</u> in this monograph are the following. X is a nonempty set, elements of which are called *alternatives*, and are usually denoted by x, y, v, w, s, t, z, and sometimes by a, b, c, d. Usually a binary relation $\succcurlyeq$, called *preference relation*, is present on X. Then $x \succcurlyeq y$ is pronounced as: "x is weakly preferred to y", or: "x is at least as good as y"; $x \succ y$: "x is strictly preferred to y", or: "x is strictly better than y". And $x \approx y$: "x is equivalent to y" (even though in general $\approx$ does not have to be an equivalence relation), or: "x and y are equally good." In Chapters II to VII, X is a cartesian product $X_{i\in I}C_i$, and with the exception of Chapter V, I is the finite set $\{1,\ldots,n\}$, for some $n \in \mathbb{N}$. Often all C_i's equal a set C; then we also write α, β, γ, δ, and sometimes μ, ν, σ, τ, for elements of C. Subsets of I are usually denoted by A, B, C, D. By E, F, G, H we usually denote subsets of C, or X. Real numbers are usually denoted by Greek characters μ, ν, λ, σ, τ, or sometimes by a, b, c, d.

CHAPTER I

FROM CHOICE FUNCTIONS

TO BINARY RELATIONS

I.1. CHOICE FUNCTIONS, THEIR USE, AND INTERPRETATIONAL COMPLICATIONS

The following simple example of a choice problem will illustrate several questions to be addressed in the sequel.

I.1.1. EXAMPLE

Suppose a consumer T is in a fruit-store, and has to decide whether to buy nothing (n), an apple (a), or a pear (p). It is his custom to buy an apple if only apples are available (so to choose a from {a,n}), because he thinks apples look nice. Furthermore T prefers buying a pear to buying an apple (so he chooses p from {p,a}), because pears are more juicy than apples. Hence his first inclination is to buy a pear (so to choose p from {n,a,p}).

However, not sure about his true motives, T strongly imagines what his choice would be from {n,p}. There is no doubt: it would be n, T would not buy the pear, he does not like pears enough. T's point of view is: if from {n,a,p} I actually choose p, then from {n,p} I should also choose p! (I.e., T wants to satisfy IIA, see Definition I.2.8.) An introspection follows, and the conclusion is that the choice of a from {a,n} was not truly motivated. T rather chooses n from {a,n}. Hence finally n is chosen from {a,p,n}.

I.1.2. ELEMENTARY FORMALIZATIONS AND SOME ASSUMPTIONS

By T we denote a *decision maker*, T is usually assumed to be a single person. But also T may stand for an animal, a computer, an extraterrestrial being, a firm, a society, etc. In the example of subsection I.1.1, T was a consumer.

We study models for situations where from some nonempty set D of *(available) alternatives*, T chooses exactly one element. (This is modified in subsection I.1.4, to simplify work.) It is intended that T is completely *free* to choose the alternative which he wants. In the example D was $\{n,a,p\}$. In several special contexts there are special terms for alternatives, such as: options, prospects, acts, securities, allocations, strategies, commodity bundles, tests, estimators, responses, etc. If there is a possibility "choosing nothing", then we just represent this by an element of D, such as n above.

We shall not use sequential models. If analogous, or other, choice situations will (repeatedly) occur, and have significance for the one choice situation presently considered, then this significance should appear in the appropriate places, such as in descriptions and valuations of the alternatives. We neither assume, nor exclude, repetitions; it is only that they are not central in our study.

I.1.3. THOUGHT EXPERIMENTS

Although our work is intended to be applicable if decision maker T one time has to choose one element from one set D, this one choice is not enough to build a meaningful theory. To show the meaning of entities such as preference relations and utility functions, more decision situations must be considered, at least as thought experiments, and comparisons between them must be made. This is in fact what we do by working with choice functions, (and by considering binary relations as representations for choice functions).

It is very useful to imagine what would have happened if some actual problem at hand would have been different in this or that respect, to compare it to other analogous problems, and to base a model on this. This is a common practice in many sciences, it teaches one what the essential *parameters* of the problem are.

In the example of subsection I.1.1, not only the actual decision situation, with available alternatives {n,a,p}, is considered, but also situations with available alternatives {a,n}, {p,a}, {n,p}, and comparisons between these are made. If n is chosen from {n,p}, and p from {p,a}, then n should be chosen from {n,a,p}, so was supposed there. From the reasoning used here, and Corollary I.2.12, one may conclude that the preference relation of T, a weak order, is an essential *parameter*.

In this chapter we shall concentrate on decision situations that differ from the actual one with respect to the set of available alternatives. Usually in the hypothetical decision situations the set of available alternatives, D', is a subset of D, the one for the actual situation.

As usual in science, a "ceteris paribus" assumption must be made. We assume that the (hypothetical) cause, restricting D to D', does not change other relevant exogeneous aspects of the situation. For instance in subsection I.1.1 the restriction of {n,a,p} to {n,a} (say often the fruit-store has no pears in store) should not change the person that T is, his desires, his knowledge, etc. We consider IIA (see Definition I.2.8) and monotonicity (Chapter II) as concrete expressions of the ceteris paribus condition.

As usual, the supposed changes are described accurately, but the relevant things that should not be changed remain, at least for a part, unspecified. The more science proceeds, the more can be said about the "relevant things" to be controlled for the ceteris paribus condition.

Let us compare the above to classical mechanics. The formula of Newton, F = m.a (F force, m mass, a accelleration) is intended to be applicable in every single situation. Essential for its significance are comparisons to (hypothetical) analogous situations such as: if some (hypothetical) cause would make F twice as big, then also accelleration a should become twice as big. The ceteris paribus condition should anyway entail that m is kept constant.

Not always does the above doubling of F have to be only a hypothetical experiment. Sometimes it really can be achieved in an experiment. Such an experiment is a different event, happening at another time and/or place. Not only is F doubled, and a too; there is an infinity of other differences. These must then be assumed to concern irrelevant matters.

Also, for our work the other considered choice situations are not always only thought experiments. Also here they may have really occurred, or have been achieved in experiments. Still we use the term "thought experiment". A reason for this is to avoid confusion with *repetitions* of choice situations. The difference between thought experiments and repetitions is exposed in section I.3.

For the derivation of mathematical results it is often convenient to use infinite alternative sets (usually endowed with a topology) such as $\mathbb{R}^n_+$. Continuity assumptions can then be made to simplify the technical work and to give convenient uniqueness results. Thus sometimes hypothetical alternatives which were not present in the actual D, but which have informative properties, are introduced. Then the set X of all considered alternatives contains more elements than only those alternatives that are actually available in D.

Also it will sometimes be of use to assume that other exogeneous aspects of the choice situation can be varied. For instance for the binary relations $\succ_A$, to be introduced in Chapter II, it is useful to imagine that certain coordinates of the alternatives can be ignored. This may be because a consumer is completely satisfied with respect to the "commodities" corresponding to these coordinates; or because the extra information is obtained that the "states of nature", corresponding to these coordinates, are untrue.

I.1.4. THE PRELIMINARY-CHOICE-PROBLEM

For theoretical purposes it is convenient to consider the case where T may choose a nonempty subset from D, instead of just one element. Such a choice is called a *preliminary choice*, or just *choice* if no confusion arises. Thus for a choice function C, the C(D)'s may contain more than one element. C(D) is interpreted as the set of all elements from D, which T would be willing to choose. His finally chosen alternative is one arbitrary element from D, say $C_f(D)$. C_f is called a "selection function" in Basu (1980, p.50). See also Richter

(1971, page 31, third paragraph).

We shall be interested in C(D), and shall represent this, in the sequel. This meets the, admitted, problem, that not C(D), but only $C_f(D)$, is observable.

In normative applications of representation results, the consequences of the preliminary-choice-problem are not serious. A representation yielding the prescription to choose an alternative from C(D), without specifying which one, is not seriously deficient in this, because it does not matter which element is chosen. All elements of C(D) are equally good.

Far more serious are the consequences of the preliminary-choice-problem for descriptive applications. Here it can never be falsified from observed choice making, that T was completely indifferent ($C(D) = D$ for all $D \in \mathcal{D}$) and made all his choices arbitrarily. Here is a subject for further investigation, to derive "sensible" preference relations from observed choices $C_f(D)$, and to find out in how far the choices must have been arbitrary. Work like Cooke and Draaisma (1984), comparing numbers of arbitrary preference relations to numbers of preference relations with "nice" properties, can be useful for this. For predictive applications it is a disadvantage to obtain only the prediction that T will choose an element from C(D), and not the prediction which element that will be.

A way to circumvent the problem of preliminary choice is to simply communicate with T, and ask him what his C(D)'s are. This approach falls outside the scope of this monograph. We shall base our representations solely on choice behaviour.

I.2. <u>FROM CHOICE FUNCTIONS TO BINARY RELATIONS</u>

In this section we indicate how to represent choice functions by binary ("preference") relations.

I.2.1. THE CONGRUENCY PROPERTY

Let X be the nonempty set of all considered *alternatives*, $\mathcal{D} \subset 2^X \setminus \{\emptyset\}$ the nonempty collection of all considered *choice situations*. For $D \in \mathcal{D}$, elements of D are *available alternatives (with respect to* D*)*. We assume that $C : \mathcal{D} \to 2^X$ is a *choice function*, (see Chapter 0). C(D), the *choice set for* D, contains exactly those elements of D that T is willing to choose from D. Elements of C(D) are called *chosen alternatives (from* D*)*.

An example of this can be found in consumer demand theory. There T is a consumer, $X = \mathbb{R}^{\ell}_{+}$, alternatives are commodity bundles, choice situations are budget sets, the choice function is the demand multi-function, and the choice set is the demand set.

DEFINITION I.2.1. A binary relation $\succ$ on X *represents* C if $C(D) = \{x \in D : x \succ y \text{ for all } y \in D\}$ for all $D \in \mathcal{D}$.

We have chosen the term "represent" instead of the more customary term "rationalize" for the sake of unity of terminology in this monograph.

In the following chapters binary relations will be assumed to represent choice functions, and will be called *preference relations*. They may be interpreted to stand for T's opinion about alternatives. In literature it is custom to let a choice function stand for choice behaviour of T, more or less intended to actually take place, and to consider the possibility that T's preference relation does *not* represent his choice behaviour. If then the preference relation (notation $\succ$) does represent T's choice behaviour, $(X, \succ, \mathcal{D})$ can be called a "rationalization" of T's choice behaviour. In Ruys (1981) rationalizability is proposed as criterion for calling choice behaviour "rational". In von Wright (1963), preference relations are placed *between* the "anthropological" (acting) level and the "axiological" (assessing) level.

The following definition shows a way to derive binary relations from choice functions. Such relations are called "revealed preference relations".

DEFINITION I.2.2. We write xRy if there is a $D \in \mathcal{D}$ such that $x \in C(D)$, $y \in D$, or if $x = y$; we write xPy if there is a $D \in \mathcal{D}$ such that $x \in C(D)$, $y \in D\backslash C(D)$; we write xIy if there is a $D \in \mathcal{D}$ such that x and $y \in C(D)$, or if $x = y$.

The above definition does not forbid occurrence of both xRy and yPx. In example I.1.1,T originally considered $C\{a,n\} = \{a\}$, $C\{p,a\} = p$, $C\{p,a,n\} = p$, and $C\{n,p\} = n$. The last two choices give pPn, and nRp (even nPp).

There are many other ways to derive binary relations from C, see Sen (1971). Often first a relation analogous to R above is defined, and then P and I are defined as the asymmetric, respectively symmetric, part of R, see for instance Weddepohl (1970). We have chosen the above definitions to achieve maximal operationality. As soon as we observe $x \in C(D)$, $y \in D\backslash C(D)$ for some $D \in \mathcal{D}$, we can now conclude xPy. Had we defined xPy by "xRy and not yRx", then for verification of "not yRx" we would have had to observe the choices from *all* $D \in \mathcal{D}$, containing both x and y. This may be an impossible task if most of the choice situations,involved, are hypothetical (see subsection I.1.3). In the sequel we adapt the results of literature to our deviating definitions. Theorem I.2.5 (vi $\Leftrightarrow$ i there, and iv $\Leftrightarrow$ i) shows that one way to characterize the desired representation in (i) there, is to require that our deviating definition of P leads to the same $\bar{P}$ as in Weddepohl (1970), where $\bar{P}$ is *defined* to be the asymmetric part of $\bar{R}$.

DEFINITION I.2.3. We write $x\bar{R}y$ if there exists a finite sequence $(x^0, x^1, \ldots, x^n)$ such that $x^0 = x$, $x^n = y$, $x^j R\, x^{j+1}$ for all $0 \leq j \leq n-1$. We write $x\bar{P}y$ if a sequence $(x^j)_{j=0}^n$ as above exists, with furthermore $x^j P x^{j+1}$ for at least one $0 \leq j \leq n-1$. Finally, we write $x\bar{I}y$ if $x\bar{R}y$ and $y\bar{R}x$.

So $\bar{R}$ is the smallest transitive extension of R, $\bar{P}$ and $\bar{I}$ are transitive extensions of P, respectively I, but usually not the smallest.

DEFINITION I.2.4. The choice function C is *congruent* if, for all $x,y \in X$: $[x\bar{R}y \Rightarrow \text{not } yPx]$.

The congruency property, and the main result (i) ⇔ (iii) below, were first obtained by Richter (1966, Theorem 1).

THEOREM I.2.5. *For the choice function* C *the following six statements are equivalent:*

(i) *There exists a weak order* $\succ$, *representing* C.

(ii) *There exists a transitive* $\succ'$, *representing* C.

(iii) C *is congruent.*

(iv) $\bar{R}$ *represents* C, $\bar{P}$ *is the asymmetric,* $\bar{I}$ *the symmetric part of* $\bar{R}$.

(v) $\bar{R}$ *represents* C.

(vi) $\bar{P}$ *is the asymmetric part of* $\bar{R}$.

Furthermore, $\succ$ *of* (i), *and the smallest reflexive extension of* $\succ'$ *of* (ii), *are extensions of* $\bar{R}$, *their asymmetric parts are extensions of* $\bar{P}$, *and their symmetric parts of* $\bar{I}$.

PROOF. First the furthermore-statement. By the definition of R, $\succ$ of (i) and the reflexive extension of $\succ'$ of (ii) are extensions of R. By transitivity they are of $\bar{R}$. So their symmetric parts extend $\bar{I}$, the symmetric part of $\bar{R}$.

Now suppose $x\bar{P}y$. To prove $x \succ y$ and $x \succ' y$.
Let $x = x^0Rx^1 \ldots Rx^jPx^{j+1}R \ldots Rx^n = y$. We write $\succcurlyeq^*$ both for $\succ$ of (i) and for the reflexive extension of $\succ'$ of (ii). $x^j \succcurlyeq^* x^{j+1}$ follows for all $0 \le j \le n-1$, and $x^k \succcurlyeq^* x^\ell$ for all $0 \le k < \ell \le n$. Were now $y \succcurlyeq^* x$, then by transitivity $x^k \approx^* x^\ell$ for all $0 \le k < \ell \le n$, contradicting x^jPx^{j+1}. So $x \succ^* y$, and $\succ^*$ must extend $\bar{P}$.

The equivalence of (i), (ii), (iii) and (v) is derived in Richter (1971, Theorems 5 and 8). For (vi) ⇒ (iii), suppose xPy. Then $x\bar{P}y$, so by (vi) not $y\bar{R}x$. So (iii) follows.

For (i) ⇒ (vi), first note that by (i), $x\bar{P}y$ implies, by the furthermore-statement, $x \succ y$, so not $y \succcurlyeq x$. Hence, again by the furthermore-statement, not $y\bar{R}x$. Since $x\bar{P}y \Rightarrow x\bar{R}y$ is always true,

$x\bar{P}y \Rightarrow x\bar{R}y$ and not $y\bar{R}x$ follows. And if $x\bar{R}y$ and not $y\bar{R}x$, then $x^0 = x$ $Rx^1 \ldots Rx^n = y$ can be arranged. Now to prevent $y = x^n\ Rx^{n-1} \ldots Rx^0 = x$, there must be j such that x^jPx^{j+1}. So $x\bar{P}y$. So (vi) is derived, (vi) is equivalent to (i), (ii), (iii), (v).

Of course (iv) $\Rightarrow$ (v) is direct. For (v) $\Rightarrow$ (iv), note that $\bar{I}$ by definition is the symmetric part of $\bar{R}$, and that (v) implies (vi).

□

I.2.2. OTHER PROPERTIES OF CHOICE FUNCTIONS

The characterization by means of congruency, obtained in Theorem I.2.5, was completely general. In this subsection we consider properties for choice functions, simpler than congruency. We show that, under certain restrictions, they imply the existence of a representing weak order; by relating them to congruency.

DEFINITIONS.

I.2.6. C satisfies the *strong axiom of revealed preference (SARP)* if no sequence $(x^j)_{j=0}^n$ exists such that x^jPx^{j+1} for all $0 \leq j \leq n-1$, and x^nPx^0.

I.2.7. C satisfies the *weak axiom of revealed preference (WARP)* if $xRy \Rightarrow$ not yPx.

I.2.8. C satisfies *independence of irrelevant alternatives (IIA)* if for all $D_1, D_2 \in \mathcal{D}$ with $D_1 \subset D_2$, $C(D_2) \cap D_1 = \emptyset$ or $C(D_2) \cap D_1 = C(D_1)$.

WARP has been introduced in Samuelson (1938), and SARP in Houthakker (1950) and Ville (1951-1952, earlier 1946). These authors studied the special context of consumer demand theory, the origin of revealed preference theory. There the assumption was often made that C(D) contains exactly one element, for every $D \in \mathcal{D}$. Then indeed SARP implies WARP. The extension of these notions to choice functions C with not always $||C(D)|| = 1$, is not unique, and has been done in several ways in literature. In the above way SARP does not imply WARP anymore. To the author's knowledge, Arrow (1948) was the first to introduce IIA; see C4 in Arrow (1959). (Arrow himself uses the term IIA for *another* property, in his impossibility theorem in Arrow, 1978.)

Other early references are Nash (1950a, 1950b), and Luce (1959, section I.C.1.c.).

LEMMA I.2.9. *The congruency property implies SARP, WARP, and IIA. WARP implies IIA.*

PROOF. Congruency forbids the existence of $(x^j)_{j=0}^n$, $(D^j)_{j=0}^n$, such that $x^j \in C(D^{j+1})$, $x^{j+1} \in D^{j+1}$ for all $0 \leq j \leq n-1$, and $x^n \in C(D^0)$, $x^0 \in D^0 \backslash C(D^0)$. SARP forbids this only for the special case that $x^{j+1} \in D^{j+1} \backslash C(D^{j+1})$ for all $0 \leq j \leq n-1$. WARP forbids it only for the special case that $n = 1$. IIA can be seen to forbid it only for the special case that $n = 1$ and furthermore $D^0 \subset D^1$ or $D^1 \subset D^0$.

□

LEMMA I.2.10. *If* $C(D)$ *contains exactly one element for all* $D \in \mathcal{D}$, *then SARP implies congruency.*

PROOF. Assume SARP, and let $C(D)$ contain exactly one element, for all $D \in \mathcal{D}$. Let $x^0 R x^1 \ldots, R x^n P x^0$. We derive a contradiction. Let $j_0, \ldots, j_k$ be such that $x^0 = x^{j_0} = x^1 = \ldots = x^{j_1 - 1} \neq x^{j_1} = x^{j_1+1} =$ $= \ldots = x^{j_2 - 1} \neq x^{j_2} \ldots = x^{j_k - 1} \neq x^{j_k} = \ldots = x^n$. We now simply leave out subsequent identical alternatives, to obtain $x^{j_0} R x^{j_1} \ldots R x^{j_k} P x^{j_0}$. Since $C(D)$ contains only one element for all $D \in \mathcal{D}$, we must now in fact have $x^{j_0} P x^{j_1} \ldots P x^{j_k} P x^{j_0}$. This contradicts SARP.

□

LEMMA I.2.11. *If* $\mathcal{D}$ *contains all two- and three-point subsets of* X, *or if* $\mathcal{D}$ *is union-closed, then IIA implies congruenty.*

PROOF. Assume IIA, and let $x^0 R x^1 \ldots R x^n$. We prove that *not* $x^n P x^0$. As in the above Lemma, we may assume $x^j \neq x^{j+1}$ for all $0 \leq j \leq n-1$ (otherwise take again $(x^{j_l})_{l=0}^k$ instead of $(x^j)_{j=0}^n$). Hence $D^1, \ldots, D^n$ exist such that $x^j \in C(D^{j+1})$, $x^{j+1} \in D^{j+1}$ for all $0 \leq j \leq n-1$. If $x^n = x^0$, or not $x^n R x^0$, then also not $x^n P x^0$. So let us suppose

$x^n \neq x^0$ and $x^n Rx^0$, i.e. D^0 exists with $x^n \in C(D^0)$, $x^0 \in D^0$. To prove is that $x^0 \in C(D^0)$.

First for the case that $\mathcal{D}$ contains all two- and three-point subsets of X. Since $\{x^j, x^{j+1}\} \subset D^{j+1}$, by IIA : $x^j \in C\{x^j,x^{j+1}\}$ for all $0 \leq j \leq n-1$. In particular $x^0 \in C\{x^0,x^1\}$, $x^1 \in C\{x^1,x^2\}$. Consider $C\{x^0,x^1,x^2\} \neq \emptyset$. If x^2 is in it, then by IIA and $\{x^1,x^2\} \subset \{x^0,x^1,x^2\}$, also $x^1 \in C\{x^0,x^1,x^2\}$. If $x^1 \in C\{x^0,x^1,x^2\}$, then by IIA and $\{x^0,x^1\} \subset \{x^0,x^1,x^2\}$, also $x^0 \in C\{x^0,x^1,x^2\}$. So always $x^0 \in C\{x^0,x^1,x^2\}$. By IIA, $x^0 \in C\{x^0,x^2\}$. Analogously we obtain $x^0 \in C\{x^0,x^j\}$ for $j = 3,4,\ldots,n$. Since $\{x^0,x^n\} \subset D^0$, IIA and $x^n \in C(D^0)$ imply $x^0 \in C(D^0)$.

Next for the case that $\mathcal{D}$ is union-closed. Consider $C(D^1 \cup D^2) \neq \emptyset$. If there is $y^2 \in D^2$ such that $y^2 \in C(D^1 \cup D^2)$, then by IIA and $D^2 \subset D^1 \cup D^2$, $x^1 \in C(D^1 \cup D^2)$. So always there is $y^1 \in D^1$ such that $y^1 \in C(D^1 \cup D^2)$. By IIA and $D^1 \subset D^1 \cup D^2$, hence always $x^0 \in C(D^1 \cup D^2)$. Analogously (substitute $D^1 \cup \ldots \cup D^{j-1}$ for D^1, D^j for D^2, x^{j-1} for x^1 above, etc.) we obtain $x^0 \in C((D^1 \cup D^2 \cup \ldots \cup D^{j-1}) \cup D^j)$ for $j = 2,3,\ldots,n$, and $x^0 \in C((D^1 \cup \ldots \cup D^n) \cup D^0)$. Since $D^0 \subset (D^1 \cup \ldots \cup D^n \cup D^0)$, by IIA : $x^0 \in C(D^0)$.

□

COROLLARY I.2.12. *If $\mathcal{D}$ contains all two- and three-point subsets of* X, *or if $\mathcal{D}$ is union-closed, then the following four statements are equivalent:*

(i) *There exists a representing weak order for* C.

(ii) C *is congruent.*

(iii) C *satisfies WARP.*

(iv) C *satisfies IIA.*

If C(D) *contains exactly one element for every* $D \in \mathcal{D}$, *then the following three statements are equivalent:*

(v) *There exists a representing weak order for* C.

(vi) C *is congruent.*

(vii) C *satisfies SARP.*

PROOF. By the previous theorems, lemmas and propositions in this section.

□

For the case where $\mathcal{D}$ contains all two- and three-point subsets of X, the equivalence of (i) and (iv) above can also be obtained from the proof in Arrow (1959), which was meant only for the case where $\mathcal{D}$ contains all finite subsets of X. Sen (1971, bottom of page 312), noted that this proof remains valid in our case. For the case where $\mathcal{D}$ is union-closed, the equivalence of (i) and (iv) above is given in Theorem 15.4 in Fishburn (1973), or Hansson (1968), or Weddepohl (1970, Theorem 3.9.6, without K5 and K7).

In the following chapters we shall work with binary relations, intended to represent C, and called "preference relations". Note that binary relations do not specify the domain of C. Also note that representing weak orders, as in (i) of Theorem I.2.5, do not have to be uniquely determined. Hence properties, characteristic for such a weak order, do not have to be characteristic for C, see page 48 of Richter (1971). If $\mathcal{D}$ is rich enough, for instance contains all 2-point subsets of X, then $\succ$ of (i) of Theorem I.2.5 equals $\bar{R}$ (even R) of (v) there, and is uniquely determined.

I.3. COMPARISON WITH OTHER SET-UPS

In Luce and Suppes (1965) a distinction is made between "probabilistic" (= "stochastic") and "algebraic" approaches. In the first approach there is randomness in the choices of T, for example it is considered that T chooses $C(D) = D_1 \subset D$ from D with probability $\frac{1}{3}$ and $C(D) = D_2 \subset D$ from D with probability $\frac{2}{3}$. Our approach is algebraic, T's choices do not involve random mechanisms.

Also there is no randomness or uncertainty in the alternatives that result from T's choices. T can choose any available alternative he wants, and then be sure to obtain this alternative. We dò consider uncertainty in the sequel, in fact that will be the major subject of this monograph. The uncertainty, made explicit and studied by us in the sequel, will concern what "consequence" will result from an alternative, see Example II.1.1. There may be further, "implicit", uncertainty in such consequences. We neither assume, nor exclude, the existence of such uncertainty, only we do not study it. As an illustration, suppose a person T can choose a bet in a boxing-match, such that he gains \$3 if boxer 1 wins, and he gains \$-7 (i.e. looses \$7) if boxer 2 wins or the match is a tie. Then other approaches may call the amounts of money \$3 and \$-7 alternatives, and say T is uncertain about which alternative will result from his choice. For the set-up of this monograph it is more convenient to call the bet alternative , the amounts of money \$3 and \$-7 "consequence" (or coordinate, see section II.1 and Example II.1.1). We do not exclude or assume the existence of uncertainty about what will result from a consequence "gain \$3"; only such uncertainty will not be central in our study.

Our set-up is ordinal in the sense that everything in the sequel will be derived solely from the preference relation of T on the set of alternatives (where the preference relation again is derived from the choice function), and structure of the set of alternatives. Nothing cardinal-like has been introduced "from outside". No strength of preference relation is presupposed. Also no addition-like operation on alternatives is used. For example we do not use repetitions.

A typical thought experiment for the repetitions approach, as for instance in Shapiro (1979) or Camacho (1980; see also Wakker, 1985 c) is as follows. Let $D_1 = \{a,p\}$, $D_2 = \{p,n\}$, $D_3 = \{a,n\}$. It is now assumed that T has to deal with *all* three of these choice situations, and for instance he must choose between two "possibilities". The first is that he obtains a from D_1, n from D_2, and a from D_3; the second that he obtains p from D_1, p from D_2, a from D_3. The first possibility could then be denoted as $a \oplus n \oplus a$, or $(2 \otimes a) \oplus n$, the second as $p \oplus p \oplus a$, or $(2 \otimes p) \oplus a$. Here $\oplus$ and $\otimes$ are formal operations. One sees that here not in each one of the choice situations D_1, D_2, D_3, T is free to choose. If T wants a from D_1, then he must take n from D_2.

In our set-up T in each single situation chooses what he thinks best there. For instance, if in the transitivity assumption we assume that choices a from D_1 and p from D_2 should imply the choice a from D_3, then all these choices are intended to agree with T's freedom of choice in each single choice situation.

Also we do not use lotteries on alternatives. For the approach with lotteries see for instance Fishburn (1970, 1982).

CHAPTER II

CARTESIAN PRODUCT STRUCTURE,

MONOTONICITY, AND INDEPENDENCE

II.1. CARTESIAN PRODUCT STRUCTURE

In this section we introduce on X, the set of alternatives, the main structure of interest in this monograph. We shall assume throughout the sequel that X is a cartesian product $X_{i \in I} C_i$, with I an index set. We shall nearly always, with Chapter V excepted, assume that I is a finite set $\{1,\ldots,n\}$, $n \in \mathbb{N}$. Many definitions and results of this chapter are directly applicable to infinite I's.

The idea is that every alternative is described by a list of properties, indexed by I. For instance alternative $x = (x_1, x_2, x_3, x_4)$ may describe a car, where x_1 is the maximum speed, x_2 the price, x_3 a description of what the car looks like, x_4 the fuel consumption; $x \succsim y$ means that x is thought at least as good as y. Let us emphasize that no physical quantification of the coordinates is needed for our work. What the car looks like may be described in non-quantitative terms.

In applications, one of the central matters is to find an appropriate list of properties, to be indexed by I. The list should be large enough to contain all relevant aspects of the alternatives; and small enough to be tractable. Also, in our set-up, each property should have a meaning on its own. If in the above example it were impossible to give a meaningful description of x_3, what the car looks like, independent of maximum speed, price, and fuel consumption, then the list of indexed properties used above would not be well-suited for our set-up. Throughout this monograph we shall assume that the cartesian product structure has already been obtained.

In the sequel of this chapter, we shall study monotonicity properties. These may be considered formal reflections of the requirements mentioned in the above paragraph.

For many fields of (economic) science the cartesian product structure is a central matter of study. Examples:

EXAMPLE II.1.1. *Decision making under uncertainty (DMUU).*
Here x is an *act*, I a *state space*, $i \in I$ a *(possible) state (of nature)*. Exactly one state is the true state, the others are untrue. Act x yields *consequence* x_i if i is the true state. T, the decision maker, is uncertain about which of the states is true. Usually in this context $C_i = C_1 =: C$ for all i. As an example one may think of a horse race. Of n participating horses exactly one will win. Here i indicates the "possible state of nature" that horse i will win. $C_i = \mathbb{R}$ for all i, and $x = (x_1,\ldots,x_n)$ is a gamble (= act) that will leave T with $\$x_j$ if the j-th horse wins. See Savage (1954).

EXAMPLE II.1.2. *Consumer Theory.*
Here x is a *commodity bundle*, i indicates a kind of *commodity*, $x_i \in \mathbb{R}_+$ the amount of commodity i in x; $x \succcurlyeq y$: *consumer* T thinks x at least as good as y. See Katzner (1970).

EXAMPLE II.1.3. *Producers Theory.*
Here x is an *input vector*, i indicates a production factor, x_i is the input (rate) of production factor i (also x_i may refer to output). $V : X \to \mathbb{R}$ is a *production function*, assigning to every x the (maximally attainable, one-dimensional) output V(x). $x \succcurlyeq y$: x gives at least as much output as y. See Shepard (1970).

EXAMPLE II.1.4. *Dynamic Applications.*
Here x is a consumption/production path, stream of income, etc. Every i indicates a *point of time*, x_i is the consumption/production/income at point of time i. See Koopmans (1972).

EXAMPLE II.1.5. *Welfare Theory.*
Here x is an *allocation* or social situation, I is a society or group of agents/players, every $i \in I$ is an *agent*/player, and x_i indicates the wealth or utility for agent i under x. See Harsanyi (1955).

EXAMPLE II.1.6. *Price Indexes.*
Here every i indicates a good or service, x_i is the price of good or service i at the time, or in the place, described by x. Here a *price index* V, assigning to every x a measure for the level of prices, is usually the primitive. $x \succcurlyeq y$: the level of prices in time or place x is at least as high as that in y. See Fisher (1927 b).

Of course, numerous other examples can be thought of. The modelling of uncertainty, as in Example II.1.1, has been introduced in economic literature by Savage (1953) and Arrow (1953). Note that in Examples II.1.3 and II.1.6, it is custom to take a quantitative (representing; see Definition III.2.2) V as primitive, instead of $\succcurlyeq$. The relation between such quantitative representations, and $\succcurlyeq$, is the central topic of this monograph.

II.2. ALTERNATIVES, SUBALTERNATIVES, CONSEQUENCES, AND PREFERENCES BETWEEN THEM

The remainder of this chapter, with the exception of the definition and notations of this section, and Definitions II.6.2, II.6.3, and Theorem II.6.4, is not needed for understanding of the follow chapters.

NOTATION II.2.1. For $x \in X_{i\in I}C_i$, and $A \subset I$, x_A is the element of $X_{i\in A}C_i$ with i-th coordinate x_i, for all $i \in A$. We call x_A a *subalternative*.

If one considers x as a map from I to $\cup_{i\in I}C_i$, assigning x_i to every $i \in I$, then one can consider x_A as the restriction of x to A. Of course $x_{\{i\}} = x_i$, $x_I = x$; coordinates and alternatives are special forms of subalternatives. We assume throughout this section, as well as in sections II.3, II.4, and II.5:

ASSUMPTION II.2.2. For every $A \subset I$ a reflexive transitive binary relation $\succcurlyeq_A$ on $X_{i\in A} C_i$ is given. These $\succcurlyeq_A$'s are called *subpreference relations*. We often write $\succcurlyeq_i$ instead of $\succcurlyeq_{\{i\}}$, and also call these binary relations *coordinate preference relations*; further we also write $\succcurlyeq$ instead of $\succcurlyeq_I$, this is the usual *preference relation*. $\succ_A$, $\prec_A$, $\preccurlyeq_A$, $\approx_A$ are as usual (see Chapter 0). For all these binary relations, we often leave out index A if no confusion is likely to arise.

Note that, for the time being, we do not assume any connection between different $\succcurlyeq_A$'s. They do not have to be derived from $\succcurlyeq_I$ in any way. The monotonicity properties, considered in the sequel, will enable such a derivation, see Proposition II.6.1. Also note that we emphatically do *not* assume completeness for the $\succcurlyeq_A$'s. (Recall that in (v) of Theorem I.2.5, we found a representing $\bar{R}$, that was transitive, reflexive, but not necessarily complete.) Thus, the assumption of the presence of all these $\succcurlyeq_A$'s does not have to be considered a serious restriction: some of them may simply be the identity relation.

The assumption that all these $\succcurlyeq_A$'s are given, deviates from the main strategy in this monograph, to consider only $\succcurlyeq$ on X as given. One reason for this deviation is that the work under this assumption serves as a preparation for the work in section II.6, where we again assume that only $\succcurlyeq$ is given. But we also hope that our work under this assumption has interest on its own.

The interpretation of $x_A \succcurlyeq_A y_A$ is something like: for as far as only the coordinates with indices from A are concerned, alternative x is weakly preferred to alternative y. In consumer theory, one may imagine that attention can indeed be restricted to the coordinates with indices from A, if the coordinates with indices from A^c (say in a thought experiment) are fixed at some standard level, for example a level of total satisfaction. In DMUU, coordinates with indices from A^c can be left out of consideration if A^c is untrue (i.e. every state of nature in A^c is untrue) and furthermore this has become known to T by the acquisition of extra information. The ceteris paribus assumption should entail that the fixation at A^c, or the extra information that A^c is untrue, does not affect other essential matters.

DEFINITION II.2.3. Let $A_1, \ldots, A_k$ be mutually disjoint subsets of I, $x^1_{A_1}, \ldots, x^k_{A_k}$ subalternatives. The subalternative, *compounded of* $x^1_{A_1}, \ldots, x^k_{A_k}$, notation $x^1_{A_1} x^2_{A_2} \ldots x^k_{A_k}$, is the subalternative assigning x^j_i to $i \in A_j$, $j = 1, \ldots, k$.

NOTATION II.2.4. Let $A_1, \ldots, A_k$ be mutually disjoint subsets of I, x an alternative, $x^1_{A_1}, \ldots, x^k_{A_k}$ subalternatives. Then we write $x_{-A_1,\ldots,A_k}$ for $x_{(A_1 \cup \ldots \cup A_k)^c}$, and $x_{-A_1,\ldots,A_k}\ x^1_{A_1}, \ldots, x^k_{A_k}$ for the alternative $x_{(A_1 \cup \ldots \cup A_k)^c}\ x^1_{A_1} \ldots x^k_{A_k}$.

If necessary, we add parentheses in the above notations. And as often, we write i instead of $\{i\}$. Thus for instance:

$$x_{-i}v_i \text{ is } (x \text{ with } x_i \text{ replaced by } v_i). \qquad \text{(II.2.1)}$$

and, for $i \neq j$,

$$(x_{-i,j}\ v_i, w_j) \text{ is } (x \text{ with } x_i \text{ replaced by } v_i,\ x_j \text{ by } w_j). \qquad \text{(II.2.2)}$$

II.3. TERMINOLOGY FOR MONOTONICITY

Throughout literature one finds very many forms of monotonicity properties, and properties closely related to them, with widely varying terminologies and meanings. We think it would be useful if a unifying terminology for these would be developed, and if the several logical relations for them would be mapped out.

The terminology, developed below, should be considered only as a first indication that such a unification may be possible. We would welcome alternative approaches from other authors. In special contexts one may adhere to (small) deviations from a unified terminology, to increase tractability. For instance in a context where never any form of monotonicity occurs other than cA monotonocity (see (II.3.1), and Definitions II.3.7 and II.3.8) one may for convenience leave out "cA" from terminology.

Let us first give the most simple and well-known example of monotonicity:

$$\text{If } x_i \succcurlyeq y_i \text{ for all } i, \text{ then } x \succcurlyeq y. \qquad \text{(II.3.1)}$$

We shall vary this monotonicity in three aspects. Firstly, the involved preferences can be varied. We can have strict preferences instead of weak preferences, etc. Secondly, we can replace coordinates and/or alternatives by subalternatives. Thirdly, we can vary what may be called the "direction of aggregation". We can for instance assume that [$x_i \succcurlyeq y_i$ for all $i \geq 2$, and $x \prec y$] implies [$x_1 \prec y_1$]. Then the preference concerning x and y (or, more generally, the "longest", "most aggregated", subalternatives) is not in the conclusion, but in the premise.

The abbreviations that will be used in the terminologies, are:

ABBREVIATIONS: c stands for coordinate, A for alternative, s for ("short") subalternative, and S for ("long") subalternative; mon stands for monotonicity.

We also use capital A to denote subsets of I; this is unlikely to give confusion. The general form of the terms, introduced in subsection II.3.1 below, is

$$(\underline{d}\underline{p}\underline{l})\text{mon} \qquad \text{(II.3.2)}$$

Here d is the generic variable for "*direction* of aggregation". This is either aggregated, or disaggregated. Further p is the generic variable for the kind(s) of involved *preferences*, weak, strict, or equivalence; p may also stand for "strong". Finally l refers to the *length* of subalternatives, and stands for sS, cS, sA, cA, or s^2S.

In the aggregated monotonicities we often leave out the term aggregate. Also we often leave out the term disaggregate, and then show this by replacing cA by Ac, sA by As, and sS by Ss.

Further we often use symbols instead of words for $\underline{p}$ above; then disaggregated monotonicities are distinguished from the aggregated ones by a dash through a symbol.

II.3.1. DEFINITIONS

First we give the strongest monotonicities, with sS for $\underline{1}$. We start with aggregated for $\underline{d}$ in (II.3.2).

DEFINITIONS II.3.1. *(Aggregated) s(ubalternative) s(ubalternative) monotonicities*. Add after every definition below:

for all $B \subset I$, partitions $(B_1, \ldots, B_m)$ of B, (II.3.3)
alternatives x, y.

We say $\{\succcurlyeq_A : A \subset I\}$ satisfies:

(a) ($\succ$ *sS) mon* (or *strict sS mon*) if:

$x_{B_k} \succ y_{B_k}$ for all $k \Rightarrow x_B \succ y_B$

(b) ($\succcurlyeq$ *sS) mon* (or *weak sS mon*) if:

$x_{B_k} \succcurlyeq y_{B_k}$ for all $k \Rightarrow x_B \succcurlyeq y_B$

(c) ($\approx$ *sS) mon* (or *equivalence sS mon*) if:

$x_{B_k} \approx y_{B_k}$ for all $k \Rightarrow x_B \approx y_B$

(d) ($\succcurlyeq\succ$ *sS) mon* (or *strong sS mon*) if:

$x_{B_k} \succcurlyeq y_{B_k}$ for all k, $x_{B_k} \succ y_{B_k}$ for some $k \Rightarrow x_B \succ y_B$

(e) *total sS mon* if:

(a), ..., (d) above are all satisfied

One may add "aggregated" before every definition above. Next we will let $\underline{d}$ in (II.3.2) be disaggregated. Each disaggregated monotonicity property is closely related to the corresponding aggregated monotonicity property. The only difference between the two can be caused by incomparability, as will be demonstrated in Proposition II.4.1. This may have been a reason that the disaggregated monotonicities, to the author's knowledge, have not yet appeared in literature. Still, they will be an indispensable tool for our work in the sequel (as we shall see in the comment after Theorem II.6.5).

DEFINITIONS II.3.2. *Disaggregated s(ubalternative) s(ubalternative) monotonicities*. Add after every definition below:

for all $B \subset I$, partitions $(B_1,\ldots,B_m)$ of B , $1 \leq j \leq m$, and alternatives x,y. (II.3.4)

We say $\{\succcurlyeq_A : A \subset I\}$ satisfies:

(a) *($\nsucc$ Ss) mon* (or *strict Ss mon*, or *disaggregated strict sS mon*) if:
$x_{B_k} \succ y_{B_k}$ for all $k \neq j$, $x_B \preccurlyeq y_B \Rightarrow x_{B_j} \preccurlyeq y_{B_j}$

(b) *($\nsucccurlyeq$ Ss) mon* (or *weak Ss mon*, or *disaggregated weak sS mon*) if:
$x_{B_k} \succcurlyeq y_{B_k}$ for all $k \neq j$, $x_B \prec y_B \Rightarrow x_{B_j} \prec y_{B_j}$

(c) *($\approx$ Ss) mon* (or *equivalence Ss mon*, or *disaggregated equivalence sS mon*) if:
$x_{B_k} \approx y_{B_k}$ for all $k \neq j$, $[x_B \succ y_B$ or $x_B \prec y_B] \Rightarrow [x_{B_j} \succ y_{B_j}$ or $x_{B_j} \prec y_{B_j}]$

(d) *($\succcurlyeq\nsucc$ Ss) mon* (or *strong Ss mon*, or *disaggregated strong sS mon*) if:
$x_{B_k} \succcurlyeq y_{B_k}$ for all $k \neq j$, $x_B \preccurlyeq y_B \Rightarrow x_{B_j} \preccurlyeq y_{B_j}$

(e) *total Ss mon* (or *disaggregated total sS mon*) if:
(a), ..., (d) above are all satisfied

Pronounciation does not distinguish between sS and Ss, hence we think for spoken language the second terms in (a) and (d), and the first term in (e), are less suited.

The following, weaker, versions of monotonicity are straightforward variations on the previous ones, so are not written out. The idea is, to replace in the ($\underline{\text{dp}}$(sS)) monotonicities above s by c, and / or S by A.

DEFINITIONS II.3.3,(a) to (e). *(Aggregated) c(oordinate) s(ubalternative) monotonicities*. Obtained from Definitions II.3.1 by substitution everywhere of c for s, and by restriction to B_k's with $||B_k|| = 1$, so to *coordinates* x_{B_k}, y_{B_k}.

DEFINITIONS II.3.4, (a) to (e). *Disaggregated c(oordinate) s(ubalternative) monotonicities.* Obtained from Definitions II.3.2 in the same way as Definitions II.3.3 have been obtained from Definitions II.3.1.

DEFINITIONS II.3.5, (a) to (e). *(Aggregated) s(ubalternative) a(lternative) monotonicities.* Obtained from Definitions II.3.1 by substitution everywhere of A for S, and by restriction to B equal I, so to *alternatives* $x_B = x$, $y_B = y$.

DEFINITIONS II.3.6, (a) to (e). *Disaggregated s(ubalternative) a(lternative) monotonicities.* Obtained from Definitions II.3.2 in the same way as Definitions II.3.5 have been obtained from Definitions II.3.1.

Since the monotonicities, introduced in the following two definitions, only involve the $\succ_i$'s, and $\succ$, we sometimes ascribe them to $\{\succ_i : i \in I\} \cup \{\succ\}$, instead of to all of $\{\succ_A : A \subset I\}$. Definition II.3.7.b equals (II.3.1).

DEFINITIONS II.3.7, (a) to (e). *(Aggregated) c(oordinate) a(lternative) monotonicities.* Obtained from Definitions II.3.1 by substitution everywhere of c for s, A for S, and by restriction to B equal I, $m = n$, and $B_k = \{k\}$ for all k; i.e. by restriction to coordinates $x_{B_k} = x_k$, $y_{B_k} = y_k$, and alternatives $x_B = x$, $y_B = y$.

DEFINITIONS II.3.8, (a) to (e). *Disaggregated c(oordinate) a(lternative) monotonicities.* Obtained from Definitions II.3.2 in the same way as Definitions II.3.7 have been obtained from Definitions II.3.1.

We shall show in Proposition II.4.3 that the (dp(sS)) monotonicities are implied by those that are restricted to $m = 2$. So we define:

DEFINITIONS II.3.9, (a) to (e). *(Aggregated) twofold s(ubalternative) s(ubalternative) monotonicities.* Obtained from Definitions II.3.1 by restriction to m = 2, and with this indicated by an index 2 above the small s, so ($\succ s^2S$) mon, etc.

DEFINITIONS II.3.10, (a) to (e). *Disaggregated twofold s(ubalternative) s(ubalternative) montonicities.* Obtained from Definitions II.3.2 in the same way as Definitions II.3.9 have been obtained from Definitions II.3.1; this gives ($\nsucc Ss^2$) mon, etc.

In following chapters we shall deal with cases where all monotonicity properties, introduced so far, are satisfied. Hence we define:

DEFINITION II.3.11. $\{\succ_A : A \subset I\}$ satisfies *total monotonicity* if Definitions II.3.1 to II.3.10 are all satisfied.

II.4. ELEMENTARY CONNECTIONS BETWEEN MONOTONICITIES

In this section some elementary logical relations between the several monotonicity properties are given. It is not our plan to elaborate this extensively. We mainly aim at minimal assumptions to guarantee total monotonicity. *Let us repreat that throughout we make Assumption II.2.2, i.e. every $\succ_A$ is transitive, reflexive, not necessarily complete.* First we relate aggregated monotonicities to disaggregated monotonicities.

PROPOSITION II.4.1. *Let every $\succ_A$ be complete. Then (aggregated $\underline{p}\underline{l}$) mon holds if and only if (disaggregated $\underline{p}\underline{l}$) mon holds.*

Here one can substitute strict, weak, equivalence, strong, or total for $\underline{p}$; and for $\underline{l}$ one can substitute sS, cS, sA, cA, or s^2S.

PROOF. We only give the proof for $\underline{p}$: strong, and $\underline{l}$: cA. The other cases are similar. In the definition of $(\succcurlyeq \not\succ Ac)$mon, let $x_i \succcurlyeq y_i$ for all $i \neq j$. Now $[x \preccurlyeq y \Rightarrow x_j \preccurlyeq y_j]$ is equivalent to $[\text{not } x_j \preccurlyeq y_j \Rightarrow \text{not } x \preccurlyeq y]$. By completeness this is equivalent to $[x_j \succ y_j] \Rightarrow [x \succ y]$. This gives $(\succcurlyeq \succ cA)$mon.

□

Since every coordinate and alternative is a subalternative, we immediately have:

PROPOSITION II.4.2. *($\underline{dp}$(sS))mon implies ($\underline{dl}$)mon, for every $\underline{l} \in \{sS, cS, sA, cA\}$.*

Here one can substitute aggregated or disaggregated for $\underline{d}$, and for $\underline{p}$ one can substitute $\succ, \succcurlyeq, \approx, \succcurlyeq\succ$, or total.

□

PROPOSITION II.4.3. *($\underline{dp}$ s^2S)mon holds if and only if ($\underline{dp}$ sS)mon holds.*

Here one can substitute aggregated or disaggregated for $\underline{d}$, and for $\underline{p}$ one can substitute $\succ$, $\succcurlyeq$, $\approx$, $\succcurlyeq\succ$, or total.

PROOF. That an sS mon implies the corresponding s^2S mon, is direct. So we assume ($\underline{dp}s^2S$)mon, and derive ($\underline{dp}$sS)mon. We do it for two cases only: $\underline{d}$ is aggregated and $\underline{p}$ is strong; or $\underline{d}$ is disaggregated, and $\underline{p}$ is equivalence. In either case, let $(B_1,\ldots,B_m)$ be a partition of $B \subset I$.

