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Publication date:
1994

Citation for published version (APA):
Feltkamp, V., Tijs, S. H., & Muto, S. (1994). *Bird's tree allocations revisited*. (CentER Discussion Paper; Vol. 1994-35). Unknown Publisher.

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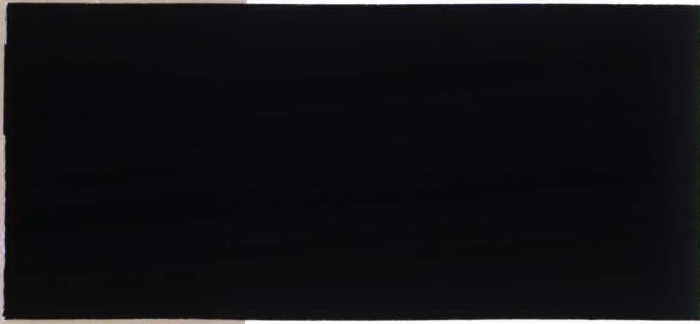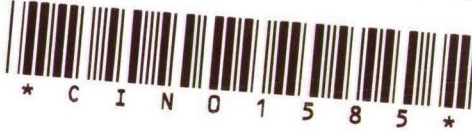
Center
for
Economic Research

No. 9435

BIRD'S TREE ALLOCATIONS
REVISITED

by Vincent Feltkamp,
Stef Tijs and Shigeo Muto

May 1994

Bird's tree allocations revisited

by

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April 25, 1994

Abstract

Minimum cost spanning tree (mcst) construction and cost allocation problems have been studied extensively in the literature, though usually not together. Bird (1976) proposes an allocation rule of which Granot and Huberman (1981) prove that it lies in the core of the associated mcst game. We show that the problems of finding an mcst and allocating its cost can be integrated. Furthermore, we provide an axiomatic characterization of the set of all Bird tree allocations using consistency and converse consistency, and give a strategic form game of which the set of Nash equilibria contains Bird's tree allocations.

1   Introduction

Consider a group of villages, each of which needs to be connected directly or via other villages to a source. Such a connection needs costly links. Each village could connect itself directly to the source, but by cooperating costs might be reduced. This cost minimization problem is an old problem in Operations Research, and Borůvka (1926) came up with algorithms to construct a tree connecting every village to the source with minimal total cost. Later, Kruskal (1956), Prim (1957) and Dijkstra (1959) found similar algorithms. A historic overview of this minimization problem can be found in Graham and Hell (1985).

However, finding a minimal cost spanning tree (mcst) is only part of the problem : if the cost of this tree has to be borne by the villages, then a cost allocation problem has to be addressed as well. Claus and Kleitman (1973) introduced this cost allocation problem, whereupon Bird (1976) treated this problem with game-theoretic methods and for each minimum cost spanning tree proposed a cost allocation associated to it. We call the allocations yielded by this rule *Bird's tree allocations*. As there can be more than one mcst for a given problem, Bird's rule can yield more than one allocation. However, generically, there is only one mcst and then this rule yields a unique allocation.

Granot and Huberman (1981) proved that Bird's tree allocations are extremal points of the core of the associated minimum cost spanning tree game. This game is defined as follows : the players are the villages and the worth of a coalition is the minimal cost of connecting this coalition to the source via links between members of this coalition. Not being satisfied with only one extremal point of the core, Granot and Huberman then provide the weak and strong demand operations, which yield more core elements when applied to Bird's tree allocations. Aarts (1992) found other extreme points of the core in case the mcst problem has an mcst that is a *chain*, i.e. a tree with only two leaves. Kuipers (1993) investigated the core of information graph games. These are games arising from mcst situations in which the costs of links are either one or zero.

The reason for looking at other core allocations than those yielded by Bird's rule, is that although core elements are stable against defection by subcoalitions, an extremal point of the core discriminates against some players. For example, Bird's tree allocations discriminate against the players closest to the root. Granot and Huberman's demand operations remedy this problem by allowing players to demand contributions from players that 'need' them.

