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Probabilistic graphs in cooperative games: a model inspired by voting situations[1]

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Abstract

In this paper a method is offered for evaluating the power of political parties in a parliament. The method is inspired by the game theoretic approach taken by Shapley and Shubik (1954). The main difference between our method and theirs is that with our method it is possible to take sociological, political, or ideological aspects into account when determining the power of the parties. The model we develop is an extension of the model of games with communication restrictions by Myerson (1977) and it is in fact applicable to situations that are much more general than voting situations.

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1 Introduction

In order to get a measure of the power[4] of different agents in majority voting situations, these situations can be modelled as cooperative (voting) games and then some solution concept from cooperative game theory can be used to assign an index of power to each party in the voting situation. Well known power indices that stem from game theory are the Shapley-Shubik index (cf. Shapley and Shubik (1954)) and the Banzhaf index (cf. Banzhaf (1965)). Both forementioned papers show that simply counting the number of votes of each party does not provide a reliable indication of the power of the parties in general. In order to get a good indication of the power of each party it is important to consider so called minimal winning coalitions, i.e., coalitions of parties that hold a majority of the votes and that are minimal with respect to this property. Then, instead of merely counting votes, it is investigated what other parties and how many of them a specific party needs in order to form a majority coalition.

This approach, however, fails to take into account sociological, political, and ideological aspects that influence the degree of compatibility of the objectives of different parties. These aspects may obviously influence the power of a party, because a party that has sociological, political, or ideological objectives that are opposite of the objectives of most other parties may often vote differently than most parties and this will deminish the chances this party has for being critical to the success of a winning coalition. In this paper we will present a theoretical model that will enable us to analyze voting situations in such a way that the degrees of compatibility of cooperation of the parties are taken into account.

The model that we present is a generalization of the model of games with communication restrictions as introduced by Myerson (1977). This model was introduced to capture non-transitive communication structures among the agents and it uses (deterministic) graphs to describe these structures. For an overview of the line of research on games with communication restrictions we refer to Borm, van den Nouweland and Tijs (1994) and van den Nouweland (1993). We extend the model of Myerson (1977) and associate with each pair of parties a probability that reflects the *a priori* degree of compatibility of both parties. An extension of the Shapley-Shubik index to this type of model is defined and this generalized index provides an *a priori* indication of the power of political parties when sociological, political, and ideological aspects are taken into consideration.

[4]We follow Shapley and Shubik (1954) and define the power of an agent to be the chance this agent has of being critical to the success of a winning coalition.

The structure of the paper is as follows. In section 2 we describe voting situations and we recall the well-known Shapley-Shubik index. In section 3 we define a model that can be used to model voting situations and that has the possibility of taking into account sociological, political, and ideological considerations of the parties. We define a generalization of the Shapley-Shubik index for this model. In section 4 we apply the theory developped in section 3 to voting situations and in section 5 we prove that the generalization of the Shapley-Shubik index defined in section 3 can characterized by two of its properties. Section 6 contains concluding remarks.

2 Voting situations and the Shapley-Shubik power index

In this section we will referesh the reader's memory on voting games and the Shapley-Shubik power index.

Consider a majority voting situation among n parties, $1, 2, \ldots,$ and n, where each party i holds v_i votes and a coalition of parties needs more than half of the total number of votes $\sum_{i=1}^{n} v_i$ in order to be able to force a decision. Such a situation can be modeled as a *voting game* (N, w) with *player set* $P = \{1, 2, \ldots, n\}$ and *characteristic function* w that is defined by

$$w(S) := \begin{cases} 1 & \text{if } \sum_{i \in S} v_i > \frac{1}{2} \sum_{i \in P} v_i \\ 0 & \text{else} \end{cases}$$

for all subsets of parties $S \subseteq P$. A coalition $S \subseteq P$ is called *minimal winning* (in the game (P, w)), if $w(S) = 1$ and $w(T) = 0$ for all (strict) subcoalitions T of S.

The Shapley-Shubik index is based on the idea of a marginal voter in a majority voting situation. Suppose that the parties are ordered in order of decreasing willingness to vote in favor of some specific topic. For this order there is one party that is marginal in the sense that if this party votes with its predecessors in favor of the topic, then the topic will be accepted, and if this party votes with its succesors against the topic, then it will be abandoned. This marginal party in effect holds all the power with respect to the topic at hand. The Shapley-Shubik index is based on the idea that all orders of the parties are equally possible, and it assigns to each party the probability that this party is the marginal one.[5] Note that the probability of being the marginal party (weakly) increases

[5] Mathematically, the Shapley-Shubik index is the Shapley value of the voting game (N, w).

with the number of votes and, hence, the Shapley-Shubik index (weakly) increases with the number of votes. To illustrate the Shapley-Shubik index we depict in table 2.1 the situation in the Spanish parliament after the elections of June 1993.