Now assume first that $(\succcurlyeq \succ s^2S)$ holds, and assume $x_{B_k} \succcurlyeq y_{B_k}$ for all k, $x_{B_k} \succ y_{B_k}$ for some k, say k = 1. To prove is, for $(\succcurlyeq \succ sS)$mon, that $x_B \succ y_B$. By $(\succcurlyeq \succ s^2S)$mon, $x_{B_1}x_{B_2} \succ y_{B_1}y_{B_2}$ follows. If now for $i < m$ we have proved that $x_{B_1} \ldots x_{B_i} \succ y_{B_1} \ldots y_{B_i}$, then we take $((B_1 \cup \ldots \cup B_i), B_{i+1})$ as partition in two parts of $B_1 \cup \ldots \cup B_{i+1}$, apply $(\succcurlyeq \succ s^2S)$mon, and obtain that $x_{B_1} \ldots x_{B_{i+1}} \succ y_{B_1} \ldots y_{B_{i+1}}$. We end up with $x_B \succ y_B$, which is what $(\succcurlyeq \succ sS)$mon requires.

For the second case we first give a new notation, for this proof only. For any $s,t \in X$, $C \subset I$, we write $s_C\underline{q}t_C$ if $[s_C \succ t_C$ or $s_C \prec t_C]$. We now assume that $(\not\approx s^2S)$mon holds, and want to derive $(\not\approx sS)$mon.

So let $x_{B_k} \approx y_{B_k}$ for all $k \neq j$, and $x_B \underline{q} y_B$. To prove is $x_{B_j} \underline{q} y_{B_j}$. Suppose $j = 1$. Since $B = (B_1 \cup \ldots \cup B_{m-1}) \cup B_m$, $x_{B_m} \approx y_{B_m}$, and $x_B \underline{q} y_B$, $(\approx s^2S)$mon gives that $(x_{B_1} \ldots x_{B_{m-1}}) \underline{q} (y_{B_1} \ldots y_{B_{m-1}})$. If now for $i > 1$ we have proved that $x_{B_1} \ldots x_{B_i} \underline{q} y_{B_1} \ldots y_{B_i}$, then we write $(B_1 \cup \ldots \cup B_i) = (B_1 \cup \ldots \cup B_{i-1}) \cup B_i$; $x_{B_i} \approx y_{B_i}$, and $(\not\approx s^2S)$mon give that $x_{B_1} \ldots x_{B_{i-1}} \underline{q} y_{B_1} \ldots y_{B_{i-1}}$. We end up with $x_{B_1} \underline{q} y_{B_1}$, which is what $(\not\approx sS)$mon requires.

$\square$

We now turn to the logical relations between the $\underline{dpl}$ monotonicities that differ with respect to $\underline{p}$, and have $\underline{d}$ and $\underline{l}$ the same.

PROPOSITION II.4.4.

(a) *($\underline{d}$ strong $\underline{l}$)mon implies ($\underline{d}$ strict $\underline{l}$)mon.*

(b) ($\underline{d}$ weak $\underline{l}$)mon implies ($\underline{d}$ equivalence $\underline{l}$)mon.

(c) ($\underline{d}$ strong $\underline{l}$)mon and ($\underline{d}$ equivalence $\underline{l}$)mon together imply ($\underline{d}$ weak $\underline{l}$) mon.

Here one can substitute aggregated or disaggregated for $\underline{d}$, and for $\underline{l}$: sS, cS, sA, cA, or s^2S.

PROOF. (a) is trivial. For (b), we consider first (aggregated) weak cA mon, and derive equivalence cA mon. If now $x_k \approx y_k$ for all k then $[x_k \succcurlyeq y_k$ and $y_k \succcurlyeq x_k]$ for all k so, by twofold application of weak cA mon, $[x \succcurlyeq y$ and $y \succcurlyeq x]$, i.e. $x \approx y$. This is what equivalence cA mon requires.

The second, and final, version of (b) that we derive, is the version where $\underline{d}$ stands for disaggregated, and $\underline{l}$ again for cA. Let disaggregated weak cA mon be satisfied. Let $x_k \approx y_k$ for all $k \neq j$, $[x \succ y$ or $x \prec y]$. Say $x \succ y$. Then by disaggregated weak cA mon $x_j \succ y_j$ follows. So certainly $[x_j \succ y_j$ or $y_j \succ x_j]$, which is all that disaggregated equivalence monotonicity requires.

For (c), we again consider two cases, again both with $\underline{l}$ = cA. First we assume (aggregated) strong cA mon and equivalence cA mon. To derive is weak cA mon. So let $x_k \succcurlyeq y_k$ for all k. If $x_i \succ y_i$ for some i, then by strong cA mon $x \succ y$. So certainly then $x \succcurlyeq y$, which is what weak cA mon requires. If $x_i \succ y_i$ for no i, then $x_i \approx y_i$ for all i. Here we apply equivalence cA mon, to obtain $x \approx y$. So certainly $x \succcurlyeq y$, which is what weak cA mon requires.

Finally, as a second case of (c), we assume disaggregated strong cA mon and disaggregated equivalence cA mon, and derive disaggregated weak cA mon. So let $x_k \succcurlyeq y_k$ for all $k \neq j$, $x \prec y$. Of course $x \preccurlyeq y$, so disaggregated strong cA mon gives $x_j \preccurlyeq y_j$. If now $x_k \approx y_k$ for all $k \neq j$, then disaggregated equivalence mon (since $x_j \succ y_j$ cannot hold) gives $x_j \prec y_j$, which is what disaggregated weak cA mon requires. So suppose $x_i \succ y_i$ for some i. Now $x_j \approx y_j$ cannot hold: then we would have $x_k \succcurlyeq y_k$ for all $k \neq i$ (also for $k = j$), which together with $x \preccurlyeq y$ by disaggregated strong cA mon would imply $x_i \preccurlyeq y_i$. This contradicts $x_i \succ y_i$. Apparently $x_j \approx y_j$ does not hold, and $x_j \prec y_j$ follows. This is what disaggregated weak cA mon requires.

□

II.5. TOTAL MONOTONICITY

In this section we give sets of monotonicity properties, sufficient to imply total monotonicity (i.e. all other monotonicity properties). *Again, throughout we make Assumption* II.2.2. First one preparatory result, less elementary than those of the previous section.

PROPOSITION II.5.1.

(a) *($\succcurlyeq$ sA)mon and ($\succcurlyeq \not\succ$ As)mon together imply ($\succcurlyeq$ sS)mon and ($\succcurlyeq \not\succ$ Ss)mon.*

(b) ($\succcurlyeq \succ$ sA)mon and ($\not\succcurlyeq$ As)mon together imply ($\succcurlyeq \succ$ sS)mon and ($\not\succcurlyeq$ Ss)mon.

PROOF. Throughout let $(B_1, \ldots, B_m)$ be a partition of $B \subset I$. We write $B_0 := B^c$. Always z is an arbitrary fixed alternative. In the proof we shall often change subalternatives into alternatives by compounding them with pieces of z.

For (a), we first derive $(\succ sS)$mon from the assumptions there. So let $x_{B_1} \succ y_{B_1}, \ldots, x_{B_m} \succ y_{B_m}$.

By $(\succ sA)$mon, $(z_{B_0} x_{B_1} \ldots x_{B_m}) \succ (z_{B_0} x_{B_1} \ldots x_{B_m})$. Now, since $z_{B_0} \prec z_{B_0}$, $(\succ \nsucc As)$mon implies $(x_{B_1} \ldots x_{B_m}) \succ (y_{B_1} \ldots y_{B_m})$, which is what $(\succ sS)$mon required.

Next, for (a), we derive $(\succ \nsucc Ss)$ from the assumptions there. So let $x_{B_2} \succ y_{B_2}, \ldots, x_{B_m} \succ y_{B_m}$, and $x_B \prec y_B$. This latter, and $z_{B_0} \prec z_{B_0}$, by $(\succ sA)$mon implies $z_{B_0} x_B \prec z_{B_0} y_B$. This, and $x_{B_k} \succ y_{B_k}$ for $k = 2,\ldots,m$, implies by $(\succ \nsucc As)$mon $z_{B_0} x_{B_1} \prec z_{B_0} y_{B_1}$. From this, $z_{B_k} \prec z_{B_k}$ for all $k \geq 2$, and $(\succ sA)$mon, follows $z_{B_0} x_{B_1} z_{B_2} \ldots z_{B_n} \prec z_{B_0} y_{B_1} z_{B_2} \ldots z_{B_n}$. Finally, this, $z_{B_k} \succ z_{B_k}$ for all $k \neq 1$, and $(\succ \nsucc As)$mon, give $x_{B_1} \prec y_{B_1}$. This is what $(\succ \nsucc Ss)$ required.

The proof of (b) is analogous, and left to the reader.

□

THEOREM II.5.2. *The following four (sets of) conditions for* $\{\succ_A : A \subset I\}$ *are equivalent:*

(i) *total monotonicity.*

(ii) $(\succ > s^2S)$ *–,* $(\succ \nsucc Ss^2)$ *–,* $(\approx s^2S)$ *–, and* $(\not\precsim Ss^2)$*mon.*

(iii) $(\succ sA)$ *–,* $(\nsucc As)$ *–,* $(\succ > sA)$ *–, and* $(\succ \nsucc As)$*mon.*

(iv) $(\approx sA)$ *–,* $(\not\precsim As)$ *–,* $(\succ > sA)$ *–, and* $(\succ \nsucc As)$*mon.*

PROOF. (iv) ⇒ (iii) is by Proposition II.4.4.c. (iii) ⇒ (ii) (even the stronger version of (ii) without indices 2) is by Propositions II.5.1, and II.4.4.b.

For (ii) ⇒ (i), first we see that by Proposition II.4.3, (ii) implies its stronger version without indices 2. That this implies *all* sS monotonicities, is by Proposition II.4.4. This of course (Proposition II.4.2) implies (i).

Of course, (i) ⇒ (iv) is by definition.

□

Under special circumstances it may be possible to weaken the properties in (ii), (iii), (iv) above, such that (i) is still implied. For instance, if all $\succcurlyeq_A$'s are known to be complete, then Proposition II.4.1 enables us to leave out the disaggregated monotonicities. We give a result, useful for the case of antisymmetry:

PROPOSITION II.5.3. *If every* $\succcurlyeq_A$ *is antisymmetric, then (d strong l)mon implies d weak l mon.*

Here one can substitute aggregated or disaggregated for d, and for l: sS, sA, cS, cA, s^2S.

PROOF. We consider only the case where l = cA. First we assume aggregated strong cA mon. To derive is aggregated weak cA mon. So assume $x_k \succcurlyeq y_k$ for all k. If $x_k = y_k$ for all k, then x = y, and $x \succcurlyeq y$ follows. If $x_k \neq y_k$ for some k, then by antisymmetry of $\succcurlyeq_k$ we have in fact $x_k \succ y_k$. By aggregated strong cA mon then $x \succ y$, so certainly again $x \succcurlyeq y$. $x \succcurlyeq y$, as required by aggregated weak cA mon, always follows.

Next we assume disaggregated strong cA mon. To derive is disaggregated weak cA mon. So assume $x_k \succcurlyeq y_k$ for all $k \neq j$, $x \prec y$. Then $x \preccurlyeq y$, and by disaggregated strong cA mon $x_j \preccurlyeq y_j$ follows. The proof is completed if we derive contradiction from the assumption that not $x_j \prec y_j$. If not $x_j \prec y_j$, then $x_j \approx y_j$, i.e. $x_j = y_j$. Further then for any i, from $x \preccurlyeq y$, $x_k \succcurlyeq y_k$ for all $k \neq i$ (also k = j) and disaggregated strong cA mon, $x_i \preccurlyeq y_i$ follows. So $x_i \succcurlyeq y_i$ and $x_i \preccurlyeq y_i$ for all i. Then apparently $x_i = y_i$ for all i, so x = y, in contradiction with $x \prec y$.

□

The above proposition shows that antisymmetry enables one to leave out, in Theorem II.5.2, the weak and equivalent monotonicities in (ii), (iii) and (iv).

Note that it depends on the involved cartesian product, whether a subalternative can be called consequence or not. Suppose that $(B_1, \ldots, B_m)$ is a partition of I. We can write $\times_{i\in I} C_i$ as $\times_{k=1}^{m}(\times_{i\in B_k} C_i)$, and consider only the cartesian product over k. Then

x_{B_k} is considered a consequence, whereas originally it was not if $||B_k|| > 1$. In contexts, where any cartesian product structure $X_{k=1}^{m}(X_{i\in B_k} C_i)$ is as natural as the original $X_{i\in I}C_i$, the subalternative alternative monotonicity properties may be considered to be as natural as the consequence alternative monotonicities. For those contexts (iii) and (iv) of Theorem II.5.2 are useful characterizations of total monotonicity. This is the more so in view of the main strategy in this monograph, to formulate as much as possible all conditions in terms of the preference relation $\succ$ on the alternatives: (iii) and (iv) at least partly involve $\succ$.

II.6. COORDINATE INDEPENDENCE AS THE OBSERVABLE CONTENT OF TOTAL MONOTONICITY

In this section we want to return to the main strategy of this monograph, to consider only the preference relation $\succ$ on the set of alternatives X as observable, together with structure of X. Then the $\succ_A$'s, for $A \subsetneq I$, are not directly given. The most we can do is *derive* them from $\succ$, under the assumption of total monotonicity (, see Proposition II.6.1). And the most we can do about verification or falsification of total monotonicity, is to find properties of $\succ$ that enable a verification or falsification of the *existence* of $\succ_A$'s, for all $A \subsetneq I$, such that $\{\succ_A: A \subset I\}$ satisfies total monotonicity. The necessary and sufficient property of $\succ$ for the existence of such $\succ_A$'s, is "coordinate independence", see Definition II.6.3, and Theorem II.6.5. The following proposition shows that, under total monotonicity, all $\succ_A$'s can be derived from $\succ$.

PROPOSITION II.6.1. *Let* $\{\succ_A: A \subset I\}$ *satisfy total monotonicity. Let* z *be an arbitrarily fixed element of* X. *Then* $x_A \succ y_A$ *if and only if* $x_A z_{A^c} \succ y_A z_{A^c}$.

PROOF. Since $z_{A^c} \succ z_{A^c}$, $[x_A \succ y_A]$ by $(\succ s^2S)$mon implies $[x_A z_{A^c} \succ y_A z_{A^c}]$.

For the converse implication, since $z_{A^c} \succ z_{A^c}$, $[y \prec x]$ by ($\succ \not\succ Ss^2$) mon implies $[y_A \prec x_A]$.

□

Since $\succ_A$ is independent of the particular z_{A^c} that we fixed in the above proposition, we see that the following property of $\succ$ is necessary for total monotonicity:

DEFINITION II.6.2. $\succ$ satisfies *independence of equal subalternatives* if:

$$[x_A v_{A^c} \succ y_A v_{A^c} \Leftrightarrow x_A w_{A^c} \succ y_A w_{A^c}]$$

for all $x, y, v, w \in X$, $A \subset I$.

For the idea of the above definition let $s := x_A v_{A^c}$, $t := y_A v_{A^c}$. Then, as soon as we know that s and t are identical in A^c, we do not have to consider the particular common value $s_{A^c} = t_{A^c} = v_{A^c}$ in A^c any further. It does not matter if this is v_{A^c} or w_{A^c}, or whatever. *The preference between s and t is independent of the $A^c \subset I$ where s and t are identical.*

For finite cartesian products, there is a simpler, equivalent, formulation for independence of equal subalternatives.

DEFINITION II.6.3. $\succ$ satisfies *independence of equal coordinates*, or shortly *coordinate independence (CI)*, if:

$$[x_{-i}v_i \succ y_{-i}v_i \Leftrightarrow x_{-i}w_i \succ y_{-i}w_i]$$

for all $x, y \in X$, $v_i, w_i \in C_i$, $i \in I$. Also we then say that $\succ$ is *coordinate independent (CI)*.

THEOREM II.6.4. $\succ$ *satisfies independence of equal subalternatives if and only if it satisfies CI.*

PROOF. Since any coordinate is a subalternative, independence of equal subalternatives implies CI. For the converse implication, assume $\succ$ is CI. Let x, y, v, w, A be as in Definition II.6.2. Let

$||A^c|| = m \leq n$. For m = 1 the result is direct. Now suppose $m \geq 2$, and for m-1 the result is proved. Let $j \in A$. Then $[x_A v_{A^c} \succ y_A v_{A^c}] \Leftrightarrow [((x_A w_{A^c})_{-j} v_j) \succ ((y_A w_{A^c})_{-j} v_j)]$. Now by CI the latter preference holds if and only if $x_A w_{A^c} \succ y_A w_{A^c}$.

□

The above two properties are central in our work and will nearly always hold in following chapters, Chapter VI excepted. A related property was introduced in Sono (1945, 1961) and Leontief (1947 a, 1947 b) in terms of derivatives of a (presupposed, representing (see Definition III.2.2)) function. See also Samuelson (1947, pp 174-180). Already Fleming (1952), for the context of welfare theory, formulated essentially the independence of equal subalternatives of length n-2 in terms of a (presupposed, representing) function, but without using derivatives. In Debreu (1960) CI was formulated in its present, more appealing, form, in terms of the preference relation, thus again without differentiability assumptions. Before, Savage (1954) had introduced the "sure-thing principle" for DMUU. This principle is in fact identical to independence of equal subalternatives, as is well known nowadays. It can be seen to underly the "likelihood principle" in statistics, which is central in the discussion about Bayesian statistics. See Berger and Wolpert (1984). Debreu, and some other authors, have used the term independence. A further usual term is (strong/strict) separability. Katzner (1970) uses the term additivity. For an extensive study of generalizations, and many applications of CI, see Blackorby, Primont, and Russell (1978). See also Mak (1984, 1985). Gorman (1976, p. 212, 224) argues for the importance of CI in economic theory. Krantz et al.(1971) mention Fisher (1927 a, p.175 ff) as an early place where the basic idea of CI can be recognized.

THEOREM II.6.5. *Let $\succ$ on X be transitive and reflexive. Then there exist transitive reflexive $\succ_A$'s on $X_{i \in A} C_i$ $(A \subset I)$ such that $\{\succ_A : A \subset I\}$ satisfies total monotonicity, if and only if $\succ$ is CI.*

PROOF. The only-if part is by Proposition II.6.1, Theorem II.6.4, and the remark above Definition II.6.2. So, we next assume that $\succ$ is

CI, and we derive existence of $\succcurlyeq_A$'s as in the theorem. Let z be an arbitrarily fixed element of X. We *define* $x_A \succcurlyeq_A y_A$ whenever $x_A z_{A^c} \succcurlyeq y_A z_{A^c}$. Note that $\succcurlyeq_I$ thus coincides with $\succcurlyeq$. Reflexivity and transitivity of $\succcurlyeq$ imply the same for any $\succcurlyeq_A$. Finally we must derive total monotonicity. By Theorem II.6.4 we have independence of equal subalternatives.

First note that $x_A \succ_A y_A$ iff $x_A z_{A^c} \succ y_A z_{A^c}$, $x_A \approx_A y_A$ iff $x_A z_{A^c} \approx y_A z_{A^c}$. By Theorem II.5.2, (ii) ⇒ (i), it is sufficient to derive $(\succcurlyeq > s^2S)$ -, $(\succcurlyeq \not\succ Ss^2)$ -, $(\approx s^2S)$ -, and $(\not\precsim Ss^2)$mon. So let $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = B \subset I$. We write $B_0 = B^c$.

For $(\succcurlyeq > s^2S)$mon, suppose that $x_{B_1} \succcurlyeq_{B_1} y_{B_1}$, $x_{B_2} \succ_{B_2} y_{B_2}$. To prove: $x_{B_1} x_{B_2} \succ_{B_1 \cup B_2} y_{B_1} y_{B_2}$. We have $x_{B_1} \succcurlyeq_{B_1} y_{B_1} \Rightarrow z_{B_0} x_{B_1} z_{B_2} \succcurlyeq z_{B_0} y_{B_1} z_{B_2} \Rightarrow$ (by independence of equal subalternatives):

$$z_{B_0} x_{B_1} x_{B_2} \succcurlyeq z_{B_0} y_{B_1} x_{B_2}. \tag{II.6.1}$$

Further , we have $x_{B_2} \succ_{B_2} y_{B_2} \Rightarrow z_{B_0} z_{B_1} x_{B_2} \succ z_{B_0} z_{B_1} y_{B_2} \Rightarrow$ (by independence of equal subalternatives)

$$z_{B_0} y_{B_1} x_{B_2} \succ z_{B_0} y_{B_1} y_{B_2}. \tag{II.6.2}$$

(II.6.1) and (II.6.2) imply $z_{B_0} x_{B_1} x_{B_2} \succ z_{B_0} y_{B_1} y_{B_2}$, i.e. $x_{B_1} x_{B_2} \succ_{B_1 \cup B_2} y_{B_1} y_{B_2}$, as desired.

$(\approx s^2S)$mon is analogous, and not elaborated.

For $(\succcurlyeq \not\succ Ss^2)$mon, suppose $x_{B_1} \succcurlyeq_{B_1} y_{B_1}$, $x_{B_1} x_{B_2} \preccurlyeq_{B_1 \cup B_2} y_{B_1} y_{B_2}$. To prove is $x_{B_2} \preccurlyeq_{B_2} y_{B_2}$. We have $y_{B_1} \preccurlyeq_{B_1} x_{B_1} \Rightarrow z_{B_0} y_{B_1} z_{B_2} \preccurlyeq z_{B_0} x_{B_1} z_{B_2} \Rightarrow$ (by independence of equal subalternatives)

$$z_{B_0} y_{B_1} y_{B_2} \preccurlyeq z_{B_0} x_{B_1} y_{B_2}. \tag{II.6.3}$$

Further, we have $x_{B_1} x_{B_2} \preccurlyeq_{B_1 \cup B_2} y_{B_1} y_{B_2} \Rightarrow z_{B_0} x_{B_1} x_{B_2} \preccurlyeq z_{B_0} y_{B_1} y_{B_2}$. This and (II.6.3) imply $z_{B_0} x_{B_1} x_{B_2} \preccurlyeq z_{B_0} x_{B_1} y_{B_2}$. By independence of equal subalternatives, $z_{B_0} z_{B_1} x_{B_2} \preccurlyeq z_{B_0} z_{B_1} y_{B_2}$ results, i.e. $x_{B_2} \preccurlyeq_{B_2} y_{B_2}$, as desired.

($\not\lesssim Ss^2$)mon is analogous, and not elaborated.

□

Observe in the above theorem that the disaggregated monotonicity properties are essential. For any arbitrary binary relation $\tilde{\succcurlyeq}$ on X that is reflexive, there exist $\succcurlyeq_A$'s to make $\{\succcurlyeq_A : A \subset I\}$ satisfy all *aggregated* monotonicity properties, with $\succcurlyeq_I = \succcurlyeq$: simply let for any $A \neq I$, $x_A \succcurlyeq_A y_A$ if and only if $x_A = y_A$.

Of course, should all $\succcurlyeq_A$'s be complete, then matters are different. By Proposition II.4.1 the disaggregated monotonicity properties can then be left out. That independence of equal subalternatives then is sufficient for the existence of $\succcurlyeq_A$'s to fulfil aggregated monotonicities, (and that independence of equal subalternatives for finite cartesian products is then equivalent to CI,) is known, see Krantz et al. (1971, Lemma 6.1.4.1, (iv) there resembles ($\succcurlyeq \not\succ$ cA mon)), or, when a representing function (Definition III.2.2) is presupposed, see Blackorby, Primont and Russell (1978, section 3.3).

CHAPTER III

ADDITIVE VALUE FUNCTIONS

III.1. INTRODUCTION

This chapter, and following chapters, can be read independently of the previous two chapters. Only some definitions of the previous two chapters are used. When needed, we shall mention these.

In the first three sections of this chapter we give some well-known results from literature. In section III.4 new results will be presented.

As before X, the set of alternatives, is a cartesian product $X_{i\in I}C_i$. With the exception of Chapter V, I will be a finite set $\{1,\ldots,n\}$. By $\succcurlyeq$, a binary relation on X, we express the "preference relation" of decision maker T on X. As before, $\succcurlyeq$ shall always, in our main results, be transitive, and from now on always complete, either as an assumption, or as consequence of other assumptions.

Furthermore, we shall from now on assume that every C_i is a connected topological space. E.g. C_i is a convex subset of a Euclidean space, such as $\mathbb{R}_+^{m_i}$, or $\mathbb{R}$. X is always endowed with the product topology, hence is connected too (see Kelley, 1955, Chapter 3, problem O). In our main results $\succcurlyeq$ will be continuous (Definition III.2.1), either as explicit assumption, or as consequence of other assumptions.

In section I.1.3 we indicated that the set X would sometimes contain hypothetical alternatives, not present in actual situations. In the set-up of Chapters I and II it was not harmful to let X be "too" large. We could then simply let the preference relation ignore the redundant part of X, by letting every redundant alternative of X be incomparable to every other alternative, or by adding only those preferences, involving redundant alternatives, that are necessary to maintain monotonicity and transitivity. Since we, from now on, usually deal with complete preference relations, "ignoring by incomparability" is no longer possible.

A consequence of our topological assumptions is that, if not all of the alternatives are equivalent, then X must be uncountable. This will follow from the remark after Theorem III.3.1 (,combined with the fact that the Y there is separable, if countable).

The above two paragraphs indicate that the combination of completeness and continuity of $\succ$ can be a serious restriction. In Schmeidler (1971) it is shown that transitivity of $\succ$, and continuity (defined appropriately) with respect to a connected topology imply completeness or symmetry of $\succ$. Sonnenschein (1965) gives conditions under which completeness and continuity imply transitivity. A further indication of the restrictiveness of completeness and continuity of $\succ$ may be the implication (ii) $\Rightarrow$ (i) in Theorem III.3.7 in the sequel; this usually is conceived as a surprisingly strong result.

In Krantz et al. (1971), instead of topological assumptions two other assumptions are made, the so-called "Archimedean" and "restricted solvability" (see Definition III.2.12) assumptions. These are less restrictive than our topological assumptions, but still allow the derivation of the results in the sequel of this monograph. We have chosen to use the topological assumptions because they are more customary in literature. Our Proposition III.2.15 will enable the application of the theorems of Krantz et al. (1971) in our topological set-up.

III.2. ELEMENTARY DEFINITIONS AND RESULTS

In this section we give elementary definitions and results from literature. Since we will sometimes use them for other binary relations than just the preference relation $\succ$ on X, we formulate some of them for a general binary relation $\succ'$ on a general set Y.

DEFINITION III.2.1. A weak order $\succcurlyeq'$ on a topological space Y is *continuous* if $\{x \in Y : x \succ' y\}$ and $\{x \in Y : x \prec' y\}$ are open for all $y \in Y$.

A weak order $\succcurlyeq'$ is, of course, continuous if and only if $\{x \in Y : x \preccurlyeq' y\}$ and $\{x \in Y : x \succcurlyeq' y\}$ are closed for all $y \in Y$; this follows by taking complements.

DEFINITION III.2.2. A function $V : Y \to \mathbb{R}$ *represents* a binary relation $\succcurlyeq'$ on Y if, for all $x, y \in Y$, $[x \succcurlyeq' y \Leftrightarrow V(x) \geq V(y)]$.

For the above function V, the term utility function is most usual in literature. We shall however reserve this term for a somewhat different notion in decision making under uncertainty. (See Definition IV.2.2.) Throughout the sequel of this monograph we shall study preference relations for which (special kinds of) representing functions exist. Obviously these preference relations must be weak orders.

We shall almost exclusively study representing functions of the following kind:

DEFINITION III.2.3. A function $V : X_{i=1}^{n} C_i \to \mathbb{R}$ is *additively decomposable* if there exist $V_i : C_i \to \mathbb{R}$, $i = 1,\ldots,n$, such that $V(x) = \Sigma_{i=1}^{n} V_i(x_i)$ for all $x \in X_{i=1}^{n} C_i$. If this V represents $\succcurlyeq$, then $(V_i)_{i=1}^{n}$ are called *additive value functions (for $\succcurlyeq$).*

Usually we are not only interested in the *existence* of a(n array of) function(s) having certain properties, (such as being representing, continuous, additively decomposable, or whatever a context requires), but we are also interested in uniqueness results.

TERMINOLOGY III.2.4. A function V is *ordinal* [respectively *continuously ordinal*] (*with respect to* some properties) if the class of all functions having these properties, consists of all strictly increasing [respectively continuous, strictly increasing] transformations of V.

TERMINOLOGY III.2.5. A function V is *cardinal (with respect to* some properties) if the class of all functions having these properties, consists of all positive affine transformations of V.

An array of functions $(V_j)_{j=1}^n$ is *simultaneously cardinal (with respect to* some properties) if the class of all arrays of functions $(W_j)_{j=1}^n$ having these properties, consists of those $(W_j)_{j=1}^n$, for which real τ_j, $j = 1,\ldots,n$, and positive σ exist such that $W_j = \tau_j+\sigma V_j$ for all j.

To give examples, we define $V : \mathbb{R}^2 \to \mathbb{R}$ by $V : (x_1,x_2) \mapsto x_1+x_2$, and we let $\succ$ on $\mathbb{R}^2$ be represented by V. We shall refer in these examples to theorems, given in the next section. V is ordinal with respect to the property of being representing, as is easily derived from the observations $V(x) \geq V(y) \Leftrightarrow x \succ y$, and $x \succ y \Leftrightarrow W(x) \geq W(y)$, for any representing W. V is continuously ordinal with respect to the properties of being continuous and representing, as follows from Theorem III.3.1. V is cardinal with respect to the properties of being continuous, representing, and additively decomposable, as can easily be derived from Theorem III.3.6. Finally, with V_1, $V_2 : \mathbb{R} \to \mathbb{R}$ being identity, (V_1,V_2) is simultaneously cardinal with respect to the properties of being continuous and being additive value functions, again by Theorem III.3.6.

In the sequel we shall use Notations II.2.1 (subalternative x_A, for $A \subset I$), and II.2.4 ($x_{-A}y_A$, etc., and lines below this Notation); Definitions II.2.3 ($x_{A_1}y_{A_2}$ is compounded of x_{A_1} and y_{A_2}), II.6.2 (independence of equal subalternatives), II.6.3 (CI); and Theorem II.6.4 (equivalence of last two notions). With these we define, deviating in a harmless way from Chapter II, some binary relations on $\times_{i\in A}C_i$.

DEFINITION III.2.6. For every $A \subset I$, and x_A, y_A in $\times_{i\in A}C_i$, we write $x_A \succcurlyeq_A y_A$ [respectively $x_A \succ_A y_A$, or $x_A \prec_A y_A$, or $x_A \preccurlyeq_A y_A$, or $x_A \approx_A y_A$] if there exists a z_{A^c} such that $z_A z_{A^c} \succcurlyeq y_A z_{A^c}$ [respectively $x_A z_{A^c} \succ y_A z_{A^c}$, or $x_A z_{A^c} \prec y_A z_{A^c}$, or $x_A z_{A^c} \preccurlyeq y_A z_{A^c}$, or $x_A z_{A^c} \approx y_A z_{A^c}$].

If $\succcurlyeq$ is CI, the main case of interest in this monograph, then the binary relations defined above coincide with those in Chapter II, denoted in the same way (by Theorem II.6.5 and Proposition II.6.1).

Furthermore, $>_A$ is then the asymmetric, and $\approx_A$ the symmetric, part of $\succcurlyeq_A$, as usual. As always, $\succcurlyeq_i = \succcurlyeq_{\{i\}}$, and $\succcurlyeq_I = \succcurlyeq$.

LEMMA III.2.7. *Let $\succcurlyeq$ be CI. If $\succcurlyeq$ is a weak order, then so is any $\succcurlyeq_A$. If further $\succcurlyeq$ is continuous, then so is any $\succcurlyeq_A$.*

PROOF. Let $A \subset I$, $z \in X$ is arbitrarily fixed. Let $\succcurlyeq$ be CI. By CI, $x_A \succcurlyeq_A y_A$ iff $x_A z_{A^c} \succcurlyeq y_A z_{A^c}$. If $\succcurlyeq$ is complete, then $x_A z_{A^c} \succcurlyeq y_A z_{A^c}$ or $y_A z_{A^c} \succcurlyeq x_A z_{A^c}$, so $x_A \succcurlyeq_A y_A$ or $y_A \succcurlyeq_A x_A$, for all x_A, y_A. Completeness of $\succcurlyeq_A$ follows.

If $\succcurlyeq$ is transitive, then $x_A \succcurlyeq_A y_A$ and $y_A \succcurlyeq_A v_A$ (i.e. $x_A z_{A^c} \succcurlyeq y_A z_{A^c}$ and $y_A z_{A^c} \succcurlyeq v_A z_{A^c}$) together imply $x_A z_{A^c} \succcurlyeq v_A z_{A^c}$, i.e. $x_A \succcurlyeq_A v_A$. Transitivity of $\succcurlyeq_A$ follows.

The above observations about completeness and transivity show that $\succcurlyeq$ being a weak order implies that any $\succcurlyeq_A$ is a weak order.

Next suppose that $\succcurlyeq$ is continuous. We derive continuity of $\succcurlyeq_A$. $\{x_A : x_A \succcurlyeq_A y_A\} = \{x_A : x_A z_{A^c} \succcurlyeq y_A z_{A^c}\} = \{x_A : x_A z_{A^c} \in C\}$, with C the closed set $\{w : w \succcurlyeq y_A z_{A^c}\}$.

By Lemma 0.1 the set $\{x_A : x_A \succcurlyeq_A y_A\}$ must be closed too. Analogously $\{x_A : x_A \preccurlyeq_A y_A\}$ is closed. Continuity of $\succcurlyeq_A$ follows.

□

The following definition of inessentiality of i expresses the idea that a coordinate has no influence on the "desirability" of any alternative, so that this coordinate may just as well be ignored for the preference relation.

DEFINITION III.2.8. Coordinate (or index) i is *inessential (with respect to $\succcurlyeq$)* if $x_{-i}v_i \approx x_{-i}w_i$ for all $x \in X$, v_i, $w_i \in C_i$. Otherwise i is *essential (with respect to $\succcurlyeq$)*.

For a weak order $\succcurlyeq$, i is essential if and only if $v_i >_i w_i$ for some v_i, w_i.

LEMMA III.2.9. *Let $\approx$ be an equivalence relation. Let $x_j = y_j$ for all essential j. Then $x \approx y$.*

PROOF. Let there be k inessential coordinates, say $\{1,\dots,k\}$. Then $x \approx (x_{-1}y_1) \approx (x_{-1,2}y_1,y_2) \approx \dots \approx (x_{-1,\dots,k}y_1,\dots,y_k) = y$. Apply transitivity of $\approx$.

□

The above Lemma shows that the inessential coordinates may just as well be suppressed from notation. That we shall sometimes do.

LEMMA III.2.10. *Let $\approx$ be an equivalence relation. Then $\succ$ is trivial if and only if no coordinate is essential.*

PROOF. If no coordinate is essential, then Lemma III.2.9 gives triviality of $\succ$. If $\succ$ is trivial, then $x_{-i}v_i \approx x_{-i}w_i$ for all x, i, v_i, w_i : no i is essential.

□

We now formulate the topological assumption that we shall mostly use in the sequel.

ASSUMPTION III.2.11. (*Topological Assumption.*)

Every C_i is a connected topological space.

$X = \times_{i=1}^{n} C_i$ is endowed with the product topology.

If exactly one coordinate i is essential, then furthermore C_i is topologically separable.

DEFINITION III.2.12. $\succ$ satisfies *restricted solvability* if, for every $x_{-i}s_i \succ y \succ x_{-i}t_i$, there exists z_i such that $x_{-i}z_i \approx y$.

LEMMA III.2.13. *Let the topological assumption III.2.11 hold. Let $\succ$ be a continuous weak order. Then $\succ$ satisfies restricted solvability.*

PROOF. Let $x_{-i}s_i \succ y \succ x_{-i}t_i$. Let $V := \{v_i \in C_i : x_{-i}v_i \succ y\}$, and $W := \{w_i \in C_i : x_{-i}w_i \prec y\}$. Then $s_i \in V$ and $t_i \in W$, so V and W are nonempty. By Lemma 0.1 they are closed. Their union is C_i. By connectedness of C_i, $V \cap W \neq \emptyset$. Let $z_i \in V \cap W$.

□

LEMMA III.2.14. *Let the topological assumption III.2.11 hold. Let $\succ$ be a continuous weak order. Let $x_{-A}s_A \succ y \succ x_{-A}t_A$. Then z_A exists with $x_{-A}z_A \approx y$.*

PROOF. Apply Lemma III.2.13 to the cartesian product

$$(X_{j \in A} C_j) \times (X_{i \notin A} C_i).$$

□

The following proposition will be used as a supplement to results of Krantz et al. (1971, Chapter 6), so that we can use their results in our topological set-up, where we need continuity.

PROPOSITION III.2.15. *Let the topological assumption III.2.11 hold. Let $\succ$ be continuous, and let at least two coordinates be essential. Let $(V_j)_{j=1}^n$ be additive value functions. Then all V_j's are continuous.*

PROOF. Suppose V_1 is *not* continuous. Contradiction will follow. Say $V_1^{-1}(]\mu, \infty[)$ is not open; then neither it is empty, nor does it equal C_1. Also there can be no sequence $(V_1(x_1^j))_{j=1}^{\infty}$ in $]\mu, \infty[$, converging to μ, because then $V_1^{-1}(]\mu, \infty[)$ would equal $\cup_j\{z_1 : z_1 \succ_1 x_1^j\}$ and so be open by the easily verified CI, and Lemma III.2.7. So $\inf(V_1(C_1) \cap]\mu, \infty[) =: \nu \in \mathbb{R}$ must be greater than μ. We now show:

$$0 < V_j(x_j) - V_j(y_j) < \nu - \mu \text{ for no } j \neq 1, x_j, y_j. \qquad \text{(III.2.1)}$$

If, to the contrary, $j \neq 1$ and $0 < V_j(x_j) - V_j(y_j) < \nu-\mu$, then, with $z \in X$ arbitrarily fixed, and a_1 such that $V_j(x_j)-V_j(y_j) > V_1(a_1)-\nu \geq 0$, c_1 such that $V_1(c_1) \leq \mu$, we obtain:

$$(z_{-1,j}c_1,x_j) \prec (z_{-1,j}a_1,y_j) \prec (z_{-1,j}a_1,x_j),$$

by substitutions of inequalities in terms of the V_j's. By restricted solvability (Lemma III.2.13), $(z_{-1,j}b_1,x_j) \approx (z_{-1,j}a_1,y_j)$ for some b_1. This would imply $\nu > V_1(b_1) > \mu$, in contradiction with the definition of ν. (III.2.1) is derived.

Now $j \neq 1$ exists that is essential, so has V_j not constant. Say $V_j(x_j) > V_j(y_j)$. There must, as a consequence of (III.2.1), be a z_j such that $V_j(x_j) > V_j(z_j)$, and $V_j(x_j) > V_j(v_j) > V_j(z_j)$ for no $v_j \in C_j$. This finally gives a partition $(\{v_j : v_j \preccurlyeq_j z_j\}, \{v_j : v_j \succcurlyeq_j x_j\})$ of C_j, consisting of two nonempty closed sets. This contradicts connectedness of C_j.

□

III.3. BASIC RESULTS ON ADDITIVE DECOMPOSABILITY

The representation theorems from literature, given in this section, underly all results in the sequel.

III.3.1. LESS THAN TWO ESSENTIAL COORDINATES

The following theorem, proved in Debreu (1954, 1964) does not consider cartesian product structure.

THEOREM III.3.1. (Debreu). *Let* Y *be a connected separable topological space. For a binary relation* $\succcurlyeq'$ *on* Y *the following two statements are equivalent:*

(i) : *There exists a continuous representing function* $\phi : Y \to \mathbb{R}$.

(ii) : $\succcurlyeq'$ *is a continuous weak order.*

Furthermore, ϕ *in* (i) *is continuously ordinal.*

□

From this one sees that, if ϕ is not constant, then $\phi(Y)$ is a nondegenerate interval, and Y must be uncountable. That the connectedness condition above cannot be dispensed with, is indicated in Fleischer (1961) and Wakker (1985a). We shall use the following small variation of the above Theorem.

COROLLARY III.3.2. *Let at most one coordinate be essential. Let the topological assumption III.2.11 hold. For the binary relation* $\succ$ *on* $\times_{i=1}^{n} C_i$ *, the following two statements are equivalent:*

(i) *There exist continuous additive value functions* $(V_j)_{j=1}^{n}$ *for* $\succ$.

(ii) $\succ$ *is a continuous weak order.*

Furthermore, the V_j*'s in* (i) *are continuously ordinal, and* $\succ$ *in* (ii) *satisfies CI.*

PROOF. (i) $\Rightarrow$ (ii) is straightforward. So we assume (ii), and derive (i) and the Furthermore-statement. If no coordinate is essential, then $\succ$ is trivial by Lemma III.2.10. Then we can, and must, let all V_j's be arbitrary constant functions, and everything follows.

So suppose one coordinate i is essential. By Lemma III.2.7, $\succ_i$ is a continuous weak order. By Theorem III.3.1 there exists a continuous, and continuously ordinal, function $\phi : C_i \to \mathbb{R}$ that represents $\succ_i$. We can let $V_i = \psi \circ \phi$ for any continuous strictly increasing $\psi : \phi(C_i) \to \mathbb{R}$; and for all $j \neq i$ let V_j be any constant function. Then, for any $x, y \in X$, $\Sigma_{j=1}^{n} V_j(x_j) \geq \Sigma_{j=1}^{n} V_j(y_j)$ iff $V_i(x_i) \geq V_i(y_i)$, which is iff $x_i \succ_i y_i$. The latter is iff there exists z such that $z_{-i}x_i \succ z_{-i}y_i$. Inessentially of all $j \neq i$, by Lemma III.2.9, gives $z_{-i}x_i \approx x$ and $z_{-i}y_i \approx y$. We conclude that $\Sigma_{j=1}^{n} V_j(x_j) \geq \Sigma_{j=1}^{n} V_j(y_j) \Leftrightarrow x \succ y$. Indeed, $(V_j)_{j=1}^{n}$ are additive value functions. They are continuous too, and (i) follows.

For the Furthermore-statement, note that any V_j must represent $\succ_j$. Hence for all $j \neq i$, V_j must be constant; and V_i must be a strictly increasing, by Lemma VIII.5 continuous, transformation of ϕ. Finally, that $\succ$ in (ii) satisfies CI, follows from (i), and the observation that $x_{-i}\alpha \succ y_{-i}\alpha \Leftrightarrow \Sigma_{j \neq i} V_j(x_j) \geq \Sigma_{j \neq i} V_j(y_j) \Leftrightarrow x_{-i}\beta \succ y_{-i}\beta$.

□

III.3.2. EXACTLY TWO ESSENTIAL COORDINATES

The previous subsection, with at most one essential coordinate, hardly dealt with the cartesian product structure. In essence, we only had to deal with the essential coordinate. In this subsection we consider the case of two essential coordinates. It turns out, as in Krantz et al. (1971), that then the requirement of topological separability can be dropped. In this section we give the, completely straightforward, adaptation of results in literature for the case of exactly two coordinates, both essential, to the case of n coordinates, two of them essential. The following property is illustrated in Figure III.3.1.

DEFINITION III.3.3. If $\succ$ has exactly two essential coordinates i,j, then $\succ$ satisfies the *Thomsen condition* if $(x_{-i,j}a_i,t_j) \approx (x_{-i,j}b_i,s_j)$ & $(x_{-i,j}b_i,v_j) \approx (x_{-i,j}c_i,t_j)$ together imply $(x_{-i,j}a_i,v_j) \approx (x_{-i,j}c_i,s_j)$; for every $x \in X$; $a_i, b_i, c_i \in C_i$; $s_j, t_j, v_j \in C_j$.

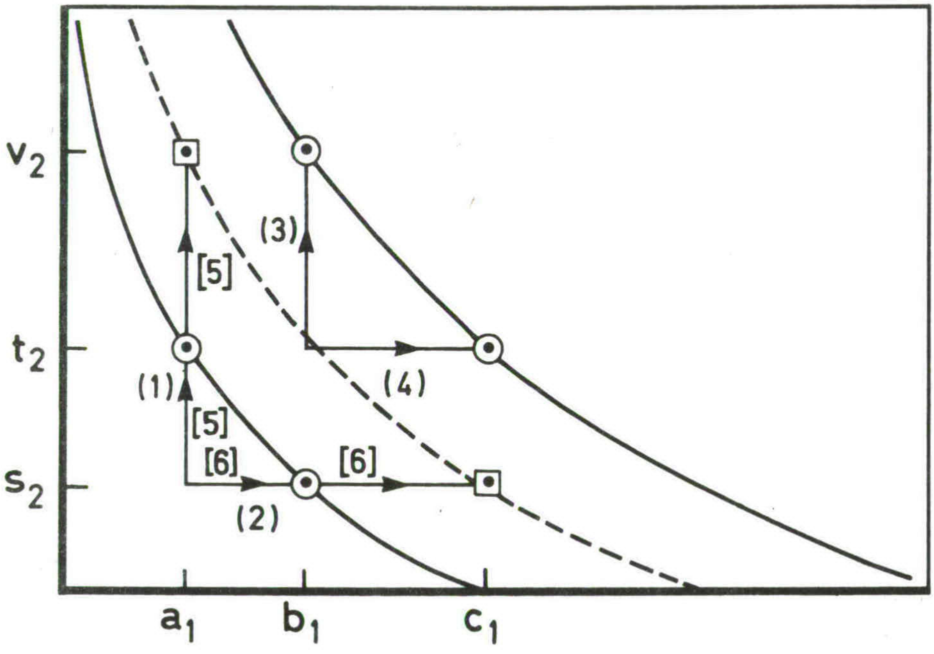


FIGURE III.3.1. The *Thomsen condition* for n = 2, i = 1, j = 2. Curves indicate equivalence classes. The solid curves through O-points are presumed; the broken curve through □-points is implied. One can interpret $(a_1,t_2) \approx (b_1,s_2)$ to mean: substitution (1) of t_2 for s_2 is as good as substitution (2) of b_1 for a_1. And $(b_1,v_2) \approx (c_1,t_2)$: substitution (3) of v_2 for t_2 is as good as substitution (4) of c_1 for b_1. The conclusion $(a_1,v_2) \approx (c_1,s_2)$: the "concatenated" substitution [5] of v_2 for s_2, is as good as the "concatenated" substitution [6] of c_1 for a_1.

At first sight the above definition may seem asymmetric for i and j. Interchanging i and j, and interchanging the first two equivalences $\approx$, shows that symmetry for i and j does hold.

LEMMA III.3.4. *If $\succ$ has exactly two essential coordinates* i *and* j, *and if additive value functions* $(V_k)_{k=1}^n$ *exist for* $\succ$, *then* $\succ$ *satisfies the Thomsen condition.*

PROOF. $\Sigma_{k\neq i,j}V_k(x_k) + V_i(a_i) + V_j(t_j) = \Sigma_{k\neq i,j}V_k(x_k) + V_i(b_i) + V_j(s_j)$ and $\Sigma_{k\neq i,j}V_k(x_k) + V_i(b_i) + V_j(v_j) = \Sigma_{k\neq i,j}V_k(x_k) + V_i(c_i) + V_j(t_j)$ together imply

$V_i(a_i) + V_j(t_j) = V_i(b_i) + V_j(s_j)$ and $V_i(b_i) + V_j(v_j) = V_i(c_i) + V_j(t_j)$.

Summing and cancelling gives:

$V_i(a_i) + V_j(v_j) = V_i(c_i) + V_j(s_j)$, or :

$\Sigma_{k\neq i,j}V_k(x_k) + V_i(a_i) + V_j(v_j) = \Sigma_{k\neq i,j}V_k(x_k) + V_i(c_i) + V_j(s_j)$.

□

The following property is a preparation for cardinal coordinate independence (Definition IV.2.4) and is illustrated in Figure III.3.2.

DEFINITION III.3.5. If $\succ$ has exactly two essential coordinates i, j, then $\succ$ satisfies *triple cancellation* if $(s_{-i,j}a_i,x_j) \prec (s_{-i,j}b_i,y_j)$ & $(s_{-i,j}c_i,x_j) \succ (s_{-i,j}d_i,y_j)$ & $(s_{-i,j}a_i,v_j) \succ (s_{-i,j}b_i,w_j)$ together imply $(s_{-i,j}c_i,v_j) \succ (s_{-i,j}d_i,w_j)$.

Again, the property can be seen to be symmetric in i and j, by interchanging second and third preference.

