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ALTERNATING BID BARGAINING WITH A SMALLEST MONEY UNIT R20

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Alternating Bid Bargaining with a Smallest Money Unit

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Abstract

In a seminal paper, Ariel Rubinstein has shown that impatience implies determinateness of the 2-person bargaining problem. In this note we show that this result depends also on the assumption that the set of alternatives is a continuum. If the pie can be divided only in finitely many different ways, (for example, because the pie is an amount of money and there is a smallest money unit), any partition can be obtained as the result of a subgame perfect equilibrium if the time interval between successive offers is sufficiently small.

1 Introduction

A natural way of modelling two-person bargaining as an extensive game makes use of a game structure in which two players take turns in making bids. In each round one player makes a bid; then the other player either accepts or rejects this offer; in case of rejection the rejector makes the next bid, etc. We use the term "alternating bid models" for bargaining games of this structure. A pioneering investigation of alternating bid models is due to Ingolf Ståhl (1972). Later Ariel Rubinstein has created a very influential bargaining theory based on an alternating bid model (Rubinstein (1982)).

The bargaining problem considered here is the division of a fixed amount of money. Rubinstein's model permits arbitrary divisions and thereby provides a continuum of possible agreements. An alternative to the assumption of infinite divisibility of money is the introduction of a smallest money unit. It is our aim to explore the consequences of this assumption for alternating bid bargaining. As far as other assumptions are concerned our analysis is based on Rubinstein's framework.

Rubinstein's theory specifies a unique solution, the uniquely determined subgame perfect equilibrium of his model. It will be shown that the introduction of a smallest money unit destroys Rubinstein's uniqueness result. If both players are risk neutral, the amount of money to be distributed is $ 50,000, the smallest money unit is 1c, the yearly interest rate is 10% and one bargaining round takes 1 minute, then all divisions of the $ 50,000 are supported by subgame perfect equilibria of the modified model (see proposition 1 and the explanation of table 1 in section 4).

Ingolf Ståhl already investigated alternating bid models with a finite number of agreements. (Ståhl (1972), also see Ståhl (1988) in which Ståhl's original model is compared to that of Rubinstein). He examined models of finite and of infinite length. However, his research questions were different from ours. He aimed at sufficient conditions for uniqueness. We are interested in non-uniqueness as a consequence of the presence of a smallest money unit.

It is impossible to construct an absolutely realistic bargaining model. Every real bargaining situation has many special features which are minor influences on the bargaining result. Idealization is an unavoidable ingredient of model construction. Is it really necessary to model a relatively inconspicuous institutional detail like the presence of a smallest money unit? Maybe the correct answer is no. However, Rubinstein's theory heavily relies on very small time costs due to discounting. It should not take more than one minute to make a bid. The bidder does not have to do more than pronounce a number. Even for quite sizable amounts of money the interest for one minute at a reasonable yearly rate is very small. Why should very small interest losses be modelled explicitely? Maybe also here the answer is no.

The smallest money unit and the time discount are both minor strategic influences, but these forces interact. Therefore, either both should be considered or both should be neglected. In this sense Rubinstein's model is an imbalanced idealization. His theory relies on an explicitely modelled weak influence and ignores a weak counteracting force.

Assume that both bargainers are risk neutral; let A be the amount of money to be distributed and let δ be the discount rate for one bargaining round. We assume that both players have the same discount rate. In Rubinstein's theory the solution is of the following type: A player who makes a bid asks for x. A player accepts every offer which gives him at least $A - x$

and rejects everything else. The solution requires indifference between $A-x$ and δx. This yields $x = A/(1+\delta)$.

Rubinstein's theory excludes strategy combinations of the following type as possible solutions: Player 1 always asks for y and player 2 always asks for $A-y$. Both accept any offer which gives them at least what they ask for and reject all others. If player 1 asks for a little more, say $y+\epsilon$, then player 2 faces a choice between $A-y-\epsilon$ for acceptance and $\delta(A-y)$ for rejection. Obviously he must accept, if we have $\epsilon < (1-\delta)(A-y)$. This contradicts the assumption of a subgame perfectness.

Now consider the consequences of the introduction of a smallest money unit g. Agreement payoffs must be integer multiples of g. If now player 1 wants to ask for more than y, he has to demand at least $y+g$. If g is greater than $(1-\delta)A$ player 1 cannot give an incentive to player 2 to accept less than $A-y$. The smallest money unit prevents him from increasing his demand by an amount ϵ which is smaller than the interest loss $(1-\delta)(A-y)$. This heuristic argument indicates why Rubinstein's uniqueness result is destroyed by the introduction of a smallest money unit. The size of the smallest money unit puts a lower bound on exploitable interest losses on the other side. Note that decreasing the time between offers corresponds to increasing δ and that for δ sufficiently large always $g > (1-\delta)A$. Hence, for short intervals between offers it may be expected that indeed any distribution can be obtained by some subgame perfect equilibrium. (See Proposition 1).

