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FACULTEIT DER ECONOMISCHE WETENSCHAPPEN

KATHOLIEKE HOGESCHOOL TILBURG

REEKS "TER DISCUSSIE"

No. 77.058

June 1977

CAPITAL MARKET MODELS FOR PORTFOLIO SELECTION
(A REVISED VERSION)

R. Heuts

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T Portfolio Investment

FACULTEIT DER ECONOMISCHE WETENSCHAPPEN

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Chapter 1: Capital market models for portfolio selection

§ 1.1. Introduction

Starting with the traditional Markowitz portfolio analysis, we then introduce the capital market theory and show its importance for the portfolio theory. Therefore we shall review the development of the modern capital market theory and some empirical evidence bearing on this theory.

We shall see that the rates of return of the so called "market portfolio" plays an important role in two of those models and that information about this factor is of importance in some portfolio models. In the next chapter we shall then test if essential information can be gained about the future behavior of the rates of return of this market portfolio.

§ 1.2. The portfolio problem

We say that we have a portfolio problem if:

- a. A finite number of investment opportunities is available.
- b. The available capital can be divided up in any way among the investments.
- c. All investments must be held for a certain period of time at the end of which they must be sold or might be sold because the transaction cost of sale is negligible.
- d. The return per dollar on the i 'th investment does not depend on the amount invested in the i 'th or any other investment.
- e. The preference depends only on the return on capital over the period of time in question. For investments in securities, return consists of capital payments plus appreciation in price. Let ℓ be the number of investments available and let \underline{r}_i be the rate of return per dollar of the i 'th investment, $i = 1, \dots, \ell$. Since \underline{r}_i is not assumed known a priori, it is a stochastic variable.

The decision maker must choose the fraction of the available capital to be invested in each investment. Thus an act leads to a portfolio that can be viewed as a vector $X' = (x_1, \dots, x_\ell)$, where x_i is the fraction invested in the i 'th investment.

We have:

$$x_i \geq 0, \quad \sum_{i=1}^{\ell} x_i = 1.$$

The portfolio has a rate of return per dollar invested of

$$(1.2.1) \quad \underline{p} = X' \underline{R}, \text{ where } \underline{R}' = (\underline{r}_1, \dots, \underline{r}_\ell).$$

The optimal act maximizes $\mathcal{E}\{u(\underline{p})\} = \mathcal{E}\{u(X' \underline{R})\}$, where $u(\cdot)$ is the decision maker's utility function for return.

It is natural to consider the mean and variance of the rate of return \underline{p} of the portfolio. They are

$$(1.2.2) \quad \begin{aligned} \mu_{\underline{p}} &= \mathcal{E}(\underline{p}) = X' E \\ \sigma_{\underline{p}}^2 &= \text{Var}(\underline{p}) = X' \Sigma X, \end{aligned}$$

where $E' = (\mu_1, \dots, \mu_\ell)$, $\mu_i = \xi(\underline{r}_i)$, $i \neq p$, and Σ is the covariance matrix of investment rates of return.

Without some assumption, such as joint normality of the rates of return \underline{r}_i , we cannot assert that the mean and the variance of the portfolio rate of return \underline{p} completely characterise its probability distribution.

§ 1.3. The Markowitz mean-variance approach (determining efficient portfolios)

Accepting the expected utility maximum as the objective function, Markowitz [34, 35] and Tobin [48, 49] have shown that diversification is a logical consequence for risk-averse investors. In particular the expected utility maximizing portfolio will be Markowitz efficient¹⁾ if

(1) the decision maker's expected utility is equal to:

$$\xi(u(p)) = \psi(\mu_p, \sigma_p),$$

where $\psi(\mu_p, \sigma_p)$ is a preference function which is defined in terms of expectation and standard deviation of p .

(2) the statistical distributions of the rates of return of all possible portfolios belong to the same class of two-parameter distributions (see e.g. Samuelson [40] and Feldstein [18]) or where the utility function to be maximized is quadratic (see e.g. Tobin [48]);

(3) the investor's preference function $\psi(\mu_p, \sigma_p)$ satisfies the conditions $\frac{\partial \psi(\mu_p, \sigma_p)}{\partial \mu_p} \geq 0$ and $\frac{\partial \psi(\mu_p, \sigma_p)}{\partial \sigma_p} \leq 0$ (corresponds with a risk-averse, see e.g. van Lieshout [29]).

Chipman [8] has shown that it is necessary to impose certain conditions on the utility function to ensure that the expected utility exists. He has specified these conditions for certain probability distributions.

Although there may be distributions other than the normal²⁾ which satisfy the first part of requirement (2), it is unreasonable to assume that investor's rates of return distributions are necessarily of such a form.

1) A portfolio X_* is Markowitz efficient if:

- a. of all possible portfolios with the same or greater expected rate of return than portfolio X_* , this portfolio has smallest variance; and at the same time;
- b. of all possible portfolios with the same or smaller variance of the rate of return than portfolio X_* , this portfolio has greatest expected rate of return.

2) e.g. the two-parameter Cauchy-distribution.

Markowitz and Tobin realized that their results are crude approximations, but they also realized that approximate and computable results are better than none.

Figure 1 gives a graphical representation of the Markowitz approach in the mean-standard deviation plane.

The shaded area in Figure 1 represents all possible combinations of portfolio risk and expected rates of return from investments in risk-bearing securities. The portfolios on the boundary a b c d represent the set of Markowitz efficient portfolios, since they all represent possible investments yielding maximum expected rates of return for given risk and minimum risk for given expected rates of return.

As for a risk averse person $\frac{\partial \psi(\mu_p, \sigma_p)}{\partial \mu_p} \geq 0$ and $\frac{\partial \psi(\mu_p, \sigma_p)}{\partial \sigma_p} \leq 0$ hold, the preference increases toward the east and south in Figure 1.

To explain the pattern of $\psi(\mu_p, \sigma_p) = c$ in the $\mu_p - \sigma_p$ space we can proceed as follows. Assume that the expected utility is linear in the expected portfolio rate of return and the variance of the portfolio rate of return:

$$\xi(u(\underline{p})) = \psi(\mu_p, \sigma_p) = \mu_p - \lambda \sigma_p^2,$$

where λ is a parameter for risk orientation.³⁾ In the $\mu_p - \sigma_p$ space in Figure 1 $\sigma_p = f(\mu_p)$ is a parabola for constant expected utility.⁴⁾ Inspection of Figure 1 shows that the $\mu_p - \sigma_p$ space, the constant utility parabolas become relatively close together at higher values of μ_p .

3) It can be shown (e.g. Chipman [8], Schneeweiss [41] that in the case of a family of normal distributions for the portfolio rate of return \underline{p} , and with expected utility function

$\xi(u(\underline{p})) = \psi(\mu_p, \sigma_p) = \mu_p - 1/2 \lambda \sigma_p^2, \lambda > 0$, the utility function $u(\underline{p})$ is up to a monotonic non decreasing transformation of the form: $u(\underline{p}) = -e^{-\lambda \underline{p}}$. The coefficient λ is called the risk aversion coefficient by Farrar [16].

4) More generally Fama and Miller [15] have shown the following. Given positive but diminishing marginal utility of wealth and a multivariate normal distribution for the rates of return to the available assets, ψ is a concave function in the (μ_p, σ_p) plane. Feldstein [18] has presented an example for a twoparameter distribution (lognormal distribution) and an utility function which has the above properties, for which ψ is not everywhere concave in the (μ_p, σ_p) plane.

The shaded area in Figure 1 represents the opportunity set in the absence of a riskless asset, and the boundary abcd represents the set of Markowitz efficient portfolios.

An investor, who is a risk-averter and who has the preference functions as indicated in Figure 1, and only invests in risky assets, will maximize his expected future utility with an investment in portfolio b, yielding μ_{p_b} and σ_{p_b} with expected utility $\psi_1(\mu_{p_b}, \sigma_{p_b})$.

Assume now the existence of a risk-free asset f, yielding a certain future rate of return r_f .

Let's assume that the investor invests a fraction α in the portfolio p_c and the rest, $(1-\alpha)$, in the risk-free asset. The expected value and the variance of the rate of the combined portfolio are $\mu_p = \alpha \mu_{p_c} + (1-\alpha)r_f$, $\sigma_p^2 = \alpha^2 \sigma_{p_c}^2$. So now all portfolios lying below point c along abcd in Figure 1 are inefficient, since any point on the line

$$(1.3.3) \quad \mu_p = r_f + \frac{\mu_{p_c} - r_f}{\sigma_{p_c}} \sigma_p, \quad \sigma_p < \sigma_{p_c},$$

represents a feasible solution.

The investor will now distribute his funds between portfolio c and the risk-free asset f, such that his combined portfolio yields him maximum expected utility $\psi_1(\mu_{p_b}, \sigma_{p_b}) < \psi_2(\mu_p, \sigma_p)$.

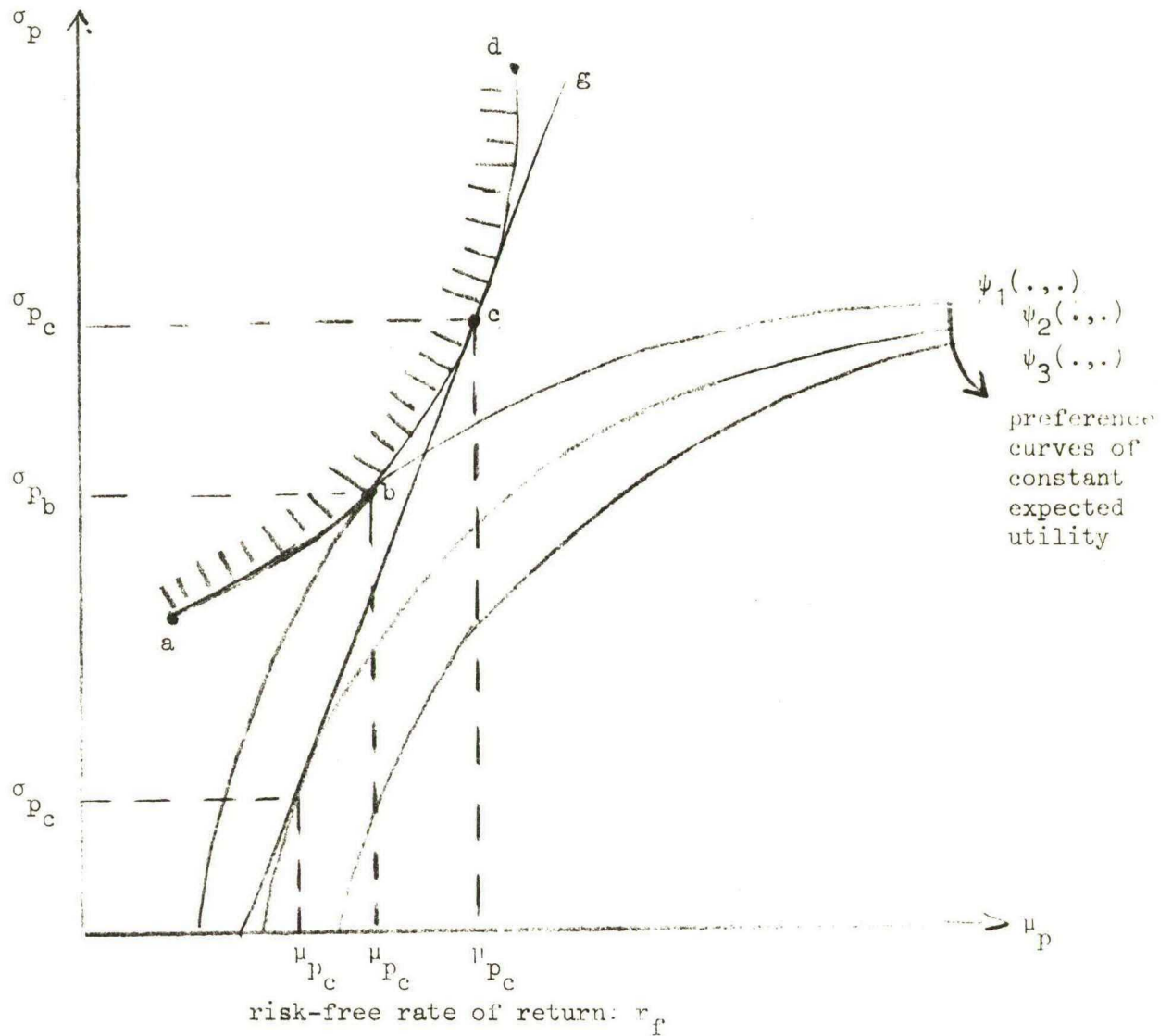


Fig.1. The maximization of utility, with and without the existence of a risk-free asset.

To find Markowitz efficient portfolios the following data are required:

- a. the expected security rates of return, μ_i , $i = 1, \dots, l$;
- b. the variances of the security rates of return, $\sigma_{r_i}^2$, $i = 1, \dots, l$;
- c. the covariances of the security rates of return r_i and r_j , $\sigma_{i,j} := \text{cov}(r_i, r_j)$, $i, j = 1, \dots, l$.

From the definition in footnote 1 a pair (μ_p, σ_p^2) is called Markowitz efficient if σ_p^2 has minimum variance, given μ_p , and μ_p is the maximum expected rate of return, given σ_p^2 . We would like to determine all the efficient pairs (μ_p, σ_p^2)

and the corresponding X which yield them. It can be shown that the following computational technique can be used to obtain them.

