Tilburg University

Dynamic programming solution of incentive constrained problems

Rustichini, A.

Publication date:
1996

entER for nomic Research

Discussion paper
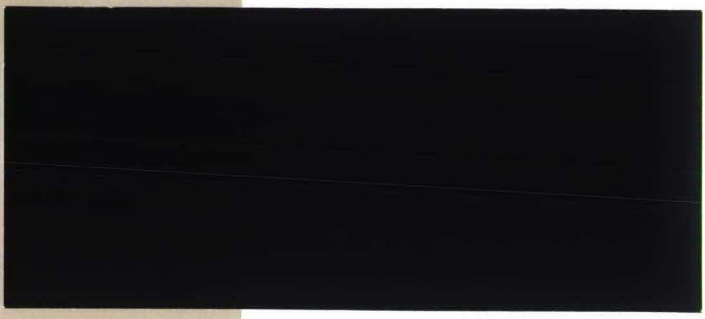
Tilburg University

Center
for
Economic Research

90

No. 9697

DYNAMIC PROGRAMMING SOLUTION OF INCENTIVE CONSTRAINED PROBLEMS

By Aldo Rustichini

R 35

+ dynamic programming

October 1996

Dynamic Programming Solution of Incentive Constrained Problems *

Aldo Rustichini

CentER

Tilburg University

September 1996

Abstract

Several problems in economic theory can be formulated as a dynamic programming problem, plus an additional *incentive compatibility* constraint. This constraint requires the continuation value along the chosen sequence, at any point in time, to be larger than some prescribed function of the state, the control, or perhaps both. This constraint changes the nature of the problem in a substantial way: for instance, even if the problem is discounted, standard arguments based on contraction principles do not apply.

In this paper we show how to reduce this class of problems to a simple variation of standard dynamic programming techniques. In particular the value function for the problem is shown to be the fixed point of an appropriately defined operator.

---

1 Introduction

In recent years there have been several problems in economic theory which have a common structure. They are all maximisation problems over an infinite horizon, subject to two types of constraints. The first is the standard set of constraints: the control variable may have to be chosen in a subset that depends on the state variable, and the state variable itself has to follow a law of motion which depends on the past value of the state and of the control. The second type has the nature of an incentive compatibility constraint; more precisely, the continuation value from the plan, at any period, must be larger than some prescribed value, which may depend on the state variable, and on the choice of the control variable prescribed by the plan.

We have referred to this additional constraint (and we shall use this name in the paper) as an *incentive compatibility constraint* because we may think of the agent who is solving the problem as having an outside option available, with some alternative value; if the value of the problem is less than that, then he may refuse the problem itself, and rather choose the outside option.

This additional constraint seems to preclude the use of traditional Dynamic Programming (DP) techniques. In fact, an extension is necessary: but we plan to show that a simple modification of standard techniques is sufficient. These problems have been discussed and analysed in Marcet and Marimon [11]; see also their previous paper [10], which provides an interesting example of a dynamic incentive constrained problem. The main idea in their paper is to introduce the Lagrange multipliers in the state space, and apply DP techniques to this enlarged state space. In the present paper the state space will be unchanged, simplifying the analysis.

A problem that may look similar, but is in reality quite different, goes usually under the name of *optimal stopping*. Since the similarity with the problem we analyse may be confusing, we will discuss this issue briefly. In an optimal stopping problem an agent is deriving in each period a payoff from a control variable, that he may choose, and a state variable, which he may be able to control. In addition he has at any period the option of stopping the process, and getting a termination payoff that may depend on the final value of the state variable. Of course in this problem the continuation value from any *optimal* policy must be at least as large as the value of choosing to terminate the process, since this option is always available to the agent. It is however a perfectly *admissible* policy for the agent to let the process continue at any state, even if the future value from doing so is lower than the termination value. As a consequence, the value (from an optimal policy) of a problem where the termination option is available is higher than in the similar problem where this option is not available.

This is quite different from the constrained problem that we consider in this paper. Here every admissible policy, and not just the optimal ones, must satisfy the constraint. If this is not possible, the value of the problem is set to minus infinity. As a consequence the value from an optimal policy of a problem where the incentive constraint has to be satisfied in addition to the others is smaller than in the similar problem where this constraint is not present.

Before we proceed we present some examples to illustrate the nature of the problem.

1.1 Examples of Incentive Constrained Problems

Our first example is a game of joint exploitation of a renewable resource. In this game two players have concave, strictly increasing utility functions and a common discount factor. The two utilities depend on the consumption of a common resource; the stock of it which is left in each period determines the quantity available in the next period. Each player may attempt to consume part of the output. He faces the following tradeoff: the less he consumes, the more stock of the common good is left for production in the next period. The saving, however, is beneficial to both players, and this externality typically produces a consumption rate higher than the optimal. It is interesting in this model to characterise the second best equilibria, that is the subset of subgame perfect equilibria that are also Pareto efficient. Due to the specific structure of the game, this analysis can be reduced to the study of a constrained maximisation problem of the type we have described in the previous section. Here are some details.

The value of the outside option is defined as follows. Each player at the moment of deciding his consumption has the alternative of following the prescribed second best plan, or defecting. If he defects, he knows that the equilibrium after defection is determined as the equilibrium (in that subgame) where both players are consuming maximally. It is fairly easy to check that this is indeed a subgame perfect equilibrium. So the best he can derive from the defection is the maximum over consumption today, different from the prescribed consumption, plus the discounted continuation value of the maximal consumption equilibrium from next period. The second best solution is therefore characterised as the solution of the problem of maximising the welfare of the two players, subject to the constraint that the continuation value for each player is larger than the defection value. For details, the reader is referred to Benhabib and Rustichini [3].

This simple example cannot be analysed with the techniques in Marcet and Marimon [11]. In fact they impose conditions on the problem that make the value function continuous: but as we see more in detail later, the value function in this problem is, in general, discontinuous.

The second example is the classical optimal taxation problem in a general equilibrium model with infinitely lived agents (see for instance Chamley [5] but with a twist. As usual in the standard second best problem the government is taking the first order conditions of the agents as given. At the moment of announcing the sequence of future taxes, however, the government has to insure that one additional constraint is satisfied: the continuation value on the competitive equilibrium from any point in the future determined by the announced plan has to be larger than some value, which may depend on the capital stock of the economy at that time. We may think of this value as the value of a deviation option for the government itself. The third-best taxation problem is the problem of maximising the welfare of

the representative agent, subject to this incentive compatibility constraint. For a detailed discussion see Benhabib and Rustichini [4]

2 The general incentive constrained problem.

We let S denote the *state space*, and A the *action set*. S and A are topological Hausdorff, locally compact spaces. The *state and action space* is $X \equiv S \times A$; its infinite product is denoted $X^\infty \equiv \times_{t=0}^\infty X$; we take on it the product topology. For $\mathbf{x} \in X^\infty, \mathbf{x} \equiv (x_0, x_1, \ldots)$ we denote $_t\mathbf{x} \equiv (x_t, x_{t+1}, \ldots)$ where $x_t = (s_t, a_t)$. Both X and X^∞ are also Hausdorff; and so is any subspace of it. There is some gain in avoiding the restriction of S and A to finite dimensional spaces: this issue is discussed later, in section 2.1.

For any set X, $\mathcal{P}(X)$ is the power set of X. The correspondence $G : S \to \mathcal{P}(A)$ defines the set of *feasible actions*, and the function $F : X \to S$ defines the *transition to the new state*. F and G jointly define a transition correspondence $Q : X \to \mathcal{P}(X)$ as:

$$Q(s, a) \equiv \{(u, b) : u = F(s, a), b \in G(u)\} \tag{2.1}$$

The iterates of Q are denoted by Q^i, for $i = 0, 1, \ldots$; $Q^0(s, a) \equiv (s, a)$.

