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ABSTRACT

Within the last decade, several estimation techniques have been proposed for estimation of series which exhibit some form of long-term persistence, or long memory, in their conditional mean. Among these techniques are two-step estimation procedures based on the spectrum of the series, and maximum likelihood estimation (MLE) procedures based in both the frequency and time domains. This paper presents the theoretical derivation of a generalized method of moments (GMM) estimator for long-memory time series. The GMM estimation technique provides an attractive alternative estimation procedure since it does not require the distributional assumptions necessary under maximum likelihood estimation techniques.

The moment conditions exploited by the GMM estimator make use of the theoretical and estimated autocorrelation functions of the long-memory process. The paper presents numerical results from variance calculations using all available moment conditions as well as groups of moment conditions to examine the efficiency of the GMM estimator relative to that of the MLE.

1. INTRODUCTION

Many economic and financial time series are characterized by strong persistence in their mean such that when expressed in levels the series appear to be nonstationary, or contain a unit root, yet when expressed in first differences the series appear to be overdifferenced. The traditional nonstationary Autoregressive-Integrated Moving Average ARIMA(p,d,q) model of Box and Jenkins (1976), where the parameter of integration can take on only integer values, is not able to account for the long-term persistence which is characteristic of these series. This can be especially relevant in applied analysis since empirical investigation of such series must be done within the framework of a model which is able to account for the substantially high order of correlation present in these series.

Several estimation techniques have been developed within the last decade to estimate the degree of persistence of a fractionally integrated, long-memory process. Among these techniques is the frequency-domain based, two-step estimation procedure of Geweke and Porter-Hudak (1983) which utilizes the spectrum of a series in estimating the parameter of fractional integration. Alternatively, the maximum likelihood estimation procedures of Hosking (1984b), Fox and Taqqu (1986), and Sowell (1992), which have been based in both the time and frequency domains, utilize standard first-order conditions in maximizing the log of the likelihood function of the fractionally integrated process. This paper presents the derivation of an alternative estimation procedure to estimate the degree of persistence of a long-memory process, which is based on the technique of generalized method of moments (GMM).

The generalized method of moments estimation technique is an attractive alternative framework in which to estimate the parameter of fractional integration of a long-memory process since it does not require the distributional assumptions necessary under maximum likelihood estimation techniques and consequently offers the advantage of robustness in parameter estimation. In addition, approximate MLE can often involve numerically cumbersome techniques which may be avoided, in some part, with the technique of generalized method of moments. For the fractionally integrated process, the GMM estimation technique exploits the set of moment conditions that equate the expected value of the sample autocorrelations to the corresponding population autocorrelations, evaluated at the true parameter values. In this way a consistent estimate of the parameters can be obtained.

The plan of the rest of the paper is as follows. The following section discusses the GMM estimation technique in the context of the fractionally integrated model and presents the motivation for the use of GMM in this context. Section 3 presents the derivation of the asymptotic distribution of the estimated autocorrelations under specified assumptions. This section also presents the derivation of the asymptotic variance of the GMM estimator. Section 4 provides an investigation of the estimation procedure by examining the asymptotic efficiency of the estimator for a range of values of the parameter d using various moment conditions as well as subsets of moment conditions. The paper ends with a brief summary and concluding section.

2. GMM ESTIMATION IN THE CONTEXT OF THE FRACTIONALLY INTEGRATED MODEL

The estimation technique of generalized method of moments makes use of a set of orthogonality conditions that are implied by the model to be estimated such that the expected value of the orthogonality condition is equal to zero at the true parameter value. For the case of the fractionally integrated model, consider the non-zero mean, stationary time series $\{y_t\}$ expressed in ARFIMA(p,d,q) form as introduced in Chapter II as

$$(1.) \qquad (1 - L)^d (y_t - \mu) = \theta(L)/\phi(L)\, \epsilon_t = u_t$$

where for $-\frac{1}{2} < d < \frac{1}{2}$, y_t is said to be fractionally integrated of order d, the polynomials $\theta(L)$ and $\phi(L)$ are as defined in Chapter II, and u_t is a stationary and invertible error process. Following the framework of Hansen's (1982) GMM estimator, estimation of a $(p \times 1)$ parameter vector λ via the GMM estimation technique involves the use of m orthogonality restrictions where m is at least as great as p. Defining the $(m \times 1)$ vector of orthogonality conditions as some function $g(y_t, \lambda)$, the GMM estimator of λ is given as that value of the parameter vector which satisfies

$$(2.) \qquad \min_{\lambda} \ \bar{g}(y,\lambda)' \ W \ \bar{g}(y,\lambda),$$

where $\bar{g}(y,\lambda)$ is the standard expression[1] for the orthogonality condition of the GMM estimator written in the form of an average as