Under the assumption that $\mu_p = X'E$ and $\sigma_p^2 = X'\Sigma X$ are known as function of the portfolio X , $X = X_*$ is then and only then a Markowitz-efficient portfolio, when there is a $\lambda > 0$, such that $X = X_*$ is the solution of $\max_{X \in \mathcal{X}} \{\lambda X'E - X'\Sigma X\}$, with the restriction that the feasible region \mathcal{X} is a convex set, so that the Kuhn-Tucker theory is applicable.

However, to reach a solution it is devised that the conditions on the variables are linear. Mostly the conditions which are supposed are

1.
$$\sum_{i=1}^{\ell} x_i = 1$$

2.
$$x_i \geq 0, \text{ for all } i.$$

For an elegant proof of the above, where Σ is assumed to be positive semi-definite, see for example lemma 2.1. in the Ph. D. Thesis of van Lieshout [29]. The algorithm which Hadley [21] suggests to find the efficient portfolios is not correct, because it also delivers inefficient portfolios.

To solve the above quadratic programming problems, the security analyst must provide estimates of ℓ expected rates of return, ℓ variances and $\frac{\ell(\ell-1)}{2}$ covariances of rates of return (beside historical information probably also using subjective information).

When partly risk-free portfolios are also possible, r_f is assumed to be known. So when ℓ is large, many parameters have to be estimated and finding efficient portfolios is also time consuming. To solve this problem the so-called market or single index model is introduced. To better understand the background of this type of portfolio models, we first review and comment on some descriptive models for the capital market.

§ 1.4. Capital asset rate pricing models

§ 1.4.1. The one-factor capital asset pricing model

The general equilibrium models of asset prices derived by Sharpe [44], Lintner [30,31], Mossin [37], and Fama [14] are an investigation of the implications of the Markowitz model for the equilibrium structure of asset prices. In short, market equilibrium simply requires a set of asset prices such that supply equals demand for every asset. The above authors all consider the relationship in equilibrium between a measure of asset risk and its one-period expected rate of return. In equilibrium, capital asset prices have adjusted so that the investor, if he follows rational procedures (primarily diversification), is able to attain any desired point along a capital market line, which relates the expected value and the standard deviation of a portfolio rate of return. He may obtain a higher expected rate of return on his holdings only by taken additional risk.

According to Sharpe the market for capital assets is out of equilibrium unless the portfolio c is Figure 1 is the "market portfolio", that is, a portfolio which consists of all assets in the market, each entering the portfolio with weight equal to the ratio of its total market value to the total market value of assets.

If the market were out of equilibrium, the prices of assets in portfolio c would be bid up and the prices of assets not in c would fall.

The above authors all involve either explicitly or implicitly the following assumptions.

1. All investors maximize the single period expected utility of terminal wealth and choose among alternative portfolios on the basis of expected value and variance of rates of return.
2. All investors can borrow or lend an unlimited amount at a given risk-free rate of interest.
3. All investors have identical estimates of the expected values, variances and covariances of the rates of return of all assets.
4. There are no transaction costs, securities are completely divisible and information is costless available for everyone.
5. The tax rate is constant for all investor in the market.
6. Everyone in the market has the same opportunities to invest, although the amount to be invested may differ from person to person.

The assumptions underlying the model seem to be restrictive; however, the model is useful to understand some forces which affect asset prices. At best this world is an idealization of the actual case, but it may serve as a useful approximation of reality and will permit us to focus later on the implications of departures from the ideal case.

We shall now briefly present the reasoning behind the results of the equilibrium models.

Portfolios along $r_f c$ are formed according to

$$p = x r_f + (1-x) p_c, \quad x \leq 1.$$

Thus when it is possible both to borrow and lend at the rate r_f , the only difference between any two efficient portfolios is in the proportion x invested in the riskless asset f . More risky efficient portfolios - those above c on the efficient set line in Figure 1 - involve borrowing ($x < 0$) and investing all available funds, including borrowings, in the risky combination c . Less risky portfolios - those along the line segment $r_f c$ - involve lending ($1 \geq x \geq 0$) some funds at r_f and investing remaining funds in c .

The particular portfolio that an investor chooses depends on this attitudes toward risk and expected rate of return, but efficient portfolios for all investors are just combinations of f and c .

Under the assumption that all investors in the market have the same expectations regarding the risk and expected rate of return from portfolios and that they can freely borrow and lend at a rate r_f , the portfolio c is the same for all investors.

Everyone will risk to hold c in some combination, since everyone agrees that c is best, except those highly risk-averse persons who will only invest in the riskless asset.

If everyone wants to hold the same portfolio c , in order for the market to be in equilibrium, that portfolio must contain all the securities in the market, because all securities must be held by someone. If some securities were not in c , then their prices would fall, thereby increasing their expected rate of return, until they became desirable and were included in c . Since in equilibrium all securities in the market are in c , the weight of each security must be equal to the ratio of its total market value to the total market value of all assets. If the market is to be in equilibrium so that no one wishes to change this holdings of any security, c must be the market portfolio.

In other words, in equilibrium the only risky asset held by investors is a portfolio m which contains all the risky securities in the market in a proportion as given above.

In addition, in the equilibrium situation the risk-free rate of interest r_f must be such that net borrowings are 0; that is, at the rate r_f the total quantity of funds that people want to borrow is equal to the quantity that others want to lend.

The line passing through r_f and m as shown in Figure 2, will be called the capital market line (CML) and can be expressed as (see equation (1.3.3))

$$(1.4.1.1) \quad \mu_{r_i} = r_f + \lambda \sigma_{r_i},$$

or

$$(1.4.1.2) \quad \sigma_{r_i} = \frac{1}{\lambda} \mu_{r_i} - \frac{r_f}{\lambda},$$

where $\frac{1}{\lambda}$ is slope of the CML in Figure 2.

Since the CML passes through the point $(\mu_{p_m}, \sigma_{p_m})$, $\frac{1}{\lambda} = \frac{\sigma_{p_m}}{\mu_{p_m} - r_f}$ in equilibrium.

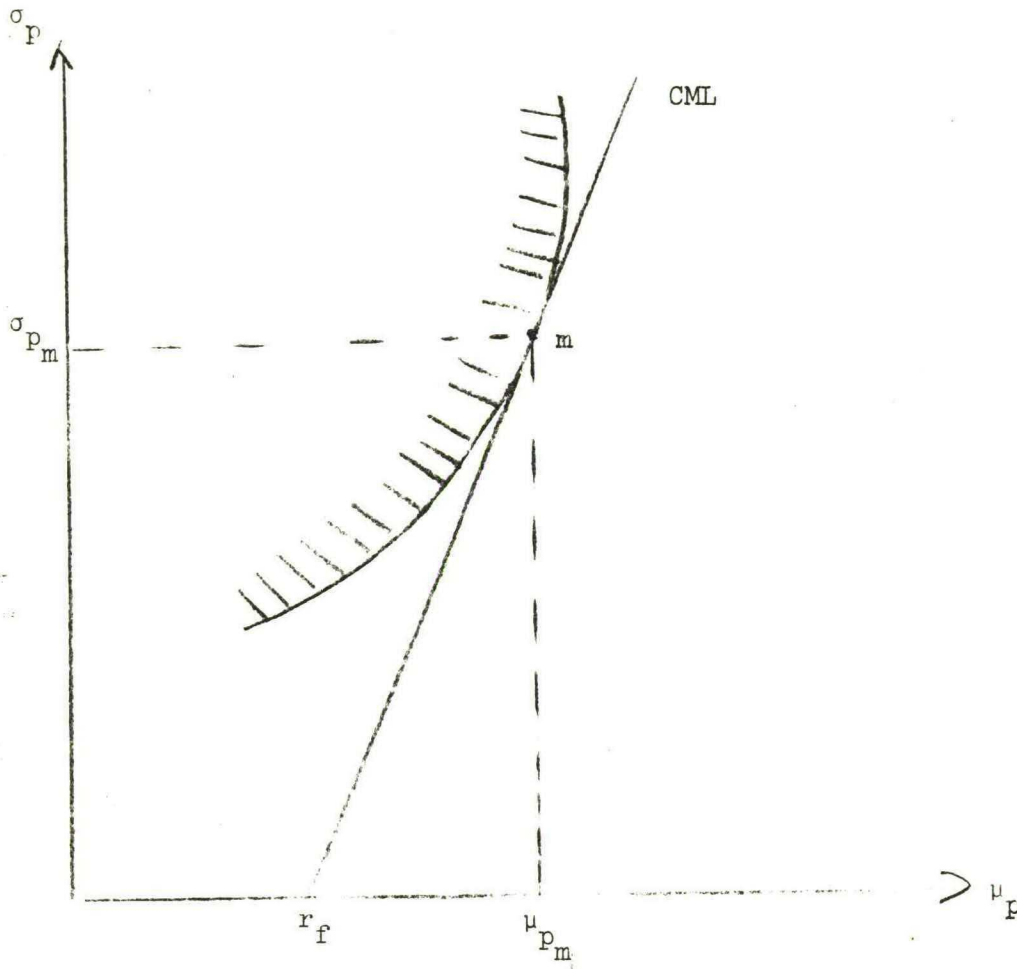


Figure 2: Capital market line.

The CML provides the equilibrium relationship for efficient combinations, but does not say anything about the expected rate of return on inefficient portfolios or individual securities.

The equilibrium conditions for inefficient portfolios are essentially a logical extension of the preceding ideas about the capital market line. A formal derivation will be given now.

The capital market line is as we have seen tangent at the point m to the frontier ($a m a'$) of risky efficient portfolios as shown in Figure 3.

Let us start by putting a fraction x_i of funds into an arbitrary asset i and the rest $(1-x_i)$ into the market portfolio m . The expected value and variance of the rate of return of the portfolios along ($b m b'$) can be expressed as

$$(1.4.1.3) \quad \mu_p = x_i \mu_{r_i} + (1-x_i) \mu_{p_m}$$

$$(1.4.1.4) \quad \sigma_p^2 = x_i^2 \sigma_{r_i}^2 + (1-x_i)^2 \sigma_{p_m}^2 + 2x_i(1-x_i) \sigma_{r_i, p_m},$$

where $\sigma_{r_i, p_m} = \text{cov}(r_i, p_m)$.

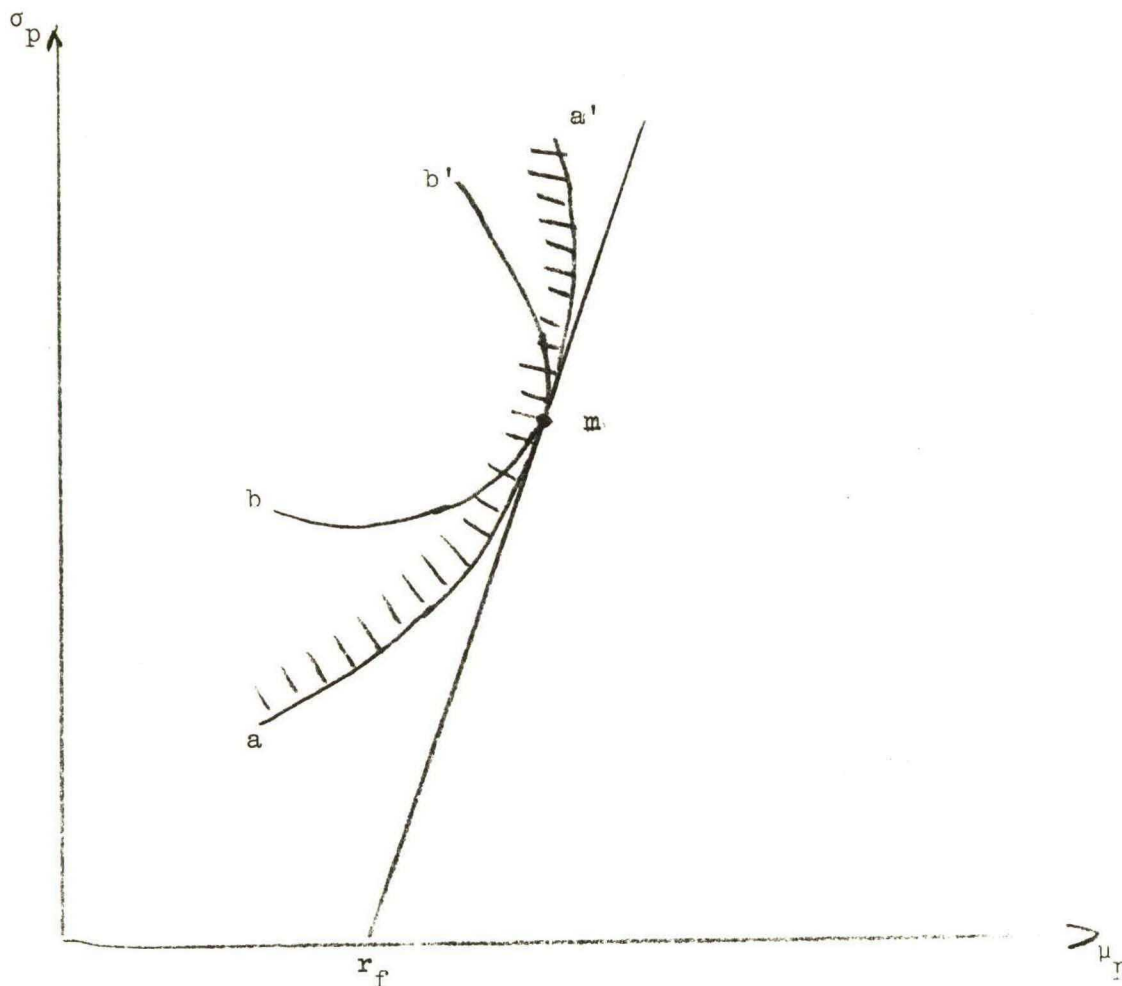


Figure 3: Security market line.