After the elections the 350 seats in the Spanish parliament are divided among 9 parties. In order to pass a bill, 176 votes are needed. The division of the seats and the corresponding Shapley-Shubik indices of the parties are as follows.

Party		Seats	S-S Index
PSOE	(Partido Socialista Obrero Español)	159	0.5000
PP	(Partido Popular)	141	0.1667
IU	(Izquierda Unida)	18	0.1667
CIU	(Covergencia i Unió)	17	0.1667
PNV	(Partido Nacionalista Vasco)	5	0.0000
CC	(Coalición Canaria)	4	0.0000
HB	(Herri Batasuna)	2	0.0000
EA	(Eusko Alkartsuna)	1	0.0000
ERC	(Esquerra Republicana de Catalunya)	1	0.0000
PAR	(Partido Aragonés Regionalista)	1	0.0000
UV	(Unión Valenciana)	1	0.0000

Table 2.1

We distinguish two big parties, PSOE and PP, two parties of intermediate size, IU and CIU, and several small parties in the parliament. We see in table 2.1 that party PNV has no power at all, because for every coalition of parties not including PNV it holds that either this coalition holds a majority of the seats or that this coalition joined with PNV still only holds a minority of the seats. Also, parties CC, HB, EA, ERC, PAR, and UV are powerless.

The approach taken by the Shapley-Shubik index only takes into account the number of seats that the parties hold in parliament. However, in reality also other factors may influence the power of the parties. Parties are typically concerned with sociological, political, and ideological objectives and these objectives influence their voting behavior. Therefore, the voting behavior of parties that have similar sociological or ideological objectives will be correlated to some extend. Hence, one may expect to get more accurate indices of the power of individual parties when the sociological and ideological compatibilities of the parties incorporated into the model.

A first idea to include these compatibilities of the parties into a model is to define a graph on the set of parties in which each pair of parties i and j is joined by an edge if and only if these two parties are compatible. Then this compatibility graph will clearly have an influence on the voting game that is appropriate to describe the situation, because not all coalitions of parties can be formed due to incompatibilities. Parties can only cooperate if they are connected in the graph. That is, a coalition consisting of two parties that are not joined by an edge cannot be formed, but the same two parties can cooperate in the presence of a third party that is joined with both former parties. Hence, the third party can serve as an intermediary. In the voting game that incorporates the compatibility graph a coalition of parties is winning if (i) it holds more than half of the total number of votes and (ii) the parties in the coalition are connected in the compatibility graph. The reader might recognize the model of Myerson (1977) in what we are describing above. For an overview on the line of research that was initiated by Myerson (1977) we refer the reader to Borm, van den Nouweland and Tijs (1994) and van den Nouweland (1993).[6]

In this paper, however, we will consider a generalization of the model of Myerson (1977), because in general it will not be the case that two parties are either compatible or incompatible, but they will be compatible to a certain degree. Therefore, it is not so natural to associate with each pair of parties either a 0 (incompatible) or a 1 (compatible), but it is more appealing to associate with each pair of parties i and j a number p_{ij} between 0 and 1 that reflects the degree of compatibility of the two parties. We want to model voting situations in such a way that we can take into account the degrees of compatiblity of different parties. To do this, we develop a generalized version of Myerson's (1977) model. This is done in the following section.

3 Games with probabilistic graphs

In this section we will extend the model of Myerson (1977) in order to be able to describe voting situations in such a way that sociological, political, and ideological (in)compatibilities are included with the description. Because the ideas underlying the model that we develop in this section also might be of relevance in situations more general than voting situations (see section 6), the model will be developped in the more general setting of coalitional games. So, the analysis in this section is not restricted to voting games. We will in the current section follow Myerson's (1977) terminology and

[6] The model of Myerson (1977) is mentioned in relation to voting situations by van Damme *et al.* (1994).

use the term 'communication restrictions' instead of '(in)compatibilities'. In section 4 we will show how the model that we are going to develop in this section can be used to model voting situations.

A *game with a probabilistic graph* is a triple (N, v, p), where $N := \{1, 2, \ldots, n\}$ is the set of agents, (N, v) is a *coalitional game* with player set N and characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ that describes the economic possibilities of the agents, and $p : \{\{i, j\} \mid i, j \in N,\ i \neq j\} \to [0, 1]$ is a function that assigns to each pair of agents i and j the probability that these two agents can communicate directly. The probabilities are assumed to be independent. Sometimes we will refer to the function p as a *system of probabilities*. Further, we will often denote p_{ij} instead of $p(\{i, j\})$.

Let (N, v, p) be a game with a probabilistic graph. With this game we will associate a new coalitional game (N, v_p), called the communication game, that incorporates both, the economic possibilities of the agents described by the coalitional game (N, v), and the probabilities of bilateral communication described by the system of probabilities p. Since we are dealing with ***probabilities*** of communication, we will consider expected profits in the new game.