[1] In the context of the fractionally integrated process expressed in (1.), use of an orthogonality condition of the form $\bar{g}(y,\lambda)$ is not directly applicable due to the difficulty in expressing the orthogonality conditions in the form of an average. This problem arises because the typical orthogonality condition of the fractionally integrated process is a function of an infinite number of terms. Therefore, as an alternative, the function $g(\cdot)$ is expressed in the form of a moment condition for the fractionally integrated process, as discussed later in this section.

$$\bar{g}(y,\lambda) = 1/T \sum_{t=1}^{T} g(y_t,\lambda).$$

In this formulation, W is an (m × m) positive definite, symmetric weighting matrix defined as that matrix which has the characteristic of minimizing the sample orthogonality conditions. The minimized value of the criterion function (2.) will be asymptotically distributed as Chi-square with (m - p) degrees of freedom. Within the context of the GMM estimation procedure, the expression $\bar{g}(y,\lambda)$ should converge to zero for the true parameter vector and not for any other element of the parameter space. Additionally, the optimal weighting matrix, W, is given as

$$W = [\mathrm{cov}\ g(y_t,\lambda)]^{-1}.$$

Under weak regularity conditions, Hansen (1982) shows that the GMM estimator of the parameter vector λ satisfies

$$\sqrt{T}\ (\hat{\lambda}_{GMM} - \lambda) \sim N(0,[D'C^{-1}D]^{-1})$$

where C^{-1} is the optimal weighting matrix. In this representation, D is defined as the (m × p) matrix of partial derivatives of the moment conditions with respect to the parameter vector; that is,

$$D = \frac{\partial\ g(y_t,\lambda)}{\partial\ \lambda'}.$$

The GMM estimation procedure may be applied to many standard econometric models, each of which exploits its own unique set of moment conditions and asymptotically optimal weighting matrix. For the case of the fractionally integrated model, the moment conditions exploited make use of the theoretical and estimated autocorrelation functions of the model. Consider, for simplicity, the zero-mean ARFIMA(0,d,0) process

$$(1 - L)^d \, y_t = u_t \tag{3.}$$

where u_t is a stationary error process, $d \in (-\frac{1}{2},\frac{1}{2})$, and $\rho_j = \text{corr}(y_t, y_{t-j})$ is defined as the j^{th} autocorrelation function of the process. The simple model expressed in (3.) is a single parameter model such that λ consists of a single element, d.[2] Recall that for the model given by (3.), ρ_j may be expressed simply as a function of d, as given earlier in Chapter II, as

$$\rho_j = \frac{\Gamma(1-d)\Gamma(j+d)}{\Gamma(d)\Gamma(j+d-1)} = \prod_{i=1}^{j} \frac{(d+i-1)}{(i-d)} .$$

The moment condition exploited by the fractionally integrated model, considering the first k moments, may be expressed as $E[\hat{\rho} - \rho(d)] = 0$ where

$$\hat{\rho} = [\hat{\rho}_1, \; . \; . \; . \; , \hat{\rho}_k]' \text{ and}$$

$$\rho(d) = [\rho_1(d), \; . \; . \; . \; , \rho_k(d)]' .$$

[2] This estimation procedure may be applied to the more general, multi-parameter ARFIMA representation given by (1.) in which case λ would be a vector and would include the parameters of the autoregressive and moving average polynomials.

Within the context of the fractionally integrated model, the GMM estimator of the parameter vector λ may be expressed as that value of d which satisfies

$$(4.) \qquad S(d) = \min_{d} \, [\hat{\rho} - \rho(d)]' \, W \, [\hat{\rho} - \rho(d)]$$

and the asymptotically optimal weighting matrix, W, is given as

$$W = [\operatorname{cov}(\hat{\rho} - \rho(d))]^{-1}.$$

In considering the efficiency of the GMM estimator, it should be the case that any estimator based on all available moment conditions should be relatively more efficient than that based on only a subset of these moment conditions. However, in the case of the fractionally integrated process there will be some advantage to considering the GMM estimator based on a subset of moment conditions, especially in the case where a stationary ARMA component exists in the series. In such a model, the autocorrelation functions for the lower-order moments of the process will be a function of the autoregressive and moving average parameters of the model as well the parameter d. As such, the autocorrelation functions for the lower-order moments will be quite different from those autocorrelations that exist at higher-order moments, which are simply a function of the parameter d. In this sense the autocorrelation functions for the lower-order moments may be thought of as being "contaminated" when a stationary ARMA component exists in the series. As a result, it would be of interest in this context to determine whether the efficiency of the GMM estimator is maintained when using some subset of the moment conditions, for example moments $(r + 1)$ through $((r + 1) + k)$, such that the first r moments may be discarded. Simple asymptotic variance calculations may be employed to determine these relative efficiencies, and these operations are discussed further in section 4.