In Figure 3 we illustrate the typical relationship between a single capital asset (point b') and an efficient combination of assets (point m) of which it is a part. The curve $(b' m b)$ indicated all μ_p, σ_p values which can be obtained with feasible combinations of asset b' and the efficient portfolio m . The frontier $(b m b')$ is drawn so that it is tangent to the efficient frontier $(a m a')$ at point m . The two curves touch at m , because we look to that inefficient portfolios which are found by a feasible combination of a single

capital asset and the efficient portfolio m . However, the line $(b m b')$ can not pass through $(a m a')$ because this would mean that $(a m a')$ is not efficient. Now the capital market line is tangent to $(a m a')$ and $(b m b')$ at m . Therefore, the slope of the capital market line must equal the slope of $(b m b')$ at m .

So, we have to determine the slope of $(b m b')$ at m and set it equal to $\frac{1}{\lambda}$. The slope of $(b m b')$ is the derivative of σ_p with respect to μ_p , $\frac{d\sigma_p}{d\mu_p}$.

As

$$(1.4.1.5) \quad \frac{d\sigma_p}{d\mu_p} = \frac{\partial \sigma_p}{\partial x_i} \frac{dx_i}{d\mu_p}$$

we get from the equations (1.4.1.3) and (1.4.1.4)

$$(1.4.1.6) \quad \left\{ \begin{array}{l} \frac{\partial \sigma_p}{\partial x_i} = \frac{x_i(\sigma_{r_i}^2 + \sigma_{p_m}^2 - 2\sigma_{r_i, p_m}) + \sigma_{r_i, p_m} - \sigma_{p_m}^2}{\sigma_{p_m}} \\ \frac{dx_i}{d\mu_p} = \frac{1}{\mu_{r_i} - \mu_{p_m}} \end{array} \right.$$

We want to know the slope at m . At m , $\sigma_p = \sigma_{p_m}$ and $x_i = 0$; therefore

$$(1.4.1.7) \quad \left(\frac{\partial \sigma_p}{\partial x_i} \right)_{x_i=0} = \frac{\sigma_{r_i, p_m} - \sigma_{p_m}^2}{\sigma_{p_m}}$$

We now substitute the results of (1.4.1.6) and (1.4.1.7) into (1.4.1.5) to get the slope of $(b m b')$ at m :

$$(1.4.1.8) \quad \left(\frac{d\sigma_p}{d\mu_p} \right)_m = \frac{\sigma_{r_i, p_m} - \sigma_{p_m}^2}{\sigma_{p_m}(\mu_{r_i} - \mu_{p_m})}$$

Now the slope of the capital market line, $\frac{1}{\lambda}$, must be equal to the slope of $(b m b')$ at m , that is

$$(1.4.1.9) \quad \frac{\sigma_{r_i, P_m} - \sigma_{P_m}^2}{\sigma_{P_m} (\mu_{r_i} - \mu_{P_m})} = \frac{1}{\lambda},$$

and solving equation (1.4.1.9) for μ_{r_i} we get

$$\mu_{r_i} = \mu_{P_m} + \lambda \frac{\sigma_{r_i, P_m} - \sigma_{P_m}^2}{\sigma_{P_m}}.$$

From (1.4.1.1) we also know that

$$\mu_{P_m} = r_f + \lambda \sigma_{P_m},$$

and therefore,

$$(1.4.1.10) \quad \mu_{r_i} = r_f + \lambda \frac{\sigma_{r_i, P_m}}{\sigma_{P_m}} = r_f + \frac{(\mu_{P_m} - r_f)}{\sigma_{P_m}^2} \sigma_{r_i, P_m},$$

which will be called the security market line.

Equation (1.4.1.10) can also be written as

$$\mu_{r_i} = r_f + \lambda \rho_{r_i, P_m} \sigma_{r_i}$$

where

ρ_{r_i, P_m} = the correlation between the rate of return on security i and the rate of return on the market portfolio.

The relation in (1.4.1.10) shows that in equilibrium the expected rate of return on risky assets is a function of the risk-free rate of interest plus a premium for risk where the risk is measured by the covariance of the asset's rate of return with the market rate of return.

As efficient portfolio combinations are perfectly correlated with the market, $\rho_{r_i, P_m} = 1$, (1.4.1.11) reduces to the capital market line in equation (1.4.1.1).

In the literature equation (1.4.1.10) is often written as

$$(1.4.1.12) \quad \mu_{r_i} = r_f + \beta_i \{ \mu_{p_m} - r_f \}$$

where

$$\beta_i = \frac{\sigma_{r_i, p_m}}{\sigma_{p_m}^2}$$

is called the systematic risk or the beta-coefficient of the i-th asset, measuring the risk that can not be eliminated by diversification. Non-systematic risk can be eliminated by diversification.

Summarizing, the important implications of the above model are:

1. risk averting investors expect more return by increasing risk, so the expected risk premium is positive,
2. β_i is the only risk measure on the capital market,
3. the expected rate of return of an investment is a linear function of systematic risk.

The characteristics of the market-clearing or equilibrium set of prices are as follows. Given that there is complete agreement among investors with respect to the joint distribution of future security rates of return, when the market auctioneer calls out a tentative set of security prices and a tentative risk-free rate, there is a tangency portfolio like m in Figure 3 that all investors try to combine with f. Some investors want to combine the tangency portfolio with borrowing at the tentative risk-free rate, while others want to combine it with lending. A market equilibrium requires a market-clearing set of prices; that is, a market equilibrium requires that investors demand all securities and demand them in the proportions in which they are outstanding in the market. The market-clearing condition means that a market equilibrium is not attained until the one tangency portfolio that all investors try to combine with risk-free borrowing or lending is a portfolio of all the positive variance securities in the market where each security is weighted by the ratio of the total market value of all its outstanding units to the total market value of all outstanding units of all securities. In short market equilibrium is not reached until the tangency portfolio m is the value weighted version of the market portfolio. So a market equilibrium means a set of prices and a value of r_f that clears the borrowing-lending market

and has aggregate investor demands for securities equal to outstanding supplies and this requires that the tangency portfolio is the value weighted version of the market port-folio.

A great advantage of the result in equation (1.4.1.10) is that the market equilibrium prices (or expected rates of return) are solely a function of potentially measurable market parameters. Thus the model is potentially testable.

An important generalization of the formula for the equilibrium expected rate of return on the i 'th security is developed by Jensen [23]. After some manipulations Jensen shows that to a close approximation the equilibrium expected rate of return on the i 'th security is given by

$$(1.4.1.13) \quad E(r_i | p_m = p_m) \approx r_f(1-\beta_i) + \beta_i p_m$$

where p_m is the rate of return of the market portfolio.

Equation (1.4.1.13) gives an expression for the expected rate of return on security i conditional on the ex post realization of the rate of return on the market portfolio. The result of the capital asset pricing model in equation (1.4.1.12) provides only an expression for the expected rate of return on the i 'th security conditional on the ex ante expectation of the rate of return on the market portfolio. So, Jensen has shown that one can explicitly use the observed realization of the rate of return on the market portfolio without worrying about using it as a proxy for the expected return.

In empirical studies (Sharpe [43], Blume [4]) one uses the model

$$(1.4.1.14) \quad r_{t,i} = \alpha_i + \beta_i p_{t,m} + u_{t,i} \quad , \quad i = 1, \dots, \ell, \quad t = 1, 2, \dots$$

where

$r_{t,i}$ is the rate of return on asset i for period t ,

$p_{t,m}$ is the rate of return on the market portfolio for period t ,

$u_{t,i}$ is the residual term for security i for period t ,

and $\alpha_i \stackrel{>}{<} (1-\beta_i)r_f$.

So, α_i is unconstrained to an equilibrium value.

Black, Jensen and Scholes [3] in a recent study present time series analysis tests of the capital asset pricing model.

They argue that if the market model which is based on the capital asset pricing model, is valid, the rates of return on securities will be generated by

$$(1.4.1.15) \quad \underline{r}_{t,i} = (1-\beta_i) \underline{r}_{t,f} + \beta_i \underline{p}_{t,m} + \underline{u}_{t,i}.$$

If we subtract $\underline{r}_{t,f}$ from both sides of equation (1.4.1.15) and use primes to denote differences between the return on any asset and the risk-free rate of interest, we obtain

$$(1.4.1.16) \quad \underline{r}'_{t,i} = \beta_i \underline{p}'_{t,m} + \underline{u}_{t,i}$$

The model can be tested by running a time series regression given by (1.4.1.16) but allowing a constant term α_i to enter:

$$(1.4.1.17) \quad \underline{r}'_{t,i} = \alpha_i + \beta_i \underline{p}'_{t,m} + \underline{u}_{t,i}.$$

Black, Jensen and Scholes apply tests to ten portfolios which contain all securities on the New York Stock Exchange in the period 1931-1965.

The results indicate that the α 's are non-zero and are directly related to the risk level β : that high-beta ($\hat{\beta} > 1$) assets tend to have negative α 's, and that low-beta ($\hat{\beta} < 1$) stocks tend to have positive α 's. Thus the high-risk securities earned less on the average over the period considered than the amount predicted by the traditional form of the asset pricing model. At the same time, the low-risk securities earned more than the amount predicted by the model. Further the risk-parameters, β_i , are fairly stationary through time but there is an indication that the α 's are not.

This evidence seems to indicate that the capital asset pricing model in its most simple form does not provide an adequate description of the process generating rates of return.

§ 1.4.2. The two-factor capital asset pricing model

The principal conclusions of the capital asset pricing model previously developed are as follows:

- a. In equilibrium every investor should be expected to hold a combination of the riskless asset and the market portfolio. Such combinations of assets dominate any other alternatives in the sense that they are subject to less risk for the same level of the expected rate of return.
- b. In equilibrium the expected rate of return on each stock in excess of the risk-free rate is related only to its beta. Mathematically, the relationship for any stock can be described by equation (1.4.1.12).

The first conclusion has a normative character, because it describes how a rational investor should behave.

The second conclusion is of a descriptive nature, it predicts how an equilibrium relation would appear if the assumptions of the model are fulfilled. The principal conclusion of the study of Black, Jensen and Scholes [3] is that while the relationship between expected excess rate of return of a stock or portfolio and its systematic risk is linear, it is not directly proportional.

The data indicate that the expected rate of return on a security can be represented by a model of the form

$$(1.4.2.1) \quad \mu_{r_i} - r_f = \delta + \beta_i \{\mu_{p_m} - r_f - \delta\} = \delta(1 - \beta_i) + \beta_i \{\mu_{p_m} - r_f\}$$

where δ is a positive quantity.

If the excess rates of return on an asset are regressed against the market excess rates of return

$$(1.4.2.2) \quad \underline{r}'_{t,i} = \alpha_i + \beta_i \underline{p}'_{t,m} + \underline{u}_{t,i},$$

then the estimate of β_i is an estimate of the systematic risk, and the regression coefficient α_i can be called the abnormal rate of return, which is the additional rate of return left after the stock's rate of return is adjusted for its systematic risk by subtracting the factor $\beta_i \underline{p}'_{t,m}$ from the total excess rate of return.

The simple capital asset pricing model says that there should be no expected abnormal rates of return, or $\alpha_i = 0$.

However in reality we observe that $\alpha_i = \delta(1-\beta_i)$, where δ is a positive quantity. So the actual behavior of capital markets differs in an important aspect from what it should be by the simple efficient market model. The assumption in that model that each investor is able to borrow without limitation at a rate equal to the rate on the riskless asset, does not correspond to actual behavior.

Generally, borrowers pay more than the risk-free rate f_r . The question then arises as to whether removing this unrealistic assumption of unlimited borrowing at the risk-free rate would produce a model that is in better argument with the empirical data.

As borrowing or lending is also never completely risk-free, Black [2] has demonstrated that one can obtain an equilibrium relationship for all assets in a market where no risk-free asset or borrowing or lending opportunities exist (but there are no restrictions on shortselling).⁵⁾

He was able to show that in equilibrium the expected rate of return on an asset will be given by

$$(1.4.2.3) \quad \mu_{r_i} = \mu_{p_z} (1-\beta_i) + \beta_i \mu_{p_m}$$

where μ_{p_z} is the expected rate of return on the so-called "zero-beta" factor, since the rate of return on this factor has zero covariance with the rate of return of the market portfolio, p_m . The relation in (1.4.2.3) is often called

5) The mechanics of shortselling are as follows. To shortsell the shares of firm a, the investor borrows the shares from someone who owns them at time 1, agreeing to return the shares at time 2 along with any dividends paid at time 2. Upon borrowing the shares, the investor immediately sells them in the market and uses the proceeds from the sale to increase his investment in b. At time 2 the investor pays his debt to the lender of the shares of firm a by selling his holdings in b and using the proceeds to purchase the shares of a which are to be returned to the lender. When the investor borrows the shares of firm a and sells them in the market at time 1, he is said to have a "short" position in the shares. He "covers" his short positions when he comes back and purchases shares of a at time 2 and returns them to the lender. In contrast, an investor who owns the shares of firm a has a "long" position in the shares.

the two-factor model.