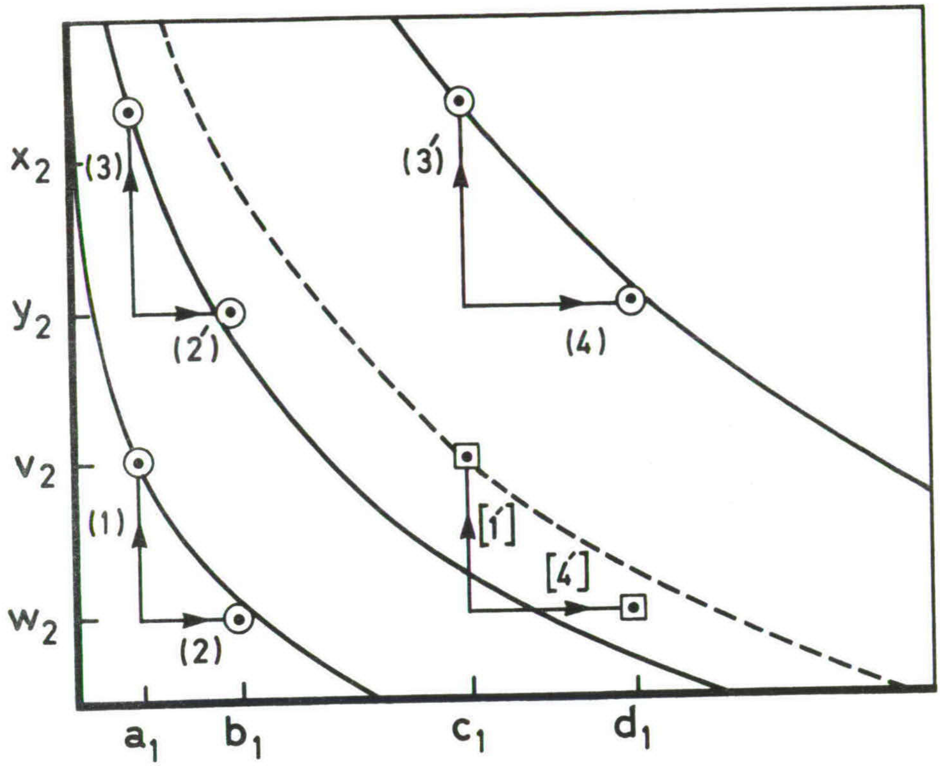


FIGURE III.3.2. *Triple cancellation* for n = 2, i = 1, j = 2. Curves indicate equivalence classes. A point above or on an equivalence class is at least as good, a point below or on it at least as bad, as the points on the equivalence class. The solid curves through and above/below O-points are presumed, the broken one through and above □-points is implied. One can interpret $(a_1,v_2) \succ (b_1,w_2)$ to mean: substitution (1) of v_2 for w_2 is at least as good as substitution (2) of b_1 for a_1. And $(a_1,x_2) \prec (b_1,y_2)$: substitution (2') of b_1 for a_1 is at least as good as substitution 3 of x_2 for y_2. Further $(c_1,x_2) \succ (d_1,y_2)$: substitution (3') of x_2 for y_2 is at least as good as substitution (4) of d_1 for c_1. The conclusion $(c_1,v_2) \succ (d_1,w_2)$: substitution [1'] of v_2 for w_2 is at least as good as substitution [4'] of d_1 for c_1.

Again it can easily be demonstrated that existence of additive value functions implies triple cancellation. The term "triple cancellation" comes from Krantz et al. (1971). In Keeney and Raiffa (1975) the term "corresponding tradeoffs condition" is used for the same property with $\approx$ instead of $\succ$ or $\prec$ everywhere. This is closely related to the "Reidemeister condition" in Blaschke and Bol (1938).

THEOREM III.3.6. *Let the topological assumption III.2.11 hold. Let exactly two coordinates be essential. For the binary relation* $\succ$ *on* $X_{i=1}^{n} C_i$, *the following three statements are equivalent:*

(i) *There exist continuous additive value functions* $(V_j)_{j=1}^{n}$ *for* $\succ$.

(ii) $\succ$ *is a continuous weak order that satisfies CI and the Thomsen condition.*

(iii) $\succ$ *is a continuous weak order that satisfies triple cancellation.*

Furthermore, $(V_j)_{j=1}^{n}$ *in* (i) *is simultaneously cardinal.*

PROOF. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are straightforward. For (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i), let i and j be the two essential coordinates. The other coordinates do not affect the preference relation, e.g. $(s_{-i,j}a_i,x_j) \succ (t_{-i,j}b_i,y_j)$ iff $(a_i,x_j) \succ_{\{i,j\}} (b_i,y_j)$, for all s,t. Hence, the Thomsen condition (respectively CI; or triple cancellation) for $\succ$ implies the same condition for $\succ_{i,j}$. So by Lemma III.2.7, (ii) (respectively (iii)) for $\succ$ implies (ii) (respectively (iii)) for $\succ_{\{i,j\}}$.

Now (ii) for $\succ_{\{i,j\}}$ implies the existence of simulaneously cardinal additive value functions (V_i, V_j) for $\succ_{\{i,j\}}$ on $C_i \times C_j$, as can be derived from Theorem 2 of section 6.2.4 of Krantz et al. (1971), in the same way that Theorem 14 of section 6.11.1 of that book is derived from Theorem 13 there. The reasoning of section 6.2.13 there applies literally for n = 2. See also exercise 34 of chapter 6 there.

Also (iii) for $\succ_{\{i,j\}}$ implies the existence of simultaneously cardinal additive value functions for $\succ_{\{i,j\}}$. A hint in this direction is given at the end of section 6.2.4 of Krantz et al. (1971).

For every $k \neq i,j$ we can, and must, let V_k be any constant function. It then follows that indeed $(V_k)_{k=1}^{n}$ are simultaneously cardinal additive value functions for $\succ$, if (V_i,V_j) are for $\succ_{i,j}$. Continuity is by Proposition III.2.15. So (ii) implies (i), (iii) implies (i) too; and the Furthermore-statement holds.

□

III.3.3. MORE THAN TWO ESSENTIAL COORDINATES

Surprisingly, when there are three or more essential coordinates, the structure turns out to be rich enough to enable a further weakening of the conditions which we met in the previous subsections.

The following theorem is essentially due to Debreu (1960). We give it in a slightly stronger form, by leaving out the assumption of topological separability, an assumption made and essentially used in the proof by Debreu. Krantz et al. (1971, Theorem 6.14) showed that without that assumption, still additive value functions exist. Combined with Proposition III.2.15, this gives:

THEOREM III.3.7. (Debreu, 1960). *Let the topological assumption III.2.11 hold. Let three or more coordinates be essential. For the binary relation* $\succcurlyeq$ *on* $X_{i=1}^{n} C_i$, *the following two statements are equivalent:*

(i) *There exist continuous additive value functions* $(V_j)_{j=1}^{n}$ *for* $\succcurlyeq$.

(ii) $\succcurlyeq$ *is a continuous CI weak order.*

Furthermore, $(V_j)_{j=1}^{n}$ *of* (i) *is simultaneously cardinal.*

PROOF. By Theorem 14 of section 6.11.1 of Krantz et al. (1971), and Proposition III.2.15.

□

III.4. SOME FURTHER RESULTS ON ADDITIVE DECOMPOSABILITY

The results of this section are from section 3 in Wakker (1985b), and will be used only in section IV.4. They may have interest of their own, since they can be considered stronger than previous results (Theorems III.3.6 and III.3.7) in this Chapter.

DEFINITION III.4.1. $\succ$ satisfies *weak separability* if $x_{-i}v_i \succ x_{-i}w_i \Rightarrow y_{-i}v_i \succ y_{-i}w_i$ for all $x, y \in X$, $1 \leq i \leq n$, $v_i, w_i \in C_i$.

The above property expresses some sort of monotonicity of $\succ$ with respect to the $\succ_i$'s ($\succ$ cA mon, in the terminology of Chapter II, see (III.4.1) in the sequel). It is well-known that the above property is necessary, and under some further assumptions sufficient, for the existence of functions $\phi_j : C_j \to \mathbb{R}$, and a function F that is strictly increasing in each variable, such that $x \mapsto F(\phi_1(x_1), \ldots, \phi_n(x_n))$ represents $\succ$. Also weak separability is implied by CI; it is in fact the independence of equal subalternatives property, restricted to equal subalternatives of length n-1.

DEFINITION III.4.2. $\succ$ satisfies *equivalence-coordinate independence (eq-CI)* if: $[x_{-i}v_i \approx y_{-i}v_i \Leftrightarrow x_{-i}w_i \approx y_{-i}w_i]$ for all x, y, i, v_i, w_i.

To see that this is implied by CI, let $x_{-i}v_i \approx y_{-i}v_i$. Then $x_{-i}v_i \succcurlyeq y_{-i}v_i$ and $y_{-i}v_i \succcurlyeq x_{-i}v_i$, by twice CI we obtain $x_{-i}w_i \succcurlyeq y_{-i}w_i$ and $y_{-i}w_i \succcurlyeq x_{-i}w_i$, i.e. $x_{-i}w_i \approx y_{-i}w_i$.

For the property defined below, in view of Definition II.6.3 (CI), III.4.2 (eq-CI), and also in view of Definitions IV.2.4 (CCI) and IV.2.6 (eq-CCI) of Chapter IV, the name "equivalence-triple cancellation", derived from "triple cancellation" (Definition III.3.5), could have been chosen for it. We deviate slightly from literature by formulating it for the case of two essential, instead of two, coordinates. Also in literature the property is usually formulated in terms of a (representing) function, instead of in terms of $\succcurlyeq$.

DEFINITION III.4.3. If $\succcurlyeq$ has exactly two essential coordinates i, j, then $\succcurlyeq$ satisfies the *Reidemeister condition* if $(s_{-i,j}a_i,x_j) \approx (s_{-i,j}b_i,y_j)$ & $(s_{-i,j}c_i,x_j) \approx (s_{-i,j}d_i,y_j)$ & $(s_{-i,j}a_i,v_j) \approx (s_{-i,j}b_i,w_j)$ together imply $(s_{-i,j}c_i,v_j) \approx (s_{-i,j}d_i,w_j)$ for all $s, a_i, b_i, c_i, d_i, x_j, y_j, v_j, w_j$.

Again, this is implied by triple cancellation: The first three equivalences $\approx$, $\approx$, $\approx$ imply $\preccurlyeq$, $\succcurlyeq$, $\succcurlyeq$, and hence, by triple cancellation, give $(s_{-i,j}c_i,v_j) \succ (s_{-i,j}d_i,w_j)$, and they also imply $\succ$, $\prec$, $\prec$, to give, by triple cancellation, $(s_{-i,j}c_i,v_j) \prec (s_{-i,j}d_i,w_j)$.

LEMMA III.4.4. *Let the topological assumption III.2.11 hold. Let $\succcurlyeq$ be a continuous weak order. The following two statements are equivalent:*

(i) $\succcurlyeq$ *satisfies CI.*

(ii) $\succcurlyeq$ *satisfies weak separability and eq-CI.*

PROOF. We have already seen above that (i) implies (ii). So we assume (ii), and derive (i).

First let us show:

$$\text{If } v_j \succcurlyeq_j w_j \text{ for all } j, \text{ then } v \succcurlyeq w. \tag{III.4.1}$$

This follows, by repeated application of weak separability, from $v \succcurlyeq v_{-1}w_1 \succcurlyeq (v_{-1}w_1)_{-2}w_2 \succcurlyeq ((v_{-1}w_1)_{-2}w_2)_{-3}w_3 \succcurlyeq \ldots \succcurlyeq w$.

Now suppose $x_{-i}v_i \succcurlyeq y_{-i}v_i$. To derive is $x_{-i}w_i \succcurlyeq y_{-i}w_i$. Let $A = \{j \neq i : x_j >_j y_j\}$. Say $A = \{1, \ldots, k\}$, and $i = n$; with $0 \leq k < n$. For all $j \notin A$, not $x_j >_j y_j$, so $z_{-j}x_j > z_{-j}y_j$ for *no* z, and $x_j \preccurlyeq_j y_j$ follows for these j. By (III.4.1) we obtain $(x_{-A}y_A)_{-n}v_n \preccurlyeq y_{-n}v_n \preccurlyeq x_{-n}v_n$. Let $x^0 = x_{-n}v_n$, $x^l = (x^{l-1}_{-l}y_l)$ for all $1 \leq l \leq k$. By weak separability $x_{-n}v_n = x^0 \succcurlyeq x^1 \succcurlyeq \ldots \succcurlyeq x^k = (x_{-A}y_A)_{-n}v_n$. Let now l be such that $x^{l-1} \succcurlyeq y_{-n}v_n \succcurlyeq x^l$. By restricted solvability (Lemma III.2.13) there exists z_l such that $x^l_{-l}z_l \approx y_{-n}v_n$. Now $x^l_{-l}z_l$ has n-th coordinate v_n, so by eq-CI we obtain $y_{-n}w_n \approx (x^l_{-l}z_l)_{-n}w_n$. That $x_l \succcurlyeq_l z_l$ follows from $x^l_{-l}x_l = x^{l-1} \succcurlyeq y_{-n}v_n \approx x^l_{-l}z_l$. Apparently $(x_{-n}w_n)_j \succcurlyeq_j ((x^l_{-l}z_l)_{-n}w_n)_j$ for all j. By (III.4.1), $(x_{-n}w_n) \succcurlyeq (x^l_{-l}z_l)_{-n}w_n$, the latter was equivalent to $y_{-n}w_n$.

□

The implication (ii) ⇒ (i) above does not have to hold if the continuity assumption does not hold. This can be seen from $\succcurlyeq$, defined as follows. First, let $V : \mathbb{R}^3 \to \mathbb{R}$ be defined by $V : x \mapsto x_1+x_2+x_3+\min\{x_1, x_2, x_3\}$. Then define $x \succ y$ whenever $V(x) > V(y)$, or $V(x) = V(y)$ and $x_1 > y_1$, or $V(x) = V(y)$ and $x_1 = y_1$ and

$x_2 > y_2$, or $x = y$. Then $x \approx y$ only if $x = y$, and eq-CI is trivially satisfied. For every i, $\succ_i = \underline{>}$, and weak separability follows. But $(9,1,1) \succ (5,5,1)$ and $(9,1,9) \prec (5,5,9)$ violate CI.

LEMMA III.4.5. *Let the topological assumption III.2.11 hold. Let* $\succ$ *be a continuous weak order, and let exactly two coordinates* i, j *be essential. The following two statements are equivalent:*

(i) $\succ$ *satisfies triple cancellation.*

(ii) $\succ$ *satisfies weak separability and the Reidemeister condition.*

PROOF. The implication (i) $\Rightarrow$ (ii) can be obtained by elementary means; or as a corollary from Theorem III.3.6. So we assume (ii), and derive (i). We suppress inessential coordinates from notation. Say the first two coordinates are essential. Let now $(a_1,x_2) \prec (b_1,y_2)$, $(c_1,x_2) \succ (d_1,y_2)$, $(a_1,v_2) \succ (b_1,w_2)$. To derive is $(c_1,v_2) \succ (d_1,w_2)$.

If we can find $a_1' \succ_1 a_1$, $b_1' \prec_1 b_1$, $c_1' \prec_1 c_1$, $d_1' \succ_1 d_1$, $v_2' \prec_2 v_2$, $w_2' \succ_2 w_2$, such that $(a_1',x_2) \approx (b_1',y_2)$, $(c_1',x_2) \approx (d_1',y_2)$, $(a_1',v_2') \approx (b_1',w_2')$, then we may conclude from the Reidemeister condition that $(c_1',v_2') \approx (d_1',w_2')$, and, by weak separability that $(c_1,v_2) \succ (c_1',v_2') \approx (d_1',w_2') \succ (d_1,w_2)$, which is what is desired. So all that remains is to find a_1', ..., w_2' as above.

First we use $(a_1,x_2) \prec (b_1,y_2)$ and $(c_1,x_2) \succ (d_1,y_2)$ to find a_1', b_1'. If $(c_1,x_2) \succ (b_1,y_2)$, then by restricted solvability (Lemma III.2.13) from $(a_1,x_2) \prec (b_1,y_2) \prec (c_1,x_2)$ we conclude that a_1' must exist such that $(a_1',x_2) \approx (b_1,y_2)$. Here $a_1' \succ_1 a_1$. We then take $b_1' = b_1$. The other case is $(c_1,x_2) < (b_1,y_2)$. Then we take $a_1' = c_1$ if $(c_1,x_2) > (a_1,x_2)$, and $a_1' = a_1$ if $(a_1,x_2) \succ (c_1,x_2)$. We then, in any way, have $(d_1,y_2) \prec (a_1',x_2) \prec (b_1,y_2)$. Restriced solvability gives existence of b_1' such that $(b_1',y_2) \approx (a_1',x_2)$. Again here $b_1' \prec_1 b_1$, also $a_1' \succ_1 a_1$. So always a_1', b_1' are found such that $a_1' \succ_1 a_1$, $b_1' \prec_1 b_1$, and $(a_1',x_2) \approx (b_1',y_2)$.

Analogously one uses $(d_1,y_2) \prec (c_1,x_2)$ and $(b_1,y_2) \succ (a_1,x_2)$ to find d_1' and c_1' as desired.

Analogously (exchange the role of the first and the second coordinate), one uses $(b_1',w_2) \prec (a_1',v_2)$ [since $a_1' \succ_1 a_1$, $b_1 \succ_1 b_1'$] and $(b_1',y_2) \succ$ (even $\approx$) (a_1',x_2) to find v_2' and w_2' as desired.

□

A straightforward consequence of the above Lemmas is the following theorem.

THEOREM III.4.6. *Let the topological assumption III.2.11 hold. For the binary relation* $\succ$ *on* $X_{i=1}^{n} C_i$, *the following two statements are equivalent:*

(i) *There exist continuous additive value functions* $(V_j)_{j=1}^{n}$ *for* $\succ$.

(ii) $\succ$ *is a continuous weak order that satisfies weak separability, eq-CI, and, in the case of exactly two essential coordinates, the Reidemeister condition.*

The following uniqueness results hold for $(V_j)_{j=1}^{n}$ *of* (i).

If two or more coordinates are essential, then $(V_j)_{j=1}^{n}$ *is simultaneously cardinal.* (III.4.2)

If exactly one coordinate is essential, then the V_j*'s are continuously ordinal.* (III.4.3)

PROOF. By Corollary III.3.2, Theorems III.3.6, III.3.7, and the Lemmas III.4.4 and III.4.5.

□

III.5. HISTORICAL REMARKS ON ADDITIVELY DECOMPOSABLE REPRESENTATIONS

In Blaschke and Bol (1938) the following problem of "web theory" was studied: suppose F_1, F_2, F_3 are three families of curves in the plane, such that through every point of the plane, for every family F_j, exactly one curve from F_j goes through this point. When can continuous transformations V_1 and V_2 be applied to the first and second coordinates, to transform the three families of curves into three families of parallel straight lines? If one now lets F_1 correspond to lines with constant first coordinate, F_2 to lines with constant second coordinate, and F_3 to equivalence classes of the preference relation, then the matter is closely related to the problem that we addressed in subsection III.3.2, for the case that $n = 2$, $C_1 = C_2 = \mathbb{R}$, and where $\succ$ should satisfy, for example, strong cA monotonicity.

In Blaschke and Bol (1938) conditions like the Thomsen condition already appeared.

Debreu (1960) showed the way, with Theorem III.3.1 as a starting point, to use the above results to obtain characterizations of continuous additively decomposable representations for binary relations on cartesian products of separable connected topological spaces. He proved that coordinate independence (together with continuity, transitivity and completeness) is the necessary and sufficient condition for the case of three or more essential coordinates. By this, Debreu also extended earlier work of Leontief (1947 a,b), who considered Euclidean spaces, presupposing the existence of "smooth" representing functions, and then obtained conditions requiring that rates of substitution of pairs of coordinates be independent of other coordinates. Results as those of Leontief had earlier been obtained by Sono (1945, 1961), but this had not been well-known. See also Samuelson (1947, pp. 174-180). Further already Fleming (1952; treated in section 4.9 of Harsanyi, 1977) had obtained a derivation of additive decomposability, on $\mathbb{R}^n_+$. His main characterizing property was even weaker than coordinate independence, it was, essentially, independence of equal subalternatives of only length n-2, formulated in terms of a (presupposed, representing) function, already without the use of derivatives; see his Postulate E.

Gorman (1968,a,b) showed, for cartesian products of topologically separable arcconnected spaces, how in fact coordinate independence can be weakened, still remaining strong enough together with the other assumptions, to imply coordinate independence. His weakening requires the independence of equal subalternatives condition for only certain subsets A of I. In Vind (1971) the extendability of Gorman's Theorem to cartesian products of separable connected (instead of arcconnected) topological spaces is indicated. See also Gorman (1971) and Murphy (1981).

Another result can be found in Krantz et al. (1971, chapter 6). They use an algebraic approach, employing a theorem of Hölder (1901) on the possibility to embed archimedean ordered groups into the reals. First they use reasonings such as those below Figures III.3.1 and

III.3.2 to derive differences or concatenation - like operations on every coordinate set C_i. Next they use this and results such as the Theorem of Hölder to construct the additive value functions on the coordinate sets. They ascribe their method of proof to Holman (1971). For further history on their approach, see section 6.2.5 of their book.

In Keeney and Raiffa (1975, sections 3.4 to 3.6) one finds, for Euclidean spaces, an appealing sketch of the main ideas of the proofs. Another appealing proof for Euclidean spaces is provided in Koopmans (1972). For the case of two essential coordinates, there is a proof in Roberts (1979).

Many results on weakenings of coordinate independence for Euclidean spaces in the spirit of Gorman's weakenings are given in Blackorby, Primont and Russell (1978).

Necessary and sufficient conditions for the existence of additively decomposable representations, without any restrictive assumptions, have been obtained in Jaffray (1974). For the case of finite C_i's, such conditions have longer been known, see Scott (1964, section 1). They can be obtained from separating hyperplane theorems, and standard ways of application of these to the solution of systems of inequalities. Jaffray used an Archimedean-like strengthening of Scott's conditions, excluding "infinitesimally small" differences.

CHAPTER IV

CARDINAL COORDINATE INDEPENDENCE

IV.1. INTRODUCTION

In this chapter we shall assume that $X = C^n$ for some connected topological space C; so, in comparison with previous chapters, we add the assumption that $C_i = C$ for all i. We shall study representations of the form $x \mapsto \Sigma_{j=1}^{n}\lambda_j U(x_j)$. Our main intended application lies in decision making under uncertainty (DMUU). Hence we shall use terminology of DMUU in this chapter, and chapters V and VI; with the exception of section IV.4 and part of section IV.5. For DMUU, Theorem IV.3.3 (given in Wakker, 1984a, for $C = \mathbb{R}$; and in Wakker, 1986 , for C any connected topological space), the central result of this and following chapters, shows for the case of a finite state space, that a person with a continuous weak order as preference relation maximizes subjective expected utility, if and only if his preference relation satisfies cardinal coordinate independence (Definition IV.2.4). The more complicated conditions for infinite state spaces are given in Chapter V. Thus we have characterized subjective expected utility maximization under only one restriction: continuity of the utility function, with respect to a connected topology (e.g. a Euclidean topology). Like Savage (1954) we derive probabilities and utilities simultaneously, without supposing that any of them are known in advance.

In section IV.4 we characterize the above representation for the case where some of the λ_j's may also be negative. This result, and applications to the theory of economic indexes, are given in Wakker (1985b). Here we indicate an application to dynamic contexts: a characterization, alternative to the one in Koopmans (1972), of a representation of the form $x \mapsto \Sigma\lambda^j U(x_j)$.

In section IV.5 we obtain a stronger result than Theorem IV.3.3, by restricting the involved indexes i and j in cardinal coordinate independence. Again we apply this to the dynamic context, to characterize a representation as in Koopmans (1972), mainly by letting every point of time be "CCI-related" to the previous point of time, and by letting the amount $\$\alpha$ at a point of time, in preference equivalent to \$1 at the previous point of time, be the same for all points of time.

In section IV.6 several other ways to strengthen Theorem IV.3.3 are suggested without elaborations. One could investigate how to combine the many ways, mentioned above, to strengthen Theorem IV.3.3. Because of the size of this task, we do not take it up.

In section IV.7 we compare our derivation of SEU maximization to the most well-known other derivations, available in literature.

IV.2. CARDINAL COORDINATE INDEPENDENCE

Let us first repeat the terminoly of DMUU.

TERMINOLOGY IV.2.1. We use the term (possible) state (of nature) instead of index, and act instead of alternative. Elements of C are called consequences, and denoted by Greek characters α, β, γ, δ; sometimes they are also called coordinates, and denoted as x_i, v_j, etc.

The following definition gives the most known approach to DMUU.

DEFINITION IV.2.2. We say $[C^n, \succcurlyeq, (p_j)_{j=1}^n, U]$ is a *subjective expected utility (SEU) model (for* $\succcurlyeq$*)* if the p_j's are nonnegative real numbers that sum to one, and $U : C \to \mathbb{R}$ is a function, such that $[x \succcurlyeq y \Leftrightarrow \Sigma_{j=1}^n p_j U(x_j) \geq \Sigma_{j=1}^n p_j U(y_j)]$ for all acts x,y. Then p_j is called the *subjective probability* for state j, U the *(subjective) utility function*, and $\Sigma_{j=1}^{n} p_j U(x_j)$ the *subjective expected utility* of act x.

A notation, only applicable in the present context, where all coordinate sets C_i are one same C:

NOTATIONS IV.2.3. For $\alpha \in C$, $\bar{\alpha} \in C^n$ is the act with all coordinates equal to α. We write $\alpha \succ \beta$ if $\bar{\alpha} \succ \bar{\beta}$.

The act $\bar{\alpha}$ with certainty gives consequence α. Note that, by the above notation, the binary relation $\succ$ on C^n induces a binary relation on C, also denoted by $\succ$. This notation will not cause confusion.

The remainder of this section is devoted to elucidations, and elementary results, for the following property.

DEFINITION IV.2.4. $\succ$ satisfies *cardinal coordinate independence (CCI)* if:

$$\begin{array}{lll} & x_{-i}\alpha \prec y_{-i}\beta \quad \text{and} & v_{-j}\alpha \succ w_{-j}\beta \\ \text{and} & x_{-i}\gamma \succ y_{-i}\delta & \\ & \qquad\qquad \text{imply} & v_{-j}\gamma \succ w_{-j}\delta \end{array}$$

for all acts x, y, v, w, all consequences α, β, γ, δ, all states j, and all essential states i.

ELUCIDATION. Replacement, in $x_{-i}\alpha \prec y_{-i}\beta$, of α,β by γ,δ, changes $\prec$ into $\succ$, to give $x_{-i}\gamma \succ y_{-i}\delta$. We imagine that replacement, in $v_{-j}\alpha \succ w_{-j}\beta$, of (α,β) by (γ,δ), should thus kind of "reinforce" $\succ$, to $v_{-j}\gamma \succ w_{-j}\delta$. So the replacement should certainly not induce a reversal of preference, into $v_{-j}\gamma < w_{-j}\delta$.

Let us emphasize that the above Definition does not have a restriction $i \neq j$, or $i = j$. If exactly two coordinates are essential, then putting $i = j$ essential, in Definition IV.2.4, gives triple cancellation. The proof of Lemma IV.2.5 in the sequel may serve as a further illustration of the meaning of CCI.

To obtain an example of a binary relation $\succ$, satisfying CCI, we let $n \in \mathbb{N}$ be arbitrary, $C = \mathbb{R}_{++}, \alpha_1, \ldots, \alpha_n \in \mathbb{R}_{++}$; further let $\succ$ be represented by the "Cobb-Douglas" production function $(x_1, \ldots, x_n) \mapsto \Pi_{j=1}^{n} x_j^{\alpha_j}$. Then, with $p_j := \alpha_j / \Sigma_{i=1}^{n} \alpha_i$, $j = 1, \ldots, n$; and $U : \mu \mapsto \log \mu$, $\succ$ is also represented by $x \mapsto \Sigma_{j=1}^{n} p_j U(x_j)$. Lemma IV.2.5 will show that indeed $\succ$ satisfies CCI.

An example of a continuous weak order, satisfying CI and triple cancellation, but violating CCI, is obtained by taking $n = 2$, $C =]0,1[$, and $\succ$ represented by $(x_1, x_2) \mapsto x_1 - x_1 x_2$. One may think of the interpretation where x_1 is a share of total income before tax, allocated to a person, x_2 is tax rate, and $x_1 - x_1 x_2$ is the share of total income after tax. Here $\succ$ is also represented by $(x_1, x_2) \mapsto \log x_1 + \log(1-x_2)$, so by Theorem III.3.6, $\succ$ is a continuous weak order satisfying CI and triple cancellation, and the Thomsen condition. We have $(\frac{1}{3}, \frac{1}{2}) \prec (\frac{2}{3}, \frac{3}{4})$, $(\frac{1}{8}, \frac{1}{2}) \succ (\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{5}, \frac{1}{3}) \succ (\frac{2}{5}, \frac{2}{3})$, but $(\frac{1}{5}, \frac{1}{8}) \prec (\frac{2}{5}, \frac{1}{4})$, so by $i = 1$, $j = 2$, $\alpha = \frac{1}{3}$, $\beta = \frac{2}{3}$, $\gamma = \frac{1}{8}$, $\delta = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $y_2 = \frac{3}{4}$, $v_1 = \frac{1}{5}$, $w_1 = \frac{2}{5}$ this gives a violation of CCI.

Lemma IV.2.5. *If an SEU model* $[C^n, \succ, (p_j)_{j=1}^n, U]$ *exists for* $\succ$, *then* $\succ$ *satisfies CCI.*

PROOF. Suppose:

$$i \text{ is essential.} \tag{IV.2.1}$$

Then there must exist z, σ, τ such that $z_{-i}\sigma \succ z_{-i}\tau$, i.e. $\Sigma_{k \neq i} p_k U(z_k) + p_i U(\sigma) > \Sigma_{k \neq i} p_k U(z_k) + p_i U(\tau)$. This can hold only if:

$$p_i > 0. \tag{IV.2.2}$$

Next let:

$$x_{-i}\alpha \prec y_{-i}\beta \text{ and } x_{-i}\gamma \succ y_{-i}\delta. \tag{IV.2.3}$$

Then $\Sigma_{k \neq i} p_k U(x_k) + p_i U(\alpha) \leq \Sigma_{k \neq i} p_k U(y_k) + p_i U(\beta)$ and $\Sigma_{k \neq i} p_k U(x_k) + p_i U(\gamma) \geq \Sigma_{k \neq i} p_k U(y_k) + p_i U(\delta)$. Taking these together: $p_i[U(\alpha) - U(\beta)] \leq \Sigma_{k \neq i} p_k [U(y_k) - U(x_k)] \leq p_i[U(\gamma) - U(\delta)]$. By (IV.2.2) we may conclude:

$$U(\alpha) - U(\beta) \leq U(\gamma) - U(\delta). \tag{IV.2.4}$$

Finally let:

$$v_{-j}\alpha \succ w_{-j}\beta. \tag{IV.2.5}$$

Then $\Sigma_{k\neq j}p_kU(v_k) + p_jU(\alpha) \geq \Sigma_{k\neq j}p_kU(w_k) + p_jU(\beta)$, or $p_j[U(\alpha) - U(\beta)] \geq \Sigma_{k\neq j}p_k[U(w_k) - U(v_k)]$. By (IV.2.4) we obtain $p_j[U(\gamma) - U(\delta)] \geq \Sigma_{k\neq j}p_k[U(w_k) - U(v_k)]$, or $\Sigma_{k\neq j}p_kU(v_k) + p_jU(\gamma) \geq \Sigma_{k\neq j}p_kU(w_k) + p_jU(\delta)$. This means:

$$v_{-j}\gamma \succcurlyeq w_{-j}\delta. \tag{IV.2.6}$$

So indeed (IV.2.1), (IV.2.3) and (IV.2.5) imply (IV.2.6), as is required by CCI.

□

Formula (IV.2.4) is an indication that comparability of utility differences underlies cardinal coordinate independence. The following property is obtained from cardinal coordinate independence by replacing $\preccurlyeq$, and all $\succcurlyeq$'s, by $\approx$.

DEFINITION IV.2.6. We say $\succcurlyeq$ satisfies *equivalence-cardinal coordinate independence (eq-CCI)* if for all acts x, y, v, w, all consequences α, β, γ, δ, all states j, and all essential states i, $x_{-i}\alpha \approx y_{-i}\beta$ & $x_{-i}\gamma \approx y_{-i}\delta$ & $v_{-j}\alpha \approx w_{-j}\beta$ imply $v_{-j}\gamma \approx w_{-j}\delta$.

This property will be studied extensively in section IV.4. For the next section, we now only need:

LEMMA IV.2.7. *CCI implies eq-CCI.*

PROOF. Replacing, in Definition IV.2.6, the first three equivalences by $\preccurlyeq$, $\succcurlyeq$, and $\succcurlyeq$, shows that, by CCI, $v_{-j}\gamma \succcurlyeq w_{-j}\delta$. Interchanging everywhere left and right sides of the equivalences, and writing again $\preccurlyeq$, $\succcurlyeq$ and $\succcurlyeq$ instead of the equivalences, gives by CCI that $w_{-j}\delta \succcurlyeq v_{-j}\gamma$.

□

IV.3. THE MAIN THEOREM

Let us first note that the existence of an SEU model $[C^n, \succ, (p_j)_{j=1}^n, U]$ for $\succ$ implies the existence of additive value functions $(V_j)_{j=1}^n$ for $\succ$, by the definition $V_j := p_j U$ for all j. In section III.3 we saw that CI, and triple cancellation for the case of exactly two essential coordinates, were necessary (and sufficient) for a continuous weak order to have additive value functions existing, under the topological assumption III.2.11. So CCI, the property of a continuous weak order that will be shown to be necessary and sufficient for the existence of an SEU model, must imply CI and triple cancellation.

LEMMA IV.3.1. *Let $\approx$ be an equivalence relation. Then CCI implies CI.*

PROOF. Let $x_{-j}\alpha \succ y_{-j}\alpha$, and $\beta \in C$. To derive is $x_{-j}\beta \succ y_{-j}\beta$. If no coordinate is essential, then by Lemma III.2.10 indeed $x_{-j}\beta \succ y_{-j}\beta$. So let i be an essential coordinate. Then, for arbitrary z, $z_{-i}\alpha \prec z_{-i}\alpha$, $z_{-i}\beta \succ z_{-i}\beta$, $x_{-j}\alpha \succ y_{-j}\alpha$, and CCI imply $x_{-j}\beta \succ y_{-j}\beta$.

□

LEMMA IV.3.2. *Let exactly two coordinates be essential. Then CCI implies triple cancellation.*

PROOF. Substitute, in Definition III.3.5, $x = (s_{-i,j}a_i,x_j)$, $\alpha = a_i$, $y = (s_{-i,j}b_i,y_j)$, $\beta = b_i$, $\gamma = c_i$, $\delta = d_i$, $v = (s_{-i,j}a_i,v_j)$, $w = (s_{-i,j}b_i,w_j)$, and let both i and j of Definition IV.2.4 correspond to the i of Definition III.3.6.

□

Before we formulate the main theorem, *let us repeat* that the topological assumption III.2.11 entails that C is a connected topological space which is topologically separable for the case of exactly one essential coordinate.

THEOREM IV.3.3. *Let the topological assumption III.2.11 hold. For the binary relation $\succcurlyeq$ on C^n, the following two statements are equivalent:*

(i) *There exists an SEU model* $[C^n, \succcurlyeq, (p_j)_{j=1}^n, U]$ *for* $\succcurlyeq$*, with* U *continuous.*

(ii) $\succcurlyeq$ *is a continuous weak order on* C^n *that satisfies CCI.*

The following uniqueness results hold for U, $(p_j)_{j=1}^n$ *of (i):*

If two or more states are essential, (IV.3.1)
then $(p_j)_{j=1}^n$ *is uniquely determined; and* U *is cardinal.*
If exactly one state i *is essential,* (IV.3.2)
then $p_i = 1$, $p_j = 0$ *for all* $j \neq i$; *and* U *is continuously ordinal.*
If no state is essential, (IV.3.3)
then $(p_j)_{j=1}^n$ *can be taken arbitrarily, as long as* $p_j \geq 0$ *for all* j, $\Sigma p_j = 1$; U *can be any constant function.*

PROOF. For (i) $\Rightarrow$ (ii), suppose (i). Then the function, assigning to every act x its expected utility $\Sigma p_j U(x_j)$, is continuous, and represents $\succcurlyeq$. So certainly $\succcurlyeq$ is a weak order. For every y, $\{x : x \succcurlyeq y\} = \{x : \Sigma p_j U(x_j) \geq \Sigma p_j U(y_j)\}$ is closed; so is $\{x : x \preccurlyeq y\}$. Consequently $\succcurlyeq$ is continuous. By Lemma IV.2.5 $\succcurlyeq$ satisfies CCI. So (ii) follows.

Next, let (ii) hold. We derive (i), and the uniqueness results. First the case of no essential i. Then by Lemma III.2.10, $\succcurlyeq$ is trivial. We can let $(p_j)_{j=1}^n$ be completely arbitrary, as long as they are nonnegative and sum to one. Further we can let U be any constant function. Also, U múst be constant, $U(\alpha) > U(\beta)$ would imply $\bar{\alpha} > \bar{\beta}$. So for the case of no essential state, (i) and the uniqueness result (IV.3.3) hold.

Next the case of exactly one essential i. By Corollary III.3.2 there exist continuous additive value functions $(V_j)_{j=1}^n$ for $\succcurlyeq$. From $x_{-k}\alpha \succ x_{-k}\beta \Leftrightarrow V_k(\alpha) > V_k(\beta)$ we see that V_i must be nonconstant, and that V_k is constant for all $k \neq i$. By the uniqueness result of Corollary III.3.2, for $p_i U$, hence for U, of (i), there must exist a continuous strictly increasing transformation ϕ such that $U = \phi \circ V_i$. So U must be nonconstant. For every $k \neq i$, $p_k U$ must be constant, so p_k

must be zero. Consequently $p_i = 1$. Further, any continuous strictly increasing transformation U of V_i , together with $p_i = 1$, $p_k = 0$ for all $k \neq i$, gives additive value functions $(p_k U)_{k=1}^{n}$ for $\succ$, so makes (i) valid. So for the case of exactly one essential coordinate, (i) and the uniqueness result (IV.3.2) are verified.

Finally, the case of two or more essential states. First we show:

There exist continuous simultaneously cardinal additive value functions $(V_j)_{j=1}^{n}$ for $\succ$. (IV.3.4)

For the case of two essential coordinates, this follows from Lemma IV.3.2 (giving triple cancellation for $\succ$) and then from Theorem III.3.6. For the case of three or more essential coordinates, it follows from Lemma IV.3.1 (giving CI for $\succ$) and then from Theorem III.3.7. *Now let* i *be essential*. We next show:

For all j : $V_j = \phi_j \circ V_i$, for a continuous nondecreasing ϕ_j. (IV.3.5)

Suppose $V_i(\alpha) \geq V_i(\beta)$. Then, for arbitrary x, v, $x_{-i}\beta \prec x_{-i}\beta$, $x_{-i}\alpha \succ x_{-i}\beta$, $v_{-j}\beta \succ v_{-j}\beta$, and CCI imply $v_{-j}\alpha \succ v_{-j}\beta$. So $V_j(\alpha) \geq V_j(\beta)$. Now (IV.3.5) follows from Lemma VIII.4.

Our following step is to show:

Every ϕ_j is affine. (IV.3.6)

If j is inessential, then V_j is constant, and affinity of ϕ_j follows. Of course, if $j = i$, then ϕ_j is identity, so affine too.

So let $j \neq i$, j essential. V_i and V_j are not constant, so the connected $V_i(C)$ [respectively $V_j(C)$] must contain an interval with length $\delta_i > 0$ [respectively $\delta_j > 0$]. Let now $V_i(\alpha) \in \mathrm{int}(V_i(C))$ be arbitrary. Since ϕ_j is continuous, there exists $\varepsilon > 0$ so small that: $\varepsilon \leq \delta_j$; $W :=]V_i(\alpha)-\varepsilon, V_i(\alpha)+\varepsilon[\subset V_i(C)$; and $\phi_j(V_i(\alpha)+\varepsilon) - \phi_j(V_i(\alpha)-\varepsilon) \leq \delta_i$.

Now let $\sigma < \tau \in W$. There exist $\beta, \gamma, \delta \in C$ such that:
$V_i(\beta) = \sigma$, $V_i(\delta) = \tau$, $V_i(\gamma) = (\sigma+\tau)/2$.
We can take $a_j, b_j \in C$ such that:
$V_j(b_j) - V_j(a_j) = V_i(\beta) - V_i(\gamma) = V_i(\gamma) - V_i(\delta) \leq \varepsilon \leq \delta_j$.
And we can take c_i, d_i such that:
$V_i(d_i) - V_i(c_i) = V_j(\beta) - V_j(\gamma) \leq \delta_i$.

All these choices lead, for arbitrary fixed $s \in C^n$, to:

$$(s_{-i,j}\beta,a_j) \approx (s_{-i,j}\gamma,b_j) \ \& \ (s_{-i,j}c_i,\beta) \approx (s_{-i,j}d_i,\gamma) \qquad \text{(IV.3.7)}$$
$$\& \ (s_{-i,j}\gamma,a_j) \approx (s_{-i,j}\delta,b_j)$$

as follows from substitution of $(V_k)_{k=1}^n$. Equivalence-CCI (Definition IV. 2.6) with $s_{-j}a_j$ in the role of x, $s_{-j}b_j$ in the role of y, $(\beta,\gamma,\gamma,\delta)$ in the role of $(\alpha,\beta,\gamma,\delta)$, $s_{-i}c_i$ in the role of v, and $s_{-i}d_i$ in the role of w, gives $(s_{-i,j}c_i,\gamma)\approx(s_{-i,j}d_i,\delta)$. This implies $V_j(\gamma) - V_j(\delta) = V_i(d_i) - V_i(c_i)$. We have chosen d_i and c_i to have the latter equal to $V_j(\beta) - V_j(\gamma)$. So $V_j(\beta) - V_j(\gamma) = V_j(\gamma) - V_j(\delta)$ has been derived. This means: $\phi_j(\sigma) - \phi_j((\sigma+\tau)/2) = \phi_j((\sigma+\tau)/2) - \phi_j(\tau)$, or: $\phi_j((\sigma+\tau)/2) = [\phi_j(\sigma) + \phi_j(\tau)]/2$. By Corollary VIII.3, with ν for $V_i(\alpha)$, $p = \frac{1}{2}$, affinity of ϕ_j follows: (IV.3.6) is demonstrated.

So now we have nonnegative $(\sigma_j)_{j=1}^n$, and real $(\tau_j)_{j=1}^n$, such that $V_j = \tau_j+\sigma_j V_i$ for all j. We can now define:

$$U := V_i;\ p_j := \sigma_j/(\Sigma_{k=1}^n \sigma_k) \text{ for all } j. \qquad \text{(IV.3.8)}$$

(Note that $\sigma_i = 1$, so $\Sigma\sigma_k > 0$.) Because of simultaneous cardinality, this gives additive value functions $(p_jU)_{j=1}^n$ for $\succcurlyeq$. Thus (i) follows.

For the uniqueness result (IV.3.1), let $[C^n, \succcurlyeq, (p_j')_{j=1}^n, U']$ be another SEU model. Then $(p_j'U')_{j=1}^n$ are additive value functions for $\succcurlyeq$ too. By simultaneous cardinality, $(\tau_j)_{j=1}^n$ and $\sigma > 0$ exist such that $p_j'U' = \sigma p_jU + \tau_j$ for all j, i.e., with α arbitrarily fixed:

$$p_j'[U'(\beta)-U'(\alpha)] = \sigma p_j[U(\beta)-U(\alpha)] \text{ for all } \beta. \qquad \text{(IV.3.9)}$$

Since U is not constant, we can take β such that $U(\beta) \neq U(\alpha)$. Then $p_j = p_j' . [U'(\beta) - U'(\alpha)] / (\sigma.[U(\beta) - U(\alpha)])$ for all j. Since $\Sigma p_j = \Sigma p_j'$, $p_j = p_j'$ for all j follows. For $p_j > 0$, (IV.3.9) now shows that $[U'(\beta)- U'(\alpha)] = \sigma[U(\beta) - U(\alpha)]$. Hence $U'(\cdot) = \sigma[U(\cdot)-U(\alpha)]+U'(\alpha)$ must hold: U' is derived from U by multiplication with a positive σ, and addition of $U'(\alpha) - \sigma U(\alpha)$, as (IV.3.1) requires it.

Conversely, that every such U' instead of U verifies (i), is straightforward.

□

IV.4. EQUIVALENCE-CARDINAL COORDINATE INDEPENDENCE

In this section we give a characterization of the representation $x \mapsto \Sigma_{j=1}^{n}\lambda_j U(x_j)$ with some λ_j's possibly negative. Our characterization is an alternative for the one in Krantz et al. (1971, Theorem 6.15). Our eq-CCI is stronger than their "standard-sequence-invariance". By this, we only have to add weak separability, instead of the stronger coordinate independence; and we do not have to treat the case of two essential coordinates separately.

For DMUU, this representation as such has little interest, since negative λ_j's are not suited to be interpreted as probabilities. For other contexts it may be desirable to allow negativity of some λ_j's, see Wakker (1985b). The interest of this representation for DMUU lies in the possibility to apply it in special contexts where the preference relation has further properties (such as monotonicity) that imply the λ_j's, mentioned above, to be nonnegative after all. We then obtain, for these special contexts, a characterization of SEU-maximization by means of weaker properties than in the previous section. See Corollary IV.4.4.c.

The main property used to derive the desired representation is eq-CCI.

LEMMA IV.4.1. *Let $\approx$ be an equivalence relation. Then eq-CCI implies eq-CI.*

PROOF. As in Lemma IV.3.1, with all preferences replaced by equivalences.

□

LEMMA IV.4.2. *Eq-CCI implies the Reidemeister condition.*

PROOF. With the same substitution as in Lemma IV.3.2.

□

The main theorem of this section:

THEOREM IV.4.3. *Let the topological assumption III.2.11 hold. For the binary relation $\succcurlyeq$ on C^n, the following two statements are equivalent:*

(i) *There exist real λ_j, $j = 1,\dots,n$, and a continuous $U : C \to \mathbb{R}$, such that $x \mapsto \Sigma_{j=1}^n \lambda_j U(x_j)$ represents $\succcurlyeq$.*

(ii) *$\succcurlyeq$ is a continuously weakly separable weak order that satisfies eq-CCI.*

The following uniqueness results hold for U, $(p_j)_{j=1}^n$ of (i):

If two or more coordinates are essential, $((\lambda_j)_{j=1}^n, U)$ can be replaced by $((\mu_j)_{j=1}^n, W)$, if and only if real ν, σ, τ exist with $\nu\sigma > 0$, such that $\mu_j = \nu\lambda_j$ for all j and $W = \tau+\sigma U$. *(IV.4.2)*

If exactly one coordinate is essential, then $((\lambda_j)_{j=1}^n, U)$ can be replaced by $((\mu_j)_{j=1}^n, W)$, if and only if positive ν and continuous strictly increasing ϕ, or negative ν and continuous, strictly decreasing ϕ, exist such that $\mu_j = \nu\lambda_j$ for all j and $W = \phi \circ U$. *(IV.4.3)*

If no coordinate is essential, then all λ_j's are 0, or U is constant. *(IV.4.4)*

PROOF. (i) $\Rightarrow$ (ii) is, as usual, straightforward, so we assume (ii), and derive (i) and the uniqueness results. The case of no essential coordinate is direct. If exactly one coordinate i is essential, then $\lambda_j = 0$ for all $j \neq i$, and everything follows from Corollary III.3.2. In the sequel we shall assume:

Two or more coordinates are essential. (IV.4.5)

There now exist simultaneously cardinal additive value functions $(V_j)_{j=1}^n$ for $\succcurlyeq$, by Lemma IV.4.1, IV.4.2, and Theorem III.4.6. As usual, we suppress inessential coordinates j from notation, they get assigned $\lambda_j = 0$. So all (remaining) coordinates are essential, and the V_j's are nonconstant. If now, for any i, $V_i(\alpha) = V_i(\beta)$, then, for any j, x, by eq-CCI, $\{x_{-i}\alpha \approx x_{-i}\alpha \;\&\; x_{-i}\alpha \approx x_{-i}\beta \;\&\; x_{-j}\alpha \approx x_{-j}\alpha\}$ imply $x_{-j}\alpha \approx x_{-j}\beta$, i.e. $V_j(\alpha) = V_j(\beta)$. This means that for any i, j, $V_i = \phi_{ij} \circ V_j$ for $\phi_{ij} : V_j(C) \to V_i(C)$. Here ϕ_{ij} is the inverse of ϕ_{ji}, so all ϕ_{ij} are bijective. By Corollary VIII.10 they are continuous. The derivation that they are affine, by Lemma VIII.8, is completely analogous to the derivation of (IV.3.6).

We can, for arbitrary $\alpha \in C$, set $V_j(\alpha) = 0$ for all j; existence of $(\lambda_j)_{j=1}^n$ such that $V_j = \lambda_j V_1$ for all j, follows. Take $U := V_1$. The uniqueness result (IV.4.2) follows from the simultaneous cardinality of the additive value functions.

