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ABSTRACT

This paper studies games where players' beliefs about their opponents' behaviour are modelled as non-additive probabilities. The concept of an *equilibrium under uncertainty* is introduced and discussed in the context of two-player games. Existence of such an equilibrium is demonstrated under usual conditions. For simple capacities equilibria under uncertainty approximate Nash equilibria as the degree of uncertainty vanishes. At the other extreme, as the degree of uncertainty goes to one, maximin equilibria appear. Finally, it can be shown that the use of non-additive probabilities can eliminate some implausible Nash equilibria.

Keywords: Uncertainty aversion, Choquet integral, non-additive probabilitiy, equilibrium concepts

JEL Classification: C72, D81

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NON-ADDITIVE BELIEFS AND GAME THEORY

Jürgen Eichberger and David Kelsey[1]

1 Introduction

Experimental evidence indicates that preferences over uncertain acts cannot be represented by expected utility functionals. In particular, Ellsberg (1961) and others have pointed out that decision makers distinguish between situations under certainty and uncertainty in a way that cannot be represented by an additive probability distribution. Recently, Schmeidler (1989) and Gilboa (1987) have axiomatised a preference functional which does not imply that beliefs are represented by additive probabilities.

With the exception of Dow and Werlang (1991), no attempt has been made to investigate the implications of a decision theory without additive probabilities for game theory. Von Neumann and Morgenstern (1944) were among the first to axiomatise a decision theory for known lotteries and, since then, this has been the dominant paradigm for the analysis of games. In a Nash equilibrium, beliefs about the behaviour of opponents were supposed to coincide with the actual behaviour of these players. Interestingly, von Neumann and Morgenstern (1944) did not propose this equilibrium concept. They worked with a decision theory of complete ignorance assuming that players choose maximin strategies. According to this behavioural assumption agents consider the worst outcome for all strategies available and choose the strategy which yields the best among these worst outcomes. Von Neumann and Morgenstern (1944) could prove however that the value concept built on maximin behaviour coincides with the equilibrium concept for the class of zero-sum games.

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[1] The first discussions on this topics between the authors took place while Jürgen Eichberger was visiting the University of Birmingham. The paper was written while Jürgen Eichberger was visiting the CentER at Tilburg University. Jürgen Eichberger would like to thank both institutions for their hospitality which made this research possible. David Kelsey would like to acknowledge a travel grant from the School of Social Sciences, University of Birmingham. We would like to thank seminar participants at Queens University, The Institute of Advanced Studies in Vienna, the universities of Exeter, Osnabrück, and Birmingham for comments and discussion.

We will show in this paper how the theory of non-additive probabilities allows to reconcile the maximin approach with the Nash equilibrium concept for general games. The notion of a *degree of confidence* can be used to relate the two approaches. If the degree of confidence in a subjective probability representation of a belief is high players will behave like expected utility maximisers whereas they will act as maximin players for a high degree of uncertainty. It can be shown that, in general, equilibrium behaviour with non-additive beliefs will be as predicted by the Nash equilibrium concept if this degree of confidence is high. For low levels of confidence, however, equilibrium behaviour under uncertainty may be maximin behaviour.

A further result is the observation that there is a generic class of games where some lack of confidence in the probability assessment will rule out play of dominated strategies in equilibrium. This provides the possibility of a refinement of Nash equilibrium that is weaker than perfectness but still eliminates dominated strategy play in many games.

The paper is organised as follows. The next section introduces notation and provides a definition of an equilibrium concept with non-additive probabilities. Section 3 presents the concept of the degree of confidence, proves existence of an equilibrium and shows the dependence of equilibrium behaviour on this degree of confidence. Section 4 relates the equilibrium notion to the concept of maximin strategies and the final section shows how robustness against lack of confidence may be used as a refinement of Nash equilibrium.

2 Equilibrium under uncertainty

Consider a game $\Gamma = (I, (S_i)_{i\in I}, (p_i)_{i\in I})$ with a finite player set I, finite pure strategy sets S_i for each player and payoff functions $p_i(s_i, s_{-i})$ which describe the payoff of player i if a strategy combination (s_i, s_{-i}) is played. The notation $s_{-i} := (s_1, \ldots s_{i-1}, s_{i+1}, \ldots s_I)$ indicates a strategy combination for all players exept player i. It is convenient to denote by $S_{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_I$ the set of strategy combinations which players other than i could choose.

Beliefs of players about opponents' behaviour are represented by non-additive probabilities (or *capacities*). A capacity assigns non-additive probability weights to subsets of a set of unknown states of the world. Let Ω be a set of states of the world, then one can formally define a capacity as follows.

Definition 2.1: *(capacity)*
A *capacity* on Ω is a real-valued function ν on the subsets of Ω which satisfies the following properties:

a) $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$;

b) $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$.

The capacity is called *convex* if $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$ holds.

Notice that, in contrast to an additive probability, it is not required that $\nu(A) + \nu(B) = \nu(A \cup B) + \nu(A \cap B)$. For a convex capacity it is possible that $\nu(A) + \nu(\Omega \backslash A) < 1$ holds, implying that not all probability mass is allocated to a set and its complement. Thus, one can define the support of a capacity either as the smallest set of states with measure one or the smallest set with a complement of measure zero[2]. We will use the latter notion which can be formally defined as follows:

Definition 2.2: *support of a capacity*
The *support of the capacity* ν is an event $E \subseteq \Omega$ such that $\nu(\Omega \backslash E) = 0$ and $\nu(F) > 0$ for all F, $\Omega \backslash E \subset F$.

---

[2] Compare Dow and Werlang (1991) for a discussion and analysis of these two notions of a support of a measure.

Notice that the support of a capacity as defined in 2.2 is not unique in general. This is a problem which will be discussed in section 3. If the set of subsets of Ω has a measurable structure, then one can use the concept of a Choquet integral to determine the average value of measurable functions.