So Black has shown that in equilibrium every investor holds a linear combination of two basic portfolios, and one of these two portfolios can be taken to be the market portfolio m , and the other portfolio is one whose rate of return has zero covariance with the market portfolio.

In addition it can be shown that this portfolio is, of all possible zero-covariance portfolios, the one with minimum variance. The reason for this separation property is that, given no constraints on shortselling, the entire efficient set of portfolios can be generated by a linear combination of these two portfolios (see Figure 4). Specifically the returns on different efficient portfolios p_c can be obtained by varying x in

$$p_c = x p_z + (1-x) p_m.$$

With $x = 1.0$ we get z , while with $x = 0.0$ we get m . Portfolios between z and m have $0 < x < 1$, that is, positive fractions of portfolio funds are invested in both z and m . Points above m on the curve involve shortselling of z , that is, $x < 0.0$.

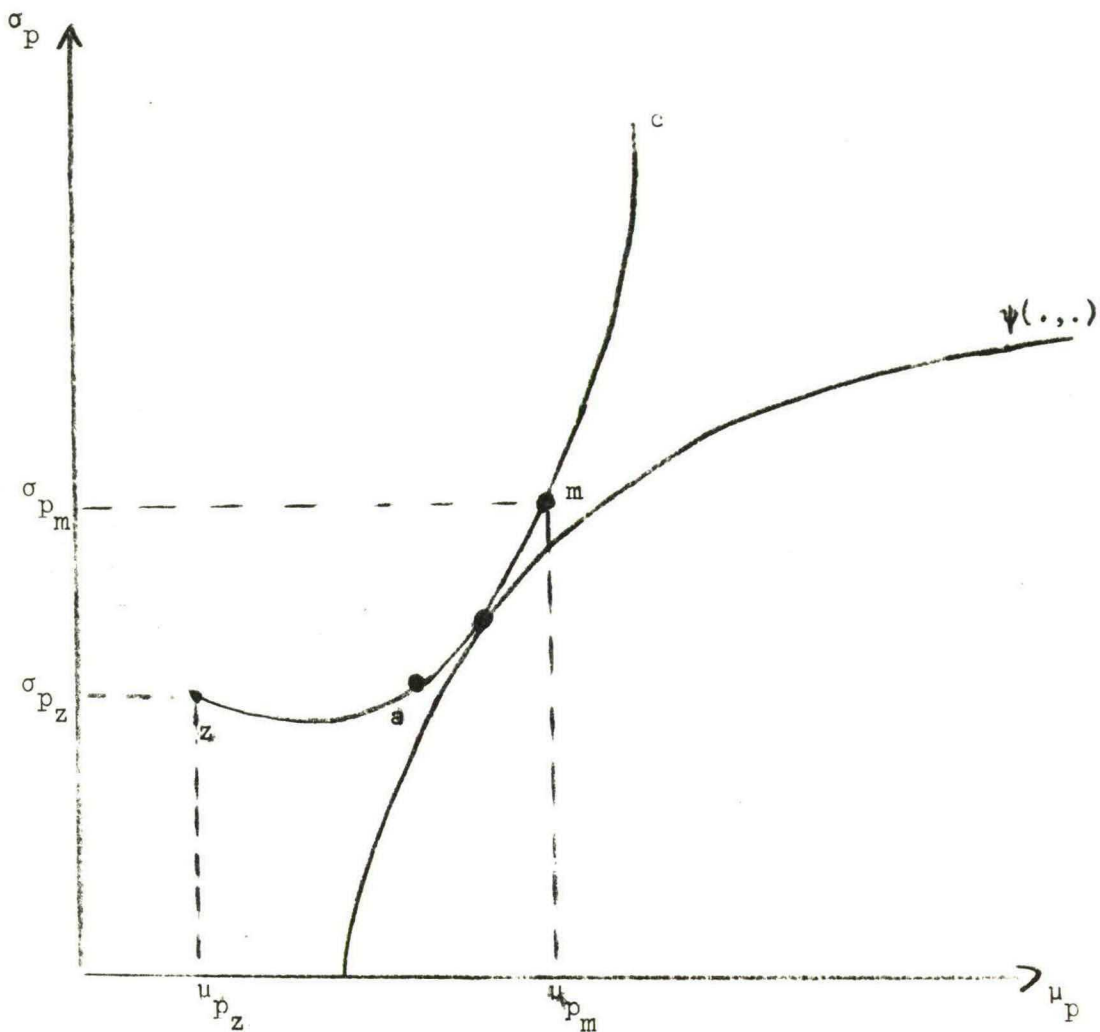


Figure 4: Equilibrium portfolio choice in the absence of risk-free borrowing or lending.

All portfolios in the range a m c are efficient in the Markowitz sense. Now each investor maximizes his utility by purchasing that combination of z and m at which his preference curve between expected return and standard deviation is just tangent to the efficient set.

Summarizing we can say that the expected rate of return on a security is still a linear function of the systematic risk factor β_i , and that the zero beta portfolio has taken on the role previously played by the risk free asset f. Black, Jensen and Scholes [3] found by time series analysis tests that a model that expresses the rate of return on a security as a linear function of the market portfolio p_m (with a coefficient of β_i) and a second factor, p_z (with a coefficient $1-\beta_i$) is a fairly accurate description of the process generating security returns.

§ 1.4.3. Further extensions of the capital asset pricing model

A number of authors have expanded the simple capital asset pricing model by relaxing some of its underlying assumptions.

Vasicek [50] dealt with the case when the riskless asset is available for investment, but investors can not borrow at the risk-free rate. He has demonstrated⁶⁾ that the equilibrium risk expected rate of return relationship for individual risky securities corresponds to that given by (1.4.2.3).

Brennan [6] derives equilibrium conditions when investors are faced with differential borrowing and lending rates.

He considers the cases: (1) all investors can borrow at a riskless rate r_b and lend at a riskless rate r_l , where $r_b > r_l$ and (2) each investor, i , is faced with different riskless borrowing and lending rates, where $r_{b_i} > r_{l_i}$.

In both cases, Brennan finds that the relationship between expected rate of return and risk remains linear. The only difference in the market equilibrium condition in going from equal borrowing and lending rates to differential rates is that the intercept of the capital market line is shifted.

Mayers [36] has studied the situation when the assumption that all assets are marketable and there are no transaction costs is violated.

This is of importance, because most investors also hold nonmarketable claims on future income (labor income, social security payments, etc.) and they cannot sell these claims in current markets. In addition there are many physical assets, such as real estate, for which transaction costs are relatively large. He considers marketable and nonmarketable assets, which means that, although all investors have the same opportunity set of marketable assets, two investors holding identical portfolios of marketable assets could have different probability distributions on total wealth.

Mayers shows that the structure of asset rates of return given the existence of non-marketable assets is identical to that we could obtain if all assets were actually marketable.

Further he demonstrates that the nature of the results remains unchanged when there is no riskless borrowing or lending.

6) For an excellent exposition of Vasicek's results we refer to Jensen [24].

Brennan⁷⁾ demonstrates under the assumptions of the simple capital asset pricing model that the equilibrium price of an asset still can be expressed as a linear function of its systematic risk β_i even when investors are faced with differential tax rates on dividend and capital gains if dividend receipts are perfectly certain.

The introduction of differential taxes on capital gains and dividends only changes the intercept of the equilibrium risk expected rate of return relationship and introduces a new variable, the dividend yield on the market portfolio.

Brennan concludes that his model fits the observed data better than does the simple model.

Lintner [32] has relaxed the assumption of complete agreement among investors with respect to the joint distribution of the rates of returns of the securities and shown that in the special case in which all investor's preference functions can be presented as

$$\psi^i = \sum (\underline{r}) - \alpha_i \sigma_r^2,$$

the structure of equilibrium prices is in many ways similar to that of the simple model.

7) The results can be found in Jensen [24].

§ 1.5. Portfolio analysis with simplified models

§ 1.5.1. Introduction

The approach of selecting a portfolio of securities on the basis of the expected rate of return and variance was introduced by Markowitz. If accurate forecasts about future expected rates of return for each security, the variance of the rates of return for each security, and the correlation of rates of return between each pair of securities could be obtained, then Markowitz's model, employing quadratic programming, would produce efficient portfolios. The problem lies in obtaining accurate forecasts (using probably also subjective information) of the three types of inputs. Forecasting the complete future correlation matrix of security rates of return is not an easy task, and therefore several authors have tried to see if the one- or two-factor capital asset pricing models in section 1.4. could be used in forecasting the future correlations (the advantage is a considerable reduction in correlations).

Sharpe [43] proposed the so called "diagonal model", which was originally suggested by Markowitz [35] as a way of reducing the number of parameter inputs required in the mean-variance portfolio model. Other authors called the model the "market model" or "single index model".

The model assumes a linear relation between the rates of return of each security and some market portfolio, and the basic feature of the model is a simplified derivation of the covariances between the stock market rates of return. We shall study Sharpe's portfolio model formulation in more detail now.

1.5.2. Portfolio selection using the one-factor capital asset pricing model (called diagonal model by Sharpe [43]): a reformulation

Sharpe's simplified method of solving the efficient set of portfolios under the assumption of a regression structure, is in fact based on the ideas in section 1.4.1. In Sharpe's original article [43] the regressor may be the level of the stock market as a whole, the Gross National Product, some price index or any other factor thought to be the most important single influence on the rates of return from securities.

However, Smith [47] and Schreiner [42] have shown that on the basis of the

relative agreement of derived covariances with corresponding true covariances, several stock market indexes are strongly preferred over several other economic indexes. This is also in accordance with the theory of equilibrium models in the preceding sections.

Also Blume [5], Cohen en Pogue [9], Sharpe and Cooper [45], Sharpe [45] and others have recently used a linear combination of the individual rates of return as regressor in the so called diagonal model.

As the formulation in the literature gives some statistical difficulties⁸⁾, we shall give an other formulation.

We assume that the rates of return of security i for period t satisfy the following relation:

$$(1.5.2.1) \quad \mathcal{E}(r_{t,i} \mid p_{t,m} = p_{t,m}) = \alpha_i + \beta_i p_{t,m} \quad (i = 1, \dots, \ell) \\ (t = 1, \dots)$$

where

$$(1.5.2.2) \quad p_{t,m} = \sum_{i=1}^{\ell} \omega_i r_{t,i} .$$

Define

$$(1.5.2.3) \quad u_{t,i} = r_{t,i} - (\alpha_i + \beta_i p_{t,m}),$$

then

$$(1.5.2.4) \quad \mathcal{E}(u_{t,i} \mid p_{t,m}) = \sum_{p_{t,m}} [\mathcal{E}(u_{t,i} \mid p_{t,m} \mid p_{t,m})] = \\ \sum_{p_{t,m}} [p_{t,m} \mathcal{E}(u_{t,i} \mid p_{t,m} = p_{t,m})] = 0 .$$

8) The model in the literature is

$$r_j = \alpha_j + \beta_j p_m + u_j \quad , \quad j = 1, \dots, \ell$$

where p_m is the rate of return of some market portfolio, u_j a stochastic variable assumed uncorrelated with p_m by most authors, and α_j and β_j are constants. However, as p_m is some average of the individual rates of return, than the assumption that u_j is uncorrelated with p_m cannot hold, since p_m contains u_j .

From (1.5.2.3) it follows that

$$(1.5.2.5) \quad \underline{r}_{t,i} = \alpha_i + \beta_i \underline{p}_{t,m} + \underline{u}_{t,i}, \quad i = 1, \dots, \ell; t = 1, 2, \dots$$

and together with (1.5.2.2) and assuming that

$$\xi(\underline{u}_{t,i}) = 0, \quad i = 1, \dots, \ell; t = 1, 2, \dots,$$

we can get

$$(1.5.2.6) \quad \left\{ \begin{array}{l} \underline{p}_{t,m} = \sum_{i=1}^{\ell} \omega_i \alpha_i + \underline{p}_{t,m} \sum_{i=1}^{\ell} \omega_i \beta_i + \sum_{i=1}^{\ell} \omega_i \underline{u}_{t,i} \\ \xi(\underline{p}_{t,m}) = \sum_{i=1}^{\ell} \omega_i \alpha_i + \xi(\underline{p}_{t,m}) \sum_{i=1}^{\ell} \omega_i \beta_i \end{array} \right.$$

From the above it follows that

$$(1.5.2.7) \quad \sum_{i=1}^{\ell} \omega_i \alpha_i = 0$$

$$(1.5.2.8) \quad \sum_{i=1}^{\ell} \omega_i \beta_i = 1$$

$$(1.5.2.9) \quad \sum_{i=1}^{\ell} \omega_i \underline{u}_{t,i} = 0.$$

When the \underline{r}_i are normally distributed we have a special case.⁹⁾

9) When the rates of return on the securities, $\underline{R} = (\underline{r}_1, \dots, \underline{r}_\ell)'$ are multivariate normally distributed with $\xi(\underline{R}) = \mu_{\underline{R}}$ and $\text{Var}(\underline{R}) = \Sigma_{\underline{R}\underline{R}}$, and $\underline{p}_m := \sum_{i=1}^{\ell} \omega_i \underline{r}_i$, then $\xi(\underline{R} | \underline{p}_m = p_m)$ is linear in p_m .