The set of sequences that are *conceivable* from s is $\times_{i=0}^\infty Q^i(s, G(s))$; this is in general a proper superset of the set of feasible sequences. Note for future use that

$$Q(s, G(s)) = (F(s, G(s)), G(F(s, G(s)))).$$

The set of *feasible sequences* from a given initial state s is

$$\Phi(s) \equiv \{\mathbf{x} \in X^\infty : x_0 \in (s, G(s)), x_{t+1} \in Q(x_t) \text{ for all } t \geq 0.\}$$

Preferences on sequences are represented by a function U taking values in the extended real line $\mathbf{R}^* \equiv \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$. The following concepts have been introduced by Koopmans [8]:

Definition 2.1 U *is after first period separable if for all $x, y \in X$ and all $\mathbf{x}, \mathbf{y} \in X^\infty$, $U(x, \mathbf{x}) \geq U(x, \mathbf{y})$ if and only if $U(y, \mathbf{x}) \geq U(y, \mathbf{y})$.*

This is his postulate (3.b) ([8], page 292).

Definition 2.2 U *is first period separable if for all $x, y \in X$ and all $\mathbf{x}, \mathbf{y} \in X^\infty$, $U(x, \mathbf{x}) \geq U(y, \mathbf{x})$ if and only if $U(x, \mathbf{y}) \geq U(y, \mathbf{y})$.*

This is his postulate (3.a) ([8], page 292). The intuitive meaning of these two postulates is clear: significantly, they go under the name *Limited Noncomplementarity* in Koopmans *et alii* ([9], page 85).

Definition 2.3 U *is stationary if for all $x \in X$ and all $\mathbf{x}, \mathbf{y} \in X^\infty$, $U(x, \mathbf{x}) \geq U(x, \mathbf{y})$ if and only if $U(\mathbf{x}) \geq U(\mathbf{y})$.*

4

This is Koopmans' *stationarity* postulate (4) ([8], page 294).

An *aggregator* is a function $W : X \times V \to V$, where V is a subset of $\mathbf{R}^*$, such that for every $\mathbf{x} \in X^\infty$,

$$U(\mathbf{x}) = W(x_0, U({}_1\mathbf{x})). \tag{2.2}$$

As Streufert ([14], page 61) notes, the proof of the classic result of Koopmans [8] can be adapted to show that if a function U is after first period separable and stationary, then there exists a unique aggregator which is strictly increasing in future utility. Let now D be the function from X to the reals, describing the incentive constraint.

The set of *feasible and incentive compatible sequences* from a given initial state s is

$$\Psi(s) \equiv \{\mathbf{x} \in X^\infty : \mathbf{x} \in \Phi(s), \text{ and for all } t \geq 0, W(x_t, U({}_{t+1}\mathbf{x})) \geq D(x_t)\}.$$

Definition 2.4 *The unconstrained problem at s is to find:*

$$\sup U(\mathbf{x}),$$

subject to :

$$\mathbf{x} \in \Phi(s) \tag{2.3}$$

We denote this problem $P(s)$, and its value $J_0(s)$.

Definition 2.5 *The incentive constrained problem at s is to find:*

$$\sup U(\mathbf{x}),$$

subject to:

$$\mathbf{x} \in \Psi(s). \tag{2.4}$$

If $\Psi(s) = \emptyset$, then $\sup U = -\infty$.

We denote this problem $CP(s)$, and its value $J^*(s)$.

The constraint $W(x_t, U({}_{t+1}\mathbf{x})) \geq D(x_t)$ will be called in the following the *incentive compatibility constraint*. We use this name because the constraint specifies that the continuation value along the sequence has to be larger than or equal to the value of a prescribed function of the state and of the control in that period. If this prescribed value is interpreted as an outside option available to the agent who is performing the maximisation, then the constraint requires that the sequence provides at any point the right incentive to the agent to follow the plan.

2.1 Stochastic Models.

The assumptions imposed on the state space S and the action set A are general enough that a rather large class of stochastic problems is included as possible applications of our results. Let us briefly discuss how this can be done.

Let Ω be a topological space, and let $S \equiv \Delta(\Omega)$ be the set of regular probability measures on Ω endowed with the Borel structure. The topology on S is the weak* topology. A transition kernel $q : \Omega \times A \to S$ which is continuous determines in turn a continuous transition function $F : S \times A \to S$ defined by:

$$F(s,a)(O) \equiv \int_\Omega q(\omega,a)(O)s(d\omega)$$

for every s, a and every Borel subset $O \subseteq \Omega$.

A typical aggregator in this situation, for example a linear aggregator, might be:

$$W((s,a),y) \equiv \int_\Omega u(\omega,a)s(d\omega) + \beta y \qquad (2.5)$$

where u is a utility function and β the discount factor. Assumptions $(A.2)$ and $(A.3)$ below are easily checked in this example.

In the general setup we have described above the feasibility constraints are described by the function G from S to subsets of A. In the case of stochastic models one might wish to express constraints that make the feasible set of actions depend on the state ω rather than on the distribution s. This type of constraints can be incorporated into the function U, or equivalently into the aggregator. For example, in the case of the linear aggregator in 2.5 we may set $u(\omega, a) \equiv -\infty$ when the action a is not feasible at the state ω.

2.2 Existence and basic properties of a solution.

In this section we collect some basic properties of the value function of the incentive constrained problem. We assume:

Assumption 2.6 $(A.1)$ *G is upper hemicontinuous and compact valued; F is continuous.*

An immediate implication of assumption 2.6 is that the correspondence Q is upper hemicontinuous and compact valued.

Assumption 2.7 $(A.2)$ *The function U is after first period separable, first period separable, and stationary. Let $W : X \times V \to V$ be the associated aggregator: W is strictly increasing in future utility, V is an interval in $\mathbf{R}^*$, and for every x, $W(x, -\infty) = -\infty$.*

Assumption 2.8 $(A.3)$ *W is upper semicontinuous.*

In Proposition 3, page 65, Streufert [14] shows how the assumption $(A.3)$ can itself be derived from assumptions imposed directly on the utility function U.

The final key concept, *biconvergence*, has been introduced by Streufert [14]. We use here a slight strengthening of his condition. Let K be any compact subset of S; then the function U is said to be *K-upper convergent* on $\times_{i=0}^{\infty}Q^i(K, G(K))$ if for every $\mathbf{x} \in \times_{i=0}^{\infty}Q^i(K, G(K))$

$$\limsup_{t \to \infty} U(x_0, \ldots, x_t, \times_{i=t+1}^{\infty}Q^i(K, G(K))) = U(\mathbf{x}). \tag{2.6}$$

Analogously, the function U is said to be *K-lower convergent* on $\times_{i=0}^{\infty}Q^i(K, G(K))$ if for every $\mathbf{x} \in \times_{i=0}^{\infty}Q^i(K, G(K))$

$$\liminf_{t \to \infty} U(x_0, \ldots, x_t, \times_{i=t+1}^{\infty}Q^i(K, G(K))) = U(\mathbf{x}). \tag{2.7}$$

The function U is said to be *K-biconvergent* if it is both K-upper and K-lower convergent.

Assumption 2.9 $(A.4)$ *For any $s \in S$ there is compact neighbourhood K of s such that U is K-biconvergent on $\times_{i=0}^{\infty}Q^i(K, G(K))$.*

This assumption is discussed in detail later (section 2.2.1). We now turn to the last component: the function D. In many applications this function gives the value of deviation from a prescribed equilibrium. As such it is derived as a solution of an optimisation problem or of an equilibrium condition, and inherits naturally from the original some specific properties. For instance, it is quite commonly a concave function. This fact may create problems, since the function D lies on the opposite side of an inequality with respect to the function W. In particular the set on incentive compatible paths may not be convex. It is therefore particularly desirable to put on it as few assumptions as possible. We only need a rather weak regularity condition, and no geometric property (like convexity):

Assumption 2.10 $(A.5)$ *For every $s \in S$, $D(s, \cdot)$ is lower semicontinuous.*

We are now ready for our preliminary results.