If Ω is finite and $f: \Omega \longrightarrow \mathbb{R}$ is a function that associates values to states of the world, then $f(\Omega)$ is a finite set of real numbers. Denote by $f_1 = \max\{f(\omega) \mid \omega \in \Omega\}$ and by $Q_1 = \operatorname{argmax}\{f(\omega) \mid \omega \in \Omega\}$. Let $T_1 = Q_1$ and define iteratively,

$Q_j = \operatorname{argmax}\{f(\omega) \mid \omega \in \Omega \setminus T_{j-1}\}$, $f_j = \max\{f(\omega) \mid \omega \in \Omega \setminus T_{j-1}\}$, and $T_j = T_{j-1} \cup Q_j$.
With the convention $Q_0 = \emptyset$, the Choquet integral can be formally defined as follows:

Definition 2.3: *(Choquet integral)*
For a finite set of states of the world Ω, the Choquet integral of a real-valued function f with regard to the capacity ν is

$$\Im(f,\nu) \equiv \sum_{i=1}^{n} f_i \cdot [\nu(T_i)) - \nu(T_{i-1})].$$

Each player $i \in I$ is supposed to hold a belief about her opponents behaviour, a strategy combination in S_{-i}, which is represented by a capacity ν_i. Given this belief, the expected payoff from a strategy s_i can be determined by the Choquet integral $P_i(s_i,\nu_i) := \Im(p_i(s_i,\cdot),\nu_i)$. Players are assumed to choose a best response given their beliefs about their opponents' behaviour. Denote by $R_i(\nu_i) = \operatorname{argmax}\{ P_i(s_i,\nu_i) \mid s_i \in S_i\}$ the best response correspondence of player i given beliefs ν_i.

To determine an equilibrium, it is necessary to relate players' beliefs to the actual behaviour. In a Nash equilibrium of a game where players' beliefs are represented by additive probabilities, expectations are assumed to be rational in the sense that each player chooses a (mixed) strategy which is a best response to the actual (mixed) strategies played by the opponents. Thus, beliefs and actual behaviour are usually not distinguished[3]. In the context of beliefs represented by non-additive probabilities, it is no longer possible to identify beliefs with actual behaviour. An important property of a Nash equilibrium is however the fact that all pure strategies

[3] Crawford (1990) considers beliefs as additive probbaility distributions over mixed strategy spaces.

that are used in equilibrium with positive probability, i.e., that are in the support of the equilibrium mixed strategy, form best responses of the respective player.

The following definition of an *equilibrium under uncertainty* generalises the notion used in Dow and Werlang (1991) to games with an arbitrary finite number of players.

Definition 2.4: *equilibrium under uncertainty*
A belief system $(\nu_1^*, \dots, \nu_I^*)$ is an *equilibrium under uncertainty* if for all $i \in I$

$$\text{supp}\ \nu_i^* \subseteq \prod_{i \neq j \in I} R_j(\nu_j^*)\ .$$

In an equilibrium under uncertainty players beliefs as represented by the capacity ν^* put positive probability only on strategies that are best responses of the opponents given their equilibrium beliefs. Equilibrium beliefs can however no longer be interpreted as equilibrium mixed strategies because a player may believe that her opponents' strategy choices are correlated. The only consistency required in an equilibrium under uncertainty is each player chooses a pure strategy that is a best response given her beliefs and beliefs give positive probability weight only to best responses. The following example will illustrate this equilibrium concept.

Example 2.1:
Consider a general 2x2 matrix game.

Player 2

		t_1	t_2
Player 1	s_1	a_{11}, b_{11}	a_{12}, b_{12}
	s_2	a_{21}, b_{21}	a_{22}, b_{22}

In this case, beliefs of each player can be represented by two numbers indicating the probability that the opponent plays either of her two strategies:

$$\nu_1 = (q_{t1}, q_{t2})\ , \qquad q_{t1} + q_{t2} \le 1,$$
$$\nu_2 = (q_{s1}, q_{s2})\ , \qquad q_{s1} + q_{s2} \le 1.$$

Notice that these probabilities need not sum to one. The support of these capacities is simply

$$\text{supp } \nu_1 = \begin{cases} \{t_1\} & \text{for } q_{t2} = 0 \\ \{t_2\} & \text{for } q_{t1} = 0 , \\ \{t_1, t_2\} & \text{otherwise} \end{cases}$$

and

$$\text{supp } \nu_2 = \begin{cases} \{s_1\} & \text{for } q_{s2} = 0 \\ \{s_2\} & \text{for } q_{s1} = 0 . \\ \{s_1, s_2\} & \text{otherwise} \end{cases}$$

Given beliefs ν_1 and ν_2, it is straightforward to compute the following Choquet expected payoff functions:

$$P_1(s_i, \nu_1) = \begin{cases} a_{i1} \cdot q_{t1} + a_{i2} \cdot (1 - q_{t1}) & \text{for } a_{i1} > a_{i2} \\ a_{i1} & \text{for } a_{i1} = a_{i2} , \\ a_{i2} \cdot q_{t2} + a_{i1} \cdot (1 - q_{t2}) & \text{for } a_{i1} < a_{i2} \end{cases}$$

$$P_2(t_i, \nu_2) = \begin{cases} b_{1i} \cdot q_{s1} + b_{2i} \cdot (1 - q_{s1}) & \text{for } b_{1i} > b_{2i} \\ b_{1i} & \text{for } b_{1i} = b_{2i} . \\ b_{2i} \cdot q_{s2} + b_{1i} \cdot (1 - q_{s2}) & \text{for } b_{1i} < b_{2i} \end{cases}$$

Consider the case[4] where the game has two Nash equilibria in pure and one in mixed strategies. This implies the following payoff parameter restrictions:

$$a_{11} > a_{12},\ a_{21} < a_{22} \quad \text{and} \quad b_{11} > b_{21},\ b_{12} < b_{22}.$$