This can be shown as follows. Define the vector \underline{Y} as $\underline{Y} := \underline{A}\underline{R}$ where

$$\underline{A} = \begin{pmatrix} \text{I} & 0 \\ \omega_1 \dots \omega_\ell \end{pmatrix} \text{ such that } \underline{Y} = (\underline{r}_1, \dots, \underline{r}_{\ell-1}, \underline{p}_m) \sim N_{\ell-1}(\underline{A}\mu_{\underline{A}}, \underline{A} \Sigma_{\underline{R}\underline{R}} \underline{A}'), \text{ where}$$

$\text{Det}(\underline{A} \Sigma_{\underline{R}\underline{R}} \underline{A}') > 0$. We are now interested in the conditional distribution of $\underline{R}^* := (\underline{r}_1, \dots, \underline{r}_{\ell-1})'$ given $\underline{p}_m = p_m$. This distribution is equal to (see e.g. Anderson [1])

$$(\underline{R}^* | \underline{p}_m = p_m) \sim N_{\ell-1} \left(\mu_{\underline{R}^*} + \Sigma_{\underline{R}^* \underline{p}_m} \frac{1}{\sigma_{\underline{p}_m}^2} (p_m - \mu_{\underline{p}_m}), \Sigma_{\underline{R}^* | \underline{p}_m} \right)$$

9) where

$$\Sigma_{R^*|P_m} = \Sigma_{R^*R^*} - \Sigma_{R^*P_m} \frac{1}{\sigma_{P_m}^2} \Sigma_{P_mR^*}, \text{ and } \Sigma_{R^*P_m}, \Sigma_{P_mR^*}$$

and $\sigma_{P_m}^2$ are defined from the partitioned matrix

$$A \Sigma_{RR} A' = \begin{pmatrix} \Sigma_{R^*R^*} & \Sigma_{R^*P_m} \\ \Sigma_{P_mR^*} & \sigma_{P_m}^2 \end{pmatrix}, \text{ where } \Sigma_{R^*R^*} \text{ is a matrix,}$$

$\Sigma_{R^*P_m}$ is a vector, and $\sigma_{P_m}^2$ a scalar.

Thus the conditional expectation is

$$E(R^*|P_m = p_m) = \mu_{R^*} - \frac{\mu_{P_m}}{\sigma_{P_m}^2} \Sigma_{R^*P_m} + \frac{1}{\sigma_{P_m}^2} \Sigma_{R^*P_m} p_m$$

or

$$E(R^*|P_m = p_m) = \alpha^* + \beta^* p_m, \text{ where } \alpha^* = (\alpha_1, \dots, \alpha_{\ell-1})', \beta^* = (\beta_1, \dots, \beta_{\ell-1})',$$

and using that $r_\ell = \frac{1}{\omega_\ell} p_m - \sum_{i=1}^{\ell-1} \frac{\omega_i}{\omega_\ell} r_i$, we get

$$E(r_\ell|P_m = p_m) = \frac{1}{\omega_\ell} p_m - \sum_{i=1}^{\ell-1} \frac{\omega_i}{\omega_\ell} (\alpha_i + \beta_i p_m) = - \sum_{i=1}^{\ell-1} \frac{\omega_i \alpha_i}{\omega_\ell} + \frac{(1 - \sum_{i=1}^{\ell-1} \omega_i \beta_i)}{\omega_\ell} p_m.$$

So we may define

$$\alpha_\ell = - \frac{\sum_{i=1}^{\ell-1} \omega_i \alpha_i}{\omega_\ell}, \quad \beta_\ell = \frac{1 - \sum_{i=1}^{\ell-1} \omega_i \beta_i}{\omega_\ell}$$

From the above it follows that

$$\sum_{i=1}^{\ell} \omega_i \alpha_i = 0, \quad \sum_{i=1}^{\ell} \omega_i \beta_i = 1.$$

By the normality assumption we can write $\xi(r_i|P_m = p_m) = \alpha_i + \beta_i p_m$ ($i=1, \dots, \ell$).

Using a result of Rao [39, see pages 441, 442] it can be shown that

$r_i - (\alpha_i + \beta_i p_m)$ and $(\alpha_i + \beta_i p_m)$ are independently distributed, when $(r_1, \dots, r_\ell, p_m)$ is multivariate normally distributed with non-negative definite covariance matrix (end footnote).

In matrix notation (1.5.2.5) can be written as

$$(1.5.2.10) \quad \underline{R}_i = \underline{X}_i \gamma_i + \underline{U}_i, \quad i = 1, \dots, \ell$$

where

$$\underline{R}_i = (r_{1,i}, \dots, r_{n,i})', \quad \underline{X}_i = \begin{pmatrix} 1 & p_{1,m} \\ \vdots & \vdots \\ \vdots & \vdots \\ 1 & p_{n,m} \end{pmatrix}$$

$$\gamma_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}, \quad \underline{U}_i = (u_{1,i}, \dots, u_{n,i})';$$

which can be written as multivariate model

$$(1.5.2.11) \quad \underline{R}^+ = (I \otimes \underline{X}_1) \gamma + \underline{U}$$

where

$$\underline{R}^+ = (\underline{R}'_1, \dots, \underline{R}'_\ell)', \quad I \text{ is a } \ell \times \ell \text{ unity matrix,}$$

$$\gamma = (\gamma'_1, \dots, \gamma'_\ell)', \quad \underline{U} = (U'_1, \dots, U'_\ell)'$$

Note that γ is subject to the linear restrictions under (1.5.2.7) and (1.5.2.8).

Further we define

$$\underline{U}^t = (u_{t,1}, \dots, u_{t,\ell})'$$

and assume that

$$(1.5.2.12) \quad \xi \{ \underline{U}^t \underline{U}^{\tau'} \} = \begin{cases} V & t = \tau \\ 0 & t \neq \tau \end{cases}, \quad \text{and } \xi (\underline{U}^t) = 0, \quad t = 1, 2, \dots$$

$\text{Cov}(\underline{U})$ can now be written as

$$\text{Cov}(\underline{U}) = (V \otimes I_n).$$

Defining $\Omega = (\omega_1, \dots, \omega_\ell)'$, we can write using (1.5.2.9)

$$(1.5.2.13) \quad \Omega' \underline{U}^t = 0$$

such that

$$(1.5.2.14) \quad \Omega' V \Omega = 0.$$

So we have a model with a singular covariance matrix.

Now $V_t : (\underline{r}_{t,1}, \dots, \underline{r}_{t,\ell-1} | p_{t,m} = p_{t,m})$ is sufficient for the quantity $(\underline{r}_{t,1}, \dots, \underline{r}_{t,\ell} | p_{t,m} = p_{t,m})$ under the given restrictions. This implies that $\underline{r}_{t,\ell}$ may be removed without loss of information in estimating $\gamma_i (i = 1, \dots, \ell)$ for given $p_{t,m}$.

The reduced model is now

$$(1.5.2.15) \quad \underline{R}^{**} = (I \otimes \underline{X}_1) \gamma^* + \underline{U}^*$$

where

$$\underline{R}^{**} = (\underline{R}'_1, \dots, \underline{R}'_{\ell-1})', \quad \gamma^* = (\gamma'_1, \dots, \gamma'_{\ell-1})', \quad \underline{U}^* = (\underline{U}'_1, \dots, \underline{U}'_{\ell-1})'$$

such that

$$\text{Cov}(\underline{U}^*) = (V_{11} \otimes I_n), \text{ and } V_{11} \text{ is assumed non-singular,}$$

and where further the restrictions (1.5.2.7) and (1.5.2.8) hold. So we can estimate $\gamma_i (i = 1, \dots, \ell-1)$ by applying G.L.S. under the restrictions, which amounts to minimizing

$$(1.5.2.16) \quad (\underline{R}^{**} - (I \otimes X_1) \gamma^*)' (V_n^{-1} \otimes I_n) (\underline{R}^{**} - (I \otimes X_1) \gamma^*)$$

under the restrictions

$$(1.5.2.17) \quad \left\{ \begin{array}{l} \sum_{i=1}^{\ell} \omega_i \alpha_i = 0 \\ \sum_{i=1}^{\ell} \omega_i \beta_i = 1 \end{array} \right.$$

There are two nice consequences:

- 1) The G.L.S. estimators of γ_i ($i = 1, \dots, \ell-1$) can be obtained without the restriction, because the G.L.S. estimator for γ_ℓ follows from (1.5.2.17).
- 2) The G.L.S. estimators are O.L.S. estimators because we have an identical regressor over all equations.

So we have

$$(1.5.2.18) \quad \hat{\underline{Y}}_i = (\underline{X}'_1 \underline{X}_1)^{-1} \underline{X}'_1 \underline{R}_i, \quad i = 1, \dots, \ell-1.$$

We can easily show that $\hat{\underline{Y}}_\ell$ is also equal to

$$(1.5.2.19) \quad \hat{\underline{Y}}_\ell = (\underline{X}'_1 \underline{X}_1)^{-1} \underline{X}'_1 \underline{R}_\ell.$$

Using (1.5.2.13), (1.5.2.10) and (1.5.2.17) we have

$$(1.5.2.20) \quad \sum_{i=1}^{\ell} \omega_i \underline{R}_i = \underline{X}_1 \sum_{i=1}^{\ell} \omega_i \gamma_i = \underline{X}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

or

$$(1.5.2.21) \quad \sum_{i=1}^{\ell-1} \omega_i \underline{R}_i = \underline{X}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \omega_\ell \underline{R}_\ell.$$

From (1.5.2.17) we find that

$$(1.5.2.22) \quad \omega_\ell \hat{\underline{Y}}_\ell = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{i=1}^{\ell-1} \omega_i \hat{\underline{Y}}_i.$$

Substituting (1.5.2.18) and (1.5.2.21) in (1.5.2.22) we get

$$\begin{aligned} \omega_\ell \hat{\underline{Y}}_\ell &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{i=1}^{\ell-1} \omega_i \left((\underline{X}'_1 \underline{X}_1)^{-1} \underline{X}'_1 \underline{R}_i \right) \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - (\underline{X}'_1 \underline{X}_1)^{-1} \underline{X}'_1 \sum_{i=1}^{\ell-1} \omega_i \underline{R}_i \\ &= \omega_\ell (\underline{X}'_1 \underline{X}_1)^{-1} \underline{X}'_1 \underline{R}_\ell \end{aligned}$$

or

$$(1.5.2.23) \quad \hat{\underline{Y}}_\ell = (\underline{X}'_1 \underline{X}_1)^{-1} \underline{X}'_1 \underline{R}_\ell \quad (\text{q.e.d.})$$

We shall now study the asymptotic properties of the generalized least squares estimators.

The O.L.S. estimators in (1.5.2.18) and (1.5.2.23) can be written as

$$(1.5.2.24) \quad \hat{\gamma}_i = \gamma_i + (\underline{X}'_1 \underline{X}_1)^{-1} \underline{X}'_1 \underline{U}_i, \quad i = 1, \dots, \ell$$

and evaluating its probability limit, we find

$$(1.5.2.25) \quad p \lim_{n \rightarrow \infty} \hat{\gamma}_i = \gamma_i + p \lim_{n \rightarrow \infty} \left(\frac{\underline{X}'_1 \underline{X}_1}{n} \right)^{-1} p \lim_{n \rightarrow \infty} \left(\frac{\underline{X}'_1 \underline{U}_i}{n} \right)$$

We assume that $p \lim_{n \rightarrow \infty} \left(\frac{\underline{X}'_1 \underline{X}_1}{n} \right)^{-1}$ exists.

If the $\underline{u}_{t,i}$ satisfy the following conditions

$$(a) \quad \xi(\underline{u}_{t,i}) = 0, \quad i = 1, \dots, \ell, \quad t = 1, 2, \dots$$

$$\xi(\underline{u}_{t,i}^2) = \sigma_{u_i}^2 \quad " \quad "$$

$$\text{Cov}(\underline{u}_{t,i}, \underline{u}_{\tau,i}) = 0, \quad i = 1, \dots, \ell, \quad t \neq \tau$$

$$(b) \quad \xi(\underline{u}_{t,i} p_{t,m}) = 0, \quad i = 1, \dots, \ell, \quad t = 1, 2, \dots$$

$$\xi[(\underline{u}_{t,i} p_{t,m})^2] = \sigma_{u_i p_m}^2, \quad i = 1, \dots, \ell, \quad t = 1, 2, \dots$$

$$\text{Cov}[(\underline{u}_{t,i} p_{t,m}), (\underline{u}_{\tau,i} p_{\tau,m})] = 0, \quad i = 1, \dots, \ell, \quad t \neq \tau$$

then (see e.g. Chebyshev's theorem in Rao [39]).

$$(1.5.2.26) \quad p \lim_{n \rightarrow \infty} \left(\frac{\underline{X}'_1 \underline{U}_i}{n} \right) = \begin{pmatrix} p \lim_{n \rightarrow \infty} \left(\frac{\sum_{t=1}^n \underline{u}_{t,i}}{n} \right) \\ p \lim_{n \rightarrow \infty} \left(\frac{\sum_{t=1}^n p_{t,m} \underline{u}_{t,i}}{n} \right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So the least squares estimators $\hat{\gamma}_i (i = 1, \dots, \ell)$ are consistent estimators for γ_i .