In Rubinstein's bargaining solution agreement is reached at once. More than one bargaining round cannot be played, unless mistakes are made. Contrary to this in the presence of a smallest money unit subgame perfect equilibrium may involve many bargaining rounds, before an agreement is reached (proposition 2) [1].

Undoubtedly Rubinstein's ingenious bargaining theory merits our admiration, but we cannot avoid the conclusion, that his model does not provide a balanced idealization of real bargaining situations. The driving force behind his uniqueness result is provided by the exploitability of small interest losses by even smaller increases of a bidder's demand. In the presence of a smallest money unit such destabilization possibilities are easily lost, since it becomes impossible to deviate sufficiently little. Needless to say that the same critique applies to more elaborate models of bargaining that also assume perfect divisibility of money, such as the various strategic models of bargaining under incomplete information. (Let us just quote Sobel and Takahashi (1982) as a representative example).

Our analysis is based on the same game theoretic rationality assumptions as Rubinstein's theory. Presumably real bargaining behaviour would not be influenced by a smallest money unit of insignificant size and equally insignificant time discounts. Nevertheless, it is worthwhile to use the concept of a subgame perfect equilibrium point in order to explore the interaction and the comparative importance of both influences.

2 The Bargaining Model

Two players, denoted by 1 and 2, have to divide an amount of money (normalised to) 1. Let $g > 0$ denote the smallest money unit. The set of possible agreements is

[1] Osborne and Rubinstein (1989, Sect. 3.9.1.) give an example to show that finitely many alternatives may lead to multiple equilibria and to delay. They do not investigate the general consequences of shrinking the time between offers.

$$X = \{(k_1 g, k_2 g) \mid k_i \in \mathbf{N}, (k_1 + k_2)g \leq 1\} \tag{1}$$

and X^e denotes the set of efficient agreements $((k_1 + k_2)g = 1)$.

Bargaining takes place over time, starts at $t = 0$, and proceeds according to the following rules:

ROUND t ($t \in \mathbf{N}, t$ even): Player 1 proposes $x \in X$; after hearing 1's proposal, player 2 either accepts or rejects. If 2 accepts, the game terminates with agreement x, otherwise the game moves to round $t + 1$.

ROUND t ($t \in \mathbf{N}, t$ odd): Player 2 proposes $x \in X$, after hearing 2's proposal, player 1 either accepts or rejects. If 1 accepts, the game ends with agreement x, otherwise the game reaches round $t + 1$.

Denote by $< x, t >$ the outcome where agreement on x is reached in round t and let D denote perpetual disagreement. Let Δ be the length of a single bargaining round. We will assume that there exist constants $r_1, r_2, > 0$, and strictly increasing concave functions U_1, U_2 (having domain [0,1]) with $U_i(0) = 0$ such that the preferences of the players can be represented by the utility functions V_i given by

$$V_i(< x, t >) = e^{-r_i \Delta t} U_i(x) \text{ and } V_i(D) = 0 \tag{2}$$

For justification of this assumption, we refer to Fishburn and Rubinstein (1982).

The above fully describes the game to be denoted $\Gamma(\Delta)$. Strategies, Nash equilibria and (subgame) perfect equilibria are defined in the standard way, hence, these definitions will not be repeated here (see Rubinstein (1982)). Rather, we directly turn to our main results.

3 Results

PROPOSITION 1. *If Δ is sufficiently small , specifically if Δ is such that for $i = 1, 2$*

$$U_i(1-g)/U_i(1) \leq e^{-r_i\Delta} \tag{3}$$

then, for any efficient agreement $x \in X^e$ there exists a subgame perfect equilibrium of $\Gamma(\Delta)$ that results in the outcome $< x, 0 >$.

PROOF. Note that (3) says that player i prefers getting the full amount 1 one period later to receiving 1-g now. Since U_i is concave this condition implies that for all $x_i \in [g, 1]$

$$U_i(x_i - g)/U_i(x_i) \leq e^{-r_i\Delta} \tag{4}$$

Let $x \in X^e$ and write $x = (x_1, x_2)$. Consider the pair of stationairy strategies $\sigma^x = (\sigma_1^x, \sigma_2^x)$ defined by

σ_i^x : Always propose x;
Accept any proposal y with $y_i \geq x_i$,
Reject any other proposal.