With these restrictions, one derives the following best response corrspondences:

$$R_1(\nu_1) = \begin{cases} \{s_1\} & \text{for } q_{t1} > \\ \{s_1, s_2\} & \text{for } q_{t1} = \dfrac{(a_{21} - a_{12})}{(a_{11} - a_{12})} + \dfrac{(a_{22} - a_{21})}{(a_{11} - a_{12})} \cdot q_{t2} \\ \{s_2\} & \text{for } q_{t1} < \end{cases}$$

$$R_2(\nu_2) = \begin{cases} \{t_1\} & \text{for } q_{s1} > \\ \{t_1, t_2\} & \text{for } q_{s1} = \dfrac{(b_{12} - b_{21})}{(b_{11} - b_{21})} + \dfrac{(b_{22} - b_{12})}{(b_{11} - b_{21})} \cdot q_{s2} \\ \{t_2\} & \text{for } q_{s1} < \end{cases}$$

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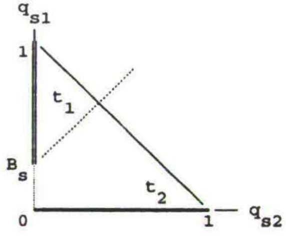
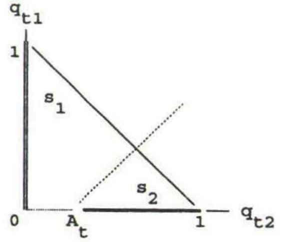
[4] The other generic payoff constellations can be analysed mutatis mutandis.

The following diagrams show the situation.





$$B_s \equiv \frac{(b_{12} - b_{21})}{(b_{11} - b_{21})} \qquad A_t \equiv \frac{(a_{12} - a_{21})}{(a_{22} - a_{21})}$$

Notice that the parameter restrictions given above guarantee that the denominators of B_s and A_t are positive. There is no restriction, however, on the sign of the numerators. The case[5] depicted in the diagram corresponds to the additional parameter constraints $a_{12} - a_{21} > 0$ and $b_{12} - b_{21} > 0$.

A belief of player 1, ν_1, is a pair (q_{t1}, q_{t2}) on or below the line (0,1) - (1,0) in the right-hand diagram. Points in the left hand diagram represent different beliefs of player 2. The broken upward sloping lines indicate the loci where players are indifferent between their two strategies. These lines are upward sloping by the assumption on parameters made above but may have positive or negative intercepts. Beliefs above these lines make it a best response for each player to choose the first strategy; beliefs below these lines make the second strategies a best response. It is easy to check that the following beliefs form equilibria of the game:

(i) (═══): $\nu_1^* = (q_{t1}, 0)$, $0 \le q_{t1} \le 1$,
$\nu_2^* = (q_{s1}, 0)$, $B_s \le q_{s1} \le 1$;

(ii) (───): $\nu_1^* = (0, q_{t2})$, $A_t \le q_{t2} \le 1$,
$\nu_2^* = (0, q_{s2})$, $0 \le q_{s2} \le 1$;

(iii) (┈┈┈): $\nu_1^* = (0, q_{t2})$, $0 \le q_{t2} \le A_t$,
$\nu_2^* = (q_{s1}, 0)$, $0 \le q_{s1} \le B_s$;

(iv) (┈┈┈┈): $\nu_1^* = (q_{t1}, q_{t2})$, $A_t \le q_{t2} \le 1$,
$\nu_2^* = (q_{s1}, q_{s2})$, $B_s \le q_{s1} \le 1$.

---

5 The other three cases are easily analysed by drawing the respective diagrams.

In an equilibrium of type (i), (s_1,t_1) will be played. This is a Nash equilibrium of this game under the assumed parameter restrictions. Similarly, type (ii) equilibria induce (s_2,t_2) as equilibrium play which corresponds to a second pure strategy Nash equilibrium of the game. Equilibria of type (iii) however lead to (s_1,t_2) being played which is not a Nash equilibrium. It is easy to check that (s_1,t_2) are the maximin strategies of the players given the payoff parameter restiction of the diagram. Finally, equilibria under uncertainty of type (iv) correspond to the mixed strategy Nash equilibrium because players are indifferent about their strategy choice. ■

Example 2.1 shows that, without further restrictions, there are many equilibria under uncertainty. Some of these equilibria induce the same equilibrium play as the Nash equilibria, while other equilibria are incompatible with Nash equilibrium play. The following section suggests a restriction on beliefs.

3 Simple capacities and the degree of confidence

In the game-theoretic context, beliefs have traditionally been represented by additive probability distributions. Identifying beliefs with probability distributions over states however makes it impossible to accommodate behaviour that distinguishes between cases where the decision maker knows the probability distribution over states and cases where s/he is ignorant about this distribution. Capacities allow us to make such a distinction.

General capacities have however little in common with probability distributions. In particular, the capacity of a set and the capacity of its complement need not sum to a number less or equal to one. Since capacities are supposed to represent beliefs, one restricts attention to convex capacities which have the property that $\nu(A) + \nu(\Omega\backslash A) \le \nu(\Omega) = 1$ holds. Even convex capacities can have the undesirable property of a non-unique support. The following definition characterises a class of capacities which is easy to interprete as beliefs and which does not suffer from these problems.

Definition 3.1: A capacity ν is called *simple* if there exists an additive probability distribution π and a real number $\gamma \in (0,1)$ such that for all events $E \subseteq S$, $\nu(E) = \gamma \cdot \pi(E)$.

A *simple capacity*[6] can be thought of as a contraction of an additive probability distribution. This could be interpreted as a lack of confidence that the decision-maker has in regard to her probability assessment. The parameter γ represents the *degree of confidence* that the agent has in the probabilistic assessment given by the additive probability distribution π. The smaller γ, the *degree of confidence*, the more uncertain is the agent about the situation. Uncertainty measured by γ can be distinguished from likelihood of a state represented by $\pi(\omega)$.

A further advantage of the representation of beliefs by simple capacities is the particularly simple form of the Choquet integral.