Lets now consider the least squares residuals

$$(1.5.2.27) \quad \hat{U}_i = R_i - X_i \hat{Y}_i = X_i \gamma_i + U_i - X_i \hat{Y}_i = U_i + X_i (\gamma_i - \hat{Y}_i), \quad i = 1, \dots, \ell.$$

Defining the matrix

$$\hat{U}^{***} = (\hat{U}_1, \dots, \hat{U}_\ell),$$

we propose as estimator for matrix V of contemporaneous disturbances

$$(1.5.2.28) \quad \hat{V} = \frac{(\hat{U}^{***})' (\hat{U}^{***})}{n}.$$

Evaluating the probability limit of element (i,j) of matrix \hat{V} demonstrates its consistency:

$$(1.5.2.29) \quad \begin{aligned} p \lim_{n \rightarrow \infty} \left(\frac{\hat{U}_i' \hat{U}_j}{n} \right) &= p \lim_{n \rightarrow \infty} \left(\frac{U_i' U_j}{n} \right) + p \lim_{n \rightarrow \infty} \left(\frac{U_i' X_1}{n} \right) * \\ &+ p \lim_{n \rightarrow \infty} (\gamma_j - \hat{Y}_j) + p \lim_{n \rightarrow \infty} (\gamma_i - \hat{Y}_i)' + p \lim_{n \rightarrow \infty} \left(\frac{X_1' U_j}{n} \right) + \\ &+ p \lim_{n \rightarrow \infty} (\gamma_i - \hat{Y}_i)' + p \lim_{n \rightarrow \infty} \left(\frac{X_1' X_1}{n} \right) p \lim_{n \rightarrow \infty} (\gamma_j - \hat{Y}_j) = v_{i,j}, \end{aligned}$$

(the (i,j)-th element of matrix V). In the above derivation we have assumed that $p \lim_{n \rightarrow \infty} \left(\frac{X_1' X_1}{n} \right)$ exists.

We shall now see how for an arbitrary portfolio $p_t = \sum_{i=1}^{\ell} x_i r_{t,i}$, the expectation and variance can be expressed as a function of the regression parameters.

As

$$(1.5.2.30) \quad p_t = X' \alpha + X' \beta p_{t,m} + X' U_t$$

where

$$X' = (\alpha_1, \dots, \alpha_\ell); \quad \beta' = (\beta_1, \dots, \beta_\ell);$$

$$\alpha' = (\alpha_1, \dots, \alpha_\ell); \quad \underline{U}'_t = (\underline{u}_{t,1}, \dots, \underline{u}_{t,\ell});$$

the expectation and variance are

$$(1.5.2.31) \left\{ \begin{array}{l} E(p_t) = X'\alpha + X'\beta E(p_{t,m}) \\ \text{Var}(p_t) = (X'\beta)^2 \text{Var}(p_{t,m}) + X'VX \end{array} \right.$$

However, in this case V is not a diagonal matrix as in Sharpe's formulation. In practice the formerly defined consistent estimates for α, β and V can be used. It should be noted that V is a positive semidefinite matrix, but the computational technique in section 1.3 of this chapter can still be used in this situation. So we have found that our formulation of the market model differs from the diagonal formulation of Sharpe (only the diagonal elements of matrix V are taken into account in Sharpe's diagonal model formulation). We are not sure that the diagonal model formulation can be used as a good approximation for our market model formulation.

The accuracy of this approximation needs further research to our opinion, as mostly the diagonal model formulation is used in the literature, which neglects the effect of the non-diagonal elements in matrix V . For 84 securities (their names are listed in the appendix of this chapter) for which monthly rates of return were readily available over the period January 1961 - December 1972, we have compared Sharpe's diagonal model formulation ¹⁰⁾ (model I) with our revised market model formulation (model II). ¹¹⁾ For the rates of return of the market portfolio we have taken the arithmetic average of the rates of return of the 84 securities. The results are presented in Table 1 and 2 and Figure 5.

From Figure 5 it can be seen that the efficient line obtained via model II is different from that obtained via model I in the following sense: for given expected rate of return of an efficient portfolio, the revised model formulation gives generally a lower variance of the rates of return of that portfolio than with Sharpe's diagonal model formulation; or for given variance of the rate of return of an efficient portfolio, the expected rate of return of that portfolio is generally higher for model formulation II.

10) Diagonal model formulations for portfolio selection are for example used by Schreiner [42] and Wallingford [51].

11) We are grateful to Mr. P. Vermeulen, who did the calculations for us.

	69	70	72	75	\bar{b}
1	0.37	20.15	1.03		0.6110
2		20.26	0.97		0.6128
3		20.41	0.83		0.6178
4		20.44	0.79		0.6192
5		20.45	0.72		0.6220
6		20.42	0.68	0.00	0.6239
7		20.39	0.63	0.02	0.6261
8		20.36	0.61	0.03	0.6273
9		20.26	0.53	0.07	0.6314
10		19.45	0.13	0.22	0.6518
11		19.16		0.27	0.6586
12		18.18		0.41	0.6795
13		16.90		0.55	0.6973
14		13.55		0.88	0.7491
15		12.52		0.98	0.7649
16		11.93		1.03	0.7710
17		9.61		1.19	0.7921
18		8.98		1.23	0.7974
19		8.00		1.28	0.8052
20		7.54		1.30	0.8072
21		3.02		1.44	0.8234
22		2.95		1.44	0.8236
23		2.20		1.46	0.8263
24		2.18		1.46	0.8263
25				1.49	0.8315
26				1.48	0.8341
27				0.80	0.8623
28				0.72	0.8642
29				0.72	0.8642
30				0.66	0.8635
31				0.64	0.8632
32					0.8587
33					0.8343

Table 2: Efficient portfolios for Sharpe's revised model (model II)

	$E(\underline{p}_t)$ in $\frac{\circ}{\circ\circ}$ per month	Var (\underline{p}_t) in $\frac{\circ}{\circ\circ\circ}$ per month	n (number of securities in the eff. portfolio)	efficient portfolios in %									
				1	3	5	12	13	16	17	20		
1	8.703	4.8283	23	1.76	3.89	0.94	2.60		8.27		5.43		
2	8.905	4.8385	24	2.04	3.48	1.09	2.35		8.42	0.00	5.51		
3	8.917	4.8394	23	2.05	3.46	1.09	2.34		8.43	0.04	5.51		
4	9.157	4.8600	22	2.11	3.21	1.16	2.18		8.46	0.97	5.49		
5	9.258	4.8705	21	2.14	3.07	1.19	2.11		8.45	1.41	5.49		
6	9.324	4.7880	20	2.15	2.97	1.20	2.06		8.42	1.70	5.49		
7	9.485	4.8990	19	2.20	2.69	1.24	1.92		8.32	2.46	5.50		
8	9.557	4.9095	20	2.21	2.57	1.25	1.87		8.26	2.80	5.49		
9	9.846	4.9590	21	2.18	2.11	1.39	1.63		7.99	4.14	5.43		
10	10.406	5.0851	22	2.12	1.27	1.62	1.25		7.28	6.56	5.31		
11	10.849	5.2120	23	2.05	0.63	1.78	0.97	0.00	6.75	8.50	5.15		
12	11.298	5.3641	22	1.96		1.89	0.61	0.25	6.28	10.36	4.98		
13	11.575	5.4705	21	1.88		2.03	0.35	0.43	5.93	11.62	4.82		
14	11.968	5.6381	20	1.79		2.22		0.68	5.40	13.41	4.58		
15	14.177	6.9692	19	1.04		3.13		1.82	1.88	23.96	3.09		
16	15.342	7.9384	18	0.64		3.58		2.42		29.55	2.31		
17	16.105	8.6760	19	0.36		3.76		2.87		33.35	1.65		
18	16.268	8.8443	18	0.29		3.79		2.96		34.17	1.50		
19	16.614	9.2145	17	0.07		3.90		3.23		35.98	1.14		
20	16.709	9.3203	16			3.90		3.31		36.51	1.04		
21	16.796	9.4173	15			3.90		3.37		36.97	0.94		
22	16.880	9.5135	14			3.89		3.44		37.41	0.83		
23	17.272	9.9763	13			3.85		3.75		39.47	0.31		
24	17.512	10.2762	12			3.46		4.02		41.02			
25	18.085	11.0590	11			2.37		4.65		44.76			
26	18.166	11.1786	10			2.19		4.74		45.31			
27	18.934	12.4095	9					5.53		50.50			
28	19.077	12.6626	8					5.65		51.51			
29	19.407	13.2758	7					5.90		53.97			
30	21.556	18.9557	6					6.05		76.70			
31	22.322	21.7601	5					5.93		85.25			
32	22.612	22.9464	4					5.73		88.54			
33	22.941	24.4114	3					4.94		92.67			
34	23.185	25.6014	2					4.27		95.73			
35	23.413	26.9153	1							100			

var σ^2 in %
per month

2.5
2.4
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1

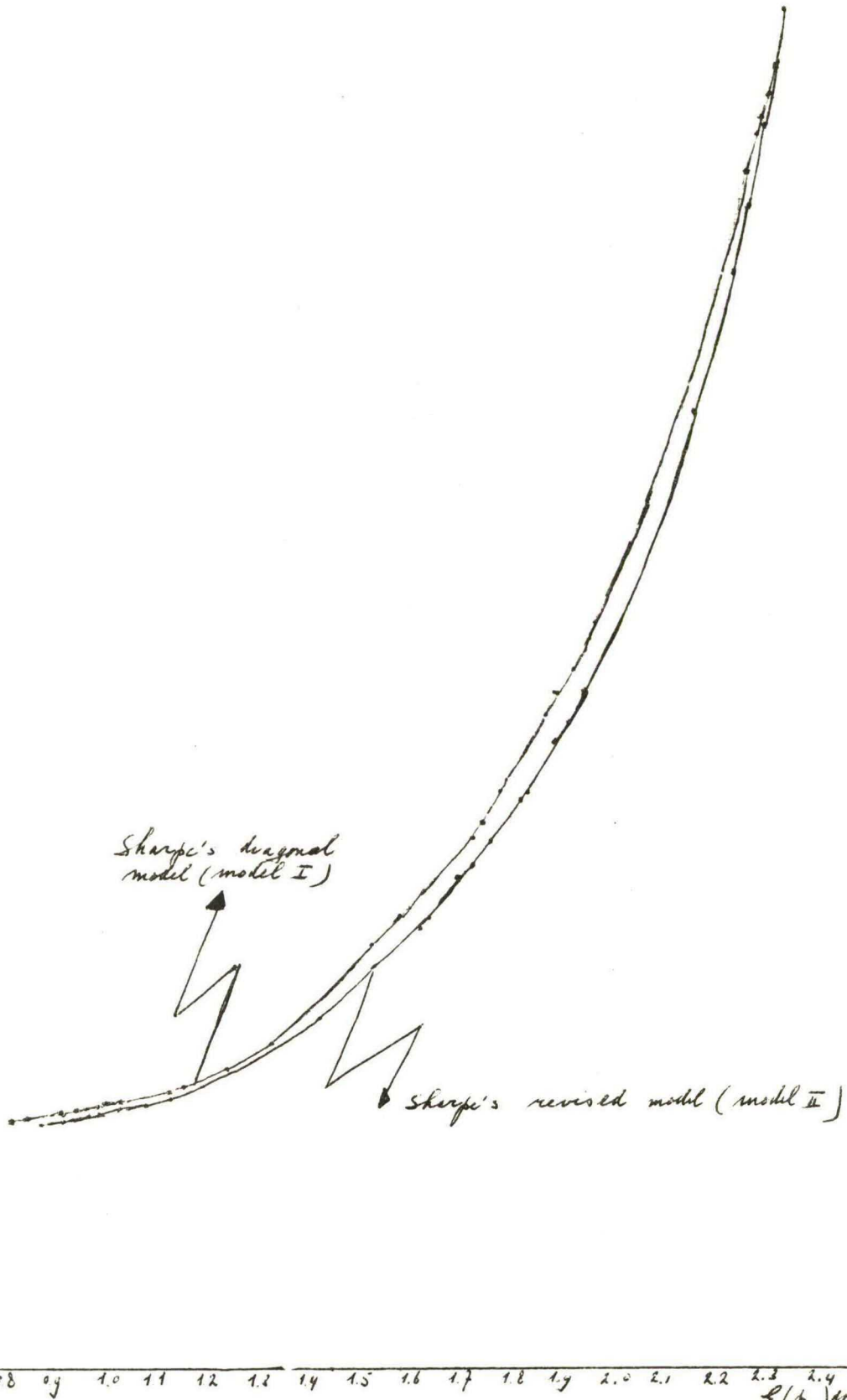


Figure 5 : Efficient lines for model I and II

Remark by Table 1 and Table 2:

The efficient portfolios in Table 1 are all determined by a choice from 28 securities. When we assume that the rates of return are independently and normally distributed, we can test if the β_i of these 28 securities differ significantly from zero. Of 28 securities the t-value is greater than 1.5, of 27 it is greater than 2, and of 24 it is greater than 3.

The efficient portfolios in Table 2 are all determined by a choice from 29 securities. In this case 29 securities have a t-value greater than 1.5, 28 are greater than 2, and 25 are greater than 3.

Further $\bar{\beta} = \frac{\sum_{i=1}^n \beta_i}{n}$, where n is the number of securities in the portfolio, and β_i is a systematic risk factor of security i defined on page 16.

§ 1.5.3. Extensions of the one-factor capital asset pricing model for portfolio selection (multi-index models)

Portfolio analysis using multi-index models is suggested by Cohen and Pogue [9], Elton and Gruber [13] and Farrell [17].