3. ASYMPTOTIC DISTRIBUTION THEORY

In order to determine the asymptotic distribution and optimal weighting matrix of the generalized method of moments estimator for the fractionally integrated process, it is necessary to derive the asymptotic distribution of the moment condition, $[\hat{\rho} - \rho(d)]$. Recall that $\hat{d}_{GMM}$ solves the operation $\partial S(d)/\partial \hat{d} = 0$ as given in equation (4.). This expression may be written in the form of its Taylor-series expansion as

$$\frac{\partial S(d)}{\partial \hat{d}} = \frac{\partial S(d)}{\partial d} + \frac{\partial^2 S(d)}{\partial d_*^2} (\hat{d} - d),$$

where d_* lies between d and $\hat{d}$. Equating the above expansion to zero and solving for $(\hat{d} - d)$ gives

$$(\hat{d} - d) = - \left[\frac{\partial^2 S(d)}{\partial d_*^2}\right]^{-1} \frac{\partial S(d)}{\partial d}$$

where $\qquad \partial S(d)/\partial d = -2\, D'W[\hat{\rho} - \rho(d)],$

$$\partial^2 S(d)/\partial d^2 = 2\ D'WD + o_p(1),$$

and D is as defined previously. It follows that the asymptotic distribution of the GMM estimator of d satisfies

$$\sqrt{T}\ (\hat{d} - d) = - \left[\frac{\partial^2 S(d)}{\partial d_*^2}\right]^{-1} \sqrt{T}\ \frac{\partial S(d)}{\partial d}$$

$$\stackrel{A}{=} (D'WD)^{-1}\ D'W\sqrt{T}[\hat{\rho} - \rho(d)].$$

The asymptotic distribution of $\sqrt{T}[\hat{\rho} - \rho(d)]$ for the fractionally integrated, long-memory process is given in Hosking (1984a). Hosking considers the fractionally integrated ARIMA(p,d,q) process as expressed in (1.) where ϵ_t is an independent and identically, but not necessarily normally, distributed white noise error process with mean zero and variance σ^2, ϵ_t has a finite fourth moment, and y_t has mean μ. The sample autocorrelation function is defined as

$$\hat{\rho}_j = \frac{\sum_{t=1}^{T-j}(y_t - \bar{y})\ (y_{t+j} - \bar{y})}{\sum_{t=1}^{T}(y_t - \bar{y})^2}$$

where $\bar{y} = 1/T \sum_{t=1}^{T} y_t$ is the sample mean of the process. For the standard, stationary, short-memory time series process where d takes on integer values, there are standard results for the asymptotic distribution of the sample autocovariance function. However, in the case of the fractionally integrated, long-memory time series process where $-\frac{1}{2} < d < \frac{1}{2}$, Hosking (1984a) shows that these standard results hold for $d \in [-\frac{1}{2},\frac{1}{4})$ but not for $d \geq \frac{1}{4}$. This discrepancy may be attributed to the treatment of the estimation of the mean of the fractionally integrated process. That is, for $\frac{1}{4} \leq d < \frac{1}{2}$ the effect of replacing μ with $\bar{y}$ is not negligible, even asymptotically, and large bias (of the same order of magnitude as that of the standard deviation) is introduced into the estimate of the autocorrelation function. Consequently, the remaining analysis of this chapter will restrict attention to the range of values of the parameter vector for which $d \in [-\frac{1}{2},\frac{1}{4})$. Within this range[3] the estimated autocorrelation functions will be distributed asymptotically normal with variance of order 1/T.