If σ^x is played, the outcome $< x, 0 >$ results. We claim that σ^x is a subgame perfect equilibrium if (3) is satisfied. Because of stationarity, it suffices to show that one-period deviations are not profitable. Hence, we must verify that it does not pay to deviate in round t when from round $t+1$ on play will always be in accordance to σ^x. Obviously, given the acceptance/rejectance decision of the other player, the best one can propose is x since x is efficient in X. Clearly, it is also optimal for player i to accept any offer that yields him at least x_i. The crucial step is to verify that, if $x_i > 0$, it is optimal for player i to reject any offer y with $y_i < x_i$. However, this is guaranteed by (4). $\square$

All equilibria constructed thus far result in an immediate efficient agreement. However, since there is a multiplicity of such equilibria, it is easy to construct alternative equilibria that do not have these nice properties. The idea is to sustain a path π in which both players have positive payoffs with the threat to continue with the equilibrium from Proposition 1 that yields player i the payoff zero if i deviates from π. Formally,this construction is carried out in Proposition 2. It is convenient to introduce the following notation: $x^1 = (1,0), x^2 = (0,1)$ and $\sigma^i = \sigma^{x^i}$. Finally, i_T denotes the player who proposes in round T (hence $i_T = T mod 2 + 1$).

PROPOSITION 2. *Let $x \in X$ with $x \neq (0,0)$ and $T \in \mathbf{N}$. If Δ satisfies (3), there exists a subgame perfect equilibrium of $\Gamma(\Delta)$ that results in the outcom e $< x, T >$.*

PROOF. We will confine ourselves to the case where $x_1 x_2 > 0$ and leave the details of the remaining cases to the reader. Consider the strategy pair $\sigma = (\sigma_1, \sigma_2)$ defined by

σ_i :In round t $(t < T)$: Propose x^i; accept x^i but reject any other proposal.
In round T : Propose x; accept x and x^i, but reject any other proposal.
In round t $(t > T)$: Play according to $\sigma_i^{i_T}$.

Let $\pi(\sigma_1, \sigma_2)$ be the path induced by (σ_1, σ_2). The strategy pair $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2)$ is constructed from σ by means of

$\bar{\sigma}_i$: Play according to σ_i, however, if in any round $t < T$ there is a deviation from $\pi(\sigma_1, \sigma_2)$ and if the first deviation is by player k, then immediately after this deviation switch to playing σ_i^{3-k} for the remainder of the game [2].

We claim that $\bar{\sigma}$ is a subgame perfect equilibrium of $\Gamma(\Delta)$ whenever (3) is satisfied. Note that $\bar{\sigma}$ results in the outcome $< x, T >$. To prove the claim, it suffices (because of Proposition 1) to show that deviating in some round $t \leq T$ is not profitable. However, this is easily verified: If a proposing player deviates he ends up with zero, hence, deviating is not profitable for him. Deviating is clearly not attractive for the responding player as long as the proposal is on the equilibium path (it will yield payoff zero). If the proposer has deviated, rejection yields the responder 1 in the next period, hence, if (3) is satisfied, he should reject anything less than 1, exactly as $\bar{\sigma}$ says that he should do. Consequently, $\bar{\sigma}$ is a subgame perfect equilibrium if Δ is small.$\square$

Proposition 2 shows that really almost anything can happen in a subgame perfect equilibrium if the time between offers is small. The only outcomes that are not covered by the theorem are (i) agreement on the outcome in which no player receives anything and (ii) perpetual disagreement. It is easy to see that none of these outcomes can arise in a subgame perfect equilibrium if there exists some $x \in X$ with $x_1 x_2 > 0$. Hence, Proposition 2 fully describes the set of all subgame perfect equilibrium outcomes.

[2] Note that, if i is the proposer in round $t \leq T$, then player j will switch to σ_j^j already in round t.

4 The Risk Neutral Case

In this section we confine ourselves to the case where players are risk neutral, i.e. $u_i(x) = x$. This case is most favorable for the point we wish to make. If both players are (equally) risk averse, the range of equilibrium payoffs will be smaller: If the utility function v displays greater risk aversion that u, then $v(x-g)/v(x) \geq u(x-g)/u(x)$ so that is becomes more difficult to obtain x in a subgame equilibrium (cf. Eq. (4)).