In this paper we provide two arguments for the defense of Bird's tree allocations : an axiomatic characterization of the set of Bird's tree allocations, and a non-cooperative game, in which Bird's tree allocations correspond to Nash equilibria.

The outline of this paper is as follows.

Section 2 presents minimum cost spanning tree problems and Bird's rule. Instead of solving the Operations Research and cost allocation problems consecutively, they can be more closely integrated : the cost of a link in an mcst can be allocated at the same moment the link is constructed in the process of forming the tree. Section 3 characterizes the set of allocations yielded by Bird's rule axiomatically, using efficiency, consistency and converse consistency. Section 4 presents a non-cooperative game, in which a strategy of a player consists of choosing how much to contribute to the cost of the links. Bird's tree allocations will turn out to be Nash equilibria of this game. Section 5 concludes.

Preliminaries and notations

We refer to any elementary textbook on graph theory for an understanding of graph theory, but recall some definitions to show the notational conventions. A graph $< V, E >$ consists of a set V of vertices and a set E of edges. An edge e incident with two vertices i and j is identified with $\{i, j\}$[3]. For a graph $< V, E >$ and a set $W \subseteq V$,

$$E(W) := \{e \in E \mid e \subseteq W\}$$

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[3]Because we do not consider multigraphs : two vertices are connected by at most one edge.

is the set of edges linking two vertices in W. For a set $E' \subseteq E$,

$$V(E') := \{v \in V \mid \text{there exists an edge } e \in E' \text{ with } v \in e\}$$

is the set of vertices incident with E'.

The complete graph on a vertex set V is the graph $K_V = <V, E_V>$, where

$$E_V := \{\{v, w\} \mid v, w \in V \text{ and } v \neq w\}.$$

A *path* from i to j in a graph $<V, E>$ is a sequence $(i = i_0, i_1, \ldots, i_k = j)$ of vertices such that for all $l \leq k$, the edge $\{i_{l-1}, i_l\}$ lies in E. A *cycle* is a path of which the begin-point coincides with the end-point. Two vertices $i, j \in V$ are *connected* in a graph $<V, E>$ if there is a path from i to j in $<V, E>$. A subset W of V is *connected* in $<V, E>$ if every two vertices $i, j \in W$ are connected in the subgraph $<W, E(W)>$. A connected set W is a *component* of the graph $<V, E>$ if no superset of W is connected. A *connected graph* is a graph $<V, E>$ with V connected in $<V, E>$. A *tree* is a connected graph that contains no cycles. A *leaf* of a tree is a vertex that is incident to only one edge of the tree.

The cardinality of a set S will be denoted by $|S|$.

With many economic situations in which costs have to be divided one can associate a *cost game* (N, c) consisting of a finite set N of players, and a *characteristic function* $c : 2^N \to \mathbf{R}$, with $c(\emptyset) = 0$. Here $c(S)$ represents the minimal cost for coalition S if it secedes, i.e. if people of S cooperate and can not count upon help from people outside S.

The economic situations in the sequel involve a set N of users of a source $*$. For a coalition $S \subseteq N$, we denote $S \cup \{*\}$ by S^*. Furthermore, for a vector $x \in \mathbf{R}^N$ and a player $i \in N$, we denote x^{-i} the restriction of x to $N \setminus \{i\}$.

The *core* of a cost game (N, c), is defined by

$$\text{Core}(c) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

2 Mcst problems and Bird's tree allocation rule

Formally, a *minimum cost spanning tree (mcst) problem* $<N, *, w>$ consists of a finite group N of agents, each of whom wants to be connected to a common source, denoted by $*$. The non-negative cost of constructing a link $\{i, j\}$ between the vertices i and j in $N^* \equiv N \cup \{*\}$ is denoted by $w(i, j)$. Because of these costs, agents have an incentive to cooperate, and to construct a minimal cost graph that connects them all to the source. If a cycle appears in such a *minimum cost spanning graph*, at least one edge in this cycle can be eliminated, to yield another minimum cost spanning graph, with less cycles. Hence, there are minimum cost spanning graphs that contain no cycles at all, i.e. that are trees. This explains the name of the problem.