Following Hosking (1984a), the estimated autocorrelation function, $\hat{\rho}$ has covariance matrix C which has i,j th element given by

$$(5.) \qquad c_{i,j} = 1/T \{\sum_{s=1}^{\infty} (\rho_{s+i} + \rho_{s-i} - 2\,\rho_i\rho_s)(\rho_{s+j} + \rho_{s-j} - 2\,\rho_j\rho_s)\}$$

and $C = \{c_{ij}\}$. In applying the GMM estimation procedure to the single parameter fractionally integrated process, then, the asymptotic distribution of $[\hat{\rho} - \rho(d)]$ will be given by $\sqrt{T}\,(\hat{\rho} - \rho(d)) \sim N(0,C)$ where the dimension of C will be defined by the number of moments used in estimation, and the asymptotic distribution of the GMM estimator will be given by

$$(6.) \qquad \sqrt{T}\,(\hat{d} - d) \sim N[0,(D'C^{-1}D)^{-1}].^{4}$$

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[3] For $d = \frac{1}{4}$, asymptotic normality is retained but the variance of the estimated autocorrelation function is of order 1/T(log T). For $d \in (\frac{1}{4},\frac{1}{2})$, asymptotic normality is not retained and the variance is of order $T^{-2(1-2d)}$.

[4] In this representation the optimal weighting matrix, defined in (2.) as W, is given by C^{-1}.

4. ASYMPTOTIC PERFORMANCE OF THE GMM ESTIMATOR FOR THE FRACTIONALLY INTEGRATED MODEL

The asymptotic performance of the generalized method of moments estimator for the fractionally integrated process is examined by calculating the large sample variance of $\hat{d}$ as given in (6.). To perform this calculation it is necessary to compute $[D'C^{-1}D]^{-1}$ where D and C are functions of the parameter d and the number of moments, k, used in estimation. In the general case, the efficiency of the GMM estimator should be greatest when calculations are performed utilizing all available moment conditions. However, in the case of the fractionally integrated process, which uses the estimated autocorrelations in calculation, the possible number of available moment conditions is infinite. Relative efficiency, then, should continue to increase as a greater number of moments are used in estimation such that more moments will always be preferred. As such, the use of any subset of moments in estimation should provide lower levels of efficiency relative to that in which a greater number of moments are employed.

The calculation of the vector of partial derivatives, D, and the covariance matrix of the estimated autocorrelation functions, C, is as follows. Recall that the $(k \times 1)$ vector $\rho(d)$ is given by

$$\rho(d) = \begin{bmatrix} \frac{d}{1-d} \\ \frac{d}{1-d}\ \frac{d+1}{2-d} \\ \vdots \\ \frac{d}{1-d}\ \frac{d+1}{2-d}\ \frac{d+2}{3-d}\ \cdots\ \frac{d+(k-1)}{k-d} \end{bmatrix}.$$

It follows then that D is given by

$$D = \begin{bmatrix} \dfrac{1}{(d-1)^2} \\ \dfrac{-2(-1-2d+2d^2)}{(d-1)^2\ (d-2)^2} \\ \dfrac{3(4+12d-9d^2-6d^3+3d^4)}{(d-1)^2\ (d-2)^2\ (d-3)^2} \\ \vdots \\ \dfrac{(-1)^{k+1}\ k * f(\cdot)}{(d-1)^2(d-2)^2(d-3)^2\ \cdots\ (d-k)^2} \end{bmatrix}$$

where $f(\cdot)$ will be a function of d^i and $i = (0, 1, 2, 3, \ldots, 2[k-1])$. From the above expressions and the formula given by (6.), the values of D and C are calculated for various values of $d \in [-\frac{1}{2},\frac{1}{4})$ taken at discrete intervals, that is $d = -.50, -.45, -.40, \ldots, .20, .24$, and various numbers of moment conditions. Relative efficiency comparisons are provided in Tables 1 through 4 which will each be discussed in turn below.

Table 1 presents the asymptotic variance calculations of the GMM estimator for given values of d using moments 1 through "n" in calculation, where n = 1, 2, 3, . . . , 20. In each case it appears that as the number of moments used in estimation increases, the asymptotic variance of the GMM estimator converges to that of the maximum likelihood estimate, $(\pi^2/6)^{-1} = .6079$, as given in Li and McLeod (1986). For positive values of d it appears quite reasonable to conclude that the relative efficiencies of the GMM estimator and the MLE are comparable when only 10 moments are used, although the efficiency of the GMM estimator decreases slightly as the absolute value of d increases. For negative values of d the convergence of the variance of the GMM estimator to that of the MLE requires the use of additional moment conditions in calculation, and the efficiency of the estimator also decreases over this range as the absolute value of d increases.