Let $i, j \in N, i \neq j$. Then, with probability p_{ij} agents i and j are able to communicate. If this is so, then they can cooperate and obtain $v(\{i, j\})$. But with probability $1 - p_{ij}$ the agents cannot communicate and in this case they cannot obtain more than $v(\{i\}) + v(\{j\})$. Therefore, the expected profit of agents i and j is

$$v_p(\{i, j\}) := p_{ij}\, v(\{i, j\}) + (1 - p_{ij})\, (v(\{i\}) + v(\{j\})).$$

Generalizing the idea that is at the basis of this definition, we can define the expected profit of arbitrary coalitions of agents. Let $S \subseteq N$ be a fixed coalition of agents and define $L(S) := \{\{i, j\} \mid i, j \in S,\ i \neq j\}$, the set of all possible communication links between agents in S. We will often denote typical links in $L(S)$ by l. For each set of links $L \subseteq L(S)$, the probability that L is the communication graph that is realized among the agents in S is equal to

$$p^S(L) := \prod_{l \in L} p_l \prod_{l \in L(S) \setminus L} (1 - p_l).$$

Now, suppose $L \subseteq L(S)$ is the set of communication links that is realized. Note that the graph (S, L) induces a partition of S into communication components in the following way: $C \subseteq S$ is a component within (S, L) if and only if $(C, L(C))$ is a maximally connected subgraph of (S, L). Here, $L(C) := \{\{i, j\} \in L \mid i, j \in C\}$. The resulting partition of S is denoted by S/L. Correspondingly, the worth obtainable by coalition S if $L \subseteq L(S)$ is realized is

$$v_L(S) := \sum_{C \in S/L} v(C).$$

Now, we can define the expected profit of coalition S, namely

$$v_p(S) := \sum_{L \subseteq L(S)} p^S(L)\, v_L(S).$$

The procedure described above is a generalization of the procedure followed by Myerson (1977). To see this, note that a deterministic communication graph (N, L) can be identified with a function $p : \{\{i,j\} \mid i,j \in N,\ i \neq j\} \to [0,1]$, defined by $p(\{i,j\}) = 1$ if $\{i,j\} \in L$ and $p(\{i,j\}) = 0$ if $\{i,j\} \notin L$. It is easily seen that for this p it holds that $v_p = v_L$.

Our interest is in allocation rules for games with probabilistic graphs, i.e., rules that associate with each game with a probabilistic graph a vector of payoffs. Formally, denoting by G_N the space of coalitional games with playerset N and by P_N the set of systems of probabilities for playerset N, an ***allocation rule*** is a function $\mathcal{R} : G_N \times P_N \to \mathbf{R}^N$. Allocation rules can be found by applying solution concepts like the Shapley value (cf. Shapley (1953)), the nucleolus (cf. Schmeidler (1969)), and others to the game (N, v_p) associated with a game with a probabilistic graph (N, v, p). In this paper we will restrict ourselves to the Shapley value[7] ($\mathcal{SH}$) and, following Aumann and Myerson (1988), we will refer to the allocation rule that we obtain in this way as the Myerson value.

Definition Let (N, v, p) be a game with a probabilistic graph. Then the ***Myerson value*** of (N, v, p), $\mathcal{M}(N, v, p) \in \mathbf{R}^N$, is defined by

$$\mathcal{M}(N, v, p) := \mathcal{SH}(N, v_p).$$

In section 5 we will show that the Myerson value can be axiomatically characterized by two of its properties, component efficiency and fairness. First, however, we will show how the model of games with probabilistic graphs can be used to model voting situations.

[7]We refer the reader to Shapley (1953) for a definition of the Shapley value.

4 The Spanish parliament

In this section we will apply the theory that we developped in the preceeding section to the situation in the Spanish parliament that we described in section 2.

Allthough we saw in section 2 that party PNV seems to be powerless, it did play a role in the course of the negotiations that took place in order to form a government. We can explain why this happened if we take into account the fact that some political parties are more compatible than others. In order to take these compatibilities into account, we use the model that we developped in the preceeding section. Hence, we define a game with a probabilistic graph. The playerset for this game is, of course, the set of parties in the Spanish parliament and the characteristic function is the voting game that assigns to each coalition of parties a 0 if they from a loosing coalition and a 1 if they form a winning coalition. The probabilities for bilateral communication are now interpreted as the *a priori* degree of compatibility of the parties.

The degrees of compatibility of the parties that we use here stem from several political analists. Prior to the elections of June 1993 we asked them to give us their estimates of the compatibilities of pairs of parties. We averaged their estimates to get the data in table 4.1. Of course, the compatibilities of the parties depend on a lot of factors like e.g. ideology, competition for votes, the major issues at a certain period, and even the personalities of the individual party members. Hence, these compatibilities are changing over time and the power that is calculated using the compatibilities at a certain moment should be interpreted as an indication of the power of the parties *at a this point in time*. When the political environment changes, then in general the compatibilities will change as well and as a result the power of the parties will also change. A nice property of the Myerson value is that it is continuous in the compatibilities. Therefore, small changes of the compatibilities (or small mistakes in the estimation of these compatibilities) do not cause dramatical changes in the power index.