Proposition 3.1: Let ν be a simple capacity. For a finite set of states Ω,

$$\mathfrak{I}(f,\nu) = \gamma \cdot [\sum_{i=1}^{n} f_i \cdot \pi(Q_i)] + (1-\gamma)\cdot[\min\{f(\omega) \mid \omega \in \Omega\}].$$

Proof: From definition 2.3 and definition 3.1, one computes:

$$\mathfrak{I}(f,\nu) = \sum_{i=1}^{n} f_i \cdot [\nu(T_i) - \nu(T_{i-1})]$$

$$= \sum_{i=1}^{n-1} f_i \cdot \nu(Q_i) + f_n \cdot [\nu(\Omega) - \nu(T_{n-1})] = \sum_{i=1}^{n-1} f_i \cdot \gamma \cdot \pi(Q_i) + f_n \cdot [1 - \gamma \cdot \pi(\Omega \backslash Q_n)]$$

$$= \gamma \cdot \sum_{i=1}^{n-1} f_i \cdot \pi(Q_i) + f_n \cdot [1 - \gamma \cdot (1 - \pi(Q_n))] = \gamma \cdot \sum_{i=1}^{n} f_i \cdot \pi(Q_i) + f_n \cdot [1-\gamma].$$

Noting that $f_n = \min\{f(\omega) \mid \omega \in \Omega\}$ completes the proof. ■

Note that, for $Q_i = \{\omega_i\}$, $i = 1,\dots n$, the Choquet integral simplifies to

$$\mathfrak{I}(f,\nu) = \gamma \cdot [\sum_{i=1}^{n} f_i \cdot \pi(s_i)] + (1-\gamma)\cdot f_n .$$

---

[6] Note that all simple capacities satisfy constant uncertainty aversion as defined in Dow and Werlang (1991). However, it is not the case that all capacities which display constant uncertainty aversion are simple. A special case of a simple capacity has been used by Dow and Werlang (1992).

The Choquet integral is a convex combination between the expected value of f given the additive measure π and the worst outcome f_n with weight $\gamma \in (0,1]$. The full weight of the non-additivity falls on the worst outcome in this case. If γ is close to zero, uncertainty dominates and the value of f is close to the worst outcome. Thus, a decision-maker who chooses an action to maximise the Choquet integral with respect to a simple capacity will

- maximise expected utility with probability distribution π for $\gamma = 1$, and
- choose a maximin action for $\gamma = 0$.

With beliefs characterised by a simple capacity with degree of confidence $\gamma \in (0,1)$, a player shows a decision-making behaviour which lies between expected utility maximisation and the extremely uncertainty-averse maximin behaviour.

Through a number of examples, Dow and Werlang (1991) have shown that the support of a capacity is, in general, not unique. An important property of simple capacities is therefore the fact that they have a unique support.

Proposition 3.2: For a simple capacity the support is unique and consists of all states with positive probability.

Proof: Let ν be a simple capacity on Ω. Since ν is simple there exists an additive probability π on Ω and $\gamma \in (0,1]$ such that for all $E \subseteq \Omega$, $\nu(E) = \gamma \cdot \pi(E)$. Let $B \subseteq \Omega$ denote the set of all states with positive measure π. By definition $\pi(\Omega \backslash B) = 0$, hence $\nu(\Omega \backslash B) = 0$. Let C be an event such that $\Omega \backslash B \subset C$. Then there exists $\omega' \in C \backslash (\Omega \backslash B) \subset B$. By monotonicity $\nu(C) \geq \nu(\{\omega'\}) = \gamma \cdot \pi(\{\omega'\}) > 0$. This demonstrates that B is the support of ν. ■

Applying the concept of a simple capacity to games, it can be shown that an equilibrium under uncertainty exists for any vector of exogenouly given degrees of confidence that the players may have[7].

Proposition 3.3: For any vector of parameters $\gamma := (\gamma_1, \ldots \gamma_I)$ there exists a Nash equilibrium under uncertainty.

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[7] Proposition 3.3 confirms a conjecture of Dow and Werlang (1991) that a *Nash equilibrium under uncertainty* will exist for any exogenously given bound on the uncertainty aversion of players.

Proof:

Let $m_i(s_i) := \min\{p_i(s_i, s_{-i}) \mid s_{-i} \in S_{-i}\}$ be the minimum payoff a player obtains from playing strategy s_i. Define new payoff functions

$$\psi_i(s_i, s_{-i}) := p_i(s_i, s_{-i}) + [(1-\gamma_i)/\gamma_i] \cdot m_i(s_i) \ .$$

The new game $\Gamma' = (I, (S_i)_{i\in I}, (\psi_i)_{i\in I})$ is well-defined and has a Nash equilibrium $\pi^* = (\pi^*_1, \ldots \pi^*_I)$ in mixed strategies. Thus, for all $i \in I$ and all $s^*_i \in S_i$ with $\pi^*_i(s^*_i) > 0$,

$$\sum_{s_{-i}\in S_{-i}} \psi_i(s^*_i, s_{-i}) \cdot \pi^*(s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} \psi_i(s_i, s_{-i}) \cdot \pi^*(s_{-i})$$

for all $s_i \in S_i$ must hold, where

$$\pi(s_{-i}) := \pi_1(s_1) \cdot \pi_2(s_2) \cdot \ldots \cdot \pi_{i-1}(s_{i-1}) \cdot \pi_{i+1}(s_{i+1}) \cdot \ldots \cdot \pi_I(s_I)$$

denotes the probability that s_{-i} is played.

Let $\nu^*_i(s_{-i}) := \gamma_i \cdot \pi^*(s_{-i})$ for all $s_{-i} \in S_{-i}$ denote the belief of player $i \in I$ with constant uncertainty aversion γ_i. It will be shown that $\nu^* = (\nu^*_1, \ldots \nu^*_I)$ forms an equilibrium under uncertainty for the original game with constant uncertainty aversion parameters γ.