As discussed in section 2, there is some interest in employing the GMM estimation technique to the fractionally integrated model since this procedure allows for calculation of the estimator based upon subsets of moments so that earlier moments may be dropped from estimation. This notion is particularly attractive within the framework of the long-memory process since the presence of autoregressive and moving average components in the process may contaminate the autocorrelation functions for lower-order moments. Tables 2 through 4 allow for an examination of the efficiency of the GMM estimator when dropping earlier moments in calculation, and the results of each table are discussed below.

Table 2 presents the asymptotic variance of $\hat{d}$ when using only moment "n" in calculation, where n = 1, 2, 3, ... , 10. In this way it will be possible to examine the contribution of each individual moment condition to the efficiency of the GMM estimator. Table 2 clearly indicates the sacrifice in efficiency for a given value of d when using only one moment condition in estimation, particularly when using any individual moment after the first moment. For negative values of d, for example, the asymptotic variance of the estimator increases dramatically when using any moment other than the first in calculation. For example, for d = -.05, the asymptotic variance of the GMM estimator based on the use of moment two only is more than five times that based on moment one only. The loss in efficiency when using only the second moment is even more dramatic as the value of d decreases to d = -.49. In addition, Table 2 indicates that similar losses in efficiency are evident when calculation is based on use of only moment three, or only moment four, and so on. The same sacrifice in efficiency in using only one moment is evident for positive values of d as well, although the magnitude of the increase in the asymptotic variance is somewhat smaller. For example, for $d = .05$, the asymptotic variance of the GMM estimator based on the use of only moment two is approximately three times that of the estimator based on only moment one; recall, as discussed above, that for $= -.05$ the variance is more than five times greater.

Recall that Table 1 illustrated the trade off that existed between the efficiency of the GMM estimator and the absolute value of the parameter d. That is, the relative efficiency of the GMM estimator based on moments 1 through n increases as the absolute value of d decreases. The same trade off is evident in Table 2. When using only a single moment to calculate the GMM estimator, the asymptotic variance of the estimator decreases as the absolute value of d decreases. This trade off may be explained for the fractionally integrated process by considering the relative contribution of successive moments to the efficiency of the estimator, for a given value of d. As expressed in (6.), the elements of the vector D represent the derivatives of the moment conditions with respect to the parameter, d. In the case of the fractionally integrated process, there is relatively little change in each element of the vector D beyond the first element. This may be attributed to the relative flatness of the autocorrelation functions beyond the first moment, for a given value of d. In addition, the diagonal elements of the matrix C, as expressed in (6.), show relatively little change beyond the first element. It appears, then, in the case of the fractionally integrated process that, for a given value of d, a significant amount of information is contained in the first moment and thus there exists a sacrifice in the efficiency of the GMM estimator when using any one moment, other than the first, in calculation.

Table 3 presents the results of using subsets of five moment conditions in calculating the asymptotic variance of the GMM estimator in which the first moment used in estimation is equal to "n", and n = 1, 2, 3, ... , 10. Again, it can be seen that, for a given value of d, the asymptotic efficiency of the GMM estimator decreases significantly when the first moment is dropped from the calculations. For example, for d = -.05, the asymptotic variance using moments 2 through 6 is more than four times that using moments 1 through 5. In addition, the results of Table 3 indicate that the estimator based on a subset of five moment conditions, dropping earlier moments in calculation, is relatively less efficient than the estimator based on more (or all available) moment conditions. As observed in Tables 1 and 2, the same trade off exists between the efficiency of the GMM estimator and the value of d when using a subset of five moments; for a given subset of five moments, the efficiency of the GMM estimator increases as the value of d approaches zero. In addition, Table 3 clearly indicates the sacrifice in the efficiency of the GMM estimator that results from dropping more and more of the earlier moments from the calculations. That is, for any given value of d, the asymptotic variance of the GMM estimator increases as more of the earlier moments are dropped from the calculations. For any given value of d when using a subset of five moment conditions, the relative efficiency of the GMM estimator is the greatest when using the first five moments.

The results of Table 3 should not be surprising given the findings of Table 2 which indicate the relative importance of the first moment condition in estimation. It appears that any calculations which omit the first moment condition result in considerable loss of efficiency.