Lemma 2.11 *For every compact subset K of S:*
(i) $\max U(\times_{i=0}^{\infty}Q^i(K, G(K))$ is achieved;
(ii) U is upper semicontinuous on $\times_{i=0}^{\infty}Q^i(K, G(K)$.

The proof of the part (i) is very similar to the proof of proposition 3, part (A.6), in page 82 of Streufert [14], once the assumption of upper convergence is replaced by the K-upper convergence condition. The proof of (ii) is similar to the proofs of lemma 2 and 3 (*ibidem*).

Lemma 2.12 *Assume A1, A2, A3, A4, A.5. Then*
(i) the correspondences Φ, Ψ are upper hemicontinuous, compact valued;
(ii) the incentive constrained problem has a solution;
(iii) the function J^* is upper semicontinuous.

7

Proof.

(i) From $(A.1)$ the correspondence Q is upper hemicontinuous and compact valued. Hence the set $R(s) \equiv \times_{i=0}^{\infty} Q^i(s, G(s))$ is compact in the product topology for every s, and so are $\Phi(s)$ and $\Psi(s)$, since it is immediate that they have a closed graph. By lemma 1 of Streufert [14] (page 76) the correspondence R is upper hemicontinuous. Since G is upper hemicontinuous and F continuous, Φ has closed graph, hence by theorem 8, page 110, of Aubin is upper hemicontinuous, since it is contained in R; it is also compact valued. Now the set

$$\{\mathbf{x} \in X^{\infty} : W(x_t, U_{(t+1}\mathbf{x})) \geq D(x_t), t = 0, 1, \ldots\}$$

is closed by assumption $(A.5)$ and the upper semicontinuity of U; hence also Ψ is upper hemicontinuous.

(ii) This claim now follows from the upper semicontinuity of U on the compact set $\times_{i=0}^{\infty} Q^i(s, G(s))$.

(iii) For any $s \in S$, let K be a compact neighbourhood of s, so that U is upper hemicontinuous on $\times_{i=0}^{\infty} Q^i(K, G(K))$ by lemma 2.11. The claim now follows from the Maximum theorem. ∎

2.2.1 Discussion of the assumptions

Assumptions $(A.1)$ to $(A.4)$ are quite standard. $(A.4)$ requires a more detailed analysis. Rather than K-convergence, Streufert [14] uses a weaker concept, *upper convergence*, which requires the equality in 2.6 to hold when K is a singleton. On the other hand to prove the upper semicontinuity of the value function for the unconstrained problem he has to assume (see his theorem C, page 69, in [14]) the existence of a pair (J^-, J^+) of super and sub solutions of the Bellman operator; *i.e.* two upper semicontinuous functions such that $J^- \leq BJ^-$, $J^+ \geq BJ^+$. So an alternative approach to obtain the upper semicontinuity of the value function for the constrained problem is the following: assume biconvergence, rather than K-biconvergence, *and* the existence of two functions (J^-, J^+) as above; conclude that the *unconstrained* value function is upper semicontinuous. Then the proof of theorem 3.6 below will give that the value function of the *constrained* problem is upper semicontinuous. Note in fact that the proof of the theorem does not require the K-biconvergence assumption, but simply biconvergence.

The concepts of upper and lower convergence impose a joint restriction on preferences and technology. See Streufert [14] for a discussion of the notions of upper and lower convergence, and in particular of the relation between these concepts and the purely ordinal, more traditional ideas of impatience, or tail insensitivity . Biconvergence and impatience turn out to equivalent, in the sense made precise by his Propositions 1 and 2, page 63.

Like biconvergence, K-biconvergence is relatively easy to check in applications. Let us mention one example. Consider a growth problem with deterministic dynamics, and state space S the finite dimensional euclidean space. In this case the function

F is typically increasing in s; that is, if we understand the inequality $\geq$ to hold componentwise, $F(s,a) \geq F(u,a)$ if $s \geq u$, for every $a \in A$; also the correspondence G is typically monotonic in s (that is, $G(s) \supseteq G(u)$ if $s \geq u$.) To check that the biconvergence assumption holds one computes the utility on a maximal accumulation path starting at the initial state s. To check that K-biconvergence holds it suffices to choose an $u > s$, and make the same computation. An extremely simple instance of this procedure can be found in the discussion of the example in section 2.3 below.

In Benhabib and Rustichini [3] an example is presented that fits into the format of our general incentive constrained problem. In the example the utility function is concave, and the production function is also concave. Still the value functions J^* is discontinuous (although it is, as follows from the above theorem upper semicontinuous.)

An intuitive reason for this discontinuity can be given if we begin by looking at the solution of the first best problem, which is a standard optimal growth problem. In this case the optimal policy requires low consumption for low levels of capital stock, in order to accumulate and reach higher levels of the stock. On the first best path the accumulation brings the stock to a "high" steady state value. This policy may not be pursued in the second best problem, because the low consumption of one of the players makes defection very attractive for the other, since there is a comparatively large amount of capital stock left available for his consumption. To make the value of defection lower than the continuation value it is therefore necessary to increase consumption. Note that in this way both the value of defection and the continuation value are decreased, since the savings are reduced compared to the optimal solution. So when the initial capital stock is very low this reduction may be so large that the capital stock in the next period is lower, rather than higher. But at that point the continuation value itself is too low, since the future benefits from accumulation are gone. Hence the fast consumption becomes the only equilibrium. The switching point between accumulation and fast consumption is the discontinuity point.

2.3   A minimum utility growth model

It is now time to present how our techniques work. To do this we introduce an example which is somewhat artificial, but also extremely simple. This should allow us to focus on the method we introduce.

Take the log-utility, linear-production version of the classical optimal growth model; i.e. let $W((s,a),u) \equiv \log a + \beta u$, $G(s) \equiv [0, \alpha s]$, $F(s,a) \equiv \alpha s - a$, and $D(x) \equiv 0$. We may think that the agent has the option always available of migrating out of the economy, and getting a non negative lifetime utility.

We assume $\alpha \geq 1$, and $\beta \in (0,1)$. Assumptions $(A.1),(A.2),(A.3)$ and $(A.5)$ are obvious. For the assumption $(A.5)$, note that for any s and any $u \geq s$, $Q^i(u, G(u)) = [0, \alpha^i u] \times [0, \alpha^{i+1} u]$ for every i; and since the series $\sum_{i=0}^{\infty} \beta^i \log(\alpha^i u)$ converges, K-biconvergence at s follows from Lebesgue dominated convergence theorem; so $(A.5)$ is satisfied for any value of α, β. Also note that the function D is a constant; still as we shall see the problem of determining the value function is non trivial.

9

The value function of the unconstrained problem is $J_0(s) = (1 - \beta)^{-1} \log s + E$, where E a constant; and the corresponding optimal stationary consumption policy is $\hat{a}(s) = \alpha(1 - \beta)s$. The capital stock in the optimal sequence is $s_t = (\alpha\beta)^t s_0$ for every t. So if

$$\alpha\beta < 1$$

the incentive constraint is eventually violated in the unconstrained optimal path. From the corollary 3.9 in the sequel we know therefore that the constrained value function is equal to $-\infty$ on some subset of the domain. In fact, this subset is a interval, since the value function is increasing. To determine the value function exactly we now introduce our basic operator. Define for any upper semicontinuous function J:

$$(TJ)(s) \equiv \max_{a \in [0, \alpha s]} \log a + \beta J(\alpha s - a) \qquad (2.8)$$

subject to $J(s) \geq 0$;

We set $(TJ)(s) \equiv -\infty$ if the constraint is not satisfied. The constraint in the problem 2.8 may seem strange, because it is satisfied or not independently of the choice of the control variable. Its role in this example will probably be clarified by the corollary 3.9 in the following. Note that T is well defined on the cone of upper semicontinuous functions on the real line, and maps this cone into itself.