The advantage of a multi-index model over a full analysis of all relations lies also in a simplified formula for portfolio selection. The multi-index models can provide substantial reduction in parameter estimation.

Cohen and Pogue [9] attempted to see if introducing industry effects into the market model of Sharpe could improve portfolio performance. Sharpe assumed that the interrelationships between the price or return movements of stocks could be expressed in terms of their movement with an index of general stock price or return movement. Cohen and Pogue added a second influence - the movements of stocks with their respective industrial average. In order to test this assumption, Cohen and Pogue employed two models.

In the first model they assumed that the returns of a security are related to its industry-index, and they allowed the industry-indexes to be correlated.

In the second model they also assumed the returns of a security to be related to its industry-index, but the industry-indexes were on their turn related to a general market index.

The results suggested that Sharpe's diagonal model formulation performed nearly as well (the efficient boundaries were close together in the $\mu_p - \sigma_p$ space) as the more costly Markowitz approach. Moreover the multi-index models did not outperform Sharpe's diagonal model formulation.

Elton and Gruber [12] questioned the use of industrial classification as a method of holding certain variables constant. They state that in finance the most common variable researchers desire to hold constant is business risk. They proposed a clustering procedure in order to form different homogeneous groups with constant business risk. For example, in seeking a group of firms which are homogeneous with respect to business risk, one can use as grouping variable: the amount of variation in past earnings.

According to Elton and Gruber, this, rather than industrial classification, should be the basis of grouping. They also proposed other financial criteria with respect to which one wants to form homogeneous groups.

However, the multi-index approach of Elton and Gruber and Farrell has the following drawbacks for portfolio selection:

- a) In a recent article Elton and Gruber [18] themselves examined how well a single index model and three versions of a multi-index model actually fore-

cast future correlation matrices. First of all they found that the market model produced better forecasts of future correlation matrices than the full historical correlation matrix. Further, the market model outperformed also the multi-index models in a 1- and 5-year ahead forecast basis. So, although the multi-index models used by Elton and Gruber are different from those of Cohen and Pogue, the results are similar.

In both cases it is found that a market model outperforms multi-index models

- b) The clustering procedures as suggested by Elton and Gruber to find homogeneous groups is complicated.

Another extension of the simple market model of Sharpe for portfolio selection is the earlier described two-factor model. In section 1.4.2. it is demonstrated that the two-factor model is characterized by

$$r_{t,i} = (1-\beta_i) p_{t,z} + \beta_i p_{t,m} + u_{t,i} ,$$

where $p_{t,z}$ is the return of the zero beta-factor for period t , and $p_{t,m}$ the return on the market portfolio for the same period.

In order to fully utilize the properties of the two factor model in portfolio evaluation (see Jensen [25]), it will first be necessary to have good estimators of the time series of returns on the beta-factor.

However, this problem will not be worked out here.

A promising two factor model, where the inter-country influence is also taken into account, is for a security i from country j of the following form:

$$r_{t,i,j} = \alpha_i + \beta_i p_{t,m,w} + \gamma_i p_{t,m,j} + u_{t,i} ,$$

where α_i , β_i and γ_i are stable parameters specific to security i and $p_{t,m,w}$ is the return on the world market portfolio and $p_{t,m,j}$ the return on the market portfolio of country i . Lessard [40] shows that the above model might be of interest for portfolio selection.

§ 1.6. Determining the optimal portfolio choice for a specified utility function and certain distribution functions

We consider the following problem: An investor with initial wealth $y > 0$ buys assets, the values of which at the end of the period (per unit of wealth invested) are given by the non-negative stochastic variables \underline{w}_i , $i = 1, \dots, \ell$.¹⁰⁾ The resulting total wealth at the end of the period of the investor is therefore described by the stochastic variable $\underline{z} := \sum_{i=1}^{\ell} x_i \underline{w}_i$ where x_i is the amount allotted to asset i .

The investor has to observe the following constraints:

$$(1.6.1) \quad \sum_{i=1}^{\ell} x_i = y \quad ; \quad x_i \geq 0 \quad , \quad i = 1, \dots, \ell$$

For a given utility function $u(z)$ the investor selects the x_i , $i = 1, \dots, \ell$ so that his expected utility $\mathcal{E}(u(\underline{z}))$ will be maximized subject to the constraints (1.6.1). In a recent paper Davies and Ronning [11] have derived explicit optimal solutions of portfolio choice under the assumption that the utility function is given by $u(z) = 1 - e^{-\eta z}$ $\eta > 0$,¹¹⁾ and that the stochastic

10) The stochastic variable \underline{w}_i can be expressed as ending wealth divided by beginning wealth taking into account dividends etc. The non-negative restriction for the variable \underline{w}_i does not mean that we cannot lose anything, but that we can not lost more than we have invested. It is clear that the \underline{w}_i can not take on nagative values with positive probability (or, equivalently, the rates of return can not take on values less than -1). In the following it will become clear why in this section we have not defined the portfolio problem for the rates of return.

11) The parameter η is the risk-aversion parameter of an investor, which can be intepreted as follows [38]: A decision taker accepts a lottery game, where he can lose or win a small amount of k dollars, only when the probability to win is at least $1/2 + \eta k/4$.

variables \underline{w}_i , $i = 1, \dots, \ell$ are distributed independently.

For gamma distributed \underline{w}_i we shall demonstrate how to find the exact optimal solution. For solutions of other distribution functions we refer to Davies and Ronning [11].

First we shall recapitulate the definition and some results of a one-sided Laplace transform of a positive stochastic variable \underline{w} . For positive λ the Laplace transform of \underline{w} will be defined by

$$(1.6.2) \quad \mathcal{L}(\lambda) := \mathbb{E}(e^{-\lambda \underline{w}}) = \int_0^{\infty} e^{-\lambda w} f(w) dw$$

where $f(w)$ is the density function of \underline{w} . As $\lambda > 0$, $\mathcal{L}(\lambda)$ exists, and is an arbitrary times differentiable, even when the moments of \underline{w} do not exist (see Feller [19]).

Assume \underline{w}_i , $i = 1, \dots, \ell$ are stochastic independent variables with Laplace transforms $\mathcal{L}_i(\lambda)$, $i = 1, \dots, \ell$. Then the Laplace transform for the sum

$$\underline{z} = \sum_{i=1}^{\ell} x_i \underline{w}_i \text{ is}$$

$$(1.6.3) \quad \mathcal{L}(\lambda) = \prod_{i=1}^{\ell} \mathcal{L}_i(\lambda x_i) .$$

When the utility function of the investor is $u(z) = 1 - e^{-\eta z}$, $\eta > 0$ and when he invests an amount x_i in asset i , then his expected utility at the end of the period is

$$(1.6.4) \quad \mathbb{E} \left\{ u \left(\sum_{i=1}^{\ell} x_i \underline{w}_i \right) \right\} = 1 - \mathbb{E} \left\{ e^{-\eta \sum_{i=1}^{\ell} x_i \underline{w}_i} \right\} = 1 - \mathcal{L}(\eta),$$

where $\mathcal{L}(\eta)$ is the Laplace transform of $\sum_{i=1}^{\ell} x_i \underline{w}_i$.

The maximizing of $\mathbb{E} \left\{ u \left(\sum_{i=1}^{\ell} x_i \underline{w}_i \right) \right\}$ is equivalent to maximizing $-\sum_{i=1}^{\ell} \log(\mathcal{L}_i(\eta x_i))$ ¹²⁾ or to minimizing

$$(1.6.5) \quad \sum_{i=1}^{\ell} \log(\mathcal{L}_i(\eta x_i))$$

under the restriction

12) As $\mathcal{L}_i(\eta x_i)$ is always positive, $\log \mathcal{L}_i(\eta x_i)$ is always defined.

$$(1.6.6) \quad \sum_{i=1}^{\ell} x_i = y$$

and

$$(1.6.7) \quad x_i \geq 0, \quad i = 1, \dots, \ell.$$

The above problem is a nonlinear programming problem for which Davies and Morton [10] have proved the following useful theorem.

Theorem

When the expected values μ_i for the stochastic variables \underline{w}_i exist, then they can be ordered as follows.

$$(1.6.8) \quad \mu(1) \geq \mu(2) \geq \mu(3) \cdots \mu(\ell).$$

There exists an integer m , $1 \leq m \leq \ell$, with the following properties:

(1) the $m+1$ equations

$$(1.6.9) \quad \sum_{i=1}^m x_i = y$$

$$(1.6.10) \quad \frac{\mathcal{L}'_i(\eta x_i)}{\mathcal{L}_i(\eta x_i)} = -x_0, \quad i = 1, \dots, m$$

have a unique solution $x_{m_i}^*$ ($i = 0, 1, \dots, m$) for which

$$(1.6.11) \quad x_{m_i}^* > 0, \quad i = 1, \dots, m;$$

(2) further

$$(1.6.12) \quad \mu(i) \leq x_{m_0}^*, \quad i = m+1, \dots, \ell;$$

(3) the solution x_i^* of (1.6.5), (1.6.6) and (1.6.7) is given by

$$(1.6.13) \quad x_i^* = \begin{cases} x_{m_i}^* & , \quad i = 1, \dots, m \\ 0 & , \quad i = m+1, \dots, \ell. \end{cases}$$

Note that the solution of (1.6.9) and (1.6.10) corresponds with a minimum as $\log \mathcal{L}_i(\lambda)$ is a convex function. From (1) and (3) we see that the non-negative restriction (1.6.7) can be build in, when we have the solution of (1.6.9) and (1.6.10) for arbitrary m .

We shall demonstrate the above using that the w_i are independently gamma distributed as follows:

$$(1.6.14) \quad f(w_i) = \begin{cases} 0 & , \quad w_i \leq 0 \\ \frac{\theta_i^{\alpha_i}}{\Gamma(\alpha_i)} w_i^{\alpha_i-1} e^{-\theta_i w_i} & , \quad w_i > 0 \end{cases}$$

where the constants α_i, θ_i are greater than zero.

The expected value of w_i is $\mu_i = \frac{\alpha_i}{\theta_i}$ and the variance is $\sigma_i^2 = \frac{\alpha_i}{\theta_i^2}$.

The Laplace transform $\mathcal{L}_i(\lambda)$ is

$$(1.6.15) \quad \mathcal{L}_i(\lambda) = \left(1 + \frac{\lambda}{\theta_i}\right)^{-\alpha_i}.$$

Equation (1.6.10) can in this case be written as

$$(1.6.16) \quad \frac{\alpha_i}{\theta_i + \eta x_i} = x_0 \quad , \quad i = 1, \dots, m$$

or written elsewise

$$x_i = \frac{1}{\eta} \left(\frac{\alpha_i}{x_0} - \theta_i \right) \quad , \quad i = 1, \dots, m .$$

Using (1.6.9) we get

$$\frac{1}{x_0} = \frac{\eta y + \sum_{j=1}^m \theta_j}{\sum_{j=1}^m \alpha_j} ,$$

so that the solution of (1.6.9) and (1.6.10) is given by

$$x_{m,i}^* = \alpha_i \frac{y + \eta^{-1} \sum_{j=1}^m \theta_j}{\sum_{j=1}^m \alpha_j} - \frac{\theta_j}{\eta}$$

$$(1.6.17) \quad \frac{\mu_i}{\sigma_i^2} \left\{ \frac{y + \eta^{-1} \left(\sum_{j=1}^m \frac{\mu_j}{\sigma_j^2} \right)}{\sum_{j=1}^m \frac{\mu_j}{\sigma_j^2}} \right\} - \frac{\mu_i}{\eta \sigma_i^2}, \quad i = 1, \dots, m$$

and

$$(1.6.18) \quad x_{m,0}^* = \frac{\sum_{j=1}^m \alpha_j}{\eta y + \sum_{j=1}^m \theta_j} = \frac{\sum_{j=1}^m \frac{\mu_j}{\sigma_j^2}}{\eta y + \sum_{j=1}^m \frac{\mu_j}{\sigma_j^2}}$$

When the expectations μ_i are ordered as indicated in the above theorem, then the optimal portfolio can be found as follows: When m ($1 \leq m \leq \ell$) is the smallest integer, for which

$$(1.6.19) \quad \mu_{(m+1)} \leq \frac{\sum_{j=1}^m \frac{\mu_j}{\sigma_j^2}}{\eta y + \sum_{j=1}^m \frac{\mu_j}{\sigma_j^2}}$$

is satisfied, then we have

$$x_i^* = \begin{cases} x_{m,i}^* & i = 1, \dots, m \\ 0 & i = m+1, \dots, \ell, \end{cases}$$

where $x_{m,i}^*$ is given by (1.6.17).

When such an integer m does not exist (which can be the case for large y), then the solution is given by (1.6.17).

From (1.6.17) we also see that

$$(1.6.20) \quad \lim_{y \rightarrow \infty} \frac{x_i^*}{y} = \frac{\mu_i^2}{\sigma_i^2} \left\{ \sum_{j=1}^{\ell} \frac{\mu_j^2}{\sigma_j^2} \right\}^{-1}.$$

The above formulation can be extended in several ways, one of which is

$$(1.6.21) \quad \min_{x_i} \sum_{i=1}^{\ell} \log(\mathcal{L}_i(\eta x_i))$$

under the restrictions

$$(1.6.22) \quad \begin{aligned} & \sum_{i=1}^{\ell} x_i \leq y \\ & c_i \leq x_i \leq d_i \end{aligned} \quad , \quad i = 1, \dots, \ell,$$

for which Castellani [7] has shown that it can be solved by the method of dynamic programming. It should be noted that the above formulations rest on the specific type of utility function which is used (with constant risk aversion¹³⁾).