Prim (1957) and Dijkstra (1959) proposed the following algorithm to find a minimum cost spanning tree given an mcst problem.

Algorithm 2.1 (Prim and Dijkstra)

> *input* : an mcst problem $\mathcal{T} \equiv <N, *, w>$
> *output* : the edge set T of a minimum cost spanning tree

1. Choose a vertex $v \in N^*$ as *root*.

2. Initialize $T = \emptyset$.

3. Find a minimal cost edge $e \in E_{N^*} \setminus T$ incident to $\{v\} \cup N^*(T)$ such that joining e to T does not introduce a cycle.

4. Join e to T.

5. If not all vertices are connected to the root in the graph $< N^*, T >$, go back to stage 3.

Prim and Dijkstra then prove that any tree resulting from the algorithm is an mcst. Note that by varying between the possible edges in step 3, this algorithm can construct all minimum cost spanning trees of this mcst problem $\mathcal{T}$.

A problem related to such a minimization problem is how to allocate the cost of the edges of a minimum cost spanning tree among the agents (users of the source) in a reasonable way. Bird (1976) associated the following transferable utility mcst-game $(N, c^{\mathcal{T}})$ to an mcst problem $\mathcal{T}$. The players are the agents and the worth $c^{\mathcal{T}}(S)$ of a coalition S is the minimal cost of a tree on $S^* := S \cup \{*\}$. In formula,

$$c^{\mathcal{T}}(S) = \min\{\sum_{e \in T} w(e) \mid T \subseteq E_S \text{ and } < S^*, T > \text{ is a tree}\}$$

for all $S \subseteq N$. Bird also proposed a cost allocation rule for the mcst problem, which he calls the *tree allocation* rule, because it associates a cost allocation to every mcst of the mcst problem. Granot and Huberman (1981) proved that Bird's tree allocation rule yields extreme points of the core of the mcst game. Given an mcst problem $< N, *, w >$ and a mcst $< N^*, T >$ for the grand coalition, Bird's tree allocation β^T is constructed by assigning to a player $i \in N$ the cost of the first edge on the unique path in the tree $< N^*, T >$ from player i to the source $*$. In fact, this allocation is intimately linked with the Prim-Dijkstra algorithm : the tree $< N^*, T >$ and the allocation β^T can be constructed together by choosing the source as root and allocating the cost of the edge added at a certain stage to the person that this edge newly connects to the the source. More formally, the algorithm is the following.

Algorithm 2.2 (Bird's rule integrated into Prim and Dijkstra's algorithm)

input : an mcst problem $< N, *, w >$
output : an edge set T of an mcst and an allocation x (Bird's tree allocation β^T)

1. Choose the source $*$ as root.

2. Initialize $T = \emptyset$.

3. Find a minimal cost edge $e = \{i, j\} \in E_{N^*} \setminus T$ incident to $\{*\} \cup N^*(T)$ such that joining e to T does not introduce a cycle.

4. One of i and j, say j, was previously connected to the source and the other vertex, i, is a player that was not yet connected to the source. Assign the cost $x_i := w(e)$ to agent i.

5. Join e to T.

6. If not all vertices are connected to the root in the graph $< N^*, T >$, go back to stage 3.

As the set of all trees obtained by Prim and Dijkstra's algorithm is independent of the root that is chosen, this algorithm yields the same trees as Prim and Dijkstra's algorithm, and for each tree $< N^*, T >$, it yields an allocation that is Bird's tree allocation β^T associated to this tree. This is easy to see : in step 4, the edge e is precisely the first edge on the unique path from agent i to the source in the tree that will be constructed. If the mcst problem contains two or more edges with the same weight, there might be more than one mcst, and for a particular mcst $< N^*, T >$, it can happen that there is more than one order in which Prim-Dijkstra's algorithm can choose the edges in T. Obviously, the order does not change the edge that a player has to pay according to Bird's tree allocation rule, so Bird's tree allocation β^T is independent of the *order* in which the edges of the tree $< N^*, T >$ are chosen. It does, however, depend on which tree is constructed. See example 2.3.
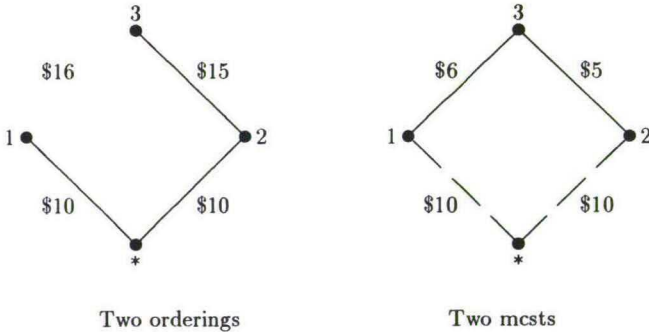
Figure 1: Edges that are not indicated cost $100.