Apply this operator to the function J_0, and iterate. We obtain a sequence of functions which is monotonically decreasing. Some easy calculation shows that this sequence of functions converges to a limit function J^* which is equal to $-\infty$ for values of $s < (\alpha - 1)^{-1}$. The interval $[(\alpha - 1)^{-1}, +\infty)$ can be written as a disjoint union of intervals I_n, and on each interval the function is logarithmic, of the form:

$$J^*(s) = \frac{1 - \beta^n}{1 - \beta} \log(\alpha^n s + \frac{\alpha^n - 1}{\alpha - 1}) + D_n;$$

(the constants D_n are more complicated and not important for our purposes.) Our general result below (theorem 3.6) shows that this is in fact the value function for the constrained problem. This function is concave where it is finite valued. The optimal policy is stationary, and is determined as usual as the solution of the maximisation problem 2.8. The optimal sequence of capital stock s_t decreases to the steady state value $(\alpha - 1)^{-1}$.

The example we have just seen contains the main ingredients. The operator T in the general case will be the natural generalisation of the one we have seen, and the value function of the problem will be determined as the decreasing limit of iterations of T, as it is in the example.

3   The constrained dynamic programming.

In this section we give a systematic treatment of the incentive constrained problem as a dynamic programming problem. We begin with the basic component, the operator

T. To avoid repetitions, we say here that all the results in this section are derived under the assumptions $(A.1)$ to $(A.5)$, and we shall not repeat this in the statement of the results, except in the statement of the main theorem 3.6.

3.1 The operator T.

We first describe the set of policies. The set of *plans* is the set of sequences $\pi = (\pi_0, \pi_1, \ldots)$ of functions, with π_t mapping S to A for $t \geq 0$. The *shift* of a plan is defined by $L(\pi_0, \pi_1, \ldots) \equiv (\pi_1, \pi_2, \ldots)$.

We define $s_0^\pi \equiv s$, $a_0^\pi \equiv \pi_0(s)$, and for every $t \geq 0$, $s_{t+1}^\pi \equiv F(s_t^\pi, a_t^\pi)$, $a_t^\pi \equiv \pi_t(s_0^\pi, a_0^\pi, \ldots, s_t^\pi)$. If $a_t^\pi \in G(s_t^\pi)$ for every $t \geq 0$ the plan is called a *policy*, so a policy is a feasible plan; the set of policies is denoted by Π. Let $x_t^\pi \equiv (s_t^\pi, a_t^\pi)$; every policy defines a sequence of state and action pairs $(x_0^\pi, x_1^\pi, \ldots)$. Also for every pair (s, π) with $\pi \in \Pi$ we can define a function $J^\pi(s)$ by:

$$J^\pi(s) = \begin{cases} U(x_0^\pi, x_1^\pi, \ldots) & \text{if } U(x_t^\pi, x_{t+1}^\pi, \ldots) \geq D(x_t), t = 0, 1, \ldots; \\ -\infty & \text{otherwise} \end{cases}$$

Note that for every π and s:

$$J^\pi(s) = W((s, \pi_0(s)), J^{L\pi}(F(s, \pi_0(s)))). \tag{3.9}$$

Clearly,

$$J^*(s) = \sup_{\pi \in \Pi} J^\pi(s). \tag{3.10}$$

Now we turn to the dynamic programming formulation of the problem. Let $C^1(S)$ denote the cone of upper semicontinuous functions from S to $\mathbf{R}^*$. We define an operator T on this space as follows: for any $J \in C^1(S)$

$$(TJ)(s) \equiv \sup_{a \in G(s)} W((s, a), J(F(s, a))) \tag{3.11}$$

$$\text{subject to } J(s) \geq D(s, a).$$

If there is no feasible a that satisfies the incentive constraint at s then we set

$$(TJ)(s) = -\infty. \tag{3.12}$$

For convenience we introduce the correspondence A from $C^1(S) \times S$ into subsets of the action set, defined for any pair (J, s) by

$$A(J, s) \equiv \{a \in G(s) : J(s) \geq D(s, a)\}. \tag{3.13}$$

Note that if

$$J \geq J'$$

11

then

$$A(J,s) \supseteq A(J',s) \text{ for every } s.$$

With this notation, the operator T can be defined as

$$(TJ)(s) = \sup_{a \in A(J,s)} W((s,a), J(F(s,a))),$$

with the convention that $\sup_{a \in \emptyset} W((s,a), J(F(s,a))) = -\infty$. This formulation should bring out clearly the similarity with the Bellman operator, as much as two crucial differences.

The first difference is the condition 3.12. This condition implies that even in well behaved cases (for instance, smooth and convex preferences and technology) the operator may not map continuous functions into continuous functions.

The second difference is already observed in Benhabib and Rustichini [3]: even in the discounted case, the operator T is not a contraction. It is easy to see why the standard proof of this property for unconstrained dynamic programming problems (see e.g. Stokey and Lucas, [12]) fails. This proof, based on Blackwell's theorem, requires that for any function J and constant c, $B(J + c) \leq BJ + \beta c$. This property does not hold in the case of the operator T because the function J appears in the constraint set.

To simplify the exposition and avoid the need to recall each time the *proviso* that the set $A(J,s)$ is non empty, we first extend the definition of W by setting: $W((s, \emptyset), y) = -\infty$ for every s and y; then to every upper semicontinuous function J we associate a function $\mu(J, \cdot)$ defined by:

$$\mu(J,s) \in \text{argmax}_{\{a \in A(J,s)\}} W((s,a), J(F(s,a))), \text{ if } A(J,s) \neq \emptyset; \qquad (3.14)$$

$$\mu(J,s) \equiv \{\emptyset\}, \text{ if } A(J,s) = \emptyset. \qquad (3.15)$$

A *fixed point* of the operator T is any upper semicontinuous function J such that $TJ = J$; it may be described by the equation

$$J(s) = W((s, \mu(J,s)), J(F(s, \mu(J,s)))) \text{ for every } s. \qquad (3.16)$$

The Bellman operator B is defined, as usual, by

$$(BJ)(s) \equiv \max_{a \in G(s)} W((s,a), J(F(s,a))). \qquad (3.17)$$

For any $J \in C^1(S)$, it is clear that

$$BJ \geq TJ.$$

For any $k = 0, 1, \ldots$ we let T^k denote the k-th iterate of T, and T^0 is the identity. In the next proposition we collect some obvious and useful facts.

Proposition 3.1   *i. The supremum in the definition 3.11, 3.12 of T is achieved;*

ii. $T : C^1(S) \rightarrow C^1(S)$;

iii. If $J \geq J'$ then $TJ \geq TJ'$;

iv. If $J \leq TJ$ then $T^k J \leq T^{k+1} J$ for every k.

We can apply now Theorem B, page 67, of Streufert [14]. This theorem imposes our assumptions $(A.1)$ to $(A.4)$ plus the conclusion of the lemma 2.11 to conclude that J_0 is an admissible solution of the Bellman's equation, *i.e.*:

$$J_0 = BJ_0.$$

Since for every J,

$$BJ \geq TJ, \tag{3.18}$$

we have that

$$J_0 \geq TJ_0. \tag{3.19}$$

Also it follows from the lemma 2.12 in the case $D \equiv -\infty$ that $J_0 \in C^1(S)$. Hence the sequence of functions

$$J_k = T^k J_0, k = 0, 1, \ldots$$

is a decreasing sequence in $C^1(S)$, and therefore the function J_∞, defined by

Definition 3.2 $J_\infty(s) \equiv \lim_{k \to \infty} J_k(s)$,

is well defined, and is an upper semicontinuous function because the limit of a decreasing sequence of upper semicontinuous functions.

We state this observations formally, together with the property that J_∞ is a fixed point of T, in the following theorem.

Theorem 3.3 *Let J_∞ be defined as in 3.2 above. Then*

i. J_∞ is an upper semicontinuous function, $J_\infty \leq J_0$;

ii. $TJ_\infty = J_\infty$.

Proof. The point *i* follows from the definition of J_∞, and the properties 3.19 and 3.18 above; point *ii* is proved in the two next lemmata.