Unfortunately, it is not possible to us to make the computations for situations with more than 8 parties (it would take us about 2 years to make the necessary computations for a situation with 9 parties on a PC-486. In the calculations the problem is not to compute the Shapley value of the modified game that incorporates both the number of votes of each party and the compatibilities of the parties, but to compute the modified game itself). Therefore, we have to modify the situation on the Spanish parliament. In order to force a situation with less parties, we join the smallest parties in the parliament. We consider the parties EA and ERC to be one party with two seats instead of two parties with one seat each. The same holds for parties PAR and UV.[8] Further, becasue of the same computational limitations, we set the probabilities of party HB to cooperate with another party equal to 0. This is only a minor simplification because party HB is an extrene left wing party that has probabilities almost equal to 0 to cooperate with other parties in the parliament. For more details about the real situation and on the simplifications we make here we refer the reader to Calvo and Lasaga (1995).

The system of compatibilities that resulted from the data provided by the political analists prior to the elections is the following:[9]

	PSOE	PP	IU	CIU	PNV	CC	HB	EA+ERC
PP	.2289							
IU	.4500	.1375						
CIU	.6361	.5712	.2094					
PNV	.7028	.4062	.2469	.8062				
CC	.4429	.6571	.3292	.5917	.4071			
HB	.0000	.0000	.0000	.0000	.0000	.0000		
EA+ERC	.1648	.0810	.3471	.3238	.3890	.2858	.0000	
PAR+UV	.1602	.7500	.1271	.3878	.3232	.6178	.0000	.1438

Table 4.1

Using the data in table 4.1, we can compute the power indices of the parties that evolve when we take into account the compatibilities of the parties. Note that the power indices do not sum to one anymore. This is because the Myerson value is component efficient and the voting game that incorporates ideological (in)compatibilities is computed using *expected* power. We could normalize and multiply the power of the parties by a constant, where the constant is chosen such that the total power of the parties sums up to 1. However, this would not have an influence on the relative power of the parties, which is in fact what we are interested in. Therefore, we will not normalize here.

[8]Note that the composed parties EA+ERC and PAR+UV have a Shapley-Shubik index equal to 0 in the new situation.

[9]The figures given in table 4.1 are averages of the estimates by the political analists. The analists gave us their *estimates* with a precision of two digits. However, since the averages give us more information than the individual estimates, the precision of the figures in table 4.1 are given up to four digits.

party	power	party	power
PSOE	0.4523	CC	0.0367
PP	0.1127	HB	0.0000
IU	0.1283	EA+ERC	0.0103
CIU	0.1859	PAR+UV	0.0181
PNV	0.0502		

Table 4.2

It is clear form table 4.2 that when compatibilities are taken into account, the situation changes drastically. For example, party PNV seemed to be powerless when (in)compatibilities were not taken into account, but the power of this party is nearly half of the power of the big party PP now. Further, we see that the parties PP, IU, and CIU no longer are assigned equal power, but that the the power for these three parties is *inversely* related to the number of seats that they have in the parliament.

A question that is natural to ask at this point is what will be the overall effect when the degrees of compatibility between parties change. The answer to this question is that in general it will not be possible to predict the effect of such a change without computing the power of the parties in the new situation. To illustrate this, we will increase the degree of compatibility of the parties PP and IU that was given in table 4.1. In table 4.3 we give the power of the parties in the Spanish parliament as a function of the increasing degree of compatibility of the parties PP and IU, other things being equal.

p(PP,IU)	.1375	.5000	.7000
PSOE	.4523	.4223	.4058
PP	.1127	.1265	.1341
IU	.1283	.1422	.1498
CIU	.1859	.1998	.2074
PNV	.0502	.0470	.0452
CC	.0367	.0333	.0315
HB	.0000	.0000	.0000
EA+ERC	.0103	.0087	.0078
PAR+UV	.0181	.0168	.0161

Table 4.3

The Spanish government is formed by the non-winning coalition of party PSOE only. This government has the support of two other parties, CIU and PNV. The two main parties opposing the government are PP and IU. Suppose that these two parties decide to cooperate more in order to decrease the power of the parties PSOE, CIU, and PNV. Then we see that the effect of such an intensification of cooperation is not clear beforehand. When the compatibility of parties PP and IU increases, then indeed the power of party PSOE decreases and also the power of party PNV decreases. However, the power of the other party supporting the government, CIU, increases. So, it is not clear in general whether the stability of the government decreases or increases.