Let us first illustrate the bound on Δ given in (3) by performing some numerical calculations. Take $r_1 = r_2 = 10\%$ per year, and let the smallest money unit be 1 cent (\$ 0.01). Proposition 1 implies that, if the time between offers is Δ, any efficient division of an amount up to A(Δ) (as given in Table 1) can be obtained in a subgame perfect equilibrium.

Δ	$A(\Delta)$
1 day	\$ 36.50
1 hour	\$ 876
1 min	\$ 52,576
1 sec	\$ 3,225,806

Table 1.

If the amount of money to be divided is larger than $A(\Delta)$, then the simple strategies from Proposition 1 are no longer in equilibrium if x is "sufficiently asymmetric", but, of course, there may be more sophisticated equilibria that still result in such x. For given r_i, U_i, Δ, g and an amount to be divided, A, we have not computed the (absolute) gap between the best and the worst equilibrium payoff (we conjecture that, it remains bounded away far from zero as A tends to infinity), however, it is easy to see that this gap becomes smaller relative to A. Proving the latter is equivalent to showing that the gap tends to zero when g tends to zero, with A fixed at 1 and all other parameters remaining constant as well. Assume $r_1 = r_2$ and write $\delta = e^{-r_1 \Delta}$.

Denote by M (resp. m) the supremum (resp. infimum) of the subgame perfect equilibrium payoffs of player 1 in $\Gamma(\Delta)$. Following the argument outlined in Shaked and Sutton [1984], it is easy to see that in the generic case where δM and δm are not integer multiples of g, M and m must satisfy the equations

$$M = 1 - \lceil \delta m \rceil \tag{5}$$

$$m = 1 - \lceil \delta M \rceil \tag{6}$$

where $\lceil \delta x \rceil$ denotes the smallest integer multiple of g that is at least equal to δx. The Eqs. (5) and (6) imply

$$M - \lceil \delta M \rceil = m - \lceil \delta m \rceil, \tag{7}$$

which is equivalent to saying that there is some $k \in N$ such that

$$n - (k+1)g < \delta n \leq n - kg \text{ for } n \in \{m, M\}, \tag{8}$$

Rewriting the last inequality yields

$$kg \leq n(1-\delta) < (k+1)g \text{ for } n \in \{m, M\}, \tag{9}$$

which implies

$$M - m \leq g/(1-\delta). \tag{10}$$

We see that $M - m$ tends to zero if g tends to zero. Furthermore, if g tends to zero, $\lfloor \delta m \rfloor$ converges to δm, so that the limit of M solves the equation

$$M = 1 - \delta M \tag{11}$$

an expression that is familiar from the work of Rubinstein. Hence, we may state

PROPOSITION 3. *If g tends to zero, the payoffs associated to any subgame perfect equilibrium of $\Gamma(\Delta)$ converge to the payoffs of the unique subgame perfect equilibrium of the continuum game.*

Finally, it is of some interest to see how the properties of the finite horizon bargaining game $\Gamma(\Delta, T)$, (i.e. truncate $\Gamma(\Delta)$ after T rounds) are affected by the introduction of a smallest money unit. It is trivial to see that, for a fixed horizon T, if Δ is sufficiently small, all subgame perfect equilibria of $\Gamma(\Delta, T)$ yield the player who has the final right to make a proposal almost 1. More interesting, however, is the question what happens for fixed Δ and g, when T tends to infinity. The analysis is easy if inequality (3) is satisfied. Let player 2 make the final proposal. In the last round, this player obtains at least $1-g$, so that in the second to last round he rejects any proposal that yields him less than $1-g$. Consequently, the equilibrium payoffs of player 1 in the second to last round are bounded above by g. Therefore, in the third to last round player 2's equilibrium payoff is again at least $1 - g$, and the argument can be continued to the beginning of the game: The equilibrium payoffs of the player who makes the last proposal are bounded below [3] by

[3]There exist equilibria in which, in the second to last round, player 1 makes the proposal (1,0) that is rejected and upon which player 2 continues with $(g, 1 - g)$. It is not optimal for player 1 to make an alternative proposal since player 2 would interpret this as a signal

$e^{-r_i\Delta}(1-g)$. Comparing this result with Proposition 2 we see that there is a discontinuity at $T=\infty$. This discontinuity is not present in Rubinstein's continuous specification. In that case, also the finite horizon model has a unique subgame perfect equilibrium and, as T tends to infinity, the payoffs associated with this equilibrium converge to the equilibrium payoffs of the infinite horizon game.

to continue with $(0,1)$ in the final round (and hence, would reject it unless the proposal itself was $(0,1)$).

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