□

Below we make some observations that follow straightforwardly from substitution in (i) above. First note that, in the above theorem, we can always arrange $\Sigma_{j=1}^n \lambda_j \geq 0$; if $\Sigma\lambda_j < 0$ we replace $((\lambda_j)_{j=1}^n, U)$ by $((-\lambda_j)_{j=1}^n, -U)$. We then have $\bar{\alpha} \succ \bar{\beta} \Rightarrow U(\alpha) > U(\beta)$. The assumption in the beginning of the following Corollary IV.4.4 serves to avoid uninteresting cases such as triviality of $\succcurlyeq$, or negativity of $\Sigma_{j=1}^n \lambda_j$ which would make U some kind of "anti-utility" (or "loss") function. The following corollary shows that the above theorem can be used to characterize several representations, studied in literature, which in fact are special forms of (i) above. This is done by the addition of, usually weak, conditions to (ii) above.

COROLLARY IV.4.4. *Let (i) of Theorem IV.4.3 hold. Assume that* $\alpha, \beta \in C$ *exist with* $\bar{\alpha} \succ \bar{\beta}$, *and let* $U(\alpha) > U(\beta)$. *Then we have, for every* j:

(a) $\lambda_j > 0$ *if and only if there exist* $\bar{\gamma} \succ \bar{\delta}$ *and* x *such that* $x_{-j}\gamma \succ x_{-j}\delta$.

(b) $\lambda_j \geq 0$ *if and only if there exist* $\bar{\gamma} \succ \bar{\delta}$ *and* x *such that* $x_{-j}\gamma \succcurlyeq x_{-j}\delta$.

Furthermore,

(c) *There exists an SEU-model for* $\succcurlyeq$ *if and only if* $x_{-j}\alpha \succcurlyeq x_{-j}\beta$ *for all* x,j.

□

The characterization of subjective expected utility maximization, obtainable from (c) above and Theorem IV.4.3.(ii), is preferable to the one in Theorem IV.3.3 in the sense that all conditions used here follow straightforwardly from those in (ii) of Theorem IV.3.3, whereas the converse derivation is not elementary since it essentially needs continuity, see the text after Lemma III.4.4.

Characterizations of $\succcurlyeq$ on C^2 by $(x_1,x_2) \mapsto U(x_1)-U(x_2)$, with $\succcurlyeq$ interpreted as *strength of preference relation* [i.e. $(x_1,x_2) \succcurlyeq (y_1,y_2)$: x_1 is preferred to x_2 more strongly than y_1 to y_2] have received much attention, and have often been discussed, in literature, see Frisch (1926), Lange (1934), Alt (1936), Scott and Suppes (1958), Debreu (1958), Suppes and Zinnes (1963), Fishburn (1970, Chapter 6), Krantz et al. (1971, Chapter 4), Shapley (1975), and Fuhrken and Richter (1985).

A new characterization of the above representation can be obtained as a corollary of Theorem IV.4.3:

COROLLARY IV.4.5. *Let* $n = 2$. *Let (i) in Theorem IV.4.3 hold. We can obtain* $\lambda_1 = 1$, $\lambda_2 = -1$, *if and only if one of the following holds:*

(a) $(\alpha,\beta) \succcurlyeq (\gamma,\delta) \Rightarrow (\delta,\gamma) \succcurlyeq (\beta,\alpha)$ *for all* α, β, γ, δ.

(b) $(\alpha,\beta) \succcurlyeq (\beta,\gamma) \Rightarrow (\gamma,\beta) \succcurlyeq (\beta,\alpha)$ *for all* α, β, γ.

(c) $\bar{\alpha} \approx \bar{\beta}$ *for all* α, β.

(d) *If there exist* α, β, γ *such that* $(\alpha,\gamma) \succ (\beta,\gamma)$, *then there also exist such* α, β, γ *with furthermore* $\bar{\alpha} \approx \bar{\beta}$.

□

Finally, in dynamic contexts (see Example II.1.4) representations of the form $(x_1,\ldots,x_n) \mapsto \Sigma_{j=1}^{n} \lambda^j U(x_j)$, with $0 < \lambda \leq 1$, have received attention. There λ is interpreted as a *discount factor*. A well-known characterization, for the case of an infinite cartesian product $C^{\mathbb{N}}$, by means of a "stationarity assumption", has been obtained by Koopmans (1972). We can characterize, for our finite cartesian product (so only finitely many points of time) the analogous representation.

COROLLARY IV.4.6. *Let (i) in Theorem IV.4.3 hold. There exists* $0 < \lambda \leq 1$ *such that we can take* $\lambda_j = \lambda^j$ *for all* j, *if and only if* $\succcurlyeq$ *is trivial or it satisfies a* <u>*weak stationarity assumption*</u>, *i.e. there exist* x, $\bar{\alpha} \succcurlyeq \bar{\beta} \succ \bar{\gamma}$ *such that* $x_{-i,i+1}\beta,\gamma \approx x_{-i,i+1}\gamma,\alpha$ *for all* $0 \leq i < n$.

□

The weak stationarity assumption above is weaker than the one used by Koopmans (1972), mainly because we only have "there exist x, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, ...", whereas Koopmans' stationarity assumption requires analogous things "for all ... ." Of course, this weakening is possible only because in (ii) in Theorem IV.4.3 we require properties for $\succcurlyeq$, far stronger than those which Koopmans uses next to his stationarity assumption.

IV.5. CCI-RELATED EVENTS

In this section we consider relaxations of cardinal coordinate independence that moderate the "for all i,j" part in the definition (IV.2.4) of cardinal coordinate independence. This weakening will be used to strengthen Theorem IV.3.3. The next definition will also be of use in Chapter V. First we introduce a notation in the spirit of Notation II.2.4; see also (II.2.1).

NOTATION IV.5.1. For $A \subset I$, $x \in X$, $\alpha \in C$, $x_{-A}\alpha$ denotes (x with x_i replaced by α, for all $i \in A$).

With this we can define:

DEFINITION IV.5.2. Let $A, B \subset I$. We say A is *CCI-related to* B if for all $x, y, v, w, \alpha, \beta, \gamma, \delta$: $x_{-B}\alpha \preccurlyeq y_{-B}\beta$ & $x_{-B}\gamma \succcurlyeq y_{-B}\delta$ & $v_{-A}\alpha \succcurlyeq w_{-A}\beta$ imply $v_{-A}\gamma \succcurlyeq w_{-A}\delta$.

The binary relation, introduced on the set of events by the above definition, usually is not symmetric or transitive or reflexive. With, as usual, i instead of $\{i\}$, every nonessential i is CCI-related to every essential j, and a coordinate j is CCI-related to a nonessential i if and only if j itself is nonessential, as one can see. CCI holds if and only if every j is CCI-related to every essential i.

LEMMA IV.5.3. *Let* $\succcurlyeq$ *be reflexive. Let* i *be CCI-related to some* j. *Then* $[x_{-i}\alpha \succcurlyeq y_{-i}\alpha \Leftrightarrow x_{-i}\beta \succcurlyeq y_{-i}\beta]$ *for all* x, y, α, β.

PROOF. $z_{-j}\alpha \preccurlyeq z_{-j}\alpha$ & $z_{-j}\beta \succcurlyeq z_{-j}\beta$ & $x_{-i}\alpha \succcurlyeq y_{-i}\alpha$ must imply $x_{-i}\beta \succcurlyeq y_{-i}\beta$.

□

THEOREM IV.5.4. *Let the topological assumption III.2.11 hold. Let* $n \geq 3$, *and let all coordinates be essential. For the binary relation* $\succcurlyeq$ *on* C^n, *the following two statements are equivalent:*

(i) *There exists an SEU model* $[C^n, \succcurlyeq, (p_j)_{j=1}^n, U]$ *for* $\succcurlyeq$, *with* U *continuous.*

(ii) $\succcurlyeq$ *is a continuous weak order on* C^n, *every* $i \geq 2$ *is CCI-related to* i-1, *and* 1 *is CCI-related to itself or some other* j.

PROOF. CI follows from Lemma IV.5.3. By Theorem III.3.7, additive value functions $(V_j)_{j=1}^n$ exists for $\succcurlyeq$. To show that V_{i+1} is an affine nondecreasing transformation of V_i, for $i = 1,\ldots,n-1$, is exactly as the derivation of (IV.3.5) and (IV.3.6) (take $j = i+1$ there). We can give all V_i's a common zero. $V_i = \mu_i V_{i-1}$ for some $\mu_i \geq 0$, follows for all $i \geq 2$, i.e. $V_i = \lambda_i V_1$, for some $\lambda_i \geq 0$ follows, for all $i \geq 1$. By essentially of all coordinates, $\lambda_i > 0$ for all i. We take $U = V_1$, $p_j = \lambda_j/(\Sigma_{i=1}^n \lambda_i)$ for all j.

□

The above theorem also can be derived for $n = 2$, but then a more complicated proof is needed. The main complication is that only a weak version of triple cancellation can be derived, so that the additive value functions cannot be obtained directly from Theorem III.3.6. Further the assumption of essentiality of all coordinates can be omitted, if in (ii) we require the first essential coordinate to be CCI-related to some other essential coordinate, and every other essential coordinate to the preceding essential coordinate. Also in (ii) above we could have assumed that every coordinate was CCI-related to coordinate 1, or that, for an appropriately chosen sequence of subsets $(A_1,\ldots,A_l)$ of I, A_{k+1} is CCI-related to A_k for all $k \leq l-1$. We do not elaborate these matters.

Let us formulate a corollary of Theorem IV.5.4 that gives another characterization of the representation of Koopmans (1972), adapted to our finite cartesian product. Our main requirement is that every coordinate (point of time) is CCI-related to the previous coordinate (point of time), and that the consequence α (say amount of dollars) on some point of time, equivalent to one dollar on the previous point of time, is independent of that point of time.

COROLLARY IV.5.5. *Let* $n \geq 3$. *Let* $\succcurlyeq$ *be a binary relation on* $\mathbb{R}^n$, *that is strongly cA monotonic (i.e.* $x \succ y$ *if* $x_j \geq y_j$ *for all* j *and* $x_j > y_j$ *for some* j*). The following two statements are equivalent:*

(i) *There exists* $0 < \lambda \leq 1$, *and a continuous* $U : \mathbb{R} \to \mathbb{R}$, *such that* $x \mapsto \Sigma\lambda^j U(x_j)$ *represents* $\succcurlyeq$.

(ii) $\succcurlyeq$ *is a continuous weak order, every* $i \geq 2$ *is CCI-related to* $i-1$, 1 *is CCI-related to some* i *(e.g.* $i = 1$, *or* $i = n$*), and* $\alpha \geq 1$ *exists such that* $(\bar{0}_{-i}1) \approx (\bar{0}_{-(i+1)}\alpha)$ *for all* $i \leq n-1$.

PROOF. (i) $\Rightarrow$ (ii) is straightforward. Let (ii) hold. All conditions in (ii) of Theorem IV.5.4 hold, so a representation $x \mapsto \Sigma p_j U(x_j)$ exists for $\succcurlyeq$. By strong cA monotonicity, positivity of the p_j's and strict increasingness of U can be arranged. Now $\lambda = [U(1)] / U(\alpha)$ is chosen. □

IV.6. FURTHER WAYS TO RESTRICT CARDINAL COORDINATE INDEPENDENCE

In this section we briefly suggest further ways to strengthen Theorem IV.3.3, by weakening the cardinal coordinate independence property in (ii) of Theorem IV.3.3. A first way may be to require cardinal coordinate independence only "locally", i.e. only to require that for every $x \in X$ there exists an open neighbourhood of x, such that cardinal coordinate independence holds in this neighbourhood. The main problem then seems to be, to strengthen the results on additive decomposable representations of Chapter III by considering the local versions of the involved characterizing properties. That this may be possible, has been mentioned in Debreu (1960, page 17, lines 2-4). First one uses the local properties to obtain local additively decomposable representations. Next these local representations must be made to fit together to give a global representation. This however seems to be a complicated operation (compare subsection VI.7.3) and additional requirements as local connectedness of C are maybe needed here. (Debreu, 1960, considered Euclidean spaces). Finally, proportionality of the additive value functions is obtained by using the local cardinal coordinate independence property for the certain acts $\bar{\alpha}$.

A second way to weaken cardinal coordinate independence is by weakening the part "for all α, β, γ, δ". It is for instance sufficient to require it for only one α. This does not complicate the proof of coordinate independence, so, for more or less than two essential coordinates, additive value functions must exist. For two essential coordinates matters are slightly more complicated because triple cancellation (Definition III.3.5) then no longer directly follows. Once additive value functions have been obtained, the derivation of proportionality of them is as in Theorem IV.3.3. Analogously one may restrict the β's, or γ's, or δ's, in Definition IV.2.4. Whether it is sufficient to require Definition IV.2.4 for only those α, β, γ, δ, for which $\beta = \gamma$, or $\alpha = \delta$, is an open question. In such a case no readily available results on additive decomposability are present in literature.

A third way to weaken cardinal coordinate independence is to restrict the involved x, y, v, w. Maybe it is sufficient to require matters for only a dense subset of X.

Also the question has been considered whether it is sufficient to require cardinal coordinate independence on every two dimensional subspace (obtained by keeping all but two coordinates fixed). The following example, communicated to the author by A. Tversky in 1985, shows that this does not work: Let $C = \mathbb{R}_{++}$, $n = 3$. Let $\succcurlyeq$ be represented by $x \mapsto x_1x_2 + x_2x_3 + x_1x_3 = (x_1+x_3)(x_2+x_3) - x_3^2$. Clearly, with x_3 fixed, $\succ$ is represented by $x \mapsto \log(x_1+x_3) + \log(x_2+x_3)$, so satisfies CCI. But $\succ$ has no additive value functions, since $(1,7,1) \approx (3,3,1)$ and $(1,7,2) \succ (3,3,2)$ violate coordinate independence. Requiring cardinal coordinate independence for every three - or more dimensional subspace, seems to be sufficient.

IV.7. COMPARISON OF OUR RESULT TO OTHER DERIVATIONS OF SUBJECTIVE EXPECTED UTILITY MAXIMIZATION

A really satisfactory characterization, with appealing conditions that are both necessary and sufficient, for subjective expected utility (SEU) maximization in the context of DMUU (with "unknown" probabilities) is not yet available in literature. Shapiro (1979), Richter and Shapiro (1978), and Richter (1975), indicate how difficult this may be. SEU provides however the most used (and criticized) approach in DMUU. Hence derivations (giving sufficient conditions) are useful.

The best known derivation of SEU maximization, like ours not presupposing any probabilities or utilities, is the one given in Savage (1954). Savage's assumption P3 allows the derivation of a "qualitative probability relation" ("more probable than") on the set of events, from the preference relation on the set of acts. Mainly Savage's assumption P4 (the "sure-thing principle") guarantees "additivity" (condition 2, at the top of page 32) of this qualitative probability relation. The main restrictive assumption in Savage's approach is P6, some sort of continuity condition, requiring structure for the state space. For example this must be infinite, though not necessarily uncountable, contrary to what is sometimes thought. The major step in the proof of Savage is to use this qualitative probability relation, and the structure on the state space, to derive the probability measure. (Wakker (1981) pointed out some misunderstandings in literature about this part of Savage's work.) Once the probability measure has been obtained, the utility function is derived analogously as this was done in von Neumann and Morgenstern (1947, 1953, Chapter 3 and Appendix). For the consequence space Savage hardly needs restrictions. Mainly must the utility function be bounded. (This was discovered after publication of Savage (1954), see Fishburn (1970, section 14.1).)

In economic contexts the consequence space is usually assumed to be endowed with topological structure; for example it is $\mathbb{R}_+^m$. Hence in economic contexts derivations of SEU maximization, employing this structure, such as our Theorem IV.3.3, and Theorem V.6.1 in the sequel, may be valuable. Note that we did not use a qualitative probability relation as intermediate in the derivation of the probability measure. Our probabilities resulted from the "scale parameters" σ_j in the proof of Theorem IV.3.3, (see (IV.3.8)); they are proportional to the scales of the additive value functions $(V_j)_{j=1}^n$ there.

Another derivation of the same representation as ours, in terms of a derived "mean groupoid operation" on the consequence space, is given in Grodal (1978).

An early derivation of SEU maximization has been given in de Finetti (1931; see also 1937, 1972, 1974). De Finetti assumed that consequences were real numbers (amounts of money). His "coherence condition" requires the impossibility of a "Dutch book" to be made against the decision maker, i.e. no positive linear combination of bets, favourable in the view of the decision maker, should result in a bet, giving with certainty a negative yield. This entails linearity of the utility function. A major advantage of de Finetti's approach above most others (including Savage's and ours) is that it gives useful results for preference relations that are not complete.

Other approaches assumed consequences to be lotteries, or more generally elements of a mixture space (see Definition VII.2.1). See Anscombe and Aumann (1963), Fishburn (1982). Also Ramsey (1931) can be placed in this group, if his "neutral event" is considered as a $\frac{1}{2}$ - $\frac{1}{2}$ lottery. These approaches used linear (affine; von Neumann - Morgenstern) utility. The involved mathematics is in fact quite similar to those of de Finetti. Compared to these, our Theorem IV.3.3 no longer needs lotteries on the consequence space, or linearity of the utility function.

Extensive surveys on expected utility are provided in Fishburn (1981), Schoemaker (1982), and Machina (1983).

CHAPTER V

SUBJECTIVE EXPECTED UTILITY

FOR ARBITRARY STATE SPACES

V.1. INTRODUCTION

In this chapter we extend the characterization of continuous subjective expected utility maximization, given only for finite state spaces in Theorem IV.3.3, to arbitrary state spaces. This is the only chapter where the index (= state) set I is not assumed finite. We shall only consider acts which are in some sense bounded. In this way we avoid the main complication for infinite state spaces: how to handle acts with infinite, or even undefined, expected utility. In our apprach the utility function itself is not necessarily bounded, this contrary to Savage's approach.

The present chapter closely follows Wakker (1984c). We slightly generalize the latter work by leaving out the condition that $\mathcal{D}$, the algebra on the consequence space C that we shall introduce in the sequel, should contain all one-point subsets of C. This we achieve by a small variation in the definition of "simple" acts. In this chapter terminology will be as in decision making under uncertainty, the primarily intended field of application of our present work.

The strategy in this chapter is to first, as much as possible, assume properties and derive results for $\succ$ on the "simple" acts, which have finite range. The results then are extended to acts with infinite range, mainly by "constant-continuity" and "pointwise monotonicity".

V.2. ELEMENTARY DEFINITIONS

Acts, consequences, and states of nature, are as in Example II.1.1 (DMUU; see also Terminology IV.2.1). To stay close to probability theory we generalize our set-up by introducing measure-theoretic structure. We assume that an *algebra* $\mathcal{A}$ on I is given, i.e. $\mathcal{A} \subset 2^I$ contains $\emptyset$, is closed with respect to finite union and complement taking; hence contains I, and is closed with respect to finite intersection taking. Elements A,B of $\mathcal{A}$ are called *events*. Also an algebra $\mathcal{D}$ on C is given, with generic elements E,F.

As an example, $\mathcal{A}$ may contain all subsets of I. Then all measure-theoretic requirements, made in the sequel, are satisfied, and can be ignored. This shows that the introduction of measure-theoretic structure really is a generalization.

By F we denote the set of acts x that are $(\mathcal{D}$- $\mathcal{A}$-)*measurable*, i.e. for every $E \in \mathcal{D}$, $\{i \in I : x_i \in E\} \in \mathcal{A}$. If $\mathcal{A} = 2^I$, then $F = C^I$.

We say $\succcurlyeq$ is a *weak order on* a subset F' of X, if the restriction of $\succcurlyeq$ to F', as binary relation on F', is a weak order. Then, in the same way, $\approx$ is an equivalence relation on F'.

Throughout this chapter a *partition* $P = (A_j)_{j=1}^m$ will, without further mention, be assumed to consist only of events. We then write $\Sigma_{j=1}^m \alpha_j 1_{A_j}$ for the act, assigning consequence α_j to every $i \in A_j$, $j = 1,\ldots,m$, and call such an act *simple*. Simple acts are elements of F. The notation for simple acts is just a suggestive notation; it does not designate any addition or scalar multiplication operation. F^s denotes $\{x \in F : x \text{ is simple}\}$.

By F^b we denote $\{x \in F : \mu,\nu \in C$ exist such that $\bar{x}_i \succcurlyeq \bar{\mu}$ and $\bar{\nu} \succcurlyeq \bar{x}_i$ for all i$\}$. Its elements are called *strongly bounded*. If $\succcurlyeq$ is a weak order on F, then $F^s \subset F^b$. Note that, if $I = \mathbb{N}$, $C =]0,1]$, $x_i = 1/i$ for all i, then x is bounded in the usual sense, but not strongly bounded. Also note that, for any $\alpha \in C$, $x \in F$ [respectively F^s; or F^b], and $A \in \mathcal{A}$, $x_{-A}\alpha$ (Notation IV.5.1) is an element of F [respectively F^s; or F^b if $\succcurlyeq$ is a weak order on F^b].

Next we adapt a definition, given earlier for finite I, to the present situation.

DEFINITION V.2.1. An event A is *simple-inessential*, or *s-inessential (with respect to $\succ$)* if $x \approx y$ for all $x,y \in F^S$ for which $x_i = y_i$ for every $i \in A^c$. Otherwise A is *simple-essential* (or *s-essential*) *(with respect to $\succ$)*.

The following assumption will be used throughout this chapter. For finite I (with $\mathcal{A} = 2^I$) it comes down to the topological assumption III.2.11. It adds to this some measure-theoretic structure.

ASSUMPTION V.2.2. $\mathcal{C}$ is a connected topological space. $\mathcal{D}$ contains all open subsets of $\mathcal{C}$. If I is s-essential, and no two disjoint s-essential events exist, then $\mathcal{C}$ is topologically separable.

The case where no two disjoint s-essential events exist, will be treated in Lemma V.3.1. This case can be interpreted to be the case of certainty. Of course $\mathcal{D}$ also contains all closed subsets of $\mathcal{C}$. A further adaptation of an earlier definition to the present situation:

DEFINITION V.2.3. $\succ$ is *simple-continuous*, or *s-continuous* if, for any partition $(A_j)_{j=1}^m$ and any act $x = \Sigma_{j=1}^m \beta_j 1_{A_j}$, we have closedness of $\{(\alpha_1,\ldots,\alpha_m) \in \mathcal{C}^m : \Sigma_{j=1}^m \alpha_j 1_{A_j} \succ x\}$ and $\{(\alpha_1,\ldots,\alpha_m) \in \mathcal{C}^m : \Sigma_{j=1}^m \alpha_j 1_{A_j} \prec x\}$ with respect to the product topology on $\mathcal{C}^m$.

One may formulate s-continuity as: the binary relation $\succ'$ on $\mathcal{C}^m$, defined by $(\alpha_1,\ldots,\alpha_m) \succ' (\beta_1,\ldots,\beta_m)$ if $\Sigma_{j=1}^m \alpha_j 1_{A_j} \succ \Sigma_{j=1}^m \beta_j 1_{A_j}$, is continuous with respect to the product topology on $\mathcal{C}^m$. The assumption of this "finite-dimensional" continuity is not unusually strong since a finite-dimensional product topology is not coarser than other usual topologies. If $\mathcal{C}$ is a metric (for example Euclidean) space, then the finite-dimensional product topology is equal to the sup-metric topology (for example to the usual Euclidean topology). Koopmans (1972) uses a sup-metric topology on a denumerable cartesian product.

The main topological complications occur for infinite dimensions. Then the product topology is coarser than other usual topologies, and continuity with respect to this is then too strong for our purposes. It would imply countable additivity of the probability measure P, to be derived in the sequel, and would quickly lead to boundedness of the utility function U, to be derived in the sequel, if $\succ$ is not restricted to an appropriately chosen subset of X. In section V.4 we shall deal with infinite-dimensional aspects. For that we use:

DEFINITION V.2.4. $\succ$ is *constant-continuous on* $F' \subset C^I$ if $\{\alpha \in C : \bar{\alpha} \succ x\}$ and $\{\alpha \in C : \bar{\alpha} \prec x\}$ are closed for all $x \in F'$.

Again, as s-continuity, this continuity is implied by the supmetric continuity assumption of Koopmans (1972), also by continuity of $\succ$ with respect to the producttopology on C^I. In fact the only consequence of it, that we shall use, is that there exists, for every x which has $\bar{\alpha} \succ x \succ \bar{\beta}$ for some $\alpha, \beta \in C$, a "certainty equivalent" $\bar{\gamma} \approx x$.

The main tool in this chapter for the characterization of subjective expected utility maximization is the following adaptation of the CCI-relatedness property:

DEFINITION V.2.5. Event A is *simple-cardinal coordinate independent related, or s-CCI-related, to* event B, if for all $\alpha, \beta, \gamma, \delta \in C$, and all $x, y, v, w \in F^s$: $x_{-B}\alpha \prec y_{-B}\beta$ & $x_{-B}\gamma \succ y_{-B}\delta$ & $v_{-A}\alpha \succ w_{-A}\beta$ imply $v_{-A}\gamma \succ w_{-A}\delta$.

For finite I every act is simple, and Definitions V.2.1, V.2.3, and V.2.5, without "s- ", coincide with the old ones; under the, for finite I usual, assumption that $A = 2^I$.

V.3. RESULTS FOR SIMPLE ACTS

First we handle the "degenerate" case where one state, or an "ultrafilter" of states (see (V.3.4)) is "certainly true".

LEMMA V.3.1. *Let there not exist two disjoint s-essential events. Under Assumption V.2.2 the following two statements are equivalent:*

(i) *There exists a finitely additive probability measure* P *on* A, *and a continuous* $U : C \to \mathbb{R}$, *such that* $\Sigma_{j=1}^{s}\alpha_j 1_{A_j} \succcurlyeq \Sigma_{k=1}^{t}\beta_k 1_{B_k} \Leftrightarrow$ $\Sigma_{j=1}^{s}P(A_j)U(\alpha_j) \geq \Sigma_{k=1}^{t}P(B_k)U(\beta_k)$ *for all* $\alpha_1,\ldots,\beta_t$.

(ii) $\succcurlyeq$ *is an s-continuous weak order on* F^s.

Furthermore, if (i) holds, then every event is s-CCI-related to every s-essential event.

The following uniqueness results hold for U, P *of (i):*

If $\succcurlyeq$ *is not trivial on* F^s, *then* $P(A) = 1$ *for all s-essential* A, $P(A) = 0$ *for all s-inessential* A, *and* U *is continuously ordinal.* *(V.3.1)*

If $\succcurlyeq$ *is trivial on* F^s, *then* U *must be constant, and* P *is arbitrary.* *(V.3.2)*

PROOF. (i) $\Rightarrow$ (ii) is straightforward. So we suppose (ii), and derive (i), and the results below (ii).

There exists no s-essential event iff $x \approx y$ for all $x,y \in F^s$, i.e. $\succcurlyeq$ is trivial on F^s. In this case all of (V.3.2), and (i), follow.

So from now on we assume:

There exists an s-essential event. (V.3.3)

To derive P, we show:

The collection of all s-essential events is an *ultrafilter*, i.e. (V.3.4)

(a) I is s-essential.

(b) Event A is s-essential iff A^c is s-inessential.

(c) If events A and B are s-essential, then so is $A \cap B$.

Were I s-inessential, then $\succcurlyeq$ would be trivial on F^s, contradicting (V.3.3). So (a) above follows. Were, for an event A, both A and A^c s-inessential, then $x \approx x_A y_{A^c} \approx y$ would follow for all $x,y \in F^s$, and $\succcurlyeq$ would be trivial on F^s. This cannot hold, and (b) now follows from the assumption that no two disjoint s-essential events can exist.

If events A and B are s-inessential, then so is $A \cup B$, since for all $x,y \in F^S$ with $x_{A^c \cap B^c} = y_{A^c \cap B^c}$ we have $x \approx x_{A^c}y_A \approx y$. This and (b) imply (c).

We define $P(A) = 1$ for all s-essential events A, and $P(A) = 0$ for all s-inessential events A. One easily checks that this gives a finitely additive probability measure P.

Let U represent $\succcurlyeq$ on C, as defined in Notation IV.2.3. By Theorem III.3.1 such an U indeed exists, and is continuously ordinal.

(i) is demonstrated if we show that:

$$x = \Sigma_{j=1}^{s}\alpha_j 1_{A_j} \succcurlyeq \Sigma_{k=1}^{t}\beta_k 1_{B_k} = y \Leftrightarrow \Sigma_{j=1}^{s}P(A_j)U(\alpha_j) \geq \Sigma_{k=1}^{t}P(B_k)U(\beta_k). \qquad (V.3.5)$$

Of the mutually disjoint $(A_j \cap B_k)_{j=1,k=1}^{s\ \ t}$ exactly one is s-essential, say $A_1 \cap B_1$. Then $x \approx \bar{\alpha}_1$, $y \approx \bar{\beta}_1$, $P(A_1) = 1 = P(B_1)$, and (V.3.5) follows.

Now (V.3.1) follows from the observation that a U as in (i) must represent $\succcurlyeq$ on C, and that P as in (i) must assign probability 0 to every s-inessential event, thus 1 to every s-essential event.

The "furthermore-statement" in the lemma is by simple substitution in (i).

□

The next lemma shows how, on a "finite-dimensional" subspace of the form $\{x \in C^I : x = \Sigma_{j=1}^{m}\alpha_j 1_{A_j}\}$, for a fixed partition $(A_1,\ldots,A_m)$, the results for finite cartesian products can be applied.

LEMMA V.3.2. *Let Assumption V.2.2 hold. Let* $\succcurlyeq$ *be an s-continuous weak order on* F^S. *Let every event be s-CCI-related to every s-essential event. Let* $P^1 = (A_1,\ldots,A_s)$ *be a partition with at least two s-essential events. Then there exist nonnegative* $(p_j^1)_{j=1}^s$, *summing to* 1, *and a continuous* $U^1 : C \to \mathbb{R}$, *such that:*

$$\Sigma_{j=1}^{s}\alpha_j 1_{A_j} \succcurlyeq \Sigma_{j=1}^{s}\beta_j 1_{A_j} \Leftrightarrow \Sigma_{j=1}^{s}p_j^1 U^1(\alpha_j) \geq \Sigma_{j=1}^{s}p_j^1 U^1(\beta_j).$$

The p_j^1*'s are uniquely determined, and* U^1 *is cardinal.*

PROOF. Define $\succ'$ on C^s by $(\alpha_j)_{j=1}^s \succ' (\beta_j)_{j=1}^s$ if $\Sigma_{j=1}^s \alpha_j 1_{A_j} \succ \Sigma_{j=1}^s \beta_j 1_{A_j}$. Then $\succ'$ satisfies all requirements of Theorem IV.3.3. This implies all desired results.

□

Next we show that for two finite partitions P^1 and P^2, each with at least two s-essential events, the representations resulting from the previous Lemma, "fit together", i.e. the utility functions can be taken the same and events occurring in both partitions, have the same probability in each representation. This we do by comparing P^1 and P^2 to a partition P^3, finer than P^1 and than P^2, and by showing that the representations of P^1 and P^2 "fit together" with that of P^3.

LEMMA V.3.3. *Let, under the assumptions and notations of Lemma V.3.2, $P^2 = (B_1,\dots,B_t)$ be another partition with at least two s-essential events. Let application of Lemma V.3.2 to P^2 give $(p_j^2)_{j=1}^t$ and U^2. Then $U^2 = \phi \circ U^1$ for a positive affine ϕ, and if $A_i = B_j$ for some i,j, then also $p_i^1 = p_j^2$.*

PROOF. Define $P^3 := (((A_j \cap B_k)_{k=1}^t)_{j=1}^s)$. First we show that P^3 must have two or more s-essential events. Say A_1 and A_2 are s-essential. Now s-inessentiality of all $A_1 \cap B_k$, $k = 1,\dots,s$, would imply, by a reasoning as used to derive (V.3.4.c), s-inessentiality of A_1. So of the $A_1 \cap B_k$'s, at least one is s-essential. Analogously of the $A_2 \cap B_k$'s at least one is s-essential.

So we can apply Lemma V.3.2 to P^3 instead of P^1, yielding $((p_{jk})_{k=1}^t)_{j=1}^s$ and U^3. Now, defining $p_j := \Sigma_{k=1}^t p_{jk}$ for all j, and $U := U^3$, we obtain an array $(p_j)_{j=1}^n$ and a U, that satisfy all requirements for $(p_j^1)_{j=1}^n$ and U^1 in Lemma V.3.2. The uniqueness results of that Lemma imply $p_j = p_j^1$ for all j, and $U^1 = \phi^1 \circ U^3$ for a positive affine ϕ^1.

Analogously $p_i^2 = \Sigma_{j=1}^s p_{ji}$ for all i, and $U^2 = \phi^2 \circ U^3$ for a positive affine ϕ^2. So $U^2 = \phi \circ U^1$ for a positive affine ϕ. And if $A_i = B_j$ for some i,j, then $p_{ik} = p_{lj} = 0$ for all $k \neq j$, $l \neq i$; $p_i^1 = p_j^2 = p_{ij}$ follows.

□

Now we are ready for the main result of this section, a characterization of a subjective expected utility representation on F^s.

THEOREM V.3.4. *Under Assumption V.2.2, for the binary relation* $\succcurlyeq$ *on* C^I *the following two statements are equivalent:*

(i) *There exist a finitely additive probability measure* P, *and a continuous function* $U : C \to \mathbb{R}$, *such that* $\Sigma_{j=1}^{m}\alpha_j 1_{A_j} \mapsto \Sigma_{j=1}^{m}P(A_j)U(\alpha_j)$ *represents* $\succcurlyeq$ *on* F^s.

(ii) $\succcurlyeq$ *is an s-continuous weak order on* F^s, *and every event is s-CCI-related to every s-essential event.*

The following uniqueness results hold for U, P *of (i):*

If two disjoint s-essential events exist, then P *is uniquely determined, and* U *is cardinal.* *(V.3.6)*

If I *is s-essential, but no two disjoint s-essential events exist, then* P *assigns* 1 *to every s-essential event,* 0 *to every s-inessential event, and* U *is continuously ordinal.* *(V.3.7)*

If I *is s-inessential, then* P *is arbitrary, and* U *can be any constant function.* *(V.3.8)*

PROOF. As always, (i) $\Rightarrow$ (ii) is straightforward. So we assume (ii), and derive (i) and the uniqueness results. For the case (V.3.8), I s-inessential, everything is straightforward. The case (V.3.7) is covered by Lemma V.3.1. So we assume that there exist two disjoint s-essential events. By Lemma V.3.2 there exist, for every partition $P = (A_1,\ldots,A_t)$ with at least two essential events, a probability measure P_P on the algebra of events, consisting of unions of events from P, and a utility function $U_P : C \to \mathbb{R}$, continuous, such that $\Sigma_{j=1}^{t}\alpha_j 1_{A_j} \mapsto \Sigma_{j=1}^{t}P_P(A_j)U_P(\alpha_j)$ represents $\succcurlyeq$ on the elements of F^s, that can be written as $\Sigma_{j=1}^{t}\alpha_j 1_{A_j}$. By Lemma V.3.3, P_P and U_P can be taken independent of P. That we do, and we leave out indexes P.

First we show that P is a probability measure $P(\emptyset) = 0$, $P(I) = 1$ are obvious. Let A,B be disjoint. To show is: $P(A \cup B) = P(A) + P(B)$. We define $A_1 := A$, $A_2 := B$, $A_3 := A^c \cap B^c$. Let C,D be two disjoint s-essential events. Define $B_1 := C$, $B_2 := D$, $B_3 := C^c \cap D^c$. Let $\mathcal{P} = (((A_i \cap B_j)_{i=1}^{3})_{j=1}^{3})$. This $\mathcal{P}$ has, by a reasoning as in the proof of Lemma V.3.3, at least two s-essential events. Let $(p_{ij})_{j=1}^{3}{}_{,i=1}^{3}$ and U be as resulting from Lemma V.3.2. Now $P(A) = \Sigma_{j=1}^{3} p_{1j}$, $P(B) = \Sigma_{j=1}^{3} p_{2j}$, and $P(A \cup B) = \Sigma_{j=1}^{3} (p_{1j}+p_{2j})$.

That now $\Sigma_{j=1}^{s} \alpha_j 1_{A_j} \succcurlyeq \Sigma_{k=1}^{t} \beta_k 1_{B_k} \Leftrightarrow \Sigma_{j=1}^{s} P(A_j)U(\alpha_j) \geq \Sigma_{k=1}^{t} P(B_k)U(\beta_k)$, follows from consideration of a $\mathcal{P}$, both finer than $(A_j)_{j=1}^{s}$ and $(B_k)_{k=1}^{t}$.

The uniqueness result (V.3.6) follows from Lemma V.3.2.

□

The following Corollary, a simple consequence of the above theorem, gives properties which $\succcurlyeq$ has on F^s, but in general not on all of F, or F^b.

COROLLARY V.3.5. *Let* $\succcurlyeq$ *satisfy (i) of Theorem V.3.4. Then, for all* $x,y \in F^s$, $[\bar{x}_i \succcurlyeq \bar{y}_i$ *for all* $i \in I \Rightarrow x \succcurlyeq y]$. *And* $\succcurlyeq$ *is coordinate independent on* F^s.

V.4. RESULTS FOR STRONGLY BOUNDED ACTS

In this section we want to extend the representation of Theorem V.3.4 (i) to more general acts, mainly those of F^b. We have in mind an expected utility representation by means of some sort of integral of U with respect to P. The approach to integration for measures that are only finitely additive, as adopted in section I.III.2 of Dunford and Schwarz (1958) or section 4.4 of Bhaskara Rao and Bhaskara Rao (1983) does not seem to be suited for our purposes. This is because we see no easy way to reformulate the properties of P and $U \circ x$, used there in the definition of an integral, in terms of our primitive, i.e. $\succcurlyeq$. The less general *Stieltjes type integral*, as exposed in section 4.5 of Bhaskara Rao and Bhaskara Rao (1983) does serve our purposes. In this, an integral, notation EU, of a bounded measurable function $U \circ x$ on I is obtained as a "lower integral", equal to

$\sup\{EU(f^s) : f^s : I \to \mathbb{R}$ has finite range and is measurable, $f^s \preccurlyeq^p U \circ x\}$, with $\preccurlyeq^p$ pointwise dominance, i.e. $f^s \preccurlyeq^p U \circ x$ whenever $f^s(i) \leq U(x_i)$ for all $i \in I$; or the integral is obtained as an upper integral, which is analogous and yields the same result for bounded functions. If $U \circ x$ is bounded below (above) but unbounded above (below), one may still define the lower (upper) integral, and see if this is useful. Of course, what we have in mind is to let the f^s above be of the form $U \circ x^s$ for $x^s \in F^s$. We handle pointwise dominance as follows:

DEFINITION V.4.1. $\succcurlyeq$ is *pointwise monotone* on $F' \subset C^I$ if $x \succcurlyeq y$ for all $x,y \in F'$ for which $\overline{x_i} \succcurlyeq \overline{y_i}$ for all $i \in I$.

Note that, in the terminology of Chapter II, this is weak cA monotonicity, if we take $\succcurlyeq_i = \succcurlyeq$ for all i, and allow for infinite cartesian products. Suppes (1956, A9) and Ferreira (1972, C1) also used this kind of monotonicity. Note that it uses comparisons of consequences x_i to consequences y_i, only if these consequences are assigned to the same state of nature. This differs from assumption "P7" in Savage (1954). The latter requires something like: $x \succcurlyeq y$ whenever $\overline{x_i} \succ y$ for all i, or $x \succ \overline{y_i}$ for all i. An advantage of our set-up with pointwise monotonicity, over Savage's set-up with his P7, is that in our set-up the utility function does not have to be bounded, where in Savage's set-up it must be, see section 14.1 in Fishburn (1970). An advantage of Savage's set-up is that, once utility is bounded, Savage's set-up handles all acts, whereas our's only handles all strongly bounded acts. For a further illustration of this the reader is referred to the example (1) in section 5.4 of Savage (1954), where no expected utility representation exists, but where pointwise monotonicity can be seen to be satisfied.

The following example illustrates that pointwise monotonicity on F^b is not implied by the other properties, introduced:

EXAMPLE V.4.2. Let $I =]0,1]$, $C = \mathbb{R}$, A the Borel σ-algebra on $]0,1]$, D the Borel σ-algebra on $\mathbb{R}$, U identity, and let P be Lebesgue measure. Let $\succcurlyeq$ on F^b be represented by a linear functional V from F to $\mathbb{R}$, with $V(1_A) = P(A)$ for all events A. Then $\succcurlyeq$ is a constant- and s-continuous weak order, even $\succcurlyeq$ is coordinate independent on F^b. Every event is s-CCI related to every s-essential event. Yet, without pointwise monotonicity we are still completely free to let V assign to x, with $x_i = i$ for all i, any real number, such as -1 since x is not in the linear subspace, spanned by the indicator functions. Then $\overline{x_i} \succcurlyeq 0$ for all i, but not $x \succcurlyeq \bar{0}$, so pointwise monotonicity is violated.

LEMMA V.4.3. *Let $\succcurlyeq$ be a constant-continuous pointwise monotone weak order on F^b. Then for every $x \in F^b$ there exists $\alpha \in C$ such that $x \approx \bar{\alpha}$.*

PROOF. $\{\beta \in C : \bar{\beta} \succcurlyeq x\}$ and $\{\beta \in C : \bar{\beta} \preccurlyeq x\}$ are closed by constant-continuity, and nonempty if $x \in F^b$, because then $[\bar{\mu} \succcurlyeq \overline{x_i} \succcurlyeq \bar{\nu}$ for all i] and pointwise monotonicity imply μ to be in the first, ν in the second, set above. These sets, with union C, must have nonempty intersection by connectedness of C. Let α be in this intersection.

□

We can now, for $x \in F^b$, simply take α as above, and define $EU(x) := U(\alpha)$, with U as in Theorem V.3.4, under the appropriate assumptions for $\succcurlyeq$. Then $x \succcurlyeq y \Leftrightarrow EU(x) \geq EU(y)$, and for any $x = \Sigma_{j=1}^{m}\alpha_j 1_{A_j}$, $EU(x) = \Sigma_{j=1}^{m} P(A_j)\ U(\alpha_j)$. Question remains whether EU can be considered a (Stieltjes-type) integral outside F^s. Below we shall see that it can.

THEOREM V.4.4. *Under Assumption V.2.2, for the binary relation $\succcurlyeq$ on C^I, the following two statements are equivalent:*

(i) *There exist a finitely additive probability measure P, and a continuous $U : C \to \mathbb{R}$, such that, on F^b, $x \mapsto \int U(x_i)dP(i)$ represents $\succcurlyeq$, with the integral well defined.*

(ii) *$\succcurlyeq$ is a constant- and s-continuous pointwise monotone weak order on F^b, such that every event is s-CCI-related to every s-essential event.*

Uniqueness results for (i) are as (V.3.6), (V.3.7), and (V.3.8) in Theorem V.3.4.

PROOF. We only show (ii) $\Rightarrow$ (i). Let P,U be as provided by Theorem V.3.4. Let $x \in F^b$, $\bar{\mu} > \overline{x_i} > \bar{\nu}$ for all $i \in I$. Let $\bar{\alpha} \approx x$, such an α exists by Lemma V.4.3. Let $EU(x) = U(\alpha)$. We have to show that EU is an integral. If $\bar{\mu} \approx \bar{\nu}$ then by pointwise monotonicity $x \approx \bar{\mu}$, so $\bar{\alpha} \approx \bar{\mu}$, $U \circ x$ is constant, and $EU(x) = \int U(x_i)dP(i)$.

Now suppose $\bar{\mu} > \bar{\nu}$. For notational convenience we shall suppose that $U(\mu) = 1$, $U(\nu) = 0$. We now construct a sequence of pairs of simple functions $(x^m, y^m)_{m=1}^{\infty}$, such that:

$$U(x_i)-1/m \leq U(x_i^m) \leq U(x_i) \leq U(y_i^m) \leq U(x_i)+1/m \qquad \text{(V.4.1)}$$

for all i,m.

For any m, and $0 \leq k \leq m-1$,

$A_k := \{i \in I : k/m \leq U(x_i) < (k+1)/m\}$
is an event. Since U is continuous, and $\mathcal{C}$ connected, also $U(\mathcal{C}) \subset \mathbb{R}$ is connected. So for any $0 \leq k \leq m$ there exists α_k such that $U(\alpha_k) = k/m$. We define

$x^m := \Sigma_{k=0}^{m-1}\alpha_k 1_{A_k} + \alpha_{m-1}1_{\{i:U(x_i)=1\}}$, and

$y^m := \Sigma_{k=0}^{m-1}\alpha_{k+1}1_{A_k} + \alpha_m 1_{\{i:U(x_i)=1\}}$.

We then have $U(x_i^m) \leq U(x_i) \leq U(y_i^m)$, so $\overline{x_i^m} < \overline{x_i} < \overline{y_i^m}$, for all i. By pointwise monotonicity $x^m < x < y^m$. Hence $EU(x^m) \leq U(\alpha) \leq EU(y^m)$. But also $EU(y^m) - EU(x^m) = 1/m$ for all m. (See lines above the theorem.) We conclude that $EU(x) = U(\alpha) = \lim_{m\to\infty} EU(x^m) = \lim_{m\to\infty} EU(y^m)$. Indeed EU(x) can be considered to be an integral of U w.r.t. P.

□

V.5. COUNTABLE ADDITIVITY

We shall give a continuity assumption, necessary and sufficient for countable additivity of the probability measure P of Theorems V.3.4 and V.4.4. This adapts the known results, as presented in section 6.9 of de Finetti (1972) to the more general case where

$C \neq \mathbb{R}$; with everything formulated in terms of the preference relation $\succ$. Property F7 in section 10.3 in Fishburn (1982), and the "monotone continuity" assumption of Villegas (1964), also used in Arrow (1971, Lecture 1), are analogous.

DEFINITION V.5.1. A probability measure P on an algebra $\mathcal{A}$ is *countably* (or σ-)*additive* if, for any sequence of events $(A_m)_{m=1}^{\infty}$ with $A_{m+1} \subset A_m$ for all m, and $\cap_{m=1}^{\infty} A_m = \emptyset$, we have $\lim_{m\to\infty} P(A_m) = 0$.

The following lemma gives an equivalent formulation that is well-known.

LEMMA V.5.2. *P is σ-additive if and only if, for any sequence $(B_m)_{m=1}^{\infty}$ of mutually disjoint events, with $B = \cup_{m=1}^{\infty} B_m$ in $\mathcal{A}$, we have $P(B) = \Sigma_{m=1}^{\infty} P(B_m)$.*

PROOF. Substitute $A_m := B \setminus (\cup_{k=1}^{m} B_k)$, or $B_m := A_m \setminus A_{m+1}$.

□

The following definition will only be used in the definition thereafter.

DEFINITION V.5.3. A set of acts $\{x^j\}_{j\in J}$ is *uniformly strongly bounded* if there exist μ, $\nu \in C$ such that $\bar{\mu} \succ \bar{x}_i^j \succ \bar{\nu}$ for all $i \in I$, $j \in J$.

With this we define the property, characterizing σ-additivity of P. We could below have restricted attention to all simple x^j's, and even to all acts with only two consequences in the range. This is the only thing needed in the proof of Theorem V.5.5.

DEFINITION V.5.4. $\succ$ is *boundedly strictly continuous* if for any uniformly strongly bounded sequence of acts $(x^j)_{j=1}^{\infty}$, and any pair of acts x,y, for which $x^j \succ y$ [respectively $x^j \prec y$] for all j, and $\lim_{i\to\infty} x_i^j = x_i$ for all i, we have $x \succ y$ [respectively $x \prec y$].

Note that the above definition is weaker than continuity with respect to the product topology, i.e. pointwise convergence. For we only consider *uniformly strongly bounded* converging sequences (and no uncountable converging "nets").

THEOREM V.5.5. *Let (i) in Theorem V.4.4 hold. Then* P *can be chosen* σ*-additive if and only if* $\succ$ *is boundedly strictly continuous.*

PROOF. First we assume bounded strict continuity, and derive σ-additivity. If I is s-inessential, then U is constant, and we can let P be any σ-additive probability measure, e.g. let P(A) = 1 if and only if A contains some fixed $i \in I$.

Next suppose I is s-essential. Then α,β exist such that $\bar{\alpha} > \bar{\beta}$, otherwise pointwise monotonicity (or s-CCI relatedness) would imply s-inessentiality of I. Now let $(A_m)_{m=1}^{\infty}$ be a sequence of events, such that $A_m \supset A_{m+1}$ for all m, and $\cap A_m = \emptyset$. Define $x^m := \alpha 1_{A_m} + \beta 1_{A_m^c}$, $x = \bar{\beta}$. By pointwise monotonicity $x^m \succ x^{m+1} \succ \bar{\beta}$ for all m, so $\lim_{m\to\infty} EU(x^m) \geq U(\beta)$. (EU: see above Theorem V.4.4.) We now first show that the last inequality is in fact equality.