5 An axiomatic characterization

We show in this section that the Myerson value can be axiomatically characterized using two of its properties, component efficiency and fairness. We will, in the tradition of section 3, not restrict our analysis to voting situations, but we will consider coalitional games in general. Note, however, that the results that are presented in the current section do also hold in the context of voting games. In particular, the axiomatic characterization of theorem 5.1 also holds when we restrict the scope of the theorem to voting situations.

Component efficiency is an axiom that states that the total payoff to the players in a coalition that has probability 0 of communicating with any player outside the coalition and that is minimal with respect to this property should be equal to the expected profit of the coalition. In order to introduce component efficiency formally, we need some notations. Let (N, v, p) be a game with a probabilistic graph. With this game we associate a deterministic graph (N, L_p) defined as follows: $l = \{i, j\} \in L_p$ if and only if $p_{ij} > 0$. The graph (N, L_p) induces a partition of N into ***communication components.*** We will refer to this partition by N/p. Now, we are ready to introduce the property component efficiency.

Definition An allocation rule $\mathcal{R} : G_N \times P_N \to \mathbf{R}^N$ is *component efficient* if for all games with a probabilistic graph (N, v, p) and all communication components $C \in N/p$ it holds that

$$\sum_{i \in C} \mathcal{R}_i(N, v, p) = v_p(C).$$

In the context of voting games, where it can never be the case that there are two disjoint components of parties that are both winning, component efficiency simply states that the total power is divided among the parties that are in the component consisting of a winning coalition of parties.

The fairness axiom states that when the possibility for direct communication between two players is destroyed, other things being equal, then the payoffs of both these players change with the same amount, so either they both lose the same amount or they both gain the same amount. In formula:

Definition An allocation rule $\mathcal{R} : G_N \times P_N \rightarrow \mathbf{R}^N$ is *fair* if for all games with a probabilistic graph (N, v, p) and all $i, j \in N$ it holds that

$$\mathcal{R}_i(N, v, p) - \mathcal{R}_i(N, v, p_{-ij}) = \mathcal{R}_j(N, v, p) - \mathcal{R}_j(N, v, p_{-ij}),$$

where $p_{-ij}(\{k, l\}) = p(\{k, l\})$ if $\{k, l\} \neq \{i, j\}$, and $p_{-ij}(\{i, j\}) = 0$.[10]

In the context of voting games the fairness axiom states that when a party decides to try and oppose another party to diminish its power, then this will cause an equal loss in power for this party and its opponent.

Theorem 5.1 The Myerson value is the unique allocation rule $\mathcal{R} : G_N \times P_N \rightarrow \mathbf{R}^N$ satisfying component efficiency and fairness.

Proof. To prove that the Myerson value satisfies component efficiency, let (N, v, p) be a probability communication situation and let $C \in N/p$. We split up (N, v_p) into two games, (N, v^C) and $(N, v^{N \backslash C})$, where for all $S \subseteq N$

$$v^C(S) := v_p(S \cap C) \text{ and } v^{N \backslash C}(S) := v_p(S \backslash C).$$

Since C is a component of (N, L_p) we know that

$$v_p = v^C + v^{N \backslash C}.$$

It follows from the dummy property of the Shapley value[11] that $\mathcal{SH}_i(N, v^{N \backslash C}) = 0$ for all $i \in C$. Hence,

$$\begin{aligned} \sum_{i \in C} \mathcal{M}_i(N, v, p) &= \sum_{i \in C} \mathcal{SH}_i(N, v^C) + \sum_{i \in C} \mathcal{SH}_i(N, v^{N \backslash C}) \\ &= \sum_{i \in C} \mathcal{SH}_i(N, v^C) = v^C(N) = v_p(C), \end{aligned}$$

[10] An equivalent definition of the fairness property is obtained when replacing '$p_{-ij}(\{i, j\}) = 0$' by '$p_{-ij}(\{i, j\}) \in [0, 1]$'. The statement of the axiom would then be as follows. When the probability for direct communication between two players changes, then the payoffs to both players change with the same amount.

[11] cf. Shapley (1953).

where the first[12] and the third equality follow from the definition of the Shapley value. To show fairness, let (N, v, p) be a game with a probabilistic graph and let $i, j \in N$, $i \neq j$, with $p_{ij} > 0$. Set $w := v_p - v_{p_{-ij}}$, where p_{-ij} is identical to p except for $p_{-ij}(\{i,j\}) = 0$. Note that $v_p(S) = v_{p_{-ij}}(S)$ for all $S \subseteq N$ with $\{i,j\} \not\subseteq S$. So, if $S \subseteq N$ such that $i \notin S$ or $j \notin S$, then $w(S) = 0$. So, the only coalitions with nonzero worth in the game (N, w) are coalitions containing both i and j. Hence, it follows from symmetry of the Shapley value that $\mathcal{SH}_i(N, w) = \mathcal{SH}_j(N, w)$. Using linearity of the Shapley value, we obtain

$$\mathcal{SH}_i(N, v_p) - \mathcal{SH}_i(N, v_{p_{-ij}}) = \mathcal{SH}_j(N, v_p) - \mathcal{SH}_j(N, v_{p_{-ij}}).$$