Finally, Table 4 presents the results of using a subset of ten moment conditions in calculating the asymptotic variance of the GMM estimator, where the first moment used in estimation is equal to "n" and n = 1, 2, 3, . . . , 10. The results of Table 4 are very similar to those of Table 3 in that they indicate the relative loss in efficiency in using subsets of moment conditions where earlier moments are dropped from estimation. It is evident that calculations based on a subset of ten moment conditions, especially when dropping the first moment, involve significant losses in efficiency, with the greatest loss occurring when the largest number of earlier moments are dropped from the calculations. Again, given the results of Table 2 this should not be surprising since a great deal of information is contained in the first moment. It does appear, however, that the efficiency of the GMM estimator is greater when a larger subset of moment conditions are used in the calculations. That is, for any given value of d, the asymptotic variance of the GMM estimator based on a subset of ten moments is smaller than that based on a subset of five moments. It is still the case, however, that the use of a greater number of moments in calculation of the GMM estimator, as opposed to the use of any subset of moments, dominates in terms of the asymptotic efficiency of the estimator.

5. SUMMARY AND CONCLUSION

This chapter has examined the use of the estimation technique of generalized method of moments in estimating the parameters of the fractionally integrated process. The use of this technique is particularly appealing in this context since it does not require the distributional assumptions encountered in using maximum likelihood estimation techniques, and also because it avoids the computational difficulty often encountered in employing approximate MLE techniques. In addition, the relative efficiencies of the two methods appear to be comparable, asymptotically, as the variance calculations provided in Table 1 indicate convergence of the variance of the GMM estimator to that of the MLE (a value of .6079). The GMM estimation technique appears to be a reasonable procedure to employ in the context of the simple ARFIMA(0,d,0) processes.

It does appear, however, that GMM applied to the fractionally integrated process requires the use of lower-order autocorrelations in order to avoid large losses of efficiency. The results of Tables 1 through 4 demonstrate that the relative efficiency of the GMM estimation technique, when applied to the fractionally integrated process, is greatest when using a greater number of moment conditions in estimation. Table 2 shows the significant loss in efficiency which is encountered when the first moment is dropped from estimation. This apparently is due to the relatively small contribution of information attributable to successively higher moments of the long-memory process. This observation is further confirmed in Tables 3 and 4 where there exists considerable inefficiency in using subsets of moment conditions, particularly as a greater number of the earlier moments are dropped in estimation.

These results are especially relevant if one allows for short-run dynamics in the model, as in the case of the ARFIMA(p,d,q) process. For $p > 0$ or $q > 0$, the lower-order autocorrelations may be substantially different than those for the (0,d,0) part of the process. Since it appears that these lower-order autocorrelations cannot be dropped from estimation without sacrificing efficiency, it is reasonable to consider GMM estimation of the ARFIMA(p,d,q) model in the context in which d is estimated jointly with the autoregressive and moving average parameters of the process. This is an important topic for further research.

TABLE 1

Asymptotic Variances of $\sqrt{T}\,(\hat{d} - d)$ Using Moments 1, 2, 3, . . ., n

	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
d=-.49	3.4013	2.0666	1.6219	1.3974	1.2611	1.1690	1.1037	1.0573	1.0219	.9938
d=-.45	3.1353	1.9355	1.5337	1.3303	1.2065	1.1225	1.0615	1.0149	.9780	.9480
d=-.40	2.8206	1.7792	1.4281	1.2482	1.1370	1.0610	1.0053	.9626	.9303	.9047
d=-.35	2.5256	1.6294	1.3256	1.1704	1.0752	1.0102	.9629	.9268	.8932	.8667
d=-.30	2.2507	1.4874	1.2274	1.0943	1.0125	.9567	.9159	.9105	.9035	.8977
d=-.25	1.9946	1.3527	1.1331	1.0204	.9510	.9036	.8690	.8425	.8214	.8043
d=-.20	1.7580	1.2257	1.0434	.9496	.8919	.8524	.8236	.8015	.7840	.7697
d=-.15	1.5399	1.1061	.9577	.8814	.8346	.8026	.7793	.7614	.7473	.7358
d=-.10	1.3407	.9950	.8773	.8172	.7804	.7553	.7372	.7233	.7123	.7034
d=-.05	1.1604	.8924	.8025	.7570	.7294	.7108	.6974	.6873	.6793	.6728
d=.00	1.0000	.8000	.7347	.7024	.6833	.6705	.6615	.6547	.6495	.6453
d=.05	.8613	.7202	.6772	.6561	.6444	.6369	.6317	.6279	.6251	.6229
d=.10	.7491	.6588	.6341	.6237	.6184	.6153	.6133	.6120	.6111	.6105
d=.15	.6769	.6303	.6215	.6186	.6179	.6177	.6176	.6176	.6176	.6176
d=.20	.6884	.6783	.6783	.6775	.6759	.6740	.6719	.6698	.6677	.6657
d=.24	.8629	.8594	.8432	.8252	.8088	.7945	.7822	.7714	.7620	.7538