Lemma 3.4 *Let J_∞ be defined as in the definition 3.2 above. Then $TJ_\infty \leq J_\infty$.*

13

Proof.

$$
\begin{aligned}
(TJ_\infty)(s) &= \max_{a \in A(J_\infty, s)} W((s,a), J_\infty(F(s,a))) \\
&\leq \max_{a \in A(T^k J_0, s)} W((s,a), (T^k J_0)(F(s,a))).
\end{aligned} \tag{3.20}
$$

The first equality is the fixed point property of J_∞. For the second, note first that (i) $T^k J_0 \geq J_\infty$, which also implies (ii) $A(T^k J_0, s) \supseteq A(J_\infty, s)$. (i) and (ii) together give the second inequality. But the term in 3.20 is $(T^{k+1} J_0)(s)$; we have therefore proved that $TJ_\infty \leq T^k J_0$ for every $k \geq 1$, and therefore $TJ_\infty \leq J_\infty$ as claimed. ∎

Lemma 3.5 *Let J_∞ be defined as in 3.2 above. Then $TJ_\infty \geq J_\infty$.*

Proof. If $J_\infty(s) = -\infty$ the claimed inequality is obvious; so in the rest of the proof we assume that $J_\infty(s) > -\infty$. Since $J_\infty(s)$ is the limit of the decreasing sequence $\{J_k(s)\}_{k=1}^\infty$, we have $J_k(s) > -\infty$ for every k. Now take, for every $k = 1, 2, \ldots$, an element a_k in the set of maximisers in the problem defining $J_k(s)$:

$$
\max_{a \in G(s)} W((s,a), J_{k-1}(F(s,a)))
$$

$$
\text{subject to } J_{k-1}(s) \geq D(s,a).
$$

Since $G(s)$ is a compact set by assumption $(A.1)$, there exists a subsequence $\phi : N \to N$ such that $a_{\phi(k)}$ converges to some $\hat{a} \in G(s)$ as k tends to ∞. Then (see Streufert, [14])

$$
\begin{aligned}
J_\infty(F(s,\hat{a})) &= \lim_k J_{\phi(k)}(F(s,\hat{a})) \\
&\geq \lim_k \limsup_{i \geq k} J_{\phi(k)}(F(s, a_{\phi(i)})) \\
&\geq \lim_k \limsup_{i \geq k} J_{\phi(i)}(F(s, a_{\phi(i)})) \\
&\geq \limsup_i J_{\phi(i)}(F(s, a_{\phi(i)})).
\end{aligned} \tag{3.21}
$$

The first equality follows from the definition of J_∞; the second inequality from the upper semicontinuity of $J_{\phi(k)}$, the third from the inequality $J_{\phi(i)} \leq J_{\phi(k)}$ for $i \geq k$, and the fourth follows because the index k is not appearing in the limit. We conclude:

$$
J_\infty(F(s,\hat{a})) \geq \limsup_i J_{\phi(i)}(F(s, a_{\phi(i)})), \tag{3.22}
$$

while

$$
-D(s,\hat{a}) \geq \limsup_i -D(s, a_{\phi(i)}) \tag{3.23}
$$

14

is immediate from the lower semicontinuity of $D(s,\cdot)$ (assumption $(A.5)$). Adding 3.23, 3.22 and using a basic property of the lim sup we get:

$$J_\infty(F(s,\hat{a})) - D(s,\hat{a}) \tag{3.24}$$
$$\geq \lim_i \sup[J_{\phi(i)}(F(s,a_{\phi(i)})) - D(s,a_{\phi(i)})].$$

But by definition of $a_{\phi(i)}$,

$$J_{\phi(i)}(F(s,a_{\phi(i)})) - D(s,a_{\phi(i)}) \geq 0 \tag{3.25}$$

for every i, and so from 3.24 and 3.25

$$J_\infty(F(s,\hat{a})) - D(s,\hat{a}) \geq 0. \tag{3.26}$$

Now

$$
\begin{aligned}
(TJ_\infty)(s) &= \max_{a \in G(s)} W((s,a), J_\infty(F(s,a))) \\
&\qquad \text{subject to } J_\infty(F(s,a)) \geq D(s,a) \\
&\geq W((s,\hat{a}), J_\infty(F(s,\hat{a}))) \\
&\geq \lim_{i \to \infty} \sup W((s,a_{\phi(i)}), J_{\phi(i)}(F(s,a_{\phi(i)}))) \\
&= \lim_{i \to \infty} \sup J_{\phi(i)+1}(s) \\
&= J_\infty(s).
\end{aligned}
$$

The first equality is again the fixed point property of J_∞; the second follows because $\hat{a} \in G(s)$, and 3.26; the third inequality is derived with the same argument used to prove the chain of inequalities 3.21, the last two are the definitions of J_k and J_∞ respectively. This proves our claim. ∎

We can now state and prove the theorem that characterizes the value function J^*:

Theorem 3.6 *Assume $(A.1)$ to $(A.5)$; then*
1. $J_\infty = J^*$;
2. $J^* = TJ^*$;
3. *If $J' \in C^1(S)$ is such that $J' = TJ'$ then $J' \leq J^*$.*

Proof. Claim 2 follows from 1 and the fixed point property of J_∞. Claim 1 is proved in the next two lemmata. Claim 3 is proved last.

Lemma 3.7 *Assume $(A.1)$ to $(A.5)$; then*

$$J_\infty \geq J^*.$$

Proof. It is enough to prove that

$$J_\infty(s) \geq J^\pi(s)$$

for every policy π and every $s \in S$. In turn to show this, from the definition 3.2 of J_∞ it suffices to prove that

$$(T^k J_0)(s) \geq J^\pi(s) \tag{3.27}$$

for every $k = 0, 1, \ldots$ The proof of this last claim is by induction on k. For $k = 0$,

$$(T^0 J_0)(s) = J_0(s) \geq J^\pi(s).$$

because T^0 is the identity operator, and the policy π satisfies the feasibility constraint. Assume now 3.27 for $k = n-1$, we claim 3.27 holds for $k = n$. If $J^\pi(s) = -\infty$ the claimed inequality is obvious, so in the rest of the proof we assume $J^\pi(s) > -\infty$. Recall that $\pi_0(s)$ denotes the first period choice of action according to the policy π; we have:

$$
\begin{aligned}
(T^n J_0)(s) &= T(T^{n-1} J_0)(s) \\
&= T(J_{n-1})(s) \\
&= \max_{a \in G(s)} W((s,a), J_{n-1}(F(s,a))) \\
&\qquad \text{subject to } J_{n-1}(F(s,a)) \geq D(s,a) \\
&\geq \max_{a \in G(s)} W((s,a), J_{L\pi}(F(s,a))) \\
&\qquad \text{subject to } J_\pi(F(s,a)) \geq D(s,a) \\
&\geq J^\pi(s);
\end{aligned}
\tag{3.28}
$$

where the first equality follows from the definition of T^n, the second from the definition of J_{n-1}, the third from the definition of T, the fourth from the induction hypothesis, and the last from the basic relation 3.9. ∎

Lemma 3.8 *Assume (A.1) to (A.5); then*

$$J^* \geq J_\infty.$$

Proof. Recall that, with $\nu(s) \equiv \mu(J_\infty, s)$ for every s,

$$J_\infty(s) = W((s, \nu(s)), J_\infty(F(s, \nu(s)))). \tag{3.29}$$

If $J_\infty(s) = -\infty$ the claimed inequality is obvious, so in the rest of the proof we assume $J_\infty(s) > -\infty$. Define the sequences

$$s_0 \equiv s, a_i \equiv \nu(s_i), s_{i+1} = F(s_i, a_i), x_i \equiv (s_i, a_i), i = 0, 1, \ldots \tag{3.30}$$

Iteration of 3.29 gives, for every $k = 0, 1, \ldots$

$$J_\infty(s_k) = W(x_0, W(x_1, \ldots, W(x_{k-1}, J_\infty(s_k)))). \tag{3.31}$$