Consider $s'_{-i} \in \text{supp } \nu^*_i$. By proposition 3.2 , $\nu^*_i(s'_{-i}) = \gamma_i \cdot \pi^*(s'_{-i}) =$

$$= \gamma_i \cdot [\pi^*_1(s'_1) \cdot \pi^*_2(s'_2) \cdot \ldots \cdot \pi^*_{i-1}(s'_{i-1}) \cdot \pi^*_{i+1}(s'_{i+1}) \cdot \ldots \cdot \pi^*_I(s'_I)] > 0.$$

Hence, for $j \neq i$, $\pi^*_j(s'_j) > 0$ and

$$s'_j \in \operatorname{argmax}\{ \sum_{s_{-j}\in S_{-j}} \psi_j(s_j, s_{-j}) \cdot \pi^*(s_{-j}) \mid s_j \in S_j\} \qquad (*)$$

by the definition of a Nash equilibrium.

Since the capacity ν^*_j is simple,

$$P_j(s'_j, \nu^*_j) := \sum_{s_{-j}\in S_{-j}} p_j(s'_j, s_{-j}) \cdot \nu^*(s_{-j}) + (1-\gamma_i) \cdot m_j(s'_j)$$

$$= \gamma_j \cdot \sum_{s_{-j}\in S_{-j}} [p_j(s'_j, s_{-j}) + [(1-\gamma_j)/\gamma_j] \cdot m_j(s'_j)] \cdot \pi^*(s_{-j})$$

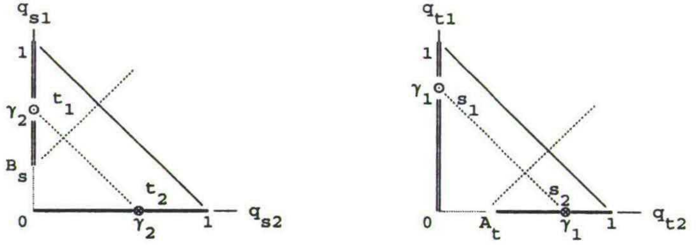
$$= \gamma_j \cdot \sum_{s_{-j}\in S_{-j}} \psi_j(s'_j, s_{-j}) \cdot \pi^*(s_{-j})$$

$$\geq \gamma_j \cdot \sum_{s_{-j}\in S_{-j}} \psi_j(s_j, s_{-j}) \cdot \pi^*(s_{-j}) = P_j(s_j, \nu^*_j)$$

follows from (*). This proves that ν^* is an equilibrium under uncertainty. ■

For the case of 2x2 matrix games which was discussed in example 2.1, the set of equilibria for a given degree of confidence of the players can be given a simple geometric interpretation.

Example 3.1: *(example 2.1 resumed)*
Recall the diagrammatic representation of beliefs in example 2.1. The exogenously given degrees of confidence γ_1 and γ_2 require beliefs to lie on the lines $(0,\gamma_1)$ - $(\gamma_1,0)$ and $(0,\gamma_2)$ - $(\gamma_2,0)$ in the respective diagrams.



If γ_1 and γ_2 are as given in the diagram above, then one checks easily that the following three equilibria under uncertainty exist:

(i) (═○═): $\nu_1^* = (\gamma_1,\ 0)$, $\nu_2^* = (\gamma_2,\ 0)$;

(ii) (—⊗—): $\nu_1^* = (0\ ,\ \gamma_1)$, $\nu_2^* = (0,\ \gamma_2)$;

(iii) $\nu_1^* = (q_{t1},\ q_{t2})$, $\nu_2^* = (q_{s1},\ q_{s2})$ with

$$q_{t1} + q_{t2} = \gamma_1, \quad q_{t1} = \frac{(a_{21} - a_{12})}{(a_{11} - a_{12})} + \frac{(a_{22} - a_{21})}{(a_{11} - a_{12})} \cdot q_{t2},$$

$$q_{s1} + q_{s2} = \gamma_2, \quad q_{s1} = \frac{(b_{12} - b_{21})}{(b_{11} - b_{21})} + \frac{(b_{22} - b_{12})}{(b_{11} - b_{21})} \cdot q_{s2}. \quad ■$$

For high degrees of confidence, equilibrium behaviour for the pure strategy equilibria ((i) and (ii)) will be exactly as in the respective Nash equilibria and the third equilibrium will have a support which coincides with the support of the mixed strategy Nash equilibrium. On the other hand, it is easy to see in this example that low[8] values of γ_1 and γ_2 lead to maximin behaviour in an equilibrium under uncertainty. The following section

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[8] For γ_1, $\gamma_2 < \min \{A_t, B_s\}$, behaviour in an equilibrium under uncertainty is maximin behaviour.

investigates whether this relationship between Nash equilibrium and equilibrium under uncertainty holds for general games as well.

4 Equilibrium under uncertainty and Nash equilibrium

This section relates equilibria under uncertainty with simple capacities to Nash equilibria on the one hand and maximin strategies on the other. It is well-known that, for zero-sum games, every Nash equilibrium strategy combination is a maximin strategy combination. This identity of prudent and equilibrium behaviour is however lost for general games. The concept of an equilibrium under uncertainty with simple capacities makes it possible to relate these two traditional methods of determining the solution of a game (compare Moulin (1986)). Indeed, the less confidence players have in their beliefs about opponents' behaviour the more likely the pure strategy cbosen in equilibrium will be a maximin strategy combination. If players are confident about their beliefs, then they will play as in a Nash equilibrium. The notion of an equilibrium under uncertainty is however general enough to allow for some players to play maximin strategies and for others to play best responses.

First, low degrees of confidence in the probability assessment are considered. Let

$$m_i(s_i) := \min\{p_i(s_i, s_{-i}) \mid s_{-i} \in S_{-i}\}$$

be the worst payoff player i can obtain from playing strategy s_i and let

$$M_i := \text{argmax}\{m_i(s_i) \mid s_i \in S_i\}.$$

be the strategies of player i that maximise these worst payoffs. Note that any strategy combination $s = (s_1, \ldots s_I) \in \prod_{i \in I} M_i$ is a maximin strategy combination. For two-player zero-sum games, any such strategy combination leads to the same payoff vector.

Definition 4.1:
An equilibrium under uncertainty with simple capacities $\nu^* = (\nu_1^*, \ldots \nu_I^*)$ *induces maximin play* if, for all $i \in I$,

$$R_i(\nu_i^*) = M_i .$$

The following result shows that, for low degrees of confidence, equilibria under uncertainty are maximin strategy combinations.