To prove uniqueness, suppose that $\mathcal{R}^1$ and $\mathcal{R}^2$ are two allocation rules that are component efficient and fair. We will prove that $\mathcal{R}^1$ and $\mathcal{R}^2$ must be identical.
First, note that component efficiency of $\mathcal{R}^1$ and $\mathcal{R}^2$ implies that $\mathcal{R}^1$ and $\mathcal{R}^2$ are identical for games with a probabilistic graphs where the probabilities of communication are zero for all pairs of players. Now, suppose that $\mathcal{R}^1$ and $\mathcal{R}^2$ are not identical. Then, let (N, v, p) be a game with a probabilistic with a minimum number of links with zero probability such that $\mathcal{R}^1(N, v, p) \neq \mathcal{R}^2(N, v, p)$. By the minimality of (N, v, p), we know that for any link $\{i, j\}$ with $p_{ij} > 0$, it holds that $\mathcal{R}^1(N, v, p_{-ij}) = \mathcal{R}^2(N, v, p_{-ij})$. Hence, fairness of both $\mathcal{R}^1$ and $\mathcal{R}^2$ implies that

$$\begin{aligned}
\mathcal{R}^1_i(N, v, p) - \mathcal{R}^1_j(N, v, p) &= \mathcal{R}^1_i(N, v, p_{-ij}) - \mathcal{R}^1_j(N, v, p_{-ij}) \\
&= \mathcal{R}^2_i(N, v, p_{-ij}) - \mathcal{R}^2_j(N, v, p_{-ij}) \\
&= \mathcal{R}^2_i(N, v, p) - \mathcal{R}^2_j(N, v, p).
\end{aligned}$$

From this we see that

$$\mathcal{R}^1_i(N, v, p) - \mathcal{R}^2_i(N, v, p) = \mathcal{R}^1_j(N, v, p) - \mathcal{R}^2_j(N, v, p)$$

whenever i and j are in the same communication component $C \in N/p$. Thus, we can find numbers d_C, $C \in N/p$, such that $\mathcal{R}^1_i(N, v, p) - \mathcal{R}^2_i(N, v, p) = d_C$ for all $i \in C$ and all $C \in N/p$. Now, we use component efficiency of both $\mathcal{R}^1$ and $\mathcal{R}^2$, which implies that for all $C \in N/p$ it holds that

$$\sum_{i \in C} \mathcal{R}^1_i(N, v, p) = \sum_{i \in C} \mathcal{R}^2_i(N, v, p) = v_p(C).$$

[12]We do indeed use additivity here. As is well-known, additivity does usually not make sense in a context of voting games. Note, however, that in the instance where we use it here there will be no problem, because the decomposition of v_p is done in such a way that one of the two games v^C and $v^{N \setminus C}$ is the zero game, and hence the sum of the two games is a well defined voting game.

Hence, we have $0 = \sum_{i \in C} \mathcal{R}_i^1(N,v,p) - \sum_{i \in C} \mathcal{R}_i^2(N,v,p) = |C|\, d_C$, and so $d_C = 0$. Therefore, we conclude that $\mathcal{R}_i^1(N,v,p) = \mathcal{R}_i^2(N,v,p)$. □

Remark 5.2 In theorem 5.1 it is possible to replace the axiom of fairness by the stronger requirement of balanced contributions.[13] An allocation rule has balanced contributions if the loss (or gain) that the isolation of a player i inflicts on a player j is equal to the effect that the isolation of player j has on player i. Formally, an allocation rule $\mathcal{R} : G_N \times P_N \to \mathbf{R}^N$ has *balanced contributions* if for all games with a probabilistic graph (N,v,p) and all $i,j \in N$ it holds that

$$\mathcal{R}_i(N,v,p) - \mathcal{R}_i(N,v,p_{-j}) = \mathcal{R}_j(N,v,p) - \mathcal{R}_j(N,v,p_{-i}),$$

where $p_{-i}(\{k,l\}) = p(\{k,l\})$ if $i \notin \{k,l\}$, and $p_{-i}(\{k,l\}) = 0$ if $k = i$ or $l = i$. The Myerson value is the unique allocation rule $\mathcal{R} : G_N \times P_N \to \mathbf{R}^N$ satisfying component efficiency and balanced contributions.

In the following proposition we show that the Myerson value for games with a probabilistic graph is stable in the sense of Myerson (1977). He defined stability to be the property that two players always (weakly) benefit from reaching a bilateral agreement whenever the game (N,v) is superadditive.[14] In a context of games with probabilistic graphs stability states that two players always (weakly) benefit when the underlying game is superadditive and the probability of communication between the two players increases, other things being equal.