Now $J_\infty(s) > -\infty$, 3.31 and assumption $(A.2)$ imply

$$J_\infty(s_k) > -\infty, \text{ for } k = 0, 1, \ldots \tag{3.32}$$

Since J_∞ is a fixed point of T, from the definition 3.30 and 3.32 we have

$$J_\infty(s_k) \geq D(x_k), \text{ for } k = 0, 1, \ldots \tag{3.33}$$

From $J_0 \geq J_\infty$ follows that for every $s \in S$:

$$\sup U(\times_{i=0}^\infty Q^i(s, G(s))) \geq J_\infty(s). \tag{3.34}$$

Now we have:

$$
\begin{aligned}
& \sup U(x_k, x_{k+1}, \ldots, x_{k+m}, \times_{i=1}^\infty Q^i(s_{k+m}, G(s_{k+m}))) \\
= \ & \sup W(x_k, W(x_{k+1}, \ldots, W(x_{k+m}, U(\times_{i=1}^\infty Q^i(s_{k+m}, G(s_{k+m})))))) \\
\geq \ & W(x_k, W(x_{k+1}, \ldots W(x_{k+m-1}, J_\infty(s_{k+m})))) \\
= \ & J_\infty(s_k) \\
\geq \ & D(x_k).
\end{aligned}
\tag{3.35}
$$

The first equality is from the basic relation 2.2 linking U and W; the second from 3.34; the third from 3.31; the fourth from 3.33. By upper convergence the above inequality implies that for every $k = 0, 1, \ldots$

$$U(x_k, x_{k+1}, \ldots) \geq J_\infty(s_k) \geq D(x_k);$$

The above inequality implies (i) that the sequence $\{x_k\}_{k=0}^\infty$ is in the set of sequences that satisfy the incentive constraints in the maximisation problem that defines $J^*(s)$; and (ii) that $U(x_0, x_1, \ldots) \geq J_\infty(s)$. The two conclusions (i) and (ii) give:

$$J^*(s) \geq J_\infty(s).$$

This concludes the proof of the lemma. ∎

To complete the proof of the theorem we turn to Claim 3. Let J' be a fixed point of T. Since $B^k J' \geq T^k J' = J'$ for every k, and $B^k J'$ converges to a limit which is less than or equal to J_0, we have $J_0 \geq J'$. Now the inequalities $T^k J_0 \geq T^k J'$ for every k imply that $J^* \geq J'$ as claimed. This concludes the proof of the theorem. ∎

We conclude with a corollary of the theorem. It shows that when the constraint function D only depends on the state variable, the problem is equivalent to the unconstrained problem if J^* is everywhere finite:

17

Corollary 3.9 *When the function D depends only on the state variable s, if $J^*(s) > -\infty$ for every $s \in S$, then $J^* = J_0$.*

Proof. As we know,

$$J_0 \geq J^* \tag{3.36}$$

holds in general. Also if $J_0(s) \geq D(s)$ for every s, then the incentive constraint is satisfied by the optimal unconstrained policy, and therefore $J^* = J_0$. Assume that this last equality, which is the conclusion of the corollary, does not hold. From our second remark we now conclude that for some s', $J_0(s') < D(s')$. But then from 3.36 $J^*(s') < D(s')$ too, and therefore $J^*(s') = -\infty$ from the definition of the operator T. ∎

4 Finite Horizon Approximations

The specific problems of the class discussed in this paper may be difficult to analyse explicitly in analytical terms: so a numerical simulation of the solution may be in some cases a useful tool. A natural way to set the problem numerically is of course the direct computation of the value function by successive iterations, as we have seen in the previous sections. A second way is to compute finite horizon approximations. In this section we study conditions under which appropriately defined approximate solutions of finite-horizon truncations are a good approximation of the infinite horizon problem. First some basic notations and definitions are introduced.

X is now assumed to be a separable metric space. (When $S \equiv \Delta(\Omega)$ as in the section 2.1, this condition follows if Ω is separable metric.) For any element $x \in X$ we define the *constant path* $_c x \in X^\infty$ as

$$(_c x)_t = x \text{ for every } t.$$

The function $v : X \to R^*$ is defined by applying the utility function U to constant paths:

$$v(x) \equiv U(_c x) \tag{4.37}$$

so that clearly

$$v(x) = W(x, v(x)).$$

We begin with the case $D \equiv -\infty$, that is the unconstrained case. This is a useful introduction, and will perhaps also be helpful to clarify the difference between the two cases.

Let X^T be the T times product of X. The *finite horizon approximation* aggregator $W^T : X^{T+1} \to R^*$ is

$$W^T(x_0, x_1, \ldots, x_T) \equiv W(x_0, W(x_1, \ldots W(x_{T-1}, v(x_T)) \ldots)). \tag{4.38}$$

18

Clearly for every T and every $(x_0, \ldots, x_T) \in X^{T+1}$

$$U^T(\mathbf{x}) \equiv U(x_0, \ldots, x_{T-1}, (x_T)) = W^T(x_0, x_1, \ldots, x_T). \tag{4.39}$$

An *finite horizon approximation* (with horizon T) to the unconstrained problem is denoted by $P^T(s)$ and is defined as the problem of finding:

$$\max W^T(x_0, \ldots, x_T), \tag{4.40}$$

subject to:

$$x_0 \in (s, G(s)), x_{t+1} \in Q(x_t) \text{ for every } t = 1, \ldots, T - 1.$$

The *optimal solution* of the T approximation is denoted by $\hat{\mathbf{x}}^T$, and the value at s by $J^{T,*}(s)$. To be useful for numerical purposes our approximation cannot require exact solutions of the truncated problems; so we introduce the concept of an ϵ-optimal solution. This is a feasible path that is not more than ϵ distant in value from the true optimal (of the truncated problem), that is a feasible path $\hat{\mathbf{y}}^T$ which satisfies

$$J^{T,*}(s) \leq W^T(\hat{y}_0, \ldots, \hat{y}_T) + \epsilon. \tag{4.41}$$

The set of ϵ-approximate solutions is denoted by ϵ-argmax$P^T(s)$.

The sequence of approximate solutions is now determined in the natural way. Find approximate solutions to the finite dimensional problem $P^T(s)$, extend the solution to an element in X^∞ setting the coordinates for $t \geq T$ equal to the T component. Now make the horizon grow, and at the same time improve the approximation. For large T, we obtain in this way a value arbitrarily close to the value of the infinite horizon problem, and a sequence of solutions that are close to an optimal solution of the infinite horizon problem. More formally we have:

Theorem 4.1 *Assume $(A.1)$ to $(A.5)$, and let $\{\epsilon^T, \mathbf{x}^T\}_{T=0}^\infty, \mathbf{x}^T \in X^\infty$ for every T be a sequence such that:*
1) $\lim_T \epsilon^T = 0$,
2) $x_t^T = x_T^T$ for every $t \geq T$, and $(x_0^T, \ldots, x_T^T) \in \epsilon^T$-argmax$P^T(s)$,
then any cluster point $\mathbf{x}$ of the sequence $\{\mathbf{x}^T\}_{T=0}^\infty$ is a solution of $P(s)$.