Proposition 4.1: Given beliefs that are represented by simple capacities, there exists $\epsilon > 0$ such that, for $\gamma_i \in [0,\epsilon]$ for all $i \in I$, every equilibrium under uncertainty induces maximin play.

Proof:

Firstly, for $M_i = S_i$, $p_i(s_i, s_{-i}) = p_i(s_i', s_{-i}')$ for all $s_i' \in S_i$ and all $s_{-i}' \in S_{-i}$. Hence, $P_i(s_i, \nu_i^*) = P_i(s_i', \nu_i^*)$ for all $s_i' \in S_i$ and $R_i(\nu_i^*) = S_i$.

Secondly, for $M_i \neq S_i$, consider any strategy $s_i \in R_i(\nu_i^*)$ such that $s_i \notin M_i$, then $m_i(s_i') > m_i(s_i)$ for all $s_i' \in M_i$ and

$$P_i(s_i, \nu_i^*) = \gamma_i \cdot [\sum_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i}) \cdot \alpha_i(s_{-i})] + (1 - \gamma_i) \cdot m_i(s_i)$$

$$\geq \gamma_i \cdot [\sum_{s_{-i} \in S_{-i}} p_i(s_i', s_{-i}) \cdot \alpha_i(s_{-i})] + (1 - \gamma_i) \cdot m_i(s_i') = P_i(s_i', \nu_i^*),$$

where α_i is an additive probability distribution on S_{-i} such that $\nu_i^*(s_{-i}) = \gamma_i \cdot \alpha_i(s_{-i})$ for all $s_{-i} \in S_{-i}$.

Since $m_i(s_i') > m_i(s_i)$ holds, there is a positive ϵ_i small enough such that $P_i(s_i', \nu_i^*) > P_i(s_i, \nu_i^*)$ for all $\gamma_i \leq \epsilon_i$ holds. For such a γ_i, $s_i \in R_i(\nu_i^*) \cap M_i$. Thus, $\epsilon = \min\{\epsilon_i \mid i \in I\}$ provides an upper bound on the degree of confidence such that $R_i(\nu_i^*) = M_i$. ■

Example 3.1 suggests that, for high degrees of confidence, equilibrium play will ressemble a Nash equilibrium. There is, however, a complication if more than two players are considered. For games with three or more agents, a player may believe that the opponents' behaviour is correlated. In a Nash equilibrium, however, players' behaviour must be uncorrelated. This motivates the following definition.

Definition 4.2:

Player i believes that the opponents *act independently* if

$$\nu_i(s_{-i}) = \prod_{j \neq i} \nu_i^j(s_j)$$

holds for some simple capacities $\nu_i^j(s_j) = \gamma_i^j \cdot \pi_i^j(s_j)$, $j \neq i$.

It is one of the advantages of simple capacities that one can define product measures using the additive structure of the defining probability distibution $\pi_i = (\pi_i^1, \ldots \pi_i^{i-1}, \pi_i^{i+1}, \ldots \pi_i^I)$. For general capacities, this is usually impossible.

Definition 4.3:
A system of beliefs that all players act independently is called *consistent* if $\nu_i^j = \nu^j$ for all $i, j \in I$, $i \neq j$.

An equilibrium under uncertainty leads to actual play which is equivalent to Nash equilibrium play if the equilibrium strategy combinations could result from a Nash equilibrium.

Definition 4.4:
An equilibrium under uncertainty with simple capacities $\nu^* = (\nu_1^*, \ldots \nu_I^*)$ induces Nash equilibrium play if there is a Nash equilibrium (in mixed strategies) $\mu^* = (\mu_1^*, \ldots \mu_I^*)$ such that, for all $i \in I$,

$$\text{supp } \mu_i^* \subseteq R_i(\nu_i^*) \quad \text{with} \quad \nu_i^*(s_{-i}) = \prod_{j \neq i} \mu_j^*(s_j) \quad \text{for all } s_{-i} \in S_{-i}.$$

The following proposition relates play in an equilibrium under uncertainty for high degrees of confidence to Nash equilibria.

Proposition 4.2: If all players believe that their opponents act independently and if their beliefs are consistent, then there exists $\epsilon > 0$ such that, for $\gamma_i^j \in [1 - \epsilon, 1]$, $i, j \in I$, $i \neq j$, every equilibrium under uncertainty induces Nash equilibrium play.

Proof:
Consider an equilibrium under uncertainty ν^* where all players believe that their opponents act independently and where beliefs are consistent, then there exists $\pi = (\pi^1, \ldots, \pi^I)$ such, that for all $i \in I$ and all $s_{-i} \in S_{-i}$,

$$\nu_i^*(s_{-i}) = \prod_{j \neq i} \gamma_i^j \cdot \pi^j(s_j), \; i \neq j .$$

By proposition 3.2, $\text{supp } \nu_i^*(s_{-i}) = \prod_{j \neq i} \text{supp } \pi^j(s_j)$, for all $i \in I$. Since ν^* is an equilibrium under uncertainty, one has

$$s_j \in \text{supp } \pi^j \text{ implies } s_j \in R_j(\nu_j^*) \text{ for all } j \in I.$$

Therefore, with $\gamma_j := \prod_{i \neq j} \gamma_j^i$ and $\pi(s_{-j}) := \prod_{i \neq j} \pi^i(s_i)$,

$$\begin{aligned} P_j(s_j, \nu_j^*) &\equiv \gamma_j \cdot [\sum_{s_{-j} \in S_{-j}} p_j(s_j, s_{-j}) \cdot \pi(s_{-j})] + (1 - \gamma_j) \cdot m_j(s_j) \\ &\geq \gamma_j \cdot [\sum_{s_{-j} \in S_{-j}} p_j(s_j', s_{-j}) \cdot \pi(s_{-j})] + (1 - \gamma_j) \cdot m_j(s_j') \equiv P_j(s_j', \nu_j^*) \end{aligned}$$

for all $s_j' \in S_j$.