Definition An allocation rule $\mathcal{R} : G_N \times P_N \to \mathbf{R}^N$ is *stable* if for all games with a probabilistic graph (N,v,p) where the coalitional game (N,v) is superadditive and for all $i,j \in N$ it holds that

$$\mathcal{R}_i(N,v,p) \geq \mathcal{R}_i(N,v,q) \text{ and } \mathcal{R}_j(N,v,p) \geq \mathcal{R}_j(N,v,q),$$

where $q \in P_N$ is such that $q(\{k,l\}) = p(\{k,l\})$ if $\{k,l\} \neq \{i,j\}$, and $q(\{i,j\}) \leq p(\{i,j\})$.

Proposition 5.3 The Myerson value is stable.

Proof. Let (N,v,p) be a game with a probabilistic graph where the game (N,v) is superadditive and let $i,j \in N$. We will prove that $\mathcal{M}_i(N,v,p)$ is an increasing function

[13] The proof of this statement is quite similar to the proofs given in Myerson (1980) and van den Nouweland (1993) (theorem 3.2.1) and we refer the reader to these papers.

[14] A game (N,v) is ***superadditive*** if $v(S \cup T) \geq v(S) + v(T)$ for all disjoint coalitions S and T.

of $p(\{i,j\})$. Using one of the definitions of the Shapley value provided in Shapley (1953), we obtain

$$\mathcal{M}_i(N,v,p) = \mathcal{SH}_i(N,v_p) = \sum_{S\subseteq N\backslash\{i\}} \frac{|S|!\,(|N|-|S|-1)!}{|N|!}(v_p(S\cup\{i\}) - v_p(S)).$$

Note that by the definition of v_p it holds that $v_p(S)$ is independent of $p(\{i,j\})$ for all $S \subseteq N$ with $i \notin S$. Using this, it follows that in order to complete the proof of this proposition it suffices to prove that $v_p(S\cup\{i\})$ is an increasing function of $p(\{i,j\})$ for all $S \subseteq N$ with $i \notin N$. So, let $S \subseteq N$ with $i \notin N$ be fixed. Note that

$$v_p(S) = \sum_{L\subseteq L(S)} p^S(L)\, v_L(S) = \sum_{L\subseteq L(S)} \prod_{l\in L} p_l \prod_{l\in L(S)\backslash L} (1-p_l)\, v_L(S).$$

Differentiating this expression to $p(\{i,j\})$ we obtain

$$\frac{\partial\, v_p(S\cup\{i\})}{\partial\, p(\{i,j\})} = \sum_{L\subseteq L(S)\backslash\{\{i,j\}\}} \prod_{l\in L} p_l \prod_{l\in L(S)\backslash(L\cup\{\{i,j\}\})} (1-p_l)\,(v_{L\cup\{\{i,j\}\}}(S) - v_L(S)).$$

Due to superadditivity of the game (N,v), $v_{L\cup\{\{i,j\}\}}(S) - v_L(S) \geq 0$ for all $S \subseteq N$. Hence, we may conclude that $v_p(S\cup\{i\})$ is an increasing function of $p(\{i,j\})$. □

It is possible to define a potential function for games with a probabilistic graph in the line of Hart and Mas-Colell (1989) and Winter (1992). Also, such a potential function is related to the Myerson value for games with a probabilistic graph in the same way as the potential function of Hart and Mas-Colell (1989) is related to the Shapley value and the potential function of Winter (1992) is related to the Myerson value for (deterministic) communication situations. Since these are straightforward generalizations, we will not include them in this paper.

6 Concluding remarks

In this paper we developed the model of games with probabilistic graphs. This model is inspired by voting situations and the (in)compatibilities of the parties in such situations. We applied the theory to the Spanish parliament and we showed that the power indices obtained using the new model are more realistic than the indices that are obtained using the Shapley-Shubik index, which only takes into account the division of the votes among the parties.

Allthough the model of games with probabilistic graphs is inspired by voting situations, the theory developed in section 3 can be applied in the broader context of coalitional games. An example where our theory could be applied is the following: consider an agent who wants to sell his house and another agent who would be interested in buying the house. It is quite usual for agents who want to buy or sell a house to go to a broker, because a broker is someone who has a lot of information concerning possible buyers and sellers of houses. So, the broker can act as an intermediary between the buyer and the seller. However, if the broker acts as an intermediary, then he has to be paid for his intermediation. Therefore, the buyer and the seller may try to find each other without the intermediation of the broker and then there is some probability, possibly very small, that they can make a transaction without intermediation of the broker.

We consider a small example. There are three agents, a seller, a buyer, and a broker. If the house of the seller can be sold to the buyer, then a surplus of say 1 unit is created. Both the seller and the buyer know the broker and hence, the probability of communication between either the seller and the broker or the buyer and the broker is taken to be equal to 1. The probability that the seller and the buyer can make a transaction without the intermediation of the broker is some $p \in [0,1]$. If we compute the Myerson value for this model, then we find that the broker gets a payoff of $\frac{1-p}{3}$ and that the seller and the buyer each get $\frac{2+p}{6}$. Hence, we see that the payoff of the broker decreases when the probability of direct cooperation between the seller and the buyer increases.