Proof. Let x^T be a sequence of ϵ^T-approximate solutions of $P^T(s)$, and let $\mathbf{x}^{T_j, T_j} \equiv \mathbf{x}^j$ be a subsequence converging to $\mathbf{x}$. Then

$$\begin{aligned}
U(\mathbf{x}) &\geq \limsup_j U^{T_j}(\mathbf{x}^j) \\
&\geq \limsup_j [J^{T_j,*}(s) - \epsilon^{T_j}] \\
&= \limsup_j J^{T_j,*}(s). \tag{4.42}
\end{aligned}$$

19

Let $\hat{\mathbf{x}}$ be a solution of the infinite horizon problem, so that:

$$J^*(s) = U(\hat{\mathbf{x}}). \tag{4.43}$$

Then

$$J^{T,*}(s) \geq U^T(\hat{\mathbf{x}}) \tag{4.44}$$

because $\hat{x}_{t+1} \in Q(\hat{x}_t)$ for every t. Since U is upper semicontinuous

$$\limsup_T U^T(\hat{\mathbf{x}}) \leq J^*(s); \tag{4.45}$$

while

$$\liminf_T U(\hat{x}_0, \ldots, \hat{x}_T, \times_{i=T+1}^\infty Q^i(s, G(s))) = J^*(s) \tag{4.46}$$

by lower convergence, so that:

$$\lim_{T_j} U^{T_j}(\hat{\mathbf{x}}) = J^*(s), \tag{4.47}$$

We conclude from 4.45, 4.46:

$$\limsup_j J^{T_j,*}(s) \geq J^*(s), \tag{4.48}$$

so the conclusion follows from the combination of 4.42 and 4.48. ■

For the general case (where D is finite) a more subtle argument is needed; in particular it is now necessary to construct a sequence of approximations of the utility function which are monotonically decreasing to U.

More precisely, let the function V be any function from X to $\mathbf{R}^*$ that satisfies:

$$V(x_0) \geq U(x_0, x_1, \ldots) \text{ for every } (x_1, \ldots) \in \Psi(F(s, a_0)). \tag{4.49}$$

Using the unconstrained value function one can find a natural candidate for the function V; but more obvious estimates may be available (for instance, if U is bounded by a constant M, then this constant is an admissible V). Also let:

$$U^T(x_0, \ldots, x_T) \equiv W(x_0, W(x_1, \ldots, W(x_{T-1}, V(x_T))). \tag{4.50}$$

For the given V, the *finite horizon approximation*, also denoted $CP^T(s)$, is defined by:

$$\max W(x_0, (W(x_1, \ldots W(x_{T-1}, V(x_T)))))), \tag{4.51}$$

subject to:

$$(i) \ x_0 \in (s, G(s)), x_{t+1} \in Q(x_t) \text{ for every } t = 1, \ldots, T-1,$$

and

$$(ii) \ W(x_t, (W(x_{t+1}, \ldots W(x_{T-1}, V(x_T))))) \geq D(x_t), t = 0, 1, \ldots, T-1.$$

20

Optimal solutions and approximate solutions are defined and denoted as for the case of the $P^T(s)$ problem. In this subsection we shall need the definition of epi-convergence of functions. Actually, since our problem is a maximization problem, we present the hypo-convergence version of the theory. For a more complete discussion the reader is referred to Attouch and Wets [2] or Attouch [1].

Definition 4.2 *A sequence of functions $\{U^T\}_{T=0}^\infty$ defined on a metric space X and with values in the extended reals hypo-converges to U at x if*
(i) For every subsequence of functions $\{U^{T_j}\}_{j=0}^\infty$ and sequence of points $\{x^j\}_{j=0}^\infty$ converging to x , we have that

$$\limsup_j U^{T_j}(x^j) \leq U(x);$$

and
(ii) there exists a sequence $\{x^T\}_{T=0}^\infty$ converging to x such that

$$\liminf_T U^T(x^T) \geq U(x).$$

With the new definition of finite approximation problem $CP^T(s)$, theorem 4.1 holds with virtually no changes. For completeness, the statement is:

Theorem 4.3 *Assume $(A.1)$ to $(A.5)$, and let $\{\epsilon^T, \mathbf{x}^T\}_{T=0}^\infty, \mathbf{x}^T \in X^\infty$ for every T be a sequence such that:*
1) $\lim_T \epsilon^T = 0$,
2) $x_t^T = x_T^T$ for every $t \geq T$, and $(x_0^T, \ldots, x_T^T) \in \epsilon^T\text{-}argmaxCP^T(s)$,
then any cluster point $\mathbf{x}$ of the sequence $\{\mathbf{x}^T\}_{T=0}^\infty$ is a solution of $CP(s)$.

Proof. For U^T defined by 4.50 we have

$$U^T \geq U^{T+1} \geq U \quad \text{for every } T; \tag{4.52}$$

so that for every T:

$$J^{T,*}(s) \geq J^*(s).$$

Also, by assumptions $(A.4)$ (in particular, from upper convergence), for every $\mathbf{x}$,

$$\lim_T U^T(\mathbf{x}) = U(\mathbf{x})$$

By the monotonicity property 4.52, the sequence of functions is equi-upper semicontinuous (in the product topology); hence by corollary 2.19, in Dolecki *et alii* [6]

$$U^T \text{ hypo converges to } U$$

We now prove that $U(\mathbf{x}) \geq J^*(s)$. Let $\{\mathbf{x}^{T_j}\}_{j=0}^\infty$ be a subsequence converging to $\mathbf{x}$. Then

21

$$U(\mathbf{x}) \geq \limsup_{j} U^{T_j}(\mathbf{x}^{T_j}) \qquad (4.53)$$

$$\geq \liminf_{j} U^{T_j}(\mathbf{x}^{T_j}) \qquad (4.54)$$

$$\geq \liminf_{T} U^{T}(\mathbf{x}^{T}) \qquad (4.55)$$

$$\geq \liminf_{T}[J^{T,*}(s) - c^{T}] \qquad (4.56)$$

$$\geq \liminf_{T} J^{T,*}(s) \qquad (4.57)$$

$$\geq J^{*}(s). \qquad (4.58)$$

The first inequality follows from the hypoconvergence of U^T, and the last by the monotonicity of the same sequence; the others are obvious.

Finally, note that the cluster point $\mathbf{x}$ is incentive compatible, that is:

$$U(_t\mathbf{x}) \geq D(x_t) \quad t = 0, 1, \ldots$$

We conclude that $\mathbf{x}$ is a solution of $CP(s)$ as claimed. ∎

5 Conclusions

Examples of applications of the general problem discussed in this paper are appearing in the literature quite frequently in the recent past particularly in the area of dynamic games and application to macroeconomics and policy issues. Examples have been provided in this paper: but see also Marimon and Marcet [11]. A very natural application is the characterization of second best equilibria: see the examples described in section 1.1. We hope to have contributed here in clarifying a few points of a technical and conceptual nature concerning these problems.

The first is that incentive constrained problems may have a natural formulation, and a natural solution, in the general framework of dynamic programming, while keeping unchanged the original state space. This is particularly clear once it is realized that the appropriate space for the value function in this case is the space of upper semicontinuous functions. Of course this space has already been used extensively for this purpose: but in the case of incentive constrained problems it is made necessary, rather than by the lack of regularity of the problem (like, for instance, an upper semicontinuous rather than continuous utility, or production) by the constraint itself.

The formulation of the value function for the problem as the solution of a fixed point should be useful in the study of analytical properties of the optimal path. Of course, one important property provided by the contraction mapping approach is lost: the uniqueness of the solution. For instance, the function which is identically equal to minus infinity is trivially always a fixed point.

The last point we want to emphasize is that the solution to incentive constrained problems is robust: in particular it is robust to numerical approximations. Consider

22

for instance the procedure, quite common in the treatment of policy problems by numerical methods, of approximating the infinite horizon problem by a finite horizon truncation. In these procedures it is usually assumed that, if the finite horizon is far enough in the future, then the solution to the finite problem is a good approximation to the true optimal solution. This assumption is legitimate in view of our theorem 4.3.

References

[1] Attouch, H. (1984), *Variational Convergence for Functions and Operators*, London, Pitman Research Notes in Mathematics.

[2] Attouch, H. and R. Wets (1983), "A Convergence Theory for Sadde Functions", *Transactions of the American Mathematical Society*, 280, 1-41.

[3] Benhabib, J. and A. Rustichini (1996), "Social Conflict and Growth", *Journal of Economic Growth*, vol. 1.

[4] Benhabib, J. and A. Rustichini (1996), "Best Sustainable Taxes in a Two Factors Model." CV Starr Center Discussion Paper, New York University.

[5] Chamley, C. (1986), "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives", *Econometrica*, 54, 3, 607-622.