Example 2.3 In the problem on the left of figure 1, whether Prim-Dijkstra's algorithm chooses the links in the unique mcst in the order $\{*, 1\}$, $\{*, 2\}$, $\{2, 3\}$ or $\{*, 2\}$, $\{*, 1\}$, $\{2, 3\}$, the link $\{*, 1\}$ is always paid by player 1 and the link $\{*, 2\}$ is always paid by player 2.

In the problem on the right of figure 1, only one of the two dashed links will be constructed. In case $\{*, 1\}$ is constructed, Bird's tree allocation is $(10, 5, 6)$ and if $\{*, 2\}$ is constructed, Bird's tree allocation is $(6, 10, 5)$.

3 An axiomatic characterization of Bird's tree allocation rule

In this section, we characterize the set of mcst and their associated Bird allocations axiomatically, using efficiency, consistency and converse consistency.

A *solution* of mcst problems is a function ψ assigning to every mcst problem $T = <N, *, w>$, a set

$$\psi(T) \subseteq \{((e^1, \ldots, e^\tau), x) \mid < N^*, \{e^1, \ldots, e^\tau\} > \text{ is connected and } \sum_{i \in N} x_i \geq \sum_{t=1}^{\tau} w(e^t)\}.$$

We mention a few properties of solutions of mcst problems.

Definition 3.1

NE A solution ψ is called *non-empty* if

$$\psi(T) \neq \emptyset \qquad \text{for all mcst problems } T.$$

Eff ψ is *efficient* if for all mcst problems T, all $((e^1, \ldots, e^\tau), x) \in \psi(T)$ are *efficient*, that is, for all $((e^1, \ldots, e^\tau), x) \in \psi(T)$, $< N^*, \{e^1, \ldots, e^\tau\} >$ is a minimal cost spanning tree and

$$\sum_{i \in N} x_i = \sum_{t=1}^{\tau} w(e^t).$$

The two properties of consistency and converse consistency use reduced mcst problems. Here, a reduced mcst problem is an mcst problem where some players have been eliminated. The idea is that solving reduced problems is easier than solving the original problem, and that the solution of the original problem should be related to the solution of the reduced problems. We only ask for this relation if one player that is a leaf is deleted, however. The idea is that a leaf is not needed by any other player to get connected to the source, so if a leaf player is missing, this should not affect the other players.

Definition 3.2 Given an mcst problem $T \equiv < N, *, w >$ and a player $i \in N$, define the *reduced* mcst problem

$$T^{-i} := < N \setminus \{i\}, *, w^{-i} >,$$

where w^{-i} is w restricted to $E_{N^* \setminus \{i\}}$.

Note that the reduced problem does not depend on any fixed solution. Note also that it is indeed an mcst problem. Using reduced mcst problems, we can define consistency and converse consistency as follows.

Definition 3.3

Cons A solution ψ of mcst problems is *consistent* if for every mcst problem $\mathcal{T}$, for every $((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{T})$ and for every player i that is a leaf in the graph $< N^*, \{e^1, \ldots, e^\tau\} >$,

$$((e^1, \ldots, e^\tau)^{-i}, x^{-i}) \in \psi(\mathcal{T}^{-i}),$$

where $(e^1, \ldots, e^\tau)^{-i}$ is obtained from $(e^1, \ldots, e^\tau)$ by deleting the unique edge incident to i and x^{-i} is the vector obtained from x by deleting the coordinate of player i.