Suppose $\lim_{m\to\infty} EU(x^m) > U(\beta)$. $U(\alpha) > U(\beta)$, so $U(\beta)$ is not maximal in $U(C)$. Since $U(C)$ is connected, a γ must exist with $\lim_{m\to\infty} EU(x^m) > U(\gamma) > U(\beta)$. Now $x^m \succ \bar{\gamma}$ for all m, so $\bar{\beta} = x \succ \bar{\gamma}$ by bounded strict continuity. This contradicts $U(\gamma) > U(\beta)$. It follows that $\lim_{m\to\infty} EU(x^m) = U(\beta)$.

The last equality, and $EU(x^m) = P(A_m)U(\alpha) + (1-P(A_m))U(\beta)$, imply $\lim_{m\to\infty} P(A_m) = 0$; as required for σ-additivity of P.

Conversely, let P be σ-additive. Then bounded strict continuity follows from continuity of U and the dominated convergence Theorem of Lebesgue (e.g. see Corollary 16 in section I.III.6.16 of Dunford and Schwartz, 1958). This theorem is usually formulated for σ-algebras. It can be applied to our context by taking the smallest σ-algebra containing $\mathcal{A}$, and taking the unique σ-additive extension of P to this, guaranteed by Royden (1963, section 12.2). The values of the involved integrals of $U \circ x_m$ and $U \circ x$ are not affected by this extension of $\mathcal{A}$ and P.

□

V.6. THE MAIN RESULT, CONCLUSIONS AND FURTHER COMMENTS

First we formulate our main result, combining the previous results. *Let us repeat* that I is a nonempty set, A an algebra on I, elements of A are "events", C is a connected topological space, $\mathcal{D}$ an algebra of subsets of C that contains all open subsets of C. $F \subset C^I$ is the set of acts that are A- $\mathcal{D}$ measurable, F^b is the set of all strongly bounded (section V.2) acts in F. $\succcurlyeq$ is a binary (preference) relation on C^I.

THEOREM V.6.1. *Under Assumption V.2.2, for the binary relation $\succcurlyeq$ on C^I, the following two statements are equivalent:*

(i) *There exists a finitely additive probability measure* P *on* A, *and a continuous* $U : C \to \mathbb{R}$, *such that, on* F^b, $x \mapsto \int U(x_i)\,dP(i)$, *(integral well-defined) represents* $\succcurlyeq$.

(ii) $\succcurlyeq$ *is a constant- and s-continuous pointwise monotone weak order on* F^b, *for which all events are s-CCI-related to all s-essential events.*

Furthermore, in (i) we may replace "finitely" by "countably", if we add in (ii) the requirement that $\succcurlyeq$ *is boundedly strictly continuous.*

Uniqueness results for (i) are as in Theorem V.3.4.

PROOF. See Theorems V.3.4, V.4.4, and V.5.5.

□

To our knowledge this is the most general characterization of subjective expected utility maximization with continuous utility, now available. The special case where $C = \mathbb{R}$, and U is identity, is treated in de Finetti (1972), a major source of inspiration for our work.

Theorem 3 of Grodal (1978) derives a representation as in (i) above, so also for a possibly infinite state space, under the supposition that a triple of disjoint s-essential events exists. The conditions used there employ a (presupposed) measure on A, and a derived mean groupoid operation. Grodal's results also treat the case where the set of acts is a *subset of* F (or F^b), as long as it is closed under a certain mixture operation, and contains the constant acts.

For not strongly bounded acts a representation as in (i) above meets new complications. Say x is not strongly bounded. Of course, if $x \approx \bar{\alpha}$ for some $\alpha \in C$, which always occurs if $\bar{\mu} \succ x \succ \bar{\nu}$ for some $\mu, \nu \in C$ (under appropriate assumptions), we would still like to define $EU(x) = U(\alpha)$. But now there is no justification to consider this as an integral of $U \circ x$. If x is strongly bounded below (there is γ such that $\overline{x_i} \succ \bar{\gamma}$ for all i) an integral value for $U \circ x$, its "lower integral", exists. This integral value is not greater than EU(x), may equal EU(x), but may also very well be smaller than EU(x). If $x \approx \bar{\alpha}$, but now x strongly bounded above , we can obtain an upper integral value that may be "too" large.

Conditions for $\succ$, strong enough to guarantee that $\succ$ can be represented by an integral for all acts, are usually undesirably strong, for instance they may simply imply boundedness of U, as turned out to be the case in Savage (1954). They may even lead to impossibility results, for instance if $C =]0,1] = I$, $\succ$ maximizes Lebesgue integral, and one would let $A = 2^I$ and require continuity of $\succ$ with respect to the product topology on C^I. Then this would require a σ-additive extension of the Lebesgue measure to $2^{]0,1]}$, which is known not to exist, see Banach and Kuratowski (1929), or Ulam (1930). Finally, such conditions for $\succ$ may restrict the set of considered acts strongly.

The integral representation càn be extended to those acts x, equivalent to some $\bar{\alpha}$, that have, for every $\bar{\beta} \prec x$, a "sufficiently good" consequence γ to ensure that the "above truncation" x' of x at γ (i.e. $x'_i = x_i$ if $\overline{x_i} \prec \bar{\gamma}$, $x'_i = \gamma$ if $\overline{x_i} \succ \bar{\gamma}$) has $\bar{\beta} \prec x$, and that have, for every $\bar{\mu} \succ x$, a "sufficiently bad" consequence ν to ensure that the "below truncation" x" of x at ν (i.e. $x''_i = x_i$ if $\overline{x_i} \succ \bar{\nu}$, $x''_i = \nu$ if $\overline{x_i} \prec \bar{\nu}$) has $x'' \prec \bar{\mu}$. This is the way to extend $\succ$ to the class of all acts with finite expected utility, a desirable result for instance for statistical applications (see De Groot, 1970, end of section 7.9). For brevity this is not elaborated here.

Other acts are difficult to handle. One quickly runs into problems, related to the "St. Petersburg paradox".

The application of our results of course is not restricted to DMUU. For instance one may think of welfare theory, with agents instead of states of nature, and with P interpreted as power index.

A major application of our results lies in dynamic contexts. Our theorems are general enough to apply both to continuous and discrete time. One may characterize "constantness" of the "discount factor" P, where P corresponds to weights of a form $ke^{-\rho t}$, by the addition of an extra stationarity assumption. Such a thing is done in Theorem 4 of Grodal (1978). Compare also Corollary IV.4.6, or Corollary IV.5.5. Drèze (1982) emphasizes the analogy between the "I=set of states" and "I=set of points of time" interpretations.

We end with two conjectures:

CONJECTURE V.6.2. If $\mathcal{C}$ is topologically separable, then s-continuity is implied by the other properties of $\succcurlyeq$ in (ii) of Theorem V.6.1.

CONJECTURE V.6.3. In Theorem V.6.1(ii) one may weaken pointwise monotonicity to: $[\overline{x_i} \succ \overline{y_i}$ for all $i \Rightarrow x \succ y$ for all $x,y \in F^b]$.

The property in Conjecture V.6.3 is more closely related to the "coherent condition" of de Finetti, see de Finetti (1974, section 3.3.6).

CHAPTER VI

SUBJECTIVE EXPECTED UTILITY

BY CHOQUET INTEGRALS

VI.1. INTRODUCTION

In this chapter we shall characterize, in Theorem VI.5.1, subjective expected utility maximization with continuous utility for the case where the probability measure no longer has to be additive. The main characterizing property will be "comonotonic cardinal coordinate independence". The "nonadditive probability measures" will be called "capacities", see Definition VI.2.1. Choquet (1953-54, 48.1) has indicated, for a special class of capacities, a way to integrate with respect to these capacities. We shall adopt this way of integration. For an alternative way to integrate with respect to capacities see Gilboa (1985a).

Capacities play a role in cooperative game theory with side payments, where I is a set of "players", a subset of I is a "coalition", and the capacity is a "characteristic function", or "game", indicating productivity, power etc. See von Neumann and Morgenstern (1944), Luce and Raiffa (1957), Driessen (1985). Capacities also play a role in the study of robustness in statistics, see Huber (1981, section 10.2), Huber and Strassen (1973).

Schmeidler (1984 a,b,c) applied capacities in decision theory. One motive was to vary on expected utility maximization so as to avoid paradoxes such as the "Ellsberg paradox" (see Ellsberg, 1961) or the "Allais paradox" (see Allais, 1953, or Savage, 1954, pp.101-103), paradoxes that are often used to criticize or falsify expected utility maximization. Another motive is the applicability in welfare theory.

Special kinds of capacities are the "belief functions" in Shafer (1976, 1979), or the "plausibility" in Reschner (1976).

In this chapter we shall again use terminology of decision making under uncertainty. Schmeidler (1984a) has characterized subjective expected utility maximization with nonadditive probabilities, for the case where consequences are lotteries. He could start with an application of the theorem of Herstein and Milnor (1953), and thus immediately obtain a cardinal representing function for the preference relation on the set of acts. (This induces "linear" utility for the consequences.) After that he could apply to this representing function the characterization of functionals that can be considered Choquet integrals, as given in Schmeidler (1984c). See also Anger (1977, Theorem 3).

We adapt, under the simplifying assumption that the state space is finite, the work of Schmeidler to the case where the consequence space is a connected topological space, and utility is continuous, not necessarily linear. In our work a (cardinal) representing function is not easily available, and a derivation of it will be the main mathematical difficulty.

One can consider Schmeidler's work the adaptation of Anscombe and Aumann's (1963) characterization of subjective expected utility, to the case of nonadditive probability, and the results of this chapter the adaptation of our characterization of subjective expected utility, given in Theorem IV.3.3. Gilboa (1985b) adapts the characterization of subjective expected utility maximization of Savage (1954) to the case of nonadditive probability.

VI.2. CAPACITIES AND THE CHOQUET INTEGRAL

Throughout this, and following, chapters, I is the finite set $\{1,\ldots,n\}$.

DEFINITION VI.2.1. A function $v : 2^I \to \mathbb{R}$ is a *capacity* if:

$$v(\emptyset) = 0. \tag{VI.2.1}$$

$$v(I) = 1. \tag{VI.2.2}$$

$$A \subset B \Rightarrow v(A) \leq v(B) \text{ (monotonicity)}. \tag{VI.2.3}$$

Note that the range of v must be a subset of [0,1]. In literature capacities are also defined when I is infinite; then usually continuity with respect to increasing and decreasing sequences of events is required. For our finite I this is trivially satisfied. Also the domain of the capacity in literature is often taken to be the collection of compact subsets of I, with I a (Hausdorff) topological space, or it is taken to be an algebra on I. To follow this, we could of course endow I with the topology, or algebra, 2^I. Finally, the normalization (VI.2.2) is sometimes left out.

The following definition was essentially first given by Choquet (1953-54, 48.1).

DEFINITION VI.2.2. Let $v : 2^I \to \mathbb{R}$ be a capacity. Then, for any $f : I \to \mathbb{R}$, the *Choquet integral* of f with respect to v, $\int_I f dv$, is:

$$\int_0^\infty v(\{i\in I : f(i) \geq \tau\})d\tau + \int_{-\infty}^0 [v(\{i\in I : f(i) \geq \tau\})-1]d\tau. \tag{VI.2.4}$$

Note that for nonnegative f the second term vanishes. And note that for additive v the Choquet integal coincides with the usual expectation of f with respect to v, as follows from integration by parts. I being finite, (VI.2.4) can be written as a sum. To this end let π be a permutation, dependent on f, on $\{1,\ldots,n\}$, such that $f(\pi(1)) \geq f(\pi(2)) \geq \ldots \geq f(\pi(n))$. So π assigns to every i, considered as a ranking number, the state of nature with this ranking number, where ranking is according to the values of f. States with

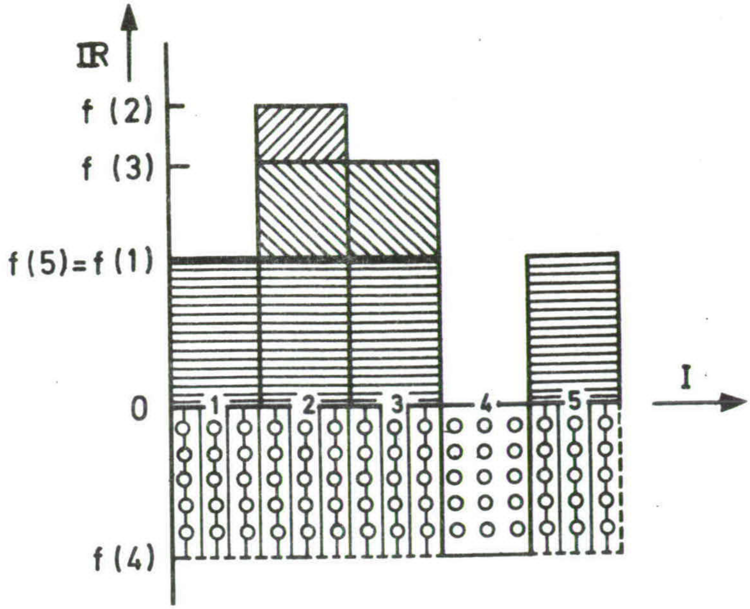


FIGURE VI.2.1.(a). $\int_I f dv$ = (VI.2.4) = A(///) + A(\\\) + A(—) + A(≡) + [A(|||) − A($^{000}_{000}$)].

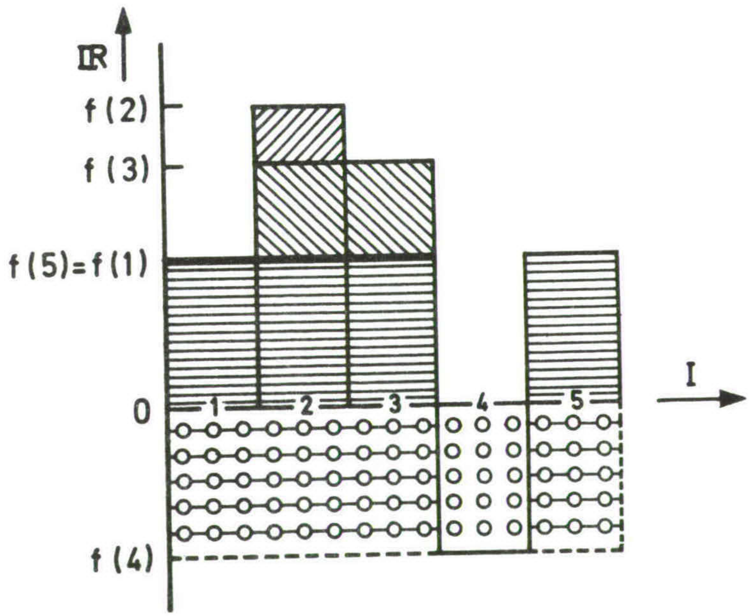
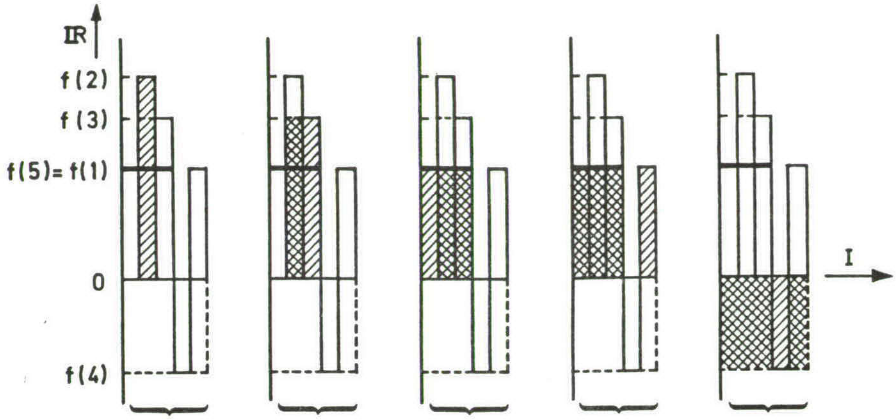


FIGURE VI.2.1.(b). $\int_I f dv$ = (VI.2.5) = A(///) + A(\\\) + A(—) + A(≡) − A($^{000}_{000}$).



A(///) + [A(///)-A(\\\)]+[A(///)-A(\\\)]+[A(///)-A(\\\)]-[A(///)-A(\\\)]

FIGURE VI.2.1.(c) $\int_I f dv$ = (VI.2.6) (Rewritten in (VI.2.7)).

FIGURE VI.2.1. *The Choquet integral.*
$I = \{1,2,3,4,5\}$, $f : I \to \mathbb{R}$, $f(2) > f(3) > f(1) = f(5) > 0 > f(4)$.
$\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = 5$, $\pi(5) = 4$. We could also have taken $\pi(3) = 5$, $\pi(4) = 1$.
A doubly marked part belongs to two areas. For example in (a) belongs both to ||| ànd to $\substack{ooo\\ooo}$.
A = "area". We always take area positive. In (a), A(///) = $[f(2)-f(3)]$ $v(\{2\})$; A(\\\) = $[f(3)-f(5)]$ $v(\{2,3\})$; $f(1) = f(5)$, hence is an empty set, A() = 0.
Area is additive in the $\mathbb{R}$-axis, so in (a), A($\equiv$ U |||) = A($\equiv$) + A(|||). Area does not have to be additive in the I-axis, so in (a), A($\equiv$) $\neq f(1)[v(\{1,2,3\}) + v(\{5\})]$ may very well hold.

equal f-value mutually can be ranked in any arbitrary way. Now (VI.2.4) can easily be seen to equal (see Figure VI.2.1, (a) and (b)) :

$$\Sigma_{j=1}^{n-1}[f(\pi(j)) - f(\pi(j+1))].\ v(\{\pi(1),\ldots,\pi(j)\}) + f(\pi(n)). \qquad \text{(VI.2.5)}$$

Note from this expression that the mutual ranking of states with equal f-value is immaterial. After a reordering of terms, (VI.2.5) becomes (see Figure VI.2.1 (c)):

$$\Sigma_{j=1}^{n} f(\pi(j))[v(\{\pi(1),\ldots,\pi(j)\}) - v(\{\pi(1),\ldots,\pi(j-1)\}]. \qquad \text{(VI.2.6)}$$

And this will lead to the expression that will be most useful for our work in the sequel. For this a new definition is needed:

DEFINITION VI.2.3. For a capacity v, and a permutation π on $\{1,\ldots,n\}$, and $1 \leq j \leq n$,
$P^{\pi}(j) := v(\{i \in I : \pi^{-1}(i) \leq \pi^{-1}(j)\}) - v(\{i \in I : \pi^{-1}(i) < \pi^{-1}(j)\})$.

Dependency of $P^{\pi}(j)$ on v is not expressed in the notation. One may interpret $P^{\pi}(j)$ as the marginal contribution in capacity of j to those states of nature which are ranked before j, by π. By this we can, with π as above formula (VI.2.5), rewrite (VI.2.6) as:

$$\Sigma_{j=1}^{n} P^{\pi}(j) f(j). \qquad \text{(VI.2.7)}$$

Note that, for fixed π (and v), the $P^{\pi}(j)$'s above are nonnegative and sum to one. One may consider $\int f dv$ as the integral of f over I with respect to the (additive) probability measure P^{π}, induced by the $P^{\pi}(j)$'s. This will lead to the main strategy of our approach to derive the main result, Theorem VI.5.1: We shall consider subsets of acts, that induce a same "ranking" permutation π. On such subsets we can proceed as if we were dealing with additive probability P^{π}, thus we can apply well-known techniques there.

Let us now give some elementary properties of the Choquet integral, that follow from the above expressions.

$\int \lambda f dv = \lambda \int f dv$ for all $\lambda \geq 0$ (*positive homogeneity*). (VI.2.8)

$\int (\lambda + f) dv = \lambda + \int f dv$ for all $\lambda \in \mathbb{R}$ (*translation invariance*). (VI.2.9)

We also have:

If $f(i) \geq g(i)$ for all i, then $\int f dv \geq \int g dv$ *(monotonicity)*. (VI.2.10)

The latter is most easily seen by taking $\lambda+f$ and $\lambda+g$ with λ so large that $\lambda+f$ and $\lambda+g$ are positive, and by applying (VI.2.9), and (VI.2.4). Finally, if we consider the Choquet integral as a function from $\mathbb{R}^n$ to $\mathbb{R}$, with $(\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n$ interpreted as the function, assigning λ_j to every j, then we obtain the well-known:

PROPOSITION VI.2.4. *The Choquet integral is continuous.*

PROOF. First we derive continuity in each variable. Let $1 \leq i \leq n$. By (VI.2.10), the Choquet integral is nondecreasing in its i-th variable. Let $x \in \mathbb{R}^n$, and $\varepsilon > 0$. Let π be as above (VI.2.5). Since the mutual ranking, by π, of states j with value x_j equal to x_i, can be chosen arbitrarily, we may assume that of these, $\pi^{-1}(i)$ is the smallest. Let $\delta = \min\{\varepsilon, x_k-x_i\}$ where k is such that $\pi^{-1}(k) = \pi^{-1}(i)-1$, if the latter is positive; let $\delta = \varepsilon$ if $\pi^{-1}(i) = 1$. Then for all $x_{-i}(x_i+\lambda)$ with $0 \leq \lambda \leq \delta$, in the calculation of the Choquet integral through (VI.2.7), we can use the same π, and thus $P^\pi(j)$'s, as for x. Thus $\int x_{-i}(x_i+\lambda)dv - \int x dv \leq P^\pi(i)\delta \leq \delta \leq \varepsilon$.

Analogously one shows that $\delta > 0$ exists such that for all $0 \leq \lambda \leq \delta$, every $x_{-i}(x_i-\lambda)$ gets assigned Choquet integral, not more than ε smaller than x. (This time let $\pi^{-1}(i)$ be as large as possible.)

The Choquet integral is nondecreasing and continuous in each variable. It must be continuous.

□

We shall need the following observation (VI.2.11) in the proof of Theorem VI.8.8. Note that the $P^\pi(i)$ of Definition VI.2.3 uniquely determine v : for any $A \subset I$, we take a π such that $A = \{\pi(1),\ldots,\pi(i)\}$, then we have $v(A) = \Sigma_{j=1}^{i} P^\pi(\pi(i))$. This also shows that if one takes an arbitrary collection of real numbers $P^\pi(j)$, $j = 1,\ldots,n$, $\pi \in$ {permutations on $\{1,\ldots,n\}$}, then: there exists a (necessarily unique) capacity v such that any $P^\pi(j)$ can be derived from v as in Definition VI.2.3, if and only if for all i, π, π':

$$P^\pi(i) \geq 0;\ \Sigma_{j=1}^{n} P^\pi(j) = 1;\ \{\pi(1),\ldots,\pi(i)\} = \{\pi'(1),\ldots,\pi'(i)\} \Rightarrow \Sigma_{j=1}^{i} P^\pi(j) = \Sigma_{j=1}^{i} P^{\pi'}(j). \qquad \text{(VI.2.11)}$$

VI.3. COMONOTONICITY

In the previous section we saw that for the Choquet integral, an ordering on the states of nature, by the "ranking" permutation π, plays a central role. Hence we define, with $\succcurlyeq$ on $\mathcal{C}$ as in Notation VI.2.3:

DEFINITION VI.3.1. For $x \in \mathcal{C}^n$, $\succcurlyeq_x$ is the binary relation on I, defined by $i \succcurlyeq_x j$ whenever $x_i \succcurlyeq x_j$.

If $\succcurlyeq$ is a weak order, then so is $\succcurlyeq_x$.

DEFINITION VI.3.2. For $S \subset \mathcal{C}^n$, $\succcurlyeq_S := \bigcap_{x \in S} \succcurlyeq_x$.

Thus $i \succcurlyeq_S j$ if and only if $x_i \succcurlyeq x_j$ for all $x \in S$. The following is a central notion:

DEFINITION VI.3.3. A set $C \subset \mathcal{C}^n$ is *comonotonic* if no $x,y \in C$, $i,j \in I$ exist such that $x_i \succ x_j$, $y_j \succ y_i$.

The following sets are "maximal" comonotonic sets, as will follow.

DEFINITION VI.3.4. For a permutation π on I,
$C^{\pi} := \{x \in \mathcal{C}^n : x_{\pi(1)} \succcurlyeq x_{\pi(2)} \succcurlyeq \ldots \succcurlyeq x_{\pi(n)}\}$.
$C^{id} = C^{\pi}$ with π identity.

We now obtain, with an *ordering* a weak order for which no different elements are equivalent, the following Lemma. We shall use only (i) and (iv) of it. Statement (iii) is added because it shows the way to proceed in case I is infinite, a case for further research. (ii) is added because it is used in the proof; and because it may be clarifying.

LEMMA VI.3.5. *Let* $S \subset \mathcal{C}^n$. *Let* $\succcurlyeq$ *be a weak order. The following four statements are equivalent:*

(i) S *is comonotonic.*

(ii) $\succcurlyeq_S$ *is a weak order.*

(iii) *There exists an ordering* $\geq^*$ *on the state space* I *such that:*

$[i \geq^* j \Rightarrow x_i \succcurlyeq x_j]$ *for all* $i,j \in I$, $x \in S$.

(iv) $S \subset C^{\pi}$ *for some permutation* π *on* I.

PROOF. (iii) follows from (ii) by letting $\geq^*$ be any ordering such that $i \geq^* j \Rightarrow i \succcurlyeq_S j$. Such an ordering exists by Szpilrajn (1930), or Richter (1966, Lemma 2) (applied to $\succ^*$).

(iii) $\Rightarrow$ (iv) follows by taking π such that $\pi(1) \geq^* \pi(2) \geq^* \ldots \geq^* \pi(n)$. If (iv) holds, then for $x,y \in S$, $x_i > x_j$ and $y_j > y_i$ would imply $\pi(i) > \pi(j)$ and $\pi(j) > \pi(i)$, which cannot hold. So (i) follows.

Finally, (i) is assumed, and (ii) is to be derived. Transitivity of $\succcurlyeq_S$ is from transitivity of $\succcurlyeq$. So completeness of $\succcurlyeq_S$ remains to be derived. If not $j \succcurlyeq_S i$, then there must be $x \in S$ with $x_i > x_j$. By comonotonicity $y_j > y_i$ for no $y \in S$, i.e. $y_i \succcurlyeq y_j$ for all $y \in S$. So $i \succcurlyeq_S j$.

□

DEFINITION VI.3.6. Let $C \subset \mathcal{C}^n$. Then i is *inessential (with respect to* $\succcurlyeq$*) on* C if $z_{-i}\alpha \approx z_{-i}\beta$ for all $z_{-i}\alpha$, $z_{-i}\beta \in C$. If i is inessential on C^{π}, then we also call i π-*inessential*. If π is the identity, we write *id-inessential*. The opposite of inessential always is *essential*.

The proof of the following lemma is more complicated than that of its "additive" analogue, Lemma III.2.9. The reason is that we are now no longer "free to cross borders" from one C^{π} to another. This is the main complication in the work of this Chapter. A preparatory notation:

NOTATION VI.3.7. For $\alpha,\beta \in \mathcal{C}$, $\alpha \vee \beta$ [respectively $\alpha \wedge \beta$] is α if $\alpha \succcurlyeq \beta$ [respectively $\alpha \preccurlyeq \beta$], β otherwise.

Note that $\alpha \vee \beta \neq \beta \vee \alpha$ if $\alpha \approx \beta$ and $\alpha \neq \beta$. If $\succcurlyeq$ is a weak order, then $\alpha \vee \beta \approx \beta \vee \alpha$ for all α, β. Same things hold for $\wedge$.

LEMMA VI.3.8. *Let* $\succcurlyeq$ *be a weak order. Let* $x,y \in C^{\pi}$, *and* $x_j = y_j$ *for all* π*-essential* j. *Then* $x \approx y$.

PROOF. Suppose $x,y \in C^{id}$. Define $x^0 := x, y^0 := y$, and inductively, for $j = 1,\ldots,n$, $x^j := x^{j-1}_{-j}(x_j \vee y_j)$, $y^j := y^{j-1}_{-j}(x_j \vee y_j)$. Note that, for all id-essential j, $x_j = y_j = x_j \vee y_j$. Note also that, for all $j > 1$, $x_j \vee y_j \preccurlyeq x^{j-1}_{j-1}$, and $x_j \vee y_j \preccurlyeq y^{j-1}_{j-1}$, so that $x^j, y^j \in C^{id}$ for all j. We conclude:
$x = x^0 \approx x^1 \approx \ldots x^n = y^n \approx y^{n-1} \approx \ldots \approx y^0 = y$.

□

LEMMA VI.3.9. *Let* $\succcurlyeq$ *be a weak order. Let all* i *be* π*-inessential for all* π. *Then* $\succcurlyeq$ *is trivial.*

PROOF. Let $x,y \in C^n$. Take any $\alpha \in C$. Since $\bar{\alpha} \in C^{\pi}$ for all π, there are π, π', such that $x,\bar{\alpha} \in C^{\pi}$, and $y,\bar{\alpha} \in C^{\pi'}$ for some π'. By the previous Lemma, $x \approx \bar{\alpha} \approx y$.

□

VI.4. COMONOTONIC CARDINAL COORDINATE INDEPENDENCE

The definition of π-essentiality, given in the previous section, is the key tool for the adaptation of cardinal coordinate independence to the present context with (nonadditive) capacities:

DEFINITION VI.4.1. $\succcurlyeq$ satisfies *comonotonic cardinal coordinate independence (Com. CCI)* if for all permutations π,π' on $\{1,\ldots,n\}$, all j and π-essential i, and all $x_{-i}\alpha, y_{-i}\beta, x_{-i}\gamma, y_{-i}\delta \in C^{\pi}$, and finally all $s_{-j}\alpha, t_{-j}\beta, s_{-j}\gamma, t_{-j}\delta \in C^{\pi'}$:
$[x_{-i}\alpha \preccurlyeq y_{-i}\beta \;\&\; x_{-i}\gamma \succcurlyeq y_{-i}\delta \;\&\; s_{-j}\alpha \succcurlyeq t_{-j}\beta] \Rightarrow [s_{-j}\gamma \succcurlyeq t_{-j}\delta]$.

A way to obtain intuitive insight into the condition, is to consider the elucidation to Definition IV.2.4 (CCI), and to study the proof in section VI.5, in the sequel. The remainder of this section is devoted to the study of consequences of Com. CCI. For this we assume throughout this section, without further mention:

ASSUMPTION VI.4.2. (For this section). $\succcurlyeq$ is a weak order that satisfies comonotonic cardinal coordinate independence.

The following is the analogue of coordinate independence. It is more convenient to formulate it now in the spirit of independence of equal subalternatives (Definition II.6.2).

DEFINITION VI.4.3. $\succcurlyeq$ satisfies *comonotonic coordinate independence (Com. CI)* if for all comonotonic $\{x_{-A}s_A, y_{-A}s_A, x_{-A}t_A, y_{-A}t_A\}$ we have $[x_{-A}s_A \succcurlyeq y_{-A}s_A \Leftrightarrow x_{-A}t_A \succcurlyeq y_{-A}t_A]$.

LEMMA VI.4.4. $\succcurlyeq$ *satisfies comonotonic coordinate independence.*

PROOF. First we consider the special case that $||A|| = 1$, say $A = \{k\}$. Let $x_{-k}s_k, y_{-k}s_k, x_{-k}t_k, y_{-k}t_k \in C^\pi$. If k is π-inessential, then $x_{-k}s_k \approx x_{-k}t_k$, and $y_{-k}s_k \approx y_{-k}t_k$, and everything follows. So let k be π-essential. Then $[x_{-k}s_k \preccurlyeq x_{-k}s_k \;\&\; x_{-k}t_k \succcurlyeq x_{-k}t_k \;\&\; x_{-k}s_k \succcurlyeq y_{-k}s_k]$ by Com. CCI imply $x_{-k}t_k \succcurlyeq y_{-k}t_k$.

Next the general case. Say $x_{-A}s_A, y_{-A}s_A, x_{-A}t_A, y_{-A}t_A \in C^{id}$. Define:

$a^0 := x_{-A}s_A$, $b^0 := y_{-A}s_A$, $c^0 := x_{-A}t_A$, $d^0 := y_{-A}t_A$.

Then define, inductively, for $j = 1,\ldots,n$:

If $j \notin A$, then $(a^j,b^j,c^j,d^j) := (a^{j-1},b^{j-1},c^{j-1},d^{j-1})$.

If $j \in A$, then $(a^j,b^j,c^j,d^j) := (a^{j-1}_{-j}\alpha, b^{j-1}_{-j}\alpha, c^{j-1}_{-j}\alpha, d^{j-1}_{-j}\alpha)$, with $\alpha = s_j \vee t_j$.

The above construction has been such that $a^j_k = c^j_k$ and $b^j_k = d^j_k$ for all $k \le j$, and such that all new acts are in C^{id}. For instance if $j \in A$, then a^{j-1}, b^{j-1}, c^{j-1}, $d^{j-1} \in C^{id}$ imply, by simple manipulations, $\alpha \preccurlyeq a^j_{j-1}, \alpha \preccurlyeq b^j_{j-1}, \alpha \preccurlyeq c^j_{j-1}, \alpha \preccurlyeq d^j_{j-1}$. Further $a^n = c^n$, $b^n = d^n$.

By repeated application of the already handled case $||A|| = 1$, we conclude that:

$$x_{-A}s_A \succcurlyeq y_{-A}s_A \Leftrightarrow a^0 \succcurlyeq b^0 \Leftrightarrow a^1 \succcurlyeq b^1 \Leftrightarrow \ldots \Leftrightarrow a^n \succcurlyeq b^n \Leftrightarrow c^n \succcurlyeq d^n$$
$$\Leftrightarrow c^{n-1} \succcurlyeq d^{n-1} \Leftrightarrow \ldots \Leftrightarrow c^0 \succcurlyeq d^0 \Leftrightarrow x_{-A}t_A \succcurlyeq y_{-A}t_A.$$

□

By the above Lemma, the following definition is useful under Assumption VI.4.2.

DEFINITION VI.4.5. We write $x_A \succ_A^{\pi} y_A$ if there exists s_{A^c} such that $x_A s_{A^c} \succ y_A s_{A^c}$, and $x_A s_{A^c}, y_A s_{A^c} \in C^{\pi}$.

LEMMA VI.4.6. *If* $x_A \succ_A^{\pi} y_A$, *then* $x_A s_{A^c} \succ y_A s_{A^c}$ *for* *__all__* s_{A^c} *for which* $x_A s_{A^c}, y_A s_{A^c} \in C^{\pi}$.

PROOF. Direct from Lemma VI.4.4.

□

The second and third consequence of comonotonic cardinal coordinate independence are, with in the terminology of Chapter II, "cA" omitted:

DEFINITION VI.4.7. $\succ$ satisfies *weak monotonicity (w.mon.)* if $x \succ y$ whenever $x_i \succ y_i$ for all i.

DEFINITION VI.4.8. $\succ$ satisfies *comonotonic strong monotonicity (com.s.mon.)* if for all comonotonic $\{x,y\} \subset C^{\pi}$ with $x_i \succ y_i$ for all i, and $x_i > y_i$ for a π-essential i, we have $x > y$.

LEMMA VI.4.9. $\succ$ *satisfies weak, and comonotonic strong, monotonicity.*

PROOF. First we derive weak monotonicity. In three steps:

Assume $y = x_{-k}\alpha$, $\{x,y\}$ comonotonic, say $x,y \in C^{id}$; $x_k \succ \alpha$ (VI.4.1)

Suppose we have $x < x_{-k}\alpha$. Contradiction is derived.

Define, for $j = 0,\ldots,n$:

z^j has $z_1^j = \ldots = z_j^j = x_k$, $z_{j+1}^j = \ldots = z_n^j = \alpha$. (VI.4.2)

Then all z^j are in C^{id}. By Com.CI, $x < x_{-k}\alpha$ implies $z_{-k}^k x_k < z_{-k}^k\alpha$, i.e. $z^k < z^{k-1}$. Each of the last three preferences implies id-essentiality of k. Thus, by Com. CCI,

$[z_{-k}^k\alpha \prec z_{-k}^k\alpha$ & $z_{-k}^k\alpha \succ z_{-k}^k x_k$ & $z_{-j}^j\alpha \succ z_{-j}^j\alpha]$

implies $z^j_{-j}\alpha \succ z^j_{-j}x_k$, i.e. $z^{j-1} \succ z^j$, for all $j \geq 1$.

Apparently $\bar{\alpha} = z^0 \succ z^1 \succ \ldots \succ z^{k-1} \succ z^k > z^{k+1} \succ \ldots \succ z^n = \bar{x}_k$. This, finally, contradicts $x_k \succ \alpha$. The case (VI.4.1) is handled. Next:

Now assume $y = x_{-k}\alpha$, $x_k \succ \alpha$; say $x \in C^{id}$. (VI.4.3)

So $\{x,y\}$ no longer has to be comonotonic. Let l be such that $x_l \succ \alpha$, $x_j < \alpha$ for all $j > l$. Then, by repeated application of the result for case (VI.4.1), $x \succ x_{-k}x_{k+1} \succ x_{-k}x_{k+2} \succ \ldots \succ x_{-k}x_l \succ x_{-k}\alpha$, since every two subsequent acts are comonotonic (e.g. $x_{-k}x_{k+2}$ and $x_{-k}x_{k+3}$ are in C^π with $\pi(k+2) = k$). The case (VI.4.3) is handled. The third case is the general case where $x_i \succ y_i$ for all i. Then, by repeated application of the above result, $x \succ x_{-1}y_1 \succ ((x_{-1}y_1)_{-2}y_2) \succ \ldots \succ (((x_{-1}y_1)_{-2}y_2) \ldots {}_{-n}\, y_n) = y$. *Weak monotonicity is proved.*

Next, to derive com.s.mon., suppose $\{x,y\}$ comonotonic, say $\{x,y\} \subset C^{id}$, further $x_j \succ y_j$ for all j, and $x_k > y_k$ for an id-essential k. To derive is $x > y$. Define:

z has $z_j = x_j$ for all $j \leq k$, $z_j = y_j$ for all $j > k$. (VI.4.4)

Then both $(z=)$ $z_{-k}x_k$ and $z_{-k}y_k$ are in C^{id}. By w.mon. $x \succ z_{-k}x_k \succ z_{-k}y_k \succ y$. It is sufficient for com.s.mon. to show that $z_{-k}x_k > z_{-k}y_k$. Suppose to the contrary that we have $z_{-k}x_k \preccurlyeq z_{-k}y_k$. We derive a contradiction.

Define $z^0,\ldots,z^n$ as in (VI.4.2), with $\alpha = y_k$. Since k is id-essential, by Com. CCI, $[z^k_{-k}y_k \preccurlyeq z^{-k}_k y_{k_1}$ & $z^k_{-k}y_k \succ z^k_{-k}x_k$ & $z^j_{-j}y_k \succ z^j_{-j}y_k]$ implies $z^j_{-j}y_k \succ z^j_{-j}x_k$, i.e. $z^{j-1} \succ z^j$, for all $j \geq 1$. So $\bar{y}_k \succ \bar{x}_k$. This contradicts $x_k > y_k$.

□

COROLLARY VI.4.10. $\succ$ *is trivial if and only if* $\alpha \succ \beta$ *for all* $\alpha,\beta \in C$.

PROOF. If $\succ$ is trivial, then $\bar{\alpha} \succ \bar{\beta}$, so $\alpha \succ \beta$, for all α,β. Next assume $\alpha \succ \beta$ for all α,β. Then for any x in any C^π, and any $\alpha \in C$, $x_i \succ \bar{\alpha}_i$ for all i, and $\bar{\alpha} \in C^\pi$, hence by w.mon. we have $x \succ \bar{\alpha}$. Analogously $x \preccurlyeq \bar{\alpha}$. So $x \approx \bar{\alpha}$. Also $x \approx \bar{\alpha} \approx y$ for all x,y,α : $\succ$ is trivial.

COROLLARY VI.4.11. *One* π *has a* π*-essential state, if and only if every* π *has a* π*-essential state.*

PROOF. If one π has a π-essential state, then $\succcurlyeq$ cannot be trivial. By Corollary VI.4.10 we have $\bar{\alpha} \prec \bar{\beta}$ for some $\alpha,\beta \in C$. Since $\bar{\alpha}$, $\bar{\beta} \in C^{\pi}$ for every π, Lemma VI.3.8 implies that every π must have a π-essential state.

□

VI.5. THE MAIN THEOREM

In this section we give the main theorem of this chapter. After the theorem we give a proof for the simplest implication (i) ⇒ (ii) in it. The proof of (ii) ⇒ (i), and of the uniqueness results, will be carried out in following sections, and completed in section VI.9. A survey is given in section VI.10.

THEOREM VI.5.1. *Let* $n \in \mathbb{N}$. *Let* C *be a connected topological space, that is separable if every permutation* π *on* $\{1,\ldots,n\}$ *has exactly one* π*-essential state. For the binary relation* $\succcurlyeq$ *on* C^n, *the following two statements are equivalent:*

(i) *There exist a capacity* v *on* $2^{\{1,\ldots,n\}}$, *and a continuous* $U : C \to \mathbb{R}$, *such that* $x \mapsto \int (U \circ x)dv$ *represents* $\succcurlyeq$.

(ii) $\succcurlyeq$ *is a continuous weak order that satisfies comonotonic cardinal coordinate independence.*

The following uniqueness results hold for U, v *of (i):*

If some π *has two or more* π*-essential states, then* U *is cardinal, and* v *is uniquely determined.* (VI.5.1)

If $\succcurlyeq$ *is not trivial, and no* π *has two or more* π*-essential states, then* U *is continuously ordinal, and* v *is uniquely determined.* (VI.5.2)

If $\succcurlyeq$ *is trivial, then* U *is any constant function, and* v *is arbitrary.* (VI.5.3)

□

PROOF OF (i) ⇒ (ii) ABOVE. Suppose (i) holds. Obviously $\succ$ is a weak order.

The map $x \mapsto (U(x_1),\ldots,U(x_n))$ is continuous, so is, by Proposition VI.2.4, the map $(U(x_1),\ldots,U(x_n)) \mapsto \int(U \circ x)dv$. The map $x \mapsto \int(U \circ x)dv$ is apparently continuous. This implies continuity of $\succ$.

All that remains is Com. CCI. For this, first suppose that:

$$x_{-i}\alpha \prec y_{-i}\beta,\ x_{-i}\gamma \succ y_{-i}\delta;\ \{x_{-i}\alpha,\ y_{-i}\beta,\ x_{-i}\gamma,\ y_{-i}\delta\} \subset C^{\pi}; \qquad \text{(VI.5.4)}$$

i is π-essential.

The two preferences give, by (VI.2.7) (with the π in (VI.2.7) identical to our present π since $x_i \succ x_j \Rightarrow U(x_i) \geq U(x_j)$):

$$\Sigma_{k\neq i}P^{\pi}(k)U(x_k) + P^{\pi}(i)U(\alpha) \leq \Sigma_{k\neq i}P^{\pi}(k)U(y_k) + P^{\pi}(i)U(\beta)$$

and

$$\Sigma_{k\neq 1}P^{\pi}(k)U(x_k) + P^{\pi}(i)U(\gamma) \geq \Sigma_{k\neq i}P^{\pi}(k)U(y_j) + P^{\pi}(i)U(\delta).$$

These two imply:

$$P^{\pi}(i)[U(\alpha) - U(\beta)] \leq P^{\pi}(i)[U(\gamma) - U(\delta)]. \qquad \text{(VI.5.5)}$$

Were $P^{\pi}(i) = 0$, then by (VI.2.7) and the representation of $\succ$ by $x \mapsto \int(U \circ x)dv$, i would be π-inessential. So:

$$P^{\pi}(i) > 0. \qquad \text{(VI.5.6)}$$

The last two numbered results imply:

$$U(\alpha) - U(\beta) \leq U(\gamma) - U(\delta). \qquad \text{(VI.5.7)}$$

Now suppose, besides (VI.5.4), also:

$$s_{-j}\alpha \succ t_{-j}\beta;\ \{s_{-j}\alpha,\ t_{-j}\beta,\ s_{-j}\gamma,\ t_{-j}\delta\} \subset C^{\pi'}. \qquad \text{(VI.5.8)}$$

The preference implies:

$\Sigma_{k\neq j}P^{\pi'}(k)U(s_k) + P^{\pi'}(j)U(\alpha) \geq \Sigma_{k\neq j}P^{\pi'}(k)U(t_k) + P^{\pi'}(j)U(\beta)$. This, and (VI.5.7), implies:

$\Sigma_{k\neq j}P^{\pi'}(k)U(s_k) + P^{\pi'}(j)U(\gamma) \geq \Sigma_{k\neq j}P^{\pi'}(k)U(t_k) + P^{\pi'}(j)U(\delta)$.

Or: $s_{-j}\gamma \succ t_{-j}\delta$. This is exactly what, by Com. CCI, should follow from (VI.5.4) and (VI.5.8).

□

Next we give some examples of decision making, discussed in Luce and Raiffa (1957, Chapter 13, for instance page 282). These examples have no expected utility representation with additive probability measures, but they can be represented by (i) in the above Theorem.

EXAMPLE VI.5.2. Let $1 \leq k \leq n$. Let $v(A) = 0$ if $||A|| < k$, $v(A) = 1$ if $||A|| \geq k$. Then $\int (U \circ x)dv = U(x_j)$, with $U(x_j)$ the k-th highest value in $U(x_1),\ldots,U(x_n)$. $P^{\pi}(\pi(k)) = 1$, for all π. The preference relation belongs to a "maximin"-decision maker if $k = n$, and to a "maximax"-decision maker if $k = 1$.

EXAMPLE VI.5.3. Let $0 < \lambda < 1$. Let $v(\emptyset) = 0$, $v(I) = 1$, $v(A) = \lambda$ for all remaining A. Here $P^{\pi}(\pi(1)) = \lambda$, $P^{\pi}(\pi(n)) = (1-\lambda)$ for all π, and $\int (U \circ x)dv = \lambda \max\{U(x_j): 1 \leq j \leq n\} + (1-\lambda)\min\{U(x_j): 1 \leq j \leq n\}$. The preference relation belongs to a decision maker, adopting the "Hurwicz criterion" with "pessimism-optimism index" $1-\lambda$, see Hurwicz (1951).

VI.6. PREPARATIONS FOR THE PROOF

LEMMA VI.6.1. *Let C be a topological space, $\succ$ a weak order on C^n, continuous with respect to the product topology. Then for all $x \in C^n$, $\{\alpha \in C : \bar{\alpha} \succ x\}$ and $\{\alpha \in C : \bar{\alpha} \prec x\}$ are open subsets of C.*

PROOF. Let $\bar{\alpha} \succ x$. Then an open neighbourhood $V \subset C^n$ of $\bar{\alpha}$ exists such that $y \succ x$ for all $y \in V$. We may assume that V is of the form $A_1 \times \ldots \times A_n$, with all A_j open subsets of C. Now $A := \cap_{j=1}^{n} A_j$ gives an open neighbourhood of α within $\{\alpha \in C : \bar{\alpha} \succ x\}$. The latter is open.

Analogously $\{\alpha \in C : \bar{\alpha} \prec x\}$ is open.

□

LEMMA VI.6.2. *Let no π have two or more π-essential states. Let the assumptions in Theorem VI.5.1, and also (ii) there, hold. Then also (i) and the uniqueness results there hold. If $\succ$ is nontrivial, then v only takes the values 0 and 1.*

PROOF. If there is a π with no π-essential state, then by Lemma VI.3.8, for all $\alpha,\beta \in C$, $\bar{\alpha} \approx \bar{\beta}$. By Corollary VI.4.10, $\succ$ is trivial. Now (VI.5.3), and (i), follow straightforwardly.

So we assume:

Every π has exactly one π-essential state. (VI.6.1)

The binary relation on C, also denoted by $\succ$, and defined by $\alpha \succ \beta$ if $\bar{\alpha} \succ \bar{\beta}$ (Notation IV..2.3) obviously is a weak order. By Lemma VI.6.1 it is continuous. By Theorem III.3.1 there exists a, continuously ordinal, $\phi : C \to \mathbb{R}$, representing $\succ$ on C. We can set $U := \phi$, as we shall see; so any continuous strictly increasing transform of U can be used.

Next we define v. Let $A \subset I$ be arbitrary. By nontriviality, there are α and β such that $\alpha > \beta$. If $\bar{\alpha}_A \bar{\beta}_{A^c} > \bar{\beta}$, we define $v(A) := 1$, otherwise $v(A) := 0$. By com.s.mon. and Lemma VI.3.8 we see that $v(A) = 1$, iff for any π with $\{\pi(1),\ldots,\pi(k)\} = A$, A contains the π-essential state. This shows that v is independent of the particular choice of α and β above. Also it follows that $P^{\pi}(j) = 0$ for all π-inessential j, and $P^{\pi}(j) = 1$ for the π-essential j.