[6] Dolecki, S.; G. Salinetti, and R. Wets, (1983) "Convergence of Functions: Equi-Semicontinuity" *Transactions of the American Mathematical Society*, 276, 1, 409-429.

[7] Fläm, S. and R. Wets (1987), "Existence Results and Finite Horizon Approximates for Infinite Horizon Optimisation Problems", *Econometrica*, 55, 5, 1187-1190.

[8] Koopmans, T.K., (1960), "Stationary Ordinary Utility and Impatience", *Econometrica*, 28, 287-309; now in *Scientific Papers of Tjalling K. Koopmans*, edited by Beckmann M., Christ C. C., Nerlove, M.; editor Springer Verlag, 1970, pages 386-409.

[9] Koopmans, T.K., P. Diamond, R. E. Williamson, (1964) "Stationary Utility and Time Perspective", *Econometrica*, 32, 82-100; now in *Scientific Papers of Tjalling K. Koopmans*, edited by Beckmann M., Christ C. C., Nerlove, M., Springer Verlag, 1970, pages 410-428.

[10] Marcet A. and R. Marimon, (1994), "Communication, Committment and Growth", *Journal of Economic Theory*, 58, 219-249.

[11] Marcet A. and R. Marimon, (1994), "Recursive Contracts", Universitat Pompeu Fabra Discussion Paper.

[12] Stokey, N. and R. E. Jr. Lucas, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.

[13] Streufert, P.A. (1990), "Stationary Recursive Utility and Dynamic Programming under the Assumption of Biconvergence", *Review of Economic Studies*, 57, 79-97.

[14] Streufert, P.A. (1992), "An Abstract Topological Approach to Dynamic Programming", *Journal of Mathematical Economics*, 21, 59-88.

No.	Author(s)	Title
9616	G. Asheim and M. Dufwenberg	Admissibility and Common Knowlegde
9617	C. Fernández, J. Osiewalski and M. Steel	On the Use of Panel Data in Bayesian Stochastic Frontier Models
9618	H. Huizinga	Intrafirm Information Transfers and Wages
9619	B. Melenberg and B. Werker	On the Pricing of Options in Incomplete Markets
9620	F. Kleibergen	Reduced Rank Regression using Generalized Method of Moments Estimators
9621	F. Janssen, R. Heuts and T. de Kok	The Value of Information in an (R,s,Q) Inventory Model
9622	F. Groot, C. Withagen and A. de Zeeuw	Strong Time-Consistency in the Cartel-Versus-Fringe Model
9623	R. Euwals and A. van Soest	Desired and Actual Labour Supply of Unmarried Men and Women in the Netherlands
9624	M. Khanman, M. Perry and P.J. Reny	An Ex-Post Envy-Free and Efficient Allocation Mechanism: Imperfect Information Without Common Priors
9625	C. Eaves, G. van der Laan D. Talman and Z. Yang	Balanced Simplices on Polytopes
9626	H. Bloemen and A. Kalwij	Female Employment and Timing of Births Decisions: a Multiple State Transition Model
9627	P. Bolton	Strategic Experimentation: a Revision
9628	C. Fershtman and Y. Weiss	Social Rewards, Externalities and Stable Preferences
9629	P.M. Kort, G. Feichtinger R.F. Hartl and J.L. Haunschmied	Optimal Enforcement Policies (Crackdowns) on a Drug Market
9630	C. Fershtman and A. de Zeeuw	Tradeable Emission Permits in Oligopoly
9631	A. Cukierman	The Economics of Central Banking
9632	A. Cukierman	Targeting Monetary Aggregates and Inflation in Europe
9633	F. de Roon, C. Veld and J. Wei	A Study on the Efficiency of the Market for Dutch Long Term Call Options
9634	B. van Aarle, L. Bovenberg M. Raith	Money, Fiscal Deficits and Government Debt in a Monetary Union

No.	Author(s)	Title
9635	G. van der Laan, D. Talman and Z. Yang	Existence of and Equilibrium in a Competitive Economy with Indivisibilities and Money
9636	I. Brouwer, J. van der Put C. Veld	Contrarian Investment Strategies in a European Context
9637	M. Berg, F. van der Duyn Schouten and J. Jansen	Optimal Service Policies to Remote Customers with Delay-Limits
9638	F.C. Drost and C.A.J. Klaassen	Efficient Estimation in Semiparametric GARCH Models
9639	A.G. de Kok and F.B.S.L.P. Janssen	Demand Management in Multi-Stage Distribution Chain
9640	D.A. Freund, T.J. Kniesner A.T. LoSasso	How Managed Care Affects Medicaid Utilization A Synthetic Difference-in-Differences Zero-Inflated Count Model
9641	M.J. Lee	Instrumental Variable Estimation For Linear Panel Data Models
9642	W. Härdle, E. Mammen and M. Müller	Testing Parametric versus Semiparametric Modelling in Generalized Linear Models
9643	C. Dustmann, N. Rajah and A. van Soest	Part-Time Work, School Success and School Leaving
9644	S.C.W. Eijffinger and M. Hoeberichts	The Trade Off Between Central Bank Independence and Conservativeness
9645	R. Sarin and P. Wakker	A Single-Stage Approach to Anscombe and Aumann's Expected Utility
9646	J.P. Ziliak and T.J. Kniesner	The Importance of Sample Attrition in Life Cycle Labor Supply Estimation
9647	P.M. Kort	Optimal R&D Investments of the Firm
9648	M.P. Berg	Performance Comparisons for Maintained Items
9649	H. Uhlig and Y. Xu	Effort and the Cycle: Cyclical Implications of Efficiency Wages
9650	M. Slikker and A. van den Nouweland	Communication Situations with a Hierarchical Player Partition
9651	H.L.F. de Groot	The Struggle for Rents in a Schumpeterian Economy
9652	R.M. de Jong and J. Davidson	Consistency of Kernel Estimators of heteroscedastic and Autocorrelated Covariance Matrices

No.	Author(s)	Title
9653	J. Suijs, A. De Waegenaere and P. Borm	Stochastic Cooperative Games in Insurance and Reinsurance
9654	A.N. Banerjee and J.R. Magnus	Testing the Sensitivity of OLS when the Variance Matrix is (Partially) Unknown
9655	A. Kalwij	Estimating the Economic Return to Schooling on the basis of Panel Data
9656	M. Lind and F. van Megen	Order Based Cost Allocation Rules
9657	A. van Soest, P. Fontein and Rob Euwals	Earnings Capacity and Labour Market Participation
9658	C. Fernández and M.F.J. Steel	On Bayesian Modelling of Fat Tails and Skewness
9659	R. Sarin and P. Wakker	Revealed Likelihood and Knightian Uncertainty
9660	J.R. Magnus and J. Durbin	A Classical Problem in Linear Regression or How to Estimate the Mean of a Univariate Normal Distribution with Known Variance
9661	U. Gneezy and J. Potters	An Experiment on Risk Taking and Evaluation Periods
9662	H.J. Bierens	Nonparametric Nonlinear Co-Trending Analysis, with an Application to Interest and Inflation in the U.S.
9663	J.P.C. Blanc	Optimization of Periodic Polling Systems with Non-Preemptive, Time-Limited Service
9664	M.J. Lee	A Root-N Consistent Semiparametric Estimator for Fixed Effect Binary Response Panel Data
9665	C. Fernández, J. Osiewalski and M.F.J. Steel	Robust Bayesian Inference on Scale Parameters
9666	X. Han and H. Webers	A Comment on Shaked and Sutton's Model of Vertical Product Differentiation
9667	R. Kollmann	The Exchange Rate in a Dynamic-Optimizing Current Account Model with Nominal Rigidities: A Quantitative Investigation
9668	R.C.H. Cheng and J.P.C. Kleijnen	Improved Design of Queueing Simulation Experiments with Highly Heteroscedastic Responses
9669	E. van Heck and P.M.A. Ribbers	Economic Effects of Electronic Markets
9670	F.Y. Kumah	The Effect of Monetary Policy on Exchange Rates: How to Solve the Puzzles