CoCons A solution ψ of mcst problems is *converse consistent* if for any mcst problem $\mathcal{T}$ and for any $((e^1, \ldots, e^\tau), x)$ efficient in $\mathcal{T}$, the following is satisfied : if

$$((e^1, \ldots, e^\tau)^{-i}, x^{-i}) \in \psi(\mathcal{T}^{-i})$$

for some player i that is a leaf of $< N^*, \{e^1, \ldots, e^\tau\} >$, then

$$((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{T}).$$

The converse consistency property is motivated by the idea that if a possible efficient solution is excluded, the 'reduced' solution should also be excluded as solution in a reduced problem where a leaf has been deleted. It ensures that solutions that satisfy it are as large as possible without violating efficiency and consistency.

Definition 3.4 The Bird solution of an mcst problem $\mathcal{T}$ is the set

$$\beta(\mathcal{T}) := \{((e^1, \ldots, e^\tau), \beta^T(\mathcal{T})) \mid T = \{e^1, \ldots, e^\tau\} \text{ and } < N^*, T > \text{ is an mcst of } \mathcal{T}\}$$

of sequences of edges of minimum cost spanning trees and the corresponding Bird tree allocations.

Proposition 3.5 The Bird solution satisfies NE, Eff, Cons and CoCons.

Proof : Efficiency was proven by Bird and non-emptiness is evident. In order to prove Cons, assume $((e^1, \ldots, e^\tau), \beta^T) \in \beta(\mathcal{T})$ and let player i be a leaf in the tree $< N^*, T >$, where $T = \{e^1, \ldots, e^\tau\}$. Define e to be the first edge on the unique path in $< N^*, T >$ from i to the source. Then $(e^1, \ldots, e^\tau)^{-i}$ is obtained from $(e^1, \ldots, e^\tau)$ by deleting the edge e and is a sequence obtained by applying the Prim-Dijkstra algorithm to the reduced mcst problem $\mathcal{T}^{-i}$. Hence $x^{-i} = \beta^{T \setminus \{e\}}$, and

$$((e^1, \ldots, e^\tau)^{-i}, x^{-i}) \in \beta(\mathcal{T}^{-i}).$$

In order to prove that the Bird solution satisfies CoCons, assume that $((e^1, \ldots, e^\tau), x)$ is efficient in an mcst problem $\mathcal{T}$ and assume that player i is a leaf of $< N^*, \{e^1, \ldots, e^\tau\} >$ such that

$$((e^1, \ldots, e^\tau)^{-i}, x^{-i}) \in \beta(\mathcal{T}^{-i}). \tag{3.1}$$

Define e_i to be the unique edge incident to i in $\{e^1, \ldots, e^\tau\}$. Then $\{e^1, \ldots, e^\tau\} = \{e^1, \ldots, e^\tau\}^{-i} \cup \{e_i\}$ and $< N^* \setminus \{i\}, \{e^1, \ldots, e^\tau\}^{-i} >$ is an mcst for the reduced mcst problem $\mathcal{T}^{-i}$. Hence efficiency of $((e^1, \ldots, e^\tau), x)$ and equation 3.1 imply

$$\sum_{k \in N} x_k = \sum_{e \in \{e^1, \ldots, e^\tau\}} w(e) = \sum_{e \in \{e^1, \ldots, e^\tau\}^{-i}} w(e) + w(e_i) = \sum_{k \in N \setminus \{i\}} x_k^{-i} + w(e_i),$$

which implies that $x_i = w(e_i)$. So $((e^1, \ldots, e^\tau), x) \in \beta(\mathcal{T})$. $\qquad \square$

Lemma 3.6 If a solution ϕ satisfies Eff and Cons and a solution ψ satisfies NE, Eff and CoCons, then $\phi(\mathcal{T}) \subseteq \psi(\mathcal{T})$ for all mcst problems $\mathcal{T}$.