There are several ways in which the model of section 3 can be generalized. One direction in which one can generalize the model is to consider probabilities for *coalitions* of parties. This would generate the possibility to express the fact that a certain party has a fairly high degree of compatibility with each one of two other parties, but a very low degree of compatibility with both other parties at the same time. If we generalize the model in this way, then we would get a generalization of the model of games with conferenc structures as introduced in Myerson (1980).

Another way of generalizing the model is to add a priori unions of the parties. Doing so generates the possibility to consider government coalitions as part of the description of the voting situation. Since the government will generally consist of more than one political party, and since these parties are more or less committed to some governmental policy, one can imagine that by making the a priori union of the governmental coalitions part of the model one will get even more accurate indices of the power of the parties in the parliament. Generalizing the model in this direction requires extending the models of games with a priori unions (cf. Owen (1977), see also Carreras and Owen (1988)) and of games with a priori unions and deterministic communication graphs as studied by Vázquez-Brage *et al.* (1995). One can also extend the model of Hart and Kurz (1983, 1984) in order to study endogenous formation of goverment coalitions. Part of the possible extensions described in this paragraph are considered in Calvo and Lasaga (1995). A particularly interesting result in the paper by Calvo and Lasaga is that they are able to explain the fact that sometimes government coalitions are not minimal winning coalitions, but that they include more parties than would be necessary to have a majority of the votes. This result of theirs is due to the incorporation of probabilities of cooperation in their model.

References

Aumann, R. and Myerson, R. (1988). *Endogenous formation of links between players and coalitions: an application of the Shapley value.* In: The Shapley Value (Ed. A. Roth), Cambridge University Press, Cambridge, United Kingdom, 175-191.

Banzhaf, J. (1965). *Weighted voting doesn't work: a mathematical analysis.* Rutgers Law Review 19, 317-343.

Borm, P., Nouweland, A. van den, and Tijs, S. (1994). *Cooperation and communication restrictions: A survey.* In: Imperfections and Behavior in Economic Organizations (Eds. R. Gilles and P. Ruys), Kluwer Academic Publishers, Dordrecht, The Netherlands, 195-227.

Calvo, E. and Lasaga, J. (1995). *Grafos probabilísticos e índices de poder: una aplicación al parlamento español.* Manuscript, University of the Bask Country, Bilbao, Spain. (In Spanish)

Calvo, E., Lasaga, J., and Winter, E. (1995). *The principle of balanced contributions and hierarchies of cooperation.* Manuscript, University of the Bask Country, Bilbao, Spain.

Carreras, F. and Owen, G. (1988). *Evaluation of the Catalonian parliament, 1980-1984.* Mathematical Social Sciences 15, 87-92.

Damme, E. van, Feltkamp, V., Hurkens, S., and Strausz, R. (1994). *Politieke machtsverhoudingen.* Economisch Statistische Berichten 79, 482-486. (in Dutch)

Hart, S. and Kurz, M. (1983). *Endogenous formation of coalitions.* Econometrica 51, 1047-1064.

Hart, S. and Kurz, M. (1984). *Stable coalition structures.* In: Coalitions and Collective Action (Ed. M. Holler), Physica-Verlag, Wuerzburg, Germany, 235-258.

Hart, S. and Mas-Colell, A. (1989). *Potential, value, and consistency.* Econometrica 3, 589-614.

Myerson, R. (1977). *Graphs and cooperation in games.* Mathematics of Operations Research 2, 225-229.

Myerson, R. (1980). *Conference structures and fair allocation rules.* International Journal of Game Theory 9, 169-182.

Nouweland, A. van den (1993). *Games and graphs in economic situations.* PhD Dissertation, Tilburg University, Tilburg, The Netherlands.

Owen, G. (1977). *Values of games with a priori unions.* In: Essays in Mathematical Economics and Game Theory (Eds. R. Henn and O. Moeschlin), Springer-Verlag, Berlin, Germany, 76-88.

Schmeidler, D. (1969). *The nucleolus of a characteristic function game.* SIAM Journal on Applied Mathematics 17, 1163-1170.

Shapley, L. (1953). *A value for n-person games.* Contributions to the Theory of Games II (Eds. A. Tucker and H. Kuhn), 307-317.

Shapley, L. and Shubik, M. (1954). *A method for evaluating the distribution of power in a committee system.* American Political Science Review 48, 787-792.

Vázquez-Brage, M., García-Jurado, I., and Carreras, F. (1995). *The Owen value applied to games with graph restricted communication.* Games and Economic Behavior. (to appear)

Winter, E. (1992). *The consistency and potential for values of games with coalition structures.* Games and Economic Behavior 4, 132-144.

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