TABLE 1 (Cont'd)

	n=11	n=12	n=13	n=14	n=15	n=16	n=17	n=18	n=19	n=20
d=-.49	.9712	.9527	.9373	.9244	.9135	.9043	.8963	.8894	.8834	.8783
d=-.45	.9230	.9018	.8834	.8674	.8533	.8407	.8295	.8193	.8101	.8017
d=-.40	.8837	.8663	.8515	.8388	.8280	.8185	.8101	.8026	.7960	.7901
d=-.35	.8446	.8258	.8095	.7908	.7748	.7587	.7442	.7306	.7179	.7062
d=-.30	.8930	.8855	.8806	.8768	.8737	.8712	.8691	.8664	.8643	.8625
d=-.25	.7900	.7779	.7675	.7584	.7505	.7435	.7372	.7316	.7253	.7200
d=-.20	.7579	.7478	.7392	.7317	.7251	.7193	.7140	.7094	.7051	.7013
d=-.15	.7263	.7183	.7113	.7055	.7001	.6934	.6885	.6843	.6806	.6771
d=-.10	.6970	.6910	.6859	.6814	.6775	.6740	.6768	.6764	.6742	.6728
d=-.05	.6675	.6630	.6592	.6559	.6531	.6506	.6534	.6522	.6509	.6496
d=.00	.6418	.6390	.6366	.6345	.6327	.6312	.6298	.6286	.6275	.6265
d=.05	.6212	.6197	.6185	.6175	.6167	.6160	.6153	.6148	.6143	.6139
d=.10	.6100	.6096	.6094	.6092	.6090	.6089	.6088	.6087	.6086	.6085
d=.15	.6176	.6176	.6175	.6174	.6174	.6173	.6172	.6171	.6170	.6170
d=.20	.6639	.6622	.6606	.6590	.6576	.6562	.6549	.6537	.6526	.6515
d=.24	.7464	.7399	.7340	.7287	.7239	.7195	.7155	.7118	.7084	.7052

TABLE 2

Asymptotic Variances of $\sqrt{T}\,(\hat{d} - d)$ Using Moment n Only

	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10
d=-.49	3.4013	276.54	383.92	706.60	1230.1	2029.5	3132.8	4622.3	6543.2	9126.8
d=-.45	3.1353	516.27	463.86	742.51	1199.8	1868.7	2760.0	3936.7	5430.4	7358.4
d=-.40	2.8206	2586.1	703.37	888.06	1282.2	1824.7	2546.7	3477.4	4598.8	6006.6
d=-.35	2.5256	9569.9	1619.9	1346.3	1611.0	2062.6	2681.0	3431.0	4378.1	5465.4
d=-.30	2.2507	454.92	23430.0	3452.5	2827.7	3039.3	3496.8	4117.2	4916.5	5886.4
d=-.25	1.9946	118.74	2853.1	644700.0	15100.0	8865.4	7585.4	7591.7	7987.4	8718.1
d=-.20	1.7580	46.549	357.98	2059.4	13130.0	220000.0	543400.0	87040.0	48290.0	36590.0
d=-.15	1.5399	22.004	103.09	326.53	847.18	1948.1	4164.0	8604.1	17570.0	35730.0
d=-.10	1.3407	11.620	39.894	96.409	193.24	343.76	563.21	866.31	1282.3	1816.1
d=-.05	1.1604	6.1534	18.073	36.554	63.441	99.403	145.45	202.48	271.58	352.19
d=.00	1.0000	4.0000	9.0000	16.000	25.000	36.000	49.000	64.000	81.000	100.00
d=.05	.8613	2.5586	4.8958	7.7839	11.166	15.014	19.328	24.997	29.081	34.572
d=.10	.7491	1.7451	2.9065	4.1977	5.5978	7.0958	8.7097	10.349	12.085	13.904
d=.15	.6769	1.3053	1.9431	2.5900	3.2467	3.9125	4.6027	5.2683	5.9566	6.6543
d=.20	.6884	1.1628	1.5944	2.0000	2.3914	2.7697	3.1317	3.5001	3.8552	4.2052
d=.24	.8629	1.3540	1.7723	2.1492	2.4982	2.8275	3.1409	3.4418	3.7322	4.0137