Now we show that with these constructions, (i) in Theorem VI.5.1 holds. Let x and y be two acts. Let $x \in C^{\pi}$, $y \in C^{\pi'}$. Let i be the π-essential state, j the π'-essential state. Then, by Lemma VI.3.8, $x \approx \overline{x_i}$, $y \approx \overline{y_j}$. There now follows:

$x \succ y \Leftrightarrow \overline{x_i} \succ \overline{y_j} \Leftrightarrow U(x_i) \geq U(y_j) \Leftrightarrow \Sigma P^{\pi}(k)U(x_k) \geq \Sigma P^{\pi'}(k)U(y_k) \Leftrightarrow$
$\int(U \circ x)dv \geq \int(U \circ y)dv$.

Finally we derive the uniqueness result (VI.5.2). We saw above that U can be any continuous strictly increasing transform of ϕ. Since, obviously, U has to represent $\succ$ on C, no other kind of U can be taken : U is continuously ordinal.

For uniqueness of v, we consider an arbitrary π, and show that $P^{\pi}(i) = 0$ for all π-inessential i. Then $P^{\pi}(j)$ must equal 1 for the π-essential j. These values $P^{\pi}(\cdot)$ uniquely determine v. So let, finally, $i = \pi(k)$ be π-inessential. Let $\alpha > \beta$. Let x assign α to $\pi(1),\ldots,\pi(k)$, β to $\pi(k+1),\ldots,\pi(n)$. Then x and $x_{-i}\beta$ are in C^{π}. By π-inessentiality of i, $x \approx x_{-i}\beta$. Since $U(\alpha) > U(\beta)$, by (VI.2.7) we obtain $P^{\pi}(i) = 0$.

□

In cooperative game theory with side payments v's as above are called "(monotonic) simple games", see Driessen (1985, Definition V.3.1).

DEFINITION VI.6.3. $\alpha \in \mathcal{C}$ is *maximal* [respectively *minimal*] if $\beta \succ \alpha$ [respectively $\beta \prec \alpha$] for no $\beta \in \mathcal{C}$.

VI.7. ADDITIVE VALUE FUNCTIONS ON C^{id}

In this section we derive results for C^{id}. Of course, the same results hold for any C^{π}. Without further mention, we assume throughout this section.

ASSUMPTION VI.7.1. The assumptions, and statements (ii), of Theorem VI.5.1 hold. There are at least two id-essential states. Further we assume that all states are id-essential. No maximal or minimal consequences exist.

The assumption of at least two id-essential states is essential for the sequel. The assumption that all states are id-essential is only made for convenience. By Lemma VI.3.8 id-inessential states do not affect the preference relation on C^{id}, and may just as well be suppressed from notation. They will simply get additive value functions V_j^{id} assigned that are constant, say zero.

DEFINITION VI.7.2. Let $C \subset \mathcal{C}^n$. Let $(V_j)_{j=1}^n$ be an array of functions, each from a subset of $\mathcal{C}$ to the reals. Then $(V_j)_{j=1}^n$ are *additive value functions (for $\succ$) on* C if $x \mapsto \Sigma_{j=1}^n V_j(x_j)$ is well-defined for every $x \in C$, and represents $\succ$ on C.

VI.7.1. ADDITIVE VALUE FUNCTIONS $(V_j^z)_{j=1}^n$ ON THE E^z'S.

NOTATION VI.7.3. For $z \in C^{id}$, $E^z := E_1^z \times \ldots \times E_n^z$, with $E_1^z :=$

$\{\alpha \in C : \alpha \succ z_1\}$, $E_n^z := \{\alpha \in C : \alpha \prec z_{n-1}\}$, and for all $j \neq 1$, $j \neq n$, $E_j^z := \{\alpha \in C : z_j \prec \alpha \prec z_{j-1}\}$.

Note that z_n plays no role in the above notation. And $z \in E^z \subset C^{id}$. The E^z's are cartesian products, and they are comonotonic so that on them the conditions of this chapter hold without the comonotonicity premise. That enables us to apply the theorems of Chapter III.

PROPOSITION VI.7.4. *For any $z \in C^{id}$ there exist continuous simultaneously cardinal additive value functions $(V_j^z)_{j=1}^n$ for $\succcurlyeq$ on E^z.*

PROOF. Since no maximal consequences exist, there is $\alpha \succ z_1$ in E_1^z. Since no minimal consequences exist, there is $\beta \prec z_{n-1}$ in E_n^z. Hence, by id-essentiality of 1,n, and by com.s.mon., $z_{-1}\alpha \succ z \succ z_{-n}\beta$. This shows that 1 and n are essential on E^z. Since E^z, and any subset of it, is comonotonic, the properties of Com. CCI and Com. CI all hold without the comonotonicity restrictions. The topological assumption III.2.11 on E^z will be guaranteed in the next subsection. Hence, for the case of three or more essential states on E^z, Theorem III.3.7 gives all desired results. Otherwise only 1 and n are essential on E^z. Then triple cancellation follows from (Com.) CCI, exactly as in Lemma IV.3.2. And then Theorem III.3.6 gives all desired results.

□

In the proof of the above Proposition we have postponed one matter: the topological assumption III.2.11. The problem is that, if we take the restriction to E_j^z of the topology on C, then E_j^z will possibly be no longer connected. For instance let (i) of Theorem VI. 5.1 hold, where $C = \mathbb{R}$ with the usual Euclidean topology, $n = 2$, v is the additive probability measure assigning $||A||/2$ to every $A \subset I$, and $U : \alpha \mapsto \alpha \sin \alpha$. Let $z = (0,0)$. Then $E_1^z = \{\alpha : U(\alpha) \geq 0\}$, is not connected.

VI.7.2. THE TOPOLOGICAL ASSUMPTION FOR PROPOSITION VI.7.4

NOTATION VI.7.5. The topology on C is denoted as T. By $T(\succcurlyeq)$ we denote the coarsest topology on C with respect to which $\succ$ on C is continuous. By ... $|E$ we denote: "restricted to E".

Of course $T(\succ)$ is coarser than T, so is connected too.

LEMMA VI.7.6. *Any* $E \subset C$ *of the form* $\{\alpha \in C : \sigma \succ \alpha \succ \tau\}$, $\{\alpha \in C : \sigma \succ \alpha > \tau\}$, $\{\alpha \in C : \sigma > \alpha \succ \tau\}$, $\{\alpha \in C : \sigma > \alpha > \tau\}$, $\{\alpha \in C : \sigma \succ \alpha\}$, $\{\alpha \in C : \sigma > \alpha\}$, $\{\alpha \in C : \alpha \succ \tau\}$, *or* $\{\alpha \in C : \alpha > \tau\}$, *is connected with respect to* $T(\succ)|E$.

PROOF. Throughout this proof, "open" always refers to $T(\succ)$. Let E have a form as above. Let F_1, F_2 be open in C. Let $E_1 = E \cap F_1$, $E_2 = E \cap F_2$. Suppose $E_1 \neq \emptyset \neq E_2$, $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = E$. We derive a contradiction.

Let $\alpha_1 \in E_1$, $\alpha_2 \in E_2$. $T_{\succ}$ does not separate between equivalent consequences, so $\alpha_1 \approx \alpha_2$ cannot hold. Say $\alpha_1 < \alpha_2$. Define:
$G_1 := [F_1 \cap \{\alpha : \alpha_1 < \alpha < \alpha_2\}] \cup [\{\alpha : \alpha \prec \alpha_1\}]$, and
$G_2 := [F_2 \cap \{\alpha : \alpha_1 < \alpha < \alpha_2\}] \cup [\{\alpha : \alpha \succ \alpha_2\}]$.
Then $G_1 \cap G_2 = \emptyset$, $G_1 \neq \emptyset \neq G_2$, and $G_1 \cup G_2 = C$ since $\{\alpha : \alpha_1 < \alpha < \alpha_2\} \subset E$.

First we derive openness of G_1. For any element of G_1, an open neighbourhood H of it within G_1 must be found. Let $\delta \in G_1$. If $\delta < \alpha_1$, take $H = \{\alpha : \alpha < \alpha_1\}$, if $\delta > \alpha_1$, $H = F_1 \cap \{\alpha : \alpha_1 < \alpha < \alpha_2\}$ is taken. So finally let $\delta \approx \alpha_1$. There must be an open neighbourhood H' of δ within F_1 of the form $\{\alpha : \alpha > \mu\}$, or $\{\alpha : \nu > \alpha > \mu\}$, or $\{\alpha : \nu > \alpha\}$ for some $\mu,\nu \in C$. The first case is impossible since $\alpha_2 \notin F_1$. So, finally, $F = \{\alpha : \nu > \alpha\}$ can be taken, in both other cases.

Analogously openness of G_2 is derived. Openness of G_1 and G_2 contradicts connectedness of C.

□

The above Lemma shows that, if we use $T(\succ)$ instead of T, then every E_j^z is connected. Next we show:

LEMMA VI.7.7. *For any* $z \in C^{id}$, $\succ$, *restricted to* E^z, *is continuous with respect to the product topology of the* $T(\succ)|E_j^z$'s.

PROOF. Let $x, y \in E^z$, $x \succ y$. We construct an auxiliary $\tilde{x}$ such that $\tilde{x} \succ y$, and by means of this a subset $F_1 \times \ldots \times F_n$ of $\{v \in E^z : v \succ y\}$, containing x, and with every $F_j \subset E_j^z$ open w.r.t. $T(\succcurlyeq)|E_j^z$. For the construction of $\tilde{x}_1$, consider:

$V := \{\alpha \in C : (\alpha, x_2, \ldots, x_n) \succ y\}$.

By Lemma 0.1 this is open w.r.t. T. V contains x_1 so is nonempty. If V contains z_1, then $\tilde{x}_1 = z_1$ and $F_1 = E_1^z$ is taken.

If V does not contain z_1, then by connectedness of C w.r.t. T, V cannot be closed w.r.t. T, so not of the form $\{\alpha : \alpha \succcurlyeq x_1\}$, by continuity of $\succcurlyeq$ on C (Lemma VI.6.1) with respect to T. Since V, by w.mon., contains all $\alpha \succcurlyeq x_1$, V must contain an $\alpha \prec x_1$. This α cannot be $\preccurlyeq z_1$ (that, by w.mon., would imply $z_1 \in V$). So $z_1 \prec \alpha \prec x_1$: $\alpha \in E_1^z$. Take $\tilde{x}_1 = \alpha$, $F_1 = E_1^z \cap \{\beta \in C : \beta \succ \alpha\}$.

Anyway, we have $(\tilde{x}_1, x_2, \ldots, x_n) \succ y$, and F_1 is open w.r.t. $T(\succcurlyeq)|E_1^z$.

By analogous constructions we obtain $\tilde{x}_2, F_2, \ldots, \tilde{x}_n, F_n$, such that: $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_j, x_{j+1}, \ldots, x_n) \succ y$ for all j, $F_j = E_j^z$ if $\tilde{x}_j = z_j$, otherwise $z_j \prec \tilde{x}_j \prec x_j$ and $F_j = \{\alpha : \alpha \succ \tilde{x}_j\} \cap E_j^z$. Finally, $(\tilde{x}_1, \ldots, \tilde{x}_n) \succ y$. For every $w \in F_1 \times \ldots \times F_n$, in particular $w = x$, $w_j \succcurlyeq \tilde{x}_j$ for all j. By w.mon.: $w \succcurlyeq \tilde{x} \succ y$.

So indeed, if $x \succ y$, we can construct $F_1 \times \ldots \times F_n \subset [E^z \cap \{w : w \succ y\}]$, containing x, and open w.r.t. the product topology of $T(\succcurlyeq)|E_j^z, j = 1, \ldots, n$. Hence $\{x \in E^z : x \succ y\}$ is open w.r.t. the latter product topology, for all $y \in E^z$. Analogously $\{x \in E^z : x \prec y\}$ is open, for all y. Continuity of $\succcurlyeq$ w.r.t. the product topology of the $T(\succcurlyeq)|E_j^z$, follows.

□

We can now take care of the topological assumption III.2.11 for Proposition VI.7.4. On every E_j^z we take $T(\succcurlyeq)|E_j^z$. By Lemma VI.7.6, E_j^z is connected. On E^z we take the product topology. By Lemma VI.7.7, $\succcurlyeq$ on E^z is continuous w.r.t. this topology. So indeed we can apply the theorems, mentioned in the proof of Proposition VI.7.4. These yield additive value functions, continuous w.r.t. the $T(\succcurlyeq)|E_j^z$'s; so certainly w.r.t. the $T|E_j^z$'s.

VI.7.3. FITTING THE FUNCTIONS V_j^z TOGETHER ON C^{id}

Our next step is to show that there exist $V_j^{id} : C \to \mathbb{R}$, $j = 1,\dots,n$, such that for every z and j, V_j^z can be taken to be the restriction of V_j^{id} to E_j^z. This of course could never be done if there were $A \subset I$, and s, t $\in C^{id}$, such that $(V_j^s)_{j\in A}$ and $(V_j^t)_{j\in A}$ would be additive value functions for different binary relations on the "common domain" $X_{j\in A}(E_j^s \cap E_j^t)$. By comonotonic coordinate independence (Lemma VI.4.4) that never happens. Both $(V_j^s)_{j\in A}$ and $(V_j^t)_{j\in A}$ are additive value functions for $\succcurlyeq_A^{id}$, on appropriate domains.

LEMMA VI.7.8. *There exist, simultaneously cardinal,* $V_j^{id} : C \to \mathbb{R}$, $j = 1,\dots,n$, *that are additive value functions on* E^z *for every* $z \in C^{id}$.

PROOF. On every E^z we are given additive value functions $(V_j^z)_{j=1}^n$ for $\succcurlyeq$, that are simultaneously cardinal. So we may add to every V_j^z an arbitrary "location" constant $\tau_j(z)$, and multiply the V_j^z's by one common positive "scale" constant $\sigma(z)$, to obtain again additive value functions. Our plan in the sequel is to choose, in 5 stages, scales and locations such that all V_j^z's will "fit together", i.e. be the same on common domains. They can then be considered the restrictions of one array $(V_j^{id})_{j=1}^n$.

There must exist $\beta^1, \beta^0 \in C$ such that $\beta^1 > \beta^0$. We shall arrange $V_j^{id}(\beta^0) = 0$ for all j, and $V_1^{id}(\beta^1) = 1$.

STAGE 1. Choice of scale and location on E^r with $r = \overline{\beta^0}$.

Let r ("reference point") $= \overline{\beta^0}$. $E_1^r = \{\alpha : \alpha \succ \beta^0\}$, contains β^1. $E_n^r = \{\alpha : \alpha \prec \beta^0\}$. For all $j \neq 1$, $j \neq n$, $E_j^r = \{\alpha : \alpha \approx \beta^0\}$. Of course we choose scale and locations such that:

$$V_j^r(\beta^0) = 0 \text{ for all j, } V_1^r(\beta^1) = 1. \qquad \text{(VI.7.1)}$$

STAGE 2. Choice of scale on all E^z, and location for all V_1^z, V_n^z.

Let now $z \in C^{id}$ be arbitrary. By Com. CI, (V_1^r, V_n^r) and (V_1^z, V_n^z) are additive value functions for the same $\succcurlyeq_{\{1,n\}}^{id}$ on $(E_1^r \cap E_1^z) \times (E_n^r \cap E_n^z)$. Note that both 1 and n are essential on $(E_1^r \cap E_1^z \cap E_1^z) \times (E_n^r \cap E_n^z)$ w.r.t. $\succcurlyeq_{\{1,n\}}^{id}$. By Lemma VI.7.6, $E_1^r \cap E_1^z$ and $E_n^r \cap E_n^z$ are connected w.r.t. the restrictions of $T(\succcurlyeq)$, and by subsection VI.7.2 we may use the uniqueness result of Theorem III.3.6. So we can choose the scale for (V_1^z, V_n^z), (and hence for all $(V_j^z)_{j=1}^n$,) and the locations for (V_1^z, V_n^z), such that $V_1^r = V_1^z$ on $E_1^r \cap E_1^z$, and $V_n^r = V_n^z$ on $E_n^r \cap E_n^z$. Thus we have, even stronger:

$V_1^s = V_1^t$ and $V_n^s = V_n^t$ on common domain (VI.7.2) for all $s, t \in C^{id^n}$.

This follows since, on $(E_1^s \cap E_1^t) \times (E_n^s \cap E_n^t)$, (V_1^s, V_n^s) and (V_1^t, V_n^t) are additive value functions for the same $\succcurlyeq^{id}_{\{1,n\}}$, hence they can only differ w.r.t. their locations, and a common scale. However, for $j = 1,n$, V_j^s and V_j^t coincide (with V_j^r) on $E_j^s \cap E_j^t \cap E_j^r$; hence they coincide on common domains.

STAGE 3. Intermediate observation.

In fact, for all s, t, j, V_j^s and V_j^t now have the same scale, and only differ w.r.t. their location, as we shall show:

There exist constants $\tau_j(s,t)$ such that on $E_j^s \cap E_j^t$, (VI.7.3) $V_j^s = \tau_j(s,t) + V_j^t$; for all $s, t \in C^{id}$, $1 \leq j \leq n$.

For $j = 1$ or $j = n$, by (VI.7.2), in fact $\tau_j(s,t) = 0$. So let $1 \neq j \neq n$. Then (V_1^s, V_j^s, V_n^s) and (V_1^t, V_j^t, V_n^t) are additive value functions for the same $\succcurlyeq^{id}_{\{1,j,n\}}$ on $(E_1^s \cap E_1^t) \times (E_j^s \cap E_j^t) \times (E_n^s \cap E_n^t)$. So they can only differ by location, and common scale. However, V_1^s and V_1^t, and V_n^s and V_n^t, coincide on their common domain (which contains more than one element). The common scales must be the same.

STAGE 4. Choice of location for all V_j^z's ($j \neq 1,n$), having β^0 in their domain.

Of course for all V_j^z's as above we choose location such that $V_j^z(\beta^0) = 0$. Then we have not only (VI.7.1) to (VI.7.3), but also:

If V_j^s and V_j^t have β^0 in their domain then they (VI.7.4) coincide on common domain.

This is direct from (VI.7.3).

STAGE 5. Choice of location for remaining V_j^z's.

Let now $z \in C^{id}$ and j be such that $\beta^0 \notin E_j^z$, $j \neq 1$, $j \neq n$. $E_j^z = \{\alpha : z_{j-1} \succ \alpha \succ z_j\}$. Say $z_j \succ \beta^0$ ($z_{j-1} \prec \beta^0$ is analogous). Let $r(z) \in C^{id}$ be such that $(r(z))_i = z_{j-1}$ for all $i < j$, $(r(z))_i = \beta^0$ for all $i \geq j$. Then $E_j^z \subset E_j^{r(z)} = \{\alpha : z_{j-1} \succ \alpha \succ \beta^0\}$. In Stage 4 we arranged $V_j^{r(z)}(\beta^0) = 0$. We now choose location of V_j^z such that $V_j^z = V_j^{r(z)}$ on E_j^z. We now shall derive:

For all $s,t \in C^{id}$, $1 \leq j \leq n$, V_j^s and V_j^t coincide on common domain. (VI.7.5)

We check this only for the case where $1 \neq j \neq n$, β^0 is neither in the domain of V_j^s, nor in that of V_j^t (other cases are treated before, or are analogous), and $s_j \succ \beta^0$. Here E_j^s is of the form $\{\alpha : s_{j-1} \succ \alpha \succ s_j\}$. For E_j^t to have nonempty intersection with E_j^s , we must have $t_j \succ \beta^0$. Now V_j^s and $V_j^{r(s)}$ coincide on $E_j^s \cap E_j^{r(s)}$, so do V_j^t and $V_j^{r(t)}$ on $E_j^t \cap E_j^{r(t)}$; so do, by (VI.7.4), $V_j^{r(s)}$ and $V_j^{r(t)}$ on $E_j^{r(s)} \cap E_j^{r(t)}$. The latter contains $E_j^s \cap E_j^t$.

We can now define $(V_j^{id})_{j=1}^n$. For any $\alpha \in C$, and $1 \leq j \leq n$, we take any $z \in C^{id}$ such that $\alpha \in E_j^z$, $z_j = \alpha$ suffices. Then we define $V_j^{id}(\alpha) := V_j^z(\alpha)$. By (VI.7.5), this does not depend on the particular choice of z; and every V_j^z is now the restriction of V_j^{id} to E_j^z.

Finally the uniqueness result. Any $(W_j^{id})_{j=1}^n$, for which real τ_j , $j = 1,\ldots,n$, and positive σ exist such that $W_j^{id} = \tau_j + \sigma V_j^{id}$ for all j, satisfy the requirements of the Lemma. Conversely, if $(W_j^{id})_{j=1}^n$ satisfy all the requirements of the Lemma, then so do $U_j^{id} := [W_j^{id} - W_j^{id}(\beta^0)] / [W_1^{id}(\beta^1) - W_1^{id}(\beta^0)]$.

From $U_j^{id}(\beta^0) = 0$, $U_1^{id}(\beta^1) = 1$, and from rereading the proof, the reader will see that this uniquely determines U_j^{id}, $U_j^{id} = V_j^{id}$ must hold for all j.

□

Note that we may not yet conclude that $(V_j^{id})_{j=1}^n$ are additive value functions on *all* C^{id}.

VI.7.4. THE FUNCTIONS $(V_j^{id})_{j=1}^n$ ARE ADDITIVE VALUE FUNCTIONS ON C^{id}

LEMMA VI.7.9. *For all* (id-*essential*) k : $\alpha \succcurlyeq \beta \Leftrightarrow V_k^{id}(\alpha) \geq V_k^{id}(\beta)$. *Hence* $\bar{\alpha} \succcurlyeq \bar{\beta} \Leftrightarrow \Sigma V_k^{id}(\alpha) \geq \Sigma V_k^{id}(\beta)$.

PROOF. Let α, β, k be arbitrary. First suppose $\alpha \succcurlyeq \beta$. Let $x_j = \alpha$ for all $j < k$, $x_j = \beta$ for all $j \geq k$. Then $x = x_{-k}\beta$ and $x_{-k}\alpha \in C^{id}$, and $x_{-k}\beta$ and $x_{-k}\alpha \in E^x$. By w.mon. and Lemma VI.7.8:
$\alpha \approx \beta \Rightarrow \alpha \succcurlyeq \beta$ and $\beta \succcurlyeq \alpha \Rightarrow x_{-k}\alpha \approx x_{-k}\beta \Rightarrow V_k^{id}(\alpha) = V_k^{id}(\beta)$.

By com.s.mon. and Lemma VI.7.8:
$\alpha \succ \beta \Rightarrow x_{-k}\alpha \succ x_{-k}\beta \Rightarrow V_k^{id}(\alpha) > V_k^{id}(\beta)$.

Analogously:
$\alpha \prec \beta \Rightarrow V_k^{id}(\alpha) < V_k^{id}(\beta)$.

All of this together implies that V_k^{id} represents $\succcurlyeq$ on C.

□

LEMMA VI.7.10. *Let* $x \in C^{id}$, $x \approx \bar{\alpha}$. *Then* $\Sigma_{j=1}^n V_j^{id}(x_j) = \Sigma_{j=1}^n V_j^{id}(\alpha)$.

PROOF. The case $x_j \approx \alpha$ for all j is direct. The case $x_j \succ \alpha$ for some j and $x_j \prec \alpha$ for no j, and the case $x_j \prec \alpha$ for some j and $x_j \succ \alpha$ for no j, are excluded by com.s.mon.

So suppose $j < i$ exist such that $x_j \succ \alpha$, $x_{j+1} \approx \dots \approx x_{i-1} \approx \alpha$, $x_i \prec \alpha$. We define x^0 such that $x_k^0 = x_k$ for all $x_k \not\approx \alpha$, and $x_k^0 = \alpha$ for all $x_k \approx \alpha$.

Now suppose, for some $0 \leq l \leq n-2$, $x^l \in C^{id}$ has been defined such that $x^l \approx \bar{\alpha}$, and $\Sigma V_k^{id}(x_k^l) = \Sigma V_k^{id}(x_k)$, with at least l coordinates of x^l equal to α, and no coordinate equivalent but unequal to α. If in fact x^l has $l+1$ or more coordinates equal α, define $x^{l+1} := x^l$. If not, then, say:
$x_a^l \succ \alpha$, $x_{a+1}^l = \dots = x_{b-1}^l = \alpha$, $x_b^l \prec \alpha$, with $b = a+l+1$.

If now $(x_{-a}^l\alpha)_{-b}\alpha \approx \bar{\alpha}$, define $x^{l+1} := (x_{-a}^l\alpha)_{-b}\alpha$ $(= x^{l+2})$. If $(x_{-a}^l\alpha)_{-b}\alpha \prec \bar{\alpha}$, define $\alpha \prec x_a^{l+1} \preccurlyeq x_a^l$ such that:
$x^{l+1} := (x_{-b}^l\alpha)_{-a}(x_a^{l+1}) \approx \bar{\alpha}$.

(Take x_a^{l+1} in $\{\beta \in C : (x_{-b}^l\alpha)_{-a}\beta \succcurlyeq \bar{\alpha}\} \cap \{\beta \in C : (x_{-b}^l\alpha)_{-a}\beta \preccurlyeq \bar{\alpha}\}$, both involved sets are nonempty; closed by Lemma 0.1; they intersect by connectedness.)

If $(x^l_{-a}\alpha)_{-b}\alpha > \bar{\alpha}$, define $\alpha > x^{l+1}_b \succ x^l_b$ such that $x^{l+1} := (x^l_{-a}\alpha)_{-b}x^{l+1}_b \approx \bar{\alpha}$.

In any case, for $z = x^l_{-a}\alpha$, both x^{l+1} and x^l are in E^z, their a-th coordinate is "between" x^l_{a-1} and α, their b-th coordinate "between" α and x^l_b. Hence by Lemma VI.7.8, $x^l(\approx \bar{\alpha}) \approx x^{l+1}$ implies $\Sigma V^{id}_k(x^{l+1}_k) = \Sigma V^{id}_k(x^l_k)$.

Finally we end up with $x^{n-1} \approx \bar{\alpha}$, with n-1 coordinates equal to α. Then by com.s.mon. the remaining coordinate of x^{n-1} must also be equivalent, so equal, to α. And:
$\Sigma_k V^{id}_k(x_k) = \Sigma_k V^{id}_k(x^0_k) = \ldots = \Sigma_k V^{id}_k(x^{n-1}_k) = \Sigma_k V^{id}_k(\alpha)$ follows.

□

Now, finally, to show that the $(V^{id}_j)^n_{j=1}$ are additive value functions on C^{id}, let $x,y \in C^{id}$ be arbitrary. First we find "certainty equivalents."

LEMMA VI.7.11. *For every* $z \in \mathcal{C}^n$ *there exists* α *such that* $z \approx \bar{\alpha}$.

PROOF. For $z \in \mathcal{C}^n$ there exist i,j such that $x_i \succ x_k \succ x_j$ for all $k \in I$. Let $V := \{\alpha \in \mathcal{C} : \bar{\alpha} > x\}$, $W := \{\beta \in \mathcal{C} : x > \bar{\beta}\}$. Then $V \cap W = \emptyset$. V and W are open by Lemma VI.6.1. Now $x_i \notin W$ and $x_j \notin V$ by w.mon. By connectedness of $\mathcal{C}$, there is an $\alpha \notin V \cup W$; so $\bar{\alpha} \approx x$.

□

We now give the main result of this section:

THEOREM VI.7.12. *There exist continuous simultaneously cardinal additive value functions* $(V^{id}_j)^n_{j=1}$ *for* $\succ$ *on* C^{id}.

PROOF. Let $x,y \in C^{id}$ be arbitrary. Let $(V^{id}_j)^n_{j=1}$ be as constructed above. Let α,β be such that $x \approx \bar{\alpha}$, $y \approx \bar{\beta}$ (Lemma VI.7.11). Then $x \succ y$ iff $\bar{\alpha} \succ \bar{\beta}$, which by Lemma VI.7.9 is iff $\Sigma_k V^{id}_k(\alpha) \geq \Sigma_k V^{id}_k(\beta)$. The latter by Lemma VI.7.10 holds iff $\Sigma_k V^{id}_k(x_k) \geq \Sigma_k V^{id}_k(y_k)$.

□

The following Corollary is not needed for the sequel, but may have some interest of its own. It considers, as all of this section does, an example of an additively decomposable representation on a set that is *not* a cartesian product, but only a subset of that. The only literature on this subject, known to the author, is Krantz et al. (1971, section 6.5.5); and Fishburn (1967, 1971) for the case where coordinate sets are mixture spaces (see Definition VII.2.1).

COROLLARY VI.7.13. *Let* $\succcurlyeq$ *be a continuous weak order on* $X := \{x \in \mathbb{R}^n_{++} : x_1 \geq x_2 \geq \ldots \geq x_n\}$, *such that* $[x_j \geq y_j$ *for all* j *and* $x \neq y]$ *implies* $[x \succ y]$. *Let* $n \geq 3$. *The following two statements are equivalent:*

(i) *There exist continuous simultaneously cardinal additive value functions for* $\succcurlyeq$ *on* X.

(ii) $\succcurlyeq$ *satisfies (comonotonic) coordinate independence.*

PROOF. As Theorem VI.7.12. Weak, and (comonotonic) strong, monotonicity are easily verified. We have all, so certainly three or more, coordinates essential. For this case the only consequence of Com. CCI, (apart from the monotonicities,) used in the proof of Theorem VI.7.12, is Com. CI. Theorem VI.7.12 only considered C^{id}, and did not need any assumption "outside" C^{id}.

□

VI.8. COMPLETION OF THE PROOF OF THEOREM VI.5.1 UNDER ABSENCE OF MAXIMAL OR MINIMAL CONSEQUENCES

Throughout this section, with Theorem VI.8.8 excepted, we shall assume:

ASSUMPTION VI.8.1. The assumptions, and statement (ii), of Theorem VI.5.1 hold. There exists π with two π-essential states, say

π = identity. By m we denote an id-essential state. No maximal or minimal consequences exist. Let $\beta^1 \succ \beta^0$ be two fixed consequences. For every π with two or more π-essential states, the continuous simultaneously cardinal additive value functions $(V_j^\pi)_{j=1}^n$ for $\succcurlyeq$ on C^π (that exist according to the previous section) are chosen such that $V_j^\pi(\beta^0) = 0$ for all j, and $\Sigma_{j=1}^n V_j^\pi(\beta^1) = 1$.

Note that we have changed "scale", as compared to the previous section. There we had $V_1^{id}(\beta^1) = 1$, now $\Sigma_{j=1}^n V_j^{id}(\beta^1) = 1$. Note also that, at present, we may not yet conclude for different π, π', and $x \in C^\pi$, $y \in C^{\pi'}$, that $x \succcurlyeq y \Leftrightarrow \Sigma_{j=1}^n V_j^\pi(x_j) \geq \Sigma_{j=1}^n V_j^{\pi'}(y_j)$. The only consequences of comonotonic cardinal coordinate independence that we used in the previous section (comonotonic coordinate independence, weak monotonicity, and comonotonic strong monotonicity) probably do not suffice for this purpose. We shall essentially use:

LEMMA VI.8.2. *Let there be at least two π-essential, and two π'-essential, states. Let k be π'-essential. Then for all $l \in I$, $V_l^\pi = \phi_l \circ V_k^{\pi'}$ for a constant or positive affine ϕ_l.*

PROOF. Say π' is identity. We write ϕ for ϕ_l. If l = π-inessential, then V_l^π is constant, and ϕ is the same constant. So assume:

l is π-essential.

By Lemma VI.7.9, (which applies to all essential k) V_l^π and V_k^{id} represent the same $\succcurlyeq$, hence $V_l^\pi = \phi \circ V_k^{id}$ for a continuous strictly increasing ϕ.

Note first that Com. CCI (Definition VI.4.1) implies the same property with all preferences replaced by equivalences (compare Lemma IV.2.7). This we write out with additive value functions brought in, and with $\phi \circ V_k^{id}$ for V_l^π everywhere, to give:

$$V_k^{id}(\alpha) - V_k^{id}(\beta) \overset{(1)}{=} \Sigma_{j\neq k}[V_j^{id}(y_j) - V_j^{id}(x_j)] \overset{(2)}{=} V_k^{id}(\gamma) - V_k^{id}(\delta) \tag{VI.8.1}$$

and

$$\phi \circ V_k^{id}(\alpha) - \phi \circ V_k^{id}(\beta) \overset{(3)}{=} \Sigma_{j\neq l}[V_j^\pi(t_j) - V_j^\pi(s_j)] \tag{VI.8.2}$$

imply

$$\Sigma_{j\neq 1}[V_j^{\pi}(t_j) - V_j^{\pi}(s_j)] \overset{(4)}{=} \phi\circ V_k^{id}(\gamma) - \phi\circ V_k^{id}(\delta) \text{ for all} \qquad (VI.8.3)$$

$x_{-k}\alpha$, $y_{-k}\beta$, $x_{-k}\gamma$, $y_{-k}\delta \in C^{id}$; $s_{-1}\alpha$, $t_{-1}\beta$, $s_{-1}\gamma$, $t_{-1}\delta \in C^{\pi}$.

Now let $V_k^{id}(\mu)$ be an arbitrary element of $int(V_k^{id}(C))$. There can be seen to be an interval S around $V_k^{id}(\mu)$, so small that for all $V_k^{id}(\alpha)$, $V_k^{id}(\beta)$, $V_k^{id}(\gamma)$, and $V_k^{id}(\delta) \in S$, there exist x,y such that $x_{-k}\alpha$, $y_{-k}\beta$, $x_{-k}\gamma$, $y_{-k}\delta$ are in C^{id}, and such that $\overset{(1)}{=}$ is satisfied. For this we use the existence of an id-essential state $i \neq k$, which implies nondegenerateness of the interval $V_i^{id}(C)$. Of course, if $i < k$, then $x_i \succ \alpha$, $x_i \succ \gamma$, $y_i \succ \beta$, $y_i \succ \delta$ will have to hold. If $i > k$, the converse has to hold. Furthermore, by continuity of ϕ, S can be taken so small that $\phi(S)$ is small enough to guarantee existence of s and t such that $s_{-1}\alpha$, $t_{-1}\beta$, $s_{-1}\gamma$, $t_{-1}\delta$ are in C^{π}, and such that $\overset{(3)}{=}$ holds.

We conclude for all $V_k^{id}(\alpha)$, $V_k^{id}(\beta)$, $V_k^{id}(\gamma)$, $V_k^{id}(\delta) \in S$:

$$V_k^{id}(\alpha) - V_k^{id}(\beta) = V_k^{id}(\gamma) - V_k^{id}(\delta) \Rightarrow \qquad (VI.8.4)$$

$$\phi\circ V_k^{id}(\alpha) - \phi\circ V_k^{id}(\beta) = \phi\circ V_k^{id}(\gamma) - \phi\circ V_k^{id}(\delta).$$

This is now shown by choosing x,y,s,t as above. (VI.8.4), only for the case where $\beta = \gamma$, already suffices to show that on S, ϕ satisfies: $\phi((\tilde{\alpha} + \tilde{\delta})/2) = [\phi(\tilde{\alpha}) + \phi(\tilde{\delta})]/2$. Corollary VIII.3 gives affinity of ϕ.

□

For all π with two or more π-essential states, we can, by Lemma VI.8.2, and the fact that all $V_j^{\pi}(\beta^0)$ equal 0, define $\lambda_j^{\pi} \in \mathbb{R}_+$ such that, with m id-essential:

$$V_j^{\pi} = \lambda_j^{\pi} V_m^{id}. \qquad (VI.8.5)$$

We define for all these π:

$$p_j^{\pi} := \lambda_j^{\pi} / \Sigma_{i=1}^{n} \lambda_i^{id}. \qquad (VI.8.6)$$

For π with exactly one π-essential state, say 1, we define:

$$p_1^{\pi} := 1,\ p_i^{\pi} := 0 \text{ for all } i \neq 1. \qquad (VI.8.7)$$

We now define $U : C \to \mathbb{R}$.

DEFINITION VI.8.3. For all $\alpha \in C$, $U(\alpha) := \Sigma_{j=1}^{n} V_j^{id}(\alpha)$.

LEMMA VI.8.4. *For all* π *with two or more* π*-essential states, and all* α, $V_j^\pi(\alpha) = p_j^\pi U(\alpha)$. *For all* π, $\Sigma p_j^\pi = 1$.

PROOF. Let π have two π-essential states. Then $p_j^\pi U(\alpha) = [\lambda_j^\pi / \Sigma_{i=1}^n \lambda_i^{id}][\Sigma_{i=1}^n V_i^{id}(\alpha)] = [\lambda_j^\pi / \Sigma_{i=1}^n \lambda_i^{id}][\Sigma_{i=1}^n \lambda_i^{id} V_m^{id}(\alpha)] = V_j^\pi(\alpha)$. For such π, $\Sigma p_j^\pi = \Sigma\lambda_j^\pi / \Sigma\lambda_i^{id} = \Sigma V_j^\pi(\beta^1) / \Sigma V_i^{id}(\beta^1) = 1/1 = 1$.

For other π, with only one π-essential state, $[\Sigma p_j^\pi = 1]$ is direct.

□

LEMMA VI.8.5. *Let* $x \in C^\pi$, $x \approx \bar{\alpha}$. *Then* $\Sigma p_j^\pi U(x_j) = U(\alpha)$.

PROOF. If there are two or more π-essential states, then by Lemma VI. 7.10, adapted to C^π, $\Sigma V_j^\pi(x_j) = \Sigma V_j^\pi(\alpha)$. Hence $\Sigma p_j^\pi U(x_j) = \Sigma p_j^\pi U(\alpha) = U(\alpha)$.

If π has exactly one π-essential state, say k, then by Lemma VI. 3.8, $x \approx \overline{x_k}$. Hence by Lemma VI.7.9, $U(\overline{x_k}) = U(\alpha)$, i.e. $\Sigma p_j^\pi U(x_j) = U(\alpha)$.

□

LEMMA VI.8.6. *Let* $x \in C^\pi$, $y \in C^{\pi'}$. *Then* $x \succcurlyeq y \Leftrightarrow \Sigma_{j=1}^n p_j^\pi U(x_j) \geq \Sigma_{j=1}^n p_j^{\pi} U(y_j)$.

PROOF. Let (Lemma VI.7.11) $x \approx \bar{\alpha}$, $y \approx \bar{\beta}$. Then $x \succcurlyeq y$ iff $\bar{\alpha} \succcurlyeq \bar{\beta}$, which by Lemma VI.7.9 is iff $U(\alpha) \geq U(\beta)$. By Lemma VI.8.5 the latter holds iff $\Sigma p_j^\pi U(x_j) \geq \Sigma p_j^\pi U(y_j)$.

□

LEMMA VI.8.7. *Let* $A \subset I$. *Let* $A = \{\pi(1),\ldots,\pi(k)\} = \{\pi'(1),\ldots,\pi'(k)\}$. *Then* $\Sigma_{j=1}^k p_j^\pi = \Sigma_{j=1}^k p_j^{\pi'}$.

PROOF. Let $x_j = \beta^1$ for all $j \in A$, $x_j = \beta^0$ for all $j \notin A$. Then $x \in C^\pi$ and $x \in C^{\pi'}$. Apply the above Lemma with $y = x$.

□

The purpose of the last two sections has been to derive the following result:

THEOREM VI.8.8. *Let the assumptions of Theorem VI.5.1 hold. Let (ii) there hold. Furthermore, let no maximal or minimal consequences exist, and let there be a π with two or more π-essential states. Then (i), and (VI.5.1), of Theorem VI.5.1 hold.*

PROOF. According to Lemma VI.8.7, and formula (VI.2.11), with $P^{\pi}(j) := p_j^{\pi}$ for all π,j, there exists a unique capacity v in accordance with Definition VI.2.3. Lemma VI.8.6, and formula (VI.2.7) now verify (i) of Theorem VI.5.1.

To derive (VI.5.1), say there are two id-essential states. Then the fact that $(P^{id}(j)U)_{j=1}^{n}$ are additive value functions for $\succ$ on C^{id}, and simultaneous cardinality of $(V_j^{id})_{j=1}^{n}$ in Theorem VI.7.12, give cardinality of U, and together with $[\Sigma P^{id}(j) = 1]$ uniquely determine $(P^{id}(j))_{j=1}^{n}$. Analogously $(P^{\pi}(j))_{j=1}^{n}$ are uniquely determined for any π with two or more π-essential states. If π has exactly one π-essential state k, then $P^{\pi}(k) = 1$ must hold, and $P^{\pi}(j) = 0$ for all $j \neq k$.

□

VI.9. MAXIMAL AND/OR MINIMAL CONSEQUENCES

In this section we derive the implication (ii) ⇒ (i), and the uniqueness result (VI.5.1) in Theorem VI.5.1, for the case where maximal and/or minimal consequences may exist, and where furthermore, as we assume throughout this section without further mention:

ASSUMPTION VI.9.1. The assumptions of Theorem VI.5.1 hold. Also (ii) there holds. There exists π with two or more π-essential states, say π = identity.

LEMMA VI.9.2. *Let $\alpha, \gamma \in C$ be such that $\alpha \succ \gamma$. Then there exists $\beta \in C$ such that $\alpha \succ \beta \succ \gamma$.*

PROOF. $E := \{\beta : \beta \succ \gamma\}$ and $F := \{\beta : \beta \prec \alpha\}$ are open and nonempty. Their union is C, for if $\beta \in F^c$ then $\beta \succcurlyeq \alpha$ so $\beta \succ \gamma$. Hence by connectedness of C, E and F must have nonempty intersection.

□

NOTATION VI.9.3. $C^* := \{\alpha \in C : \alpha$ is neither maximal nor minimal$\}$. $C^{\pi *} := C^{\pi} \cap (C^*)^n$.

Since π = id has a π-essential state (even more than one), there exists $\alpha \succ \beta$. By Lemma VI.9.2, C^* is nonempty, and has no ("new") maximal or minimal consequences itself.

LEMMA VI.9.4. *If* i *is essential on* C^{π} (i.e. π-*essential), then it is on* $C^{\pi *}$.

PROOF. Say π is identity. There exist $\alpha, \beta \in C$ such that $\alpha \succ \beta$. By Lemma VI.9.2, there exists γ such that $\alpha \succ \gamma \succ \beta$, again Lemma VI.9.2 gives δ such that $\gamma \succ \delta \succ \beta$. Let $x \in C^{id}$ have $x_k = \gamma$ for all $k \leq i$, $x_k = \delta$ for all $k > i$. Then $x_{-i}\gamma$, $x_{-i}\delta$ are in C^{id}, even in $C^{id\,*}$. By com.s.mon. $x_{-i}\gamma \succ x_{-i}\delta$.

□

Next we show that, on $(C^*)^n$, (i) in Theorem VI.5.1 is satisfied.

PROPOSITION VI.9.5. *There exist a capacity* v, *and a continuous* $U^* : C^* \to \mathbb{R}$, *such that* $x \mapsto \int (U^* \circ x)dv$ *represents* $\succcurlyeq$ *on* $(C^*)^n$.

PROOF. By Lemma VI.9.2, C^* itself has no maximal or minimal consequences. By Lemma VI.9.4, essentiality of states on $(C^*)^n$ is as on C^n. The proposition now follows from Theorem VI.8.8, if the topological assumptions in it can be guaranteed. This is done analogously to subsection VI.7.2. $T(\succcurlyeq)|C^*$ is taken as topology on C^*. Mainly by Lemma VI.7.6 this preserves connectedness. Continuity of $\succcurlyeq$ on $(C^*)^n$ w.r.t. the product topology of the $T(\succcurlyeq)|C^*$'s, differs only in details from Lemma VI.7.7:

Let again $x \succ y$, for $x,y \in (C^*)^n$. We construct $\tilde{x} \succ y$, and by means of this a subset $F_1 \times \ldots \times F_n$ of $\{w \in (C^*)^n : w > y\}$, containing x, and with every $F_j \subset C^*$, open w.r.t. $T_{\succ}|C^*$. For the construction of $\tilde{x}_1$, consider:

$V := \{\alpha \in C : (\alpha, x_2,\ldots, x_n) > y\}$. By Lemma 0.1 this is open w.r.t. T, the "old" topology on C. V contains x_1 so is nonempty. If $V = C$, then $x_1 \in C^*$ not being minimal, we take $\tilde{x}_1 = \alpha$ for any $\alpha \in C^*$ with $\alpha < x_1$. If $V \neq C$, then by connectedness of C w.r.t. T, V cannot be closed w.r.t. T, so not of the form $\{\alpha : \alpha \succ x_1\}$, by continuity of $\succ$ on C (Lemma VI.6.1) w.r.t. T. And since, by w.mon., V contains all $\alpha \succ x_1$, V must contain an $\alpha < x_1$. Now take (Lemma VI.9.2) $\tilde{x}_1 = \beta$ for any $\alpha < \beta < x_1$. Then $\tilde{x}_1 \in C^*$.

So always $\tilde{x}_1 \in C^*$ is found with $\tilde{x}_1 < x_1$, $(\tilde{x}_1,x_2,\ldots,x_n) > x$. Let $F_1 := \{\alpha : \alpha > \tilde{x}_1\}$.

Further we proceed as in the proof of Lemma VI.7.7.

□

We plan to define $U(\alpha) := \sup(U^*(C^*))$ [respectively $\inf(U^*(C^*))$] for maximal [respectively minimal] α. Hence:

LEMMA VI.9.6. *If α is maximal [respectively minimal], then $U^*(C^*)$ is bounded above [respectively below].*

PROOF. Only for maximal α. Let $i < j$ be two id-essential states. Let, only in this proof, $(\bar{\beta},\bar{\gamma})$ denote the act z with $z_k = \beta$ for $k \leq i$, $z_k = \gamma$ for $k > i$, for all β, $\gamma \in C$. By com.s.mon., for all $\gamma \in C^*$, $\bar{\alpha} > (\bar{\alpha},\bar{\gamma})$.

Let $\gamma \in C^*$ be fixed, let β (by Lemma VI.7.11) be such that $(\bar{\alpha},\bar{\gamma}) \approx \bar{\beta}$ (so $\beta \in C^*$). Now for all $\mu \in C^*$ with $\mu \succ \gamma$, $(\bar{\mu},\bar{\gamma})$ is in $C^{id\,*}$, and $(\bar{\mu},\bar{\gamma}) < (\bar{\alpha},\bar{\gamma}) \approx \bar{\beta}$, so:

$$v(\{1,\ldots,i\})U^*(\mu) + [v(N) - v(\{1,\ldots,i\})]U^*(\gamma) < U^*(\beta). \qquad \text{(VI.9.1)}$$

Since i is essential on $C^{id\,*}$, $v(\{1,\ldots,i\})$ is positive, and (VI.9.1) gives an upper bound for $\{U^*(\mu) : \mu \in C^*, \mu \succ \gamma\}$, thus for $U^*(C^*)$.

□

DEFINITION VI.9.7. If $\alpha \in C$ is maximal, then $U(\alpha) := \sup(U^*(C^*))$. If $\alpha \in C$ is minimal, then $U(\alpha) := \inf(U^*(C^*))$. If $\alpha \in C^*$, then $U(\alpha) := U^*(\alpha)$.

As we saw above, $U(\alpha) \in \mathbb{R}$ for all α. We denote:

NOTATION VI.9.8. $C^+ := C^* \cup \{\alpha \in C : \alpha \text{ is maximal}\}$.

With this we obtain:

LEMMA VI.9.9. *For all* $x \in (C^+)^n$, *and* $\gamma \in C$ *with* $x \approx \bar{\gamma}$, $\int(U \circ x)dv = U(\gamma)$.

PROOF. Say $x \in C^{id}$. By com.s.mon., γ is not minimal, so $\gamma \in C^+$. If no maximal α exists, Proposition VI.9.5 gives the desired result. So let α be maximal. Let $0 \leq k \leq n$ be such that $x_1 \approx \alpha, \ldots, x_k \approx \alpha$, $x_{k+1} \prec \alpha$, $\ldots$, $x_n \prec \alpha$. If γ is maximal, then $\gamma \approx \alpha$, and by com.s.mon. $k+1,\ldots,n$ must be id-inessential. Then $\int(U \circ x)dv = U(\gamma)$ follows.

There remains the most complicated case, where γ is not maximal, so, neither being minimal, is in C^*. First we show that $\int(U \circ x)dv \leq U(\gamma)$. By w.mon., for all $\mu \in C^*$ with $(\alpha \succ)\mu \succcurlyeq x_{k+1}$, we have $(\mu,\ldots,\mu,x_{k+1},\ldots,x_n) \preccurlyeq \bar{\gamma}$, i.e. $\int(U \circ (\mu,\ldots,\mu,x_{k+1},\ldots,x_n))dv \leq U(\gamma)$. Writing for all $1 \leq j \leq k$, $U(x_j) = U(\alpha) = \sup\{U(\mu) : \mu \in C^*, \mu \succcurlyeq x_{k+1}\}$ shows that $\int(U \circ x)dv \leq U(\gamma)$.