Such an equilibrium under uncertainty exists for all degrees of confidence γ (proposition 3.3). Hence, there must be an ϵ_j such that, for all $\gamma_j \geq 1-\epsilon_j$ and the associated probability distribution π,

$$[\sum_{s_{-j}\in S_{-j}} p_j(s_j,s_{-j}) \cdot \pi(s_{-j})] \geq [\sum_{s_{-j}\in S_{-j}} p_j(s_j',s_{-j}) \cdot \pi(s_{-j})] .$$

Thus, π is a Nash equilibrium (in mixed strategies) and $\mu^* := \pi$ satisfies the claim of the proposition. ■

In two-player games, the consistency and independence condition of proposition 4.2 are trivially satisfied.

5 Robustness against lack of confidence

As the previous section has shown, for γ_i close to 1 for all $i \in I$, equilibrium behaviour under uncertainty is the same as in a Nash equilibrium. This raises the question whether all Nash equilibria have a nearby equilibrium under uncertainty with the same equilibrium play. If this is the case for high degrees of confidence, then one can consider the Nash equilibrium as robust against some lack of confidence. On the other hand, if there are games where, for some Nash equilibrium, there is no equilibrium under uncertainty inducing this Nash equilibrium play, then such a Nash equilibrium is not robust against some lack of confidence.

The following proposition shows that there are games with Nash equilibria whose equilibrium play occurs in no equilibrium under uncertainty.

Proposition 5.1:
Consider games such that $m_i(s_i) \neq m_i(s_i')$ for all $s_i \neq s_i'$ holds. If beliefs are represented by simple capacities, then an equilibrium under uncertainty does not use dominated strategies.

Proof:
Let $\nu^* = (\nu_1^*, \ldots . \nu_I^*)$ be an equilibrium under uncertainty. Since ν_j^* are simple capacities there must be an additive probability distribution α^j on S_{-j} such that $\nu_j^* = \gamma_j \cdot \alpha^j$ for all $j \in I$. Let s_i be a strategy of some player $i \neq j$ which forms part of a strategy combination $s_{-j} \in \text{supp } \nu_j^*$. By the definition of an equilibrium under uncertainty, one has

$$P_i(s_i, \nu_i^*) \equiv \gamma_i \cdot [\sum_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i}) \cdot \alpha^i(s_{-i})] + (1-\gamma_i) \cdot m_i(s_i)$$

$$\geq \gamma_i \cdot [\sum_{s_{-i} \in S_{-i}} p_i(s_i', s_{-i}) \cdot \alpha^i(s_{-i})] + (1-\gamma_i) \cdot m_i(s_i) \equiv P_i(s_i', \nu_i^*)$$

for all $s_i' \in S_i$.

Suppose that s_i is dominated by strategy s_i'', i.e. $p_i(s_i'', s_{-i}) \geq p_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ with strict inequality for some s_{-i}. Clearly,

$$\sum_{s_{-i} \in S_{-i}} p_i(s_i'', s_{-i}) \cdot \alpha^i(s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i}) \cdot \alpha^i(s_{-i})$$

and

$$P_i(s_i'', \nu_i^*) \equiv \gamma_i \cdot [\sum_{s_{-i} \in S_{-i}} p_i(s_i'', s_{-i}) \cdot \alpha^i(s_{-i})] + (1-\gamma_i) \cdot m_i(s_i'')$$

$$> \gamma_i \cdot [\sum_{s_{-i} \in S_{-i}} p_i(s_i, s_{-i}) \cdot \alpha^i(s_{-i})] + (1-\gamma_i) \cdot m_i(s_i) \equiv P_i(s_i, \nu_i^*)$$ in

this case because

$$m_i(s_i'') = p_i(s_i'', \tilde{s}_{-i}) \geq p_i(s_i, \tilde{s}_{-i}) \geq m_i(s_i).$$

By the premise $m_i(s_i) \neq m_i(s_i'')$ and, hence,

$$m_i(s_i'') > m_i(s_i).$$

This contradicts the required optimality of s_i. ■

We conclude this section with an example which illustrates the necessity of the condition $m_i(s_i) \neq m_i(s_i'')$.

Example 5.1:

Consider the following 2x2 matrix game.

Player 2

Player 1		t_1	t_2
	s_1	1,1	0,-1
	s_2	-1,0	0,0

It is easy to check that

$$m_1(s_1) = 0 \neq m_1(s_2) = -1 \quad \text{and}$$
$$m_2(t_1) = 0 \neq m_2(t_2) = -1$$

holds. There are two Nash equilibria in pure strategies (s_1, t_1) and (s_2, t_2). The second Nash equilibrium (s_2, t_2) uses dominated strategies. According to proposition 5.1 there is therefore no equilibrium under uncertainty with this support. To check this claim, consider player 1. The Choquet expected payoff of strategy 1 is $P_1(s_1, \nu_1) = q_{t1}$ and for strategy 2 $P_1(s_2, \nu_1) = q_{t2} - 1$. For any degree of confidence less than 1, it is impossible that

$$P_1(s_1, \nu_1) = q_{t1} \leq P_1(s_2, \nu_1) = q_{t2} - 1 < 1.$$

Since this is necessary for an equilibrium under uncertainty with support (s_2, t_2), no equilibrium under uncertainty can have this Nash equilibrium play.

Consider on the other hand the following modification of this game.