Proof : We proceed by induction on the cardinality of N. Let $|N| = 1$ and denote by e the edge between the unique player and the source. By efficiency of both solutions and non-emptiness of ψ, we obtain $\phi(\mathcal{T}) \subseteq \{((e), w(e))\} = \psi(\mathcal{T})$. Take an mcst problem $\mathcal{T}$ with $k > 1$ players, and suppose that for all mcst problems $\mathcal{T}'$ with less than k players, $\phi(\mathcal{T}') \subseteq \psi(\mathcal{T}')$. Take $((e^1, \ldots, e^\tau), x) \in \phi(\mathcal{T})$ and choose a leaf $i \neq *$ of the tree T induced by $(e^1, \ldots, e^\tau)$. Then by consistency of ϕ, $((e^1, \ldots, e^\tau)^{-i}, x^{-i}) \in \phi(\mathcal{T}^{-i}) \subseteq \psi(\mathcal{T}^{-i})$. Now $((e^1, \ldots, e^\tau), x)$ is efficient, hence $((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{T})$ by converse consistency of ψ. $\qquad \square$

Theorem 3.7 The unique solution that satisfies NE, Eff, Cons, and CoCons is the Bird solution.

Proof : The Bird solution has the properties, and if another solution has the properties, by lemma 3.6, it coincides with the Bird solution. $\qquad \square$

The properties used to characterize the Bird solution are logically independent. We show this by giving examples of solutions that satisfy three of the four properties.

Example 3.8 If we leave out the non-emptiness property, the empty solution that assigns the empty set to every mcst problem satisfies Eff, Cons and CoCons.

Example 3.9 If we leave out the efficiency property, the solution that assigns $((e)_{e \in E_{N^*}}, (a, \ldots, a))$ to every mcst problem, satisfies the other three properties. Here $(e)_{e \in E_{N^*}}$ denotes the sequence of all edges of the complete graph on N^* ordered by non-decreasing magnitude, and $a = sum_{e \in E_{N^*}} w(e)$. Notice that there are no leaves in the complete graph, except if there is only one player, so the consistency property is satisfied vacuously.

Example 3.10 If we leave out consistency, the solution that assigns to an mcst problem $< N, *, w >$ the set of all efficient outcomes

$$\{((e^1, \ldots, e^\tau), x) \mid < N^*, \{e^1, \ldots, e^\tau\} > \text{ is an mcst, } x \in \mathbf{R}^N \text{ and } x(N) = \sum_{t=1}^\tau w(e^t)\}$$

satisfies the three other properties.

For the last example, we assume there is a total ordering $<$ on the set of all possible players. This is a reasonable assumption : usually, names of players are finite strings in some finite alphabet, which can be alphabetically ordered. Define the lexicographical order on elements of a solution to an mcst problem $< N, *, w >$ by $((e^1, \ldots, e^\tau), x) \prec_L ((\hat{e}^1, \ldots, \hat{e}^{\hat{\tau}}), y)$ if there exists a $k \in N$ such that $x_i = y_i$ for $i < k$ and $x_k < y_k$.

Example 3.11 If we leave out converse consistency, the solution that assigns to every mcst problem the set of lexicographically smallest elements of the Bird solution satisfies the three other properties, but does not coincide with the Bird solution on all mcst problems, so it does not satisfy converse consistency.

4 Sustaining the Bird tree allocations by Nash equilibria

In the previous sections we studied mcst problems by means of cooperative games. In this section, we model the problems by strategic games, in which an action of a player consists of a specification of the edges to which this player will contribute, and which amounts he will contribute.