TABLE 3

Asymptotic Variances of $\sqrt{T}\,(\hat{d} - d)$ Using Five Moments

	1-5	2-6	3-7	4-8	5-9	6-10	7-11	8-12	9-13	10-14
d=-.49	1.2611	18.783	30.082	50.677	81.124	123.23	178.13	247.71	336.24	445.14
d=-.45	1.2065	25.396	35.049	54.530	82.519	119.99	168.15	228.73	303.11	392.54
d=-.40	1.1370	43.514	47.144	65.464	93.257	129.49	174.34	226.67	285.86	360.89
d=-.35	1.0752	104.33	79.417	93.092	118.20	151.85	193.27	242.42	300.21	365.64
d=-.30	1.0125	243.61	205.21	233.60	293.37	352.72	368.21	315.24	373.03	439.79
d=-.25	.9510	80.067	1100.9	770.22	568.87	528.54	540.65	580.11	635.27	703.36
d=-.20	.8919	23.526	178.61	1179.6	7417.6	7995.3	4483.7	3238.2	2735.5	2517.4
d=-.15	.8346	9.7678	38.425	108.16	259.33	569.49	1197.7	2485.3	5223.9	11492.0
d=-.10	.7804	5.0525	14.130	29.573	53.255	87.237	136.46	201.61	286.87	393.32
d=-.05	.7294	3.0305	6.7501	12.000	18.781	27.314	37.697	50.014	64.412	80.942
d=.00	.6833	1.5866	2.4957	6.0128	8.6091	11.604	15.003	18.808	23.010	27.614
d=.05	.6444	1.5008	2.4652	3.5344	4.7070	5.9816	7.3544	8.8226	10.383	12.034
d=.10	.6184	1.2080	1.7925	2.3886	3.0021	3.6355	4.2890	4.9633	5.6568	6.3707
d=.15	.6179	1.0710	1.4693	1.8734	2.2523	2.6272	2.9992	3.3702	3.7398	4.1180
d=.20	.6759	1.1052	1.4667	1.7913	2.0937	2.3809	2.6566	2.9257	3.1872	3.4432
d=.24	.8088	1.3202	1.7423	2.1097	2.4411	2.7470	3.0341	3.3067	3.5676	3.8189

TABLE 4

Asymptotic Variances of $\sqrt{T}\,(\hat{d} - d)$ Using Ten Moments

	1-10	2-11	3-12	4-13	5-14	6-15	7-16	8-17	9-18	10-19
d=-.49	.9938	7.6420	11.667	18.301	27.366	39.155	53.889	72.010	94.265	121.09
d=-.45	.9480	9.5782	13.336	19.683	28.216	39.070	52.494	68.820	88.312	111.19
d=-.40	.9047	14.678	17.880	24.570	33.483	46.352	57.196	71.978	88.853	109.44
d=-.35	.8667	26.757	26.655	32.747	40.604	53.169	62.260	76.579	94.173	116.78
d=-.30	.8977	101.99	88.877	89.724	99.346	110.97	114.12	107.84	120.12	135.37
d=-.25	.8043	72.637	211.98	173.74	162.58	166.95	178.92	195.72	215.98	236.49
d=-.20	.7697	22.335	175.82	870.72	1398.0	1128.2	924.65	825.65	783.05	770.91
d=-.15	.7358	8.4917	32.553	90.117	214.99	475.75	1013.1	2167.8	4756.9	10826.
d=-.10	.7034	4.2714	11.275	22.508	38.964	61.773	92.229	146.48	216.36	311.04
d=-.05	.6728	2.5812	5.3867	9.0495	13.592	19.045	25.452	37.398	49.635	54.364
d=.00	.6453	1.7927	3.1769	4.7666	6.5621	8.5587	10.758	13.159	15.758	18.558
d=.05	.6229	1.3797	2.1680	2.9939	3.8625	4.7770	5.7375	6.7442	7.7965	8.8950
d=.10	.6105	1.1597	1.6728	2.1727	2.6698	3.1693	3.6742	4.1864	4.7057	5.2343
d=.15	.6176	1.0629	1.4420	1.8127	2.1530	2.4822	2.8034	3.1193	3.4308	3.7462
d=.20	.6657	1.0989	1.4644	1.7894	2.0880	2.3677	2.6330	2.8890	3.1356	3.3751
d=.24	.7538	1.2490	1.6753	2.0540	2.3980	2.7154	3.0119	3.2918	3.5580	3.8126

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