Player 2

Player 1		t_1	t_2
	s_1	1,1	0,0
	s_2	0,0	0,0

Notice that this modification did not change the set of pure Nash equilibria nor the fact that s_2 and t_2 are dominated strategies. The condition of proposition 5.1 is however no longer satisfied, since

$$m_1(s_1) = m_1(s_2) = 0 \quad \text{and} \quad m_2(t_1) = m_2(t_2) = 0.$$

The condition on the Choquet expected utilities of player 1 for an equilibrium under uncertainty is now

$$P_1(s_1, \nu_1) = q_{t1} \leq P_1(s_2, \nu_1) = 0,$$

which can be satisfied for $q_{t1} = 0$. Similarly, one can show that for $q_{s1} = 0$ player 2 may find it optimal to choose t_2. Thus, in this case, there is an equilibrium under uncertainty compatible with this Nash equilibrium play. ∎

Proposition 5.1 and the example show that 'robustness against some lack of confidence', does not coincide with refinements like perfectness or iterated deletion of dominated strategies. The necessary condition of proposition 5.1, that there be no ties between the minimum payoffs achieved with different pure strategies of a player, is however not a generic property for two-player games with finite pure strategy sets. Thus, there may be a closer relationship between these refinements for generic games.

The paper concludes with an example which illustrates how the notion of a degree of confidence may be used to derive new results in economic applications.

Example 5.2:

Consider n communities that have to cooperate in the prevention of pollution. The quality of the environment x depends on the contribution of each community v_i, $i = 1,\dots,n$, *according to the production function* $x = \min\{v_i \mid i = 1,\dots n\}$. *Thus, the minimal care taken determines the overall outcome. All communities have the same preferences over environmental quality and effort given by the utility function* $u_i(x,v_i) = 2\cdot x - v_i$. *To simplify the exposition assume that effort levels can only take the values 1 and 2.*

For the case of two communities the following payoff matrix arises from this scenario.

		Player 2: 2	Player 2: 1
Player 1	2	2,2	0,1
	1	1,0	1,1

It is straightforward to compute the following Choquet integral of player i:

$\Im(p_i(2,\cdot),\gamma_i\cdot\pi_i) = \gamma_i\cdot[2\cdot\pi_i(2) + 0\cdot\pi_i(1)] + (1-\gamma_i)\cdot 0 = 2\cdot\gamma_i\cdot\pi_i(2)$

$\Im(p_i(1,\cdot),\gamma_i\cdot\pi_i) = 1.$

Clearly, player i will choose effort level 2 if and only if $\gamma_i\cdot\pi_i(2) \geq 0.5$. For an equilibrium under uncertainty where players believe that the opponent plays never strategy 1, i.e. with $\pi_i \equiv (\pi_i(2),\pi_i(1)) = (1,0)$, $\gamma_i \geq 0.5$ is a necessary condition. In fact, it is easy to see that for $\gamma_i < 0.5$ there is a unique equilibrium under uncertainty where both players choose a low effort level, $\pi_i = (0,1)$.

Consider now the case where the number of players increases. For three players the Choquet integral of player 1 becomes

$$\Im(p_1(2,\cdot),\nu_1) = \gamma_1^2\cdot\gamma_1^3\cdot[2\cdot\pi_1^2(2)\cdot\pi_1^3(2)] \quad \text{and} \quad \Im(p_1(1,\cdot),\nu_1) = 1.$$

Assuming all players believe that opponents play independently and beliefs that are consistent, an equilibrium under uncertainty in which (2,2) is played requires now that $\gamma_2\cdot\gamma_3 \geq 0.5$ holds. Assuming further the same degree of confidence for each player, $\gamma \geq \sqrt{0.5} > 0.5$ is required for coordination on the equilibrium with contribution 2. Thus, one can see that an increase in the number of participants reduces the possibility of the good coordination equilibrium (2,2).

In fact, it is easy to check that, with n players, $\gamma^{n-1} < 0.5$ implies a unique equilibrium under uncertainty with contribution level 1. Thus, for any lack of confidence $1-\gamma > 0$, no matter how small, there is a number of players N large enough to make contributing only one unit the only equilibrium under uncertainty, since $\lim_{n\to\infty} \sqrt[n-1]{[0.5]} = 1$. One can therefore conclude that the larger the number of players of this game the more likely is the equilibrium with the smallest possible contribution level. ■

6 Concluding remarks

This paper has introduced an equilibrium concept for games with a finite number of players whose beliefs are represented by non-additive probabilities. In general, relaxing the restrictions imposed on beliefs by the probabilistic nature of mixed strategies and the consistency of beliefs required in a Nash equilibrium will increase the number of equilibria substantially. The minimal deviation from a probabilistic representation of beliefs that is implied by the concept of a simple capacity makes it possible to parametrise the degree of deviation. This degree of deviation from a representation of beliefs by a probability distribution can be interpreted as a player's lack confidence in the assessment of the opponents' play.

With the help of the concept of the degree of confidence, it is possible to parametrise equilibria under uncertainty. It could be shown that, without further assumptions, equilibria under uncertainty will coincide with maximin strategies of the players if the degree of confidence of all players is low.

On the other hand, for high degrees of confidence, equilibria under uncertainty are similar to Nash equilibria only if all players believe that their opponents act independently and if these beliefs are mutually consistent. This result shows the importance of these two properties for the Nash equilibrium concept. In addition, the notion of an equilibrium under uncertainty allows the analyst of a game to test the robustness of Nash equilibrium in regard to these two implicit assumptions.

It could be shown as well that robustness against small degrees of uncertainty provides a refinement of the Nash equilibrium concept that does not coincide with perfectness or robustness against iterated deletion of dominated strategies. In this area, however, more research is necessary.

We conclude the paper with a brief discussion of other game-theoretic equilibrium concepts where the assumption of expected-utility maximising agents is abandoned. Crawford (1990) introduces the notion of an *equilibrium in beliefs*. In Crawford (1990), beliefs are probability distributions over mixed strategies, not capacities over pure strategies as in this paper. Nevertheless, there is a noteworthy relationship between the concept of an equilibrium in beliefs and an equilibrium under uncertainty. An equilibrium in beliefs requires a player's belief to be concentrated on the set of best-reply mixed strategies of the other player given that player's beliefs. In this regard, equilibria in beliefs are similar to equilibria under uncertainty. In particular, strategies that are actually played need not coincide with the strategy that the opponent expects as long as it is part of the support of the opponent's belief. Since Crawford (1990) introduces the concept of an equilibrium in beliefs for two-player games only, the issues of players' believing that their opponents act independently and of consistency of beliefs do not arise.

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