Definition 4.1 To an minimum cost spanning tree problem $< N, *, w >$, we associate the strategic game $< N, (A^i)_{i \in N}, (u_i)_{i \in N} >$ in normal form with player set N, and in which an action $a^i = (a^i_j)_{j \in N^* \setminus \{i\}} \in A^i \equiv \mathbf{R}_+^{N^* \setminus \{i\}}$ of a player i specifies for each other vertex j (j can be a player or the source), the non-negative amount a^i_j that player i is willing to contribute to the cost of the edge $\{i, j\}$. The utility that player i derives from a strategy profile $a = (a^i)_{i \in N}$ is determined in the following way. We assume that players dislike making contributions, but they absolutely have to be connected to the source. So the utility function is linear in the contributions, and a big penalty is subtracted if the player is not connected to the source in the graph resulting from the contributions of all players. More precisely, for a strategy profile a, the set C_a of edges that have been completely paid for and that will be constructed is defined as

$$C_a := \{\{i, j\} \in E_{N^*} \mid a^i_j + a^j_i \geq w(e)\}$$

and the utility of player i is defined as

$$u_i(a^1, \ldots, a^n) := \begin{cases} -\displaystyle\sum_{j \in N^* \setminus \{i\}} a^i_j & \text{if } i \text{ is connected to the source in } < N^*, C_a > \\ -\displaystyle\sum_{j \in N^* \setminus \{i\}} a^i_j - P & \text{otherwise} \end{cases}$$

where P is a large number ($P > \displaystyle\sum_{e \in E_{N^*}} w(e)$).

We proceed to establish a relationship between the Bird solution presented in section 2 and Nash equilibria of the above strategic mcst game.

Theorem 4.2 Each element $((e^1, \ldots, e^\tau), x)$ of the Bird solution of an mcst problem corresponds to a Nash equilibrium of the associated strategic mcst game, in which the strategy of a player i is to construct the first edge on the unique path from i to the source in the tree $< N^*, \{e^1, \ldots, e^\tau\} >$ and his payoff equals $-x_i$.

Proof : Let $\mathcal{T} = < N, *, w >$ be an mcst problem and let $((e^1, \ldots, e^\tau), x)$ be an element of the Bird solution $\beta(\mathcal{T})$. The corresponding strategy a^i for a player i is to contribute only to the first edge e_i that lies on the unique path from i to the source in the tree $< N^*, \{e^1, \ldots, e^\tau\} >$ and to pay the cost $w(e_i)$ of this edge completely. If every player plays this strategy, the resulting set C_a of constructed edges is precisely $\{e^1, \ldots, e^\tau\}$, which implies that all players are connected to the source. So the payoff to player i equals $-w(e_i)$. Hence $u_i(a) = -w(e_i) = -x_i$.

To prove that a is a Nash equilibrium, suppose that a player i deviates from a. It is clear that contributing more to the edge e_i does not improve i's payoff. If i contributes less than $w(e_i)$ to the edge e_i, then e_i is not paid completely, so it will not be constructed. Now player i wants to avoid the penalty, which is larger than $w(e_i)$, so i has to pay at least one other edge e' that connects the component of i in the graph $< N^*, C_a \setminus \{e_i\} >$ to the component of the source. Because $< N^*, \{e^1, \ldots, e^\tau\} >$ is an mcst, such an edge e' has to be at least as costly as the edge e_i. Hence i is not better off. □

If the costs of all edges are positive, it can be proved that in all Nash equilibria of the strategic game a spanning tree is formed, although it does not have to be a mcst.

Theorem 4.3 If in an mcst problem the costs of all edges are positive, each Nash equilibrium a of the strategic mcst game specifies a spanning tree $< N^*, C_a >$ for the mcst problem, and the payoff vector is $-\beta^{C_a}$. Here, for any spanning tree $< N^*, T >$ we again denote β^T the allocation assigning to every player the cost of the first edge on the unique path from the player to the source in this tree.

Proof : Let $< N, *, w >$ be an mcst problem in which the costs of all edges is positive and let $< N, (A^i)_{i \in N}, (u_i)_{i \in N} >$ be the associated strategic mcst game. Let $a = (a^1, \ldots, a^n)$ be a Nash equilibrium and consider the set C_a of edges that have been completely paid. If a player i is not connected to the source in the graph $< N^*, C_a >$, then by deviating and using the strategy $\hat{a}$ in which he pays precisely the link $\{i, *\}$, he can improve his payoff. So in a Nash equilibrium, every player is connected to the root. Furthermore, if a cycle were present in the graph $< N^*, C_a >$, there is a player i that contributes a positive amount to an edge of this cycle. Then i can improve his payoff by not contributing to this edge. Hence $< N^*, C_a >$ is a tree.