To see that $\int(U \circ x)dv \geq U(\gamma)$, we consider δ such that $\gamma \succ \delta$, so $x \succ \bar{\delta}$. By standard arguments continuity of $\succcurlyeq$, Lemma 0.1, and connectedness of C, imply existence of μ_k such that $x_k \succ \mu_k \succcurlyeq x_{k+1}$, and $x_{-k}\mu_k \succ \bar{\delta}$. Also, μ_{k-1} exists such that $x_{k-1} \succ \mu_{k-1} \succcurlyeq \mu_k$ and $(x_{-k,k-1}\mu_k, \mu_{k-1}) \succ \bar{\delta}$. Finally we end up with $\alpha \succ \mu_1 \succcurlyeq \mu_2 \succcurlyeq \ldots \mu_k$ such that $(\mu_1,\ldots,\mu_k,x_{k+1},\ldots,x_n) \succ \bar{\delta}$. Hence, for all $\mu \in C$ such that $\alpha \succ \mu \succcurlyeq \mu_1 (\succcurlyeq \ldots \succcurlyeq \mu_k)$, we obtain $\int(U \circ (\mu,\ldots,\mu,x_{k+1},\ldots,x_n))dv > U(\delta)$.

Substituting, for $1 \leq j \leq k$, $U(x_j) = U(\alpha) = \sup\{U(\mu) : \mu \in C^*, \mu \succcurlyeq \mu_1\}$, shows that $\int(U \circ x)dv \geq U(\delta)$. This holds for all $\delta \prec \gamma$. Hence $\int(U \circ x)dv \geq U(\gamma)$.

□

LEMMA VI.9.10. *The map* $x \mapsto \int(U \circ x)dv$ *represents* $\succcurlyeq$ *on* $(C^+)^n$.

PROOF. First for constant acts. Suppose $\bar{\gamma} \succ \bar{\delta}$, with γ maximal. Then, by Lemma VI.9.2, $\bar{\gamma} \succ \bar{\alpha} \succ \bar{\delta}$ for some $\alpha \in C$. So $U(\gamma) \geq U(\alpha) > U(\delta)$ follows, the latter strict inequality by Proposition VI.9.5. All other cases of $\bar{\gamma} \succcurlyeq \bar{\delta} \Leftrightarrow U(\gamma) \geq U(\delta)$ are straightforward.

Next let $x,y \in (C^+)^n$ be arbitrary. Let $x \approx \bar{\gamma}$, $y \approx \bar{\delta}$ (Lemma VI.7.11). Then $x \succcurlyeq y \Leftrightarrow \bar{\gamma} \succcurlyeq \bar{\delta} \Leftrightarrow U(\gamma) \geq U(\delta) \Leftrightarrow \int(U\circ x)dv \geq \int(U\circ y)dv$, the latter by Lemma VI.9.9.

□

Next we must turn to $(C^+ \cup \{\alpha \in C : \alpha \text{ is minimal}\})^n = C^n$, and show that also here $x \mapsto \int(U\circ x)dv$ represents $\succcurlyeq$. This is very analogous to the above, elaboration is left out. We conclude that the implication (ii) $\Rightarrow$ (i) in Theorem VI.5.1 is now also proved if maximal and/or minimal consequences exist. For the uniqueness result (VI.5.1) in Theorem VI.5.1, we must show that for maximal [respectively minimal] α no other choice for $U(\alpha)$, than $\sup(U(C^*))$ [or $\inf(U(C^*))$] can be made. This can for instance be seen from the proof of Lemma VI.9.9. Let $i > j$ be id-essential states. Then, with α maximal, $x_1 = \ldots = x_i = \alpha$, $\alpha \succ x_{i+1} \succcurlyeq \ldots \succcurlyeq x_n$, the formula $\int(U\circ x)dv = U(\gamma)$ there uniquely determines $U(\alpha)$. For minimal consequences matters are analogous.

VI.10. SURVEY OF THE PROOF OF THEOREM VI.5.1

The implication (i) $\Rightarrow$ (ii) in Theorem VI.5.1 has been demonstrated directly below the Theorem. The proof of (ii) $\Rightarrow$ (i) for the case where no π has two or more π-essential states, and the proof of the uniqueness results (VI.5.2) and (VI.5.3), have been given in Lemma VI.6.2. There remains the case where one π has two or more π-essential states. The case of no maximal or minimal consequences is handled in Theorem VI.8.8, the existence of maximal consequences is handled in Lemma VI.9.10, the general case in the final lines of section VI.9.

VI.11. STRONG SUB- AND SUPERADDITIVITY

In this section we study the following properties of capacities:

DEFINITION VI.11.1. A capacity $v : 2^I \to \mathbb{R}$ is *strongly superadditive* [respectively *strongly subadditive*] if for all $A, B \subset I$:
$v(A \cup B) + v(A \cap B) \geq$ [respectively $\leq$] $v(A) + v(B)$.

Other terms for strong superadditivity are 2-monotonicity, or (strong) convexity. This property has received much attention as it is a sufficient property for v to be the infimum of all additive probability measures, dominating v; and even stronger, this property of v is necessary and sufficient for the Choquet integral with respect to v, to be the infimum of all integrals with respect to the additive probability measures which dominate v (see Huber, 1981, Propositions 10.2.5, and 10.2.1 applied to $v^*(A) := 1-v(A^c)$; or, for arbitrary state spaces I, Schmeidler, 1984b, Proposition 3; or Anger, 1977). Such dominating additive probability measures are called "core-elements" in cooperative game theory with side payments. For strongly superadditive (= "convex") v's [that do not have to satisfy (VI.2.2) or (VI.2.3)], core-elements are studied in Shapley (1972). For strong subadditivity, other common terms are 2-alternating, or (strong) concavity.

The following lemma reflects ideas of nondecreasing (or non-increasing) marginal measure, and is like (6) in Shapley (1972). $P_\pi(i)$ is as in Definition VI.2.3.

LEMMA VI.11.2. *For a capacity* $v : 2^I \to \mathbb{R}$ *the following four statements are equivalent:*

(i) v *is strongly superadditive.*

(ii) $v(A_0 \cup A_1 \cup A_2) - v(A_1 \cup A_2) \geq v(A_0 \cup A_1) - v(A_1)$ *for all disjoint* $A_0, A_1, A_2 \subset N$.

(iii) $v(\{i\} \cup A \cup \{j\}) - v(A \cup \{j\}) \geq v(\{i\} \cup A) - v(A)$ *for all disjoint* $\{i\}, A, \{j\}$.

(iv) *Let* $1 \leq k < n$, *and let* π, π' *be two permutations on* N, *such that* $\pi = \pi'$ *on* $N \setminus \{k,k+1\}$, $\pi(k) = \pi'(k+1)$, $\pi(k+1) = \pi'(k)$. *Then* $P_{\pi'}(\pi(k)) \geq P_{\pi}(\pi(k))$.

The same holds if superadditive in (i) is replaced by subadditive, and $\geq$ *by* $\leq$ *everywhere.*

PROOF. Only for strong superadditivity, and $\geq$ everywhere. v is strongly superadditive iff $v(A \cup B) - v(A) \geq v(B) - v(A \cap B)$ for all A, B. This is equivalent to (ii) above by the substitution $A_0 = B\setminus A$, $A_1 = A \cap B$, $A_2 = A\setminus B$. The implication (ii) ⇒ (iii) is by $\{i\} = A_0$, $A = A_1$, $\{j\} = A_2$. So suppose (iii), to derive is (ii).

Let there be given disjoint $A_0 = \{i_a\}_{a=1}^{k}$, A_1, and $A_2 = \{j_b\}_{b=1}^{l}$. We write $A_1^{a,b} := \{i_1,\ldots,i_a\} \cup A_1 \cup \{j_1,\ldots,j_b\}$. So $A_1^{0,0} = A_1$. Furthermore:

$$v(A_0 \cup A_1 \cup A_2) - v(A_1 \cup A_2) = v(A_1^{k,l}) - v(A_1^{0,l}) = \Sigma_{a=1}^{k}[v(A_1^{a,l})-v(A_1^{a-1,l})].$$

Now for every $a \geq 1$, by (iii):

$$v(A_1^{a,l}) -v(A_1^{a-1,l}) \geq v(A_1^{a,l-1}) - v(A_1^{a-1,l-1}) \geq \ldots \geq v(A_1^{a,0}) - v(A_1^{a-1,0}).$$

So the above summation is:

$$\geq \Sigma_{a=1}^{k}[v(A_1^{a,0}) - v(A_1^{a-1,0}) = v(A_1^{k,0}) - v(A_1^{0,0}) = v(A_0 \cup A_1) - v(A_1).$$

(iii) ⇔ (iv) is by taking $A = \pi(1), \ldots, \pi(k-1)$, $i = \pi(k)$, $j = \pi(k+1)$.

□

In section VI.1.2 we chose, for the calculation of the Choquet integral of $U \circ x$ (where x is an act) a permutation π such that a low value $\pi^{-1}(j)$ indicated that state j was "favourable", i.e. had a relatively highly-preferred consequence x_j. With this in mind, one may formulate (iv) in the above lemma as: the weight $P.(j)$ of state j ($j = \pi(k)$, in (iv)) does not decrease if j becomes less favourable. This indicates a kind of pessimism.

DEFINITION VI.11.3. $\succ$ is *pessimistic* [respectively *optimistic*] if for all $i \neq j$, $\alpha > \beta > \gamma > \delta$ [respectively $\alpha > \gamma > \beta > \delta$], and comonotonic $\{(x_{-i,j}\beta,\delta), (y_{-i,j}\gamma,\delta)\}$ and $\{(x_{-i,j}\beta,\alpha), (y_{-i,j}\gamma,\alpha)\}$ for which $\alpha > x_k > \delta$ for no $i \neq k \neq j$, and $\alpha > y_k > \delta$ for no $i \neq k \neq j$, we have:

$$(x_{-i,j}\beta,\delta) \succ (y_{-i,j}\gamma,\delta) \Rightarrow$$
$$(x_{-i,j}\beta,\alpha) \succ (y_{-i,j}\gamma,\alpha). \qquad \text{(VI.11.1)}$$

For an elucidation of the pessimism definition, note that in both preferences, the i-th state assigns a better consequence to the left act than to the right act, so may be interpreted as a positive argument for preferring the left act. Further the j-th state may be interpreted as a neutral argument. For the lower acts, state i is less favourable than for the upper ones, it no longer being more favourable than state j. So a pessimistic person will give at least as much weight to state i when he is dealing with the lower acts, as when he deals with the upper ones.

With this we obtain:

THEOREM VI.11.4. *Let every* π *have at least three* π*-essential states. Let the assumptions, and statements (i) and (ii), of Theorem VI.5.1 hold. Then* v *is stronly superadditive if and only if* $\succ$ *is pessimistic;* v *is strongly subadditive if and only if* $\succ$ *is optimistic, and* v *is additive if and only if* $\succ$ *is both optimistic and pessimistic.*

PROOF. First suppose v is strongly superadditive. Let $(x_{-i,j}\beta,\delta) \succ (y_{-i,j}\gamma,\delta)$, where all conditions in the definition of pessimism, apart from the implication there, are assumed to be satisfied. To derive is $(x_{-i,j}\beta,\alpha) \succ (y_{-i,j}\gamma,\alpha)$.

Let π be such that $(x_{-i,j}\beta,\delta), (y_{-i,j}\gamma,\delta) \in C^{\pi}$, and for some k, $\pi(k) = i$, $\pi(k+1) = j$. Let $\pi' = \pi$ on $N\setminus\{k,k+1\}$, $\pi'(k) = j$, $\pi'(k+1)=i$. Then $(x_{-i,j}\beta,\alpha), (y_{-i,j}\gamma,\alpha) \in C^{\pi'}$. The first preference above implies:

$$\Sigma_{m\neq i,j}P_{\pi}(m)U(x_m)+P_{\pi}(i)U(\beta)+P_{\pi}(j)U(\delta) \geq \qquad \text{(VI.11.2)}$$
$$\Sigma_{m\neq i,j}P_{\pi}(m)U(y_m)+P_{\pi}(i)U(\gamma)+P_{\pi}(j)U(\delta).$$

This, $P_{\pi'}(m) = P_{\pi}(m)$ for all $m \neq i,j$, and (Lemma VI.11.2.(iv)) $P_{\pi'}(i) \geq P_{\pi}(i)$, together implies:

$$\Sigma_{m\neq i,j}P_{\pi'}(m)U(x_m)+P_{\pi'}(i)U(\beta)+P_{\pi'}(j)U(\alpha) \geq$$
$$\Sigma_{m\neq i,j}P_{\pi'}(m)U(y_m)+P_{\pi'}(i)U(\gamma)+P_{\pi'}(j)U(\alpha), \qquad \text{(VI.11.3)}$$

i.e. $(x_{-i,j}\beta,\alpha) \succ (y_{-i,j}\gamma,\alpha)$. Indeed $\succ$ is pessimistic.

Next suppose $\succ$ is pessimistic. We derive (iv) in Lemma VI.11.2.

Let k, π, π' be as given there, $i = \pi(k)$, $j = \pi(k+1)$. Because of the essentiality assumption in the Theorem, there exists $m \neq i,j$ with $P_\pi(m) > 0$, and $U(C)$ is an interval consisting of more than one point. So we can find x, y, α, β, γ, δ such that (VI.11.2) holds *with equality*, and such that:

$U(x_{\pi(1)}) \geq \ldots \geq U(x_{\pi(k-1)}) \geq U(\alpha) > U(\beta) > U(\gamma) > U(\delta) \geq$
$U(x_{\pi(k+2)}) \geq \ldots \geq U(x_{\pi(n)})$,

and such that the same holds with y instead of x. Hence $(x_{-i,j}\beta,\delta)$, $(y_{-i,j}\gamma,\delta) \in C^\pi$, $(x_{-i,j}\beta,\alpha)$, $(y_{-i,j}\gamma,\alpha) \in C^{\pi'}$. By pessimism of $\succcurlyeq$ we may conclude that (VI.11.3) holds. This, (VI.11.2) with equality, and $U(\beta) > U(\gamma)$, imply $P_{\pi'}(i) \geq P_\pi(i)$. By Lemma VI.11.2, (iv) $\Rightarrow$ (i) there, v is strongly superadditive.

Analogously equivalence of strong subadditivity of v, and optimism of $\succcurlyeq$, is derived. The last statement of the theorem holds because additivity of v is equivalent to the combination of strong sub- and superadditivity of v.

□

Note that the last statement in the above theorem gives a further way to characterize subjective expected utility maximization with (additive) probability. Finally we give an example to show that the condition of the three π-essential states in the above theorem cannot be omitted.

EXAMPLE VI.11.5. Let $N = \{1,2\}$, $C = \mathbb{R}$, $0 \leq v(\{1\}) = v(\{2\}) \leq 1$, U is identity. Let $\succcurlyeq$ be represented by $x \mapsto \int (U \circ x)dv$. Then $\succcurlyeq$ is both optimistic and pessimistic; v is strongly superadditive and *not* strongly subadditive for $v(\{1\}) < \frac{1}{2}$, v is strongly subadditive and *not* strongly superadditive for $v(\{1\}) > \frac{1}{2}$.

CHAPTER VII

CONCAVITY ON MIXTURE SPACES

VII.1. INTRODUCTION

In this chapter we shall assume that X, the set of alternatives, is a cartesian product of "mixture spaces", i.e. spaces endowed with some sort of convex combination operation. Two main examples of mixture spaces are, firstly, convex subsets of Euclidean spaces, and secondly, sets of probability distributions, "lotteries" over a given set of "certain outcomes". Mixture spaces have been introduced in von Neumann and Morgenstern (1944), mainly as generalizations of lotteries, and have almost exclusively been studied with the purpose to obtain results, useful for lotteries. Fishburn (1982) contains many results. See also Luce and Suppes (1965). The applicability of mixture spaces to fields such as quantum mechanics, and colour perception in psychology, is indicated in Gudder (1977) and Gudder and Schroeck (1980).

We shall study mixture spaces mainly as generalization of convex subsets of Euclidean spaces. We shall also study concave and convex (representing) functions on them. To the best of our knowledge concavity and/or convexity of functions on mixture spaces have not yet been studied in literature, whereas mixture spaces do have the natural structure for the study of these notions.

The first five sections of this chapter closely follow Wakker (1986). The first four sections study *(quasi)concave* additively decomposable representing functions. (Quasi)concavity is a very usual assumption in consumer and production theory, see section 1 in Debreu and Koopmans (1982). The recent study Crouzeix and Lindberg (1985) mentions usefulness of quasiconcave additively decomposable functions in mathematical programming.

Section 5 applies results to decision making under uncertainty, where concavity is associated with risk aversion. Arrow (1953) has already noted the importance of the assumption of risk aversion in the analysis of equilibrium with uncertainty. Shubik (1975) remarked that also without uncertainty the assumption of concavity of the utility function (to be used in expected utility) is important. Without it, in a Walras allocation the risk-loving agents would "create markets for lotteries". (See Debreu, 1976, footnote 1.) See further Drèze (1971).

The final section follows Wakker (1984b). It considers decision making under uncertainty with monetary consequences, and characterizes the most usual special case of expected utility maximization with risk aversion: that with nonincreasing risk aversion. We shall see that this further behavioural assumption simplifies the derivation of expected utility maximization, and makes it possible to dispense with the cardinal coordinate independence condition. Arrow (1971, Essay 3, page 96) states that nonincreasing (in fact, decreasing) risk aversion seems supported by everyday observation. Comments are given in Stiglitz (1969a, 1969b). See also section 3 in Bernoulli (1738). An empirical study, finding nonincreasing risk aversion, is Binswanger (1981). Many more references are given in Machina (1983). The case of state-dependent utility functions is studied in Karni (1985).

VII.2. PRODUCT TOPOLOGICAL MIXTURE SPACES

The notations for mixture spaces that we shall adopt below will as much as possible be as in Euclidean spaces, to be of most convenience for readers interested only in this special case.

DEFINITION VII.2.1. Let C be a nonempty set, and θ a map from $C \times [0,1] \times C$ to C. Let $\lambda\alpha + (1-\lambda)\beta$ denote $\theta(\alpha,\lambda,\beta)$. θ is a *mixture operation* if for all $\alpha,\beta \in C$, $\lambda,\mu \in [0,1]$:

$$\lambda\alpha + (1-\lambda)\beta = (1-\lambda)\beta + \lambda\alpha \text{ (commutativity)}. \tag{VII.2.1}$$

$$\mu(\lambda\alpha+(1-\lambda)\beta)+(1-\mu)\beta = (\mu\lambda)\alpha +(1-\mu\lambda)\beta \text{ (associativity)}. \tag{VII.2.2}$$

$$1\alpha + 0\beta = \alpha \text{ (identity)}. \tag{VII.2.3}$$

Here (C,θ), or simply C, is called a *mixture space*.

We write α/μ for $(1/\mu)\alpha$, and $\lambda\alpha/\mu$ for $(\lambda/\mu)\alpha$. We say γ *is between* α and β if $\lambda \in [0,1]$ exists such that $\gamma = \lambda\alpha + (1-\lambda)\beta$.

The following result is proved in Fishburn (1970, section 8.4).

LEMMA VII.2.2. *If C is a mixture space, then for all $\alpha,\beta \in C$, $\lambda,\mu,\nu \in [0,1]$:*

$$\mu\alpha + (1-\mu)\alpha = \alpha. \tag{VII.2.4}$$

$$\lambda(\mu\alpha + (1-\mu)\beta) + (1-\lambda)(\nu\alpha + (1-\nu)\beta) = \tag{VII.2.5}$$
$$(\lambda\mu + (1-\lambda)\nu)\alpha + (\lambda(1-\mu) + (1-\lambda)(1-\nu))\beta.$$

□

Some examples of mixture spaces:

EXAMPLE VII.2.3. C is a convex subset of a linear space over $\mathbb{R}$. θ is the usual convex combination operation.

EXAMPLE VII.2.4. C is a set of probability distributions ("lotteries") over a measure space. For every $P_1,P_2 \in C$, and $0 \leq \lambda \leq 1$, the probability distribution $\lambda P_1 + (1-\lambda)P_2$, assigning $\lambda P_1(A) + (1-\lambda)P_2(A)$ to every A, is in C too.

One can consider Example VII.2.4 as a special case of Example VII.2.3. As Gudder (1977) indicated, not all mixture spaces are isomorphic to convex subsets of linear spaces:

EXAMPLE VII.2.5. Let $C = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0, -1 \leq x_1 \leq 0\} \cup \{(x_1,x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, -x_1 \leq x_2 \leq x_1\}$. Let θ be as follows:

(i) If $x_1y_1 \geq 0$, then $\theta((x_1,x_2),\lambda,(y_1,y_2)) = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2)$.

(ii) If $x_1y_1 < 0$, then $\theta((x_1,x_2),\lambda,(y_1,y_2))$ is the unique point in $\{\mu x : 0 \leq \mu \leq 1\} \cup \{\mu y : 0 \leq \mu \leq 1\}$ with first coordinate $\lambda x_1 + (1-\lambda)y_1$.

In the above example every "line segment" $\{z : z \text{ is between } x,y\}$ can be considered isomorphic to $\{z_1 \in \mathbb{R} : z_1 \text{ is between } x_1 \text{ and } y_1\}$. Still C is not isomorphic to a convex subset of a linear space as follows from $\theta((-1,0),1/2,(1,1)) = (0,0) = \theta((-1,0),\frac{1}{2},(1,-1))$ whereas of course $(1,1) \neq (1,-1)$.

EXAMPLE VII.2.6. Let $C = \{g,u,b\}$, were g stands for "good", u for "undetermined", and b for "bad". Let $\lambda x + (1-\lambda)y = :$
g if $x = y = g$, if $\lambda = 1$ and $x = g$, or if $\lambda = 0$ and $y = g$;
b if $x = y = b$, if $\lambda = 1$ and $x = b$, or if $\lambda = 0$ and $y = b$;
u in all remaining cases.

The following adaptations of well-known notions for linear spaces to mixture spaces are straightforward. Let C be a mixture space. A subset E of C is *convex* if $\lambda\alpha + (1-\lambda)\beta \in E$ for all $\alpha,\beta \in E$, $0 \leq \lambda \leq 1$. A function $V : E \to \mathbb{R}$ is *concave* if $V(\lambda\alpha + (1-\lambda)\beta) \geq \lambda V(\alpha) + (1-\lambda)V(\beta)$ for all $\alpha,\beta \in E$, $0 \leq \lambda \leq 1$. V is *convex* if $-V$ is concave, and V is *affine* if it is both convex and concave. We prefer the term affine to the often used term linear. Finally, V is *quasiconcave* if $V(\lambda\alpha + (1-\lambda)\beta) \geq \min\{V(\alpha), V(\beta)\}$ for all $\alpha,\beta \in C$, $0 \leq \lambda \leq 1$. The latter holds if and only if, for every $\mu \in \mathbb{R}$, $\{\alpha \in C : V(\alpha) \geq \mu\}$ is convex. Every concave function is quasiconcave.

DEFINITION VII.2.7. A triple (C, T, θ), is a *topological mixture space* if C is a nonempty set, T a topology on C, and θ a mixture operation which is continuous (with respect to the product topology on $C \times [0,1] \times C$).

Often we simply write C instead of (C, T, θ). Again, any convex subset of a Euclidean space is a topological mixture space. The following lemma will be used for Corollary VII.2.9, and in the proof of Lemma VII.2.10.

LEMMA VII.2.8. *Let C be a topological mixture space. Let $\alpha,\beta \in C$. Then $\phi : [0,1] \to C$, defined by $\phi : \lambda \mapsto \lambda\alpha + (1-\lambda)\beta$, is continuous.*

PROOF. Let $E \subset C$ be open. By continuity of θ, $\{\gamma,\lambda,\delta) \in C \times [0,1] \times C : \lambda\gamma + (1-\lambda)\delta \in E\}$ is open. By Lemma 0.1, $\{\lambda \in [0,1] : \lambda\alpha + (1-\lambda)\beta \in E\}$ is open. Continuity of ϕ follows.

□

A direct consequence of Lemma VII.2.8:

COROLLARY VII.2.9. *A topological mixture space C is arcconnected, hence connected.*

□

The following lemma is the straightforward generalization of related results for linear spaces (compare Lemma VIII.2), and will be used in the proof of Theorem VII.3.5.

LEMMA VII.2.10. *Let V be a continuous function from a mixture space C to $\mathbb{R}$. Let there exist $\eta > 0$ such that for all $\alpha,\beta \in C$ with $0 \leq V(\alpha) - V(\beta) \leq \eta$, there exists $0 < \lambda < 1$ for which $V(\lambda\alpha + (1-\lambda)\beta) \geq \lambda V(\alpha) + (1-\lambda)V(\beta)$. Then V is concave.*

PROOF. Let $\gamma,\delta \in C$ be arbitrary. We must show that $V(\lambda\gamma + (1-\lambda)\delta) \geq \lambda V(\gamma) + (1-\lambda)V(\delta)$ for all $0 \leq \lambda \leq 1$. By Lemma VII.2.8, $\phi : \lambda \mapsto \lambda\gamma+(1-\lambda)\delta$ is continuous. So $W = V\circ\phi$ is also continuous. The proof is complete if we show that W is concave.

Let $\mu \in]0,1[$ be arbitrary. W being continuous, there is an open interval S around μ within $[0,1]$, such that $|W(\sigma) - W(t)| \leq \eta$ for all σ,τ in S. So for all $\sigma,\tau \in S$, with, say, $W(\sigma) \geq W(\tau)$, $0 \leq V(\sigma\gamma + (1-\sigma)\delta) - V(\tau\gamma + (1-\tau)\delta) \leq \eta$. Hence $0 < \lambda < 1$ exists such that:

$$V(\lambda[\sigma\gamma+(1-\sigma)\delta]+(1-\lambda)[\tau\gamma+(1-\tau)\delta]) \geq \lambda V(\sigma\gamma+(1-\sigma)\delta)+(1-\lambda)V(\tau\gamma+(1-\tau)\delta).$$

To the left side of this inequality we apply (VII.2.5), to obtain:

$$V([\lambda\sigma+(1-\lambda)\tau]\gamma+[\lambda(1-\sigma)+(1-\lambda)(1-\tau)]\delta) \geq \lambda V(\sigma\gamma+(1-\sigma)\delta)+(1-\lambda)V(\tau\gamma+(1-\tau)\delta).$$

Next we substitute W:

$W(\lambda\sigma+(1-\lambda)\tau) \geq \lambda W(\sigma)+(1-\lambda)W(\tau)$.

By Lemma VIII.2, W is concave.

□

As in linear spaces, a binary relation $\succcurlyeq$ on a mixture space C is *convex* if $\{x \in C : x \succcurlyeq y\}$ is convex for every $y \in C$. A weak order $\succcurlyeq$ is convex if and only if $[x \succcurlyeq y]$ implies $[\lambda x+(1-\lambda)y \succcurlyeq y]$ for all x,y,λ. This holds if and only if $\lambda x+(1-\lambda)y \succcurlyeq x \wedge y$ ($\wedge$: see Notation VI.3.7). If a function V represents $\succcurlyeq$, then $\succcurlyeq$ is convex if and only if V is quasiconcave.

LEMMA VII.2.11. *Let $\succcurlyeq$ be a continuous weak order on a topological mixture space C. Let, for all $x \succ y$, $0 < \lambda < 1$ exist such that $\lambda x+(1-\lambda)y \succcurlyeq y$. Then $\succcurlyeq$ is convex.*

PROOF. Let $s,t \in C$ be arbitrary. Let $s \succcurlyeq t$. We shall demonstrate that $S := \{\mu \in [0,1] : \mu s+(1-\mu)t \succcurlyeq t\}$ equals $[0,1]$.

By continuity of $\succcurlyeq$, $\{z \in C : z \succcurlyeq t\}$ is closed. By continuity of θ, $\{(v,\mu,w) \in C \times [0,1] \times C : \mu v+(1-\mu)w \succcurlyeq t\}$ is closed. By Lemma 0.1, S is closed.

Let $\sigma,\tau \in S$, $\sigma \neq \tau$. Say $\sigma s+(1-\sigma)t \succcurlyeq \tau s+(1-\tau)t \succcurlyeq t$. There exists $0 < \lambda < 1$ such that:

$\lambda[\sigma s+(1-\sigma)t]+(1-\lambda)[\tau s+(1-\tau)t] \succcurlyeq \tau s+(1-\tau)t$.

By (VII.2.5) and transitivity this gives:

$[\lambda\sigma+(1-\lambda)\tau]s+[\lambda(1-\sigma)+(1-\lambda)(1-\tau)]\tau \succcurlyeq t$.

So S is a closed subset of $[0,1]$, containing 0 and 1, and containing, for every $\sigma \neq \tau$ in S, an element between σ and τ, and different from σ and τ. $S = [0,1]$ follows.

□

The terminology in the following definition will be justified by Theorem VII.2.13.

DEFINITION VII.2.12. For a sequence of mixture spaces $(C_i,\theta_i)_{i=1}^n$, the *product mixture operation* $\theta : \times_{i=1}^n C_i \times [0,1] \times \times_{i=1}^n C_i \to \times_{i=1}^n C_i$, is defined by:

$\theta : (x,\lambda,y) \mapsto (\lambda x_1+(1-\lambda)y_1,\ldots,\lambda x_n+(1-\lambda)y_n) =: \lambda x+(1-\lambda)y$,
where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$. We then call $(X_{i=1}^n C_i,\theta)$, or simply $X_{i=1}^n C_i$, the *product mixture space*.

If the C_i's are topological mixture spaces, then $X_{i=1}^n C_i$, endowed with the product topology, is the *product topological mixture space*.

THEOREM VII.2.13. *A product mixture space is a mixture space. A product topological mixture space is a topological mixture space.*

PROOF. Let $(C_i,\theta_i)_{i=1}^n,\theta$ be as in Definition VII.2.12. It is straightforward that θ is a mixture operation. Now let T_i be a topology on C_i, $i = 1,\ldots,n$, let every θ_i be continuous. We derive continuity of θ.

Let $E_1 \in T_1$. Then $\theta^{-1}(E_1 \times C_2 \times \ldots \times C_n)$ equals, after a reordering of the coordinates of $(X_{i=1}^n C_i) \times [0,1] \times (X_{i=1}^n C_i)$, the set $(\theta^{-1}(E_1)) \times (X_{i=2}^n C_i) \times (X_{i=2}^n C_i)$, which is open. This can be shown, not only for E_1, but, mutatis mutandis, for any $E_i \in T_i$. Continuity of θ follows.

□

VII.3. THE CONCAVITY ASSUMPTION

In this section we shall assume without further mention:

ASSUMPTION VII.3.1. $X_{i=1}^n C_i$ is a product topological mixture space.

Further, as throughout this monograph, $\succ$ is a binary ("preference") relation on $X_{i=1}^n C_i$. The following property is a generalization of "Axiom Q" in Yaari (1978, p. 109) which was formulated for the case where $C_i = \mathbb{R}_+$ for all i, and for this case by some elementary analysis can be seen to be equivalent to our present definition. (See also Corollary VII.3.7.(ii) below.)

DEFINITION VII.3.2. $\succ$ satisfies the *concavity assumption* if for all x, y, i, v_i, w_i, and λ:

$$x_{-i}v_i \succ y_{-i}(\lambda v_i+(1-\lambda)w_i) \Rightarrow x_{-i}(\lambda w_i+(1-\lambda)v_i) \succ y_{-i}w_i.$$

If in the above definition the second preference $\succ$ were replaced by $\prec$, then it would seem that the "extreme" coordinates v_i and w_i were coming off relatively better than the "intermediate" coordinates $\lambda v_i+(1-\lambda)w_i$ and $\lambda w_i+(1-\lambda)v_i$. This seems not in accordance with concavity, a concave function assigning relatively high values to intermediate arguments, as in the sequel can be inferred from (VII.3.1). The following Lemmas adapt to the present context some results of Yaari (1978; the Remark at section 4, and the Lemma 2 of section 5 and, by that, the implication of "axiom D" through "axiom Q").

LEMMA VII.3.3. *The concavity assumption implies coordinate independence.*

PROOF. Let $\lambda = 1$ in the definition of the concavity assumption.

□

LEMMA VII.3.4. *Let $\succ$ be a continuous weak order, that satisfies the concavity assumption. Then $\succ$ is convex.*

PROOF. By Lemma VII.2.11, it is sufficient to prove that $v \succ w$ implies $v/2 + w/2 \succ w$. For this it is sufficient to prove that even $v/2 + w/2 \succ v$, under the assumption:

$v/2 + w/2 \prec w \prec v$.

We define, inductively, for $0 \leq j \leq n$:

$v^0 = v/2 + w/2$, $v^j = v^{j-1}_{-j}v_j$; $w^0 = w$, $w^j = w^{j-1}_{-j}(v_j/2 + w_j/2)$.

This gives $v^n = v$, $w^n = v/2 + w/2$. For $j = 0$ we have, by assumption, $w^0 \succ v^0$. Now suppose $w^{j-1} \succ v^{j-1}$, for some $1 \leq j \leq n$. Then:

$(w^{j-1} =)\ w^j_{-j}w_j \succ v^j_{-j}(v_j/2 + w_j/2)\ (= v^{j-1})$.

By the concavity assumption with $\lambda = 1/2$, this implies

$w^j_{-j}(v_j/2 + w_j/2) \succ v^j_{-j}v_j$, i.e. $w^j \succ v^j$.

By repeated application, $w^n \succ v^n$ follows, i.e. $v/2 + w/2 \succ v$.

□

If three or more coordinates are essential, the above Lemma can also be obtained as a corollary of Theorem VII.3.5 below. We are now ready for the main result of this section:

THEOREM VII.3.5. *Let the binary relation* $\succ$ *on the product topological mixture space* $X_{i=1}^{n} C_i$ *have at least two essential coordinates. Then the following two statements are equivalent:*

(i) *There exist continuous concave additive value functions* $(V_j)_{j=1}^{n}$ *for* $\succ$.

(ii) $\succ$ *is a continuous weak order that satisfies the concavity assumption, and furthermore the Thomsen condition of exactly two coordinates are essential.*

Furthermore, $(V_j)_{j=1}^{n}$ *of (i) is simulaneously cardinal.*

PROOF. Suppose (i). Then all of (ii), except the concavity assumption follows straightforwardly, see Theorems III.3.6 and III.3.7.

For the concavity assumption, first note that twofold application of concavity of V_i , and addition of inequalities, gives:

$$V_i(\lambda v_i+(1-\lambda)w_i)+V_i(\lambda w_i+(1-\lambda)v_i) \geq V_i(v_i)+V_i(w_i). \qquad \text{(VII.3.1)}$$

If we now had:

$x_{-i}v_i \succ y_{-i}(\lambda v_i+(1-\lambda)w_i)$ and $y_{-i}w_i \succ x_{-i}(\lambda w_i+(1-\lambda)v_i)$,
then we could express these two preferences in inequalities of sums of additive value functions, add up these two inequalities, cancel all terms $V_j(x_j)$ and $V_j(y_j)$ $(j \neq i)$, and end up with formula (VII.3.1) with "<" instead of "$\geq$" : contradiction!

Next we assume (ii) above. To derive is (i), and the uniqueness result. The existence of continuous additive value functions $(V_j)_{j=1}^{n}$, simultaneously cardinal, directly follows from Lemma VII.3.3, and Theorems III.3.6 and III.3.7. So only concavity of the V_j's remains to be proved.

Rewriting the definition of the concavity assumption in terms of additive value functions, with $\lambda = 1/2$, gives:

$$V_i(v_i)-V_i(v_i/2 + w_i/2) \overset{(1)}{\geq} \Sigma_{j\neq i}[V_j(y_j)-V_j(x_j)] \qquad \text{(VII.3.2)}$$

$$\Rightarrow \Sigma_{j\neq i}[V_j(y_j)-V_j(x_j)] \overset{(2)}{\leq} V_i(v_i/2 + w_i/2)-V_i(w_i).$$

This means that, for all v_i, w_i, for which x,y can be found to make $\overset{(1)}{\geq}$ hold with equality, we have:

$$[V_i(v_i) + V_i(w_i)]/2 \leq V_i(v_i/2 + w_i/2). \qquad \text{(VII.3.3)}$$

At least one coordinate $j \neq i$ is essential, so $V_j(C_j)$ (, by connectedness of C_j and continuity of V_j an interval,) must have length greater than η for some $\eta > 0$. For any v_i and w_i with $0 \leq V_i(v_i)-V_i(w_i) \leq \eta$, we can find x_j, y_j with $V_j(y_j)-V_j(x_j) = V_i(v_i)-V_i(w_i)$. Taking $x_k = y_k$ for all $k \neq i$, $k \neq j$, gives $\overset{(1)}{\geq}$ with equality.

Now concavity of V_i, analogously of any V_j , follows from Lemma VII.2.10.

□

The statement (i) above is equivalent to the statement that there exists a cardinal concave continuous additively decomposable representing function $V : X_{i=1}^n C_i \to \mathbb{R}$, as mainly follows from the following result.

PROPOSITION VII.3.6. *Let* $V_j : C_j \to \mathbb{R}$ *for all* $1 \leq j \leq n$. *Let* $V : x \mapsto \Sigma_{j=1}^n V_j(x_j)$. *Then* V *is concave if and only if every* V_j *is concave.*

PROOF. Let V be concave. $V_1(\lambda x_1+(1-\lambda)y_1)$ equals, for any arbitrary z, $V(\lambda(z_{-1}x_1) + (1-\lambda)(z_{-1}y_1)) - \Sigma_{j\neq 1}V_j(z_j)$.

By concavity of V this is greater/equal $\lambda V(z_{-1}x_1) + (1-\lambda)V(z_{-1}y_1) - \Sigma_{j\neq 1}V_j(z_j)$.

The latter equals $\lambda V_1(x_1) + (1-\lambda)V_1(y_1)$. Concavity of V_1, analogously of any V_j, follows.

Next assume: Every V_j is concave. Then every V_j', assigning $V_j(x_j)$ to every x, is concave. V is a sum of concave functions V_j' , so V itself is concave.

□

In Yaari (1977) and Debreu and Koopmans (1982, Theorem 2, and end of section 4) it is demonstrated that a quasiconcave additively decomposable function has all but one of its terms concave. By Lemma

VII.3.4, (ii) in Theorem VII.3.5 implies convexity of the preference relation. This in turn implies quasiconcavity of the representing additively decomposable function V, that exists according to section III.3. So now, by the theorem of Debreu and Koopmans, all but one of the additive value functions in (ii) above are concave. At this stage, we do not see an easy way to proceed to derive concavity of the remaining additive value function. Hence we have chosen a proof, which does not employ the results of Debreu and Koopmans.

Also, from the above observations, one may divine that in (ii) above we might replace the concavity assumption by three conditions, as follows. First one uses coordinate independence (and the Thomsen condition) to guarantee the existence of additive value functions. Next one uses convexity of $\succ$ to guarantee quasiconcavity of the sum of the additive value functions, which by the result of Koopmans and Debreu implies concavity of all but one of the additive value functions. Thirdly, one adds one weak condition for $\succ$ to guarantee concavity of the one remaining additive value function. We have not been able to find a weak condition for $\succ$ as described after "thirdly" above. Hence we have taken our alternative approach. Figure VII.4.1 (, mainly f^3 there,) will show that a further (weak) condition as after "thirdly" above, cannot be dispensed with. The earliest reference for this observation, given in Debreu and Koopmans (1982), is Slutsky (1915).

The following Corollary applies Theorem VII.3.5 to the case where $C_i = \mathbb{R}_{++}$ for all i, and $\succ$ is *weakly cA monotonic* ($x_i \geq y_i$ for all i, then $x \succ y$; see Definition II.3.7.b). The property after "furthermore" in (ii) below is simply a reformulation of the concavity assumption, so of Yaari's axiom Q, which may appeal to the idea of nonincreasing marginal utility.

COROLLARY VII.3.7. *Let* $n \geq 3$, *and let* $\succ$ *be a binary relation on* $\mathbb{R}^n_{++}$. *The following two statements are equivalent:*

(i) *There exist concave (so continuous) nondecreasing nonconstant additive value functions* $(V_j)_{j=1}^n$.

(ii) $\succ$ *is a continuous weak order, weakly cA monotonic, every coordinate is essential, and furthermore:*

$x_{-i}\alpha \succ y_{-i}\beta \Rightarrow x_{-i}(\alpha-\varepsilon) \succ y_{-i}(\beta-\varepsilon)$ *whenever* $(\alpha-\beta)\varepsilon \geq 0$.

□

Of course the results in this section can easily be adapted to deal with convex additive value functions; e.g. by replacing everywhere $\succ$ by $\prec$, and V_j by $-V_j$. Also results on concavity and results on convexity can be combined, to obtain results for affine additive value functions. This, under the addition of continuity conditions, gives characterizations, alternative to those in Fishburn (1965), Pollak (1967), and Keeney and Raiffa (1976, Theorem 6.4).

VII.4. SOME COUNTEREXAMPLES

In this section we give all logical relations between the statements (VII.4.1) through (VII.4.4) in Figure VII.4.1. Throughout we assume:

ASSUMPTION VII.4.1. $\succ$ is a continuous weak order on a product topological mixture space $X_{i=1}^{n}C_i$. Further $m \leq n$ is the number of *essential* coordinates.

In the sequel of this section we shall give elucidations to the seven counterexamples of Figure VII.4.1.

Counterexample (1). For m = 1, statement (VII.4.2) does not imply (VII.4.1), even if a representing function V exists. This follows from Kannai (1977, p.17), or from f^5 in the Figure. This function f^5 is straightforwardly seen to represent a binary relation $\succ$, satisfying the concavity assumption. Our f^5 is a minor variation on the example of Artstein in Kannai (1981 , p.562), where it is shown not to be "concavifiable", i.e. $\succ$, represented by f^5, has no concave representation.

There exists an array of continuous concave additive value functions for $\succ$. (VII.4.1)

$$\Uparrow : \left\{\begin{array}{ll} m=1: \text{ counterexample (1), } V = f^5 & \text{By Th.VII.3.5} \\ m=2: \text{ counterexample (2), } V = f^2 & \text{for } m \geq 2; \\ m \geq 3: \text{ correct by Theorem VII.3.5} & \text{analogously for } m=1 \end{array}\right\} : \Downarrow$$

$\succ$ *satisfies the concavity assumption.* (VII.4.2)

$$\Uparrow : \left\{\begin{array}{ll} m=1: \text{ counterexample (3), } V = f^1 & \text{by} \\ m=2: \text{ counterexample (4), } V = f^3 & \text{Lemmas} \\ m \geq 3: \text{ counterexample (5), } V = f^3 & \text{VII.3.3 \& VII.3.4} \end{array}\right\} : \Downarrow$$

$\succ$ *is convex and CI.* (VII.4.3)

$$\Uparrow : \left\{\begin{array}{ll} m=1: \text{ correct} & \\ m=2: \text{ counterexample (6), } V = f^4 & \text{direct} \\ m \geq 3: \text{ counterexample (7), } V = f^2 & \end{array}\right\} : \Downarrow$$

$\succ$ *is convex.* (VII.4.4)

FIGURE VII.4.1. $\succ$ is a continuous weak order on $\mathbb{R}^n_{++}$, with m essential coordinates. In the counterexamples the function V represents $\succ$. The solid arrows downwards indicate implications that hold, the broken arrows upwards indicate implications that do not always hold. For all $1 \leq k \leq 5$,

f^k is a function from $\mathbb{R}^n_{++}$ to $\mathbb{R}$:

$f^1(x) = 1$ if $x_1 \leq 1$, $f^1(x) = x_1$ if $x_1 \geq 1$;

$f^2(x) = \Sigma^n_{j=1} x_j + \min(\{x_j\}^n_{j=1})$;

$f^3(x) = (n-1)\, e^{x_1} + \Sigma^n_{j=2} \log x_j$;

$f^4(x) = -(\Sigma^n_{j=1}(x_j-2))^2$;

$f^5(x) = x_1 - 1$ for $0 < x_1 < 1$, $f^5(x) = (x_1-1)^2$ for $1 \leq x_1 < 2$,
$f^5(x) = 3 - x_1$ for $x_1 \geq 2$.

Counterexample (2). For $m = 2$, (VII.4.2) does imply (VII.4.1) if and only if $\succ$ satisfies the Thomsen condition. That $\succ$, represented by f^2, does not satisfy this for $m = 2$, hence has no additive value functions, can be seen from:
$(1,4,9,\ldots,9) \approx (2,2,9,\ldots,9)$, $(2,8,9,\ldots,9) \approx (4,4,9,\ldots,9)$, $(1,8,9,\ldots,9) > (4,2,9,\ldots,9)$. Still this $\succ$ by some elementary arguments can be seen to satisfy the concavity assumption.

Counterexample (3). That $\succ$, represented by f^1, does not satisfy the concavity assumption, follows from $(1/2,1,\ldots,1) \succ (1,\ldots,1) < (3/2,1,\ldots,1)$.

Counterexamples (4) and (5). That f^3 is quasiconcave, thus represents a convex $\succ$, can be derived from 6.28 of Arrow and Enthoven (1961). Here f^3 is a sum of additive value functions of which the first is not concave. For $m \geq 2$ any additive value functions are positive affine transformations of the above ones, so have the first one not concave. So $\succ$, represented by f^3 (,satisfying the Thomsen condition for $m = 2$,) must violate the concavity assumption.

The observation that (VII.4.3) does not imply (VII.4.2) for $m \geq 2$, is closely related to the observation that quasiconcavity and additive decomposability of V do not imply (VII.4.1), i.e. concavity of V. This latter observation has some times been made in literature. The earliest reference to this, given in Debreu and Koopmans (1982), is Slutsky (1915).

Counterexample (6). That for $m \geq 2$, $\succ$ as represented by f^4, is not coordinate independent, follows from $(2,\ldots,2) > (2,3,2,\ldots,2)$ and $(1,2,\ldots,2) < (1,3,2,\ldots,2)$.

Counterexample (7). That for $m \geq 3$, the $\succ$ as represented by f^2, is not coordinate independent, follows from $(1,6,1,\ldots,1) > (3,3,1,\ldots,1)$, $(1,6,3,\ldots,3) < (3,3,\ldots,3)$.

VII.5. SUBJECTIVE EXPECTED UTILITY WITH RISK AVERSION

In this section we assume $C_i = C$ for all i. So we have:

ASSUMPTION VII.5.1. C^n is a product topological mixture space.

In this section we again adopt the terminology of decision making under uncertainty. We combine cardinal coordinate independence and the concavity assumption to obtain a concise characterization of subjective expected utility maximization with "risk aversion", which here is simply defined to mean concavity of the utility function.

DEFINITION VII.5.2. $\succcurlyeq$ satisfies *concave cardinal coordinate independence* if for all acts x,y,v,w, all consequences $\alpha,\beta,\gamma,\delta$, every $\lambda \in [0,1]$, every state j, and every essential state i:

$$x_{-i}\,\alpha \preccurlyeq y_{-i}\,\beta \quad\text{and}\quad v_{-j}\,\alpha \succcurlyeq w_{-j}\beta$$
$$x_{-i}\,\gamma \succcurlyeq y_{-i}(\lambda\gamma+(1-\lambda)\delta)$$
$$\text{imply } v_{-j}(\lambda\delta+(1-\lambda)\gamma) \succcurlyeq w_{-j}\delta.$$

LEMMA VII.5.3. *Let $\succcurlyeq$ be a continuous weak order. Let $\succcurlyeq$ satisfy concave cardinal coordinate independence. Then $\succcurlyeq$ satisfies cardinal coordinate independence, and the concavity assumption.*

PROOF. That CCI holds can be seen by setting $\lambda = 0$ in Definition VII. 5.2. So only the concavity assumption remains to be derived. Let:

$$x_{-i}\gamma \succcurlyeq y_{-i}(\lambda\gamma + (1-\lambda)\delta). \qquad \text{(VII.5.1)}$$

To prove is:

$$x_{-i}(\lambda\delta + (1-\lambda)\gamma) \succcurlyeq y_{-i}\delta. \qquad \text{(VII.5.2)}$$

If i is inessential this is immediate. So let i be essential. Suppose there are η, $\zeta \in C$ with $x_{-i}\eta \prec y_{-i}\zeta$; if no such η,ζ should exist (VII.5.2) would be direct. Our plan is to find α, β in C such that:

$$x_{-i}\,\alpha \approx y_{-i}\,\beta. \qquad \text{(VII.5.3)}$$

If we succeed in this, then we can apply concave cardinal coordinate independence, with $i = j$, $x = v$, $y = w$, to obtain (VII.5.2). So finally, by means of η, ζ as above, we derive (VII.5.3) for some α, β.

Suppose firstly that $y_{-i}(\lambda\gamma + (1-\lambda)\delta) \succcurlyeq x_{-i}\eta$. Then $x_{-i}\gamma \succcurlyeq y_{-i}(\lambda\gamma + (1-\lambda)\delta) \succcurlyeq x_{-i}\eta$. By restricted solvability (Lemma III.2.13), with $\beta := \lambda\gamma + (1-\lambda)\delta$, we obtain an α such that (VII.5.3) holds.

Secondly, suppose $y_{-i}(\lambda\gamma + (1-\lambda)\delta) \prec x_{-i}\eta$. Then $y_{-i}(\lambda\gamma + (1-\lambda)\delta) \prec x_{-i}\eta \prec y_{-i}\zeta$. By restricted solvability, with $\alpha := \eta$, we obtain β such that (VII.5.3) holds.