A utility function which has decreasing local risk aversion is

$$(1.6.23) \quad u(x) = \frac{1 - e^{-\eta x} + x}{2 - e^{-\eta}} \quad , \quad x \geq 0 \quad , \quad \eta > 0$$

with properties

$$a) \quad u'(x) = \frac{1 + \eta e^{-\eta x}}{2 - e^{-\eta}} \quad , \quad \text{which is positive for } \eta > 0,$$

$$b) \quad u''(x) = \frac{-\eta^2 e^{-\eta x}}{2 - e^{-\eta}} < 0, \quad \text{which means diminishing marginal utility}$$

13) Pratt [38] has introduced the function

$$r(x) = - \frac{u'(x)}{u''(x)}$$

which is a measure of local risk aversion.

c) the premium on individual is willing to pay to insure against risk, diminishes with the size of his assets. It has been shown that if this is true, his utility function must exhibit risk aversion:

$$r(x) = \frac{\eta^2 e^{-\eta x}}{1 + \eta e^{-\eta x}} \quad \text{and}$$

$$r'(x) = \frac{-\eta^3 e^{-\eta x}}{(1 + \eta e^{-\eta x})^2} < 0$$

Under the same assumptions as before, except for the type of utility function in (1.6.23), equation (1.6.4) is now

$$(1.6.24) \quad \mathcal{E} \left\{ u \left(\sum_{i=1}^{\ell} x_i w_i \right) \right\} = \beta \left[1 - \mathcal{E} \left\{ e^{-\eta \sum_{i=1}^{\ell} x_i w_i} \right\} \right] + \mathcal{E} \left\{ \sum_{i=1}^{\ell} x_i w_i \right\} =$$

$$\beta \left[1 - \prod_{i=1}^{\ell} \mathcal{L}_i(\eta x_i) + \sum_{i=1}^{\ell} x_i \mathcal{E}(w_i) \right],$$

where

$$\beta = \frac{1}{2 - e^{-\eta}}.$$

Maximizing (1.6.24) under the restriction (1.6.22) leads to a nonlinear programming formulation.

§ 1.7. Concluding remarks

In the preceding sections we reviewed and commented on equilibrium models with regard to the capital market and its use for portfolio selection. We reformulated the so-called market model of Sharpe and illustrated its difference with the so-called diagonal model for Dutch securities. Besides the determination of efficient portfolios we also showed how to obtain optimal portfolios, which however depend primarily on the investor's utility function.

As in the single and multi-index models the rate of return of the market portfolio is of great importance, we shall in the next chapters try to define and test several hypotheses, concerning the stochastic dependence structure in the rates of return of the market portfolio itself or in relation with other economic variables.

This possible information of the stochastic structure of the rates of return of the market portfolio might then be used in the portfolio selection problem.

§ 1.8. Appendix: Dutch securities listed on the Amsterdam Stock Exchange,
which are used to calculate the efficient portfolios in section
1.5.2

- | | |
|---------------------------------------|-------------------------------------|
| 1. Ahold | 35. K.N.P. |
| 2. A.R.M. | 36. Krasnapolsky |
| 3. Arnhemsche Scheepsbouw | 37. Lindeteves |
| 4. Ballast - Nedam | 38. Gelatine - Delft |
| 5. Batenburg | 39. Meneba |
| 6. Begemann | 40. De Meteor |
| 7. Van Berkel | 41. Mulder's Rollend Mate-
riëel |
| 8. De Boer | 42. Naarden |
| 9. Bols | 43. Naeff |
| 10. Braat | 44. Nedap |
| 11. Bredero | 45. Nederlandsche Crediet-
bank |
| 12. Deli - Atjeh | 46. N.D.U. |
| 13. Desseaux | 47. N.M.B. |
| 14. Dijkers | 48. N.S.U. |
| 15. Van Dorp | 49. Van Nelle |
| 16. Drentsch-Overijselsche Houthandel | 50. Norit |
| 17. Elsevier | 51. Nijverdal |
| 18. Emba | 52. Ogem |
| 19. Fokker | 53. Van Ommeren |
| 20. Furness | 54. Orenstein |
| 21. Van Gelder | 55. Palthe |
| 22. Giessen-De Noord | 56. Philips |
| 23. Grasso | 57. Pont Houthandel |
| 24. Van der Grinten | 58. Proost |
| 25. Hagemeyer | 59. Reesink |
| 26. Hero | 60. Van Reeuwijk |
| 27. Hoek | 61. Reiss |
| 28. H.A.L. | 62. Riva |
| 29. H.B.G. | 63. Rommenhüller |
| 30. Houtvaart | 64. Sanders |
| 31. Zeeland | 65. Schokbeton |
| 32. Jean Heybroek | 66. K.S.H. |
| 33. Kloos | 67. Schuitema |
| 34. K.N.S.M. | 68. Van Schuppen Sajet |

- | | |
|------------------------------|------------------------|
| 69. Schuttersveld | 77. Veneta |
| 70. Slavenburg | 78. Ver. Glasfabrieken |
| 71. Tilburgsche Waterleiding | 79. Verto |
| 72. Tricotbest | 80. Vihamij |
| 73. Stoomspinnerij Twenthe | 81. V.M.F. |
| 74. Twentsche Kabelfabriek | 82. V.N.U. |
| 75. Ubbink | 83. Wessanen |
| 76. Unilever | 84. Wijers |

§ 1.9. References

- [1] T.W. Anderson, An introduction to multivariate statistical analysis, Wiley & Sons, New-York, 1958.
- [2] F. Black, "Capital market equilibrium with restricted borrowing", The Journal of Business, 45 (1972), July, pp. 444-454.
- [3] F. Black, M.C. Jensen and M. Scholes, "The capital asset pricing model:some empirical tests", in: M.C. Jensen ed., Studies in the theory of capital markets, Praeger Publishers, New-York, 1972.
- [4] M. Blume, The assessment of portfolio performance, unpublished Ph.D. Thesis, University of Chicago, Chicago, 1968.
- [5] M. Blume, "Portfolio theory: a step toward its practical application", The Journal of Business, 43 (1970), April, pp. 152-173.
- [6] M.J. Brennan, "Capital market equilibrium with divergent borrowing and lending rates", Journal of Financial and Quantitative Analysis, 6 (1971), December, pp. 1197-1205.
- [7] G. Castellani, "A mathematical model on the selection of investments in conditions of risk", in: G.P. Szegö and K. Shell, editors, Mathematical methods in investments and finance, North-Holland Publishing Company, Amsterdam, 1972.
- [8] J.S. Chipman, "The ordering of portfolios in terms of mean and variance", The Review of Economic Studies, 40 (1973), April, pp. 167-190.
- [9] K. Cohen and G. Pogue, "An empirical evaluation of alternative portfolio selection models", The Journal of Business, 40 (1967), April, pp. 166-193.
- [10] P.L. Davies and G. Morton, Maximization of a separable utility function under budget constraints, Diskussionsbeiträge des Fachbereichs Statistik der Universität Konstanz, No. 1, Konstanz, 1974.

- [11] P.L. Davies and G. Ronning, "Einige exakte und asymptotische Ergebnisse für das Standardmodell der Portefeuille - Auswahl innerhalb einer Periode", Statistische Hefte, 16 (1975), Heft 3 , pp. 188-212.

- [12] E.J. Elton and M.J. Gruber, "Improved forecasting through the design of homogeneous groups", The Journal of Business, 44 (1971), October, pp. 432-450.

- [13] E.J. Elton and M.J. Gruber, "Estimating the dependence structure of share prices - implications for portfolio selection", The Journal of Finance, 28 (1973), December, pp. 1203-1232.

- [14] E.F. Fama, "Risk, return and equilibrium: some clarifying comments", The Journal of Finance, 23 (1968), March, pp. 29-40.

- [15] E.F. Fama and M.H. Miller, The theory of finance, Holt, Rinehart and Winston, New-York, 1972.

- [16] D.E. Farrar, The investment decision under uncertainty, Prentice-Hall Inc., Englewood Cliffs, 1962.

- [17] J.L. Farrell, "Analyzing covariation of returns to determine homogeneous stock groupings", The Journal of Business, 47 (1974), April, pp. 186-207.

- [18] M.S. Feldstein, "Mean-variance analysis in the theory of liquidity preference and portfolio selection", The Review of Economic Studies, 36 (1969), January, pp. 5-11.

- [19] W. Feller, An introduction to probability theory and its applications, Vol. II, Wiley & Sons, New-York, 1966.

- [20] G.M. Frankfurter, H.E. Phillips and J.P. Seagle, "Performance of the Sharpe portfolio selection model: a comparison", Journal of Financial and Quantitative Analysis, 11 (1976), June, pp. 195-204.

- [21] G. Hadley, Non-linear and dynamic programming, Addison-Wesley Publishing Company, Reading, 1964.

- [22] M.C. Jensen , "The performance of mutual funds in the period 1945-1964", The Journal of Finance, 23 (1968), May, pp. 389-416.
- [23] M.C. Jensen, "Risk, the pricing of capital assets and the evaluation of investment portfolios", The Journal of Business, 42 (1969), April, pp. 167-247.
- [24] M.C. Jensen , Studies in the theory of capital markets, Praeger Publishers, New-York, 1972.
- [25] M.C. Jensen, Optimal utilization of market forecasts and the evaluation of investment performance, (Working Paper No. 7109, University of Rochester, School of Management, September, 1971).
- [26] M. Kendall , The advanced theory of statistics, Vol. 3, Charles Griffin & Company Limited, London, 1966.
- [27] B.F. King , "Market and industry factors in stock price behaviour", The Journal of Business, 39 (1966), Part II, January, pp. 139-190.
- [28] D. Lessard, "World, country and industry relationship in equity returns", in: International capital markets , eds.: E.J. Elton and M.J. Gruber, North-Holland Publishing Company, Amsterdam, 1975.
- [29] J.Th.H.C. van Lieshout , Investerings-financieringsselectie en wiskundige optimalisering, Ph.D. Thesis, Tilburg School of Economics, Tilburg, 1973.
- [30] J. Lintner, "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets", The Review of Economics and Statistics, 47 (1965), February, pp. 13-37.
- [31] J. Lintner, "Security prices, risks, and maximal gains from diversification", The Journal of Finance, 20 (1965), December, pp. 587-616.
- [32] J. Lintner, "The aggregation of investors' judgement and preferences in purely competitive securities markets", Journal of Financial and Quantitative Analysis, 4 (1969), December, No. 4, pp. 347-400.

- [33] E. Malinvaud, Statistical methods of econometrics, North-Holland Publishing Company, Amsterdam, 1970.
- [34] H.M. Markowitz, "Portfolio selection", The Journal of Finance, 7 (1952), March, pp. 77-91.
- [35] H.M. Markowitz, 'Portfolio selection, efficient diversification of investments, Wiley & Sons, New-York, 1959.
- [36] D. Mayers, "Non-marketable assets and capital market equilibrium under uncertainty", in: M.C. Jensen, ed., Studies in the theory of capital markets (see [24.f]).
- [37] J. Mossin, "Equilibrium in a capital asset market", Econometrica, 34 (1966), October, pp.768-783.
- [38] J.W. Pratt, "Risk aversion in the small and the large", 32 (1964), January-April, pp. 122-136.
- [39] R. Rao, Linear statistical inference and its applications, Wiley & Sons, New-York, 1965.
- [40] P.A. Samuelson, "General proof that diversification pays", Journal of Financial and Quantitative Analysis, 2 (1967), March, pp. 1-13.
- [41] H. Schneeweiss, 'Entscheidungskriterien bei Risiko, Springer Verlag, Berlin, 1967.
- [42] J.Chr. Schreiner, 'Historically successful portfolio selection rules, Ph.D. Thesis, University of California, Los Angeles, 1970.
- [43] W.F. Sharpe, "A simplified model for portfolio analysis", Management Science, 9 (1963), January, pp. 277-293.
- [44] W.F. Sharpe, "Capital asset prices: a theory of market equilibrium under conditions of risk", The Journal of Finance, 19 (1964), September, pp. 425-442.

- [45] W.F. Sharpe and G.M. Cooper, "Risk-return classes of New York Stock Exchange common stocks, 1931-1967", *Financial Analysis Journal*, 28 (1972), March-April, pp. 46-81.
- [46] W.F. Sharpe, "Imputing expected security returns from portfolio composition", *Journal of Financial and Quantitative Analysis*, 9 (1974), June, pp. 463-472.
- [47] K.V. Smith, "Stock price and economic indexes for generating efficient portfolios", *The Journal of Business*, 42 (1969), July, pp. 326-336.
- [48] J. Tobin, "Liquidity preference as behaviour towards risk", *The Review of Economic Studies*, 25 (1958), February, pp. 65-85.
- [49] J. Tobin, "The theory of portfolio selection", in: F.H. Hahn and F.P.R. Brechling (eds.), *The theory of interest rates*, St. Martin's Press, New York, 1965.
- [50] O.A. Vasicek, "Capital asset pricing model with no riskless borrowing (unpublished manuscript), Wells Fargo Bank, March, 1971.
- [51] B.A. Wallingford, "A survey and comparison of portfolio selection models", *Journal of Financial and Quantitative Analysis*, 2 (1967), June, pp. 85-106.

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