It is clear that no edge of the constructed tree will be 'overpaid' and that no other edge will be contributed to. Furthermore, every player i contributes only to edges that lie on the path in the tree from i to the source. If this were not true, some player could reduce his contribution to an edge that does not lie on the path from himself to the source, without incurring the penalty. Now there is only one edge in the path from i to the source in the tree that i can contribute to, and that is the edge $e = \{i, j\}$ in the tree, that is the first edge on the unique path in the tree from i to the source. By an induction

argument one sees that no other player contributes anything to this edge, which implies that i pays $w(e)$ alone. So $a_j^i = w(e)$ and $a_k^i = 0$ for all other $k \in N^*$, and the payoff is $u_i(a) = -w(e) = \beta^{C_a}$. □

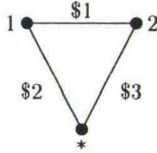Not all Nash equilibria correspond to mcsts, as is shown by the next example.

Figure 2: A problem with a Nash equilibrium that does not correspond to an mcst.

Example 4.4 Consider the problem drawn in figure 2. The strategy pair where player 1 pays the edge $\{1,2\}$ and player 2 pays $\{2,*\}$ is a Nash equilibrium in the associated strategic game, but the unique mcst uses the edge $\{1,*\}$ instead of $\{2,*\}$.

We call a Nash equilibrium *total payoff maximizing* if the sum of the payoffs to the players in this equilibrium is maximal among all Nash equilibria.

Theorem 4.5 In an mcst problem with positive weights, the total payoff maximizing Nash equilibria correspond to mcsts.

Proof : We know that a Nash equilibrium corresponds to a tree, and that the payoffs of the players correspond to costs of edges in the tree. If the total of the payoffs is maximal, the cost of the tree is minimal, hence the tree is a mcst. □

Remark Total payoff maximizing Nash equilibria are strong Nash equilibria (private communication by Gert-Jan Otten). In a strong equilibrium, deviations by coalitions of players do not strictly improve the payoffs of all players in the coalition. Hence, theorem 4.2 can be replaced by a stronger theorem, viz.,

Theorem 4.6 Each element $((e^1, \ldots, e^r), x)$ of the Bird solution of an mcst problem corresponds to a strong Nash equilibrium of the associated strategic mcst game, in which the strategy of a player i is to construct the first edge on the unique path from i to the source in the tree $< N^*, \{e^1, \ldots, e^r\} >$ and his payoff equals $-x_i$.

However, the non-total-payoff-maximizing equilibrium in example 4.4 is a strong Nash equilibrium. Hence, the set of all strong equilibria does not coincide with the set of all total payoff maximizing equilibria in all strategic mcst games.

5 Conclusion

In this paper, we have reconsidered Bird's tree allocations for mcst problems, and have studied them from two points of view. First we gave an axiomatic characterization which permits us to determine whether the allocation rule is applicable. Second we introduced

a non-cooperative game, of which the total payoff maximizing Nash equilibria coincide with Bird's tree allocations.

Instead of using Bird's tree allocation rule, one could use elements of the irreducible core of a mcst problem to evaluate it. The irreducible core was also introduced by Bird (1976) and studied by Granot and Huberman (1982) and Aarts and Driessen (1993). We axiomatically characterize the irreducible core of a generalization of mcst problems in Feltkamp, Muto and Tijs (1994a), and also prove that it can be obtained as the set of all allocations that are associated to Kruskal's (1956) algorithm to construct minimum cost spanning trees.

Knowing that there are several algorithms to construct minimum cost spanning trees, one can consider associating other allocations to them. This is done in Feltkamp, Tijs and Muto (1994a) and (1994b) for Kruskal's algorithm, and an older algorithm described in Borůvka (1926).

Acknowledgements : Inspiring discussions with Peter Borm, Michael Maschler, Gert-Jan Otten, Jos Potters, Hans Reijnierse and Oscar Volij about the subject of the paper are gratefully acknowledged.

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