□

With the above lemma we obtain:

THEOREM VII.5.4. *Let at least two states be essential with respect to the binary relation $\succcurlyeq$ on the product topological mixture space C^n. The following three statements are equivalent:*

(i) *There exists a SEU model $[C^n, \succcurlyeq, (p_j)_{j=1}^n, U]$ for $\succcurlyeq$, with U concave and continuous.*

(ii) *$\succcurlyeq$ is a continuous CCI weak order; $\succcurlyeq$ satisfies the concavity assumption, or $\succcurlyeq$ is convex.*

(iii) *$\succcurlyeq$ is a continuous weak order, satisfying concave cardinal coordinate independence.*

PROOF. We derive (i) ⇒ (iii) ⇒ (ii) ⇒ (i). First assume (i). Obviously $\succcurlyeq$ is a continuous weak order. For concave cardinal coordinate independence, let i be essential. Now $x_{-i}\alpha \preccurlyeq y_{-i}\beta$ and $x_{-i}\gamma \succcurlyeq y_{-i}(\lambda\gamma + (1-\lambda)\delta)$ imply $U(\alpha) - U(\beta) \le U(\gamma) - U(\lambda\gamma + (1-\lambda)\delta)$, compare (IV.2.4) in the proof of Lemma IV.2.5. By concavity of U, the latter righthand side is smaller/equal $U(\lambda\delta + (1-\lambda)\gamma) - U(\delta)$. Now $U(\alpha) - U(\beta) \le U(\lambda\delta + (1-\lambda)\gamma) - U(\delta)$ and $v_{-j}\alpha \succcurlyeq w_{-j}\beta$, imply $v_{-j}(\lambda\delta + (1-\lambda)\gamma) \succcurlyeq w_{-j}\delta$. Concave cardinal coordinate independence is derived, hence (iii).

By Lemma (VII.5.3 the implication (iii) ⇒ (ii) follows. So finally we assume (ii), and derive (i). By Theorem IV.3.3 there exists a SEU model for $\succ$, with U continuous. Of course $(p_jU)_{j=1}^n$ are additive value functions for $\succ$. If now $\succ$ satisfies the concavity assumption, then, $\succ$ satisfying the Thomsen condition if exactly two states are essential, by Theorem VII.3.5 there must exist simultaneously cardinal concave additive value functions $(V_j)_{j=1}^n$ for $\succ$. Further every V_j then is a positive affine transformation of p_jU, and since at least one p_j is positive, U must be concave.

If $\succ$ is convex, then it is well-known that U must be concave, see for example Debreu and Koopmans (1982, near the end of section 1).

□

A derivation of the SEU model with concave utility, using differentiability conditions, is given in Stigum (1972).

VII.6. SUBJECTIVE EXPECTED UTILITY WITH NONINCREASING RISK AVERSION

In this section we consider again the context of decision making under uncertainty. In Theorem IV.3.3, we have characterized (roughly) the class of all preference relations, representable by subjective expected utility with continuous utility. Usually one is not interested in all of this class, but only in a subclass of those preference relations that furthermore have certain "desirable behavioural properties". For instance, in the previous section we considered "risk averse" preference relations. For such further "desirable properties" of the preference relation then necessary and/or sufficient properties of the probabilities and/or utilities are searched, usually under the *presupposition* that a subjective expected utility model exists.

In this section the further "desirable property" of the preference relation that we shall consider, is (mainly) nonincreasing risk aversion. Necessary and sufficient properties of the utility function for this are well known, from the work of Pratt (1964) and Arrow (1965, 1971). Our aim in this section is to show a, surprising, extra implication of nonincreasing risk aversion for an earlier part of the characterization work: together with, mainly, the concavity assumption, nonincreasing risk aversion implies subjective expected utility maximization, and makes cardinal coordinate independence superfluous. This applies to the context where consequences are real numbers (say, amounts of money):

ASSUMPTION VII.6.1. Let in this section $C \subset \mathbb{R}$ be a nondegenerate interval. Let $C_i = C$ for all i.

VII.6.1. PREPARATORY RESULTS

DEFINITION VII.6.2. Let $[C^n, \succ, (p_j)_{j=1}^n, U]$ be a SEU model for $\succ$. Then $\succ$ is *risk averse* if $x \prec \bar{\alpha}$ for all x and α with $\alpha = \Sigma p_j x_j$.

If a decision maker T (i.e. his preference relation) is risk averse, then T will never strictly prefer an act x to its "expected value" $\Sigma_{j=1}^n p_j x_j$. The characterization (i) $\Leftrightarrow$ (ii) below of risk aversion is well known.

PROPOSITION VII.6.3. *Let* $n \geq 2$. *Let* $[C^n, \succ, (p_j)_{j=1}^n, U]$ *be a SEU model for* $\succ$, *with all* $p_j > 0$, *and* U *continuous. Then the following three statements are equivalent:*

(i) U *is concave.*

(ii) $\succ$ *is risk averse.*

(iii) $\succ$ *satisfies the concavity assumption.*

PROOF. (i) $\Leftrightarrow$ (iii) is by Theorem VII.5.4, and (i) $\Rightarrow$ (ii) is straightforward. Next assume (ii).

Let $\gamma = p_1\alpha + (1-p_1)\beta$. By risk aversion, $\bar{\gamma} \succ (\alpha,\beta,\ldots,\beta)$, so $U(p_1\alpha + (1-p_1)\beta) \geq p_1 U(\alpha) + (1-p_1)U(\beta)$. By Lemma VIII.1, U is concave.

□

One may argue that the Definition VII.6.2 of risk aversion reflects more decision maker T's attitude towards the (linear structure of) money, than his attitude towards risk or uncertainty. Statement

(i) above supports this. Some authors, inspired by Kahneman and Tversky (1979), have introduced new definitions of "risk aversion", reflecting more T's attitude towards risk and probability, see Quiggin (1982) and Yaari (1984).

DEFINITION VII.6.4. $\succ$ has *nonincreasing* [respectively *nondecreasing*; or *constant*] *(absolute) risk aversion* if for all $\varepsilon \geq 0$ [respectively $\varepsilon \leq 0$; or $\varepsilon \in \mathbb{R}$], and for all $x, x+\bar{\varepsilon}$ in C^n, $\alpha, \alpha + \varepsilon \in C$, we have:
$x \succ \bar{\alpha} \Rightarrow x + \bar{\varepsilon} \succ \bar{\alpha} + \bar{\varepsilon}$.

Say a decision maker T has a preference relation with nonincreasing risk aversion. If then he is willing to take a (possibly) risky act x, instead of a certain amount α of money, then certainly he is willing to do so if his wealth is increased by an amount ε.

DEFINITION VII.6.5. $\succ$ has *constant relative risk aversion* if for all $\lambda \in \mathbb{R}_{++}$, $x, \lambda x$ in C^n, $\alpha, \lambda\alpha \in C$, we have:
$x \succ \bar{\alpha} \Rightarrow \lambda x \succ \lambda\bar{\alpha}$.

Now let decision maker T have a preference relation with constant relative risk aversion. Say he is willing to invest an amount α into a risky undertaking, instead of keeping amount α for himself; where the risky undertaking gives him in return x_j/α per invested unit, if state of nature j is the true state. Then, if the amount to be invested is $\lambda\alpha$ instead of α, he is still willing to invest it in the risky undertaking. In other contexts than decision making under uncertainty, the above property of preference relations is often called "homotheticity".

With $\succ$ *strongly cA monotonic* if $x > y$ whenever $x_j \geq y_j$ for all j, and $x_j > y_j$ for some j, we have the following result, mainly due to Pratt (1964) and Arrow (1965, 1971).

THEOREM VII.6.6. *Let* $n \geq 2$. *The following three statements are equivalent for the nondegenerate interval* C, *and the binary relation* $\succ$ *on* C^n:

(i) *There exists a SEU model* $[C^n, \succ, (p_j)_{j=1}^n, U]$ *for* $\succ$, *with all* $p_j > 0$, *and with* U *continuous, strictly increasing, concave, and* *for all* $\alpha \geq \beta \geq \gamma > \delta$ *in* C:

$$\varepsilon \mapsto [U(\alpha+\varepsilon) - U(\beta+\varepsilon)] \;/\; [U(\gamma+\varepsilon) - U(\delta+\varepsilon)] \qquad \text{(VII.6.1)}$$

is a nondecreasing function on its domain.

(ii) *There exists a SEU model* $[C^n, \succ, (p_j)_{j=1}^n, U]$ *for* $\succ$, *with all* $p_j > 0$, U *continuous, strictly increasing. Further* $\succ$ *is risk averse, and has nonincreasing risk aversion.*

(iii) $\succ$ *is a continuous strongly cA monotonic CCI weak order, it satisfies the concavity assumption, and has nonincreasing risk aversion.*

PROOF. Apart from the statements on nonincreasing risk aversion, and the statement on the function defined in (VII.6.1), everything is straightforward from Theorem IV.3.3, and Proposition VII.6.3. The remaining statements do not immediately follow from, mainly, (e) in Theorem 1 of Pratt (1964), because there U was assumed twice continuously differentiable, and because here we only have a fixed and finite number of probabilities $p_1,\ldots,p_n$. The present results follow from Wakker, Peters, and Van Riel (1985, Theorem 4.1 and Lemma A.7.4), mainly by comparing $\succ$ with $\succ'$, defined by $x \succ' y$ if $x-\bar{\varepsilon} \succ y-\bar{\varepsilon}$. For brevity, we omit elaboration.

□

We added the formulation in (iii) to give a "complete" characterization of (i), i.e. a formulation of necessary and sufficient conditions, completely in terms of properties of the preference relation. Hence we could not use the property of risk aversion in it, as this needs the probabilities for its definition.

One can replace nonincreasing risk aversion by nondecreasing risk aversion in (ii) and (iii) above, if one replaces nondecreasingness of the function defined in (VII.6.1) by nonincreasingness. Analogously one can of course substitute "constant risk aversion" in (ii) and (iii), and constantness of the function, defined in (VII.6.1). In the latter case either U is affine or exponential ($\alpha \mapsto \tau + \lambda\alpha^{\rho}$; concavity implies $\rho \leq 1$), as can be derived from Theorem VII.6.12 in the sequel. Finally, if $C = \mathbb{R}_{++}$, one can replace "nonincreasing risk aversion" in (ii) and

(iii) above by "constant relative risk aversion", if one replaces the statement on the function defined in (VII.6.1) by the statement that $U : \alpha \mapsto \log \alpha$, or $U : \alpha \mapsto \lambda\alpha^{\rho}$, as can be derived from Theorem VII.6.11 in the sequel.

VII.6.2. REMOVING CARDINAL COORDINATE INDEPENDENCE

The major mathematical difficulty of this section is dealt with in the following lemma.

LEMMA VII.6.7. *Let* $(V_j)_{j=1}^n$ *be continuous nondecreasing additive value functions for* $\succ$, *with* V_1, *and at least one other of them, nonconstant. Let, for* $j = 1,\dots,n$, *there exist* $f_j : C \to \mathbb{R}$ *such that* $V_j(\alpha) - V_j(\beta) = \int_{]\beta,\alpha[} f_j(\tau)d\tau$ *(Lebesgue integral) for all* $\alpha > \beta$ *in* C. *Let* $\succ$ *have nonincreasing, or nondecreasing, risk aversion. Then there exist* $\tau_j \in \mathbb{R}$, *and* $\sigma_j \in \mathbb{R}_+$, *such that* $V_j = \tau_j + \sigma_j V_1$ *for all* $j \geq 2$.

PROOF. First the case where $\succ$ has nonincreasing risk aversion. By Theorem 6 of Chapter VI of Hartman and Mikusińsky (1961), every function which can be written as an integral, so also every V_j, is Lebesgue almost everywhere differentiable on every $[\alpha,\beta] \subset C$; hence on C. So there is a subset E of C, with Lebesgue measure zero, such that for every j, V_j is differentiable on $C\setminus E$. We may assume that E includes boundary points of C, and that f_j vanishes on E, $f_j = V_j'$ on $C\setminus E$, for every j, by the above-mentioned theorem. Note that V_j', hence f_j, is nonnegative.

First we derive an auxiliary result:

$$f_i(\alpha)f_j(\beta) = f_j(\alpha)f_i(\beta) \text{ for all } i,j, \text{ and } \alpha,\beta \in C. \tag{VII.6.2}$$

Because of symmetry in i and j, it is sufficient to prove:

$$\text{If } i \neq j, \text{ and } \alpha > \beta, \text{ then } f_i(\alpha)f_j(\beta) \geq f_j(\alpha)f_i(\beta). \tag{VII.6.3}$$

The result is direct if α or $\beta \in E$, then $f_i(\alpha)f_j(\beta) = 0 = f_j(\alpha)f_i(\beta)$. So let α,β in $C\setminus E$, i.e. the f_k's are derivatives of the V_k's and α,β are in $\text{int}(C)$.

First we derive (VII.6.3) for those β for which $\delta > 0$ exists such that $\beta - \delta \in C$, and $V_j(\beta-\delta) = V_j(\beta)$. Then V_j is constant on $[\beta-\delta,\beta]$, and $f_j(\beta) = V_j'(\beta) = 0$. Also then $\bar{\beta}_{-j}(\beta-\delta) \approx \bar{\beta}$. We add $\alpha-\beta$, apply nonincreasing risk aversion, and get $\bar{\alpha}_{-j}(\alpha-\delta) \succ \bar{\alpha}$. Consequently $V_j(\alpha-\delta) \geq V_j(\alpha)$, i.e. V_j is constant on $[\alpha-\delta,\alpha]$. Also $f_j(\alpha) = 0$, and $f_i(\alpha)f_j(\beta) = 0 = f_j(\alpha)f_i(\beta)$: (VII.6.3) holds.

Next we derive (VII.6.3) for those β for which $\delta > 0$ exists such that $\beta + \delta \in C$, and $V_i(\beta+\delta) = V_i(\beta)$. We can have δ so small that $\alpha + \delta \in C$. Now $\bar{\beta} + \bar{\delta} \approx (\bar{\beta}+\bar{\delta})_{-i}\beta$. By nonincreasing risk aversion $\bar{\alpha} + \bar{\delta} \prec (\bar{\alpha}+\bar{\delta})_{-i}\alpha$. Consequently $V_i(\alpha+\delta) = V_i(\alpha)$, and $f_i(\alpha) = 0 = f_i(\beta)$, and again (VII.6.3) follows.

Remains the case where $V_j(\gamma) < V_j(\beta)$ for all $\gamma < \beta$, and $V_i(\gamma) > V_i(\beta)$ for all $\gamma > \beta$. For this case we first take $(\sigma^k)_{k=1}^{\infty}, (\tau^k)_{k=1}^{\infty} \in \mathbb{R}_{++}$ such that

$$\sigma^k \downarrow 0,\ \tau^k \downarrow 0,\ V_i(\beta+\tau^k) - V_i(\beta) = V_j(\beta) - V_j(\beta-\sigma^k) \qquad \text{(VII.6.4)}$$

for all k.

By continuity of V_i, V_j, indeed such σ^k, τ^k exist. Now, for all k, $(\bar{\beta}_{-i,j}(\beta+\tau^k),(\beta-\sigma^k)) \approx \bar{\beta}$ follows. By nonincreasing risk aversion $(\bar{\alpha}_{-i,j}(\alpha+\tau^k),(\alpha-\sigma^k)) \succ \bar{\alpha}$, hence $V_i(\alpha+\tau^k) - V_i(\alpha) \geq V_j(\alpha) - V_j(\alpha-\sigma^k)$, for all k. We obtain:

$$f_i(\alpha)f_j(\beta) = \lim_{k\to\infty}[V_i(\alpha+\tau^k) - V_i(\alpha)][V_j(\beta) - V_j(\beta-\sigma^k)]/\tau^k\sigma^k \geq$$
$$\lim_{k\to\infty}[V_j(\alpha) - V_j(\alpha-\sigma^k)][V_i(\beta+\tau^k) - V_i(\beta)]/\sigma^k\tau^k = f_j(\alpha)f_i(\beta).$$

So (VII.6.3) always holds, hence (VII.6.2) holds. Now we use this, with $j = 1$. Since V_1 is not constant, $f_1(\eta) > 0$ for some η. We define $\sigma_i := f_i(\eta) / f_1(\eta)$ for all i. By (VII.6.2), with $\beta = \eta$, $j = 1$, we have $f_i(\alpha) = \sigma_i f_1(\alpha)$ for all $\alpha \in C$. So $V_i(\alpha) - V_i(\beta) = \int_{(\beta,\alpha)} f_i(\tau)d\tau = \int_{(\beta,\alpha)} \sigma_i f_1(\tau)d\tau = \sigma_i[V_1(\alpha) - V_1(\beta)]$ follows. Of course now $\tau_i := V_i(\eta) - \sigma_i V_1(\eta)$.

For the case where $\succ$ has nondecreasing, instead of nonincreasing, risk aversion, the proof is like above, with minor changes, mainly reversals of inequalities and preferences. Let then $\alpha < \beta$ in (VII.6.3), next let δ always be negative, etc.

□

With this we obtain the main mathematical result of this section:

THEOREM VII.6.8. *Let* $\succ$ *have continuous nondecreasing additive value functions* $(V_j)_{j=1}^n$, *such that, for* $j = 1,\ldots,n$, *there exist* f_j *with* $V_j(\alpha) - V_j(\beta) = \int_{]\beta,\alpha[} f_j(\tau)d\tau$, *for all* $\alpha > \beta$. *Let at least two states be essential, and let* $\succ$ *either have nonincreasing, or nondecreasing, absolute risk aversion. Then there exists a SEU model* $(C^n, \succ, (p_j)_{j=1}^n, U)$ *for* $\succ$.

PROOF. Say state 1 is essential, so V_1 is not constant. Apply Lemma VII.6.7, let $U := V_1$, $\sigma_1 := 1$, and $p_j := \sigma_j/\Sigma_{i=1}^n \sigma_i$ for all j.

□

In all characterization theorems of this monograph after Chapter II, it has been our aim to use in the characterizing statements (mostly numbered (ii)) only conditions directly in terms of the preference relation. The above theorem as such is not well suited to be considered a characterization theorem, because the assumption on the existence of the f_j's has to the author's knowledge no equivalent formulation in terms of simple appealing properties of the preference relation. It does however serve as a starting point to derive characterization theorems.

COROLLARY VII.6.9. *In (iii) of Theorem VII.6.6, for* $n \geq 3$ *the CCI assumption may be omitted.*

PROOF. The strong cA monotonicity assumption there implies that every, so ($n \geq 3$) at least three, states are essential. By Theorem VII.3.5, the concavity assumption implies existence of continuous concave additive value functions $(V_j)_{j=1}^n$. Strong cA monotonicity implies non-decreasingness, even strict increasingness, of every V_j. By concavity of every V_j, Corollary 24.2.1 of Rockafellar (1970) implies existence of f_j such that $V_j(\alpha) - V_j(\beta) = \int_{]\beta,\alpha[} f_j(\tau)d\tau$ for all $\alpha > \beta$ in $\mathrm{int}(C)$, e.g. f_j may be the right or left derivative of V_j. By continuity of V_j this also holds for α and/or β boundary points of C, e.g. let $f_j := 0$ in boundary points. Theorem VII.6.8 gives existence of a SEU model, which implies CCI.

□

Of course, the same as above holds with nondecreasing, instead of nonincreasing, risk aversion. For characterization purposes, the following conjecture, if true, would be useful. It would show equivalence of (i) and (iii) in Theorem VII.6.6, if concavity of U in (i) was left out, and CCI in (iii) was left out, further in (iii) the concavity assumption was replaced by coordinate independence (= sure-thing principle); for $n \geq 3$.

CONJECTURE VII.6.10. In Theorem VII.6.8, existence of the f_j's can be left out.

We do not need the "f_j-condition" in Theorem VII.6.8 if $C = \mathbb{R}$ and we have constant absolute risk aversion, or if $C = \mathbb{R}_{++}$ and we have constant relative risk aversion. First we give the latter result, this being directly derivable from Stehling (1975).

THEOREM VII.6.11. *Let* $C = \mathbb{R}_{++}$. *The following two statements are equivalent for the binary relation* $\succcurlyeq$ *on* C^n:

(i) *There exists a SEU model* $[C^n, \succcurlyeq, (p_j)_{j=1}^n, U]$ *for* $\succcurlyeq$, *with all* $p_j > 0$, *and either* $U : \alpha \mapsto \lambda\alpha^\rho$ *for some* $\lambda, \rho \in \mathbb{R}$ *with* $\lambda\rho > 0$, *or* $U : \alpha \mapsto \log \alpha$.

(ii) $\succcurlyeq$ *is a continuous strongly cA monotonic coordinate independent weak order, satisfying the Thomsen condition if* $n = 2$; $\succcurlyeq$ *has constant relative risk aversion.*

PROOF. Suppose (i). Then, for any $\mu > 0$, $x \in C^n$, for the expected utility EU, $EU(\mu x) = \mu^\rho EU(x)$ or $EU(\mu x) = \mu + EU(x)$. From this, constant relative risk aversion, and all of (ii) follows straightforwardly. So we suppose (ii), and derive (i).

If $n = 1$, the choice $p_1 = 1$ and U = identity, by strong cA monotonicity gives (i). So let $n \geq 2$. By strong cA monotonicity every state is essential. By Theorems III.3.6 and III.3.7, there exist continuous additive value functions $(V_j)_{j=1}^n$ for $\succcurlyeq$. By strong monotonicity, every V_j is strictly increasing. Define $V : C^n \to \mathbb{R}$, $\phi : C \to \mathbb{R}$, $W : C^n \to \mathbb{R}$ by:

$V : x \mapsto \Sigma V_j(x_j)$, $\phi : \alpha \mapsto V(\bar{\alpha})$, $W : x \mapsto \phi^{-1} \circ V(x)$.

Then V and W represent $\succcurlyeq$, $W(\bar{\alpha}) = \alpha$, $[W(x) = \alpha \Rightarrow x \approx \bar{\alpha}]$, $W(\mu x) = \mu W(x)$ for $\mu > 0$ (W is "linearly homogeneous", so V is "homothetic"). By Stehling (1975, Theorem 2), or Eichhorn (1978, Theorem 2.5.2), either:

$$V : x \mapsto \psi\,[\mu(\Pi_{j=1}^{n} x_j^{p_j})] \text{ for a continuous strictly} \tag{VII.6.5}$$

increasing ψ, positive μ, and nonzero $p_1,\ldots,p_n$ that sum to one,

or:

$$V : x \mapsto \psi[(\Sigma_{j=1}^{n} \sigma_j x_j^{\rho})^{1/\rho}] \text{ for a continuous strictly} \tag{VII.6.6}$$

increasing ψ, positive $\sigma_1,\ldots,\sigma_n$, and nonzero ρ.

In case of (VII.6.5), V is a strictly increasing transform of $x \mapsto \Pi x_j^{p_j}$, so, by taking logarithms, of $x \mapsto \Sigma p_j \log(x_j)$. By strict increasingness of every V_j, every p_j is positive. So indeed we have a SEU model for $\succcurlyeq$, with $U : \alpha \mapsto \log \alpha$.

Next suppose (VII.6.6). First assume $\rho > 0$. Then V is a strictly increasing transform of $\Sigma\sigma_j x_j^{\rho}$. So we have a SEU model for $\succcurlyeq$, with $p_j := \sigma_j/\Sigma_{i=1}^{n}\sigma_i$ for every j, and $U : \alpha \mapsto \alpha^{\rho}$, so $\lambda = 1$ in (i) above.

Finally, suppose (VII.6.6), with $\rho < 0$. Then V is a strictly *decreasing* transform of $x \mapsto \Sigma\sigma_j x_j^{\rho}$, so a strictly increasing transform of $x \mapsto \Sigma\sigma_j(-(x_j^{\rho}))$. We have a SEU model for $\succcurlyeq$, with $p_j := \sigma_j/\Sigma_{i=1}^{n}\sigma_i$ for every j, and $U : \alpha \mapsto -(\alpha^{\rho})$, so in (i) above, $\lambda = -1$.

□

From this we derive:

THEOREM VII.6.12. *Let* $C = \mathbb{R}$. *The following two statements are equivalent for the binary relation* $\succcurlyeq$ *on* C^n :

(i) *There exists a SEU model* $[C^n, \succcurlyeq, (p_j)_{j=1}^{n}, U]$ *for* $\succcurlyeq$, *with all* $p_j > 0$, *and* $U : \alpha \mapsto \lambda e^{\rho\alpha}$ *for some* $\lambda,\rho \in \mathbb{R}$ *with* $\lambda\rho > 0$, *or* U *identity.*

(ii) $\succcurlyeq$ *is a continuous strongly cA monotonic coordinate independent weak order, satisfying the Thomsen condition if* $n = 2$; $\succcurlyeq$ *has constant absolute risk aversion.*

PROOF. Suppose (i). Then, for any $\mu > 0$, $x \in C^n$, the expected utility $EU(x)$ has $EU(x+\mu) = e^{\rho\mu}EU(x)$ or $EU(x+\mu) = \mu+EU(x)$. From this constant absolute risk aversion, and all of (ii), follows straightforwardly. So we suppose (ii), and derive (i).

Define $L : \mathbb{R}^n_{++} \to \mathbb{R}^n$ by $L : (x_1,\ldots,x_n) \mapsto (\log(x_1),\ldots,\log(x_n))$, and define $\succcurlyeq'$ on $\mathbb{R}^n_{++}$ by $x \succcurlyeq' y$ iff $L(x) \succcurlyeq L(y)$. Then it follows straightforwardly that $\succcurlyeq'$ satisfies (ii) of Theorem VII.6.11. We obtain, for all $x,y \in \mathbb{R}^n$:

$x \succcurlyeq y \Leftrightarrow L^{-1}(x) \succcurlyeq' L^{-1}(y) \Leftrightarrow \Sigma p_j U(e^{x_j}) \geq \Sigma p_j U(e^{y_j})$, with U, p_j , and also λ, ρ as in (i) of Theorem VII.6.10.

□

Most probably the last two theorems also hold for any interval $C \subset \mathbb{R}_{++}$, respectively $C \subset \mathbb{R}$, but we do not know of a reference where the analogue of Stehling's (1975) theorem, needed to prove that, is readily available. For the study of the extendability of the above results, for the case of constant risk aversion, to multidimensional consequences, Rothblum (1975) may be useful.

CHAPTER VIII

CONTINUOUS FUNCTIONS ON INTERVALS

In this chapter we derive some elementary properties of functions from nondegenerate intervals to the reals. These properties have been used, and referred to, in many places in this monograph.

VIII.1. GENERALIZATIONS OF MIDPOINT CONVEXITY

The first two results give conditions sufficient for convexity of a function ϕ, by means of properties that are variations on "midpoint-convexity" $[\phi((\mu+\nu)/2) \leq \phi(\mu)/2 + \phi(\nu)/2)]$. When formulated for $-\phi$, these conditions of course are sufficient for concavity of ϕ, and when formulated both for ϕ and $-\phi$, they are sufficient for affinity of ϕ, see for example Corollary VIII.3. The first Lemma, and its elegant proof, are due to Hardy, Littlewood and Pólya (1959, Theorem 88).

LEMMA VIII.1. *Let* $S \subset \mathbb{R}$ *be a nondegenerate interval. Let* $\phi : S \to \mathbb{R}$ *be continuous. For all* $\sigma < \tau \in S$ *let there exist* $0 < p < 1$ *such that* $\phi(p\sigma + (1-p)\tau) \leq p\phi(\sigma) + (1-p)\phi(\tau)$. *Then* ϕ *is convex.*

PROOF. Suppose ϕ were not convex. Then we had $\lambda < \mu < \nu$ in S such that the point $(\mu,\phi(\mu))$ of the graph G of ϕ lies strictly above the straight line l through $(\lambda,\phi(\lambda))$ and $(\nu,\phi(\nu))$. Let then $(\sigma,\phi(\sigma))$ and $(\tau,\phi(\tau))$ be the points of intersection of G and l, closest to $(\mu,\phi(\mu))$, with $\sigma < \mu < \tau$. Then $\lambda \leq \sigma < \mu < \tau \leq \nu$. Between σ and τ all of G lies above l, contradicting the existence of the p as in the Lemma.

□

LEMMA VIII.2. *Let* $S \subset \mathbb{R}$ *be a nondegenerate interval. Let* $\phi : S \to \mathbb{R}$ *be continuous. Let, for every* $\nu \in int(S)$*, an open neighbourhood* W *of* ν *within* S *be given such that for all* $\sigma < \tau$ *in* W*, there exists* $0 < p < 1$ *such that* $\phi(p\sigma + (1-p)\tau) \leq p\phi(\sigma) + (1-p)\phi(\tau)$. *Then* ϕ *is convex.*

PROOF. For every ν in int(S) there must exist an interval $]\nu-\delta, \nu+\delta[$ around ν within S, such that for all σ,τ within this interval, a p as in the Lemma exists. By Lemma VIII.1, ϕ is convex on $]\nu-\delta, \nu+\delta[$. This implies convexity of ϕ on all of int(S), for instance because ϕ has a nondecreasing right derivative. By continuity, ϕ is convex on all of S.

□

COROLLARY VIII.3. *Let* $S \subset \mathbb{R}$ *be a nondegenerate interval. Let* $\phi : S \to \mathbb{R}$ *be continuous. Let, for every* $\nu \in int(S)$*, an open neighbourhood* W *of* ν *within* S *be given such that for all* $\sigma < \tau$ *in* W*, there exists* $0 < p < 1$ *such that* $\phi(p\sigma + (1-p)\tau) = p\phi(\sigma) + (1-p)\phi(\tau)$. *Then* ϕ *is affine.*

PROOF. Apply Lemma VIII.2 to ϕ and $-\phi$.

□

VIII.2. CONTINUITY OF TRANSFORMATIONS

The following results consider transformations ϕ, such that $f = \phi \circ g$ for two functions f, g.

LEMMA VIII.4. *Let* C *be a connected topological space. Let* $f, g : C \to \mathbb{R}$ *be continuous. The following three statements are equivalent:*

(i) $f = \phi \circ g$ *for a nondecreasing* ϕ.

(ii) $f = \phi \circ g$ *for a nondecreasing continuous* ϕ.

(iii) $g(\alpha) \geq g(\beta) \Rightarrow f(\alpha) \geq f(\beta)$ *for all* $\alpha, \beta \in C$.

PROOF. (ii) ⇒ (i) and (i) ⇒ (iii) are obvious. So we assume (iii). To derive is (ii). If $g(\alpha) = g(\beta)$, then $g(\alpha) \geq g(\beta)$ and $g(\beta) \geq g(\alpha)$, so

$f(\alpha) \geq f(\beta)$ and $f(\beta) \geq f(\alpha)$. $f(\alpha) = f(\beta)$ follows. Hence $f = \phi \circ g$ for some ϕ. By (iii), ϕ must be nondecreasing. Continuity is postponed to the next lemma.

□

Throughout the sequel we assume:

ASSUMPTION. $\mathcal{C}$ is a connected topological space. Further f and g are continuous functions from $\mathcal{C}$ to $\mathbb{R}$, and $f = \phi \circ g$ for a transformation ϕ.

We now investigate the kinds of properties that ϕ may have, such as continuity.

LEMMA VIII.5. *If ϕ is nonincreasing or nondecreasing, then it is continuous.*

PROOF. ϕ is a nondecreasing or nonincreasing function from the connected $g(\mathcal{C})$ *onto* the connected $f(\mathcal{C})$, hence must be continuous. (It cannot make "jumps".)

□

The following results are only used in section IV.4. Lemmas VIII.6 and VIII.8, and Example VIII.7, were found, and communicated to the author, by A.C.M. van Rooij in 1985.

LEMMA VIII.6. *ϕ has the intermediate value property.*

PROOF. Let $G = \{g(\alpha), f(\alpha) : \alpha \in \mathcal{C}\} \subset \mathbb{R}^2$. G is the graph of ϕ. Since f and g, thus $\alpha \mapsto (g(\alpha), f(\alpha))$, are continuous, G is connected. Now let $\mu < \nu$, and let $\phi(\mu) < \phi(\nu)$ [$\phi(\mu) > \phi(\nu)$ is analogous]. Let $\phi(\mu) < \lambda < \phi(\nu)$ for some λ. Let $V = \{(\sigma,\tau) \in G : \sigma \leq \mu$, or $\mu \leq \sigma \leq \nu$ and $\tau \leq \lambda\}$, and $W = \{(\sigma,\tau) \in G : \sigma \geq \nu$, or $\mu \leq \sigma \leq \nu$ and $\tau \geq \lambda\}$. Then $V \cup W = G$, $(\mu,\phi(\mu)) \in V \neq \emptyset$, $(\nu,\phi(\nu)) \in W \neq \emptyset$, V and W are closed subsets of G. By connectedness of G, $V \cap W \neq \emptyset$. Let $(\sigma,\tau) \in V \cap W$. There follows $\tau = \lambda$ and $\mu < \sigma < \nu$. Since $\phi(\sigma) = \tau$, the intermediate value property has been obtained.

□

EXAMPLE VIII.7. ϕ is not necessarily continuous: Let $\phi : \mathbb{R} \to \mathbb{R}$ assign 0 to 0, and $\sin(\frac{1}{\mu})$ to every $\mu \neq 0$. Let G be the graph of ϕ. Let $C = G$. Let f be the projection on the second coordinate, g that on the first. Then indeed C is connected, f and g are continuous, $f = \phi \circ g$; ϕ is not continuous in 0.

□

LEMMA VIII.8. *If ϕ is bijective, then it is strictly increasing or strictly decreasing.*

PROOF. It is sufficient to show, for any $\lambda < \mu < \nu$ in the domain of ϕ, that either $\phi(\lambda) < \phi(\mu) < \phi(\nu)$, or $\phi(\lambda) > \phi(\mu) > \phi(\nu)$. Say, for $\lambda < \mu < \nu$, $\phi(\lambda) < \phi(\nu)$. Now were $\phi(\mu) < \phi(\lambda)$, then by Lemma VIII.6 any value between $\phi(\lambda)$ and $\phi(\mu)$, would be taken by ϕ at least two times: once between λ and μ, and once between μ and ν. By bijectivity this cannot hold. An analogous violation of bijectivity occurs if $\phi(\mu) > \phi(\nu)$. Also $\phi(\mu) = \phi(\lambda)$ or $\phi(\mu) = \phi(\nu)$ violates bijectivity. Hence $\phi(\lambda) < \phi(\mu) < \phi(\nu)$ follows.

□

The following lemma shows that in the main case of interest for us, where C is a convex subset of a Euclidean space, ϕ must be continuous.

LEMMA VIII.9. *If C is arcconnnected, then ϕ is continuous.*

PROOF. It is sufficient to show that any sequence $(\mu_j)_{j=1}^{\infty}$ in $g(C)$, converging to μ in $g(C)$, has a subsequence $(\mu_{j_i})_{i=1}^{\infty}$ such that $\lim_{i\to\infty} \phi(\mu_{j_i}) = \phi(\mu)$. So let (μ_j) converge to μ. We may assume $\mu_j \neq \mu$ for all j. There must exist a subsequence $(\nu_i)_{i=1}^{\infty}$ of $(\mu_j)_{j=1}^{\infty}$ that either strictly increases or strictly decreases; say the first. Now take arbitrary α_1, α in C such that $g(\alpha_1) = \nu_1$, $g(\alpha) = \mu$. Of course $\alpha_1 \neq \alpha$. We use arcconnectedness by taking an arc λ from α_1 to α, i.e. $\lambda : [0,1] \to C$ is continuous, with $\lambda(0) = \alpha_1$, $\lambda(1) = \alpha$. Now $g \circ \lambda$ is continuous, $(g\circ\lambda)(0) = \nu_1$, $(g\circ\lambda)(1) = \mu$. By the intermediate value property, $(\sigma_j)_{j=1}^{\infty}$ in $[0,1]$ exists such that $(g\circ\lambda)(\sigma_j) = \nu_j$ for all j.

So $(\tau_j)_{j=1}^{\infty}$ on λ $([0,1])$ exist with $\tau_j := \lambda(\sigma_j)$ for all j, $g(\tau_j) = \nu_j$ for all j. Since $\lambda([0,1])$ is compact, $(\tau_j)_{j=1}^{\infty}$ has a convergent subsequence $(\tau_{j_i})_{i=1}^{\infty}$, with limit say τ. Also $(g(\tau_{j_i}))_{i=1}^{\infty}$ and $(f(\tau_{j_i}))_{i=1}^{\infty}$ must converge to $g(\tau)$, respectively $f(\tau)$. This can only hold if $g(\tau) = \mu$, and $\lim_{i\to\infty} \phi(\nu_{j_i}) = \lim_{i\to\infty} \phi(g(\tau_{j_i})) = \lim_{i\to\infty} f(\tau_{j_i}) = f(\tau) = \phi(g(\tau)) = \phi(\mu)$.

□

COROLLARY VIII.10. *ϕ is continuous if it is nonincreasing, nondecreasing, bijective, or if C is arcconnected.*

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AUTHOR INDEX

SUBJECT INDEX

REPRESENTATIES VAN KEUZESITUATIES

SAMENVATTING

In deze monografie worden representatiestellingen voor beslissingstheorie afgeleid. Nadruk zal daarbij liggen op stellingen die toepasbaar zijn op beslissen bij onzekerheid.

Allereerst worden in hoofdstuk 0 enige elementaire definities gegeven.

Vervolgens geven we in hoofdstuk I aan wanneer er verband bestaat tussen preferentierelaties en keuzegedrag, en bespreken we enige intuïtieve veronderstellingen. In hoofdstuk I veronderstellen we nog niet dat structuur op de verzameling alternatieven gegeven is (behalve verzamelingstheoretische structuur). In volgende hoofdstukken zal steeds meer structuur op de verzameling alternatieven worden ingevoerd.

In hoofdstuk II voeren we de belangrijkste structuur van deze monografie in: we veronderstellen dat de verzameling van alternatieven een cartesisch product is. Iedere coordinaat van een alternatief beschrijft een relevant aspect van het alternatief, en bij een keuze tussen alternatieven moeten de voor- en nadelen betreffende de diverse aspecten tegen elkaar worden afgewogen. Deze benadering is dusdanig algemeen dat toepassing op velerlei gebied mogelijk is. De zes belangrijkste economische toepassingsgebieden in deze monografie worden gegeven in paragraaf II.1. In de daaropvolgende paragrafen worden diverse monotoniciteitseigenschappen behandeld. Met behulp van de in deze paragrafen verkregen resultaten tonen we in paragraaf II.6 aan dat "coordinaat onafhankelijkheid" de enige waarneembare implicatie is van de monotoniciteitseigenschappen, onder de in het vervolg van deze monografie steeds gemaakte veronderstelling dat slechts de preferentierelatie op de verzameling alternatieven waarneembaar is.

In hoofdstuk III, en alle volgende hoofdstukken, veronderstellen we dat de verzameling alternatieven voorzien is van een samenhangende producttopologie. Met behulp van deze kunnen we in het vervolg continuïteitsveronderstellingen formuleren. In de paragrafen III.3 en

III.4 geven we veralgemeniseringen van bekende stellingen over het bestaan van representerende functies die als som van coordinaatfuncties te verkrijgen zijn.

We nemen vanaf hoofdstuk IV (, met uitzondering van de paragrafen VII.1 tot en met VII.4,) steeds aan dat alle coordinaatverzamelingen identiek zijn. Stelling IV.3.3 geeft een hoofdresultaat van deze monografie: een karakterisering van subjectief verwacht nut maximalisatie met behulp van een nieuwe eigenschap voor preferentierelaties, namelijk cardinale coordinaat onafhankelijkheid. Dit gebeurt onder restricties die in economische contexten gewoonlijk vervuld zijn. Verder worden in de hoofdstukken IV, V en VI vele veralgemeniseringen van stelling IV.3.3 gegeven. Ook geven we toepassingen aan voor andere contexten dan beslissen bij onzekerheid; dynamische contexten vooral.

In hoofdstuk V breiden we het resultaat van stelling IV.3.3 uit naar willekeurige, mogelijkerwijs oneindige, toestandsruimten. We bekijken dan zowel σ-additieve, als eindig additieve, kansmaten.

Hoofdstuk VI breidt stelling IV.3.3 uit tot "capaciteiten", dat wil zeggen "niet-additieve kansmaten". Deze zijn in beslissingstheorie ingevoerd door Schmeidler (1984 a,b). Sub- en superadditiviteit zijn veel bestudeerde eigenschappen van capaciteiten; ze worden gekarakteriseerd in paragraaf VI.11.

In hoofdstuk VII wordt weer een nieuwe structuur op de verzameling alternatieven ingevoerd. We veronderstellen dat de coordinaatverzamelingen zogenaamde "mengruimten" (mixture spaces) zijn. Standaardvoorbeelden van mengruimten zijn convexe deelverzamelingen van lineaire ruimten. We karakteriseren dan *concave* representerende functies die te schrijven zijn als som van coordinaatfuncties. In paragraaf VII.6 veronderstellen we dat de coordinaatverzamelingen convexe deelverzamelingen zijn van de verzameling van reële getallen. Hier hebben we te maken met de meest gestructureerde verzameling van alternatieven in deze monografie. In paragraaf VII.6 laten we dan zien dat veronderstellingen over (nietstijgende) risicoafkerigheid op een verrassende wijze de karakterisering van verwacht nut maximalisatie vereenvoudigen.

Hoofdstuk VIII tenslotte geeft enige wiskundige resultaten betreffende functies op intervallen. In vorige hoofdstukken is al vaak naar deze resultaten verwezen.

CURRICULUM VITAE

Peter Wakker werd op 16 maart 1956 geboren te Lobith-Tolkamer. Hij behaalde in 1972 het diploma Gymnasium-β aan het Thomas à Kempiscollege te Arnhem, en begon in datzelfde jaar met de studie wiskunde aan de Universiteit te Nijmegen. Hij volgde in de doctoraalfase colleges in de statistiek bij Professor Dr. F.H. Ruymgaart, colleges in de stochastiek bij Professor Dr. W. Vervaat, en enige colleges in de speltheorie bij Professor Dr. S.H. Tijs. Na een doctoraalscriptie onder begeleiding van Professor Dr. W. Vervaat geschreven te hebben, studeerde hij cum laude af in maart 1979. (Inmiddels heeft hij ook de eerstegraadsonderwijsbevoegdheid in de wiskunde verkregen.)

Van april 1979 tot april 1983 was hij, onder begeleiding van Professor Dr. J. Fabius, werkzaam als wetenschappelijk assistent aan het Instituut voor Toegepaste Wiskunde en Informatica van de Rijksuniversiteit te Leiden. Vanaf augustus 1983 heeft hij een deeltijdbaan als wetenschappelijk assistent aan het Mathematisch Instituut van de Katholieke Universiteit te Nijmegen, onder begeleiding van Professor Dr. S.H. Tijs.

Van Z.W.O., de Nederlandse organisatie voor zuiver wetenschappelijk onderzoek, verkreeg hij een beurs voor het maken van een studiereis naar Israël, van 8 januari 1985 tot 20 februari 1985. Hier werd hoofdzakelijk samengewerkt met Professor Dr. D. Schmeidler van de Tel Aviv Universiteit.

STELLINGEN

behorende bij het proefschrift

REPRESENTATIONS OF CHOICE SITUATIONS

van

Peter WAKKER

STELLING 1. De benadering van speltheorie, waarbij wordt verondersteld dat de uitbetaling niet in ("von Neumann-Morgenstern") nut is, maar in reële grootheden als geld of goederen, levert interessante onderzoeksproblemen op.

Wakker, P.P. (1983), "The Existence of Utility Functions in the Nash Solution for Bargaining". Forthcoming in Paelinck, J.H.P., and P.H. Vossen (Eds., 1986), "*Axiomatics and Pragmatics of Conflict Analysis*" (Studies in Interdisciplinary Issues), Grower Press, Aldershot.

STELLING 2. De onmogelijkheidsstelling van Arrow (zie Arrow, 1978) is geen verrassend resultaat als men bedenkt dat transitiviteit van een groepspreferentierelatie alleen redelijk is wanneer de groepspreferenties tussen alternatieven x,y, tussen alternatieven y,z, en tussen alternatieven x,z, onder "overigens gelijke omstandigheden" tot stand komen, wat onder andere inhoudt dat de groep bij de totstandkoming van de drie preferenties steeds over dezelfde informatie beschikt, terwijl de "independence of irrelevant alternatives" conditie juist inhoudt dat de drie preferenties op verschillende informatie gebaseerd zijn.

Arrow, K.J. (1978), "*Social Choice and Individual Values*," 9th edition. Yale University Press, New Haven.

STELLING 3. Met behulp van optimaliseringstheorie kan men bewijzen dat een niet-expanderende afbeelding van een deelverzameling van $\mathbb{R}^n$, naar $\mathbb{R}^n$, kan worden uitgebreid tot een niet-expanderende afbeelding van $\mathbb{R}^n$ naar $\mathbb{R}^n$.

Wakker, P.P. (1985), "Extending Monotone and Non-Expansive Mappings by Optimization," *Cahiers du C.E.R.O.* 27, 141-151.

STELLING 4. Het statistische toetsen met behulp van significantie-toetsen voldoet niet aan het "sure-thing principle."

Wakker, P.P. (1981), "The Additivity Principle in Decision Making under Uncertainty," Report 81-35, Department of Mathematics, University of Leiden.

STELLING 5. De eerste Remark in paragraaf 2 van Wakker (1981) geeft aan dat het deel van paragraaf III.4, bovenaan pagina 43 in Savage (1954), weinig gelezen is.

Savage, L.J. (1954), "*The Foundations of Statistics.*" Wiley, New York.

Wakker, P.P. (1981), "Agreeing Probability Measures for Comparative Probability Structures," *The Annals of Statistics* 9, 658-662.

STELLING 6. De "weerlegging" van het scepticisme die gebaseerd is op de redenering dat een scepticus meent te weten dat hij niets weet, en zodoende toch iets meent te weten (zie O'Connor en Carr, 1982, bovenaan pagina 3), is niet juist omdat een scepticus alleen maar, uitgaande van de veronderstelling dat hij iets kan weten, tot de conclusie komt dat hij niets weet, en in de daaruit voortvloeiende tegenspraak niet een weerlegging, doch integendeel een bekrachtiging, van zijn houding ziet.

O'Connor, D.J. en B. Carr (1982), "*Introduction to the Theory of Knowledge.*" University of Minnesota Press, Minneapolis.

STELLING 7. Volgens het criterium dat empirische wetenschappers zich alleen moeten bezighouden met zaken die tot waarneembare, dat wil zeggen verifieerbare of falsifieerbare, resultaten leiden, moeten empirische wetenschappers zich niet bezighouden met het criterium, dat empirische wetenschappers zich alleen moeten bezighouden met zaken die tot waarneembare, dat wil zeggen verifieerbare of falsifieerbare, resultaten leiden.

STELLING 8. Wanneer men denksporten als het schaken als wetenschappen wil beschouwen, en men bijvoorbeeld de opvatting in het schaken dat een voor wit gewonnen stelling ontstaat als in de beginstelling de zwarte dame en een wit paard worden verwijderd, als een wet uit deze wetenschappen wil beschouwen, dan horen deze wetenschappen bij de inductieve, en niet bij de deductieve, wetenschappen.

STELLING 9. In het schaakspel staat na de beginzetten 1.e4 - e6.2.d4 - d5.3.Pd2 - Pf6.4.e5 - Pfd7 5.Ld3 - c5.6.c3 - Pc6.7.P1f3 - f6.8.Pg5 - fg5! niet wit gewonnen, zoals veel theorieboeken beweren (zie Matanovic, 1981, voetnoot 109 bij variant C05-21), doch zwart, omdat zwart na 9.Dh5† - g6.10.Lg6† - hg6.11.Dg6† - Ke7 over het tegenoffer Pd7 xe5 beschikt, bijvoorbeeld 12.Pc4 - Pd7e5!, als in de partij H. Otten - P. Wakker (1982, Leiden, 3e ronde Notenboomtoernooi), of 12.Pf3 - Pd7e5! 13.Lg5† - Kd7.14.de5 - Le7.15.h4 - Dg8, of 12.Pe4!? (H.J. Goeman) - Pd7e5.13.Lg5† - Kd7.14.Pf6† - Kc7.15.Pe8† (15.de5 - Pe5) - Kd7. 16.Pf6† - Kc7.17.Pe8† - De8! 18.De8 - Lg7.

Matanovic, A. (Ed., 1981), "*Encyclopedia of Chess Openings C, Vol I.*" Batsford, Londen.

Met dank aan internationaal schaakgrootmeester John van der Wiel voor het controleren en goedbevinden van bovenstaande stelling (en voor het weerleggen van enige andere "nieuwtjes").

STELLING 10. Voor het verkrijgen van zelfkennis is het bezit van een geweten een hinderpaal.